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Title: Positive representations on ordered Banach spaces

Issue Date: 2013-11-27

Chapter 1

Introduction

Many dynamical systems from nature must comply with certain ‘positivity constraints’ to make sense. For instance, in population dynamics negative populations do not make sense. Neither do negative values of concentration profiles of some material diffusing in a fluid within a sealed container. Furthermore, often some conservation principle governs the system, e.g., the total amount of material diffusing in a fluid within a sealed container remains constant in time.

Translating systems with such ‘positivity constraints’ into mathematical language usually yield pre-ordered or partially ordered vector spaces. By a pre-ordered vector space, we mean a vector space V over \mathbb{R} with a pre-order \leq that is compatible with the vector space structure in the sense that the pre-order is invariant under translations and multiplication by positive scalars. Such a relation defines a special subset $C := \{x \in V : x \geq 0\}$ of V , which satisfies $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. Such a set C is called a *cone*, (a *proper cone* if $C \cap (-C) = \{0\}$), and the elements of C are called the *positive* elements of V . Conversely, every cone C in a vector space V defines a such a pre-order (partial order, if C is proper) on V when, for $x, y \in V$, defining $x \leq y$ to mean $y - x \in C$.

We may then study dynamical systems with such ‘positivity constraints’ through group representations (or semigroups) acting on the space V (which may have a norm), leaving the cone (and perhaps the norm) invariant. Such actions are called *positive*, since they preserve the positive elements of V . A natural setting for studying such systems is that of pre-ordered Banach spaces. A simple example is the semigroup $(S_t)_{t \geq 0}$ acting on \mathbb{R}^3 with norm $\|\cdot\|_\infty$ and cone $C := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$, defined by

$$S : t \mapsto \begin{bmatrix} e^{-t} \cos t & -e^{-t} \sin t & 0 \\ e^{-t} \sin t & e^{-t} \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In natural systems there is quite often a symmetry group of the underlying space that acts canonically on the associated vector spaces, and in such a way that it leaves the cone of positive elements invariant. For example, the rotation group $SO(3, \mathbb{R})$

acts on the unit sphere (or the Earth's surface), and the canonically associated action on functions (think of temperature profiles) on the unit sphere, which rotates the function as a whole, obviously leaves positive functions positive. In this way, one obtains group homomorphisms from symmetry groups into pre-ordered vector spaces, such that the groups act as positive operators. These positive group representations, as they are called, are therefore quite common, but they have not been studied systematically, contrary to the case of unitary representations which have enjoyed attention for nearly a century. In studying positive group representations on pre-ordered Banach spaces, we draw much inspiration from the success of the theory of unitary representations on Hilbert spaces.

Since the early 1900's, motivated by quantum theory, much work has gone into the study of unitary representations of groups on Hilbert spaces. The decomposability of unitary representations into irreducible representations is a particularly interesting feature, in that the study of unitary representations can to some extent be reduced to studying the simplest 'building blocks'.

An example of this is the description of a particle trapped in an infinite well. The particle's dynamics (in time) is then determined by a unitary representation of the group \mathbb{R} on the complex Hilbert space $L^2([0, 1])$, in which the particle's wave function lives. The space $L^2([0, 1])$ can be written as direct sum of one-dimensional subspaces (spanned by the countable orthonormal basis of normalized solutions to the time-independent Schrödinger equation), with each subspace being invariant under the group representation. Decomposing an arbitrary wave function with respect to this basis allows one to describe the particle's motion by merely knowing how the group acts on these one-dimensional subspaces.

This decomposition is an example of a more general phenomenon for unitary representations of locally compact groups on Hilbert spaces. In 1927 Peter and Weyl proved:

Theorem 1. ([17, Theorem 7.2.4]) *Let G be a compact group and $U : G \rightarrow B(H)$ a unitary representation of G on a Hilbert space H . Then there exists a family of mutually orthogonal finite dimensional subspaces $\{H^{(i)}\}_{i \in I}$ of H , each invariant under U , such that the restriction of U to each $H^{(i)}$, denoted $U^{(i)} : G \rightarrow B(H^{(i)})$, is irreducible (has no non-trivial invariant subspaces), $H = \bigoplus_{i \in I} H^{(i)}$ and $U = \bigoplus_{i \in I} U^{(i)}$.*

Following on the work of von Neumann it was shown that this result can be extended to locally compact groups through a generalization of the concept of a direct sum to what is called a direct integral, in the same vein as a summation is a specific example of an integral over a measure space with respect to the counting measure:

Theorem 2. ([20, Theorem 8.5.2, Remark 18.7.6]) *Let G be separable locally compact group, H a separable Hilbert space and $U : G \rightarrow B(H)$ a strongly continuous unitary representation of G on H . Then there exists a standard Borel space Ω , a bounded measure μ on Ω , a measurable family of Hilbert spaces $\{H^{(\omega)}\}_{\omega \in \Omega}$, a measurable family unitary representations $\{U^{(\omega)} : G \rightarrow B(H^{(\omega)})\}_{\omega \in \Omega}$ such that*

$U^{(\omega)}$ is irreducible for almost every $\omega \in \Omega$. Furthermore, the representation $U : G \rightarrow B(H)$ is unitarily equivalent to the strongly continuous unitary representation $\int_{\Omega}^{\oplus} U^{(\omega)} d\mu(\omega)$ of G on the Hilbert space $\int_{\Omega}^{\oplus} H^{(\omega)} d\mu(\omega)$ through an isometric isomorphism between H and $\int_{\Omega}^{\oplus} H^{(\omega)} d\mu(\omega)$.

