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Positive representations on ordered Banach spaces

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Positive representations on ordered Banach spaces

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Chapter 1

Introduction

Many dynamical systems from nature must comply with certain 'positivity constraints' to make sense. For instance, in population dynamics negative populations do not make sense. Neither do negative values of concentration profiles of some material diffusing in a fluid within a sealed container. Furthermore, often some conservation principle governs the system, e.g., the total amount of material diffusing in a fluid within a sealed container remains constant in time.

Translating systems with such 'positivity constraints' into mathematical language usually yield pre-ordered or partially ordered vector spaces. By a pre-ordered vector space, we mean a vector space V over $\mathbb{R}$ with a pre-order $\leq$ that is compatible with the vector space structure in the sense that the pre-order is invariant under translations and multiplication by positive scalars. Such a relation defines a special subset $C := \{x \in V : x \geq 0\}$ of V, which satisfies $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. Such a set C is called a *cone,* (a *proper cone* if $C \cap (-C) = \{0\}$), and the elements of C are called the *positive* elements of V. Conversely, every cone C in a vector space V defines a such a pre-order (partial order, if C is proper) on V when, for $x, y \in V$, defining $x \leq y$ to mean $y - x \in C$.

We may then study dynamical systems with such 'positivity constraints' through group representations (or semigroups) acting on the space V (which may have a norm), leaving the cone (and perhaps the norm) invariant. Such actions are called *positive,* since they preserve the positive elements of V. A natural setting for studying such systems is that of pre-ordered Banach spaces. A simple example is the semigroup $(S_t)_{t\geq 0}$ acting on $\mathbb{R}^3$ with norm $\|\cdot\|_\infty$ and cone $C := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$, defined by

$$S : t \mapsto \begin{bmatrix} e^{-t}\cos t & -e^{-t}\sin t & 0 \\ e^{-t}\sin t & e^{-t}\cos t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In natural systems there is quite often a symmetry group of the underlying space that acts canonically on the associated vector spaces, and in such a way that it leaves the cone of positive elements invariant. For example, the rotation group $SO(3, \mathbb{R})$

acts on the unit sphere (or the Earth's surface), and the canonically associated action on functions (think of temperature profiles) on the unit sphere, which rotates the function as a whole, obviously leaves positive functions positive. In this way, one obtains group homomorphisms from symmetry groups into pre-ordered vector spaces, such that the groups act as positive operators. These positive group representations, as they are called, are therefore quite common, but they have not been studied systematically, contrary to the case of unitary representations which have enjoyed attention for nearly a century. In studying positive group representations on pre-ordered Banach spaces, we draw much inspiration from the success of the theory of unitary representations on Hilbert spaces.

Since the early 1900's, motivated by quantum theory, much work has gone into the study of unitary representations of groups on Hilbert spaces. The decomposability of unitary representations into irreducible representations is a particularly interesting feature, in that the study of unitary representations can to some extent be reduced to studying the simplest 'building blocks'.

An example of this is the description of a particle trapped in an infinite well. The particle's dynamics (in time) is then determined by a unitary representation of the group $\mathbb{R}$ on the complex Hilbert space $L^2([0,1])$, in which the particle's wave function lives. The space $L^2([0,1])$ can be written as direct sum of one-dimensional subspaces (spanned by the countable orthonormal basis of normalized solutions to the time-independent Schrödinger equation), with each subspace being invariant under the group representation. Decomposing an arbitrary wave function with respect to this basis allows one to describe the particle's motion by merely knowing how the group acts on these one-dimensional subspaces.

This decomposition is an example of a more general phenomenon for unitary representations of locally compact groups on Hilbert spaces. In 1927 Peter and Weyl proved:

Theorem 1. *([17, Theorem 7.2.4]) Let G be a compact group and $U : G \to B(H)$ a unitary representation of G on a Hilbert space H. Then there exists a family of mutually orthogonal finite dimensional subspaces $\{H^{(i)}\}_{i\in I}$ of H, each invariant under U, such that the restriction of U to each $H^{(i)}$, denoted $U^{(i)} : G \to B(H^{(i)})$, is irreducible (has no non-trivial invariant subspaces), $H = \bigoplus_{i\in I} H^{(i)}$ and $U = \bigoplus_{i\in I} U^{(i)}$.*

Following on the work of von Neumann it was shown that this result can be extended to locally compact groups through a generalization of the concept of a direct sum to what is called a direct integral, in the same vein as a summation is a specific example of an integral over a measure space with respect to the counting measure:

Theorem 2. *([20, Theorem 8.5.2, Remark 18.7.6]) Let G be separable locally compact group, H a separable Hilbert space and $U : G \to B(H)$ a strongly continuous unitary representation of G on H. Then there exists a standard Borel space Ω, a bounded measure μ on Ω, a measurable family of Hilbert spaces $\{H^{(\omega)}\}_{\omega\in\Omega}$, a measurable family unitary representations $\{U^{(\omega)} : G \to B(H^{(\omega)})\}_{\omega\in\Omega}$ such that*

$U^{(\omega)}$ is irreducible for almost every $\omega \in \Omega$. Furthermore, the representation $U : G \to B(H)$ is unitarily equivalent to the strongly continuous unitary representation $\int_{\Omega}^{\oplus} U^{(\omega)} d\mu(\omega)$ of G on the Hilbert space $\int_{\Omega}^{\oplus} H^{(\omega)} d\mu(\omega)$ through an isometric isomorphism between H and $\int_{\Omega}^{\oplus} H^{(\omega)} d\mu(\omega)$.

An important question that still remains open, is whether similar results holds for positive group representations and pre-ordered Banach algebras on pre-ordered Banach spaces. Orthogonality plays a crucial part in the the theory developed for unitary representations. Some natural partially ordered vector spaces, like the (real) vector spaces $L^p([0,1])$ for $1 \leq p \leq \infty$, have similarities with Hilbert spaces through notions defined by their partial order that imply orthogonality in the Hilbert space case $p = 2$. Some work in this direction has been done by de Jeu and Wortel in the case of positive representations of finite groups on Riesz spaces [16] and positive representations of compact groups on Banach sequence spaces [15], but much still remains to be investigated in more general cases.

This thesis is a contribution to the study of positive representations of groups and pre-ordered Banach algebras on pre-ordered Banach spaces. It is mainly concerned with the investigation of positive representations on pre-ordered Banach spaces through the study of structures called *crossed products of Banach algebras*, which are themselves Banach algebras and encode information on covariant representations of Banach algebra dynamical systems on Banach spaces into information on their algebra representations on Banach spaces. Their construction is inspired by group C^*-algebras and crossed products of C^*-algebras. Group C^*-algebras play a crucial role in the proof of Theorem 2 and crossed products of C^*-algebras provide a satisfying conceptual framework for studying induction of unitary representations on Hilbert spaces. It is hoped that crossed products of Banach algebras will enable the establishment of generalizations or analogies of such results outside the C^*- and Hilbert space framework.

1.1 Pre-ordered Banach spaces

During the investigation into crossed products of Banach algebras in the ordered context, fundamental questions concerning general pre-ordered Banach spaces, interesting in their own right, reared their head and also warranted investigation to provide better insight into the main line of investigation. The following two sections of this introduction will explain these general questions.

1.1.1 Continuous generation

Let X be a Banach space and $C \subseteq X$ a closed cone in X. One says that C is *generating* in X, if $X = C - C$, i.e., every element from X can be written as a difference from elements of the cone C. With Ω a compact Hausdorff space, let $C(\Omega, X)$ denote the Banach space of continuous X-valued functions with the uniform norm. This space also becomes a pre-ordered Banach space when endowed with the closed cone $C(\Omega, C)$. An immediate question that can be raised is the following:

Question 3. If C is a closed generating cone in a Banach space X, does that necessarily imply that the closed $C(\Omega, C)$ cone is generating in the space $C(\Omega, X)$ of continuous X-valued functions?

The resolution of this question provides insight into one aspect of the order structure of crossed products of pre-ordered Banach algebras studied in Chapter 5.

In the case that X is a Banach lattice, Question 3 has an easy solution. The maps $x \mapsto x \vee 0$ and $x \mapsto (-x) \vee 0$ $(x \in X)$ are uniformly continuous on X. Therefore the maps $f^{\pm} : \omega \mapsto (\pm f(\omega)) \vee 0$ are indeed continuous, are elements of $C(\Omega, C)$, and satisfy $f = f^{+} - f^{-}$. Hence $C(\Omega, C)$ is generating in $C(\Omega, X)$.

In the general case where X is a Banach space with closed generating cone $C \subseteq X$, with the lack of lattice operations, this line of reasoning is not available. Still, by the axiom of choice, we can define functions $(\cdot)^{\pm} : X \to C$, such that $x = x^{+} - x^{-}$ for all $x \in X$. Hence, for any $f \in C(\Omega, X)$ and $\omega \in \Omega$, we have $f(\omega) = f(\omega)^{+} - f(\omega)^{-}$. However, the functions $\omega \mapsto f(\omega)^{\pm}$ $(\omega \in \Omega)$, of course, need to be continuous, and hence are not generally elements of the cone $C(\Omega, C)$. Therefore, this reasoning brings one no closer to answering Question 3 in its most general form. What is needed here is a continuous version of the axiom of choice...

We translate this problem into a more geometric version. For $x \in X$, consider the set valued map, called a *correspondence*, $\varphi : X \to 2^{X}$ defined by $\varphi : x \mapsto C \cap (x + C)$. For example, consider $\mathbb{R}^3$ with the cone $C := \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^2 + y^2}\}$ (see Figure 1.1). Intuitively, considering the map $\varphi : x \mapsto C \cap (x + C)$ in this example, one can see that there is a certain sense of continuity to this set-valued map when varying $x \in \mathbb{R}^3$.

This raises the question of whether this sense of continuity of set-valued maps can be defined precisely. Furthermore, if this can be done, can one exploit this to construct a continuous function $f : X \to C$, such that $f(x) \in \varphi(x)$ for every $x \in X$? Remarkably, the answer to this question is affirmative! In the 1950's Michael published a landmark series of papers [31, 32, 33] outlining the theory of continuous selections, which included the following:

Theorem 4. (*Michael Selection Theorem [1, Theorem 17.66]*) *If $\varphi : \Omega \to 2^{F}$ is a lower hemicontinous correspondence from a paracompact space Ω into a Fréchet space F, with non-empty closed convex values, then there exists a continuous function $f : \Omega \to F$, such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.*

Therefore, to resolve Question 3 affirmatively, it is sufficient to prove that the correspondence $\varphi : X \to 2^{X}$ defined by $\varphi : x \mapsto C \cap (x + C)$ is lower hemicontinuous (which we will not define here). This is indeed the case, as can be shown through invoking a theorem due to Andô [3, Lemma 1]. However, more can be said. In Chapter 2 we show that Andô's Theorem is a special case of the following version of the Open Mapping Theorem:

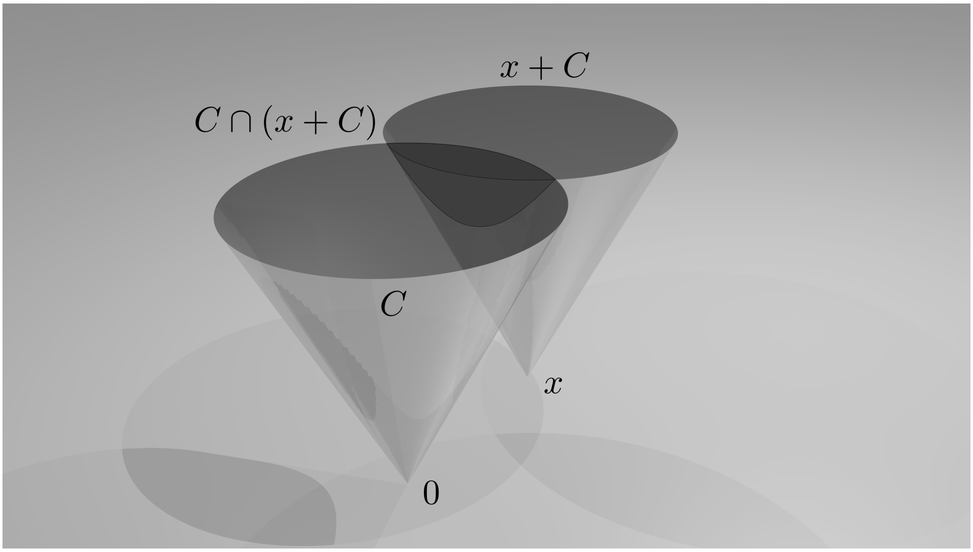

Figure 1.1

Theorem 5. (*Open Mapping Theorem*) *Let C be a closed cone in a real or complex Banach space, not necessarily proper. Let X be a real or complex Banach space, not necessarily over the same field as the surrounding space of C, and $T : C \to X$ a continuous additive positively homogeneous map. Then the following are equivalent:*

(1) T is surjective;

(2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $\|c\| \leq K\|x\|$;

(3) T is an open map;

(4) 0 is an interior point of $T(C)$.

This theorem together with the Michael Selection Theorem allows us to resolve a more general problem than what is stated in Question 3 through the following theorem of Chapter 2:

Theorem 6. *Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i\in I}$ be a collection of closed cones in X, such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i\in I} c_i$, where $c_i \in C_i$, for all $i \in I$. Then, there exist a constant $\alpha > 0$ and continuous positively homogeneous maps $\gamma_i : X \to C_i$ $(i \in I)$, such that:*

(1) $x = \sum_{i\in I} \gamma_i(x)$, for all $x \in X$;

(2) $\sum_{i\in I} \|\gamma_i(x)\| \leq \alpha\|x\|$, for all $x \in X$.

With Ω a compact Hausdorff space, X a Banach space and $C \subseteq X$ a closed generating cone in X, Question 3 is therefore resolved through this theorem by taking $C_1 := C$ and $C_2 := -C$. Then, invoking the above theorem, every function $f \in C(\Omega, X)$ can be written as $f = \gamma_1 \circ f - (-\gamma_2 \circ f)$, where both $\gamma_1 \circ f$ and $-\gamma_2 \circ f$ are elements of $C(\Omega, C)$.

As is clear from the above theorem, we need not restrict ourselves to a single closed generating cone in X, but similar results hold when X is generated by a number of unrelated closed cones. To the author's knowledge, such spaces have never been investigated, and this provides an avenue along which more research can be done.

1.1.2 Normality of spaces of operators

Let X and Y be Banach lattices (with cones denoted by X_+ and Y_+). The space $B(X,Y)$ of all bounded linear operators from X to Y becomes a pre-ordered Banach space when endowed with the cone

$$B(X,Y)_+ := \{T :\in B(X,Y) : TX_+ \subseteq Y_+\}.$$

Elementary properties of Banach lattices then imply that the operator norm on $B(X,Y)$ and the cone $B(X,Y)_+$ interact in the following way: If $T, S \in B(X,Y)$ satisfy $\pm T \leq S$, then $\|T\| \leq \|S\|$.

In Chapter 5, with X and Y pre-ordered Banach spaces, we will see that similar interactions of the cone $B(X,Y)_+$ and the operator norm determines certain aspects of the order structure of crossed products of pre-ordered Banach algebras. This motivates the following question investigated in Chapter 3:

Question 7. For general pre-ordered Banach spaces X and Y (with cones denoted by X_+ and Y_+), what properties should X and Y have so that the operator norm on $B(X,Y)$ and the cone $B(X,Y)_+$ interact in a similar fashion as described above? Do there exist examples of spaces X and Y that are not Banach lattices which have these properties?

These properties turn out to be the so-called *normality* and *conormality* properties which describe possible interactions of the cone of a pre-ordered Banach space with its norm. There are numerous variations of such properties that occur scattered throughout the literature. They usually appear in dual pairs, in the sense that a space has a normality property if and only if its dual space has the paired conormality property, and vice versa. An example of such a normality-conormality dual pair is the following:

Definition 8. Let X be a pre-ordered Banach space with a closed cone X_+ and $\alpha > 0$.

- The space X is said to be *α-absolutely normal* if, for every $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \alpha\|y\|$.
- The space X is said to be *α-absolutely conormal* if, for every $x \in X$, there exists some $y \in X_+$ such that $\pm x \leq y$ and $\|y\| \leq \alpha\|x\|$.

Roughly speaking, a normality property encodes, through the magnitude of α, how obtuse/blunt the cone X_+ is. On the other hand, a conormality property encodes, through the magnitude of α, how acute/sharp the cone X_+ is. This is illustrated in Figure 1.2 with $\mathbb{R}^2$ endowed with the $\|\cdot\|_2$-norm and two different cones. The space on the left will be α-absolutely normal for a larger value of α than the space on the right. The space on the right will be α-absolutely conormal for a larger value of α than the space on the left.

