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Chapter 3

Normality of spaces of operators and quasi-lattices

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3.1 Introduction

This paper's main aim is to investigate normality and monotonicity (defined in Section 3.3) of pre-ordered spaces of operators between pre-ordered Banach spaces. This investigation is motivated by the relevance of this notion in the theory of positive semigroups on pre-ordered Banach spaces [6], and in the positive representation theory of groups and pre-ordered algebras on pre-ordered Banach spaces [12].

If X and Y are Banach lattices an elementary calculation shows that the space $B(X,Y)$ is absolutely monotone, i.e., for $T,S \in B(X,Y)$, if $\pm T \leq S$, then $\|T\| \leq \|S\|$. If X and Y are general pre-ordered Banach spaces the situation is not so clear, and raises a number of questions: If $B(X,Y)$ is, e.g., absolutely monotone, does this necessarily imply that X and Y are Banach lattices? If not, what are examples of pre-ordered Banach spaces X and Y, not being Banach lattices, such that $B(X,Y)$ is absolutely monotone? What are the more general necessary and/or sufficient conditions X and Y have to satisfy for $B(X,Y)$ to be absolutely monotone? This paper will attempt to answer such questions through an investigation of the notions of normality and conormality of pre-ordered Banach spaces which describe various ways in which cones interact with norms.

A substantial part will devoted to introducing a class of ordered Banach spaces, called quasi-lattices, which will furnish us with many examples that are not necessarily Banach lattices. Quasi-lattices occur in two slightly different forms, one of which includes all Banach lattices (cf. Proposition 3.5.2). We give a brief sketch of their construction.

There are many pre-ordered Banach spaces with closed proper generating cones that are not normed Riesz spaces, e.g., the finite dimensional spaces $\mathbb{R}^n$ (with $n \geq 3$) endowed with Lorentz cones or endowed with polyhedral cones whose bases (in the sense of [2, Section 1.7]) are not $(n-1)$-simplexes. Although there is often an abundance of upper bounds of arbitrary pairs of elements, none of them is a least upper bound with respect to the ordering defined by the cone. An interesting situation arises when one takes the norm into account when studying the set of upper bounds of arbitrary pairs of elements. Even though there might not exist a least upper bound with respect to the ordering defined by the cone, there often exists a unique upper bound, called the *quasi-supremum*, which minimizes the sum of the distances from this upper bound to the given two elements. This allows us to define what will be called a *quasi-lattice structure* on certain ordered Banach spaces which might not be lattices (cf. Definition 3.5.1). Surprisingly, many elementary vector lattice properties for Riesz spaces carry over nearly verbatim to such spaces (cf. Theorem 3.5.8), and in the case that a space is a Banach lattice, its quasi-lattice structure and lattice structure actually coincide (cf. Proposition 3.5.2).

Quasi-lattices occur in relative abundance, in fact, every strictly convex reflexive ordered Banach space with a closed proper generating cone is a quasi-lattice (cf Theorem 3.6.1). This will be used to show that every Hilbert space $\mathcal{H}$ endowed with a Lorentz cone is a quasi-lattice (which is not a Banach lattice if $\dim(\mathcal{H}) \geq 3$). Such spaces will serve as examples of spaces, which are not Banach lattices, such that the spaces of operators between them are absolutely monotone (cf. Theorem 3.7.10), hence resolving the question of the existence of such spaces as posed above.

We briefly describe the structure of the paper.

After giving preliminary definitions and terminology in Section 3.2, we introduce various versions of the concepts of normality and conormality of pre-ordered Banach spaces with closed cones in Section 3.3. Normality is a more general notion than monotonicity, and roughly is a measure of 'the obtuseness/bluntness of a cone' (with respect to the norm). Conormality roughly is a measure of 'the acuity/sharpness of a cone' (with respect to the norm). Normality and conormality properties often occur in dual pairs, where a pre-ordered Banach space with a closed cone has a normality property precisely when its dual has the appropriate conormality property (cf. Theorem 3.3.7). The terms 'monotonicity' and 'normality' are fairly standard throughout the literature. However, the concept of conormality occurs scattered under many names throughout the literature (chronologically, [23, 7, 3, 21, 11, 34, 45, 35, 47, 44, 39, 48, 6, 9, 37]). Although the definitions and results in Section 3.3 are not new, they are collected here in an attempt to give an overview and to standardize the terminology.

In Section 3.4, with X and Y pre-ordered Banach spaces with closed cones, we investigate the normality of $B(X, Y)$ in terms of the normality and conormality of X and Y. Roughly, excluding degenerate cases, some form of conormality of X and normality of Y is necessary and sufficient for having some form of normality of the pre-ordered Banach space $B(X, Y)$ (cf. Theorems 3.4.1 and 3.4.2). Again, certain results are not new, but are included for the sake of completeness.

In Section 3.5 we introduce quasi-lattices, a class of pre-ordered Banach spaces spaces that strictly includes the Banach lattices. We establish their basic properties, in particular, basic vector lattice identities which carry over from Riesz spaces to quasi-lattices (cf. Theorem 3.5.8).

In Section 3.6 we prove one of our main results: Every strictly convex reflexive pre-ordered Banach space with a closed proper and generating cone is a quasi-lattice. Hence there are many quasi-lattices.

Finally, in Section 3.7, we show that real Hilbert spaces endowed with Lorentz cones are quasi-lattices and satisfy an identity analogous to the elementary identity $\||x|\| = \|x\|$ which holds for all elements x of a Banach lattice. This is used to show, for real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ endowed with Lorentz cones, that $B(\mathcal{H}_1, \mathcal{H}_2)$ is absolutely monotone.

3.2 Preliminary definitions and notation

Let X be a Banach space over the real numbers. Its topological dual will be denoted by X'. A subset $C \subseteq X$ will be called a *cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. If a cone C satisfies $C \cap (-C) = \{0\}$, it will be called a *proper cone*, and if $X = C - C$, it will be said to be *generating* (*in* X).

Definition 3.2.1. A pair (X, C), with X a Banach space and $C \subseteq X$ a cone, will be called a *pre-ordered Banach space*. If C is a proper cone, (X, C) will be called an *ordered Banach space*. We will often suppress explicit mention of the pair and merely say that X is a (pre-)ordered Banach space. When doing so, we will denote the implicit cone by X_+ and refer to it as *the cone of* X. For any $x, y \in X$, by $x \geq y$ we will mean $x - y \in X_+$. We do not exclude the possibilities $X_+ = \{0\}$ or $X_+ = X$, and we do not assume that X_+ is closed.

Let X and Y be pre-ordered Banach spaces. The space of bounded linear operators from X to Y will be denoted by $B(X, Y)$ and by $B(X)$ if $X = Y$. Unless otherwise mentioned, $B(X, Y)$ is always endowed with the operator norm. The space $B(X, Y)$ is easily seen to be a pre-ordered Banach space when endowed with the cone $B(X, Y)_+ := \{T \in B(X, Y) : TX_+ \subseteq Y_+\}$. In particular, the topological dual X' also becomes a pre-ordered Banach space when endowed with the *dual cone* $X'_+ := B(X, \mathbb{R})_+$. For any $f \in X'$ and $y \in Y$, we will define the operator $f \otimes y \in B(X, Y)$ by $(f \otimes y)(x) := f(x)y$ for all $x \in X$. It is easily seen that $\|f \otimes y\| = \|f\|\|y\|$.

3.3 Normality and Conormality

In the current section we will define some of the possible norm-cone interactions that may occur in pre-ordered Banach spaces, and investigate how they relate to norm-cone interactions in the dual. Historically, these properties have been assigned to either the norm or the cone (e.g., ‘a cone is normal’ and ‘a norm is monotone’). We will not follow this convention and rather assign these labels to the pre-ordered Banach space as a whole to emphasize the norm-cone interaction.

We attempt to collect all known results and to standardize the terminology. The definitions and results in the current section are essentially known, but are scattered throughout the literature under quite varied terminology[1]. References are provided when known to the author.

Definition 3.3.1. Let X be a pre-ordered Banach space with a closed cone and $\alpha > 0$.

We define the following ***normality properties****:*

(1) We will say X is *α-max-normal* if, for any $x, y, z \in X$, $z \le x \le y$ implies $\|x\| \le \alpha \max\{\|y\|, \|z\|\}$.

(2) We will say X is *α-sum-normal* if, for any $x, y, z \in X$, $z \le x \le y$ implies $\|x\| \le \alpha(\|y\| + \|z\|)$.

(3) We will say X is *α-absolutely normal* if, for any $x, y \in X$, $\pm x \le y$ implies $\|x\| \le \alpha\|y\|$. We will say X is *absolutely monotone* if it is 1-absolutely normal.

(4) We will say X is *α-normal* if, for any $x, y \in X$, $0 \le x \le y$ implies $\|x\| \le \alpha\|y\|$. We will say X is *monotone* if it is 1-normal.

We define the following ***conormality properties****:*

(1) We will say X is *α-sum-conormal* if, for any $x \in X$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| \le \alpha\|x\|$. We will say X is *approximately α-sum-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \alpha\|x\| + \varepsilon$.

(2) We will say X is *α-max-conormal* if, for any $x \in X$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \le \alpha\|x\|$. We will say X is *approximately α-max-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} < \alpha\|x\| + \varepsilon$.

(3) We will say X is *α-absolutely conormal* if, for any $x \in X$, there exist some $a \in X_+$ such that $\pm x \le a$ and $\|a\| \le \alpha\|x\|$. We will say X is *approximately α-absolutely conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a \in X_+$ such that $\pm x \le a$ and $\|a\| < \alpha\|x\| + \varepsilon$.

(4) We will say X is *α-conormal* if, for any $x \in X$, there exist some $a \in X_+$ such that $0, x \le a$ and $\|a\| \le \alpha\|x\|$. We will say X is *approximately α-conormal* if,

[1] A note on terminology: The terms 'normality' (due to Krein [28]) and 'monotonicity' are fairly standard terms throughout the literature. Our consistent use of the adjective 'absolute' is inspired by [47] and mimics its use in the term 'absolute value'.

The concept that we will call 'conormality' has seen numerous equivalent definitions and the nomenclature is rather varied in the existing literature. The term 'conormality' is due to Walsh [44], who studied the property in the context of locally convex spaces. What we will call '1-max-conormality' occurs under the name 'strict bounded decomposition property' in [8]. The properties that we will call 'approximate 1-absolute conormality' and 'approximate 1-conormality', were first defined (but not named) respectively by Davies [11] and Ng [34]. Batty and Robinson give equivalent definitions for our conormality properties which they call 'dominating' and 'generating' [6].

for any $x \in X$ and $\varepsilon > 0$, there exist some $a \in X_+$ such that $\{0, x\} \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$.

The following two results show the relationship between different (co)normality properties and for the most part are immediate from the definitions.

Proposition 3.3.2. *For any fixed $\alpha > 0$, the following implications hold between normality properties of a pre-ordered Banach space X with a closed cone:*

$$\begin{array}{ccccc}
\alpha\textit{-max-normality} & \Longrightarrow & \alpha\textit{-absolute-normality} & & \\
\Downarrow & & \Downarrow & & \\
\alpha\textit{-sum-normality} & \Longrightarrow & \alpha\textit{-normality} & \Longrightarrow & (\alpha+1)\textit{-sum-normality} \\
\Downarrow & & \Downarrow & & \\
2\alpha\textit{-max-normality} & & X_+ \textit{ is proper} & &
\end{array}$$

Proof. The only implication that is not immediate from the definitions is that α-normality implies $(\alpha+1)$-sum-normality. As to this, let X be an α-normal pre-ordered Banach space with a closed cone and $x, y, z \in X$ such that $z \leq x \leq y$. Then $0 \leq x - z \leq y - z$, so that, by α-normality and the reverse triangle inequality,

$$\|x\| - \|z\| \leq \|x - z\| \leq \alpha\|y - z\| \leq \alpha(\|y\| + \|z\|).$$

Hence $\|x\| \leq \alpha\|y\| + (\alpha+1)\|z\| \leq (\alpha+1)(\|y\| + \|z\|)$. ☐

Similar relationships hold between conormality properties as do between normality properties. All implications follow immediately from the definitions, with one exception. That is, if a pre-ordered Banach space has a closed generating cone, then there exists a constant $\beta > 0$ such that it is β-max-conormal (and hence 2β-sum-conormal). This is a result due to Andô [3, Lemma 1], and is the bottom implication in the following proposition (although [3, Lemma 1] assumes the cone to be proper, this is not necessary for its statement to hold, cf. Theorem 3.3.6).

