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Chapter 2

Right inverses of surjections from cones onto Banach spaces

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2.1 Introduction

Consider the following question, that arose in other research of the authors: Let X be a real Banach space, ordered by a closed generating proper cone X^+, and let Ω be a topological space. Then the Banach space $C_0(\Omega, X)$, consisting of the continuous X-valued functions on Ω vanishing at infinity, is ordered by the natural closed proper cone $C_0(\Omega, X^+)$. Is this cone also generating? If X is a Banach lattice, then the answer is affirmative. Indeed, if $f \in C_0(\Omega, X)$, then $f = f^+ - f^-$, where $f^\pm(\omega) := f(\omega)^\pm$ $(\omega \in \Omega)$. Since the maps $x \mapsto x^\pm$ are continuous, $f^\pm$ is continuous, and since $\|f^\pm(\omega)\| \leq \|f(\omega)\|$ $(\omega \in \Omega)$, $f^\pm$ vanishes at infinity. Thus a decomposition as desired has been obtained. For general X, the situation is not so clear. The natural approach is to consider a pointwise decomposition as in the Banach lattice case, but for this to work we need to know that at the level of X the constituents $x^\pm$ in a decomposition $x = x^+ - x^-$ can be chosen in a continuous and simultaneously also bounded (in an obvious terminology) fashion, as x varies over X. Boundedness is certainly attainable, due to the following classical result:

Theorem 2.1.1. (*Andô's Theorem [3]*) *Let X be a real Banach space ordered by a closed generating proper cone $X^+ \subseteq X$. Then there exists a constant $K > 0$ with the property that, for every $x \in X$, there exist $x^\pm \in X^+$ such that $x = x^+ - x^-$ and $\|x^\pm\| \leq K\|x\|$.*

Continuity is not asserted, however. Hence we are not able to settle our question in the affirmative via Andô's Theorem alone and stronger results are needed. With

Ω a compact Hausdorff space, it is a consequence of a result due to Asimow and Atkinson [4, Theorem 2.3] that $C(\Omega, X^+)$ is generating in $C(\Omega, X)$ when X^+ is closed and generating in X. A similar result due to Wickstead [45, Theorem 4.4] establishes this for $C_0(\Omega, X)$ when Ω is locally compact (cf. Remark 2.4.6). We will also retrieve these results, but by a different method, namely by establishing the general existence of a continuous bounded decompositions, analogous to that for Banach lattices, as a special case of Theorem 2.4.5 below.

In fact, although the above question and results are in the context of ordered Banach spaces, it will become clear in this paper that for these spaces one is merely looking at a particular instance of more general phenomena. In short: if $T : C \to X$ is a continuous additive positively homogeneous surjection from a closed not necessarily proper cone in a Banach space onto a Banach space, then T has a well-behaved right inverse, and (stronger) versions of theorems such as Andô's, where several cones in one space are involved, are then almost immediately clear. We will now elaborate on this, and at the same time explain the structure of the proofs.

The usual notation to express that X^+ is generating is to write $X = X^+ - X^+$, but the actually relevant point turns out to be that $X = X^+ + (-X^+)$ is a sum of two closed cones: the fact that these are related by a minus sign is only a peculiarity of the context. In fact, if X is the sum of possibly uncountably many closed cones, which need not be proper (this is redundant in Andô's Theorem), then it is possible to choose a bounded decomposition: this is the content of the first part of Theorem 2.4.1. However, this is in itself a consequence of the following more fundamental result, a special case of Theorem 2.3.2. Since a Banach space is a closed cone in itself, it generalizes the usual Open Mapping Theorem for Banach spaces, which is used in the proof.

Theorem 2.1.2. (*Open Mapping Theorem*) *Let C be a closed cone in a real or complex Banach space, not necessarily proper. Let X be a real or complex Banach space, not necessarily over the same field as the surrounding space of C, and $T : C \to X$ a continuous additive positively homogeneous map. Then the following are equivalent:*

(1) T is surjective;

(2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $\|c\| \leq K\|x\|$;

(3) T is an open map;

(4) 0 is an interior point of $T(C)$.

As an illustration of how this can be applied, suppose $X = \sum_{i \in I} C_i$ is the sum of a finite (for the ease of formulation) number of closed not necessarily proper cones. We let Y be the sum of $|I|$ copies of X, and let $C \subset Y$ be the direct sum of the C_i's. Then the natural summing map $T : C \to X$ is surjective by assumption, so that part (2) of Theorem 2.1.2 provides a bounded decomposition. Andô's Theorem corresponds to the case where X is the image of $X^+ \times (-X^+) \subset X \times X$ under the summing map.

In this fashion, generalizations of Andô's Theorem are obtained as a consequence of an Open Mapping Theorem. However, this still does not resolve the issue of a decomposition that is not only bounded, but continuous as well. A possible attempt to obtain this would be the following: if $T : Y \to X$ is a continuous linear surjection between Banach spaces (or even Fréchet spaces), then T has a continuous right inverse, see [1, Corollary 17.67]. The proof is based on Michael's Selection Theorem, which we will recall in Section 2. Conceivably, the proof as in [1] could be modified to yield a similar statement for a continuous surjective additive positively homogeneous $T : C \to X$ from a closed cone C in a Banach space onto X. In that case, if $X = \sum_{i \in I} C_i$ is a finite (say) sum of closed not necessarily proper cones, the setup with product cone and summing map would yield the existence of a continuous decomposition, but unfortunately this time there is no guarantee for boundedness. Somehow the generalized Open Mapping Theorem as in Theorem 2.1.2 and Michael's Selection Theorem must be combined. The solution lies in a refinement of the correspondences to which Michael's Selection Theorem is to be applied, and take certain subadditive maps on C into account from the very beginning. In the end, one of these maps will be taken to be the norm on C, and this provides the desired link between the generalized Open Mapping Theorem and Michael's Selection Theorem, cf. the proof of Proposition 2.3.5. It is along these lines that the following is obtained. It is a special case of Theorem 2.3.6 and, as may be clear by now, it implies the existence of a continuous bounded (and even positively homogeneous) decomposition if $X = \sum_{i \in I} C_i$. It also shows that, if $T : Y \to X$ is a continuous linear surjection between Banach spaces, then it is not only possible to choose a bounded right inverse for T (a statement equivalent to the usual Open Mapping Theorem), but also to choose a bounded right inverse that is, in addition, continuous and positively homogenous.

Theorem 2.1.3. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Then there exists a constant $K > 0$ and a continuous positively homogeneous map $\gamma : X \to C$, such that:

(1) $T \circ \gamma = id_X$;

(2) $\|\gamma(x)\| \leq K\|x\|$, *for all* $x \in X$.

The underlying Proposition 2.3.5 is the core of this paper. It is reworked into the somewhat more practical Theorems 2.3.6 and 2.3.7, but this is all routine, as are the applications in Section 2.4. For example, the following result (Corollary 2.4.2) is virtually immediate from Section 2.3. We cite it in full, not only because it shows concretely how Andô's Theorem figuring so prominently in our discussion so far can be strengthened, but also to enable us to comment on the interpretation of the various parts of this result and similar ones.

Theorem 2.1.4. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+. Let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : X \times X \to \mathbb{R}$ be a continuous seminorm or a continuous linear functional. Then:*

(1) There exist a constant $K > 0$ and continuous positively homogeneous maps $\gamma^\pm : X \to X^+$, such that:

(a) $x = \gamma^+(x) - \gamma^-(x)$, for all $x \in X$;

(b) $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exist maps $\gamma_\varepsilon^\pm : S_X \to X^+$, where $S_X := \{x \in X : \|x\| = 1\}$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in S_X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)$, for all $x \in S_X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) For every $\varepsilon > 0$, there exist continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)\|x\|$, for all $x \in X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)\|x\|$, for all X and $j \in J$.

The existence of a bounded continuous positively homogeneous decomposition in part (1) is of course a direct consequence of Theorem 2.1.3. Naturally, the argument as for Banach lattices then shows that $C_0(\Omega, X) = C_0(\Omega, X^+) - C_0(\Omega, X^+)$ for an arbitrary topological space Ω, so that our original question has been settled in the affirmative.