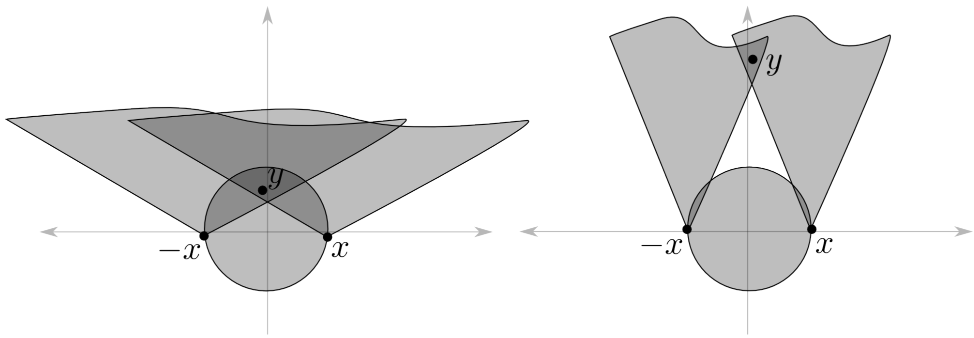

Figure 1.2

How normality and conormality of pre-ordered Banach spaces X and Y influence interaction of the operator norm on $B(X,Y)$ and the cone $B(X,Y)_+$ is described in Chapter 3 and follows the work of Yamamuro [48], Wickstead [45] and Batty and Robinson [6].

Knowledge of these interactions in spaces of bounded linear operators is required to describe the order structure of pre-ordered crossed product algebras, and will be discussed in the final section of this introduction.

The the second part of Question 7 remains: whether there exist examples of spaces that are not-Banach lattices and also have the properties described. This will be discussed in the next section.

1.1.3 Quasi-lattices

Finite dimensional Banach lattices can be shown to always be isomorphic to $\mathbb{R}^n$, for some $n \in \mathbb{N}$, with the cone $C := \{x \in \mathbb{R}^n : x_j \geq 0,\ j \in \{1, \ldots, n\}\}$, i.e., for any n, there is essentially only one cone which makes $\mathbb{R}^n$ into a Banach lattice. Even in the case $n = 3$, this excludes a great multitude of possible cones that define partial orders on $\mathbb{R}^3$. For example, for every $m \geq 4$, every cone $C \subseteq \mathbb{R}^3$ such that the intersection with the plane $\{(x,y,z) \in \mathbb{R}^3 : z = 1\}$ is a regular m-sided polygon. Figure 1.3 shows examples for $m \in \{4, 5, 6, 7\}$.

Let X be a pre-ordered Banach space with a closed generating cone C. For every pair of elements $x, y \in X$, the set of their upper bounds $(x + C) \cap (y + C)$ is non-empty, but in general there need not exist a supremum of x and y in $(x+C)\cap(y+C)$ (with respect to the ordering defined by C on X). Equivalently: there need not exist

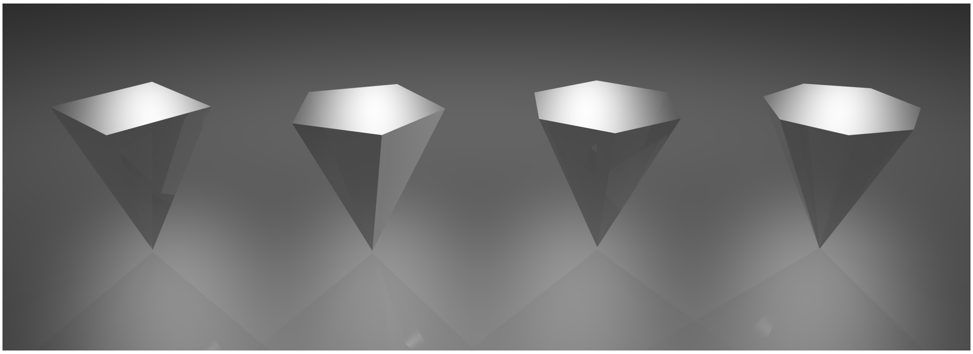

Figure 1.3

a point in the set of upper bounds $(x+C)\cap(y+C)$ which is smaller than all other elements from $(x+C)\cap(y+C)$, in contrast to when X is a Riesz space or a Banach lattice. For example, consider $\mathbb{R}^3$ with the cone $C := \{(x,y,z) \in \mathbb{R}^3 : z \geq \sqrt{x^2+y^2}\}$ (see Figure 1.1).

Still, taking the norm on the Banach space into account, there often exists a unique element in $(x+C)\cap(y+C)$ which is "the closest" to the points x and y. This enables one to define what we will call a *quasi-lattice* structure on X, as follows:

Definition 9. Let X be an ordered Banach space with a closed generating proper cone C. If, for every pair of elements $x, y \in X$, there exists a unique point $z_0 \in (x+C)\cap(y+C)$ such that z_0 minimizes the function

$$\sigma_{x,y}(z) := \|x-z\| + \|y-z\|$$

on $(x+C)\cap(y+C)$, then X is called a *quasi-lattice*, and z_0 is called the *quasi-supremum* of x and y, denoted by $x\tilde{\vee}y$. We define the following notation $x\tilde{\wedge}y := -((-x)\tilde{\vee}(-y))$, $\lceil x \rceil := x\tilde{\vee}(-x)$ and $x^{\pm} := 0\tilde{\vee}(\pm x)$.

Many spaces are in fact quasi-lattices. In Chapter 3 we will prove:

Theorem 10. *Every reflexive Banach space with a strictly convex norm ordered by a closed generating proper cone is a quasi-lattice.*

This, of course, includes all spaces $\mathbb{R}^n$ with a $\|\cdot\|_p$-norm for $1<p<\infty$ ordered by a closed generating proper cone. Furthermore, through a slightly altered definition of quasi-lattice which we will not discuss here, every Banach lattice can be shown to be a quasi-lattice, and the true lattice structure coincides with the quasi-lattice structure.

Quite surprisingly, many elementary vector lattice identities carry over verbatim from Riesz spaces to quasi-lattices. The following list of identities illustrates the similarity between quasi-lattices and Riesz spaces. Every symbol $\tilde{\vee}$, $\tilde{\wedge}$, and $\lceil\cdot\rceil$ may be replaced by $\vee$, $\wedge$, and $|\cdot|$ respectively, and each identity again holds true if X is replaced by a Riesz space.

Theorem 11. *Let X be a quasi-lattice and $x, y, z \in X$, $\alpha \geq 0$, $\beta < 0$, and $\gamma \in \mathbb{R}$. Then,*

(1) $x \tilde{\vee} x = x \tilde{\wedge} x = x$.

(2) $(\alpha x) \tilde{\vee} (\alpha y) = \alpha(x \tilde{\vee} y)$ *and* $(\alpha x) \tilde{\wedge} (\alpha y) = \alpha(x \tilde{\wedge} y)$.

(3) $(\beta x) \tilde{\vee} (\beta y) = \alpha(x \tilde{\wedge} y)$ *and* $(\beta x) \tilde{\wedge} (\beta y) = \beta(x \tilde{\vee} y)$.

(4) $(x \tilde{\vee} y) + z = (x + z) \tilde{\vee} (y + z)$ *and* $(x \tilde{\wedge} y) + z = (x + z) \tilde{\wedge} (y + z)$.

(5) $x^{\pm} \geq 0$, $x^- = (-x)^+$.

(6) $\lceil x \rceil \geq 0$ *and* $\lceil \gamma x \rceil = |\gamma| \lceil x \rceil$.

(7) $x = x^+ - x^-$; $\lceil x \rceil = x^+ + x^-$ *and* $x^+ \tilde{\vee} x^- = 0$.

(8) If $x \geq 0$, *then* $x = x^+ = \lceil x \rceil$.

(9) $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.

(10) $x \tilde{\vee} y + x \tilde{\wedge} y = x + y$ *and* $x \tilde{\vee} y - x \tilde{\wedge} y = \lceil x + y \rceil$.

(11) $x \tilde{\vee} y = \frac{1}{2}(x + y) + \frac{1}{2} \lceil x - y \rceil$ *and* $x \tilde{\wedge} y = \frac{1}{2}(x + y) - \frac{1}{2} \lceil x - y \rceil$.

Returning to the second part of Question 7 posed in the previous section (as to whether there exist pre-ordered Banach spaces X and Y that are not Banach lattices, such that, for $T, S \in B(X, Y)$, the inequalities $\pm T \leq S$ imply $\|T\| \leq \|S\|$) we prove in Chapter 3, using the theory of quasi-lattices, that the following family furnishes us with examples of such spaces:

Example 12. Let $\mathcal{H}$ be a real Hilbert space, $v \in \mathcal{H}$ any element with norm one, and P the orthogonal projection onto the hyperplane $\{v\}^{\perp}$. Then $\mathcal{H}$, ordered by the *Lorentz cone* $\mathcal{L}_v := \{x \in \mathcal{H} : \langle x|v \rangle \geq \|Px\|\}$, is a quasi-lattice, but not a Banach lattice when $\dim \mathcal{H} \geq 3$. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are such spaces, then $B(\mathcal{H}_1, \mathcal{H}_2)$ is such that, for $T, S \in B(\mathcal{H}_1, \mathcal{H}_2)$, $\pm T \leq S$ implies $\|T\| \leq \|S\|$.

1.2 Crossed products

1.2.1 Crossed products of Banach algebras

When studying representations of a group on vector spaces, it is often useful to study algebras related to the group which encode information of the group's representations. For example, if G is a group and k is a field, there is a bijection between the representations of G on vector spaces over k and representations of the group algebra $k[G]$ on such spaces. In this way questions pertaining to representations of a group can be translated into questions pertaining to representations of a related algebra and vice versa.

One example of how this paradigm is used with success is in the proof of Theorem 2 above. If G is a locally compact group, there exists a related C^*-algebra

$C^*(G)$, called the *group C^*-algebra*. The algebra $C^*(G)$ is such that there exists a bijection between the strongly continuous unitary representations of G on Hilbert spaces and the non-degenerate *-representations of $C^*(G)$ on Hilbert spaces. Theorem 2 is then proven through proving that direct integral decompositions of non-degenerate *-representations of $C^*(G)$ on Hilbert spaces exist. Subsequently, one transforms a unitary representation of G into a *-representation of $C^*(G)$, decomposes, and transforms the decomposed *-representation of $C^*(G)$ back into a (now decomposed) unitary representation of G.

Group C^*-algebras are specific examples of more general objects called *crossed products of C^*-algebras*. Let the triple (A, G, α) be such that A is a C^*-algebra, G a locally compact group and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous *-representation of G on A (where $\mathrm{Aut}(A)$ denotes the *-automorphism group of A). Such a triple is called a *C^*-algebra dynamical system*. A pair (π, U), where π is a *-representation of A on a Hilbert space H, and U a strongly continuous unitary representation of G on H, such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1} \quad (a \in A,\ s \in G), \tag{1.2.1}$$

is called a *covariant representation* of (A, G, α) on H. The *crossed product $A \rtimes_\alpha G$ associated with* (A, G, α), is a C^*-algebra such that there exists a bijection between the non-degenerate covariant representations of (A, G, α) on Hilbert spaces, and the non-degenerate *-representations of $A \rtimes_\alpha G$ on Hilbert spaces. In the case where $A = \mathbb{C}$, the crossed product $A \rtimes_\alpha G$ reduces to the group C^*-algebra $C^*(G)$.

Although notationally intimidating, C^*-algebra dynamical systems and covariant representations occur quite naturally, in that every group acting in a measure preserving way on a standard probability space easily generates such structures. For example, let the circle group $\mathbb{T} \subseteq \mathbb{C}$ act on the closed unit disc $\mathbb{D} \subseteq \mathbb{C}$, with the normalized Lebesgue measure, through rotation (complex multiplication). Then, with $\alpha_t(f)(s) := f(t^{-1}s)$ $(f \in C(\mathbb{T}),\ t \in \mathbb{T},\ s \in \mathbb{D})$, the triple $(C(\mathbb{D}), \mathbb{T}, \alpha)$ is a C^*-algebra dynamical system. Furthermore, with $\pi : C(\mathbb{D}) \to B(L^2(\mathbb{D}))$ and $U : \mathbb{T} \to B(L^2(\mathbb{D}))$ defined by $\pi(f)g := fg$ $(f \in C(\mathbb{D}),\ g \in L^2(\mathbb{D}))$ and $(U_t g)(s) := g(t^{-1}s)$ $(g \in L^2(\mathbb{D}),\ t \in \mathbb{T},\ s \in \mathbb{D})$, the pair (π, U) is a non-degenerate covariant representation of $(C(\mathbb{D}), \mathbb{T}, \alpha)$ on $L^2(\mathbb{D})$. One immediately observes that the same construction is also valid when the Hilbert space $L^2(\mathbb{D})$ is replaced with the Banach spaces $L^p(\mathbb{D})$ where $1 \leq p < \infty$, and justifies the investigation of such kinds of objects in the more general Banach algebra and Banach space setting. Moreover, restricting oneself in this example to spaces over the real numbers and subsequently endowing them with the standard (pointwise) partial order, we see that all actions of the group $\mathbb{T}$ are in fact positive, and hence justifies the investigation of such objects in the ordered context as well.

A *Banach algebra dynamical system* is a triple (A, G, α) where A is a Banach algebra, G a locally compact group and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous representation of G on A (where $\mathrm{Aut}(A)$ denotes the automorphism group of A). A covariant representation (π, U) in this case is a pair such that both π and U are continuous representations respectively of A and G on a Banach space instead of a Hilbert space, and satisfy (1.2.1).

Our aim in Chapters 4 and 5 is, building on work by Dirksen, de Jeu and Wortel on crossed products associated with Banach algebra dynamical systems [19], to construct a pre-ordered Banach algebra, in analogy with the crossed product $A \rtimes_\alpha G$ associated with a C^*-dynamical system. For this construction to be a meaningful analogy, this pre-ordered Banach algebra should then encode (in its positive representation theory) information on the positive continuous covariant representations of the 'pre-ordered Banach algebra dynamical system' it is associated with.

One immediate difference between the C^*- and Banach algebra cases is that representations of Banach algebras on Banach spaces need not be contractive, as in the *-representation case of C^*-algebras on Hilbert spaces. Also, unitary representations of a group on a Hilbert space are automatically uniformly bounded, which is not necessarily the case for general strongly continuous group representations on Banach spaces. In the construction of the crossed product algebra associated with a Banach algebra dynamical system, as opposed to the C^*-case, this necessitates the making of a choice, depending on the situation, of what one considers "good" continuous covariant representations and collecting them in a so-called *uniformly bounded class* $\mathcal{R}$ of covariant representations. The condition is that all elements $(\pi, U) \in \mathcal{R}$ should satisfy the uniform bounds $\|\pi\| \leq C$ and $\|U_s\| \leq \nu(s)$ for all $s \in G$, were $C \geq 0$ and $\nu : G \to \mathbb{R}_{\geq 0}$ is a function that is bounded on compact subsets of G. One example of choosing "good" continuous covariant representations, would be to choose all continuous covariant representations (π, U) of (A, G, α) on Banach spaces with $\|\pi\| \leq 1$ and $\|U_s\| = 1$ for all $s \in G$.

With (A, G, α) a Banach algebra dynamical system and a uniformly bounded class $\mathcal{R}$ of continuous covariant representations, one can then construct a Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$, called the *crossed product associated with* (A, G, α) *and* $\mathcal{R}$. In the presence of a bounded approximate left identity of A, there then exists a bijection between so called $\mathcal{R}$*-continuous* non-degenerate continuous covariant representations of (A, G, α) on Banach spaces and non-degenerate bounded representations of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ on Banach spaces.

In Chapter 4 we develop the theory of crossed products of Banach algebras further. Amongst others, we prove that (under mild assumptions) $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra, up to topological isomorphism, such that there exists a bijection between its non-degenerate bounded representations on Banach spaces and the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on Banach spaces. Furthermore, we show, through a particular choice of $\mathcal{R}$, that the crossed product algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is topologically (and in some cases isometrically) isomorphic to a generalized Beurling algebra, which is introduced in this chapter. Through this, classical results, like the relation between uniformly bounded representations of a locally compact group G on Banach spaces and non-degenerate bounded representations of $L^1(G)$ on Banach spaces, are shown to follow as special cases from the theory of crossed products of Banach algebras.