Proposition 3.3.3. *For any fixed $\alpha > 0$, the following implications hold between conormality properties of a pre-ordered Banach space X with a closed cone:*

$$
\begin{tikzcd}
\alpha\text{-sum-conorm.} \arrow[rrr, Rightarrow] \arrow[dr, Rightarrow] \arrow[dd, Rightarrow] & & & \text{approx. } \alpha\text{-sum-conorm.} \arrow[d, Rightarrow] \arrow[dddll, Rightarrow] \\
& \alpha\text{-abs.-conorm.} \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \text{approx. } \alpha\text{-abs.-conorm.} \arrow[dd, Rightarrow] \\
\alpha\text{-max-conorm.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, Rightarrow] & \alpha\text{-conorm.} \arrow[drr, Rightarrow] & & \\
2\alpha\text{-sum-conorm.} & \text{approx. } \alpha\text{-max-conorm.} \arrow[rr, Rightarrow] & & \text{approx. } \alpha\text{-conorm.} \arrow[d, Rightarrow] \\
& & \exists \beta > 0 : \ \beta\text{-sum-conorm.} & X_+ \text{ is generating} \arrow[l, Rightarrow]
\end{tikzcd}
$$

Remark 3.3.4. The direct analogue to Andô's Theorem [3, Lemma 1] (the bottom implication in Proposition 3.3.3) in Proposition 3.3.2 would be that having X_+ proper implies that X is β-max-normal for some $\beta > 0$. This is false. Example 3.6.7 gives a space which has a proper cone but is not α-normal for any $\alpha > 0$.

Remark 3.3.5. For the sake of completeness, we note that Andô's Theorem [3, Lemma 1] (the bottom implication in Proposition 3.3.3) can be improved, in that the decomposition of elements into a difference of elements from the cone can be chosen in a continuous, as well as bounded and positively homogeneous manner. The following result is a special case of [13, Theorem 4.1], which is a general principle for Banach spaces that are the sum of (not necessarily countably many) closed cones. Its proof proceeds through an application of Michael's Selection Theorem [1, Theorem 17.66] and a generalization of the usual Open Mapping Theorem [13, Theorem 3.2]:

Theorem 3.3.6. *Let X be a pre-ordered Banach space with a closed generating cone. Then there exist continuous positively homogeneous functions $(\cdot)^\pm : X \to X_+$ and a constant $\alpha > 0$ such that $x = x^+ - x^-$ and $\|x^\pm\| \le \alpha \|x\|$ for all $x \in X$.*

Normality and conormality properties often appear in dual pairs. Roughly, a pre-ordered Banach space has a normality property if and only if its dual has a corresponding conormality property, and vice versa. The following theorem provides an overview of these normality-conormality duality relationships as known to the author.

Theorem 3.3.7. *Let X be a pre-ordered Banach space with a closed cone.*

(1) The following equivalences hold:

(a) For $\alpha > 0$, the space X is α-max-normal if and only if X' is α-sum-conormal.

(b) For $\alpha > 0$, the space X is α-sum-normal if and only if X' is α-max-conormal.

(c) For $\alpha > 0$, the space X is α-absolutely normal if and only if X' is α-absolutely conormal.

(d) For $\alpha > 0$, the space X is α-normal if and only if X' is α-conormal.

(e) There exists an $\alpha > 0$ such that X is α-max-normal if and only if X'_+ is generating.

(2) The following equivalences hold:

(a) For $\alpha > 0$, the space X is approximately α-sum-conormal if and only if X' is α-max-normal.

(b) For $\alpha > 0$, the space X is approximately α-max-conormal if and only if X' is α-sum-normal.

(c) For $\alpha > 0$, the space X is approximately α-absolutely conormal if and only if X' is α-absolutely normal.

(d) For $\alpha > 0$, the space X is approximately α-conormal if and only if X' is α-normal.

(e) The cone X_+ is generating if and only if there exists an $\alpha > 0$ such that X' is α-max-normal.

The result (1)(a) was first proven by Grosberg and Krein in [23] (via [21, Theorem 7]). The result (2)(a) was established by Ellis [21, Theorem 8]. For $\alpha = 1$, the results (1)(c),(d), (2)(c) and (d) are due to Ng [34, Proppositions 5, 6; Theorems 6, 7]. The fully general results (1)(d) and (2)(d) appear first in [39, Theorem 1.1] by Robinson and Yamamuro, and later in [37, Theorems 1,2] by Ng an Law. Proofs of (1)(a) (again), (1)(b), (1)(c), (1)(d) (again), and (2)(a) (again), (2)(b), (2)(c), and (2)(d) (again) are due to Batty and Robinson in [6, Theorems 1.1.4, 1.3.1, 1.2.2]. The results (1)(e) and (2)(e) are due to Andô [3, Theorem 1].

Bonsall proved an analogous duality result for locally convex spaces in [7, Theorem 2].

The following lemma shows that conormality properties and approximate conormality properties of dual spaces are equivalent. Ng proved (3) for the case $\alpha = 1$ in [34, Theorem 6]:

Lemma 3.3.8. *Let X be a pre-ordered Banach space with a closed cone. Then the following equivalences hold:*

(1) For $\alpha > 0$, the space X' is approximately α-sum-conormal if and only if X' is α-sum-conormal.

(2) For $\alpha > 0$, the space X' is approximately α-max-conormal if and only if X' is α-max-conormal.

(3) For $\alpha > 0$, the space X' is approximately α-absolutely conormal if and only if X' is α-absolutely conormal.

(4) For $\alpha > 0$, the space X' is approximately α-conormal if and only if X' is α-conormal.

Proof. That a conormality property implies the associated approximate conormality property is trivial. We will therefore only prove the forward implications.

We prove (1). Let X' be approximately α-sum-conormal. Then, for any $\beta > \alpha$ and any $0 \neq f \in X$, by taking $\varepsilon = (\beta - \alpha)\|f\| > 0$, we have that there exist $g, h \in X'_+$ such that $f = g - h$ and $\|g\| + \|h\| \leq \alpha\|f\| + (\beta - \alpha)\|f\| = \beta\|f\|$. Therefore, X' is β-sum-conormal for every $\beta > \alpha$. Now, by part (1)(a) of Theorem 3.3.7, X is β-max-normal for every $\beta > \alpha$. Therefore, if $x, y, z \in X$ are such that $z \leq x \leq y$, then $\|x\| \leq \beta \max\{\|y\|, \|z\|\}$ for all $\beta > \alpha$, and hence $\|x\| \leq \inf_{\beta>\alpha} \beta \max\{\|y\|, \|z\|\} = \alpha \max\{\|y\|, \|z\|\}$. We conclude that X is α-max-normal, and, again by part (1)(a) Theorem 3.3.7, that X' is α-sum-conormal.

The assertions (2), (3) and (4) follow through similar arguments. □

By Theorem 3.3.7 and Lemma 3.3.8, a pre-ordered Banach space with a closed cone possesses both a normality property and its paired approximate conormality property (with the same constant) if and only if its dual possesses the same properties (cf. Corollary 3.3.11). Such spaces are called regular and were first studied by Davies in [11] and Ng in [34].

Definition 3.3.9. Let X be a pre-ordered Banach space with a closed cone. We define the following ***regularity properties***:[2]

(1) For $\alpha > 0$, we will say X is *α-Ellis-Grosberg-Krein regular* if X is both α-max-normal and approximately α-sum-conormal.

(2) For $\alpha > 0$, we will say X is *α-Batty-Robinson regular* if X is both α-sum-normal and approximately α-max-conormal.

(3) For $\alpha > 0$, we will say X is *α-absolutely Davies-Ng regular* if X is both α-absolutely normal and approximately α-absolutely conormal.

(4) For $\alpha > 0$, we will say X is *α-Davies-Ng regular* if X is both α-normal and approximately α-conormal.

[2] The term 'regularity' is due to Davies [11]. Our naming convention is to attach the names of the persons who (to the author's knowledge) first proved the relevant normality-conormality duality results of the defining properties (cf. Theorem 3.3.7).

(5) We will say X is *Andô regular* if X_+ is generating and there exists an $\alpha > 0$ such that X is α-max-normal.

It should be noted that every Banach lattice is 1-absolutely Davies-Ng regular.

The following result combines Propositions 3.3.2 and 3.3.3 to provide relationships that exist between regularity properties.

Proposition 3.3.10. *For any fixed $\alpha > 0$, the following implications hold between regularity properties of a pre-ordered Banach space with a closed cone:*

$$\begin{array}{ccc}
\alpha\text{-Ellis-Grosberg-Krein regularity} & \Longrightarrow & \alpha\text{-Batty-Robinson regularity} \\
\Downarrow & & \Downarrow \\
\alpha\text{-absolute Davies-Ng regularity} & \Longrightarrow & \alpha\text{-Davies-Ng regularity} \\
 & & \Downarrow \\
\exists \beta > 0 : \ \beta\text{-Ellis-Grosberg-Krein regularity} & \Longleftarrow & \text{Andô regularity}
\end{array}$$

Proof. The only implication that does not follow immediately from Propositions 3.3.2 and 3.3.3, is that Andô regularity implies β-Ellis-Grosberg-Krein regularity for some $\beta > 0$. As to this, let X be an Andô regular ordered Banach space with a closed cone. By Proposition 3.3.3, since X_+ is generating, there exists some $\delta > 0$ such that X is δ-sum-conormal. By assumption, there exists an $\alpha > 0$, such that X is α-max-normal. By taking $\beta := \max\{\delta, \alpha\}$, we see that X is also β-max-normal and (approximately) β-sum-conormal. We conclude that X is β-Ellis-Grosberg-Krein regular. $\square$

A straightforward application of Theorem 3.3.7 and Lemma 3.3.8 then yields:

Corollary 3.3.11. *Let X be a pre-ordered Banach space with a closed cone. Then the following equivalences hold:*

(1) For $\alpha > 0$, the space X is α-Ellis-Grosberg-Krein regular if and only if X' is α-Ellis-Grosberg-Krein regular.

(2) For $\alpha > 0$, the space X is α-Batty-Robinson regular if and only if X' is α-Batty-Robinson regular.

(3) For $\alpha > 0$, the space X is α-absolutely Davies-Ng regular if and only if X' is α-absolutely Davies-Ng regular.

(4) For $\alpha > 0$, the space X is α-Davies-Ng regular if and only if X' is α-Davies-Ng regular.

(5) The space X is Andô regular if and only if X' is Andô regular.

3.4 The normality of pre-ordered Banach spaces of bounded linear operators

If X and Y are pre-ordered Banach spaces with closed cones, we investigate necessary and sufficient conditions for the pre-ordered Banach space $B(X,Y)$ to have a normality property. Where results are known to the author from the literature, references are provided.

We begin, in the following result, by investigating necessary conditions for $B(X,Y)$ to have a normality property. Parts (2) and (3) in the special case $X = Y$ and $\alpha = 1$ in the following theorem are due Yamamuro [48, 1.2–3]. Batty and Robinson also proved part (2) for $X = Y$ and $\alpha = 1$, and part (3) for $\alpha = \beta = 1$ [6, Corollary 1.7.5, Proposition 1.7.6]. Part (5) is due to Wickstead [45, Theorem 3.1].

Theorem 3.4.1. *Let X and Y be non-zero pre-ordered Banach spaces with closed cones and $\alpha > 0$.*

(1) The cone $B(X,Y)_+$ is proper if and only if $X = \overline{X_+ - X_+}$ and Y_+ is proper.

(2) Let $B(X,Y)$ be α-normal. If $Y_+ \neq \{0\}$, then X is approximately α-conormal. If $X'_+ \neq \{0\}$, then Y is α-normal.

(3) Let $B(X,Y)$ be α-absolutely normal. If $Y_+ \neq \{0\}$, then X is approximately α-absolutely conormal. If $X'_+ \neq \{0\}$, then Y is α-absolutely normal.

(4) Let $B(X,Y)$ be α-sum-normal. If $Y_+ \neq \{0\}$, then X is approximately α-max-conormal. If $X'_+ \neq \{0\}$, then Y is α-sum-normal.

(5) Let $B(X,Y)$ be α-max-normal. If $Y_+ \neq \{0\}$, then X is approximately α-sum-conormal. If $X'_+ \neq \{0\}$, then Y is α-max-normal.