The equivalence under (2) has the following consequence: If there exist maps $\gamma^\pm : S_X \to X^+$, such that $x = \gamma^+(x) - \gamma^-(x)$, $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K$, and $\rho_j((\gamma^+(x), \gamma^-(x)) \leq \alpha_j$, for all $x \in S_X$ and $j \in J$, then certainly maps as under (2)(a) exist (take $\gamma_\varepsilon^\pm = \gamma^\pm$, for all $\varepsilon > 0$), and hence a family of much better behaved global versions exists as under (2)(b), at an arbitrarily small price in the constants.

The possibility to include the ρ_j's in part (2) (with similar occurrences in other results) is a bonus from the refinement of the correspondences to which Michael's Selection Theorem is applied. For several issues, such as our original question concerning $C_0(\Omega, X)$, it will be sufficient to use part (1) and conclude that a continuous bounded decomposition exists. In this paper we also include some applications of part (2) with non-empty J. Corollary 2.4.3 shows that approximate α-conormality of a (pre)-ordered Banach space is equivalent with continuous positively homogeneous approximate α-conormality, and Corollary 2.4.9 shows that approximate α-conormality of X is inherited by various spaces of continuous X-valued functions on a topological space.

We emphasize that, although Banach spaces that are a sum of cones, and ordered Banach spaces in particular, have played a rather prominent role in this introduction, the actual underlying results are those in Section 2.3, valid for a continuous additive positively homogeneous surjection $T : C \to X$ from a closed not necessarily proper cone C in a Banach space onto a Banach space X. That is the heart of the matter.

This paper is organized as follows.

Section 2.2 contains the basis terminology and some preliminary elementary results. The terminology is recalled in detail, in order to avoid a possible misunderstanding due to differing conventions.

In Section 2.3 the Open Mapping Theorem for Banach spaces and Michael's Selection Theorem are used to investigate surjective continuous additive positively homogeneous maps $T : C \to X$.

Section 2.4 contains the applications, rather easily derived from Section 2.3. Banach spaces that are a sum of closed not necessarily proper cones are approached via the naturally associated closed cone in a Banach space direct sum and the summing map. The results thus obtained are then in turn applied to a (pre)-ordered Banach space X and to various spaces of continuous X-valued functions.

2.2 Preliminaries

In this section we establish terminology, include a few elementary results concerning metric cones for later use, and recall Michael's Selection Theorem.

If X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ denotes its unit sphere.

2.2.1 Subsets of vector spaces

For the sake of completeness we recall that a non-empty subset A of a real vector space X is *star-shaped with respect to 0* if $\lambda x \in A$, for all $x \in A$ and $0 \leq \lambda \leq 1$, and that it is *balanced* if $\lambda x \in A$, for all $x \in A$ and $-1 \leq \lambda \leq 1$. A is *absorbing in X* if, for all $x \in X$, there exists $\lambda > 0$ such that $x \in \lambda A$. A is *symmetric* if $A = -A$.

The next rather elementary property will be used in the proof of Proposition 2.3.1.

Lemma 2.2.1. *Let X be a real vector space and suppose $A, B \subseteq X$ are star-shaped with respect to 0 and absorbing. Then $A \cap B$ is star-shaped with respect to 0 and absorbing.*

Proof. It is clear that $A \cap B$ is star-shaped with respect to 0. Let $x \in X$, then, since A is absorbing, $x \in \lambda A$ for some $\lambda > 0$. The fact that A is star-shaped with respect to 0 then implies that $x \in \lambda' A$ for all $\lambda' \geq \lambda$. Likewise, $x \in \mu B$ for some $\mu > 0$, and then $x \in \mu' B$ for all $\mu' \geq \mu$. Hence $x \in \max(\lambda, \mu)(A \cap B)$. ☐

A subset C of the real or complex vector space X is called a *cone in X* if $C + C \subseteq C$ and $\lambda C \subseteq C$, for all $\lambda \geq 0$. We note that we do not require C to be a proper cone, i.e., that $C \cap (-C) = \{0\}$.

2.2.2 Cones

The cones figuring in the applications in Section 2.4 are cones in Banach spaces, but one of the two main results leading to these applications, the Open Mapping Theorem (Theorem 2.3.2), can be established for the following more abstract objects.

Definition 2.2.2. Let C be a set equipped with operations $+ : C \times C \to C$ and $\cdot : \mathbb{R}_{\geq 0} \times C \to C$. Then C will be called an *abstract cone* if there exists an element $0 \in C$, such that the following hold for all $u, v, w \in C$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$:

(1) $u + 0 = u$;

(2) $(u + v) + w = u + (w + v)$;

(3) $u + v = v + u$;

(4) $u + v = u + w$ implies $v = w$;

(5) $1u = u$ and $0u = 0$;

(6) $(\lambda\mu)u = \lambda(\mu u)$;

(7) $(\lambda + \mu)u = \lambda u + \mu u$;

(8) $\lambda(u + v){=}\lambda u + \lambda v$.

Here we have written $\lambda \cdot u$ as λu for short, as usual.

The natural class of maps between two cones C_1 and C_2 consists of the *additive and positively homogeneous* maps, i.e., the maps $T : C_1 \to C_2$ such that $T(u + v) = Tu + Tv$ and $T(\lambda u) = \lambda u$, for all $u, v \in C$ and $\lambda \geq 0$.

Definition 2.2.3. A pair (C, d) will be called a *metric cone* if C is an abstract cone and $d : C \times C \to \mathbb{R}_{\geq 0}$ is a metric, satisfying

$$d(0, \lambda u) = \lambda d(0, u), \tag{2.2.1}$$

$$d(u + v, u + w) \leq d(v, w), \tag{2.2.2}$$

for every $u, v, w \in C$ and $\lambda \geq 0$. A metric cone (C, d) is a *complete metric cone* if it is a complete metric space.

Remark 2.2.4. (1) Once Michael's Selection Theorem is combined with the Open Mapping Theorem (Theorem 2.3.2), C will be a closed not necessarily proper cone in a Banach space, and the metric will be induced by the norm. In that case it is translation invariant, but for the Open Mapping Theorem as such requiring (2.2.2) is already sufficient. The natural similar requirement $d(0, \lambda u) \leq \lambda d(0, u)$, which is likewise sufficient for the proofs, is easily seen to be actually equivalent to requiring equality as in (2.2.1) above.

(2) Although we will not use this, we note that, if (C, d) is a metric cone, then $+ : C \times C \to C$ is easily seen to be continuous, as is the map $\lambda \to \lambda u$ from $\mathbb{R}_{\geq 0}$ into C, for each $u \in C$.

The following elementary results will be needed in the proof of Proposition 2.3.1.

Lemma 2.2.5. *Let (C, d) be a metric cone as in Definition 2.2.2.*

(1) If $c_1, \ldots, c_n \in C$, then $d(0, \sum_{i=1}^n c_i) \leq \sum_{i=1}^n d(0, c_i)$.

(2) Let X be a real or complex normed space and suppose $T : C \to X$ is positively homogeneous and continuous at 0. Then T maps metrically bounded subsets of C to norm bounded subsets of X

Proof. For the first part, using the triangle inequality and (2.2.2) we conclude that $d(0, \sum_{i=1}^n c_i) \leq d(0, c_n) + d(c_n, \sum_{i=1}^n c_i) \leq d(0, c_n) + d(0, \sum_{i=1}^{n-1} c_i)$, so the statement follows by induction.

As to the second part, by continuity of T at zero there exists some $\delta > 0$ such that $\|Tc\| < 1$ holds for all $c \in C$ satisfying $d(0, c) < \delta$. If $U \subseteq C$ is bounded, choose $r > 0$ such that $U \subseteq \{c \in C : d(0, c) < r\}$. Since $d(0, \lambda u) = \lambda d(0, u)$, for all $u \in C$ and $\lambda \geq 0$, $\delta r^{-1} U \subseteq \{c \in C : d(0, c) < \delta\}$. Then by positive homogeneity of T, $\sup_{u \in U} \|Tu\| \leq \delta^{-1} r < \infty$. □

2.2.3 Correspondences

Our terminology and definitions concerning correspondences follow that in [1]. Let A, B be sets. A map φ from A into the power set of B is called a *correspondence from A into B*, and is denoted by $\varphi : A \twoheadrightarrow B$. A *selector* for a correspondence $\varphi : A \twoheadrightarrow B$ is a function $\sigma : A \to B$ such that $\sigma(x) \in \varphi(x)$ for all $a \in A$. If A and B are topological spaces, we say a correspondence φ is *lower hemicontinuous* if, for every $a \in A$ and every open set $U \subseteq B$ with $\varphi(a) \cap U \neq \emptyset$, there exists an open neighborhood V of a in A such that $\varphi(a') \cap U \neq \emptyset$ for every $a' \in V$. The following result is the key to the proof of Proposition 2.3.4 concerning the existence of continuous sections for surjections of cones onto normed spaces.