1.2.2 Crossed products of pre-ordered Banach algebras

First attempts at specializing the theory of crossed products of Banach algebras to the ordered case aimed at leveraging the well-developed theory of Banach lattices. For example, let (A, G, α) be a Banach algebra dynamical system, where A is a Banach lattice algebra, by which we mean A is a Banach algebra, a Banach lattice with cone A_+, and satisfies $A_+ \cdot A_+ \subseteq A_+$. Furthermore α is assumed to be positive (for each $s \in G$, α_s maps the cone A_+ into A_+), and $\mathcal{R}$ consists of positive continuous covariant representations (π, U) of (A, G, α) on Banach lattices, i.e., π maps positive elements of A to positive operators, and U maps G to positive invertible operators. Taking this route, however, one runs into technical difficulties in the construction of the crossed product. Intermediate objects in the construction of the crossed product are not always structured in such a way that one can conclude from known Banach lattice theory that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a Banach lattice algebra in general (cf. Example 5.3.10). Attempts at forcing further structure on these intermediate objects, so that the crossed product is indeed a Banach lattice algebra, had the undesirable effect of leaving the crossed product synthetically enlarged, and thereby with a possibly altered representation theory. The construction of the sought bijection between positive continuous covariant representations of (A, G, α) and positive bounded representations of thus constructed Banach lattice algebras met with serious obstacles which the author and his collaborators were unable to surmount.

The Banach lattice setting, it would seem, is a too restrictive setting for studying ordered versions of crossed products of Banach algebras. A more suited setting in which to study ordered versions of crossed products of Banach algebras, turned out to be that of pre-ordered Banach algebras and pre-ordered Banach spaces. This allows for a wider range of structures for objects to roam in, which includes, but is not restricted to, Banach lattice algebras and Banach lattices.

In Chapter 5 we develop the theory along this line. A *pre-ordered Banach algebra dynamical system* is a triple (A, G, α) where A is a pre-ordered Banach algebra with a closed cone A_+, (by which we mean A is a Banach algebra pre-ordered by a cone A_+ which satisfies $A_+ \cdot A_+ \subseteq A_+$), G a locally compact group and $\alpha : G \to \mathrm{Aut}(A)$ a positive strongly continuous representation of G on A. With $\mathcal{R}$ a uniformly bounded class of (not necessarily positive) continuous covariant representations, through an identical construction as in the unordered case, the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ can be shown to inherit a natural cone, denoted $(A \rtimes_\alpha G)^{\mathcal{R}}_+$, from the cone of A. Furthermore, in the presence of a positive bounded approximate left identity of A, this pre-ordered Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ then has the desired property that there exists a bijection between the positive non-degenerate $\mathcal{R}$-continuous covariant representations (π, U) of (A, G, α) on pre-ordered Banach spaces with closed cones, and the positive non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces. Using a similar argument as for the unordered case in Chapter 4, $(A \rtimes_\alpha G)^{\mathcal{R}}$, thus constructed, is shown (under mild conditions) to be the unique pre-ordered Banach algebra with this property, up to order preserving topological isomorphism.

In studying the order structure of the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ deriving from the cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$, the work done in Chapters 2 and 3 can be applied.

To establish whether or not the cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is (topologically) generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$, one is required to know whether the cone $C_c(G, A_+)$ of continuous compactly supported A_+-valued functions is generating in the space $C_c(G, A)$ of all continuous compactly supported A-valued functions (see Question 3 in Section 1.1.1 above) and motivated the investigation in Chapter 2.

If $\mathcal{R}$ consists of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, the normality (see Definition 8) of the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is determined by the normality of all the pre-ordered operator algebras $B(X)$, where X is ranges over the pre-ordered Banach spaces acted on by the covariant representations in $\mathcal{R}$ (see Question 7 in Section 1.1.2 above). This motivated our investigation, done in Chapter 3, into the normality of spaces of operators and into quasi-lattices which give examples of pre-ordered Banach spaces X (that are not necessarily Banach lattices) where $B(X)$ is normal.

It is hoped that the theory of crossed products of pre-ordered Banach algebras as established in this thesis can sensibly be used in further study of positive group representations on Riesz spaces, Banach lattices and pre-ordered Banach spaces. In particular it is hoped that it can provide insights into possible future decomposition theories and induction of positive group representations as the group C^*-algebra and crossed products of C^*-algebras did for unitary representations.

However, as is usually the case, more questions have been raised than have been answered during the time spent investigating the structures contained in the chapters that will soon follow. We pose a few of these questions, all in the context of ordered Banach spaces (which as of printing of this manuscript still remain open), in the hope that they may pique the reader's interest:

Question 13. Are quasi-lattice operations ever/always (uniformly) continuous?

Question 14. Can the functions $\gamma_i : X \to C_i \, (i \in I)$ figuring in Theorem 6 be chosen so as to be uniformly continuous (as is the case for the functions $x \mapsto x^\pm := (\pm x) \vee 0$ on Banach lattices)?

Question 15. Currently, Banach spaces generated by a arbitrary collection of closed cones (and their continuous decomposition) is a curiosity which just so happens to be a generalization of pre-ordered Banach spaces with closed generating cones (cf. Theorem 6). Do there exist applications from economics (or any other field) of this theory? In other words, do there exist problems that translate to the study of a collection of different interacting pre-orders defined on a Banach space?

Question 16. (de Pagter) Can the definitions of normality and conormality be extended to Banach spaces X with arbitrary collections of closed cones $\{C_i\}_{i \in I}$ in X, so that they reduce to the classical definitions in the case when X is a pre-ordered Banach space with closed cone C, and taking $I = \{1, 2\}$ with $C_1 := C$ and $C_2 := -C$? And, can a duality relationship for these definitions be established, as exists for normality and conormality of usual pre-ordered Banach space with closed cones? (cf. Theorem 6 and Section 1.1.2).

Chapter 2

Right inverses of surjections from cones onto Banach spaces

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2.1 Introduction

Consider the following question, that arose in other research of the authors: Let X be a real Banach space, ordered by a closed generating proper cone X^+, and let Ω be a topological space. Then the Banach space $C_0(\Omega, X)$, consisting of the continuous X-valued functions on Ω vanishing at infinity, is ordered by the natural closed proper cone $C_0(\Omega, X^+)$. Is this cone also generating? If X is a Banach lattice, then the answer is affirmative. Indeed, if $f \in C_0(\Omega, X)$, then $f = f^+ - f^-$, where $f^\pm(\omega) := f(\omega)^\pm$ $(\omega \in \Omega)$. Since the maps $x \mapsto x^\pm$ are continuous, $f^\pm$ is continuous, and since $\|f^\pm(\omega)\| \leq \|f(\omega)\|$ $(\omega \in \Omega)$, $f^\pm$ vanishes at infinity. Thus a decomposition as desired has been obtained. For general X, the situation is not so clear. The natural approach is to consider a pointwise decomposition as in the Banach lattice case, but for this to work we need to know that at the level of X the constituents $x^\pm$ in a decomposition $x = x^+ - x^-$ can be chosen in a continuous and simultaneously also bounded (in an obvious terminology) fashion, as x varies over X. Boundedness is certainly attainable, due to the following classical result:

Theorem 2.1.1. (*Andô's Theorem [3]*) *Let X be a real Banach space ordered by a closed generating proper cone $X^+ \subseteq X$. Then there exists a constant $K > 0$ with the property that, for every $x \in X$, there exist $x^\pm \in X^+$ such that $x = x^+ - x^-$ and $\|x^\pm\| \leq K\|x\|$.*

Continuity is not asserted, however. Hence we are not able to settle our question in the affirmative via Andô's Theorem alone and stronger results are needed. With

Ω a compact Hausdorff space, it is a consequence of a result due to Asimow and Atkinson [4, Theorem 2.3] that $C(\Omega, X^+)$ is generating in $C(\Omega, X)$ when X^+ is closed and generating in X. A similar result due to Wickstead [45, Theorem 4.4] establishes this for $C_0(\Omega, X)$ when Ω is locally compact (cf. Remark 2.4.6). We will also retrieve these results, but by a different method, namely by establishing the general existence of a continuous bounded decompositions, analogous to that for Banach lattices, as a special case of Theorem 2.4.5 below.

In fact, although the above question and results are in the context of ordered Banach spaces, it will become clear in this paper that for these spaces one is merely looking at a particular instance of more general phenomena. In short: if $T : C \to X$ is a continuous additive positively homogeneous surjection from a closed not necessarily proper cone in a Banach space onto a Banach space, then T has a well-behaved right inverse, and (stronger) versions of theorems such as Andô's, where several cones in one space are involved, are then almost immediately clear. We will now elaborate on this, and at the same time explain the structure of the proofs.

The usual notation to express that X^+ is generating is to write $X = X^+ - X^+$, but the actually relevant point turns out to be that $X = X^+ + (-X^+)$ is a sum of two closed cones: the fact that these are related by a minus sign is only a peculiarity of the context. In fact, if X is the sum of possibly uncountably many closed cones, which need not be proper (this is redundant in Andô's Theorem), then it is possible to choose a bounded decomposition: this is the content of the first part of Theorem 2.4.1. However, this is in itself a consequence of the following more fundamental result, a special case of Theorem 2.3.2. Since a Banach space is a closed cone in itself, it generalizes the usual Open Mapping Theorem for Banach spaces, which is used in the proof.

Theorem 2.1.2. (*Open Mapping Theorem*) *Let C be a closed cone in a real or complex Banach space, not necessarily proper. Let X be a real or complex Banach space, not necessarily over the same field as the surrounding space of C, and $T : C \to X$ a continuous additive positively homogeneous map. Then the following are equivalent:*

(1) T is surjective;

(2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $\|c\| \leq K\|x\|$;

(3) T is an open map;

(4) 0 is an interior point of $T(C)$.

As an illustration of how this can be applied, suppose $X = \sum_{i \in I} C_i$ is the sum of a finite (for the ease of formulation) number of closed not necessarily proper cones. We let Y be the sum of $|I|$ copies of X, and let $C \subset Y$ be the direct sum of the C_i's. Then the natural summing map $T : C \to X$ is surjective by assumption, so that part (2) of Theorem 2.1.2 provides a bounded decomposition. Andô's Theorem corresponds to the case where X is the image of $X^+ \times (-X^+) \subset X \times X$ under the summing map.

In this fashion, generalizations of Andô's Theorem are obtained as a consequence of an Open Mapping Theorem. However, this still does not resolve the issue of a decomposition that is not only bounded, but continuous as well. A possible attempt to obtain this would be the following: if $T : Y \to X$ is a continuous linear surjection between Banach spaces (or even Fréchet spaces), then T has a continuous right inverse, see [1, Corollary 17.67]. The proof is based on Michael's Selection Theorem, which we will recall in Section 2. Conceivably, the proof as in [1] could be modified to yield a similar statement for a continuous surjective additive positively homogeneous $T : C \to X$ from a closed cone C in a Banach space onto X. In that case, if $X = \sum_{i\in I} C_i$ is a finite (say) sum of closed not necessarily proper cones, the setup with product cone and summing map would yield the existence of a continuous decomposition, but unfortunately this time there is no guarantee for boundedness. Somehow the generalized Open Mapping Theorem as in Theorem 2.1.2 and Michael's Selection Theorem must be combined. The solution lies in a refinement of the correspondences to which Michael's Selection Theorem is to be applied, and take certain subadditive maps on C into account from the very beginning. In the end, one of these maps will be taken to be the norm on C, and this provides the desired link between the generalized Open Mapping Theorem and Michael's Selection Theorem, cf. the proof of Proposition 2.3.5. It is along these lines that the following is obtained. It is a special case of Theorem 2.3.6 and, as may be clear by now, it implies the existence of a continuous bounded (and even positively homogeneous) decomposition if $X = \sum_{i\in I} C_i$. It also shows that, if $T : Y \to X$ is a continuous linear surjection between Banach spaces, then it is not only possible to choose a bounded right inverse for T (a statement equivalent to the usual Open Mapping Theorem), but also to choose a bounded right inverse that is, in addition, continuous and positively homogenous.

Theorem 2.1.3. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Then there exists a constant $K > 0$ and a continuous positively homogeneous map $\gamma : X \to C$, such that:

(1) $T \circ \gamma = id_X$*;*

(2) $\|\gamma(x)\| \leq K\|x\|$*, for all* $x \in X$.

The underlying Proposition 2.3.5 is the core of this paper. It is reworked into the somewhat more practical Theorems 2.3.6 and 2.3.7, but this is all routine, as are the applications in Section 2.4. For example, the following result (Corollary 2.4.2) is virtually immediate from Section 2.3. We cite it in full, not only because it shows concretely how Andô's Theorem figuring so prominently in our discussion so far can be strengthened, but also to enable us to comment on the interpretation of the various parts of this result and similar ones.

Theorem 2.1.4. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+. Let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : X \times X \to \mathbb{R}$ be a continuous seminorm or a continuous linear functional. Then:*

(1) There exist a constant $K > 0$ and continuous positively homogeneous maps $\gamma^{\pm} : X \to X^+$, such that:

(a) $x = \gamma^+(x) - \gamma^-(x)$, for all $x \in X$;

(b) $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exist maps $\gamma_\varepsilon^{\pm} : S_X \to X^+$, where $S_X := \{x \in X : \|x\| = 1\}$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in S_X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)$, for all $x \in S_X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) For every $\varepsilon > 0$, there exist continuous positively homogeneous maps $\gamma_\varepsilon^{\pm} : X \to X^+$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)\|x\|$, for all $x \in X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)\|x\|$, for all X and $j \in J$.

The existence of a bounded continuous positively homogeneous decomposition in part (1) is of course a direct consequence of Theorem 2.1.3. Naturally, the argument as for Banach lattices then shows that $C_0(\Omega, X) = C_0(\Omega, X^+) - C_0(\Omega, X^+)$ for an arbitrary topological space Ω, so that our original question has been settled in the affirmative.

The equivalence under (2) has the following consequence: If there exist maps $\gamma^{\pm} : S_X \to X^+$, such that $x = \gamma^+(x) - \gamma^-(x)$, $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K$, and $\rho_j((\gamma^+(x), \gamma^-(x)) \leq \alpha_j$, for all $x \in S_X$ and $j \in J$, then certainly maps as under (2)(a) exist (take $\gamma_\varepsilon^{\pm} = \gamma^{\pm}$, for all $\varepsilon > 0$), and hence a family of much better behaved global versions exists as under (2)(b), at an arbitrarily small price in the constants.

The possibility to include the ρ_j's in part (2) (with similar occurrences in other results) is a bonus from the refinement of the correspondences to which Michael's Selection Theorem is applied. For several issues, such as our original question concerning $C_0(\Omega, X)$, it will be sufficient to use part (1) and conclude that a continuous bounded decomposition exists. In this paper we also include some applications of part (2) with non-empty J. Corollary 2.4.3 shows that approximate α-conormality of a (pre)-ordered Banach space is equivalent with continuous positively homogeneous approximate α-conormality, and Corollary 2.4.9 shows that approximate α-conormality of X is inherited by various spaces of continuous X-valued functions on a topological space.

We emphasize that, although Banach spaces that are a sum of cones, and ordered Banach spaces in particular, have played a rather prominent role in this introduction, the actual underlying results are those in Section 2.3, valid for a continuous additive positively homogeneous surjection $T : C \to X$ from a closed not necessarily proper cone C in a Banach space onto a Banach space X. That is the heart of the matter.

This paper is organized as follows.

Section 2.2 contains the basis terminology and some preliminary elementary results. The terminology is recalled in detail, in order to avoid a possible misunderstanding due to differing conventions.

In Section 2.3 the Open Mapping Theorem for Banach spaces and Michael's Selection Theorem are used to investigate surjective continuous additive positively homogeneous maps $T : C \to X$.

Section 2.4 contains the applications, rather easily derived from Section 2.3. Banach spaces that are a sum of closed not necessarily proper cones are approached via the naturally associated closed cone in a Banach space direct sum and the summing map. The results thus obtained are then in turn applied to a (pre)-ordered Banach space X and to various spaces of continuous X-valued functions.

2.2 Preliminaries

In this section we establish terminology, include a few elementary results concerning metric cones for later use, and recall Michael's Selection Theorem.

If X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ denotes its unit sphere.

2.2.1 Subsets of vector spaces

For the sake of completeness we recall that a non-empty subset A of a real vector space X is *star-shaped with respect to 0* if $\lambda x \in A$, for all $x \in A$ and $0 \leq \lambda \leq 1$, and that it is *balanced* if $\lambda x \in A$, for all $x \in A$ and $-1 \leq \lambda \leq 1$. A is *absorbing in X* if, for all $x \in X$, there exists $\lambda > 0$ such that $x \in \lambda A$. A is *symmetric* if $A = -A$.