Proof. We prove (1). Let $B(X,Y)_+$ be proper. Suppose $X \neq \overline{X_+ - X_+}$. By the Hahn-Banach Theorem there exists a non-zero functional $f \in X'$ such that $f|_{\overline{X_+ - X_+}} = 0$. Let $0 \neq y \in Y$, then $\pm f \otimes y \geq 0$ since $f \otimes y|_{X_+} = 0$. Therefore $B(X,Y)_+$ is not proper, contradicting our assumption. Suppose Y_+ is not proper. Let $0 \neq y \in Y_+ \cap (-Y_+)$ and $0 \neq f \in X'$. Then $\pm f \otimes y \geq 0$, and hence $B(X,Y)_+$ is not proper, contradicting our assumption.

Let $X = \overline{X_+ - X_+}$ and Y_+ be proper. If $T \in B(X,Y)_+ \cap (-B(X,Y)_+)$, then, since Y_+ is proper, $TX_+ = \{0\}$. Hence $T(X_+ - X_+) = \{0\}$, and by density of $X_+ - X_+$ in X, we have $T = 0$.

We prove (2). Let $B(X,Y)$ be α-normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, to conclude that X is approximately α-conormal, it is sufficient to prove that X' is α-normal. Let $f, g \in X'$ satisfy $0 \leq f \leq g$, and let $0 \neq y \in Y_+$. Then $0 \leq f \otimes y \leq g \otimes y$, and by the α-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|g \otimes y\| = \alpha\|g\|\|y\|.$$

Therefore $\|f\| \leq \alpha\|g\|$, and hence X' is α-conormal. With $X'_+ \neq \{0\}$, let $0 \neq f \in X'_+$ be arbitrary, and $y, z \in Y$ such that $0 \leq y \leq z$. Then $0 \leq f \otimes y \leq f \otimes z$ in $B(X,Y)$, and by the α-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|f \otimes z\| = \alpha\|f\|\|z\|.$$

Hence $\|y\| \leq \alpha\|z\|$ and we conclude that Y is α-normal.

We prove (3). Let $B(X,Y)$ be α-absolutely normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, to conclude that X is approximately α-absolutely conormal, it is sufficient to prove that X' is α-absolutely normal. Let $f, g \in X'$ satisfy $\pm f \leq g$, and let $0 \neq y \in Y_+$. Then $\pm f \otimes y \leq g \otimes y$, and by the α-absolute normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|g \otimes y\| = \alpha\|g\|\|y\|.$$

Therefore $\|f\| \leq \alpha\|g\|$, and hence X' is α-absolutely normal. With $X'_+ \neq \{0\}$, let $0 \neq f \in X'_+$ be arbitrary, and $y, z \in Y$ such that $\pm y \leq z$. Then $\pm f \otimes y \leq f \otimes z$ in $B(X,Y)$, and by the α-absolute normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|f \otimes z\| = \alpha\|f\|\|z\|$$

Hence $\|y\| \leq \alpha\|z\|$ and we conclude that Y is α-absolutely normal.

We prove (4). Let $B(X,Y)$ is α-sum-normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, it is sufficient to prove that X' is α-sum-normal to conclude that X is approximately α-max-conormal. Let $0 \neq y \in Y_+$ and $f, g, h \in X'$ satisfy $g \leq f \leq h$. Then $g \otimes y \leq f \otimes y \leq h \otimes y$ in $B(X,Y)$, and by the α-sum-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\left(\|g \otimes y\| + \|h \otimes y\|\right) = \alpha(\|g\| + \|h\|)\|y\|.$$

Hence $\|f\| \leq \alpha(\|g\| + \|h\|)$ and X' is α-sum-normal. With $X'_+ \neq \{0\}$, to prove that Y is α-sum-normal, let $u, v, y \in Y$ satisfy $u \leq y \leq v$ and let $0 \neq f \in X'_+$. Then $f \otimes u \leq f \otimes y \leq f \otimes v$ in $B(X,Y)$, and hence, $\|y\| \leq \alpha(\|u\| + \|v\|)$ as before.

The proof of (5) is analogous to that of (4). □

Converse-like implications to the previous result also hold, giving sufficient conditions for $B(X,Y)$ to have a normality property. Part (1) and the case $\alpha = \beta = 1$ of part (3) are due to Batty and Robinson Batty and Robinson [6, Proposition 1.7.3, Corollary 1.7.5]. The special case $X = Y$ and $\alpha = \beta = 1$ of part (3) is due to Yamamuro [48, 1.3]. The case where X is approximately α-sum-conormal and Y is β-max-normal of part (4) is due to Wickstead [45, Theorem 3.1].

Theorem 3.4.2. *Let X and Y be pre-ordered Banach spaces with closed cones and $\alpha, \beta > 0$.*

(1) If X_+ is generating and Y is α-normal, then there exists some $\gamma > 0$ for which $B(X,Y)$ is γ-normal.

(2) If X is approximately α-conormal and Y is β-normal, then $B(X,Y)$ is $(2\alpha + 1)\beta$-normal.

(3) If X is approximately α-absolutely conormal and Y is β-absolutely normal, then $B(X,Y)$ is $\alpha\beta$-absolutely normal.

(4) If X is approximately α-sum-conormal and Y is β-normal (β-absolutely normal, β-max-normal, β-sum-normal respectively), then $B(X,Y)$ is $\alpha\beta$-normal ($\alpha\beta$-absolutely normal, $\alpha\beta$-max-normal, $\alpha\beta$-sum-normal respectively)

Proof. We prove (1). By Andô's Theorem [3, Lemma 1], the fact that X_+ is generating in X implies that there exists some $\beta > 0$ such that X is β-max-conormal. Let $T, S \in B(X,Y)$ be such that $0 \leq T \leq S$. Then, for any $x \in X$, let $a, b \in X_+$ be such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$, so that $0 \leq Ta \leq Sa$ and $0 \leq Tb \leq Sb$. By α-normality of Y,

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \alpha(\|Sa\| + \|Sb\|) \leq \alpha\|S\|(\|a\| + \|b\|) \leq 2\alpha\beta\|S\|\|x\|,$$

hence $\|T\| \leq 2\alpha\beta\|S\|$.

We prove (2). Let $T, S \in B(X,Y)$ be such that $0 \leq T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ such that $\{0, x\} \leq a$ and $\|a\| \leq \alpha\|x\| + \varepsilon$. Since $x = a - (a - x)$ and $a, a - x \geq 0$, we obtain $0 \leq Ta \leq Sa$ and $0 \leq T(a - x) \leq S(a - x)$, and hence

$$\begin{aligned}
\|Tx\| &= \|Ta - T(a-x)\| \\
&\leq \|Ta\| + \|T(a-x)\| \\
&\leq \beta\|Sa\| + \beta\|S(a-x)\| \\
&\leq \beta\|S\|(\alpha\|x\| + \varepsilon) + \beta\|S\|(\alpha\|x\| + \varepsilon + \|x\|) \\
&= (2\alpha + 1)\beta\|S\|\|x\| + 2\varepsilon\beta\|S\|.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\| \leq (2\alpha + 1)\beta\|S\|$.

We prove (3). Let $T, S \in B(X,Y)$ satisfy $\pm T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ satisfying $\pm x \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$. Then

$$Tx = T\left(\frac{a+x}{2}\right) - T\left(\frac{a-x}{2}\right),$$

and hence,

$$\pm Tx = \pm T\left(\frac{a+x}{2}\right) \mp T\left(\frac{a-x}{2}\right).$$

Since $(a+x)/2 \geq 0$, $(a-x)/2 \geq 0$ and $\pm T \leq S$, we find

$$\pm Tx \leq S\left(\frac{a+x}{2}\right) + S\left(\frac{a-x}{2}\right) = Sa.$$

Now, because Y is β-absolutely normal, we obtain

$$\|Tx\| \leq \beta\|Sa\| \leq \beta\|S\|\|a\| \leq \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X,Y)$ is $\alpha\beta$-absolutely normal.

We prove (4). Let X be approximately α-sum-conormal and let Y be β-normal. Let $T, U \in B(X,Y)$ satisfy $0 \leq T \leq U$ and let $x \in X$ be arbitrary. Then, for every

$\varepsilon > 0$, there exist $x_1, x_2 \in X_+$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| < \alpha\|x\| + \varepsilon$. Also, $0 \le Tx_i \le Ux_i$ implies $\|Tx_i\| \le \beta\|Ux_i\|$ for $i = 1, 2$. Therefore,

$$\begin{aligned} \|Tx\| &\le \|Tx_1\| + \|Tx_2\| \\ &\le \beta\|Ux_1\| + \beta\|Ux_2\| \\ &\le \beta\|U\|(\|x_1\| + \|x_2\|) \\ &\le \alpha\beta\|U\|\|x\| + \varepsilon\beta\|U\|. \end{aligned}$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, we may conclude that $B(X, Y)$ is $\alpha\beta$-normal. The case where X is approximately α-sum-conormal and Y is β-absolutely normal follows similarly.

Let X be approximately α-sum-conormal and let Y be β-max-normal. Let $T, U, V \in B(X, Y)$ satisfy $U \le T \le V$ and let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exist $x_1, x_2 \in X_+$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| < \alpha\|x\| + \varepsilon$. Also, $Ux_i \le Tx_i \le Vx_i$ implies $\|Tx_i\| \le \beta \max\{\|Ux_i\|, \|Vx_i\|\}$ for $i = 1, 2$. Therefore,

$$\begin{aligned} \|Tx\| &\le \|Tx_1\| + \|Tx_2\| \\ &\le \beta \max\{\|Ux_1\|, \|Vx_1\|\} + \beta \max\{\|Ux_2\|, \|Vx_2\|\} \\ &\le \beta \max\{\|U\|, \|V\|\}(\|x_1\| + \|x_2\|) \\ &\le \alpha\beta \max\{\|U\|, \|V\|\}\|x\| + \varepsilon\beta \max\{\|U\|, \|V\|\}. \end{aligned}$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, we may conclude that $B(X, Y)$ is $\alpha\beta$-max-normal. The case where X is approximately α-sum-conormal and Y is β-sum-normal follows similarly. □

If one has further knowledge of the behavior of the positive bounded linear operators, specifically that their norms are determined by their behavior on the cone, then one can improve the constant in (2) of the above theorem. This will be discussed in the rest of this section.

Definition 3.4.3. Let X be a pre-ordered Banach space with a closed cone and Y a Banach space. For $T \in B(X, Y)$, we define $\|T\|_+ := \sup\{\|Tx\| : x \in X_+,\ \|x\| = 1\}$.

If $X = \overline{X_+ - X_+}$, then $\|\cdot\|_+$ is a norm on $B(X, Y)$, called the *Robinson norm* (as named by Yamamuro in [48]). We will say that the operator norm on $B(X, Y)$ is *positively attained* (as named by Batty and Robinson in [6]) if $\|T\| = \|T\|_+$ for all positive operators $T \in B(X, Y)_+$.

If X_+ is closed and generating, $\|\cdot\|_+$ is in fact equivalent to the usual operator norm on $B(X, Y)$. The following result is a slight refinement of a remark by Batty and Robinson [6, p. 248].

Proposition 3.4.4. *If X is a pre-ordered Banach space with a closed generating cone and Y a Banach space, then the Robinson norm is equivalent to the operator norm on $B(X, Y)$.*

Proof. By Andô's Theorem [3, Lemma 1], X is α-max-conormal for some $\alpha > 0$. Let $x \in X$ and $T \in B(X,Y)$ be arbitrary, then there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \le \alpha\|x\|$. Hence

$$\begin{aligned} \|Tx\| &= \|Ta - Tb\| \\ &\le \|Ta\| + \|Tb\| \\ &\le \|T\|_+(\|a\| + \|b\|) \\ &\le 2\alpha\|T\|_+\|x\|. \end{aligned}$$

Therefore, $\|T\|_+ \le \|T\| \le 2\alpha\|T\|_+$. □

Part (2) of Theorem 3.4.2 can be improved if we know that the operator norm is positively attained.