Theorem 2.2.6. (*Michael's Selection Theorem [1, Theorem 17.66]*) *Let $\varphi : A \twoheadrightarrow Y$ be a correspondence from a paracompact space A into a real or complex Fréchet space Y. If φ is lower hemicontinuous and has non-empty closed convex values, then it admits a continuous selector.*

2.3 Main results

In this section we establish our main results, Theorems 2.3.2, 2.3.6 and 2.3.7. Theorem 2.3.2 is an Open Mapping Theorem for surjections from complete metric cones onto Banach spaces; its proof is based on the usual Open Mapping Theorem. Together with the technical Proposition 2.3.4 (based on Michael's Selection Theorem) it yields the key Proposition 2.3.5. This is then reworked into two more practical results. The first of these, Theorem 2.3.6, guarantees the existence of continuous bounded positively homogeneous right inverses, while the second, Theorem 2.3.7, shows that the existence of a family of possibly ill-behaved local right inverses implies the existence of a family well-behaved global ones.

As before, if X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ is its unit sphere.

We start with the core of the proof of the Open Mapping Theorem, which employs a certain Minkowski functional. The use of such functionals when dealing with cones and Banach spaces goes back to Klee [27] and Andô [3].

Proposition 2.3.1. *Let (C, d) be a complete metric cone as in Definition 2.2.2. Let X be a real Banach space and $T : C \to X$ a continuous additive positively homogeneous surjection. Let $B := \{c \in C : d(0, c) \leq 1\}$ denote the closed unit ball around zero in C, and define $V := T(B) \cap (-T(B))$. Then V is an absorbing convex balanced subset of X and its Minkowski functional $\|\cdot\|_V : X \to \mathbb{R}$, given by $\|x\|_V := \inf\{\lambda > 0 : x \in \lambda V\}$, for $x \in X$, is a norm on X that is equivalent to the original norm on X.*

Proof. It follows from Lemma 2.2.5 and Definition 2.2.3 that $B := \{c \in C : d(0, c) \leq 1\}$ is convex. Hence $T(B)$ is convex and contains zero, since T is additive and positively homogeneous. Since $0 \in T(B)$, its convexity implies that $T(B)$ is star-shaped with respect to 0. Furthermore, $T(B)$ is absorbing, as a consequence of the surjectivity and positive homogeneity of T and (2.2.1). Thus $T(B)$ is star-shaped with respect to 0 and absorbing, and since this implies the same properties for $-T(B)$, Lemma 2.2.1 shows that $V := T(B) \cap (-T(B))$ is star-shaped with respect to 0 and absorbing. As V is clearly symmetric, its star-shape with respect to 0 implies that it is balanced. Furthermore, the convexity of $T(B)$ implies that V is convex. All in all, V is an absorbing convex balanced subset of the real vector space X, and hence its Minkowski functional $\|\cdot\|_V$ is a seminorm by [40, Theorem 1.35]. Because T is continuous at 0, Lemma 2.2.5 implies that $\sup_{y \in V} \|y\| \leq M$ for some $M > 0$. If $x \in X$ and $\lambda > \|x\|_V$, then the definition of $\|\cdot\|_V$ and the star-shape of V with respect to 0 imply that $x \in \lambda V$, so that $\|x\| \leq \lambda M$. Hence

$$\|x\| \leq M \|x\|_V \quad (x \in X). \tag{2.3.1}$$

We conclude that $\|\cdot\|_V$ is a norm on X. In view of (2.3.1), the equivalence of $\|\cdot\|_V$ and $\|\cdot\|$ is an immediate consequence of the Bounded Inverse Theorem for Banach spaces, once we know that $(X, \|\cdot\|_V)$ is complete. We will now proceed to show this, using the completeness of (C, d).

To this end, it suffices to show $\|\cdot\|_V$-convergence of all $\|\cdot\|_V$-absolutely convergent series. Let $\{x_i\}_{i=1}^\infty$ be a sequence in X such that $\sum_{i=1}^\infty \|x_i\|_V < \infty$. Since $\|x\| \leq M\|x\|_V$ for all $x \in X$, $\sum_{i=1}^\infty \|x_i\| < \infty$ also holds, hence by completeness of X the $\|\cdot\|$-sum $x_0 := \sum_{i=1}^\infty x_i$ exists. We claim that $\left\|x_0 - \sum_{i=1}^{N-1} x_i\right\|_V \to 0$ as $N \to \infty$, i.e., that x_0 is also the $\|\cdot\|_V$-sum of this series.

In order to establish this, we start by noting that, for $x \in X$, there exists $x' \in V$ such that $x = 2\|x\|_V x'$. This is clear if $\|x\|_V = 0$. If $\|x\|_V \neq 0$, then $2\|x\|_V > \|x\|_V$ and, as already observed earlier, this implies that $x \in 2\|x\|_V V$. Therefore for every $i \in \mathbb{N}$ there exists $x_i' \in V$ satisfying $x_i = 2\|x_i\|_V x_i'$. Since $V \subset T(B)$, for every $i \in \mathbb{N}$ there exists $b_i \in B$ such that $Tb_i = x_i'$, so that $x_i = 2\|x_i\|_V Tb_i = T(2\|x_i\|_V b_i)$. Note that $d(0, 2\|x_i\|_V b_i) = 2\|x_i\|_V d(0, b_i) \leq 2\|x_i\|_V$.

From (2.2.2) and Lemma 2.2.5 it then follows that, for any fixed $N \in \mathbb{N}$ and all $n, m \in \mathbb{N}$ with $N \leq m \leq n$,

$$\begin{aligned} d\left(\sum_{i=N}^{m} 2\|x_i\|_V b_i, \sum_{i=N}^{n} 2\|x_i\|_V b_i\right) &\leq d\left(0, \sum_{i=m+1}^{n} 2\|x_i\|_V b_i\right) \\ &\leq \sum_{i=m+1}^{n} d(0, 2\|x_i\|_V b_i) \\ &\leq \sum_{i=m+1}^{n} 2\|x_i\|_V. \end{aligned}$$

We conclude that, for any fixed $N \in \mathbb{N}$, $\{\sum_{i=N}^{n} 2\|x_i\|_V b_i\}_{n=N}^{\infty}$ is a Cauchy sequence in (C, d) and hence, by completeness, converges to some $c_N \in C$. Using Lemma 2.2.5 again we find that

$$d(0, c_N) = \lim_{n\to\infty} d\left(0, \sum_{i=N}^{n} 2\|x_i\|_V b_i\right) \leq \limsup_{n\to\infty} \sum_{i=N}^{n} d(0, 2\|x_i\|_V b_i) \leq \sum_{i=N}^{\infty} 2\|x_i\|_V,$$

so that $c_N \in (\sum_{i=N}^{\infty} 2\|x_i\|_V)\, B$, as a consequence of (2.2.1). By the continuity, additivity and positive homogeneity of T, $Tc_N = \lim_{n\to\infty} T\left(\sum_{i=N}^{n} 2\|x_i\|_V b_i\right) = \lim_{n\to\infty} \sum_{i=N}^{n} T(2\|x_i\|_V b_i) = \lim_{n\to\infty} \sum_{i=N}^{n} x_i = \sum_{i=N}^{\infty} x_i$ with respect to the $\|\cdot\|$-topology. Hence

$$x_0 - \sum_{i=1}^{N-1} x_i = \sum_{i=N}^{\infty} x_i = Tc_N \in \left(\sum_{i=N}^{\infty} 2\|x_i\|_V\right) T(B).$$

Similarly, the inclusion $V \subset -T(B)$ implies that, for every $i \in \mathbb{N}$, there exists $\tilde{b}_i \in B$ such that $-T\tilde{b}_i = x_i'$. Then $\tilde{c}_N = \lim_{n\to\infty} \sum_{i=N}^{n} 2\|x_i\|_V \tilde{b}_i$ exists for all $N \in \mathbb{N}$, $\tilde{c}_N \in (\sum_{i=N}^{\infty} 2\|x_i\|_V)\, B$, and $T\tilde{c}_N = -\sum_{i=N}^{\infty} x_i$, so that

$$x_0 - \sum_{i=1}^{N-1} x_i = \sum_{i=N}^{\infty} x_i = -T\tilde{c}_N \in \left(\sum_{i=N}^{\infty} 2\|x_i\|_V\right) (-T(B)).$$

We conclude that $x_0 - \sum_{i=1}^{N-1} x_i \in (\sum_{i=N}^{\infty} 2\|x_i\|_V)\, V$. Therefore,

$$\left\| x_0 - \sum_{i=1}^{N-1} x_i \right\|_V \leq \sum_{i=N}^{\infty} 2\|x_i\|_V \to 0$$

as $N \to \infty$, and hence $(X, \|\cdot\|_V)$ is complete. $\square$

The Open Mapping Theorem is now an easy consequence.