An important question that still remains open, is whether similar results holds for positive group representations and pre-ordered Banach algebras on pre-ordered Banach spaces. Orthogonality plays a crucial part in the the theory developed for unitary representations. Some natural partially ordered vector spaces, like the (real) vector spaces $L^p([0, 1])$ for $1 \leq p \leq \infty$, have similarities with Hilbert spaces through notions defined by their partial order that imply orthogonality in the Hilbert space case $p = 2$. Some work in this direction has been done by de Jeu and Wortel in the case of positive representations of finite groups on Riesz spaces [16] and positive representations of compact groups on Banach sequence spaces [15], but much still remains to be investigated in more general cases.

This thesis is a contribution to the study of positive representations of groups and pre-ordered Banach algebras on pre-ordered Banach spaces. It is mainly concerned with the investigation of positive representations on pre-ordered Banach spaces through the study of structures called *crossed products of Banach algebras*, which are themselves Banach algebras and encode information on covariant representations of Banach algebra dynamical systems on Banach spaces into information on their algebra representations on Banach spaces. Their construction is inspired by group C^* -algebras and crossed products of C^* -algebras. Group C^* -algebras play a crucial role in the proof of Theorem 2 and crossed products of C^* -algebras provide a satisfying conceptual framework for studying induction of unitary representations on Hilbert spaces. It is hoped that crossed products of Banach algebras will enable the establishment of generalizations or analogies of such results outside the C^* - and Hilbert space framework.

1.1 Pre-ordered Banach spaces

During the investigation into crossed products of Banach algebras in the ordered context, fundamental questions concerning general pre-ordered Banach spaces, interesting in their own right, reared their head and also warranted investigation to provide better insight into the main line of investigation. The following two sections of this introduction will explain these general questions.

1.1.1 Continuous generation

Let X be a Banach space and $C \subseteq X$ a closed cone in X . One says that C is *generating* in X , if $X = C - C$, i.e., every element from X can be written as a difference from elements of the cone C . With Ω a compact Hausdorff space, let $C(\Omega, X)$ denote the Banach space of continuous X -valued functions with the uniform

norm. This space also becomes a pre-ordered Banach space when endowed with the closed cone $C(\Omega, C)$. An immediate question that can be raised is the following:

Question 3. If C is a closed generating cone in a Banach space X , does that necessarily imply that the closed $C(\Omega, C)$ cone is generating in the space $C(\Omega, X)$ of continuous X -valued functions?

The resolution of this question provides insight into one aspect of the order structure of crossed products of pre-ordered Banach algebras studied in Chapter 5.

In the case that X is a Banach lattice, Question 3 has an easy solution. The maps $x \mapsto x \vee 0$ and $x \mapsto (-x) \vee 0$ ($x \in X$) are uniformly continuous on X . Therefore the maps $f^\pm : \omega \mapsto (\pm f(\omega)) \vee 0$ are indeed continuous, are elements of $C(\Omega, C)$, and satisfy $f = f^+ - f^-$. Hence $C(\Omega, C)$ is generating in $C(\Omega, X)$.

In the general case where X is a Banach space with closed generating cone $C \subseteq X$, with the lack of lattice operations, this line of reasoning is not available. Still, by the axiom of choice, we can define functions $(\cdot)^\pm : X \rightarrow C$, such that $x = x^+ - x^-$ for all $x \in X$. Hence, for any $f \in C(\Omega, X)$ and $\omega \in \Omega$, we have $f(\omega) = f(\omega)^+ - f(\omega)^-$. However, the functions $\omega \mapsto f(\omega)^\pm$ ($\omega \in \Omega$), of course, need to be continuous, and hence are not generally elements of the cone $C(\Omega, C)$. Therefore, this reasoning brings one no closer to answering Question 3 in its most general form. What is needed here is a continuous version of the axiom of choice...

We translate this problem into a more geometric version. For $x \in X$, consider the set valued map, called a *correspondence*, $\varphi : X \rightarrow 2^X$ defined by $\varphi : x \mapsto C \cap (x + C)$. For example, consider \mathbb{R}^3 with the cone $C := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$ (see Figure 1.1). Intuitively, considering the map $\varphi : x \mapsto C \cap (x + C)$ in this example, one can see that there is a certain sense of continuity to this set-valued map when varying $x \in \mathbb{R}^3$.