Proposition 3.4.5. *Let X and Y be pre-ordered Banach spaces with closed cones, with Y α-normal for some $\alpha > 0$. If the operator norm on $B(X,Y)$ is positively attained, then $B(X,Y)$ is α-normal.*

Proof. Let $T, S \in B(X,Y)$ satisfy $0 \le T \le S$. Then, for any $x \in X_+$, $0 \le Tx \le Sx$, and hence $\|Tx\| \le \alpha\|Sx\|$. We then see that

$$\begin{aligned} \|T\| &= \|T\|_+ \\ &= \sup\{\|Tx\| : x \in X_+,\ \|x\| \le 1\} \\ &\le \alpha \sup\{\|Sx\| : x \in X_+,\ \|x\| \le 1\} \\ &= \alpha\|S\|_+ \\ &= \alpha\|S\|, \end{aligned}$$

and conclude that $B(X,Y)$ is α-normal. □

The following theorem gives one necessary condition and some sufficient conditions to have that an operator norm is positively attained. The sufficiency of (1)[3], (2), and the necessity of approximate 1-conormality in the following theorem are due to Batty and Robinson in [6, Proposition 1.7.8.].

Theorem 3.4.6. *Let X and Y be pre-ordered Banach spaces with closed cones.*

If $Y_+ \neq \{0\}$ and the operator norm on $B(X,Y)$ is positively attained, then X is approximately 1-conormal.

Any of the following conditions is sufficient for the operator norm on $B(X,Y)$ to be positively attained:

(1) The space X is approximately 1-max-conormal and Y is 1-max-normal.

(2) The space X is approximately 1-absolutely conormal and Y is absolutely monotone (i.e., if $X = Y$, X is 1-absolutely Davies-Ng regular).

[3] There is a small error in the statement of (1) in [6, Proposition 1.7.8.]. We give its correct statement and proof.

(3) *The space X is approximately* 1*-sum-conormal* (*in which case* $\|T\| = \|T\|_+$ *even holds for all* $T \in B(X,Y)$).

Proof. We prove the necessity of approximate 1-conormality of X when $Y_+ \neq \{0\}$ and the operator norm on $B(X,Y)$ is positively attained. Let $f \in X'_+$ be arbitrary and let $0 \neq y \in Y_+$. Then, since the operator norm on $B(X,Y)$ is positively attained,

$$\|f\|\|y\| = \|f \otimes y\| = \|f \otimes y\|_+ = \|f\|_+\|y\|,$$

so that $\|f\| = \|f\|_+$. For all $f, g \in X'$ satisfying $0 \leq f \leq g$, we obtain $\|f\| = \|f\|_+ \leq \|g\|_+ = \|g\|$. Therefore X' is monotone, and by part (2)(d) of Theorem 3.3.7, X is approximately 1-conormal.

We prove the sufficiency of (1). Let $T \in B(X,Y)_+$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Then, since X is 1-max-conormal, there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} < \|x\| + \varepsilon$. We notice that $-b \leq x \leq a$ and $T \geq 0$ imply that $-Tb \leq Tx \leq Ta$. Then, since Y is 1-max-normal,

$$\|Tx\| \leq \max\{\|Ta\|, \|Tb\|\} \leq \|T\|_+ \max\{\|a\|, \|b\|\} \leq \|T\|_+(\|x\| + \varepsilon).$$

Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\|_+ \leq \|T\| \leq \|T\|_+$.

We prove the sufficiency of (2). Let $T \in B(X,Y)_+$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary, then there exists a $z \in X_+$ such that $\{-x, x\} \leq z$ and $\|z\| < \|x\| + \varepsilon$. Then, since $T \geq 0$, we see that $\{-Tx, Tx\} \leq Tz$, and therefore, since Y is absolutely monotone,

$$\|Tx\| \leq \|Tz\| \leq \|T\|_+\|z\| \leq \|T\|_+(\|x\| + \varepsilon).$$

Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\|_+ \leq \|T\| \leq \|T\|_+$.

We prove the sufficiency of (3). Let $x \in X$ be arbitrary. Since X is approximately 1-sum-conormal, for every $\varepsilon > 0$, there exist $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \|x\| + \varepsilon$. For any $T \in B(X,Y)$, we have

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \|T\|_+(\|a\| + \|b\|) \leq \|T\|_+(\|x\| + \varepsilon).$$

Since $\varepsilon > 0$ and $x \in X$ were chosen arbitrarily, we obtain $\|T\|_+ \leq \|T\| \leq \|T\|_+$. □

3.5 Quasi-lattices and their basic properties

In this section we will define quasi-lattices, establish their basic properties and provide a number of illustrative (non-)examples.

Let X be a pre-ordered Banach space and A any subset of X. For $x \in X$, by $A \leq x$ we mean that $a \leq x$ for all $a \in A$ and say *x is an upper bound of A*. We will use the Greek letter 'upsilon' to denote the set of all upper bounds of A, written as $\upsilon(A)$. If $x \in X$ is such that $A \leq x$ and, for any $y \in X$, $A \leq y \leq x$ implies $x = y$, we say that x is a *minimal upper bound of A*. We will use the Greek letter 'mu' to denote the set of all minimal upper bounds of A, written as $\mu(A)$. We note that $\upsilon(A)$ and $\mu(A)$ could be empty for some $A \subseteq X$.

For any fixed $x, y \in X$, we define the function $\sigma_{x,y} : X \to \mathbb{R}_{\geq 0}$ by $\sigma_{x,y}(z) := \|z - x\| + \|z - y\|$ for all $z \in X$, and note that $\sigma_{x,y}(z) \geq \|x - y\|$ for all $x, y, z \in X$. We will refer to $\sigma_{x,y}$ as *the distance sum to x and y.*

We introduce the following definitions and notation:

Definition 3.5.1. Let X be a pre-ordered Banach space with a closed cone.

(1) We say that X is an *υ-quasi-lattice* if, for every pair of elements $x, y \in X$, $\upsilon(\{x, y\})$ is non-empty and there exists a unique element $z \in \upsilon(\{x, y\})$ minimizing $\sigma_{x,y}$ on $\upsilon(\{x, y\})$. The element z will be called the *υ-quasi-supremum* of $\{x, y\}$.

(2) We say that X is a *μ-quasi-lattice* if, for every pair of elements $x, y \in X$, $\mu(\{x, y\})$ is non-empty and there exists a unique element $z \in \mu(\{x, y\})$ minimizing $\sigma_{x,y}$ on $\mu(\{x, y\})$. The element z will be called the *μ-quasi-supremum* of $\{x, y\}$.

We immediately note that all Banach lattices are μ-quasi-lattices:

Proposition 3.5.2. *If X is a lattice ordered Banach space with a closed cone* (*in particular, if X is a Banach lattice*), *then X is a μ-quasi-lattice and its lattice structure coincides with its μ-quasi-lattice structure.*

Proof. Since for every $x, y \in X$, $\mu(\{x, y\}) = \{x \vee y\}$ is a singleton, this is clear. ☐

Remark 3.5.3. If X is a pre-ordered Banach space with a closed cone, then, for $x, y \in X$, the set $\upsilon(\{x, y\})$ is closed and convex, and hence techniques from convex optimization can be used to establish whether a pre-ordered Banach space is an υ-quasi-lattice (cf. Theorem 3.6.1). The set $\mu(\{x, y\})$ need not be convex in general (cf. Example 3.5.9), and hence it is usually more difficult to determine whether or not a space is a μ-quasi-lattice than an υ-quasi-lattice.

Except in the case of monotone υ-quasi-lattices which are also μ-quasi-lattices with coinciding $\upsilon-$ and μ-quasi-lattice structures (cf. Theorem 3.5.12), no further relationship is known between $\upsilon-$ and μ-quasi-lattices. Example 3.5.5 will provide a Banach lattice, and hence μ-quasi-lattice, that is not an υ-quasi-lattice. Furthermore, Example 3.5.13 will provide a non-monotone υ-quasi-lattice, which exhibits υ-quasi-suprema that are not minimal, hence if this space were a μ-quasi-lattice (which is currently not known), then its $\upsilon-$ and μ-quasi-lattice structures will not coincide.

To avoid repetition, we will often use the term quasi-lattice when it is unimportant whether a space is an $\upsilon-$ or μ-quasi-lattice, i.e., a quasi-lattice is either an $\upsilon-$ or μ-quasi-lattice. In such cases we will refer to the relevant $\upsilon-$ or μ-quasi-supremum as just the quasi-supremum. When it is indeed important whether a space is an $\upsilon-$ or μ-quasi-lattice, we will mention it explicitly.

The following notation will be used for both $\upsilon-$ and μ-quasi-lattices. Let X be a quasi-lattice and $x, y \in X$ arbitrary. We will denote the quasi-supremum of $\{x, y\}$ by $x \tilde{\vee} y$. This operation is symmetric, i.e., $x \tilde{\vee} y = y \tilde{\vee} x$. We define the *quasi-infimum* of $\{x, y\}$ by $x \tilde{\wedge} y := -((-x) \tilde{\vee} (-y))$. It is elementary to see that $x \tilde{\wedge} y \leq \{x, y\}$. We define the *quasi-absolute value* of x by $\lceil x \rceil := (-x)\tilde{\vee}(x)$. We will often use the notation $x^+ := 0\tilde{\vee}x$ and $x^- := 0\tilde{\vee}(-x)$.

Before establishing the basic properties of quasi-lattices, we will give a few examples of spaces that are (not) quasi-lattices.

The following is an example of a quasi-lattice that is not a Riesz space, and hence not a Banach lattice:

Example 3.5.4. The space $\{\mathbb{R}^3, \|\cdot\|_2\}$, endowed with the Lorentz cone

$$C := \{(x_1, x_2, x_3) : x_1 \geq (x_2^2 + x_3^2)^{1/2}\}.$$

There are many minimal upper bounds of, e.g., $\{(0,0,0),(0,0,2)\}$ (cf. Example 3.5.9 and Proposition 3.7.5). Hence no supremum exists, and this space is not a Riesz space. Another method to establish this would be to note that C has more than distinct 3 extreme rays, while every lattice cone in $\mathbb{R}^3$ has at most 3 disinct extreme rays [2, Theorem 1.45].

This space is (simultaneously an υ-quasi-lattice and) a μ-quasi-lattice. Intuitively, this can be seen by taking arbitrary elements, $x, y \in \mathbb{R}^3$, and seeing that there exists a unique element in $\mu(\{x,y\})$ with least first coordinate, which is then the quasi-supremum. It is possible give a more explicit proof, but this is not needed in view of the general Theorem 3.7.10 which is applicable to this example.

The following is an example of a Banach lattice, hence a μ-quasi-lattice, that is not an υ-quasi-lattice:

Example 3.5.5. Consider the space $\{\mathbb{R}^3, \|\cdot\|_\infty\}$ with the standard cone. Let $x := (1,-1,0)$. Then

$$\upsilon(\{0,x\}) = \{(z_1,z_2,z_3) \in \mathbb{R}^3 : z_1 \geq 1,\ z_2, z_3 \geq 0\},$$

and hence, for all $z \in \upsilon(\{0,x\})$, we see $\sigma_{0,x}(z) = \|z\|_\infty + \|(z_1-1, z_2+1, z_3)\|_\infty \geq 1+1 = 2$. But, for every $t \in [0,1]$, $\{0,x\} \leq z_t := (1,0,t)$ is such that

$$\sigma_{x,0}(z_t) = \|z_t - x\|_\infty + \|z_t - 0\|_\infty = \|(0,1,t)\|_\infty + \|(1,0,t)\|_\infty = 2,$$

so that there exists no unique upper bound of $\{x,0\}$ minimizing the distance sum to x and 0. We conclude that this space is not an υ-quasi-lattice.

There do exist ordered Banach spaces endowed with closed proper generating cones that are not normed Riesz spaces, nor μ-quasi-lattices or υ-quasi-lattices:

Example 3.5.6. Let $\{\mathbb{R}^3, \|\cdot\|_\infty\}$ be endowed with the cone defined by the four extreme rays $\{(\pm 1, \pm 1, 1)\}$. Let $x := (0,0,0)$ and $y := (2,0,0)$. It can be seen that $\mu(\{x,y\}) = \{(1,t,1) \in \mathbb{R}^3 : t \in [-1,1]\}$. Since this set is not a singleton, this space is not a Riesz space. Moreover, $\sigma_{x,y}$ takes the constant value 2 on $\mu(\{x,y\})$, and hence there does not exist a unique element minimizing $\sigma_{x,y}$ on $\mu(\{x,y\})$. Therefore this space is not a μ-quasi-lattice. Furthermore, if $z \in \upsilon(\{x,y\})$ and $z_3 > 1$, then $\sigma_{x,y}(z) > 2$, and since $\upsilon(\{x,y\}) \cap \{z \in \mathbb{R}^3 : z_3 \leq 1\} = \mu(\{x,y\})$, all minimizers of $\sigma_{x,y}$ in $\upsilon(\{x,y\})$ must be elements of $\mu(\{x,y\})$. Since $\sigma_{x,y}$ is constant on $\mu(\{x,y\})$, this space is not an υ-quasi-lattice.