Theorem 2.3.2. (*Open Mapping Theorem*) *Let (C, d) be a complete metric cone as in Definition 2.2.2; for example, C could be a closed not necessarily proper cone in a Banach space. Let X be a real or complex Banach space and $T : C \to X$ a continuous additive positively homogeneous map. Then the following are equivalent:*

(1) T is surjective;

(2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $d(0, c) \leq K\|x\|$;

(3) T is an open map;

(4) 0 is an interior point of $T(C)$.

Proof. Given the nature of the statements in (1)-(4), we may assume that X is a real Banach space, by viewing a complex one as such if necessary. We first prove that (1) implies (2). By Proposition 2.3.1, there exists a constant $L > 0$ such that $\|x\|_V \leq L\|x\|$, for all $x \in X$. If $\|x\| \neq 0$, then $2L\|x\| > L\|x\| \geq \|x\|_V$. Hence $x \in 2L\|x\|V$, which is also trivially true if $x = 0$. In particular, for all $x \in X$ there exists some $c \in B$ such that $x = 2L\|x\|T(c)$. Then $x = T(2L\|x\|c)$, and $d(0, 2L\|x\|c) = 2L\|x\|d(0, c) \leq 2L\|x\|$.

Next, we prove that (2) implies (3). Let $U \subseteq C$ be an open set, and let $x \in T(U)$ be arbitrary with $b \in U$ satisfying $Tb = x$. Since U is open, there exists some $r > 0$ such that $W := \{c \in C : d(b, c) < r\}$ is contained in U. We define $W' := \{c \in C : d(0, c) < r\}$. Then $b + W' \subseteq W$, since $d(b, b + w') \leq d(0, w') < r$ for all $w' \in W'$. Now, by hypothesis, for every $x' \in X$ with $\|x'\| < rK^{-1}$ there exists some $w' \in W'$ with $Tw' = x'$. With $B_X := \{x \in X : \|x\| < 1\}$, by additivity of T, it follows that $x + rK^{-1}B_X \subseteq T(b + W') \subseteq T(W) \subseteq T(U)$. We conclude that $T(U)$ is open.

That (3) implies (4) is trivial, and (4) implies (1) by the positive homogeneity of T. □

Remark 2.3.3. (1) Since a real or complex Banach space is a complete metric cone, Theorem 2.3.2 generalizes the Open Mapping Theorem for Banach spaces that was used in the proof of the preparatory Proposition 2.3.1.

(2) If C is a closed cone in a Banach space, X is a topological vector space, and $T : C \to X$ is continuous, additive and positively homogeneous, then we can conclude that T is an open map, provided that we know beforehand that the closure of $\{Tc : c \in C, \|c\| \leq 1\}$ is a neighborhood of 0 in X. This follows from [36, Theorem 1] . Since we do not have such a hypothesis, this result does not imply ours. The difference is not only that in our case T is assumed to be surjective, but, more fundamentally, that our image space is in fact a Banach space, for which an Open Mapping Theorem is already known to hold that serves as a stepping stone for the more general result.

(3) In [43] an Open Mapping Theorem is established for maps between two abstract cones in a certain class, provided that we know beforehand that these maps satisfy a so-called almost-openess condition. Since we do not have such a hypothesis, this result does not imply ours. Again the difference lies in the image space: in our context this is not just a cone, but actually a full Banach space with accompanying Open Mapping Theorem.

We will now proceed with the second basic result, Proposition 2.3.4, which is concerned with families of continuous right inverses for a surjective (this follows from the hypotheses) map.

Proposition 2.3.4. *Let X be a real or complex normed space and let Y be a real or complex topological vector space, not necessarily Hausdorff and not necessarily over the same field as X, with $C \subseteq Y$ a closed not necessarily proper cone. Let I be a finite set, possibly empty. For each $i \in I$, let $\alpha_i \in \mathbb{R}$ and let $\rho_i : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map.*

Suppose that $T : C \to X$ is a continuous additive positively homogeneous map with the property that, for every $\varepsilon > 0$, there exists a map $\sigma_\varepsilon : S_X \to C$, such that:

(1) $T \circ \sigma_\varepsilon = id_{S_X}$;

(2) $\rho_i(\sigma_\varepsilon(x)) \leq \alpha_i + \varepsilon$, for all $x \in S_X$ and $i \in I$;

(3) $\sigma_\varepsilon(S_X)$ is bounded in Y.

Then, for every $\varepsilon > 0$, the correspondence $\varphi_\varepsilon : S_X \twoheadrightarrow Y$, defined by

$$\varphi_\varepsilon(x) := \{y \in C : Ty = x,\ \rho_i(y) \leq \alpha_i + \varepsilon \text{ for all } i \in I\} \quad (x \in S_X),$$

has non-empty closed convex values, and is lower hemicontinuous on S_X.

If Y is a Fréchet space, there exist continuous maps $\sigma'_\varepsilon : S_X \to C$, for all $\varepsilon > 0$, satisfying:

(a) $T \circ \sigma'_\varepsilon = id_{S_X}$;

(b) $\rho_i(\sigma'_\varepsilon(x)) \leq \alpha_i + \varepsilon$, for all $x \in S_X$ and $i \in I$.

If $\varepsilon > 0$ and $\sigma'_\varepsilon(S_X)$ is bounded in Y in the sense of topological vector spaces, then σ'_ε can be extended to a continuous positively homogeneous map $\sigma'_\varepsilon : X \to C$ on the whole space, satisfying:

(a) $T \circ \sigma'_\varepsilon = id_X$;

(b) $\rho_i(\sigma'_\varepsilon(x)) \leq (\alpha_i + \varepsilon)\|x\|$, for all $x \in X$ and $i \in I$.

Before embarking on the proof, let us point out that the salient point lies in the fact that the right inverses σ'_ε of T on the unit sphere of X are continuous, whereas this is not required for the original family of the σ_ε's, and that this extra property can be achieved retaining the relevant inequalities. It is for this that Michael's Selection Theorem is used. The subsequent conditional extension of such a σ'_ε to the whole space is rather trivial.

Furthermore, we note that, although in the applications we have in mind the constants α_i will be positive and each ρ_i will be the restriction to C of a continuous seminorm or (if Y is a Banach space) a continuous real-linear functional on the whole space Y, the present proof does not require this.

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\sigma_\varepsilon(x) \in \varphi_\varepsilon(x)$, for all $x \in S_X$, φ_ε is non-empty-valued. By continuity of T and the ρ_i's, φ_ε is closed-valued. Since T is affine on the convex set C, and each ρ_i, if any, is subadditive and positively homogeneous, φ_ε is convex-valued.

We will now show that φ_ε is lower hemicontinuous, for any fixed $\varepsilon > 0$. To this end, let $x \in S_X$ be arbitrary, and let $U \subseteq Y$ be open such that $\varphi_\varepsilon(x) \cap U \neq \emptyset$.