This raises the question of whether this sense of continuity of set-valued maps can be defined precisely. Furthermore, if this can be done, can one exploit this to construct a continuous function $f : X \rightarrow C$, such that $f(x) \in \varphi(x)$ for every $x \in X$? Remarkably, the answer to this question is affirmative! In the 1950's Michael published a landmark series of papers [31, 32, 33] outlining the theory of continuous selections, which included the following:

Theorem 4. (*Michael Selection Theorem [1, Theorem 17.66]*) *If $\varphi : \Omega \rightarrow 2^F$ is a lower hemicontinuous correspondence from a paracompact space Ω into a Fréchet space F , with non-empty closed convex values, then there exists a continuous function $f : \Omega \rightarrow F$, such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.*

Therefore, to resolve Question 3 affirmatively, it is sufficient to prove that the correspondence $\varphi : X \rightarrow 2^X$ defined by $\varphi : x \mapsto C \cap (x + C)$ is lower hemicontinuous (which we will not define here). This is indeed the case, as can be shown through invoking a theorem due to Andô [3, Lemma 1]. However, more can be said. In Chapter 2 we show that Andô's Theorem is a special case of the following version of the Open Mapping Theorem:

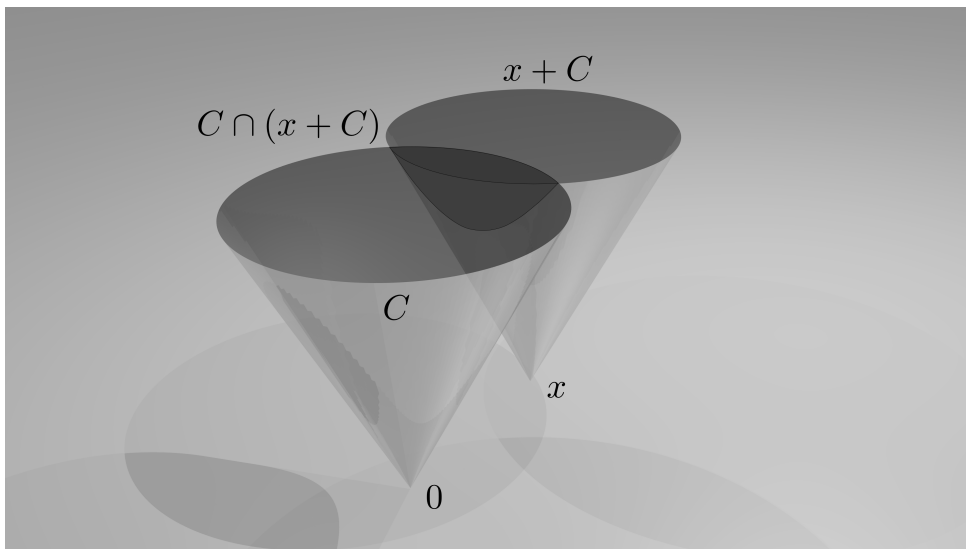


Figure 1.1

Theorem 5. (*Open Mapping Theorem*) Let C be a closed cone in a real or complex Banach space, not necessarily proper. Let X be a real or complex Banach space, not necessarily over the same field as the surrounding space of C , and $T : C \rightarrow X$ a continuous additive positively homogeneous map. Then the following are equivalent:

- (1) T is surjective;
- (2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $\|c\| \leq K\|x\|$;
- (3) T is an open map;
- (4) 0 is an interior point of $T(C)$.

This theorem together with the Michael Selection Theorem allows us to resolve a more general problem than what is stated in Question 3 through the following theorem of Chapter 2:

Theorem 6. Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i \in I}$ be a collection of closed cones in X , such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i \in I} c_i$, where $c_i \in C_i$, for all $i \in I$. Then, there exist a constant $\alpha > 0$ and continuous positively homogeneous maps $\gamma_i : X \rightarrow C_i$ ($i \in I$), such that:

- (1) $x = \sum_{i \in I} \gamma_i(x)$, for all $x \in X$;
- (2) $\sum_{i \in I} \|\gamma_i(x)\| \leq \alpha\|x\|$, for all $x \in X$.

With Ω a compact Hausdorff space, X a Banach space and $C \subseteq X$ a closed generating cone in X , Question 3 is therefore resolved through this theorem by taking $C_1 := C$ and $C_2 := -C$. Then, invoking the above theorem, every function $f \in C(\Omega, X)$ can be written as $f = \gamma_1 \circ f - (-\gamma_2 \circ f)$, where both $\gamma_1 \circ f$ and $-\gamma_2 \circ f$ are elements of $C(\Omega, C)$.

As is clear from the above theorem, we need not restrict ourselves to a single closed generating cone in X , but similar results hold when X is generated by a number of unrelated closed cones. To the author's knowledge, such spaces have never been investigated, and this provides an avenue along which more research can be done.

1.1.2 Normality of spaces of operators

Let X and Y be Banach lattices (with cones denoted by X_+ and Y_+). The space $B(X, Y)$ of all bounded linear operators from X to Y becomes a pre-ordered Banach space when endowed with the cone

$$B(X, Y)_+ := \{T \in B(X, Y) : TX_+ \subseteq Y_+\}.$$

Elementary properties of Banach lattices then imply that the operator norm on $B(X, Y)$ and the cone $B(X, Y)_+$ interact in the following way: If $T, S \in B(X, Y)$ satisfy $\pm T \leq S$, then $\|T\| \leq \|S\|$.