The following results establish some basic properties of quasi-lattices.

Proposition 3.5.7. *If X is a quasi-lattice, then X_+ is a proper and generating cone.*

Proof. If X_+ is not proper, there exists an $x \in X$ such that $x > 0$ and $-x > 0$. Let $z \geq \{0, x\}$ be arbitrary. Then $z - x > z \geq 0 > x$, so that $z > z - x \geq \{0, x\}$. Hence no upper bound of $\{0, x\}$ is minimal and therefore X cannot be a μ-quasi-lattice.

Moreover, every $z \in \{\lambda x : \lambda \in [-1, 1]\}$ minimizes $\sigma_{-x,x}$ on $\upsilon(\{x, -x\})$, therefore X cannot be an υ-quasi-lattice either.

For all $x \in X$, since $\upsilon(\{x, 0\})$ is non-empty, taking any $z \in \upsilon(\{x, 0\})$ and writing $x = z - (z - x)$ shows that X_+ is generating in X. $\square$

Surprisingly, many elementary Riesz space properties have direct analogues in quasi-lattices. Many of the proofs below follow arguments from [49, Sections 5, 6] nearly verbatim.

Theorem 3.5.8. *Let X be a quasi-lattice, and $x, y, z \in X$ arbitrary. Then:*

(1) $x \tilde{\vee} x = x \tilde{\wedge} x = x$.

(2) For $\alpha \geq 0$, $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x \tilde{\vee} y)$ *and* $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x \tilde{\wedge} y)$.

(3) For $\alpha \leq 0$, $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x \tilde{\wedge} y)$ *and* $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x \tilde{\vee} y)$.

(4) $(x \tilde{\vee} y) + z = (x + z)\tilde{\vee}(y + z)$ *and* $(x \tilde{\wedge} y) + z = (x + z)\tilde{\wedge}(y + z)$.

(5) $x^{\pm} \geq 0$, $x^- = (-x)^+$.

(6) $\lceil x \rceil \geq 0$ *and, for all* $\alpha \in \mathbb{R}$, $\lceil \alpha x \rceil = |\alpha| \lceil x \rceil$. *In particular* $\lceil -x \rceil = \lceil x \rceil$.

(7) $x = x^+ - x^-$; $x^+ \tilde{\wedge} x^- = 0$ *and* $\lceil x \rceil = x^+ + x^-$.

(8) If $x \geq 0$, *then* $x \tilde{\wedge} 0 = 0$ *and* $x = x^+ = \lceil x \rceil$.

(9) $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.

(10) $x \tilde{\vee} y + x \tilde{\wedge} y = x + y$ *and* $x \tilde{\vee} y - x \tilde{\wedge} y = \lceil x - y \rceil$.

(11) $x \tilde{\vee} y = \frac{1}{2}(x + y) + \frac{1}{2} \lceil x - y \rceil$ *and* $x \tilde{\wedge} y = \frac{1}{2}(x + y) - \frac{1}{2} \lceil x - y \rceil$.

Proof. Assertion (1) follows from $x \leq x$ and the fact that $\sigma_{x,x}(x) = 0$ and $\sigma_{x,x}(y) > 0$ for all $y \neq x$.

We prove the assertion (2) for μ-quasi-lattices. The case $\alpha = 0$ follows from (1), hence we assume $\alpha > 0$. By definition, $x \tilde{\vee} y$ is a minimal upper bound of $\{x, y\}$. Since $\alpha > 0$, the element $\alpha(x \tilde{\vee} y)$ is then a minimal upper bound of $\{\alpha x, \alpha y\}$. Suppose that $\alpha(x \tilde{\vee} y) \neq (\alpha x)\tilde{\vee}(\alpha y)$, then there exists a minimal upper bound of $\{\alpha x, \alpha y\}$, say z_0, such that

$$\sigma_{\alpha x, \alpha y}(z_0) = \|z_0 - \alpha x\| + \|z_0 - \alpha y\| < \|\alpha(x \tilde{\vee} y) - \alpha x\| + \|\alpha(x \tilde{\vee} y) - \alpha y\|.$$

But then $\alpha^{-1}z_0$ is a minimal upper bound for $\{x,y\}$, and

$$\sigma_{x,y}(\alpha^{-1}z_0) = \|\alpha^{-1}z_0 - x\| + \|\alpha^{-1}z_0 - y\| < \|(x\tilde{\vee}y) - x\| + \|(x\tilde{\vee}y) - y\|,$$

contradicting the definition of $x\tilde{\vee}y \in \mu(\{x,y\})$ as the unique element minimizing $\sigma_{x,y}$ on $\mu(\{x,y\})$. We conclude that $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x\tilde{\vee}y)$. The same argument holds for υ-quasi-lattices by ignoring the word 'minimal' in the previous argument. By using what was just established, showing that $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x\tilde{\wedge}y)$ holds is an elementary calculation.

The assertion (3) follows from applying (2) with $\beta := -\alpha \geq 0$.

The assertion (4) follows from the translation invariance of both the metric defined by the norm and the partial order, and (5) is immediate from the definitions.

To establish (6), we notice that $\{x,-x\} \leq \lceil x \rceil$ implies $0 \leq x - x \leq 2\lceil x \rceil$. The second part follows by noticing that $(-\alpha x)\tilde{\vee}(\alpha x) = (-|\alpha|x)\tilde{\vee}(|\alpha|x)$ and applying (2).

We prove (7). By (4), $x^+ - x = (x\tilde{\vee}0) - x = (x-x)\tilde{\vee}(-x) = 0\tilde{\vee}(-x) = x^-$, so $x = x^+ - x^-$. By this, we then have $0 = -x^- + x^- = x\tilde{\wedge}0 + x^- = (x+x^-)\tilde{\wedge}(x^-) = (x^+)\tilde{\wedge}(x^-)$. By (2) and (4), $\lceil x \rceil = (-x)\tilde{\vee}x = (-x)\tilde{\vee}x + x - x = 0\tilde{\vee}(2x) - x = 2x^+ - x^+ + x^- = x^+ + x^-$.

We prove (8). Let $x \geq 0$, then 0 is an upper bound of $\{0,-x\}$. Moreover, since the cone is proper, 0 is a minimal upper bound for $\{0,-x\}$. But, for any $z \in X$ (and in particular all (minimal) upper bounds of $\{0,-x\}$), we have

$$\sigma_{-x,0}(0) = \|0 - (-x)\| + \|0 - 0\| = \|0 - (-x)\| \leq \sigma_{-x,0}(z).$$

Hence we have $0 = (-x)\tilde{\vee}0 = x^-$, and hence, by (7), $x = x^+ = \lceil x \rceil$.

The assertion (9) follows from (6) and (8).

We prove (10). We observe that $x\tilde{\vee}y = ((x-y)\tilde{\vee}0) + y = (x-y)^+ + y$, and $x\tilde{\wedge}y = x + (0\tilde{\wedge}(y-x)) = x - (x-y)^+$. Adding these two equations yields $x\tilde{\vee}y + x\tilde{\wedge}y = x+y$, and subtracting gives $x\tilde{\vee}y - x\tilde{\wedge}y = 2(x-y)^+ + y - x = (2(x-y)\tilde{\vee}0) - (x-y) = ((x-y)\tilde{\vee}(-(x-y)) = \lceil x-y \rceil$.

The assertion (11) follows by adding and subtracting the equations established in (10). □

In a sense the more interesting results concerning quasi-lattices are ones outlining how they may differ from Riesz spaces and Banach lattices. An important remark, that may at first sight be counterintuitive, is the following: For elements x,y,z in a quasi-lattice, $x \leq z$ and $y \leq z$ does not, in general, imply that $x\tilde{\vee}y \leq z$. The following example shows how this may happen:

Example 3.5.9. We continue with Example 3.5.4. Let $x = (0,0,0)$ and $y = (0,0,2)$, then $x\tilde{\vee}y = (1,0,1)$. The set of minimal upper bounds of $\{x,y\}$ forms a branch of a hyperbola. Choosing z from this hyperbola such that z and $x\tilde{\vee}y$ are not comparable, say any $z = (\sqrt{t^2+1}, \pm t, 1)$ with $t > 0$, we see that, although $x \leq z$ and $y \leq z$, it does not hold that $x\tilde{\vee}y \leq z$.

The previous example shows how it may sometimes happen in quasi-lattices that the quasi-supremum operation is not monotone: $x \leq y$ does not necessarily imply

$x^+ \leq y^+$. We can therefore not expect distributive laws, Birkhoff type inequalities or the Riesz decomposition property to hold in general quasi-lattices.

The following example shows how a quasi-supremum operation need not even be associative:

Example 3.5.10. Let $\{\mathbb{R}^3, \|\cdot\|_2\}$ be endowed with a 'half Lorentz cone'

$$C := \{(x_1, x_2, x_3) : x_1 \geq (x_2^2 + x_3^2)^{1/2}, x_2 \geq 0\}.$$

By Corollary 3.6.2, this space is a μ-quasi-lattice.

For any $x, y \in \mathbb{R}^3$, we claim that $(x\tilde{\vee}y)_2 = \max\{x_2, y_2\}$. To this end, let $z \geq \{x, y\}$ be arbitrary and define $z' := (z_1, \max\{x_2, y_2\}, z_3)$. We first show that $z' \geq \{x, y\}$. Firstly, $z'_2 - x_2 = \max\{x_2, y_2\} - x_2 \geq 0$ and $z'_2 - y_2 = \max\{x_2, y_2\} - y_2 \geq 0$. Since $z_2 - x_2 \geq 0$ and $z_2 - y_2 \geq 0$, we also have $z_2 \geq z'_2$. Also, where we use the fact that $(z_2 - z'_2)(z_2 - x_2) \geq 0$ and $(z_2 - z'_2)^2 \geq 0$ in the last step,

$$\begin{aligned}
z'_1 - x_1 &= z_1 - x_1 \\
&\geq \sqrt{(z_2 - x_2)^2 + (z_3 - x_3)^2} \\
&= \sqrt{(z_2 - z'_2 + z'_2 - x_2)^2 + (z_3 - x_3)^2} \\
&= \sqrt{(z_2 - z'_2)^2 + 2(z_2 - z'_2)(z_2 - x_2) + (z'_2 - x_2)^2 + (z'_3 - x_3)^2} \\
&\geq \sqrt{(z'_2 - x_2)^2 + (z'_3 - x_3)^2}.
\end{aligned}$$

Similarly $z'_1 - y_1 \geq \sqrt{(z'_2 - y_2)^2 + (z'_3 - y_3)^2}$, so that $z' \geq \{x, y\}$. We claim that $\sigma_{x,y}(z) \geq \sigma_{x,y}(z')$. Indeed, again since $(z_2 - z'_2)(z_2 - x_2) \geq 0$ and $(z_2 - z'_2)^2 \geq 0$,

$$\begin{aligned}
& \|z - x\|_2 \\
= & \sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2 + (z_3 - x_2)^2} \\
= & \sqrt{(z_1 - x_1)^2 + (z_2 - z'_2 + z'_2 - x_2)^2 + (z_3 - x_2)^2} \\
= & \sqrt{(z_1 - x_1)^2 + (z_2 - z'_2)^2 + 2(z_2 - z'_2)(z'_2 - x_2) + (z'_2 - x_2)^2 + (z_3 - x_2)^2} \\
\geq & \sqrt{(z_1 - x_1)^2 + (z'_2 - x_2)^2 + (z_3 - x_2)^2} \\
= & \sqrt{(z'_1 - x_1)^2 + (z'_2 - x_2)^2 + (z'_3 - x_2)^2} \\
= & \|z' - x\|_2.
\end{aligned}$$

Similarly we have $\|z - y\|_2 \geq \|z' - y\|_2$. Therefore $\sigma_{x,y}(z) = \|z - x\|_2 + \|z - y\|_2 \geq \sigma_{x,y}(z')$. We conclude that $(x\tilde{\vee}y)_2 = \max\{x_2, y_2\}$, else, by the above construction, there would exist an upper bound of $\{x, y\}$ different from $x\tilde{\vee}y$, but which also minimizes $\sigma_{x,y}$ on $\mu(\{x, y\})$.