We start by establishing that there exists some $y \in \varphi_\varepsilon(x) \cap U$ such that $\rho_i(y) < \alpha_i + \varepsilon$, for all $i \in I$ (if any), where the inequality that is valid for $\sigma_\varepsilon(x)$ has been improved to strict inequality for y. As to this, choose y' in the non-empty set $\varphi_\varepsilon(x) \cap U$, and define $y_t := t\sigma_{\varepsilon/2}(x) + (1-t)y'$, for $t \in [0,1]$. Then $y_t \in C$ and $Ty_t = x$, for all $t \in [0,1]$. Now, for all $t \in (0,1]$ and all $i \in I$,

$$
\begin{aligned}
\rho_i(y_t) &\leq t\rho_i(\sigma_{\varepsilon/2}(x)) + (1-t)\rho_i(y') \\
&\leq t\left(\alpha_i + \frac{\varepsilon}{2}\right) + (1-t)\left(\alpha_i + \varepsilon\right) \\
&< t\left(\alpha_i + \varepsilon\right) + (1-t)\left(\alpha_i + \varepsilon\right) \\
&= \left(\alpha_i + \varepsilon\right).
\end{aligned}
$$

Since U is open, there exists some $t_0 > 0$ such that $y := y_{t_0} \in U$. Then y is as required.

Having found and fixed this y, we define $\eta := \min_{i \in I}\{\alpha_i + \varepsilon - \rho_i(y)\} > 0$ if $I \neq \emptyset$, and $\eta := 1$ if $I = \emptyset$; here we use that I is finite.

Next, let $W \subseteq Y$ be an open neighborhood of zero such that $y + W \subseteq U$. Since $\sigma_{\eta/2}(S_X)$ is bounded by assumption, we can fix some $0 < r \leq 1$ such that $r'\sigma_{\eta/2}(S_X) \subseteq W$, for all $0 \leq r' < r$, and $\alpha_i r' < \eta/2$, for all $0 \leq r' < r$ and all $i \in I$.

Now, if $x' \in X$ satisfies $0 < \|x'\| < r$, then

$$
y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right) \in C,
$$

$$
T\left(y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) = x + x',
$$

and

$$
y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right) \in y + \|x'\|\sigma_{\eta/2}(S_X) \subset y + W \subset U.
$$

Furthermore, for such x' and for $i \in I$ we find, using $\alpha_i\|x'\| < \eta/2$ and $\|x'\| < r \leq 1$, that

$$
\begin{aligned}
\rho_i\left(y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) &\leq \rho_i(y) + \|x'\|\rho_i\left(\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) \\
&\leq \rho_i(y) + \|x'\|\left(\alpha_i + \frac{\eta}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&< \rho_i(y) + \frac{\eta}{2} + \frac{\eta}{2} \\
&= \rho_i(y) + \eta \\
&\leq \rho_i(y) + \alpha_i + \varepsilon - \rho_i(y) \\
&= \alpha_i + \varepsilon.
\end{aligned}
$$

Therefore, if $x + x' \in S_X$ with $0 < \|x'\| < r$, we conclude that

$$y + \|x'\|\sigma_{\eta/2}(x'/\|x'\|) \in \varphi_\varepsilon(x + x') \cap U.$$

Hence φ_ε is lower hemicontinuous on S_X, as was to be proved.

If Y is a Fréchet space then, since S_X as a metric space is paracompact [42], Michael's Selection Theorem (Theorem 2.2.6), applied to each individual φ_ε, supplies a family of continuous maps $\sigma'_\varepsilon : S_X \to C$, such that $\sigma_\varepsilon(x) \in \varphi_\varepsilon(x)$, for all $\varepsilon > 0$ and all $x \in S_X$. Then the σ'_ε are as required.

If $\varepsilon > 0$ and $\sigma'_\varepsilon(S_X)$ happens to be bounded in the topological vector space Y, we extend $\sigma'_\varepsilon : S_X \to C$ to a positively homogeneous C-valued map on all of X, also denoted by σ'_ε, by defining

$$\sigma'_\varepsilon(x) := \begin{cases} 0 & \text{for } x = 0; \\ \|x\|\sigma'_\varepsilon\left(\frac{x}{\|x\|}\right) & \text{for } x \neq 0. \end{cases}$$

The continuity of σ'_ε at 0 then follows from the boundedness of $\sigma'_\varepsilon(S_X)$, and at other points it is immediate. It is easily verified that such a global σ'_ε has the properties as claimed. □

Combination of Theorem 2.3.2 and Proposition 2.3.4 yields the following key result on right inverses of surjections from cones onto Banach spaces. The structure of the proofs makes it clear that it is ultimately based on the Open Mapping Theorem for Banach spaces and Michael's Selection Theorem.

Proposition 2.3.5. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Furthermore, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map. For example, each ρ_j could be the restriction to C of a globally defined continuous seminorm or continuous real-linear functional.

(1) If ρ_j is bounded from above on $S_Y \cap C$, for all $j \in J$, then there exist constants $K > 0$, $\alpha_j \in \mathbb{R}$ ($j \in J$), and a map $\gamma : S_X \to C$, such that:

- *(a) $T \circ \gamma = id_{S_X}$;*
- *(b) $\|\gamma(x)\| \leq K$, for all $x \in S_X$;*
- *(c) $\rho_j(\gamma(x)) \leq \alpha_j$, for all $x \in S_X$ and $j \in J$.*

(2) *If* $K > 0$, $\alpha_j \in \mathbb{R}$ $(j \in J)$, *and* $\gamma : S_X \to C$ *satisfy* (a), (b) *and* (c) *in part* (1), *then, for every* $\varepsilon > 0$, *there exists a continuous positively homogeneous map* $\gamma_\varepsilon : X \to C$ *such that:*

   (a) $T \circ \gamma_\varepsilon = id_X$;

   (b) $\|\gamma_\varepsilon(x)\| \leq (K + \varepsilon)\|x\|$, *for all* $x \in X$;

   (c) $\rho_j(\gamma(x)) \leq (\alpha_j + \varepsilon)\|x\|$, *for all* $x \in X$ *and* $j \in J$.

Proof. As to the first part, we start by applying part (2) of Theorem 2.3.2 and obtain $K > 0$ and a map $\gamma : S_X \to C$, such that $T \circ \gamma = \mathrm{id}_{S_X}$ and $\|\gamma(x)\| \leq K$ $(x \in S_X)$.

If $j \in J$, and $\beta_j \in \mathbb{R}$ is such that $\rho_j(c) \leq \beta_j$ for all $c \in S_Y \cap C$, where we may assume that $\beta_j \geq 0$, then $\rho_j(\gamma(x)) \leq K\beta_j$, for all $x \in S_X$. Indeed, this is obvious if $\gamma(x)$=0, and if $\gamma(x) \neq 0$ we have

$$\rho_j\left(\gamma(x)\right) = \|\gamma(x)\|\rho_j\left(\frac{\gamma(x)}{\|\gamma(x)\|}\right) \leq K\beta_j.$$

The existence of the $\alpha_j := K\beta_j$ is then clear.

As to the second part, suppose that $K > 0$, $\alpha_j \in \mathbb{R}$ $(j \in J)$ and $\gamma : S_X \to C$ satisfy (a), (b) and (c) in part (1). We augment J to $I := J \cup \{\| \, \|\}$, where we choose an index symbol $\| \, \| \notin J$, and let $\rho_{\| \, \|}(c) := \|c\|$, for $c \in C$, and put $\alpha_{\| \, \|} := K$. We can now apply Proposition 2.3.4 with $\sigma_\varepsilon = \gamma$ for all $\varepsilon > 0$, since its hypotheses (1), (2) and (3) are then satisfied. The continuous $\sigma'_\varepsilon : S_X \to C$ as furnished by Proposition 2.3.4 are, in particular, such that $\rho_{\| \, \|}(\sigma'_\varepsilon(x)) \leq \alpha_{\| \, \|} + \varepsilon$, i.e., such that $\|\sigma'_\varepsilon(x)\| \leq K + \varepsilon$, for all $x \in S_X$. Hence each of the sets $\sigma'_\varepsilon(S_X)$ is bounded in Y, and the last part of Proposition 2.3.4 applies, yielding global $\sigma'_\varepsilon : X \to C$ that can be taken as the required γ_ε. □

Let us remark explicitly that the α_j's need not be non-negative and that in part (2) the ρ_j's are not required to be bounded from above on $S_Y \cap C$ as in part (1), but rather on $\gamma(S_X)$ (as a consequence of the hypothesized validity of (1)(c)), which is a weaker hypothesis.