In Chapter 5, with X and Y pre-ordered Banach spaces, we will see that similar interactions of the cone $B(X, Y)_+$ and the operator norm determines certain aspects of the order structure of crossed products of pre-ordered Banach algebras. This motivates the following question investigated in Chapter 3:

Question 7. For general pre-ordered Banach spaces X and Y (with cones denoted by X_+ and Y_+), what properties should X and Y have so that the operator norm on $B(X, Y)$ and the cone $B(X, Y)_+$ interact in a similar fashion as described above? Do there exist examples of spaces X and Y that are not Banach lattices which have these properties?

These properties turn out to be the so-called *normality* and *conormality* properties which describe possible interactions of the cone of a pre-ordered Banach space with its norm. There are numerous variations of such properties that occur scattered throughout the literature. They usually appear in dual pairs, in the sense that a space has a normality property if and only if its dual space has the paired conormality property, and vice versa. An example of such a normality-conormality dual pair is the following:

Definition 8. Let X be a pre-ordered Banach space with a closed cone X_+ and $\alpha > 0$.

- The space X is said to be α -*absolutely normal* if, for every $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \alpha\|y\|$.
- The space X is said to be α -*absolutely conormal* if, for every $x \in X$, there exists some $y \in X_+$ such that $\pm x \leq y$ and $\|y\| \leq \alpha\|x\|$.

Roughly speaking, a normality property encodes, through the magnitude of α , how obtuse/blunt the cone X_+ is. On the other hand, a conormality property encodes, through the magnitude of α , how acute/sharp the cone X_+ is. This is illustrated in Figure 1.2 with \mathbb{R}^2 endowed with the $\|\cdot\|_2$ -norm and two different cones. The space on the left will be α -absolutely normal for a larger value of α than the space on the right. The space on the right will be α -absolutely conormal for a larger value of α than the space on the left.

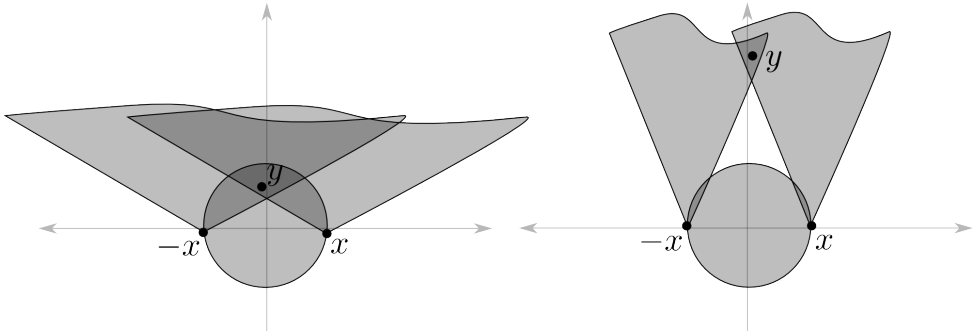


Figure 1.2

How normality and conormality of pre-ordered Banach spaces X and Y influence interaction of the operator norm on $B(X, Y)$ and the cone $B(X, Y)_+$ is described in Chapter 3 and follows the work of Yamamuro [48], Wickstead [45] and Batty and Robinson [6].

Knowledge of these interactions in spaces of bounded linear operators is required to describe the order structure of pre-ordered crossed product algebras, and will be discussed in the final section of this introduction.

The second part of Question 7 remains: whether there exist examples of spaces that are not-Banach lattices and also have the properties described. This will be discussed in the next section.

1.1.3 Quasi-lattices

Finite dimensional Banach lattices can be shown to always be isomorphic to \mathbb{R}^n , for some $n \in \mathbb{N}$, with the cone $C := \{x \in \mathbb{R}^n : x_j \geq 0, j \in \{1, \dots, n\}\}$, i.e., for any n , there is essentially only one cone which makes \mathbb{R}^n into a Banach lattice. Even in the case $n = 3$, this excludes a great multitude of possible cones that define partial orders on \mathbb{R}^3 . For example, for every $m \geq 4$, every cone $C \subseteq \mathbb{R}^3$ such that the intersection with the plane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$ is a regular m -sided polygon. Figure 1.3 shows examples for $m \in \{4, 5, 6, 7\}$.