The next rather elementary property will be used in the proof of Proposition 2.3.1.

Lemma 2.2.1. *Let X be a real vector space and suppose $A, B \subseteq X$ are star-shaped with respect to 0 and absorbing. Then $A \cap B$ is star-shaped with respect to 0 and absorbing.*

Proof. It is clear that $A \cap B$ is star-shaped with respect to 0. Let $x \in X$, then, since A is absorbing, $x \in \lambda A$ for some $\lambda > 0$. The fact that A is star-shaped with respect to 0 then implies that $x \in \lambda' A$ for all $\lambda' \geq \lambda$. Likewise, $x \in \mu B$ for some $\mu > 0$, and then $x \in \mu' B$ for all $\mu' \geq \mu$. Hence $x \in \max(\lambda, \mu)(A \cap B)$. □

A subset C of the real or complex vector space X is called a *cone in X* if $C + C \subseteq C$ and $\lambda C \subseteq C$, for all $\lambda \geq 0$. We note that we do not require C to be a proper cone, i.e., that $C \cap (-C) = \{0\}$.

2.2.2 Cones

The cones figuring in the applications in Section 2.4 are cones in Banach spaces, but one of the two main results leading to these applications, the Open Mapping Theorem (Theorem 2.3.2), can be established for the following more abstract objects.

Definition 2.2.2. Let C be a set equipped with operations $+ : C \times C \to C$ and $\cdot : \mathbb{R}_{\geq 0} \times C \to C$. Then C will be called an *abstract cone* if there exists an element $0 \in C$, such that the following hold for all $u, v, w \in C$ and $\lambda, \mu \in \mathbb{R}_{\geq 0}$:

(1) $u + 0 = u$;

(2) $(u + v) + w = u + (w + v)$;

(3) $u + v = v + u$;

(4) $u + v = u + w$ implies $v = w$;

(5) $1u = u$ and $0u = 0$;

(6) $(\lambda\mu)u = \lambda(\mu u)$;

(7) $(\lambda + \mu)u = \lambda u + \mu u$;

(8) $\lambda(u + v) = \lambda u + \lambda v$.

Here we have written $\lambda \cdot u$ as λu for short, as usual.

The natural class of maps between two cones C_1 and C_2 consists of the *additive and positively homogeneous* maps, i.e., the maps $T : C_1 \to C_2$ such that $T(u + v) = Tu + Tv$ and $T(\lambda u) = \lambda u$, for all $u, v \in C$ and $\lambda \geq 0$.

Definition 2.2.3. A pair (C, d) will be called a *metric cone* if C is an abstract cone and $d : C \times C \to \mathbb{R}_{\geq 0}$ is a metric, satisfying

$$d(0, \lambda u) = \lambda d(0, u), \tag{2.2.1}$$
$$d(u + v, u + w) \leq d(v, w), \tag{2.2.2}$$

for every $u, v, w \in C$ and $\lambda \geq 0$. A metric cone (C, d) is a *complete metric cone* if it is a complete metric space.

Remark 2.2.4. (1) Once Michael's Selection Theorem is combined with the Open Mapping Theorem (Theorem 2.3.2), C will be a closed not necessarily proper cone in a Banach space, and the metric will be induced by the norm. In that case it is translation invariant, but for the Open Mapping Theorem as such requiring (2.2.2) is already sufficient. The natural similar requirement $d(0, \lambda u) \leq \lambda d(0, u)$, which is likewise sufficient for the proofs, is easily seen to be actually equivalent to requiring equality as in (2.2.1) above.

(2) Although we will not use this, we note that, if (C, d) is a metric cone, then $+ : C \times C \to C$ is easily seen to be continuous, as is the map $\lambda \to \lambda u$ from $\mathbb{R}_{\geq 0}$ into C, for each $u \in C$.

The following elementary results will be needed in the proof of Proposition 2.3.1.

Lemma 2.2.5. *Let* (C, d) *be a metric cone as in Definition 2.2.2.*

(1) If $c_1, \ldots, c_n \in C$*, then* $d(0, \sum_{i=1}^{n} c_i) \leq \sum_{i=1}^{n} d(0, c_i)$*.*

(2) Let X *be a real or complex normed space and suppose* $T : C \to X$ *is positively homogeneous and continuous at 0. Then* T *maps metrically bounded subsets of* C *to norm bounded subsets of* X

Proof. For the first part, using the triangle inequality and (2.2.2) we conclude that $d(0, \sum_{i=1}^{n} c_i) \leq d(0, c_n) + d(c_n, \sum_{i=1}^{n} c_i) \leq d(0, c_n) + d(0, \sum_{i=1}^{n-1} c_i)$, so the statement follows by induction.

As to the second part, by continuity of T at zero there exists some $\delta > 0$ such that $\|Tc\| < 1$ holds for all $c \in C$ satisfying $d(0, c) < \delta$. If $U \subseteq C$ is bounded, choose $r > 0$ such that $U \subseteq \{c \in C : d(0, c) < r\}$. Since $d(0, \lambda u) = \lambda d(0, u)$, for all $u \in C$ and $\lambda \geq 0$, $\delta r^{-1} U \subseteq \{c \in C : d(0, c) < \delta\}$. Then by positive homogeneity of T, $\sup_{u \in U} \|Tu\| \leq \delta^{-1} r < \infty$. □

2.2.3 Correspondences

Our terminology and definitions concerning correspondences follow that in [1]. Let A, B be sets. A map φ from A into the power set of B is called a *correspondence from* A *into* B, and is denoted by $\varphi : A \twoheadrightarrow B$. A *selector* for a correspondence $\varphi : A \twoheadrightarrow B$ is a function $\sigma : A \to B$ such that $\sigma(x) \in \varphi(x)$ for all $a \in A$. If A and B are topological spaces, we say a correspondence φ is *lower hemicontinuous* if, for every $a \in A$ and every open set $U \subseteq B$ with $\varphi(a) \cap U \neq \emptyset$, there exists an open neighborhood V of a in A such that $\varphi(a') \cap U \neq \emptyset$ for every $a' \in V$. The following result is the key to the proof of Proposition 2.3.4 concerning the existence of continuous sections for surjections of cones onto normed spaces.

Theorem 2.2.6. (*Michael's Selection Theorem [1, Theorem 17.66]*) *Let* $\varphi : A \twoheadrightarrow Y$ *be a correspondence from a paracompact space* A *into a real or complex Fréchet space* Y*. If* φ *is lower hemicontinuous and has non-empty closed convex values, then it admits a continuous selector.*

2.3 Main results

In this section we establish our main results, Theorems 2.3.2, 2.3.6 and 2.3.7. Theorem 2.3.2 is an Open Mapping Theorem for surjections from complete metric cones onto Banach spaces; its proof is based on the usual Open Mapping Theorem. Together with the technical Proposition 2.3.4 (based on Michael's Selection Theorem) it yields the key Proposition 2.3.5. This is then reworked into two more practical results. The first of these, Theorem 2.3.6, guarantees the existence of continuous bounded positively homogeneous right inverses, while the second, Theorem 2.3.7, shows that the existence of a family of possibly ill-behaved local right inverses implies the existence of a family well-behaved global ones.

As before, if X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ is its unit sphere.

We start with the core of the proof of the Open Mapping Theorem, which employs a certain Minkowski functional. The use of such functionals when dealing with cones and Banach spaces goes back to Klee [27] and Andô [3].

Proposition 2.3.1. *Let (C, d) be a complete metric cone as in Definition 2.2.2. Let X be a real Banach space and $T : C \to X$ a continuous additive positively homogeneous surjection. Let $B := \{c \in C : d(0, c) \leq 1\}$ denote the closed unit ball around zero in C, and define $V := T(B) \cap (-T(B))$. Then V is an absorbing convex balanced subset of X and its Minkowski functional $\|\cdot\|_V : X \to \mathbb{R}$, given by $\|x\|_V := \inf\{\lambda > 0 : x \in \lambda V\}$, for $x \in X$, is a norm on X that is equivalent to the original norm on X.*

Proof. It follows from Lemma 2.2.5 and Definition 2.2.3 that $B := \{c \in C : d(0, c) \leq 1\}$ is convex. Hence $T(B)$ is convex and contains zero, since T is additive and positively homogeneous. Since $0 \in T(B)$, its convexity implies that $T(B)$ is star-shaped with respect to 0. Furthermore, $T(B)$ is absorbing, as a consequence of the surjectivity and positive homogeneity of T and (2.2.1). Thus $T(B)$ is star-shaped with respect to 0 and absorbing, and since this implies the same properties for $-T(B)$, Lemma 2.2.1 shows that $V := T(B) \cap (-T(B))$ is star-shaped with respect to 0 and absorbing. As V is clearly symmetric, its star-shape with respect to 0 implies that it is balanced. Furthermore, the convexity of $T(B)$ implies that V is convex. All in all, V is an absorbing convex balanced subset of the real vector space X, and hence its Minkowski functional $\|\cdot\|_V$ is a seminorm by [40, Theorem 1.35]. Because T is continuous at 0, Lemma 2.2.5 implies that $\sup_{y\in V} \|y\| \leq M$ for some $M > 0$. If $x \in X$ and $\lambda > \|x\|_V$, then the definition of $\|\cdot\|_V$ and the star-shape of V with respect to 0 imply that $x \in \lambda V$, so that $\|x\| \leq \lambda M$. Hence

$$\|x\| \leq M\|x\|_V \quad (x \in X). \tag{2.3.1}$$

We conclude that $\|\cdot\|_V$ is a norm on X. In view of (2.3.1), the equivalence of $\|\cdot\|_V$ and $\|\cdot\|$ is an immediate consequence of the Bounded Inverse Theorem for Banach spaces, once we know that $(X, \|\cdot\|_V)$ is complete. We will now proceed to show this, using the completeness of (C, d).

To this end, it suffices to show $\|\cdot\|_V$-convergence of all $\|\cdot\|_V$-absolutely convergent series. Let $\{x_i\}_{i=1}^\infty$ be a sequence in X such that $\sum_{i=1}^\infty \|x_i\|_V < \infty$. Since $\|x\| \leq M\|x\|_V$ for all $x \in X$, $\sum_{i=1}^\infty \|x_i\| < \infty$ also holds, hence by completeness of X the $\|\cdot\|$-sum $x_0 := \sum_{i=1}^\infty x_i$ exists. We claim that $\left\|x_0 - \sum_{i=1}^{N-1} x_i\right\|_V \to 0$ as $N \to \infty$, i.e., that x_0 is also the $\|\cdot\|_V$-sum of this series.

In order to establish this, we start by noting that, for $x \in X$, there exists $x' \in V$ such that $x = 2\|x\|_V x'$. This is clear if $\|x\|_V = 0$. If $\|x\|_V \neq 0$, then $2\|x\|_V > \|x\|_V$ and, as already observed earlier, this implies that $x \in 2\|x\|_V V$. Therefore for every $i \in \mathbb{N}$ there exists $x'_i \in V$ satisfying $x_i = 2\|x_i\|_V x'_i$. Since $V \subset T(B)$, for every $i \in \mathbb{N}$ there exists $b_i \in B$ such that $Tb_i = x'_i$, so that $x_i = 2\|x_i\|_V Tb_i = T(2\|x_i\|_V b_i)$. Note that $d(0, 2\|x_i\|_V b_i) = 2\|x_i\|_V d(0, b_i) \leq 2\|x_i\|_V$.

From (2.2.2) and Lemma 2.2.5 it then follows that, for any fixed $N \in \mathbb{N}$ and all $n, m \in \mathbb{N}$ with $N \leq m \leq n$,

$$\begin{aligned} d\left(\sum_{i=N}^{m} 2\|x_i\|_V b_i, \sum_{i=N}^{n} 2\|x_i\|_V b_i\right) &\leq d\left(0, \sum_{i=m+1}^{n} 2\|x_i\|_V b_i\right) \\ &\leq \sum_{i=m+1}^{n} d(0, 2\|x_i\|_V b_i) \\ &\leq \sum_{i=m+1}^{n} 2\|x_i\|_V. \end{aligned}$$

We conclude that, for any fixed $N \in \mathbb{N}$, $\{\sum_{i=N}^{n} 2\|x_i\|_V b_i\}_{n=N}^{\infty}$ is a Cauchy sequence in (C, d) and hence, by completeness, converges to some $c_N \in C$. Using Lemma 2.2.5 again we find that

$$d(0, c_N) = \lim_{n\to\infty} d\left(0, \sum_{i=N}^{n} 2\|x_i\|_V b_i\right) \leq \limsup_{n\to\infty} \sum_{i=N}^{n} d(0, 2\|x_i\|_V b_i) \leq \sum_{i=N}^{\infty} 2\|x_i\|_V,$$

so that $c_N \in (\sum_{i=N}^{\infty} 2\|x_i\|_V) B$, as a consequence of (2.2.1). By the continuity, additivity and positive homogeneity of T, $Tc_N = \lim_{n\to\infty} T(\sum_{i=N}^{n} 2\|x_i\|_V b_i) = \lim_{n\to\infty} \sum_{i=N}^{n} T(2\|x_i\|_V b_i) = \lim_{n\to\infty} \sum_{i=N}^{n} x_i = \sum_{i=N}^{\infty} x_i$ with respect to the $\|\cdot\|$-topology. Hence

$$x_0 - \sum_{i=1}^{N-1} x_i = \sum_{i=N}^{\infty} x_i = Tc_N \in \left(\sum_{i=N}^{\infty} 2\|x_i\|_V\right) T(B).$$

Similarly, the inclusion $V \subset -T(B)$ implies that, for every $i \in \mathbb{N}$, there exists $\tilde{b}_i \in B$ such that $-T\tilde{b}_i = x_i'$. Then $\tilde{c}_N = \lim_{n\to\infty} \sum_{i=N}^{n} 2\|x_i\|_V \tilde{b}_i$ exists for all $N \in \mathbb{N}$, $\tilde{c}_N \in (\sum_{i=N}^{\infty} 2\|x_i\|_V) B$, and $T\tilde{c}_N = -\sum_{i=N}^{\infty} x_i$, so that

$$x_0 - \sum_{i=1}^{N-1} x_i = \sum_{i=N}^{\infty} x_i = -T\tilde{c}_N \in \left(\sum_{i=N}^{\infty} 2\|x_i\|_V\right) (-T(B)).$$

We conclude that $x_0 - \sum_{i=1}^{N-1} x_i \in (\sum_{i=N}^{\infty} 2\|x_i\|_V) V$. Therefore,

$$\left\| x_0 - \sum_{i=1}^{N-1} x_i \right\|_V \leq \sum_{i=N}^{\infty} 2\|x_i\|_V \to 0$$

as $N \to \infty$, and hence $(X, \|\cdot\|_V)$ is complete. □

The Open Mapping Theorem is now an easy consequence.

Theorem 2.3.2. (*Open Mapping Theorem*) *Let (C, d) be a complete metric cone as in Definition 2.2.2; for example, C could be a closed not necessarily proper cone in a Banach space. Let X be a real or complex Banach space and $T : C \to X$ a continuous additive positively homogeneous map. Then the following are equivalent:*

(1) T is surjective;

(2) There exists some constant $K > 0$ such that, for every $x \in X$, there exists some $c \in C$ with $x = Tc$ and $d(0, c) \leq K\|x\|$;

(3) T is an open map;

(4) 0 is an interior point of $T(C)$.

Proof. Given the nature of the statements in (1)-(4), we may assume that X is a real Banach space, by viewing a complex one as such if necessary. We first prove that (1) implies (2). By Proposition 2.3.1, there exists a constant $L > 0$ such that $\|x\|_V \leq L\|x\|$, for all $x \in X$. If $\|x\| \neq 0$, then $2L\|x\| > L\|x\| \geq \|x\|_V$. Hence $x \in 2L\|x\|V$, which is also trivially true if $x = 0$. In particular, for all $x \in X$ there exists some $c \in B$ such that $x = 2L\|x\|T(c)$. Then $x = T(2L\|x\|c)$, and $d(0, 2L\|x\|c) = 2L\|x\|d(0, c) \leq 2L\|x\|$.

Next, we prove that (2) implies (3). Let $U \subseteq C$ be an open set, and let $x \in T(U)$ be arbitrary with $b \in U$ satisfying $Tb = x$. Since U is open, there exists some $r > 0$ such that $W := \{c \in C : d(b, c) < r\}$ is contained in U. We define $W' := \{c \in C : d(0, c) < r\}$. Then $b + W' \subseteq W$, since $d(b, b + w') \leq d(0, w') < r$ for all $w' \in W'$. Now, by hypothesis, for every $x' \in X$ with $\|x'\| < rK^{-1}$ there exists some $w' \in W'$ with $Tw' = x'$. With $B_X := \{x \in X : \|x\| < 1\}$, by additivity of T, it follows that $x + rK^{-1}B_X \subseteq T(b + W') \subseteq T(W) \subseteq T(U)$. We conclude that $T(U)$ is open.