We extract two practical consequences from Proposition 2.3.5. First of all, if the ρ_j's are bounded from above on $S_Y \cap C$ then part (1) of Proposition 2.3.5 is applicable and its conclusion shows that the hypothesis of part (2) are satisfied. Taking $\varepsilon = 1$, say, we therefore have the following.

Theorem 2.3.6. *Let* X *and* Y *be real or complex Banach spaces, not necessarily over the same field, and let* C *be a closed not necessarily proper cone in* Y. *Let* $T : C \to X$ *be a surjective continuous additive positively homogeneous map.*

Furthermore, let J *be a finite set, possibly empty, and, for all* $j \in J$, *let* $\rho_j : C \to \mathbb{R}$ *be a continuous subadditive positively homogeneous map that is bounded from above on* $S_Y \cap C$. *For example, each* ρ_j *could be the restriction to* C *of a globally defined continuous seminorm or continuous real-linear functional.*

Then there exist constants $K > 0$ and $\alpha_j \in \mathbb{R}$ $(j \in J)$ and a continuous positively homogeneous map $\gamma : X \to C$, such that:

(1) $T \circ \gamma = id_X$;

(2) $\|\gamma(x)\| \leq K\|x\|$, *for all* $x \in X$;

(3) $\rho_j(\gamma(x)) \leq \alpha_j \|x\|$, *for all* $x \in X$ *and* $j \in J$.

The next consequence of Proposition 2.3.5 states that the existence of a family of possibly ill-behaved right inverses on the unit sphere is actually equivalent with the existence of a family of well-behaved global ones. Note that, compared with Theorem 2.3.6, the boundedness assumption from above for the ρ_j's on $S_Y \cap C$ has been replaced with the assumptions (1)(c) and (2)(c) below.

Theorem 2.3.7. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Furthermore, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map. For example, each ρ_j could be the restriction to C of a globally defined continuous seminorm or continuous real-linear functional.

If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(1) For every $\varepsilon > 0$, there exists a map $\gamma_\varepsilon : S_X \to C$, such that:

  (a) $T \circ \gamma_\varepsilon = id_{S_X}$;
  (b) $\|\gamma_\varepsilon(x)\| \leq K + \varepsilon$, *for all* $x \in S_X$;
  (c) $\rho_j(\gamma_\varepsilon(x)) \leq \alpha_j + \varepsilon$, *for all* $x \in S_X$ *and* $j \in J$.

(2) For every $\varepsilon > 0$, there exists a continuous positively homogeneous map $\gamma_\varepsilon : X \to C$ such that:

  (a) $T \circ \gamma_\varepsilon = id_X$;
  (b) $\|\gamma_\varepsilon(x)\| \leq (K + \varepsilon)\|x\|$, *for all* $x \in X$;
  (c) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)\|x\|$, *for all* $x \in X$ *and* $j \in J$.

Proof. Clearly the second part implies the first. For the converse implication, let $\varepsilon > 0$ be given. Then, by assumption, there exists a map (we add accents to avoid notational confusion) $\gamma'_{\varepsilon/2} : S_X \to C$, such that:

(1) $T \circ \gamma'_{\varepsilon/2} = \mathrm{id}_{S_X}$;

(2) $\|\gamma'_{\varepsilon/2}(x)\| \leq K + \varepsilon/2$, for all $x \in S_X$;

(3) $\rho_j(\gamma'_{\varepsilon/2}(x)) \leq \alpha_j + \varepsilon/2$, for all $x \in S_X$ and $j \in J$.

We can now apply part (2) of Proposition 2.3.5, with K replaced with $K + \varepsilon/2$, α_j with $\alpha_j + \varepsilon/2$, γ with $\gamma'_{\varepsilon/2}$, and ε with $\varepsilon/2$. The map $\gamma_{\varepsilon/2}$ as furnished by part (2) of Proposition 2.3.5 can then be taken as the map γ_ε in part (2) of the present Theorem. $\square$

2.4 Applications

By varying C and the ρ_j's various types of consequences of Theorems 2.3.6 and 2.3.7 can be obtained, and we collect some in the present section, considering situations where the ρ_j's are restrictions to C of globally defined continuous seminorms or continuous real-linear functionals. This seems to be a natural context to work in, but we note that it is not required as such by these two underlying Theorems, nor by the key Proposition 2.3.5, so that applications of another type are conceivable.

As in earlier sections, if X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ is its unit sphere.

To start with, Theorems 2.3.6 and 2.3.7 are clearly applicable when $T : C \to X$ is the restriction to C of a global continuous linear map $T : Y \to X$ and (as already mentioned in these Theorems) each of the ρ_j's is the restriction of a globally defined continuous seminorm or continuous real-linear functional. Furthermore, Y is a closed cone in itself, so that these Theorems can be specialized to yield statements on well-behaved right inverses for continuous linear surjections between Banach spaces. For reasons of space, we refrain from explicitly formulating all these quite obvious special cases.

Instead, we give applications to the internal structure of a Banach space that is a sum of closed not necessarily proper cones, and to the structure of spaces of continuous functions with values in such a Banach space. Thus we return to the to the improvements of Andô's Theorem and our original motivating question alluded to in the introduction.

The following result applies, in particular, when $X = \sum_{i=1}^{n} C_i$ is the sum of a finite number of closed not necessarily proper cones. In that case, the Banach space Y in the following Theorem is the direct sum of n copies of X.

Theorem 2.4.1. *Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X, such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i\in I} c_i$, where $c_i \in C_i$, for all $i \in I$.*

Let $Y = \ell^1(I, X)$ be the ℓ^1-direct sum of $|I|$ copies of X, and let C be the natural closed cone in the Banach space Y, consisting of those elements where the i-th component is in C_i. Finally, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : Y \to \mathbb{R}$ be a continuous seminorm or a continuous real-linear functional.

Then:

(1) There exist a constant $K > 0$ and a continuous positively homogeneous map $\gamma : X \to C$ with continuous positively homogeneous component maps $\gamma_i : X \to C_i$ $(i \in I)$, such that:

(a) $x = \sum_{i\in I} \gamma_i(x)$, for all $x \in X$;

(b) $\sum_{i\in I} \|\gamma_i(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exists a map $\gamma_\varepsilon : S_X \to C$ with component maps $\gamma_{\varepsilon,i} : S_X \to C_i\,(i \in I)$, such that:

(i) $x = \sum_{i \in I} \gamma_{\varepsilon,i}(x)$, for all $x \in S_X$;

(ii) $\sum_{i \in I} \|\gamma_{\varepsilon,i}(x)\| \leq (K + \varepsilon)$, for all $x \in S_X$;

(iii) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) For every $\varepsilon > 0$, there exists a continuous positively homogeneous map $\gamma_\varepsilon : X \to C$ with continuous positively homogeneous component maps $\gamma_{\varepsilon,i} : X \to C_i\,(i \in I)$, such that:

(i) $x = \sum_{i \in I} \gamma_{\varepsilon,i}(x)$, for all $x \in X$;

(ii) $\sum_{i \in I} \|\gamma_{\varepsilon,i}(x)\| \leq (K + \varepsilon)\|x\|$, for all $x \in X$;

(iii) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)\|x\|$, for all $x \in X$ and $j \in J$.

Proof. Let $T : C \to X$ be the canonical summing map. Then Theorem 2.3.6 yields part (1), and Theorem 2.3.7 yields part (2). □

In order to illustrate Theorem 2.4.1 we consider the situation where X is a Banach space, (pre)-ordered by a closed not necessarily proper cone X^+. If X^+ is generating in the sense of (pre)-ordered Banach spaces, i.e., if $X = X^+ - X^+$, and if X^+ is proper, then Andô's Theorem (Theorem 2.1.1) applies. On the other hand, Theorem 2.4.1, also yields this result (and an even stronger one) by writing $X = X^+ + (-X^+)$ as the sum of two closed cones, coincidentally related by a minus sign. For convenience we formulate the result explicitly in the usual notation with minus signs.