Now let $a := (0, 0, 0)$, $b := (0, -1, 1)$ and $c := (0 - 1, -1)$. Using what was just proven and the fact that the space is a μ-quasi-lattice, it can be seen that $a\tilde{\vee}b$ must be an element of the plane $\{x \in \mathbb{R}^3 : x_2 = 0\}$ and must be a minimal upper bound of $\{a,b\}$. The minimal upper bounds of $\{a,b\}$ that are elements of $\{x \in \mathbb{R}^3 : x_2 = 0\}$ can be parameterized by $\gamma : t \mapsto (\sqrt{1+(1-t)^2}, 0, t)$ with $t \in (-\infty, 1]$ and the function $t \mapsto \sigma_{a,b}(\gamma(t))$ attains its minimum at $t = \sqrt{3}-1$. Therefore $a\tilde{\vee}b = (2\sqrt{2-\sqrt{3}}, 0, \sqrt{3}-1)$. Again, using similar reasoning, it can be verified (using a computer algebra system) that $(a\tilde{\vee}b)\tilde{\vee}c = (\sqrt{1+(1+\kappa)^2}, 0, \kappa)$, where $\kappa := 23^{-1}\left(-29 - 8\sqrt{2} + 9\sqrt{3} + 12\sqrt{6}\right)$. Also, since $(1,-1,0)$ is the only minimal upper bound of $\{b,c\}$ that is an element of the plane $\{x \in \mathbb{R}^3 : x_2 = -1\}$, we must have $b\tilde{\vee}c = (1,-1,0)$. It can then be verified that $a\tilde{\vee}(b\tilde{\vee}c) = (2,0,0)$. We conclude that $a\tilde{\vee}(b\tilde{\vee}c) \neq (a\tilde{\vee}b)\tilde{\vee}c$.

The triangle and reverse triangle inequality take the following form in quasi-lattices. They reduce to the familiar ones in lattice-ordered μ-quasi-lattices.

Theorem 3.5.11. (*Triangle and reverse triangle inequality*) *Let X be a quasi-lattice and $x, y \in X$ be arbitrary. Then*

$$\{x+y, -(x+y)\} \leq \lceil x \rceil + \lceil y \rceil,$$

and

$$\{x - \lceil y \rceil, -x - \lceil y \rceil, y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil.$$

Proof. By Theorem 3.5.8 (7), for all $z \in X$, we have $\lceil z \rceil \geq z^{\pm} \geq \pm z$, and hence we obtain $\lceil x \rceil + \lceil y \rceil \geq x^+ + y^+ \geq x + y$ and $\lceil x \rceil + \lceil y \rceil \geq x^- + y^- \geq -x - y$. Therefore $\lceil x \rceil + \lceil y \rceil$ is an upper bound of $\{x+y, -(x+y)\}$.

To establish the second inequality, we use what was just established to see, by Theorem 3.5.8 (6) and (9), that $\{x, -x\} = \{(x \pm y) \mp y, -((x \pm y) \mp y)\} \leq \lceil x \pm y \rceil + \lceil \mp y \rceil = \lceil x \pm y \rceil + \lceil y \rceil$. Hence $\{x - \lceil y \rceil, -x - \lceil y \rceil\} \leq \lceil x \pm y \rceil$. Similarly, $\{y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil$, and finally we conclude that $\{x - \lceil y \rceil, -x - \lceil y \rceil, y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil$. ☐

The following result allows us to conclude that monotone υ-quasi-lattices are in fact μ-quasi-lattices:

Theorem 3.5.12. *Every monotone υ-quasi-lattice is a μ-quasi-lattice, and its υ– and μ-quasi-lattice structures coincide.*

Proof. We first claim that, if X is a monotone ordered Banach space, then, for $x, y \in X$, if $z_0 \in \upsilon(\{x,y\})$ is such that $\|z - x\| + \|z - y\| > \|z_0 - x\| + \|z_0 - y\|$ for all $z \in \upsilon(\{x,y\})$ with $z \neq z_0$, then z_0 is a minimal upper bound of $\{x,y\}$.

As to this, by translating, we may assume that $y = 0$. Let $z \in X$ be any element satisfying $\{x, 0\} \leq z \leq z_0$. Then $0 \leq z \leq z_0$ and $0 \leq z - x \leq z_0 - x$. By monotonicity, $\|z\| \leq \|z_0\|$ and $\|z - x\| \leq \|z_0 - x\|$, so that $\|z\| + \|z - x\| \leq \|z_0\| + \|z_0 - x\|$. The hypothesis on z_0 then implies that $z = z_0$. Hence z_0 is a minimal upper bound of $\{x,y\}$, establishing the claim.

Let X be a monotone υ-quasi-lattice and $x, y \in X$ arbitrary. By the above claim, the υ-quasi-supremum of $\{x,y\}$ is a minimal upper bound of $\{x,y\}$. Since

$\mu(\{x,y\}) \subseteq \upsilon(\{x,y\})$ we have that the υ-quasi-supremum of $\{x,y\}$ is also the μ-quasi-supremum. We conclude that X is also a μ-quasi-lattice, and that its $\upsilon-$ and μ-quasi-lattice structures coincide. $\square$

The following example shows that there exist υ-quasi-lattices in which some υ-quasi-suprema are not minimal upper bounds.

Example 3.5.13. Consider the space $\{\mathbb{R}^3, \|\cdot\|_2\}$, endowed with the cone

$$C := \{(a,b,c) \in \mathbb{R}^3 : ax^2 + bx + c \geq 0 \text{ for all } x \in [0,1]\}.$$

By Theorem 3.6.1, this space is an υ-quasi-lattice. Let $x := (0,1,0)$, $y := (0,-1,1)$. It can be verified (using a computer algebra system) that

$$x \tilde{\vee} y = (2^{-1}(2-\sqrt{3}), -2^{-1}(2-\sqrt{3}), 1),$$

while $\{x,y\} \leq (1,-1,1) < x\tilde{\vee}y$. Therefore $x\tilde{\vee}y \notin \mu(\{x,y\})$.

By comparing the norms of the elements in $0 \leq (0,-1,1) \leq (0,0,1)$, we see that this space is not monotone. We can therefore not draw any conclusion from Theorem 3.5.12 as to whether this space is a μ-quasi-lattice. A valid conclusion we may draw is that, if this space is indeed also a μ-quasi-lattice in addition to being an υ-quasi-lattice, its $\mu-$ and υ-quasi-lattice structures will *not* coincide.

3.6 A concrete class of quasi-lattices

In the previous section we have already noted that lattice ordered Banach spaces are μ-quasi-lattices (cf. Proposition 3.5.2) and gave a number of examples of quasi-lattices. We begin this section by proving that quite a large class of (not necessarily lattice ordered) ordered Banach spaces with closed generating cones are in fact quasi-lattices. Afterwards, we briefly investigate conditions under which a space has a quasi-lattice as a dual, or is the dual of a quasi-lattice.

We recall that a normed space X is *strictly convex* or *rotund* if, for $x, y \in X$, $\|x+y\| = \|x\| + \|y\|$ implies that either x or y is a non-negative multiple of the other [29, Definition 5.1.1, Proposition 5.1.11].

The following theorem shows that there exist relatively many quasi-lattices:

Theorem 3.6.1. *Every strictly convex reflexive ordered Banach space X with a closed proper generating cone is an υ-quasi-lattice.*

Proof. We need to prove that every pair of elements $x_0, y_0 \in X$ has an υ-quasi-supremum in X.

If x_0 and y_0 are comparable, by exchanging the roles of x_0 and y_0 if necessary, we may assume $x_0 \leq y_0$. We may further assume that $x_0 = 0$ by translating over $-x_0$. We will denote the distance sum to 0 and y_0 by σ instead of σ_{0,y_0}.

If $y_0 = 0$, then $\sigma(z) = 0$ if and only if $z = 0$, so that $0\tilde{\vee}0 = 0$. If $0 \neq y_0 \geq 0$, we have that $y_0 \in \upsilon(\{0, y_0\})$ and, for all $z \in \upsilon(\{0, y_0\})$, we have $\sigma(z) = \|y_0 - z\| + \|z\| \geq \|y_0\| = \sigma(y_0)$. Suppose that $z_0 \in \upsilon(\{0, y_0\})$ is such that $\sigma(y_0) = \sigma(z_0)$. We must have $z_0 \neq 0$, else $0 \leq y_0 \leq z_0 = 0$ hence, since X_+ is proper, $y_0 = 0$, while $y_0 \neq 0$. Then, since

$$\|y_0 - z_0 + z_0\| = \|y_0\| = \sigma(y_0) = \sigma(z_0) = \|y_0 - z_0\| + \|z_0\|,$$

by strict convexity we obtain $y_0 - z_0 = \lambda z_0$ for some $\lambda \geq 0$. Hence $z_0 \geq y_0 = (1+\lambda)z_0 \geq z_0$ and then, since X_+ is proper, $y_0 = z_0$. Therefore $0\tilde{\vee} y_0 = y_0$.

We consider the case where neither $x_0 \leq y_0$ nor $y_0 \leq x_0$. Again, by translating, we may assume without loss of generality that $x_0 = 0$, and that neither $y_0 \leq 0$ nor $0 \leq y_0$. We again denote the distance sum to 0 and y_0 by σ instead of σ_{0,y_0}.

Since X_+ is generating, $\upsilon(\{y_0, 0\}) = X_+ \cap (y_0 + X_+)$ is non-empty, hence let $z_0 \in X_+ \cap (y + X_+)$. Consider the non-empty closed bounded and convex set

$$K := X_+ \cap (y_0 + X_+) \cap \{x \in X : \sigma(x) \leq \sigma(z_0)\}.$$

We note that $0, y_0 \notin K$, since we had assumed that neither $y_0 \leq 0$ nor $0 \leq y_0$ holds.

The function σ is continuous and convex and, since K is bounded closed and convex and X is reflexive, by [5, Theorem 2.11], there exists an element $z_m \in K$ minimizing σ on K. We claim that z_m is the unique minimizer of σ on K. To prove this claim it is sufficient to establish that σ is strictly convex on K, i.e., if $z, z' \in K$ with $z \neq z'$ and $t \in (0,1)$, then $\sigma(tz + (1-t)z') < t\sigma(z) + (1-t)(z')$.

We first claim that the line $\mathbb{R}y_0$ does not intersect K. Indeed, if $\lambda y_0 \in K$ for some $\lambda \in \mathbb{R}$, then we must have $\lambda \neq 0$, since $0 \notin K$. But then $\lambda y_0 \in K \subseteq X_+$ implies that either $y_0 \leq 0$ or $0 \leq y_0$, contrary to our assumption that neither $y_0 \leq 0$ nor $0 \leq y_0$.

We now prove that σ is strictly convex on K. Let $z, z' \in K$ be arbitrary but distinct and $t \in (0,1)$. If $z \neq \lambda z'$ for all $\lambda \geq 0$, then, by strict convexity of X, $\|tz + (1-t)z'\| < t\|z\| + (1-t)\|z'\|$, and hence $\sigma(tz+(1-t)z') < t\sigma(z) + (1-t)\sigma(z')$. On the other hand, if $z' = \lambda z$ for some $\lambda \geq 0$, we must have that $\lambda \neq 1$ (since $z \neq z'$) and $\lambda \neq 0$ (since $0 \notin K$). Therefore, supposing that

$$\|(1-t)(y_0 - z) + t(y_0 - z')\| = (1-t)\|y_0 - z\| + t\|y_0 - z'\|,$$

by strict convexity of X, we obtain $(1-t)(y_0 - z) = \rho t(y_0 - z')$ for some $\rho > 0$ (if $\rho = 0$, then $y_0 = z \in K$ contradicts $y_0 \notin K$). By rewriting, we obtain $((1-t)-\rho t)y_0 = ((1-t) - \rho t\lambda)z$. If $((1-t) - \rho t\lambda) = 0$, then $((1-t) - \rho t) \neq 0$ since $\lambda \neq 1$ and $\rho t \neq 0$, and hence $y_0 = 0$, contradicting the assumption that neither $y_0 \leq 0$ nor $0 \leq y_0$. Therefore $((1-t) - \rho t\lambda) \neq 0$, and $z \in K \cap \mathbb{R}y_0$, contracting the fact that K and $\mathbb{R}y_0$ are disjoint. Therefore, we must have $\|(1-t)y_0 - (1-t)z + ty_0 - tz'\| < (1-t)\|y_0 - z\| + t\|y_0 - z'\|$, and hence $\sigma(tz + (1-t)z') < t\sigma(z) + (1-t)\sigma(z')$.