That (3) implies (4) is trivial, and (4) implies (1) by the positive homogeneity of T. □

Remark 2.3.3. (1) Since a real or complex Banach space is a complete metric cone, Theorem 2.3.2 generalizes the Open Mapping Theorem for Banach spaces that was used in the proof of the preparatory Proposition 2.3.1.

(2) If C is a closed cone in a Banach space, X is a topological vector space, and $T : C \to X$ is continuous, additive and positively homogeneous, then we can conclude that T is an open map, provided that we know beforehand that the closure of $\{Tc : c \in C, \|c\| \leq 1\}$ is a neighborhood of 0 in X. This follows from [36, Theorem 1] . Since we do not have such a hypothesis, this result does not imply ours. The difference is not only that in our case T is assumed to be surjective, but, more fundamentally, that our image space is in fact a Banach space, for which an Open Mapping Theorem is already known to hold that serves as a stepping stone for the more general result.

(3) In [43] an Open Mapping Theorem is established for maps between two abstract cones in a certain class, provided that we know beforehand that these maps satisfy a so-called almost-openess condition. Since we do not have such a hypothesis, this result does not imply ours. Again the difference lies in the image space: in our context this is not just a cone, but actually a full Banach space with accompanying Open Mapping Theorem.

We will now proceed with the second basic result, Proposition 2.3.4, which is concerned with families of continuous right inverses for a surjective (this follows from the hypotheses) map.

Proposition 2.3.4. *Let X be a real or complex normed space and let Y be a real or complex topological vector space, not necessarily Hausdorff and not necessarily over the same field as X, with $C \subseteq Y$ a closed not necessarily proper cone. Let I be a finite set, possibly empty. For each $i \in I$, let $\alpha_i \in \mathbb{R}$ and let $\rho_i : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map.*

Suppose that $T : C \to X$ is a continuous additive positively homogeneous map with the property that, for every $\varepsilon > 0$, there exists a map $\sigma_\varepsilon : S_X \to C$, such that:

(1) $T \circ \sigma_\varepsilon = id_{S_X}$;

(2) $\rho_i(\sigma_\varepsilon(x)) \leq \alpha_i + \varepsilon$, for all $x \in S_X$ and $i \in I$;

(3) $\sigma_\varepsilon(S_X)$ is bounded in Y.

Then, for every $\varepsilon > 0$, the correspondence $\varphi_\varepsilon : S_X \twoheadrightarrow Y$, defined by

$$\varphi_\varepsilon(x) := \{y \in C : Ty = x,\ \rho_i(y) \leq \alpha_i + \varepsilon \text{ for all } i \in I\} \quad (x \in S_X),$$

has non-empty closed convex values, and is lower hemicontinuous on S_X.

If Y is a Fréchet space, there exist continuous maps $\sigma'_\varepsilon : S_X \to C$, for all $\varepsilon > 0$, satisfying:

(a) $T \circ \sigma'_\varepsilon = id_{S_X}$;

(b) $\rho_i(\sigma'_\varepsilon(x)) \leq \alpha_i + \varepsilon$, for all $x \in S_X$ and $i \in I$.

If $\varepsilon > 0$ and $\sigma'_\varepsilon(S_X)$ is bounded in Y in the sense of topological vector spaces, then σ'_ε can be extended to a continuous positively homogeneous map $\sigma'_\varepsilon : X \to C$ on the whole space, satisfying:

(a) $T \circ \sigma'_\varepsilon = id_X$;

(b) $\rho_i(\sigma'_\varepsilon(x)) \leq (\alpha_i + \varepsilon)\|x\|$, for all $x \in X$ and $i \in I$.

Before embarking on the proof, let us point out that the salient point lies in the fact that the right inverses σ'_ε of T on the unit sphere of X are continuous, whereas this is not required for the original family of the σ_ε's, and that this extra property can be achieved retaining the relevant inequalities. It is for this that Michael's Selection Theorem is used. The subsequent conditional extension of such a σ'_ε to the whole space is rather trivial.

Furthermore, we note that, although in the applications we have in mind the constants α_i will be positive and each ρ_i will be the restriction to C of a continuous seminorm or (if Y is a Banach space) a continuous real-linear functional on the whole space Y, the present proof does not require this.

Proof. Let $\varepsilon > 0$ be arbitrary. Since $\sigma_\varepsilon(x) \in \varphi_\varepsilon(x)$, for all $x \in S_X$, φ_ε is non-empty-valued. By continuity of T and the ρ_i's, φ_ε is closed-valued. Since T is affine on the convex set C, and each ρ_i, if any, is subadditive and positively homogeneous, φ_ε is convex-valued.

We will now show that φ_ε is lower hemicontinuous, for any fixed $\varepsilon > 0$. To this end, let $x \in S_X$ be arbitrary, and let $U \subseteq Y$ be open such that $\varphi_\varepsilon(x) \cap U \neq \emptyset$.

We start by establishing that there exists some $y \in \varphi_\varepsilon(x) \cap U$ such that $\rho_i(y) < \alpha_i + \varepsilon$, for all $i \in I$ (if any), where the inequality that is valid for $\sigma_\varepsilon(x)$ has been improved to strict inequality for y. As to this, choose y' in the non-empty set $\varphi_\varepsilon(x) \cap U$, and define $y_t := t\sigma_{\varepsilon/2}(x) + (1-t)y'$, for $t \in [0,1]$. Then $y_t \in C$ and $Ty_t = x$, for all $t \in [0,1]$. Now, for all $t \in (0,1]$ and all $i \in I$,

$$\begin{aligned}
\rho_i(y_t) &\leq t\rho_i(\sigma_{\varepsilon/2}(x)) + (1-t)\rho_i(y') \\
&\leq t\left(\alpha_i + \frac{\varepsilon}{2}\right) + (1-t)\left(\alpha_i + \varepsilon\right) \\
&< t\left(\alpha_i + \varepsilon\right) + (1-t)\left(\alpha_i + \varepsilon\right) \\
&= \left(\alpha_i + \varepsilon\right).
\end{aligned}$$

Since U is open, there exists some $t_0 > 0$ such that $y := y_{t_0} \in U$. Then y is as required.

Having found and fixed this y, we define $\eta := \min_{i\in I}\{\alpha_i + \varepsilon - \rho_i(y)\} > 0$ if $I \neq \emptyset$, and $\eta := 1$ if $I = \emptyset$; here we use that I is finite.

Next, let $W \subseteq Y$ be an open neighborhood of zero such that $y + W \subseteq U$. Since $\sigma_{\eta/2}(S_X)$ is bounded by assumption, we can fix some $0 < r \leq 1$ such that $r'\sigma_{\eta/2}(S_X) \subseteq W$, for all $0 \leq r' < r$, and $\alpha_i r' < \eta/2$, for all $0 \leq r' < r$ and all $i \in I$.

Now, if $x' \in X$ satisfies $0 < \|x'\| < r$, then

$$y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right) \in C,$$

$$T\left(y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) = x + x',$$

and

$$y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right) \in y + \|x'\|\sigma_{\eta/2}(S_X) \subset y + W \subset U.$$

Furthermore, for such x' and for $i \in I$ we find, using $\alpha_i\|x'\| < \eta/2$ and $\|x'\| < r \leq 1$, that

$$\begin{aligned}
\rho_i\left(y + \|x'\|\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) &\leq \rho_i(y) + \|x'\|\rho_i\left(\sigma_{\eta/2}\left(\frac{x'}{\|x'\|}\right)\right) \\
&\leq \rho_i(y) + \|x'\|\left(\alpha_i + \frac{\eta}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&< \rho_i(y) + \frac{\eta}{2} + \frac{\eta}{2} \\
&= \rho_i(y) + \eta \\
&\leq \rho_i(y) + \alpha_i + \varepsilon - \rho_i(y) \\
&= \alpha_i + \varepsilon.
\end{aligned}
$$

Therefore, if $x + x' \in S_X$ with $0 < \|x'\| < r$, we conclude that

$$y + \|x'\|\sigma_{\eta/2}(x'/\|x'\|) \in \varphi_\varepsilon(x + x') \cap U.$$

Hence φ_ε is lower hemicontinuous on S_X, as was to be proved.

If Y is a Fréchet space then, since S_X as a metric space is paracompact [42], Michael's Selection Theorem (Theorem 2.2.6), applied to each individual φ_ε, supplies a family of continuous maps $\sigma'_\varepsilon : S_X \to C$, such that $\sigma_\varepsilon(x) \in \varphi_\varepsilon(x)$, for all $\varepsilon > 0$ and all $x \in S_X$. Then the σ'_ε are as required.

If $\varepsilon > 0$ and $\sigma'_\varepsilon(S_X)$ happens to be bounded in the topological vector space Y, we extend $\sigma'_\varepsilon : S_X \to C$ to a positively homogeneous C-valued map on all of X, also denoted by σ'_ε, by defining

$$\sigma'_\varepsilon(x) := \begin{cases} 0 & \text{for } x = 0; \\ \|x\|\sigma'_\varepsilon\left(\frac{x}{\|x\|}\right) & \text{for } x \neq 0. \end{cases}$$

The continuity of σ'_ε at 0 then follows from the boundedness of $\sigma'_\varepsilon(S_X)$, and at other points it is immediate. It is easily verified that such a global σ'_ε has the properties as claimed. □

Combination of Theorem 2.3.2 and Proposition 2.3.4 yields the following key result on right inverses of surjections from cones onto Banach spaces. The structure of the proofs makes it clear that it is ultimately based on the Open Mapping Theorem for Banach spaces and Michael's Selection Theorem.

Proposition 2.3.5. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Furthermore, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map. For example, each ρ_j could be the restriction to C of a globally defined continuous seminorm or continuous real-linear functional.

(1) If ρ_j is bounded from above on $S_Y \cap C$, for all $j \in J$, then there exist constants $K > 0$, $\alpha_j \in \mathbb{R}$ $(j \in J)$, and a map $\gamma : S_X \to C$, such that:

(a) $T \circ \gamma = id_{S_X}$;

(b) $\|\gamma(x)\| \leq K$, for all $x \in S_X$;

(c) $\rho_j(\gamma(x)) \leq \alpha_j$, for all $x \in S_X$ and $j \in J$.

(2) *If* $K > 0$, $\alpha_j \in \mathbb{R}$ $(j \in J)$, *and* $\gamma : S_X \to C$ *satisfy* (a), (b) *and* (c) *in part* (1), *then, for every* $\varepsilon > 0$, *there exists a continuous positively homogeneous map* $\gamma_\varepsilon : X \to C$ *such that:*

(a) $T \circ \gamma_\varepsilon = id_X$;

(b) $\|\gamma_\varepsilon(x)\| \leq (K + \varepsilon)\|x\|$, *for all* $x \in X$;

(c) $\rho_j(\gamma(x)) \leq (\alpha_j + \varepsilon)\|x\|$, *for all* $x \in X$ *and* $j \in J$.

Proof. As to the first part, we start by applying part (2) of Theorem 2.3.2 and obtain $K > 0$ and a map $\gamma : S_X \to C$, such that $T \circ \gamma = \mathrm{id}_{S_X}$ and $\|\gamma(x)\| \leq K$ $(x \in S_X)$.

If $j \in J$, and $\beta_j \in \mathbb{R}$ is such that $\rho_j(c) \leq \beta_j$ for all $c \in S_Y \cap C$, where we may assume that $\beta_j \geq 0$, then $\rho_j(\gamma(x)) \leq K\beta_j$, for all $x \in S_X$. Indeed, this is obvious if $\gamma(x)$=0, and if $\gamma(x) \neq 0$ we have

$$\rho_j\left(\gamma(x)\right) = \|\gamma(x)\|\rho_j\left(\frac{\gamma(x)}{\|\gamma(x)\|}\right) \leq K\beta_j.$$

The existence of the $\alpha_j := K\beta_j$ is then clear.

As to the second part, suppose that $K > 0$, $\alpha_j \in \mathbb{R}$ $(j \in J)$ and $\gamma : S_X \to C$ satisfy (a), (b) and (c) in part (1). We augment J to $I := J \cup \{\| \, \|\}$, where we choose an index symbol $\| \, \| \notin J$, and let $\rho_{\| \, \|}(c) := \|c\|$, for $c \in C$, and put $\alpha_{\| \, \|} := K$. We can now apply Proposition 2.3.4 with $\sigma_\varepsilon = \gamma$ for all $\varepsilon > 0$, since its hypotheses (1), (2) and (3) are then satisfied. The continuous $\sigma'_\varepsilon : S_X \to C$ as furnished by Proposition 2.3.4 are, in particular, such that $\rho_{\| \, \|}(\sigma'_\varepsilon(x)) \leq \alpha_{\| \, \|} + \varepsilon$, i.e., such that $\|\sigma'_\varepsilon(x)\| \leq K + \varepsilon$, for all $x \in S_X$. Hence each of the sets $\sigma'_\varepsilon(S_X)$ is bounded in Y, and the last part of Proposition 2.3.4 applies, yielding global $\sigma'_\varepsilon : X \to C$ that can be taken as the required γ_ε. □

Let us remark explicitly that the α_j's need not be non-negative and that in part (2) the ρ_j's are not required to be bounded from above on $S_Y \cap C$ as in part (1), but rather on $\gamma(S_X)$ (as a consequence of the hypothesized validity of (1)(c)), which is a weaker hypothesis.

We extract two practical consequences from Proposition 2.3.5. First of all, if the ρ_j's are bounded from above on $S_Y \cap C$ then part (1) of Proposition 2.3.5 is applicable and its conclusion shows that the hypothesis of part (2) are satisfied. Taking $\varepsilon = 1$, say, we therefore have the following.

Theorem 2.3.6. *Let* X *and* Y *be real or complex Banach spaces, not necessarily over the same field, and let* C *be a closed not necessarily proper cone in* Y. *Let* $T : C \to X$ *be a surjective continuous additive positively homogeneous map.*

Furthermore, let J *be a finite set, possibly empty, and, for all* $j \in J$, *let* $\rho_j : C \to \mathbb{R}$ *be a continuous subadditive positively homogeneous map that is bounded from above on* $S_Y \cap C$. *For example, each* ρ_j *could be the restriction to* C *of a globally defined continuous seminorm or continuous real-linear functional.*

Then there exist constants $K > 0$ and $\alpha_j \in \mathbb{R}$ $(j \in J)$ and a continuous positively homogeneous map $\gamma : X \to C$, such that:

(1) $T \circ \gamma = id_X$;

(2) $\|\gamma(x)\| \leq K\|x\|$, *for all* $x \in X$;

(3) $\rho_j(\gamma(x)) \leq \alpha_j \|x\|$, *for all* $x \in X$ *and* $j \in J$.

The next consequence of Proposition 2.3.5 states that the existence of a family of possibly ill-behaved right inverses on the unit sphere is actually equivalent with the existence of a family of well-behaved global ones. Note that, compared with Theorem 2.3.6, the boundedness assumption from above for the ρ_j's on $S_Y \cap C$ has been replaced with the assumptions (1)(c) and (2)(c) below.

Theorem 2.3.7. *Let X and Y be real or complex Banach spaces, not necessarily over the same field, and let C be a closed not necessarily proper cone in Y. Let $T : C \to X$ be a surjective continuous additive positively homogeneous map.*

Furthermore, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : C \to \mathbb{R}$ be a continuous subadditive positively homogeneous map. For example, each ρ_j could be the restriction to C of a globally defined continuous seminorm or continuous real-linear functional.

If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(1) For every $\varepsilon > 0$, there exists a map $\gamma_\varepsilon : S_X \to C$, such that:

  (a) $T \circ \gamma_\varepsilon = id_{S_X}$;
  (b) $\|\gamma_\varepsilon(x)\| \leq K + \varepsilon$, *for all* $x \in S_X$;
  (c) $\rho_j(\gamma_\varepsilon(x)) \leq \alpha_j + \varepsilon$, *for all* $x \in S_X$ *and* $j \in J$.

(2) For every $\varepsilon > 0$, there exists a continuous positively homogeneous map $\gamma_\varepsilon : X \to C$ such that:

  (a) $T \circ \gamma_\varepsilon = id_X$;
  (b) $\|\gamma_\varepsilon(x)\| \leq (K + \varepsilon)\|x\|$, *for all* $x \in X$;
  (c) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)\|x\|$, *for all* $x \in X$ *and* $j \in J$.