Let X be a pre-ordered Banach space with a closed generating cone C . For every pair of elements $x, y \in X$, the set of their upper bounds $(x + C) \cap (y + C)$ is non-empty, but in general there need not exist a supremum of x and y in $(x + C) \cap (y + C)$ (with respect to the ordering defined by C on X). Equivalently: there need not exist

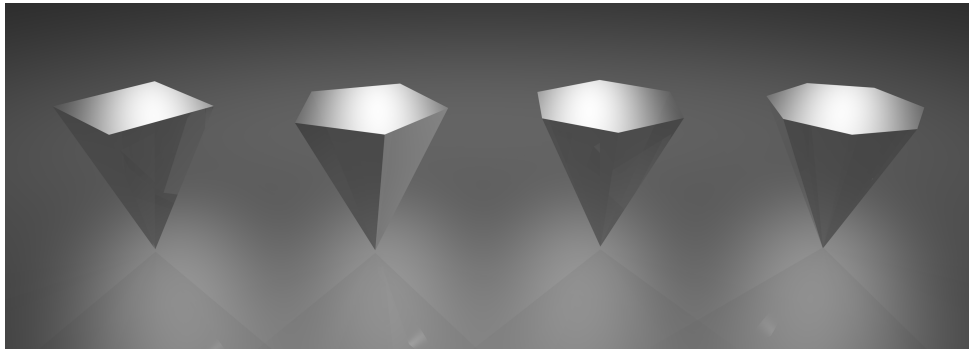


Figure 1.3

a point in the set of upper bounds $(x + C) \cap (y + C)$ which is smaller than all other elements from $(x + C) \cap (y + C)$, in contrast to when X is a Riesz space or a Banach lattice. For example, consider \mathbb{R}^3 with the cone $C := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$ (see Figure 1.1).

Still, taking the norm on the Banach space into account, there often exists a unique element in $(x + C) \cap (y + C)$ which is “the closest” to the points x and y . This enables one to define what we will call a *quasi-lattice* structure on X , as follows:

Definition 9. Let X be an ordered Banach space with a closed generating proper cone C . If, for every pair of elements $x, y \in X$, there exists a unique point $z_0 \in (x + C) \cap (y + C)$ such that z_0 minimizes the function

$$\sigma_{x,y}(z) := \|x - z\| + \|y - z\|$$

on $(x + C) \cap (y + C)$, then X is called a *quasi-lattice*, and z_0 is called the *quasi-supremum* of x and y , denoted by $x \tilde{\vee} y$. We define the following notation $x \tilde{\wedge} y := -((-x) \tilde{\vee} (-y))$, $\lceil x \rceil := x \tilde{\vee} (-x)$ and $x^\pm := 0 \tilde{\vee} (\pm x)$.

Many spaces are in fact quasi-lattices. In Chapter 3 we will prove:

Theorem 10. *Every reflexive Banach space with a strictly convex norm ordered by a closed generating proper cone is a quasi-lattice.*

This, of course, includes all spaces \mathbb{R}^n with a $\|\cdot\|_p$ -norm for $1 < p < \infty$ ordered by a closed generating proper cone. Furthermore, through a slightly altered definition of quasi-lattice which we will not discuss here, every Banach lattice can be shown to be a quasi-lattice, and the true lattice structure coincides with the quasi-lattice structure.

Quite surprisingly, many elementary vector lattice identities carry over verbatim from Riesz spaces to quasi-lattices. The following list of identities illustrates the similarity between quasi-lattices and Riesz spaces. Every symbol $\tilde{\vee}$, $\tilde{\wedge}$, and $\lceil \cdot \rceil$ may be replaced by \vee , \wedge , and $|\cdot|$ respectively, and each identity again holds true if X is replaced by a Riesz space.

Theorem 11. *Let X be a quasi-lattice and $x, y, z \in X$, $\alpha \geq 0$, $\beta < 0$, and $\gamma \in \mathbb{R}$. Then,*

- (1) $x\tilde{\vee}x = x\tilde{\wedge}x = x$.
- (2) $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x\tilde{\vee}y)$ and $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x\tilde{\wedge}y)$.
- (3) $(\beta x)\tilde{\vee}(\beta y) = \alpha(x\tilde{\wedge}y)$ and $(\beta x)\tilde{\wedge}(\beta y) = \beta(x\tilde{\vee}y)$.
- (4) $(x\tilde{\vee}y) + z = (x + z)\tilde{\vee}(y + z)$ and $(x\tilde{\wedge}y) + z = (x + z)\tilde{\wedge}(y + z)$.
- (5) $x^\pm \geq 0$, $x^- = (-x)^+$.
- (6) $\lceil x \rceil \geq 0$ and $\lceil \gamma x \rceil = |\gamma| \lceil x \rceil$.
- (7) $x = x^+ - x^-$; $\lceil x \rceil = x^+ + x^-$ and $x^+\tilde{\vee}x^- = 0$.
- (8) If $x \geq 0$, then $x = x^+ = \lceil x \rceil$.
- (9) $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.
- (10) $x\tilde{\vee}y + x\tilde{\wedge}y = x + y$ and $x\tilde{\vee}y - x\tilde{\wedge}y = \lceil x + y \rceil$.
- (11) $x\tilde{\vee}y = \frac{1}{2}(x + y) + \frac{1}{2} \lceil x - y \rceil$ and $x\tilde{\wedge}y = \frac{1}{2}(x + y) - \frac{1}{2} \lceil x - y \rceil$.