We conclude that σ is strictly convex on K, and that $z_m \in K$ is the unique minimizer of σ on K. Then clearly z_m is also the unique minimizer of σ on $\upsilon(\{0, y_0\})$. $\square$

Theorem 3.6.1 and Theorem 3.5.12 together yield the following two corollaries:

Corollary 3.6.2. *Every strictly convex reflexive monotone ordered Banach space with a closed proper generating cone is both an υ-quasi-lattice and a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).*

Corollary 3.6.3. *For $1 < p < \infty$, every L^p-space endowed with a closed proper generating cone is an υ-quasi-lattice. In particular, every ℓ^p-space and every space $\{\mathbb{R}^n, \|\cdot\|_p\}$ that is endowed with a closed proper generating cone is an υ-quasi-lattice. If, in addition, the space is monotone, it is also a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).*

Proof. That an L^p-space is strictly convex for every $1 < p < \infty$ is a consequence of [29, Theorem 5.2.11]. The result then follows from the previous theorem and Theorem 3.5.12. ☐

The remainder of this section will be devoted to dual considerations, specifically to the question of when the dual of a pre-ordered Banach space is a quasi-lattice. The following result gives necessary conditions for this to be the case.

Proposition 3.6.4. *If a pre-ordered Banach space X with a closed cone has a quasi-lattice as dual, then:*

(1) There exists an $\alpha > 0$ such that X is α-max-normal.

(2) $X_+ - X_+$ is dense in X.

Proof. By Proposition 3.5.7, the dual cone is proper and generating. By part (1)(e) of Theorem 3.3.7, there exists an $\alpha > 0$ such that X is α-max-normal. By [2, Theorem 2.13(2)], $X_+ - X_+$ is weakly dense in X. Since $X_+ - X_+$ is convex, its weak closure and norm closure coincide, and $X_+ - X_+$ is therefore norm dense in X. ☐

Corollary 3.6.5. *Let X be a pre-ordered Banach space with a closed generating cone. If X has a quasi-lattice as dual, then there exists an $\alpha > 0$ such that X is α-Ellis-Grosberg-Krein regular.*

Proof. By the previous result, there exists a $\beta > 0$ such that X is β-max-normal. The cone X_+ was assumed to be generating, and therefore X is Andô-regular. By Proposition 3.3.10 there exists an $\alpha > 0$ such that X is α-Ellis-Grosberg-Krein regular. ☐

The following theorem provides sufficient conditions for a pre-ordered Banach space to have a quasi-lattice as dual.

We recall that a normed space X is *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique element $\phi \in X'$ with $\|\phi\| = 1$ such that $\phi(x) = 1$ [29, Definition 5.4.1, Corollary 5.4.3].

Theorem 3.6.6. *If, for some $\alpha > 0$, X is an α-normal smooth reflexive pre-ordered Banach space with a closed cone such that $X_+ - X_+$ is dense in X, then its dual is an υ-quasi-lattice.*

If, in addition, X is approximately 1-conormal, its dual is a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).

Proof. By [2, Corollary 2.14, Theorem 2.40], the dual cone is proper and generating in X'. That the dual cone is closed is elementary. By [29, Proposition 5.4.7], X' is strictly convex, since X was assumed to be smooth. Therefore X' satisfies the hypotheses of Theorem 3.6.1, and is an υ-quasi-lattice.

If we make the extra assumption that X is approximately 1-conormal, then by part (2)(d) of Theorem 3.3.7, X' is monotone. Then, by Theorem 3.5.12, X' is a μ-quasi-lattice. ☐

The following example shows that there exist υ-quasi-lattices that are not α-normal for any $\alpha > 0$. It cannot have a quasi-lattice as dual, nor is it the dual of a quasi-lattice. Indeed, by Proposition 3.6.4 its dual is not a quasi-lattice. Moreover, by part (2)(e) of Theorem 3.3.7, since the space is not α-normal for any $\alpha > 0$, its cone is not the dual cone of a pre-ordered Banach space with a closed generating cone, and in particular, it is not the dual of a quasi-lattice.

Example 3.6.7. Consider the following subset of ℓ^2:

$$C := \left\{ x \in \ell^2 : x_1 \geq \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2 \right)^{1/2} \right\}.$$

Clearly, $C \cap (-C) = \{0\}$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. Also, by Minkowski's inequality, $C + C \subseteq C$ so that we may conclude that C is a proper cone. For any $x \in \ell^2$, taking $y := \lambda(1, 0, 0, \ldots) \in C$ with $\lambda \geq |x_1| + \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2\right)^{1/2}$, we see $x = y - (y - x) \in C - C$, so that C is generating in ℓ^2. Since the map $\rho_0 : \ell^2 \to \mathbb{R}$ defined by $\rho_0 : x \mapsto \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2\right)^{1/2}$ is a continuous seminorm, the map $\rho : x \mapsto x_1 - \rho_0(x)$ is also continuous. Since $C = \rho^{-1}(\mathbb{R}_{\geq 0})$, we conclude that C is closed. By Theorem 3.6.1, this space is an υ-quasi-lattice.

We claim that this space is not α-normal for any $\alpha > 0$. It is sufficient to show, for every $\alpha \geq 1$, that there exist $x, y \in \ell^2$ with $0 \leq x \leq y$, such that $\|x\| > \alpha \|y\|$. To this end, we set $y := (2, 0, \ldots)$. We define x as follows: let $\mathbb{N} \ni n_\alpha > (2\alpha)^2$ and $x = (1, 0, \ldots, 0, \sqrt{n_\alpha}, 0, \ldots)$ with $\sqrt{n_\alpha}$ occurring at the n_α-th coordinate. We then see that $0 \leq x \leq y$, while

$$\|x\| = (1 + n_\alpha)^{\frac{1}{2}} > n_\alpha^{\frac{1}{2}} > 2\alpha = \alpha \|y\|.$$

We conclude that this space is not α-normal for any $\alpha > 0$.

3.7 A class of quasi-lattices with absolutely monotone spaces of operators

In this section we show that a real Hilbert space $\mathcal{H}$ endowed with a Lorentz cone (defined below) is a 1-absolutely Davies-Ng regular μ-quasi-lattice (that is not a Banach lattice if $\dim \mathcal{H} \geq 3$). Through an application of Theorem 3.4.2, this will resolve the question posed in the introduction of whether there exist non-Banach lattice pre-ordered Banach spaces X and Y for which $B(X, Y)$ is absolutely monotone.

Results established in this section will be collected in Theorem 3.7.10. In particular it will be shown that $\|x\| = \| \lceil x \rceil \|$ for all $x \in \mathcal{H}$ (which is analogous to the identity $\|x\| = \||x|\|$ which holds for all elements x of a Banach lattice). Then, for $\alpha > 0$ and pre-ordered Banach spaces X and Y that are respectively approximately α-absolutely conormal and α-absolutely normal, the spaces of operators $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are shown to be α-absolutely normal. Furthermore, if $\alpha = 1$ (in particular if X and Y are Hilbert spaces endowed with Lorentz cones), then the operator norms of $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are positively attained.

We begin with the following lemma which outlines sufficient conditions for establishing 1-absolute Davies-Ng regularity of absolutely monotone quasi-lattices:

Lemma 3.7.1. *Let X be an absolutely monotone quasi-lattice satisfying $\|x\| = \| \lceil x \rceil \|$ for all $x \in X$. Then X is 1-absolutely Davies-Ng regular.*

Proof. The fact that $\|x\| = \| \lceil x \rceil \|$ for all $x \in X$ implies that X is 1-absolutely conormal. Therefore X is 1-absolutely Davies-Ng regular. □

Every Banach lattice satisfies the hypotheses of the previous proposition. The rest of this section will be devoted to proving that there exist quasi-lattices that are not Banach lattices, but still satisfy the hypothesis of the previous proposition.

Definition 3.7.2. Let $\mathcal{H}$ be a real Hilbert space. For a norm-one element $v \in \mathcal{H}$, let P be the orthogonal projection onto $\{v\}^{\perp}$. We define the *Lorentz cone*

$$\mathcal{L}_v := \{x \in \mathcal{H} : \langle v|x \rangle \geq \|Px\|\}.$$

As in Example 3.6.7, it is elementary to see that this cone is closed, proper and generating in $\mathcal{H}$.

It is widely known that the Hilbert space $\mathbb{R}^3$ ordered by the Lorentz cone $\mathcal{L}_{e_1} \subseteq \mathbb{R}^3$ is not a Riesz space (cf. Example 3.5.4). This is actually true for arbitrary Hilbert spaces endowed with a Lorentz cone as we will now proceed to show. The following two lemmas will be used in the proof of Proposition 3.7.5 which establishes this fact.

Lemma 3.7.3. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. If $x \in \mathcal{L}_v$ is such that $\langle x|v \rangle = \|Px\|$ and $z_1, z_2 \in \mathcal{L}_v$ are such that $x = z_1 + z_2$, then $z_1, z_2 \in \{\lambda x : \lambda \geq 0\}$.*

Proof. Let P be the orthogonal projection onto $\{v\}^{\perp}$. If $x = 0$, since $\mathcal{L}_v$ is proper, the statement is clear. Let $0 \neq x \in \mathcal{L}_v$ be such that $\langle x|v \rangle = \|Px\|$. Then $\langle x|v \rangle = \|Px\| > 0$, else $x = 0$. Suppose $z_1, z_2 \in \mathcal{L}_v$ are such that $x = z_1 + z_2$. Then

$$\langle x|v \rangle = \langle z_1 + z_2|v \rangle \geq \|Pz_1\| + \|Pz_2\| \geq \|P(z_1 + z_2)\| = \|Px\| = \langle x|v \rangle.$$

Therefore $\|Pz_1\| + \|Pz_2\| = \|Pz_1 + Pz_2\| = \|Px\| > 0$, and hence Pz_1 and Pz_2 cannot both be zero. We assume $Pz_1 \neq 0$, and then, by strict convexity of $\mathcal{H}$, $Pz_2 = \lambda Pz_1$ for some $\lambda \geq 0$. If $\langle z_1|v \rangle > \|Pz_1\|$ or $\langle z_2|v \rangle > \|Pz_2\|$, then $\langle x|v \rangle = \langle z_1|v \rangle + \langle z_2|v \rangle > \|Pz_1\| + \|Pz_2\| = \|Px\|$, contradicting the assumption that $\langle x|v \rangle = \|Px\|$. Hence, since $z_1, z_2 \in \mathcal{L}_v$, we must have $\langle z_1|v \rangle = \|Pz_1\|$ and $\langle z_2|v \rangle = \|Pz_2\|$, and therefore $\langle z_2|v \rangle = \|Pz_2\| = \lambda \|Pz_1\| = \lambda \langle z_1|v \rangle$. Now, since $\langle z_2|v \rangle = \lambda \langle z_1|v \rangle$ and $Pz_2 = \lambda Pz_1$, we obtain $z_2 = \langle z_2|v\rangle v + Pz_2 = \lambda z_1$, and hence $x = z_1 + z_2 = (1+\lambda)z_1$. We conclude that $z_1, z_2 \in \{\lambda x : \lambda \geq 0\}$. $\square$

Lemma 3.7.4. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. If $x \in \mathcal{L}_v$ is such that $\langle x|v\rangle = \|Px\|$ and $0 \leq y \leq x$, then $y \in \{\lambda x : \lambda \in [0,1]\}$.*

Proof. Since $\mathcal{L}_v$ is proper, this is clear if $x = 0$. If $0 \leq y \leq x \neq 0$, then $x = y+(x-y)$ with $y, (x-y) \in \mathcal{L}_v$, so that by the previous lemma $y = \lambda x$ for some $\lambda \geq 0$. If $\lambda > 1$, then $x \leq \lambda x = y \leq x$, since $\mathcal{L}_v$ is proper and $y = \lambda x$, implies $x = y = 0$ contradicting the assumption $x \neq 0$. We conclude that $\lambda \in [0,1]$. $\square$

Proposition 3.7.5. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ such that $\|v\| = 1$. If $\dim(\mathcal{H}) \geq 3$, then $\mathcal{H}$ is not a Riesz space (and hence not a Banach lattice).*