Proof. Clearly the second part implies the first. For the converse implication, let $\varepsilon > 0$ be given. Then, by assumption, there exists a map (we add accents to avoid notational confusion) $\gamma'_{\varepsilon/2} : S_X \to C$, such that:

(1) $T \circ \gamma'_{\varepsilon/2} = \mathrm{id}_{S_X}$;

(2) $\|\gamma'_{\varepsilon/2}(x)\| \leq K + \varepsilon/2$, for all $x \in S_X$;

(3) $\rho_j(\gamma'_{\varepsilon/2}(x)) \leq \alpha_j + \varepsilon/2$, for all $x \in S_X$ and $j \in J$.

We can now apply part (2) of Proposition 2.3.5, with K replaced with $K + \varepsilon/2$, α_j with $\alpha_j + \varepsilon/2$, γ with $\gamma'_{\varepsilon/2}$, and ε with $\varepsilon/2$. The map $\gamma_{\varepsilon/2}$ as furnished by part (2) of Proposition 2.3.5 can then be taken as the map γ_ε in part (2) of the present Theorem. □

2.4 Applications

By varying C and the ρ_j's various types of consequences of Theorems 2.3.6 and 2.3.7 can be obtained, and we collect some in the present section, considering situations where the ρ_j's are restrictions to C of globally defined continuous seminorms or continuous real-linear functionals. This seems to be a natural context to work in, but we note that it is not required as such by these two underlying Theorems, nor by the key Proposition 2.3.5, so that applications of another type are conceivable.

As in earlier sections, if X is a normed space, then $S_X := \{x \in X : \|x\| = 1\}$ is its unit sphere.

To start with, Theorems 2.3.6 and 2.3.7 are clearly applicable when $T : C \to X$ is the restriction to C of a global continuous linear map $T : Y \to X$ and (as already mentioned in these Theorems) each of the ρ_j's is the restriction of a globally defined continuous seminorm or continuous real-linear functional. Furthermore, Y is a closed cone in itself, so that these Theorems can be specialized to yield statements on well-behaved right inverses for continuous linear surjections between Banach spaces. For reasons of space, we refrain from explicitly formulating all these quite obvious special cases.

Instead, we give applications to the internal structure of a Banach space that is a sum of closed not necessarily proper cones, and to the structure of spaces of continuous functions with values in such a Banach space. Thus we return to the to the improvements of Andô's Theorem and our original motivating question alluded to in the introduction.

The following result applies, in particular, when $X = \sum_{i=1}^{n} C_i$ is the sum of a finite number of closed not necessarily proper cones. In that case, the Banach space Y in the following Theorem is the direct sum of n copies of X.

Theorem 2.4.1. *Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X, such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i\in I} c_i$, where $c_i \in C_i$, for all $i \in I$.*

Let $Y = \ell^1(I, X)$ be the ℓ^1-direct sum of $|I|$ copies of X, and let C be the natural closed cone in the Banach space Y, consisting of those elements where the i-th component is in C_i. Finally, let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : Y \to \mathbb{R}$ be a continuous seminorm or a continuous real-linear functional.

Then:

(1) There exist a constant $K > 0$ and a continuous positively homogeneous map $\gamma : X \to C$ with continuous positively homogeneous component maps $\gamma_i : X \to C_i$ $(i \in I)$, such that:

(a) $x = \sum_{i\in I} \gamma_i(x)$, for all $x \in X$;

(b) $\sum_{i\in I} \|\gamma_i(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}\,(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exists a map $\gamma_\varepsilon : S_X \to C$ with component maps $\gamma_{\varepsilon,i} : S_X \to C_i\,(i \in I)$, such that:

(i) $x = \sum_{i\in I} \gamma_{\varepsilon,i}(x)$, for all $x \in S_X$;
(ii) $\sum_{i\in I} \|\gamma_{\varepsilon,i}(x)\| \leq (K+\varepsilon)$, for all $x \in S_X$;
(iii) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) For every $\varepsilon > 0$, there exists a continuous positively homogeneous map $\gamma_\varepsilon : X \to C$ with continuous positively homogeneous component maps $\gamma_{\varepsilon,i} : X \to C_i\,(i \in I)$, such that:

(i) $x = \sum_{i\in I} \gamma_{\varepsilon,i}(x)$, for all $x \in X$;
(ii) $\sum_{i\in I} \|\gamma_{\varepsilon,i}(x)\| \leq (K+\varepsilon)\|x\|$, for all $x \in X$;
(iii) $\rho_j(\gamma_\varepsilon(x)) \leq (\alpha_j + \varepsilon)\|x\|$, for all $x \in X$ and $j \in J$.

Proof. Let $T : C \to X$ be the canonical summing map. Then Theorem 2.3.6 yields part (1), and Theorem 2.3.7 yields part (2). □

In order to illustrate Theorem 2.4.1 we consider the situation where X is a Banach space, (pre)-ordered by a closed not necessarily proper cone X^+. If X^+ is generating in the sense of (pre)-ordered Banach spaces, i.e., if $X = X^+ - X^+$, and if X^+ is proper, then Andô's Theorem (Theorem 2.1.1) applies. On the other hand, Theorem 2.4.1, also yields this result (and an even stronger one) by writing $X = X^+ + (-X^+)$ as the sum of two closed cones, coincidentally related by a minus sign. For convenience we formulate the result explicitly in the usual notation with minus signs.

Corollary 2.4.2. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+. Let J be a finite set, possibly empty, and, for all $j \in J$, let $\rho_j : X \times X \to \mathbb{R}$ be a continuous seminorm or a continuous linear functional. Then:*

(1) There exist a constant $K > 0$ and continuous positively homogeneous maps $\gamma^\pm : X \to X^+$, such that:

(a) $x = \gamma^+(x) - \gamma^-(x)$, for all $x \in X$;
(b) $\|\gamma^+(x)\| + \|\gamma^-(x)\| \leq K\|x\|$, for all $x \in X$.

(2) If $K > 0$ and $\alpha_j \in \mathbb{R}$ $(j \in J)$ are constants, then the following are equivalent:

(a) For every $\varepsilon > 0$, there exist maps $\gamma_\varepsilon^\pm : S_X \to X^+$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in S_X$;
(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K+\varepsilon)$, for all $x \in S_X$;
(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)$, for all $x \in S_X$ and $j \in J$.

(b) For every $\varepsilon > 0$, there exist continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that:

(i) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in X$;

(ii) $\|\gamma_\varepsilon^+(x)\| + \|\gamma_\varepsilon^-(x)\| \leq (K + \varepsilon)\|x\|$, for all $x \in X$;

(iii) $\rho_j((\gamma_\varepsilon^+(x), \gamma_\varepsilon^-(x)) \leq (\alpha_j + \varepsilon)\|x\|$, for all X and $j \in J$.

Proof. We apply Theorem 2.4.1 with $I = \{1, 2\}$, $C_1 = X^+$ and $C_2 := -X^+$, and then let $\gamma^+ = \gamma_1$ and $\gamma^- = -\gamma_2$ in part (1), and $\gamma_\varepsilon^+ = \gamma_{\varepsilon,1}$ and $\gamma_\varepsilon^- = -\gamma_{\varepsilon,2}$ in part (2) □

We continue in the context of a real (pre)-ordered normed space X ordered by a closed generating not necessarily proper cone X^+. If $\alpha > 0$, then we will say that X is

(1) *α-conormal* if, for each $x \in X$, there exist $x^\pm \in X^+$, such that $x = x^+ - x^-$ and $\|x^+\| \leq \alpha\|x\|$;

(2) *approximately α-conormal* if X is $(\alpha + \varepsilon)$-conormal, for all $\varepsilon > 0$.

Andô's Theorem is equivalent to asserting that every real Banach space, ordered by a closed generating proper cone, is α-conormal for some $\alpha > 0$. Clearly, α-conormality implies approximate α-conormality. What is less obvious is that approximate α-conormality is equivalent with a continuous positively homogeneous version of the same notion, as is the content of the following consequence of Corollary 2.4.2.

Corollary 2.4.3. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+, and let $\alpha > 0$. Then the following are equivalent:*

(1) X is approximately α-conormal;

(2) For every $\varepsilon > 0$, there exist continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that:

(a) $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$, for all $x \in X$;

(b) $\|\gamma_\varepsilon^+(x)\| \leq (\alpha + \varepsilon)\|x\|$, for all $x \in X$.

Proof. Clearly part (2) implies part (1). For the converse we will apply Corollary 2.4.2 with $J = \{1\}$, as follows. Let $\varepsilon > 0$ be given and fixed. For each $x \in S_X$, the $(\alpha + \varepsilon/2)$-conormality of X implies that, for each $x \in S_X$, we can choose and fix $\gamma_{\varepsilon/2}^\pm(x) \in X^+$, such that

$$x = \gamma_{\varepsilon/2}^+(x) - \gamma_{\varepsilon/2}^-(x) \ (x \in S_X),$$

and $\|\gamma_{\varepsilon/2}^+(x)\| \leq (\alpha + \varepsilon/2)$. Then $\|\gamma_{\varepsilon/2}^-(x)\| \leq (\alpha + \varepsilon/2 + 1)$, so that

$$\|\gamma_{\varepsilon/2}^+(x)\| + \|\gamma_{\varepsilon/2}^-(x)\| \leq (2\alpha + 1 + \varepsilon) \ (x \in S_X).$$

Define $\rho_1 : X \times X \to \mathbb{R}$ by $\rho_1((x_1, x_2)) := \|x_1\|$, for $x_1, x_2 \in X$, so that

$$\rho_1((\gamma^+_{\varepsilon/2}(x), \gamma^-_{\varepsilon/2}(x)) = \|\gamma^+_{\varepsilon/2}(x)\| \leq \alpha + \varepsilon/2 \leq \alpha + \varepsilon \ (x \in S_X).$$

Thus we have found constants $K = 2\alpha+1 > 0$, $\alpha_1 = \alpha$ and maps $\gamma^\pm_{\varepsilon/2} : S_X \to X_+$ satisfying (i), (ii) and (iii) in part (2)(a) of Corollary 2.4.2. Hence the continuous positively homogeneous maps as in part (2)(b) of Corollary 2.4.2 also exist, and these are as required. □

Remark 2.4.4. The term "conormality" is due to Walsh [44] and several variations of it have been studied. For example, a real normed space X is said to be *α-max-conormal* if, for each $x \in X$, there exist $x^\pm \in X^+$, such that $x = x^+ - x^-$ and $\max(\|x^+\|, \|x^-\|) \leq \alpha\|x\|$; X is *approximately α-max-conormal* if it is $(\alpha + \varepsilon)$-max-conormal, for every $\varepsilon > 0$. As another example, X is said to be *α-sum-conormal* if, for each $x \in X$, there exist $x^\pm \in X^+$, such that $x = x^+ - x^-$ and $\|x^+\| + \|x^-\| \leq \alpha\|x\|$; X is *approximately α-sum-conormal* if it is $(\alpha + \varepsilon)$-sum-conormal, for every $\varepsilon > 0$. Just as Corollary 2.4.3 shows that approximately α-conormality implies its continuous positively homogenous version, the elements $x^\pm$ figuring in the definitions of approximately α-max-conormality and approximately α-sum-conormality can be chosen in a continuous and positively homogeneous fashion. The proof is analogous to the proof of Corollary 2.4.3, but now taking $\rho_1(x_1, x_2) := \max(\|x_1\|, \|x_2\|)$ for approximately α-sum-conormality, and $\rho_1(x_1, x_2) := \|x_1\| + \|x_2\|$ for approximately α-sum-conormality.

The dual notion of conormality is normality (terminology due to Krein [28]). Several equivalences between versions of normality of an ordered Banach space X and conormality of its dual (and vice versa) are known, but are scattered throughout the literature under various names. The most complete account of normality-conormality duality relationships may be found in [6].

Finally, we return to our original motivating context in the introduction, but in a more general setting. As a rule, no additional hypotheses on the topological space Ω are necessary to pass from X to a space of X-valued functions, since the arguments are pointwise in X, but for some converse implications it is required that the vector valued function space in question is non-zero. If $C_c(\Omega) \neq \{0\}$, for example if Ω is a non-empty locally compact Hausdorff space, then this assumption is always satisfied.

Theorem 2.4.5. *Let X be a real or complex Banach space. Let I be a non-empty set, possibly uncountable, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X, such that every $x \in X$ can be written as an absolutely convergent series $x = \sum_{i\in I} c_i$, where $c_i \in C_i$, for all $i \in I$. Let Ω be a topological space. Then there exists a constant $K > 0$ with the property that, for each X-valued continuous function $f \in C(\Omega, X)$ on Ω, there exist $f_i \in C(\Omega, C_i)$ $(i \in I)$, such that*

(1) For every $\omega \in \Omega$, $f(x) = \sum_{i\in I} f_i(\omega)$, and $\sum_{i\in I} \|f_i(\omega)\| \leq K\|f(\omega)\|$;

(2) $\|f_i\|_\infty \leq K\|f\|_\infty$, for all $i \in I$, where the right hand side, or both the left hand side and the right hand side, may be infinite;

(3) The support of each f_i is contained in that of f;

(4) If f vanishes at infinity, then so does each f_i;

(5) If $\omega_1, \omega_2 \in \Omega$ and $\lambda_1, \lambda_2 \geq 0$ are such that $\lambda_1 f(\omega_1) = \lambda_2 f(\omega_2)$, then $\lambda_1 f_i(\omega_1) = \lambda_2 f_i(\omega_2)$, for all $i \in I$.

In particular, if I is finite, so that $X = \sum_{i \in I} C_i$, then we can write the following vector spaces as the sum of cones naturally associated with the C_i, where the cones are closed in the last three normed spaces:

(1) $C(\Omega, X) = \sum_{i \in I} C(\Omega, C_i)$ for the continuous X-valued functions on Ω;

(2) $C_b(\Omega, X) = \sum_{i \in I} C_b(\Omega, C_i)$ for the bounded continuous X-valued functions on Ω;

(3) $C_0(\Omega, X) = \sum_{i \in I} C_0(\Omega, C_i)$ for the continuous X-valued functions on Ω vanishing at infinity;

(4) $C_c(\Omega, X) = \sum_{i \in I} C_c(\Omega, C_i)$ for the compactly supported continuous X-valued functions on Ω.

Proof. We apply part (1) of Theorem 2.4.1 and let $f_i := \gamma_i \circ f \ (i \in I)$. This supplies the f_i as required for the first part, and the statement on the finite number of naturally associated cones is then clear. □

Clearly then, the answer to our original question in the introduction is affirmative: If X is a Banach space with a closed generating proper cone X^+, and Ω is a topological space, then $C_0(\Omega, X^+)$ is generating in $C_0(\Omega, X)$. In fact, Theorem 2.4.5 shows that X^+ need not even be proper.

Remark 2.4.6. As mentioned in the introduction, if Ω is a (locally) compact Hausdorff space and X is a (pre-)ordered Banach space with closed generating cone X^+, certain special cases of Theorem 2.4.5 also follow from [4, Theorem 2.3] and [45, Theorem 4.4]. Both of these results proceed through an application of Lazar's affine selection theorem to show that cones of continuous affine X^+-valued functions on a Choquet simplex K are generating in spaces of continuous affine X^+-valued functions on K. If Ω is a compact Hausdorff space, the fact that $C(\Omega, X^+)$ is generating in $C(\Omega, X)$ follows from [4, Theorem 2.3] by taking K to be the Choquet simplex of all regular Borel probability measures on Ω, and considering the maps $\mu \mapsto \int_\Omega f d\mu \ (\mu \in K,\ f \in C(\Omega, X))$ and $\omega \mapsto a(\delta_\omega) \ (\omega \in \Omega,\ a \in A(K, X))$, where $A(K, X)$ denotes the space of continuous affine X-valued functions on K.

The converse of the four last statements in Theorem 2.4.5 also holds provided the function spaces are non-zero, as is shown by our next (elementary) result. Note that $C(\Omega, X)$ and $C_b(\Omega, X)$ are zero only when $\Omega \neq \emptyset$ and $X = \{0\}$.

Lemma 2.4.7. *Let X be a real or complex normed space. Let I be a finite non-empty set, and let $\{C_i\}_{i \in I}$ be a collection of cones in X, not necessarily closed or proper. Let Ω be a topological space.*

If $C(\Omega, X) \neq \{0\}$ and $C(\Omega, X) = \sum_{i\in I} C(\Omega, C_i)$, then $X = \sum_{i\in I} C_i$; similar statements hold for $C_b(\Omega, X)$, $C_0(\Omega, X)$ and $C_c(\Omega, X)$.