Returning to the second part of Question 7 posed in the previous section (as to whether there exist pre-ordered Banach spaces X and Y that are not Banach lattices, such that, for $T, S \in B(X, Y)$, the inequalities $\pm T \leq S$ imply $\|T\| \leq \|S\|$) we prove in Chapter 3, using the theory of quasi-lattices, that the following family furnishes us with examples of such spaces:

Example 12. Let \mathcal{H} be a real Hilbert space, $v \in \mathcal{H}$ any element with norm one, and P the orthogonal projection onto the hyperplane $\{v\}^\perp$. Then \mathcal{H} , ordered by the Lorentz cone $\mathcal{L}_v := \{x \in \mathcal{H} : \langle x|v \rangle \geq \|Px\|\}$, is a quasi-lattice, but not a Banach lattice when $\dim \mathcal{H} \geq 3$. If \mathcal{H}_1 and \mathcal{H}_2 are such spaces, then $B(\mathcal{H}_1, \mathcal{H}_2)$ is such that, for $T, S \in B(\mathcal{H}_1, \mathcal{H}_2)$, $\pm T \leq S$ implies $\|T\| \leq \|S\|$.

1.2 Crossed products

1.2.1 Crossed products of Banach algebras

When studying representations of a group on vector spaces, it is often useful to study algebras related to the group which encode information of the group's representations. For example, if G is a group and k is a field, there is a bijection between the representations of G on vector spaces over k and representations of the group algebra $k[G]$ on such spaces. In this way questions pertaining to representations of a group can be translated into questions pertaining to representations of a related algebra and vice versa.

One example of how this paradigm is used with success is in the proof of Theorem 2 above. If G is a locally compact group, there exists a related C^* -algebra

$C^*(G)$, called the *group C^* -algebra*. The algebra $C^*(G)$ is such that there exists a bijection between the strongly continuous unitary representations of G on Hilbert spaces and the non-degenerate $*$ -representations of $C^*(G)$ on Hilbert spaces. Theorem 2 is then proven through proving that direct integral decompositions of non-degenerate $*$ -representations of $C^*(G)$ on Hilbert spaces exist. Subsequently, one transforms a unitary representation of G into a $*$ -representation of $C^*(G)$, decomposes, and transforms the decomposed $*$ -representation of $C^*(G)$ back into a (now decomposed) unitary representation of G .

Group C^* -algebras are specific examples of more general objects called *crossed products of C^* -algebras*. Let the triple (A, G, α) be such that A is a C^* -algebra, G a locally compact group and $\alpha : G \rightarrow \text{Aut}(A)$ a strongly continuous $*$ -representation of G on A (where $\text{Aut}(A)$ denotes the $*$ -automorphism group of A). Such a triple is called a *C^* -algebra dynamical system*. A pair (π, U) , where π is a $*$ -representation of A on a Hilbert space H , and U a strongly continuous unitary representation of G on H , such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1} \quad (a \in A, s \in G), \quad (1.2.1)$$

is called a *covariant representation* of (A, G, α) on H . The *crossed product $A \rtimes_\alpha G$ associated with (A, G, α)* , is a C^* -algebra such that there exists a bijection between the non-degenerate covariant representations of (A, G, α) on Hilbert spaces, and the non-degenerate $*$ -representations of $A \rtimes_\alpha G$ on Hilbert spaces. In the case where $A = \mathbb{C}$, the crossed product $A \rtimes_\alpha G$ reduces to the group C^* -algebra $C^*(G)$.

Although notationally intimidating, C^* -algebra dynamical systems and covariant representations occur quite naturally, in that every group acting in a measure preserving way on a standard probability space easily generates such structures. For example, let the circle group $\mathbb{T} \subseteq \mathbb{C}$ act on the closed unit disc $\mathbb{D} \subseteq \mathbb{C}$, with the normalized Lebesgue measure, through rotation (complex multiplication). Then, with $\alpha_t(f)(s) := f(t^{-1}s)$ ($f \in C(\mathbb{T})$, $t \in \mathbb{T}$, $s \in \mathbb{D}$), the triple $(C(\mathbb{D}), \mathbb{T}, \alpha)$ is a C^* -algebra dynamical system. Furthermore, with $\pi : C(\mathbb{D}) \rightarrow B(L^2(\mathbb{D}))$ and $U : \mathbb{T} \rightarrow B(L^2(\mathbb{D}))$ defined by $\pi(f)g := fg$ ($f \in C(\mathbb{D})$, $g \in L^2(\mathbb{D})$) and $(U_t g)(s) := g(t^{-1}s)$ ($g \in L^2(\mathbb{D})$, $t \in \mathbb{T}$, $s \in \mathbb{D}$), the pair (π, U) is a non-degenerate covariant representation of $(C(\mathbb{D}), \mathbb{T}, \alpha)$ on $L^2(\mathbb{D})$. One immediately observes that the same construction is also valid when the Hilbert space $L^2(\mathbb{D})$ is replaced with the Banach spaces $L^p(\mathbb{D})$ where $1 \leq p < \infty$, and justifies the investigation of such kinds of objects in the more general Banach algebra and Banach space setting. Moreover, restricting oneself in this example to spaces over the real numbers and subsequently endowing them with the standard (pointwise) partial order, we see that all actions of the group \mathbb{T} are in fact positive, and hence justifies the investigation of such objects in the ordered context as well.