Corollary 2.4.2. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+. Let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : X \times X \to \mathbb{R}$ be a continuous seminorm or a continuous linear functional. Then:*

(1) There exist a constant $K > 0$ and continuous positively homogeneous maps $\gamma^\pm : X \to X^+$, such that:

(a) $x = \gamma^+(x) - \gamma^-(x)$, for all $x \in X$;

(b) $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}$ $(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exist maps $\gamma_\varepsilon^\pm : S_X \to X^+$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in S_X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)$, for all $x \in S_X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) *For every* $\varepsilon > 0$*, there exist continuous positively homogeneous maps* $\gamma_\varepsilon^\pm : X \to X^+$*, such that:*

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$*, for all* $x \in X$*;*

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)\|x\|$*, for all* $x \in X$*;*

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)\|x\|$*, for all* X *and* $j \in J$*.*

Proof. We apply Theorem 2.4.1 with $I = \{1, 2\}$, $C_1 = X^+$ and $C_2 := -X^+$, and then let $\gamma^+ = \gamma_1$ and $\gamma^- = -\gamma_2$ in part (1), and $\gamma_\varepsilon^+ = \gamma_{\varepsilon,1}$ and $\gamma_\varepsilon^- = -\gamma_{\varepsilon,2}$ in part (2) □

We continue in the context of a real (pre)-ordered normed space X ordered by a closed generating not necessarily proper cone X^+. If $\alpha > 0$, then we will say that X is

(1) *α-conormal* if, for each $x \in X$, there exist $x^\pm \in X^+$, such that $x = x^+ - x^-$ and $\|x^+\| \leq \alpha\|x\|$;

(2) *approximately α-conormal* if X is $(\alpha + \varepsilon)$-conormal, for all $\varepsilon > 0$.

Andô's Theorem is equivalent to asserting that every real Banach space, ordered by a closed generating proper cone, is α-conormal for some $\alpha > 0$. Clearly, α-conormality implies approximate α-conormality. What is less obvious is that approximate α-conormality is equivalent with a continuous positively homogeneous version of the same notion, as is the content of the following consequence of Corollary 2.4.2.

Corollary 2.4.3. *Let X be a real* (*pre*)*-ordered Banach space,* (*pre*)*-ordered by a closed generating not necessarily proper cone X^+, and let $\alpha > 0$. Then the following are equivalent:*

(1) *X is approximately α-conormal;*

(2) *For every $\varepsilon > 0$, there exist continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that:*

(a) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$*, for all* $x \in X$*;*

(b) $\|\gamma_\varepsilon^+(x)\| \leq (\alpha + \varepsilon)\|x\|$*, for all* $x \in X$*.*

Proof. Clearly part (2) implies part (1). For the converse we will apply Corollary 2.4.2 with $J = \{1\}$, as follows. Let $\varepsilon > 0$ be given and fixed. For each $x \in S_X$, the $(\alpha + \varepsilon/2)$-conormality of X implies that, for each $x \in S_X$, we can choose and fix $\gamma_{\varepsilon/2}^\pm(x) \in X^+$, such that

$$x = \gamma_{\varepsilon/2}^+(x) - \gamma_{\varepsilon/2}^-(x) \ (x \in S_X),$$

and $\|\gamma_{\varepsilon/2}^+(x)\| \leq (\alpha + \varepsilon/2)$. Then $\|\gamma_{\varepsilon/2}^-(x)\| \leq (\alpha + \varepsilon/2 + 1)$, so that

$$\|\gamma_{\varepsilon/2}^+(x)\| + \|\gamma_{\varepsilon/2}^-(x)\| \leq (2\alpha + 1 + \varepsilon) \ (x \in S_X).$$

Define $\rho_1 : X \times X \to \mathbb{R}$ by $\rho_1((x_1, x_2)) := \|x_1\|$, for $x_1, x_2 \in X$, so that

$$\rho_1((\gamma^+_{\varepsilon/2}(x), \gamma^-_{\varepsilon/2}(x)) = \|\gamma^+_{\varepsilon/2}(x)\| \leq \alpha + \varepsilon/2 \leq \alpha + \varepsilon \ (x \in S_X).$$

Thus we have found constants $K = 2\alpha+1 > 0$, $\alpha_1 = \alpha$ and maps $\gamma^{\pm}_{\varepsilon/2} : S_X \to X_+$ satisfying (i), (ii) and (iii) in part (2)(a) of Corollary 2.4.2. Hence the continuous positively homogeneous maps as in part (2)(b) of Corollary 2.4.2 also exist, and these are as required. □

Remark 2.4.4. The term "conormality" is due to Walsh [44] and several variations of it have been studied. For example, a real normed space X is said to be *α-max-conormal* if, for each $x \in X$, there exist $x^{\pm} \in X^+$, such that $x = x^+ - x^-$ and $\max(\|x^+\|, \|x^-\|) \leq \alpha\|x\|$; X is *approximately α-max-conormal* if it is $(\alpha+\varepsilon)$-max-conormal, for every $\varepsilon > 0$. As another example, X is said to be *α-sum-conormal* if, for each $x \in X$, there exist $x^{\pm} \in X^+$, such that $x = x^+ - x^-$ and $\|x^+\| + \|x^-\| \leq \alpha\|x\|$; X is *approximately α-sum-conormal* if it is $(\alpha + \varepsilon)$-sum-conormal, for every $\varepsilon > 0$. Just as Corollary 2.4.3 shows that approximately α-conormality implies its continuous positively homogenous version, the elements $x^{\pm}$ figuring in the definitions of approximately α-max-conormality and approximately α-sum-conormality can be chosen in a continuous and positively homogeneous fashion. The proof is analogous to the proof of Corollary 2.4.3, but now taking $\rho_1(x_1, x_2) := \max(\|x_1\|, \|x_2\|)$ for approximately α-sum-conormality, and $\rho_1(x_1, x_2) := \|x_1\| + \|x_2\|$ for approximately α-sum-conormality.

The dual notion of conormality is normality (terminology due to Krein [28]). Several equivalences between versions of normality of an ordered Banach space X and conormality of its dual (and vice versa) are known, but are scattered throughout the literature under various names. The most complete account of normality-conormality duality relationships may be found in [6].

Finally, we return to our original motivating context in the introduction, but in a more general setting. As a rule, no additional hypotheses on the topological space Ω are necessary to pass from X to a space of X-valued functions, since the arguments are pointwise in X, but for some converse implications it is required that the vector valued function space in question is non-zero. If $C_c(\Omega) \neq \{0\}$, for example if Ω is a non-empty locally compact Hausdorff space, then this assumption is always satisfied.

Theorem 2.4.5. *Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X, such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i\in I} c_i$, where $c_i \in C_i$, for all $i \in I$. Let Ω be a topological space. Then there exists a constant $K > 0$ with the property that, for each X-valued continuous function $f \in C(\Omega, X)$ on Ω, there exist $f_i \in C(\Omega, C_i)$ $(i \in I)$, such that*

(1) For every $\omega \in \Omega$, $f(x) = \sum_{i\in I} f_i(\omega)$, and $\sum_{i\in I} \|f_i(\omega)\| \leq K\|f(\omega)\|$;

(2) $\|f_i\|_\infty \leq K\|f\|_\infty$, for all $i \in I$, where the right hand side, or both the left hand side and the right hand side, may be infinite;

(3) The support of each f_i is contained in that of f;

(4) If f vanishes at infinity, then so does each f_i;

(5) If $\omega_1, \omega_2 \in \Omega$ and $\lambda_1, \lambda_2 \geq 0$ are such that $\lambda_1 f(\omega_1) = \lambda_2 f(\omega_2)$, then $\lambda_1 f_i(\omega_1) = \lambda_2 f_i(\omega_2)$, for all $i \in I$.

In particular, if I is finite, so that $X = \sum_{i\in I} C_i$, then we can write the following vector spaces as the sum of cones naturally associated with the C_i, where the cones are closed in the last three normed spaces:

(1) $C(\Omega, X) = \sum_{i\in I} C(\Omega, C_i)$ for the continuous X-valued functions on Ω;

(2) $C_b(\Omega, X) = \sum_{i\in I} C_b(\Omega, C_i)$ for the bounded continuous X-valued functions on Ω;

(3) $C_0(\Omega, X) = \sum_{i\in I} C_0(\Omega, C_i)$ for the continuous X-valued functions on Ω vanishing at infinity;

(4) $C_c(\Omega, X) = \sum_{i\in I} C_c(\Omega, C_i)$ for the compactly supported continuous X-valued functions on Ω.