Proof. If there exists $0 \neq f \in C(\Omega, X)$, then composing f with a suitable continuous linear functional on X yields a non-zero $\varphi \in C(\Omega)$. Choose $\omega_0 \in \Omega$ such that $\varphi(\omega_0) \neq 0$; we may assume that $\varphi(\omega_0) = 1$. If $x \in X$, then (employing the usual notation) $\varphi \otimes x = \sum_{i\in I} f_i$, for some $f_i \in C(\Omega, C_i)$ $(i \in I)$ by assumption. Specializing this to the point ω_0 shows that $x = \sum_{i\in I} c_i$, for some $c_i \in C_i$ $(i \in I)$.

The proofs for $C_b(\Omega, X)$, $C_0(\Omega, X)$ and $C_c(\Omega, X)$ are similar. □

Corollary 2.4.8. *Let X be a real or complex Banach space. Let I be a non-empty finite set, and let $\{C_i\}_{i\in I}$ be a collection of closed not necessarily proper cones in X. Let Ω be a topological space. If $C_c(\Omega) \neq \{0\}$, for example if Ω is a non-empty locally compact Hausdorff space, then the following are equivalent:*

(1) $X = \sum_{i\in I} C_i$;

(2) $C(\Omega, X) = \sum_{i\in I} C(\Omega, C_i)$;

(3) $C_b(\Omega, X) = \sum_{i\in I} C_b(\Omega, C_i)$;

(4) $C_0(\Omega, X) = \sum_{i\in I} C_0(\Omega, C_i)$;

(5) $C_c(\Omega, X) = \sum_{i\in I} C_c(\Omega, C_i)$.

Proof. If $X = \{0\}$ there is nothing to prove. If $X \neq \{0\}$, then the fact that $C_c(\Omega) \neq \{0\}$ implies that $C_c(\Omega, X) \neq \{0\}$, hence that the other three spaces of X-valued functions are non-zero as well. Combining Theorem 2.4.5 and Lemma 2.4.7 therefore concludes the proof. □

Theorem 2.4.5 and Corollary 2.4.8 are based on part (1) of Theorem 2.4.1. It is also possible to take part (2) into account and, e.g., obtain results on various types of conormality for spaces of continuous functions with values in a (pre)-ordered Banach space. Here is an example, where part (2) of Theorem 2.4.1 is used via an appeal to Corollary 2.4.3. Note that, analogous to Corollary 2.4.8, the approximate α-conormality of X and of the three normed spaces of X-valued functions are all equivalent if $C_c(\Omega) \neq \{0\}$.

Corollary 2.4.9. *Let X be a real (pre)-ordered Banach space, (pre)-ordered by a closed generating not necessarily proper cone X^+, and let Ω be a topological space. Suppose that $\alpha > 0$.*

If X is approximately α-conormal, then so are $C_b(\Omega, X)$, $C_0(\Omega, X)$, and $C_c(\Omega, X)$.

If $C_b(\Omega, X) \neq \{0\}$ and $C_b(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

If $C_0(\Omega, X) \neq \{0\}$ and $C_0(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

If $C_c(\Omega, X) \neq \{0\}$ and $C_c(\Omega, X)$ is approximately α-conormal, then X is approximately α-conormal.

Proof. Let X be approximately α-conormal, and let $f \in C_b(\Omega, X)$ and $\varepsilon > 0$ be given. Corollary 2.4.3 supplies continuous positively homogeneous maps $\gamma_\varepsilon^\pm : X \to X^+$, such that $x = \gamma_\varepsilon^+(x) - \gamma_\varepsilon^-(x)$ and $\|\gamma_\varepsilon^+(x)\| \leq (\alpha + \varepsilon)\|x\|$, for all $x \in X$. Then $\gamma_\varepsilon^\pm \circ f \in C_b(\Omega, X^+)$, $f = \gamma_\varepsilon^+ \circ f - \gamma_\varepsilon^- \circ f$, and $\|\gamma_\varepsilon^\pm \circ f\|_\infty \leq (\alpha + \varepsilon)\|f\|_\infty$, so that $C_b(\Omega, X)$ is approximately α-conormal. The proof for $C_0(\Omega, X)$ and $C_c(\Omega, X)$ is similar.

If $C_b(\Omega, X) \neq \{0\}$ and $C_b(\Omega, X)$ is approximately α-conormal, let $x \in X$ and $\varepsilon > 0$ be given. As in the proof of Corollary 2.4.7 we find a non-zero $\varphi \in C_b(\Omega)$, and we may assume that $\|\varphi\|_\infty = 1$ and φ is real-valued. Passing to $-\varphi$ if necessary we obtain a sequence $\{\omega_n\} \subset \Omega$ such that $0 < \varphi(\omega_n) \uparrow 1$. Hence there exists $\omega_{n_0} \in \Omega$ such that $0 < \varphi(\omega_{n_0})$ and $(\alpha + \varepsilon/2)\varphi(\omega_{n_0})^{-1} < \alpha + \varepsilon$. By assumption, there exist $f^\pm \in C_b(\Omega, X^+)$, such that $\varphi \otimes x = f^+ - f^-$ and $\|f^+\|_\infty \leq (\alpha + \varepsilon/2)\|\varphi \otimes x\|_\infty = (\alpha + \varepsilon/2)\|x\|$. In particular, $\varphi(\omega_{n_0})x = f^+(\omega_{n_0}) - f^-(\omega_{n_0})$. Since $\varphi(\omega_{n_0})^{-1} f^\pm(\omega_{n_0}) \in X^+$, and $\|\varphi(\omega_{n_0})^{-1} f^+(\omega_{n_0})\| \leq \varphi(\omega_{n_0})^{-1}\|f^+\|_\infty \leq \varphi(\omega_{n_0})^{-1}(\alpha + \varepsilon/2)\|x\| \leq (\alpha + \varepsilon)\|x\|$, we conclude that X is approximately α-conormal.

The proofs for $C_0(\Omega, X)$ and $C_c(\Omega, X)$ are similar. □

Remark 2.4.10. In the context of Corollary 2.4.9, the conclusion that the Banach spaces $C_b(\Omega, X)$ and $C_0(\Omega, X)$ are approximately α-conormal shows that part (1) of Corollary 2.4.3 is satisfied for these (pre)-ordered Banach spaces. Hence (2) is valid as well. Therefore, if X is approximately α-conormal, then $C_b(\Omega, X)$ and $C_0(\Omega, X)$ are continuously positively homogeneously approximately α-conormal in the sense of part (2) of Corollary 2.4.3. The converse holds for $C_b(\Omega, X)$ if this space is non-zero, and similarly for $C_0(\Omega, X)$.

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Chapter 3

Normality of spaces of operators and quasi-lattices

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3.1 Introduction

This paper's main aim is to investigate normality and monotonicity (defined in Section 3.3) of pre-ordered spaces of operators between pre-ordered Banach spaces. This investigation is motivated by the relevance of this notion in the theory of positive semigroups on pre-ordered Banach spaces [6], and in the positive representation theory of groups and pre-ordered algebras on pre-ordered Banach spaces [12].

If X and Y are Banach lattices an elementary calculation shows that the space $B(X,Y)$ is absolutely monotone, i.e., for $T, S \in B(X,Y)$, if $\pm T \leq S$, then $\|T\| \leq \|S\|$. If X and Y are general pre-ordered Banach spaces the situation is not so clear, and raises a number of questions: If $B(X,Y)$ is, e.g., absolutely monotone, does this necessarily imply that X and Y are Banach lattices? If not, what are examples of pre-ordered Banach spaces X and Y, not being Banach lattices, such that $B(X,Y)$ is absolutely monotone? What are the more general necessary and/or sufficient conditions X and Y have to satisfy for $B(X,Y)$ to be absolutely monotone? This paper will attempt to answer such questions through an investigation of the notions of normality and conormality of pre-ordered Banach spaces which describe various ways in which cones interact with norms.

A substantial part will devoted to introducing a class of ordered Banach spaces, called quasi-lattices, which will furnish us with many examples that are not necessarily Banach lattices. Quasi-lattices occur in two slightly different forms, one of which includes all Banach lattices (cf. Proposition 3.5.2). We give a brief sketch of their construction.

There are many pre-ordered Banach spaces with closed proper generating cones that are not normed Riesz spaces, e.g., the finite dimensional spaces $\mathbb{R}^n$ (with $n \geq 3$) endowed with Lorentz cones or endowed with polyhedral cones whose bases (in the sense of [2, Section 1.7]) are not $(n-1)$-simplexes. Although there is often an abundance of upper bounds of arbitrary pairs of elements, none of them is a least upper bound with respect to the ordering defined by the cone. An interesting situation arises when one takes the norm into account when studying the set of upper bounds of arbitrary pairs of elements. Even though there might not exist a least upper bound with respect to the ordering defined by the cone, there often exists a unique upper bound, called the *quasi-supremum*, which minimizes the sum of the distances from this upper bound to the given two elements. This allows us to define what will be called a *quasi-lattice structure* on certain ordered Banach spaces which might not be lattices (cf. Definition 3.5.1). Surprisingly, many elementary vector lattice properties for Riesz spaces carry over nearly verbatim to such spaces (cf. Theorem 3.5.8), and in the case that a space is a Banach lattice, its quasi-lattice structure and lattice structure actually coincide (cf. Proposition 3.5.2).

Quasi-lattices occur in relative abundance, in fact, every strictly convex reflexive ordered Banach space with a closed proper generating cone is a quasi-lattice (cf Theorem 3.6.1). This will be used to show that every Hilbert space $\mathcal{H}$ endowed with a Lorentz cone is a quasi-lattice (which is not a Banach lattice if $\dim(\mathcal{H}) \geq 3$). Such spaces will serve as examples of spaces, which are not Banach lattices, such that the spaces of operators between them are absolutely monotone (cf. Theorem 3.7.10), hence resolving the question of the existence of such spaces as posed above.

We briefly describe the structure of the paper.

After giving preliminary definitions and terminology in Section 3.2, we introduce various versions of the concepts of normality and conormality of pre-ordered Banach spaces with closed cones in Section 3.3. Normality is a more general notion than monotonicity, and roughly is a measure of 'the obtuseness/bluntness of a cone' (with respect to the norm). Conormality roughly is a measure of 'the acuity/sharpness of a cone' (with respect to the norm). Normality and conormality properties often occur in dual pairs, where a pre-ordered Banach space with a closed cone has a normality property precisely when its dual has the appropriate conormality property (cf. Theorem 3.3.7). The terms 'monotonicity' and 'normality' are fairly standard throughout the literature. However, the concept of conormality occurs scattered under many names throughout the literature (chronologically, [23, 7, 3, 21, 11, 34, 45, 35, 47, 44, 39, 48, 6, 9, 37]). Although the definitions and results in Section 3.3 are not new, they are collected here in an attempt to give an overview and to standardize the terminology.

In Section 3.4, with X and Y pre-ordered Banach spaces with closed cones, we investigate the normality of $B(X,Y)$ in terms of the normality and conormality of X and Y. Roughly, excluding degenerate cases, some form of conormality of X and normality of Y is necessary and sufficient for having some form of normality of the pre-ordered Banach space $B(X,Y)$ (cf. Theorems 3.4.1 and 3.4.2). Again, certain results are not new, but are included for the sake of completeness.

In Section 3.5 we introduce quasi-lattices, a class of pre-ordered Banach spaces spaces that strictly includes the Banach lattices. We establish their basic properties, in particular, basic vector lattice identities which carry over from Riesz spaces to quasi-lattices (cf. Theorem 3.5.8).

In Section 3.6 we prove one of our main results: Every strictly convex reflexive pre-ordered Banach space with a closed proper and generating cone is a quasi-lattice. Hence there are many quasi-lattices.

Finally, in Section 3.7, we show that real Hilbert spaces endowed with Lorentz cones are quasi-lattices and satisfy an identity analogous to the elementary identity $\||x|\| = \|x\|$ which holds for all elements x of a Banach lattice. This is used to show, for real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ endowed with Lorentz cones, that $B(\mathcal{H}_1, \mathcal{H}_2)$ is absolutely monotone.

3.2 Preliminary definitions and notation

Let X be a Banach space over the real numbers. Its topological dual will be denoted by X'. A subset $C \subseteq X$ will be called a *cone* if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. If a cone C satisfies $C \cap (-C) = \{0\}$, it will be called a *proper cone*, and if $X = C - C$, it will be said to be *generating* (*in* X).

Definition 3.2.1. A pair (X, C), with X a Banach space and $C \subseteq X$ a cone, will be called a *pre-ordered Banach space*. If C is a proper cone, (X, C) will be called an *ordered Banach space*. We will often suppress explicit mention of the pair and merely say that X is a (pre-)ordered Banach space. When doing so, we will denote the implicit cone by X_+ and refer to it as *the cone of* X. For any $x, y \in X$, by $x \geq y$ we will mean $x - y \in X_+$. We do not exclude the possibilities $X_+ = \{0\}$ or $X_+ = X$, and we do not assume that X_+ is closed.

Let X and Y be pre-ordered Banach spaces. The space of bounded linear operators from X to Y will be denoted by $B(X, Y)$ and by $B(X)$ if $X = Y$. Unless otherwise mentioned, $B(X, Y)$ is always endowed with the operator norm. The space $B(X, Y)$ is easily seen to be a pre-ordered Banach space when endowed with the cone $B(X, Y)_+ := \{T \in B(X, Y) : TX_+ \subseteq Y_+\}$. In particular, the topological dual X' also becomes a pre-ordered Banach space when endowed with the *dual cone* $X'_+ := B(X, \mathbb{R})_+$. For any $f \in X'$ and $y \in Y$, we will define the operator $f \otimes y \in B(X, Y)$ by $(f \otimes y)(x) := f(x)y$ for all $x \in X$. It is easily seen that $\|f \otimes y\| = \|f\|\|y\|$.

3.3 Normality and Conormality

In the current section we will define some of the possible norm-cone interactions that may occur in pre-ordered Banach spaces, and investigate how they relate to norm-cone interactions in the dual. Historically, these properties have been assigned to either the norm or the cone (e.g., 'a cone is normal' and 'a norm is monotone'). We will not follow this convention and rather assign these labels to the pre-ordered Banach space as a whole to emphasize the norm-cone interaction.

We attempt to collect all known results and to standardize the terminology. The definitions and results in the current section are essentially known, but are scattered throughout the literature under quite varied terminology[1]. References are provided when known to the author.

Definition 3.3.1. Let X be a pre-ordered Banach space with a closed cone and $\alpha > 0$.

We define the following ***normality properties****:*

(1) We will say X is *α-max-normal* if, for any $x, y, z \in X$, $z \leq x \leq y$ implies $\|x\| \leq \alpha \max\{\|y\|, \|z\|\}$.

(2) We will say X is *α-sum-normal* if, for any $x, y, z \in X$, $z \leq x \leq y$ implies $\|x\| \leq \alpha(\|y\| + \|z\|)$.

(3) We will say X is *α-absolutely normal* if, for any $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \alpha\|y\|$. We will say X is *absolutely monotone* if it is 1-absolutely normal.

(4) We will say X is *α-normal* if, for any $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq \alpha\|y\|$. We will say X is *monotone* if it is 1-normal.

We define the following ***conormality properties****:*

(1) We will say X is *α-sum-conormal* if, for any $x \in X$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| \leq \alpha\|x\|$. We will say X is *approximately α-sum-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \alpha\|x\| + \varepsilon$.

(2) We will say X is *α-max-conormal* if, for any $x \in X$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \alpha\|x\|$. We will say X is *approximately α-max-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} < \alpha\|x\| + \varepsilon$.

(3) We will say X is *α-absolutely conormal* if, for any $x \in X$, there exist some $a \in X_+$ such that $\pm x \leq a$ and $\|a\| \leq \alpha\|x\|$. We will say X is *approximately α-absolutely conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a \in X_+$ such that $\pm x \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$.

(4) We will say X is *α-conormal* if, for any $x \in X$, there exist some $a \in X_+$ such that $0, x \leq a$ and $\|a\| \leq \alpha\|x\|$. We will say X is *approximately α-conormal* if,

[1] A note on terminology: The terms 'normality' (due to Krein [28]) and 'monotonicity' are fairly standard terms throughout the literature. Our consistent use of the adjective 'absolute' is inspired by [47] and mimics its use in the term 'absolute value'.

The concept that we will call 'conormality' has seen numerous equivalent definitions and the nomenclature is rather varied in the existing literature. The term 'conormality' is due to Walsh [44], who studied the property in the context of locally convex spaces. What we will call '1-max-conormality' occurs under the name 'strict bounded decomposition property' in [8]. The properties that we will call 'approximate 1-absolute conormality' and 'approximate 1-conormality', were first defined (but not named) respectively by Davies [11] and Ng [34]. Batty and Robinson give equivalent definitions for our conormality properties which they call 'dominating' and 'generating' [6].

for any $x \in X$ and $\varepsilon > 0$, there exist some $a \in X_+$ such that $\{0, x\} \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$.