A *Banach algebra dynamical system* is a triple (A, G, α) where A is a Banach algebra, G a locally compact group and $\alpha : G \rightarrow \text{Aut}(A)$ a strongly continuous representation of G on A (where $\text{Aut}(A)$ denotes the automorphism group of A). A covariant representation (π, U) in this case is a pair such that both π and U are continuous representations respectively of A and G on a Banach space instead of a Hilbert space, and satisfy (1.2.1).

Our aim in Chapters 4 and 5 is, building on work by Dirksen, de Jeu and Wortel on crossed products associated with Banach algebra dynamical systems [19], to construct a pre-ordered Banach algebra, in analogy with the crossed product $A \rtimes_{\alpha} G$ associated with a C^* -dynamical system. For this construction to be a meaningful analogy, this pre-ordered Banach algebra should then encode (in its positive representation theory) information on the positive continuous covariant representations of the ‘pre-ordered Banach algebra dynamical system’ it is associated with.

One immediate difference between the C^* - and Banach algebra cases is that representations of Banach algebras on Banach spaces need not be contractive, as in the $*$ -representation case of C^* -algebras on Hilbert spaces. Also, unitary representations of a group on a Hilbert space are automatically uniformly bounded, which is not necessarily the case for general strongly continuous group representations on Banach spaces. In the construction of the crossed product algebra associated with a Banach algebra dynamical system, as opposed to the C^* -case, this necessitates the making of a choice, depending on the situation, of what one considers “good” continuous covariant representations and collecting them in a so-called *uniformly bounded class* \mathcal{R} of covariant representations. The condition is that all elements $(\pi, U) \in \mathcal{R}$ should satisfy the uniform bounds $\|\pi\| \leq C$ and $\|U_s\| \leq \nu(s)$ for all $s \in G$, where $C \geq 0$ and $\nu : G \rightarrow \mathbb{R}_{\geq 0}$ is a function that is bounded on compact subsets of G . One example of choosing “good” continuous covariant representations, would be to choose all continuous covariant representations (π, U) of (A, G, α) on Banach spaces with $\|\pi\| \leq 1$ and $\|U_s\| = 1$ for all $s \in G$.

With (A, G, α) a Banach algebra dynamical system and a uniformly bounded class \mathcal{R} of continuous covariant representations, one can then construct a Banach algebra $(A \rtimes_{\alpha} G)^{\mathcal{R}}$, called the *crossed product associated with (A, G, α) and \mathcal{R}* . In the presence of a bounded approximate left identity of A , there then exists a bijection between so called \mathcal{R} -continuous non-degenerate continuous covariant representations of (A, G, α) on Banach spaces and non-degenerate bounded representations of the Banach algebra $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ on Banach spaces.

In Chapter 4 we develop the theory of crossed products of Banach algebras further. Amongst others, we prove that (under mild assumptions) $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ is the unique Banach algebra, up to topological isomorphism, such that there exists a bijection between its non-degenerate bounded representations on Banach spaces and the non-degenerate \mathcal{R} -continuous covariant representations of (A, G, α) on Banach spaces. Furthermore, we show, through a particular choice of \mathcal{R} , that the crossed product algebra $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ is topologically (and in some cases isometrically) isomorphic to a generalized Beurling algebra, which is introduced in this chapter. Through this, classical results, like the relation between uniformly bounded representations of a locally compact group G on Banach spaces and non-degenerate bounded representations of $L^1(G)$ on Banach spaces, are shown to follow as special cases from the theory of crossed products of Banach algebras.

1.2.2 Crossed products of pre-ordered Banach algebras

First attempts at specializing the theory of crossed products of Banach algebras to the ordered case aimed at leveraging the well-developed theory of Banach lattices. For example, let (A, G, α) be a Banach algebra dynamical system, where A is a Banach lattice algebra, by which we mean A is a Banach algebra, a Banach lattice with cone A_+ , and satisfies $A_+ \cdot A_+ \subseteq A_+$. Furthermore α is assumed to be positive (for each $s \in G$, α_s maps the cone A_+ into A_+), and \mathcal{R} consists of positive continuous covariant representations (π, U) of (A, G, α) on Banach lattices, i.e., π maps positive elements of A to positive operators, and U maps G to positive invertible operators. Taking this route, however, one runs into technical difficulties in the construction of the crossed product. Intermediate objects in the construction of the crossed product are not always structured in such a way that one can conclude from known Banach lattice theory that $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ is a Banach lattice algebra in general (cf. Example 5.3.10). Attempts at forcing further structure on these intermediate objects, so that the crossed product is indeed a Banach lattice algebra, had the undesirable effect of leaving the crossed product synthetically enlarged, and thereby with a possibly altered representation theory. The construction of the sought bijection between positive continuous covariant representations of (A, G, α) and positive bounded representations of thus constructed Banach lattice algebras met with serious obstacles which the author and his collaborators were unable to surmount.