Proof. Let P be the orthogonal projection onto $\{v\}^\perp$ and $\{v, e_1, e_2\} \subseteq \mathcal{H}$ be any orthonormal set. For $t \in \mathbb{R}$, we have

$$\{0, 2e_1\} \leq e_1 + te_2 + \sqrt{t^2+1}v =: z_t.$$

We claim that each z_t is a minimal upper bound of $\{0, 2e_1\}$. We have $\langle z_t|v\rangle = \|Pz_t\|$, and hence by the previous lemma, if $\{0, 2e_1\} \leq y \leq z_t$, we must have $y = \lambda z_t$ for some $\lambda \in [0,1]$. If $\lambda < 1$, then $\lambda\sqrt{t^2+1} = \langle y - 2e_1|v\rangle$ and

$$\begin{aligned}
\|P(y-2e_1)\|^2 &= \|P(\lambda e_1 + \lambda te_2 + \lambda\sqrt{t^2+1}v - 2e_1)\|^2 \\
&= \|(\lambda-2)e_1 + \lambda te_2\|^2 \\
&= (\lambda-2)^2 + \lambda^2t^2 \\
&> 1 + \lambda^2t^2 \\
&> \lambda^2 + \lambda^2t^2.
\end{aligned}$$

Hence $\langle y - 2e_1|v\rangle = \lambda\sqrt{t^2+1} < \|P(y-2e_1)\|$, contradicting $2e_1 \leq y$. Therefore we must have $\lambda = 1$, and $y = z_t$, and hence z_t is a minimal upper bound of $\{0, 2e_1\}$ for every $t \in \mathbb{R}$. Clearly all z_t are distinct, and therefore there exists no supremum of $\{0, 2e_1\}$. $\square$

Since every Hilbert space is strictly convex, and knowing that Lorentz cones are closed proper and generating, we conclude from Theorem 3.6.1 that every Hilbert space endowed with a Lorentz cone is an v-quasi-lattice. We will now proceed to show that these spaces are absolutely monotone. Once this has been established, Theorem 3.5.12 will imply that they are in fact μ-quasi-lattices.

The following lemma will be applied in Propositions 3.7.7 and 3.7.9, which together will show that Hilbert spaces endowed with a Lorentz cones are in fact 1-absolutely Davies-Ng regular.

Lemma 3.7.6. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. Let $x \in \mathcal{H}$ and Q be the orthogonal projection onto* $\operatorname{span}\{x, v\}$. *If $\{-x, x\} \leq y$, then $\{-x, x\} \leq Qy$.*

Proof. Let P be the orthogonal projection onto $\{v\}^\perp$ and Q the orthogonal projection onto $\operatorname{span}\{x, v\}$. Let $Q^\perp := \operatorname{id} - Q$. We note that $\operatorname{ran}(\operatorname{Id} - P) = \operatorname{span}\{v\} \subseteq \operatorname{ran}(Q)$, so that $\operatorname{Id}-P$ and Q commute, and hence P and Q also commute. Therefore, from

$$\begin{aligned} \langle v|Qy \pm x\rangle &= \langle v|Qy + Q^\perp y - Q^\perp y \pm x\rangle \\ &= \langle v|y \pm x\rangle - \langle v|Q^\perp y\rangle \\ &= \langle v|y \pm x\rangle \\ &\geq \|P(y \pm x)\| \\ &\geq \|QP(y \pm x)\| \\ &= \|P(Qy \pm Qx)\| \\ &= \|P(Qy \pm x)\|, \end{aligned}$$

we conclude that $Qy \geq \{-x, x\}$. □

The following proposition, together with Theorem 3.5.12, will show that every Hilbert space endowed with a Lorentz cone is in fact a μ-quasi-lattice.

Proposition 3.7.7. *A real Hilbert space endowed with a Lorentz cone is absolutely monotone.*

Proof. Let $\mathcal{H}$ be a real Hilbert space ordered by a Lorentz cone $\mathcal{L}_v$, where $v \in \mathcal{H}$ is a norm-one element. Let P be the orthogonal projection onto $\{v\}^\perp$. Let $\{-x, x\} \leq y$ and let Q denote the orthogonal projection onto $V := \operatorname{span}\{x, v\}$. By Lemma 3.7.6, $\{-x, x\} \leq Qy$.

If $x \in \operatorname{span}\{v\}$, then $Px = 0$. Also $V = \operatorname{span}\{v\}$, so that $PQ = 0$. Therefore $\{-x, x\} \leq Qy$ implies $\langle v|Qy \pm x\rangle \geq \|PQ(y \pm x)\| = \|Px\| = 0$, and hence $\langle v|Qy\rangle \geq |\langle v|x\rangle|$. Then $\|Qy\| = |\langle v|Qy\rangle| \geq \langle v|Qy\rangle \geq |\langle v|x\rangle| = \|x\|$, and hence $\|x\| \leq \|Qy\| \leq \|y\|$ as was to be shown.

If $x \notin \operatorname{span}\{v\}$, since $0 \neq Px = x - \langle v|x\rangle v \in V$, we see that

$$e_\pm := (\sqrt{2}\|Px\|)^{-1}(\pm Px + \|Px\|v)$$

are orthonormal elements of $V \cap \mathcal{L}_v$. We claim that $V \cap \mathcal{L}_v = \{\lambda e_+ + \lambda' e_- : \lambda, \lambda' \geq 0\}$. Let $a \in V \cap \mathcal{L}_v$. Since $x \notin \operatorname{span}\{v\}$, $0 \neq Px \in V$ is orthogonal to v, and hence $\{Px, v\}$ is a basis of V. Then, by writing $a = \alpha Px + \beta v$ for some $\alpha, \beta \in \mathbb{R}$, we obtain $\beta = \langle a|v\rangle \geq \|Pa\| = |\alpha|\|Px\|$. Hence, by

$$\begin{aligned} \langle a|e_\pm\rangle &= (\sqrt{2}\|Px\|)^{-1}\langle \alpha Px + \beta v| \pm Px + \|Px\|v\rangle \\ &= (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\langle Px|Px\rangle + \beta\|Px\|) \\ &= (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\|Px\|^2 + \beta\|Px\|) \end{aligned}$$

$$\begin{aligned} &\geq (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\|Px\|^2 + |\alpha|\|Px\|^2) \\ &\geq 0, \end{aligned}$$

we conclude that $V \cap \mathcal{L}_v = \{\lambda e_+ + \lambda' e_- : \lambda, \lambda' \geq 0\}$. Now $Qy \pm x \in V \cap \mathcal{L}_v$ implies $\langle Qy \pm x | e_\pm \rangle \geq 0$, so that $\langle Qy | e_\pm \rangle \geq |\langle x | e_\pm \rangle|$, and hence $\|x\| \leq \|Qy\| \leq \|y\|$ as was to be shown. ☐

Remark 3.7.8. If $x \notin \operatorname{span}\{v\}$ we note that $(V, V \cap \mathcal{L}_v)$ in the previous proposition is isometrically order isomorphic to the Banach lattice $\{\mathbb{R}^2, \|\cdot\|_2\}$ with the standard cone through mapping $e_+ \in V$ and $e_- \in V$ to $(1,0) =: e_1 \in \mathbb{R}^2$ and $(0,1) =: e_2 \in \mathbb{R}^2$ respectively.

We can now show that real Hilbert spaces endowed with Lorentz cones satisfy the hypotheses of Lemma 3.7.1:

Proposition 3.7.9. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone. Then $\|x\| = \|\lceil x \rceil\|$ for all $x \in \mathcal{H}$. Hence $\mathcal{H}$ is 1-absolutely conormal.*

Proof. Let $v \in \mathcal{H}$ be a norm one element and order $\mathcal{H}$ with the Lorentz cone $\mathcal{L}_v$. We again denote the projection onto $\{v\}^\perp$ by P. Let $x \in \mathcal{H}$ be arbitrary.

If $x \geq 0$ or $x \leq 0$, then, by Theorem 3.5.8 (6) and (8), $\lceil x \rceil = x$ or $\lceil x \rceil = -x$, respectively, so that $\|x\| = \|\lceil x \rceil\|$.

It remains to show that $\|x\| = \|\lceil x \rceil\|$ when neither $x \geq 0$ nor $x \leq 0$. Then $x \notin \operatorname{span}\{v\}$. We define the two dimensional subspace $V := \operatorname{span}\{x, v\}$, denote the orthogonal projection onto V by Q, and define $Q^\perp := \mathrm{Id} - Q$. By Lemma 3.7.6, if $\{-x, x\} \leq w$, then $\{-x, x\} \leq Qw$.

When $w \notin V$, we see that $Q^\perp w \neq 0$ implies

$$\begin{aligned} &\|w - x\| + \|w + x\| \\ &= \sqrt{\|Q(w-x)\|^2 + \|Q^\perp(w-x)\|^2} + \sqrt{\|Q(w+x)\|^2 + \|Q^\perp(w+x)\|^2} \\ &= \sqrt{\|Qw - x\|^2 + \|Q^\perp w\|^2} + \sqrt{\|Qw + x\|^2 + \|Q^\perp w\|^2} \\ &> \|Qw - x\| + \|Qw + x\|. \end{aligned}$$

We conclude that $\lceil x \rceil$ must be an element of V. Furthermore, by Proposition 3.7.7 and Theorem 3.5.12, $\mathcal{H}$ is a μ-quasi-lattice, and hence $\lceil x \rceil \in V$ is a minimal upper bound of $\{-x, x\}$.

Finally, V endowed with the cone $\mathcal{L}_v \cap V$ is seen to be isometrically order isomorphic to $\{\mathbb{R}^2, \|\cdot\|_2\}$ with the standard cone (cf. Remark 3.7.8). Viewing V as a Banach lattice, we notice that the Banach lattice absolute value $|x|$ in V is the only minimal upper bound for $\{-x, x\}$ in $\mathcal{H}$ that is also an element of V. By the argument in the previous paragraph, we conclude that $\lceil x \rceil = |x|$, and hence that $\|\lceil x \rceil\| = \||x|\| = \|x\|$, by invoking the Banach lattice property $\||x|\| = \|x\|$ in V. ☐

We collect the results established in this section and some of their consequences in the following theorem. We note that (8) below resolves the question alluded to in the introduction of the existence pre-ordered Banach spaces X and Y, which are not Banach lattices, while $B(X, Y)$ is absolutely monotone.

Theorem 3.7.10. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone. Then:*

(1) If $\dim(\mathcal{H}) \geq 3$, then $\mathcal{H}$ is not a Riesz space (and hence not a Banach lattice).

(2) $\mathcal{H}$ is an υ-quasi-lattice.

(3) $\mathcal{H}$ is absolutely monotone.

(4) $\mathcal{H}$ is a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).

(5) For every $x \in \mathcal{H}$, $\|x\| = \| \lceil x \rceil \|$. Hence $\mathcal{H}$ is 1-absolutely conormal.

(6) $\mathcal{H}$ is 1-absolutely Davies-Ng regular.

(7) If X and Y are pre-ordered Banach spaces with closed cones, with X approximately 1-absolutely conormal and Y absolutely monotone, then the operator norms of both $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are positively attained, i.e., $\|T\| = \sup\{\|Tx\| : x \geq 0,\ \|x\| = 1\}$ for $T \in B(X, \mathcal{H})_+$ or $T \in B(\mathcal{H}, Y)_+$. In particular, if $\mathcal{H}_1$ is another real Hilbert space endowed with a Lorentz cone, then the operator norm of $B(\mathcal{H}, \mathcal{H}_1)$ is positively attained.

(8) If $\alpha > 0$ and X and Y are pre-ordered Banach spaces with closed cones, with X approximately α-absolutely conormal and Y α-absolutely normal, then both $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are α-absolutely normal. In particular, if $\mathcal{H}_1$ is another real Hilbert space endowed with a Lorentz cone, then $B(\mathcal{H}, \mathcal{H}_1)$ is absolutely monotone.

Proof. The assertion (1) is Proposition 3.7.5. The assertion (2) follows from Theorem 3.6.1. The assertion (3) was established in Proposition 3.7.7 and hence (4) follows from Corollary 3.6.2. The assertion (5) was established in Proposition 3.7.9, and hence (6) follows from Lemma 3.7.1. The assertion (7) follows from (6) and part (2) of Theorem 3.4.6. The assertion (8) is then immediate from (6) and part (3) of Theorem 3.4.2. □

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