Proof. We apply part (1) of Theorem 2.4.1 and let $f_i := \gamma_i \circ f \ (i \in I)$. This supplies the f_i as required for the first part, and the statement on the finite number of naturally associated cones is then clear. □

Clearly then, the answer to our original question in the introduction is affirmative: If X is a Banach space with a closed generating proper cone X^+, and Ω is a topological space, then $C_0(\Omega, X^+)$ is generating in $C_0(\Omega, X)$. In fact, Theorem 2.4.5 shows that X^+ need not even be proper.

Remark 2.4.6. As mentioned in the introduction, if Ω is a (locally) compact Hausdorff space and X is a (pre-)ordered Banach space with closed generating cone X^+, certain special cases of Theorem 2.4.5 also follow from [4, Theorem 2.3] and [45, Theorem 4.4]. Both of these results proceed through an application of Lazar's affine selection theorem to show that cones of continuous affine X^+-valued functions on a Choquet simplex K are generating in spaces of continuous affine X^+-valued functions on K. If Ω is a compact Hausdorff space, the fact that $C(\Omega, X^+)$ is generating in $C(\Omega, X)$ follows from [4, Theorem 2.3] by taking K to be the Choquet simplex of all regular Borel probability measures on Ω, and considering the maps $\mu \mapsto \int_\Omega f d\mu \ (\mu \in K,\ f \in C(\Omega, X))$ and $\omega \mapsto a(\delta_\omega) \ (\omega \in \Omega,\ a \in A(K, X))$, where $A(K, X)$ denotes the space of continuous affine X-valued functions on K.

The converse of the four last statements in Theorem 2.4.5 also holds provided the function spaces are non-zero, as is shown by our next (elementary) result. Note that $C(\Omega, X)$ and $C_b(\Omega, X)$ are zero only when $\Omega \neq \emptyset$ and $X = \{0\}$.

Lemma 2.4.7. *Let X be a real or complex normed space. Let I be a finite non-empty set, and let $\{C_i\}_{i\in I}$ be a collection of cones in X, not necessarily closed or proper. Let Ω be a topological space.*

If $C(\Omega, X) \neq \{0\}$ and $C(\Omega, X) = \sum_{i\in I} C(\Omega, C_i)$, then $X = \sum_{i\in I} C_i$; similar statements hold for $C_b(\Omega, X)$, $C_0(\Omega, X)$ and $C_c(\Omega, X)$.

Proof. If there exists $0 \neq f \in C(\Omega, X)$, then composing f with a suitable continuous linear functional on X yields a non-zero $\varphi \in C(\Omega)$. Choose $\omega_0 \in \Omega$ such that $\varphi(\omega_0) \neq 0$; we may assume that $\varphi(\omega_0) = 1$. If $x \in X$, then (employing the usual notation) $\varphi \otimes x = \sum_{i\in I} f_i$, for some $f_i \in C(\Omega, C_i)$ $(i \in I)$ by assumption. Specializing this to the point ω_0 shows that $x = \sum_{i\in I} c_i$, for some $c_i \in C_i$ $(i \in I)$.

The proofs for $C_b(\Omega, X)$, $C_0(\Omega, X)$ and $C_c(\Omega, X)$ are similar. □

Corollary 2.4.8. *Let X be a real or complex Banach space. Let I be a non-empty finite set, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X. Let Ω be a topological space. If $C_c(\Omega) \neq \{0\}$, for example if Ω is a non-empty locally compact Hausdorff space, then the following are equivalent:*

(1) $X = \sum_{i\in I} C_i$;

(2) $C(\Omega, X) = \sum_{i\in I} C(\Omega, C_i)$;

(3) $C_b(\Omega, X) = \sum_{i\in I} C_b(\Omega, C_i)$;

(4) $C_0(\Omega, X) = \sum_{i\in I} C_0(\Omega, C_i)$;

(5) $C_c(\Omega, X) = \sum_{i\in I} C_c(\Omega, C_i)$.

Proof. If $X = \{0\}$ there is nothing to prove. If $X \neq \{0\}$, then the fact that $C_c(\Omega) \neq \{0\}$ implies that $C_c(\Omega, X) \neq \{0\}$, hence that the other three spaces of X-valued functions are non-zero as well. Combining Theorem 2.4.5 and Lemma 2.4.7 therefore concludes the proof. □

Theorem 2.4.5 and Corollary 2.4.8 are based on part (1) of Theorem 2.4.1. It is also possible to take part (2) into account and, e.g., obtain results on various types of conormality for spaces of continuous functions with values in a (pre)-ordered Banach space. Here is an example, where part (2) of Theorem 2.4.1 is used via an appeal to Corollary 2.4.3. Note that, analogous to Corollary 2.4.8, the approximate α-conormality of X and of the three normed spaces of X-valued functions are all equivalent if $C_c(\Omega) \neq \{0\}$.

Corollary 2.4.9. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+, and let Ω be a topological space. Suppose that $\alpha > 0$.*

If X is approximately α-conormal, then so are $C_b(\Omega, X)$, $C_0(\Omega, X)$, and $C_c(\Omega, X)$.

If $C_b(\Omega, X) \neq \{0\}$ and $C_b(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

If $C_0(\Omega, X) \neq \{0\}$ and $C_0(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

If $C_c(\Omega, X) \neq \{0\}$ and $C_c(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

Proof. Let X be approximately α-conormal, and let $f \in C_b(\Omega, X)$ and $\varepsilon > 0$ be given. Corollary 2.4.3 supplies continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$ and $\|\gamma_\varepsilon^+(x)\| \leq (\alpha + \varepsilon)\|x\|$, for all $x \in X$. Then $\gamma_\varepsilon^\pm \circ f \in C_b(\Omega, X^+)$, $f = \gamma_\varepsilon^+ \circ f - \gamma_\varepsilon^- \circ f$, and $\|\gamma_\varepsilon^\pm \circ f\|_\infty \leq (\alpha + \varepsilon)\|f\|_\infty$, so that $C_b(\Omega, X)$ is approximately α-conormal. The proof for $C_0(\Omega, X)$ and $C_c(\Omega, X)$ is similar.

If $C_b(\Omega, X) \neq \{0\}$ and $C_b(\Omega, X)$ is approximately α-conormal, let $x \in X$ and $\varepsilon > 0$ be given. As in the proof of Corollary 2.4.7 we find a non-zero $\varphi \in C_b(\Omega)$, and we may assume that $\|\varphi\|_\infty = 1$ and φ is real-valued. Passing to $-\varphi$ if necessary we obtain a sequence $\{\omega_n\} \subset \Omega$ such that $0 < \varphi(\omega_n) \uparrow 1$. Hence there exists $\omega_{n_0} \in \Omega$ such that $0 < \varphi(\omega_{n_0})$ and $(\alpha + \varepsilon/2)\varphi(\omega_{n_0})^{-1} < \alpha + \varepsilon$. By assumption, there exist $f^\pm \in C_b(\Omega, X^+)$, such that $\varphi \otimes x = f^+ - f^-$ and $\|f^+\|_\infty \leq (\alpha + \varepsilon/2)\|\varphi \otimes x\|_\infty = (\alpha + \varepsilon/2)\|x\|$. In particular, $\varphi(\omega_{n_0})x = f^+(\omega_{n_0}) - f^-(\omega_{n_0})$. Since $\varphi(\omega_{n_0})^{-1} f^\pm(\omega_{n_0}) \in X^+$, and $\|\varphi(\omega_{n_0})^{-1} f^+(\omega_{n_0})\| \leq \varphi(\omega_{n_0})^{-1}\|f^+\|_\infty \leq \varphi(\omega_{n_0})^{-1}(\alpha + \varepsilon/2)\|x\| \leq (\alpha + \varepsilon)\|x\|$, we conclude that X is approximately α-conormal.

The proofs for $C_0(\Omega, X)$ and $C_c(\Omega, X)$ are similar. □

Remark 2.4.10. In the context of Corollary 2.4.9, the conclusion that the Banach spaces $C_b(\Omega, X)$ and $C_0(\Omega, X)$ are approximately α-conormal shows that part (1) of Corollary 2.4.3 is satisfied for these (pre)-ordered Banach spaces. Hence (2) is valid as well. Therefore, if X is approximately α-conormal, then $C_b(\Omega, X)$ and $C_0(\Omega, X)$ are continuously positively homogeneously approximately α-conormal in the sense of part (2) of Corollary 2.4.3. The converse holds for $C_b(\Omega, X)$ if this space is non-zero, and similarly for $C_0(\Omega, X)$.

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