The Banach lattice setting, it would seem, is a too restrictive setting for studying ordered versions of crossed products of Banach algebras. A more suited setting in which to study ordered versions of crossed products of Banach algebras, turned out to be that of pre-ordered Banach algebras and pre-ordered Banach spaces. This allows for a wider range of structures for objects to roam in, which includes, but is not restricted to, Banach lattice algebras and Banach lattices.

In Chapter 5 we develop the theory along this line. A *pre-ordered Banach algebra dynamical system* is a triple (A, G, α) where A is a pre-ordered Banach algebra with a closed cone A_+ , (by which we mean A is a Banach algebra pre-ordered by a cone A_+ which satisfies $A_+ \cdot A_+ \subseteq A_+$), G a locally compact group and $\alpha : G \rightarrow \text{Aut}(A)$ a positive strongly continuous representation of G on A . With \mathcal{R} a uniformly bounded class of (not necessarily positive) continuous covariant representations, through an identical construction as in the unordered case, the crossed product $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ can be shown to inherit a natural cone, denoted $(A \rtimes_{\alpha} G)_{+}^{\mathcal{R}}$, from the cone of A . Furthermore, in the presence of a positive bounded approximate left identity of A , this pre-ordered Banach algebra $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ then has the desired property that there exists a bijection between the positive non-degenerate \mathcal{R} -continuous covariant representations (π, U) of (A, G, α) on pre-ordered Banach spaces with closed cones, and the positive non-degenerate bounded representations of $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ on such spaces. Using a similar argument as for the unordered case in Chapter 4, $(A \rtimes_{\alpha} G)^{\mathcal{R}}$, thus constructed, is shown (under mild conditions) to be the unique pre-ordered Banach algebra with this property, up to order preserving topological isomorphism.

In studying the order structure of the pre-ordered crossed product $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ deriving from the cone $(A \rtimes_{\alpha} G)_{+}^{\mathcal{R}}$, the work done in Chapters 2 and 3 can be applied.

To establish whether or not the cone $(A \rtimes_{\alpha} G)_{+}^{\mathcal{R}}$ is (topologically) generating in $(A \rtimes_{\alpha} G)^{\mathcal{R}}$, one is required to know whether the cone $C_c(G, A_+)$ of continuous compactly supported A_+ -valued functions is generating in the space $C_c(G, A)$ of all continuous compactly supported A -valued functions (see Question 3 in Section 1.1.1 above) and motivated the investigation in Chapter 2.

If \mathcal{R} consists of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, the normality (see Definition 8) of the crossed product $(A \rtimes_{\alpha} G)^{\mathcal{R}}$ is determined by the normality of all the pre-ordered operator algebras $B(X)$, where X ranges over the pre-ordered Banach spaces acted on by the covariant representations in \mathcal{R} (see Question 7 in Section 1.1.2 above). This motivated our investigation, done in Chapter 3, into the normality of spaces of operators and into quasi-lattices which give examples of pre-ordered Banach spaces X (that are not necessarily Banach lattices) where $B(X)$ is normal.

It is hoped that the theory of crossed products of pre-ordered Banach algebras as established in this thesis can sensibly be used in further study of positive group representations on Riesz spaces, Banach lattices and pre-ordered Banach spaces. In particular it is hoped that it can provide insights into possible future decomposition theories and induction of positive group representations as the group C^* -algebra and crossed products of C^* -algebras did for unitary representations.

However, as is usually the case, more questions have been raised than have been answered during the time spent investigating the structures contained in the chapters that will soon follow. We pose a few of these questions, all in the context of ordered Banach spaces (which as of printing of this manuscript still remain open), in the hope that they may pique the reader's interest:

Question 13. Are quasi-lattice operations ever/always (uniformly) continuous?

Question 14. Can the functions $\gamma_i : X \rightarrow C_i$ ($i \in I$) figuring in Theorem 6 be chosen so as to be uniformly continuous (as is the case for the functions $x \mapsto x^{\pm} := (\pm x) \vee 0$ on Banach lattices)?

Question 15. Currently, Banach spaces generated by a arbitrary collection of closed cones (and their continuous decomposition) is a curiosity which just so happens to be a generalization of pre-ordered Banach spaces with closed generating cones (cf. Theorem 6). Do there exist applications from economics (or any other field) of this theory? In other words, do there exist problems that translate to the study of a collection of different interacting pre-orders defined on a Banach space?

Question 16. (de Pagter) Can the definitions of normality and conormality be extended to Banach spaces X with arbitrary collections of closed cones $\{C_i\}_{i \in I}$ in X , so that they reduce to the classical definitions in the case when X is a pre-ordered Banach space with closed cone C , and taking $I = \{1, 2\}$ with $C_1 := C$ and $C_2 := -C$? And, can a duality relationship for these definitions be established, as exists for normality and conormality of usual pre-ordered Banach space with closed cones? (cf. Theorem 6 and Section 1.1.2).