The following two results show the relationship between different (co)normality properties and for the most part are immediate from the definitions.

Proposition 3.3.2. *For any fixed $\alpha > 0$, the following implications hold between normality properties of a pre-ordered Banach space X with a closed cone:*

$$\begin{array}{ccccc} \alpha\text{-}\textit{max-normality} & \Longrightarrow & \alpha\text{-}\textit{absolute-normality} & & \\ \Downarrow & & \Downarrow & & \\ \alpha\text{-}\textit{sum-normality} & \Longrightarrow & \alpha\text{-}\textit{normality} & \Longrightarrow & (\alpha+1)\text{-}\textit{sum-normality} \\ \Downarrow & & \Downarrow & & \\ 2\alpha\text{-}\textit{max-normality} & & X_+ \textit{ is proper} & & \end{array}$$

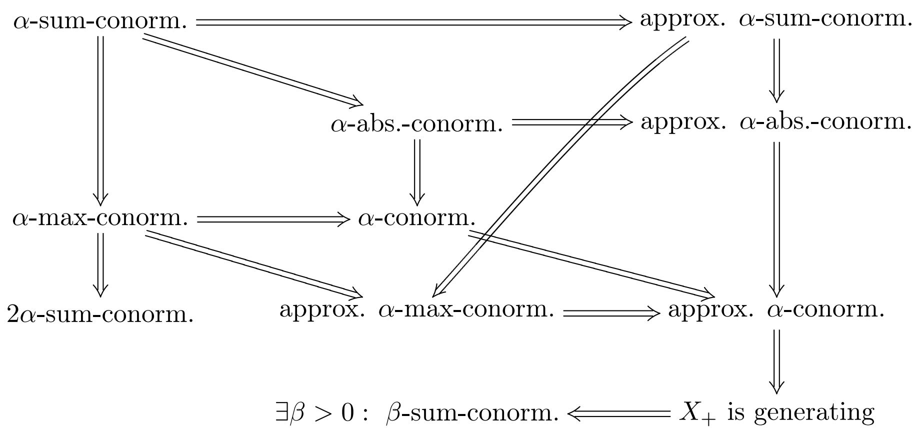
Proof. The only implication that is not immediate from the definitions is that α-normality implies $(\alpha + 1)$-sum-normality. As to this, let X be an α-normal pre-ordered Banach space with a closed cone and $x, y, z \in X$ such that $z \leq x \leq y$. Then $0 \leq x - z \leq y - z$, so that, by α-normality and the reverse triangle inequality,

$$\|x\| - \|z\| \leq \|x - z\| \leq \alpha\|y - z\| \leq \alpha(\|y\| + \|z\|).$$

Hence $\|x\| \leq \alpha\|y\| + (\alpha + 1)\|z\| \leq (\alpha + 1)(\|y\| + \|z\|)$. □

Similar relationships hold between conormality properties as do between normality properties. All implications follow immediately from the definitions, with one exception. That is, if a pre-ordered Banach space has a closed generating cone, then there exists a constant $\beta > 0$ such that it is β-max-conormal (and hence 2β-sum-conormal). This is a result due to Andô [3, Lemma 1], and is the bottom implication in the following proposition (although [3, Lemma 1] assumes the cone to be proper, this is not necessary for its statement to hold, cf. Theorem 3.3.6).

Proposition 3.3.3. *For any fixed $\alpha > 0$, the following implications hold between conormality properties of a pre-ordered Banach space X with a closed cone:*



Remark 3.3.4. The direct analogue to Andô's Theorem [3, Lemma 1] (the bottom implication in Proposition 3.3.3) in Proposition 3.3.2 would be that having X_+ proper implies that X is β-max-normal for some $\beta > 0$. This is false. Example 3.6.7 gives a space which has a proper cone but is not α-normal for any $\alpha > 0$.

Remark 3.3.5. For the sake of completeness, we note that Andô's Theorem [3, Lemma 1] (the bottom implication in Proposition 3.3.3) can be improved, in that the decomposition of elements into a difference of elements from the cone can be chosen in a continuous, as well as bounded and positively homogeneous manner. The following result is a special case of [13, Theorem 4.1], which is a general principle for Banach spaces that are the sum of (not necessarily countably many) closed cones. Its proof proceeds through an application of Michael's Selection Theorem [1, Theorem 17.66] and a generalization of the usual Open Mapping Theorem [13, Theorem 3.2]:

Theorem 3.3.6. *Let X be a pre-ordered Banach space with a closed generating cone. Then there exist continuous positively homogeneous functions $(\cdot)^{\pm} : X \to X_+$ and a constant $\alpha > 0$ such that $x = x^+ - x^-$ and $\|x^{\pm}\| \leq \alpha\|x\|$ for all $x \in X$.*

Normality and conormality properties often appear in dual pairs. Roughly, a pre-ordered Banach space has a normality property if and only if its dual has a corresponding conormality property, and vice versa. The following theorem provides an overview of these normality-conormality duality relationships as known to the author.

Theorem 3.3.7. *Let X be a pre-ordered Banach space with a closed cone.*

(1) The following equivalences hold:

(a) For $\alpha > 0$, the space X is α-max-normal if and only if X' is α-sum-conormal.

(b) For $\alpha > 0$, the space X is α-sum-normal if and only if X' is α-max-conormal.

(c) For $\alpha > 0$, the space X is α-absolutely normal if and only if X' is α-absolutely conormal.

(d) For $\alpha > 0$, the space X is α-normal if and only if X' is α-conormal.

(e) There exists an $\alpha > 0$ such that X is α-max-normal if and only if X'_+ is generating.

(2) The following equivalences hold:

(a) For $\alpha > 0$, the space X is approximately α-sum-conormal if and only if X' is α-max-normal.

(b) For $\alpha > 0$, the space X is approximately α-max-conormal if and only if X' is α-sum-normal.

(c) For $\alpha > 0$, the space X is approximately α-absolutely conormal if and only if X' is α-absolutely normal.

(d) For $\alpha > 0$, the space X is approximately α-conormal if and only if X' is α-normal.

(e) The cone X_+ is generating if and only if there exists an $\alpha > 0$ such that X' is α-max-normal.

The result (1)(a) was first proven by Grosberg and Krein in [23] (via [21, Theorem 7]). The result (2)(a) was established by Ellis [21, Theorem 8]. For $\alpha = 1$, the results (1)(c),(d), (2)(c) and (d) are due to Ng [34, Proppositions 5, 6; Theorems 6, 7]. The fully general results (1)(d) and (2)(d) appear first in [39, Theorem 1.1] by Robinson and Yamamuro, and later in [37, Theorems 1,2] by Ng an Law. Proofs of (1)(a) (again), (1)(b), (1)(c), (1)(d) (again), and (2)(a) (again), (2)(b), (2)(c), and (2)(d) (again) are due to Batty and Robinson in [6, Theorems 1.1.4, 1.3.1, 1.2.2]. The results (1)(e) and (2)(e) are due to Andô [3, Theorem 1].

Bonsall proved an analogous duality result for locally convex spaces in [7, Theorem 2].

The following lemma shows that conormality properties and approximate conormality properties of dual spaces are equivalent. Ng proved (3) for the case $\alpha = 1$ in [34, Theorem 6]:

Lemma 3.3.8. *Let X be a pre-ordered Banach space with a closed cone. Then the following equivalences hold:*

(1) For $\alpha > 0$, the space X' is approximately α-sum-conormal if and only if X' is α-sum-conormal.

(2) For $\alpha > 0$, the space X' is approximately α-max-conormal if and only if X' is α-max-conormal.

(3) For $\alpha > 0$, the space X' is approximately α-absolutely conormal if and only if X' is α-absolutely conormal.

(4) For $\alpha > 0$, the space X' is approximately α-conormal if and only if X' is α-conormal.

Proof. That a conormality property implies the associated approximate conormality property is trivial. We will therefore only prove the forward implications.

We prove (1). Let X' be approximately α-sum-conormal. Then, for any $\beta > \alpha$ and any $0 \neq f \in X$, by taking $\varepsilon = (\beta - \alpha)\|f\| > 0$, we have that there exist $g, h \in X'_+$ such that $f = g - h$ and $\|g\| + \|h\| \leq \alpha\|f\| + (\beta - \alpha)\|f\| = \beta\|f\|$. Therefore, X' is β-sum-conormal for every $\beta > \alpha$. Now, by part (1)(a) of Theorem 3.3.7, X is β-max-normal for every $\beta > \alpha$. Therefore, if $x, y, z \in X$ are such that $z \leq x \leq y$, then $\|x\| \leq \beta \max\{\|y\|, \|z\|\}$ for all $\beta > \alpha$, and hence $\|x\| \leq \inf_{\beta > \alpha} \beta \max\{\|y\|, \|z\|\} = \alpha \max\{\|y\|, \|z\|\}$. We conclude that X is α-max-normal, and, again by part (1)(a) Theorem 3.3.7, that X' is α-sum-conormal.

The assertions (2), (3) and (4) follow through similar arguments. ☐

By Theorem 3.3.7 and Lemma 3.3.8, a pre-ordered Banach space with a closed cone possesses both a normality property and its paired approximate conormality property (with the same constant) if and only if its dual possesses the same properties (cf. Corollary 3.3.11). Such spaces are called regular and were first studied by Davies in [11] and Ng in [34].

Definition 3.3.9. Let X be a pre-ordered Banach space with a closed cone. We define the following ***regularity properties***:[2]

(1) For $\alpha > 0$, we will say X is *α-Ellis-Grosberg-Krein regular* if X is both α-max-normal and approximately α-sum-conormal.

(2) For $\alpha > 0$, we will say X is *α-Batty-Robinson regular* if X is both α-sum-normal and approximately α-max-conormal.

(3) For $\alpha > 0$, we will say X is *α-absolutely Davies-Ng regular* if X is both α-absolutely normal and approximately α-absolutely conormal.

(4) For $\alpha > 0$, we will say X is *α-Davies-Ng regular* if X is both α-normal and approximately α-conormal.

[2] The term 'regularity' is due to Davies [11]. Our naming convention is to attach the names of the persons who (to the author's knowledge) first proved the relevant normality-conormality duality results of the defining properties (cf. Theorem 3.3.7).

(5) We will say X is *Andô regular* if X_+ is generating and there exists an $\alpha > 0$ such that X is α-max-normal.

It should be noted that every Banach lattice is 1-absolutely Davies-Ng regular.

The following result combines Propositions 3.3.2 and 3.3.3 to provide relationships that exist between regularity properties.

Proposition 3.3.10. *For any fixed $\alpha > 0$, the following implications hold between regularity properties of a pre-ordered Banach space with a closed cone:*

$$\begin{array}{ccc}
\alpha\text{-Ellis-Grosberg-Krein regularity} & \Longrightarrow & \alpha\text{-Batty-Robinson regularity} \\
\Downarrow & & \Downarrow \\
\alpha\text{-absolute Davies-Ng regularity} & \Longrightarrow & \alpha\text{-Davies-Ng regularity} \\
 & & \Downarrow \\
\exists \beta > 0 : \ \beta\text{-Ellis-Grosberg-Krein regularity} & \Longleftarrow & \text{Andô regularity}
\end{array}$$

Proof. The only implication that does not follow immediately from Propositions 3.3.2 and 3.3.3, is that Andô regularity implies β-Ellis-Grosberg-Krein regularity for some $\beta > 0$. As to this, let X be an Andô regular ordered Banach space with a closed cone. By Proposition 3.3.3, since X_+ is generating, there exists some $\delta > 0$ such that X is δ-sum-conormal. By assumption, there exists an $\alpha > 0$, such that X is α-max-normal. By taking $\beta := \max\{\delta, \alpha\}$, we see that X is also β-max-normal and (approximately) β-sum-conormal. We conclude that X is β-Ellis-Grosberg-Krein regular. □

A straightforward application of Theorem 3.3.7 and Lemma 3.3.8 then yields:

Corollary 3.3.11. *Let X be a pre-ordered Banach space with a closed cone. Then the following equivalences hold:*

(1) For $\alpha > 0$, the space X is α-Ellis-Grosberg-Krein regular if and only if X' is α-Ellis-Grosberg-Krein regular.

(2) For $\alpha > 0$, the space X is α-Batty-Robinson regular if and only if X' is α-Batty-Robinson regular.

(3) For $\alpha > 0$, the space X is α-absolutely Davies-Ng regular if and only if X' is α-absolutely Davies-Ng regular.

(4) For $\alpha > 0$, the space X is α-Davies-Ng regular if and only if X' is α-Davies-Ng regular.

(5) The space X is Andô regular if and only if X' is Andô regular.

3.4 The normality of pre-ordered Banach spaces of bounded linear operators

If X and Y are pre-ordered Banach spaces with closed cones, we investigate necessary and sufficient conditions for the pre-ordered Banach space $B(X,Y)$ to have a normality property. Where results are known to the author from the literature, references are provided.

We begin, in the following result, by investigating necessary conditions for $B(X,Y)$ to have a normality property. Parts (2) and (3) in the special case $X = Y$ and $\alpha = 1$ in the following theorem are due Yamamuro [48, 1.2–3]. Batty and Robinson also proved part (2) for $X = Y$ and $\alpha = 1$, and part (3) for $\alpha = \beta = 1$ [6, Corollary 1.7.5, Proposition 1.7.6]. Part (5) is due to Wickstead [45, Theorem 3.1].

Theorem 3.4.1. *Let X and Y be non-zero pre-ordered Banach spaces with closed cones and $\alpha > 0$.*

(1) The cone $B(X,Y)_+$ is proper if and only if $X = \overline{X_+ - X_+}$ and Y_+ is proper.

(2) Let $B(X,Y)$ be α-normal. If $Y_+ \neq \{0\}$, then X is approximately α-conormal. If $X'_+ \neq \{0\}$, then Y is α-normal.

(3) Let $B(X,Y)$ be α-absolutely normal. If $Y_+ \neq \{0\}$, then X is approximately α-absolutely conormal. If $X'_+ \neq \{0\}$, then Y is α-absolutely normal.

(4) Let $B(X,Y)$ be α-sum-normal. If $Y_+ \neq \{0\}$, then X is approximately α-max-conormal. If $X'_+ \neq \{0\}$, then Y is α-sum-normal.

(5) Let $B(X,Y)$ be α-max-normal. If $Y_+ \neq \{0\}$, then X is approximately α-sum-conormal. If $X'_+ \neq \{0\}$, then Y is α-max-normal.

Proof. We prove (1). Let $B(X,Y)_+$ be proper. Suppose $X \neq \overline{X_+ - X_+}$. By the Hahn-Banach Theorem there exists a non-zero functional $f \in X'$ such that $f|_{\overline{X_+ - X_+}} = 0$. Let $0 \neq y \in Y$, then $\pm f \otimes y \geq 0$ since $f \otimes y|_{X_+} = 0$. Therefore $B(X,Y)_+$ is not proper, contradicting our assumption. Suppose Y_+ is not proper. Let $0 \neq y \in Y_+ \cap (-Y_+)$ and $0 \neq f \in X'$. Then $\pm f \otimes y \geq 0$, and hence $B(X,Y)_+$ is not proper, contradicting our assumption.

Let $X = \overline{X_+ - X_+}$ and Y_+ be proper. If $T \in B(X,Y)_+ \cap (-B(X,Y)_+)$, then, since Y_+ is proper, $TX_+ = \{0\}$. Hence $T(X_+ - X_+) = \{0\}$, and by density of $X_+ - X_+$ in X, we have $T = 0$.

We prove (2). Let $B(X,Y)$ be α-normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, to conclude that X is approximately α-conormal, it is sufficient to prove that X' is α-normal. Let $f, g \in X'$ satisfy $0 \leq f \leq g$, and let $0 \neq y \in Y_+$. Then $0 \leq f \otimes y \leq g \otimes y$, and by the α-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha \|g \otimes y\| = \alpha \|g\|\|y\|.$$

Therefore $\|f\| \leq \alpha \|g\|$, and hence X' is α-conormal. With $X'_+ \neq \{0\}$, let $0 \neq f \in X'_+$ be arbitrary, and $y, z \in Y$ such that $0 \leq y \leq z$. Then $0 \leq f \otimes y \leq f \otimes z$ in $B(X,Y)$, and by the α-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|f \otimes z\| = \alpha\|f\|\|z\|.$$

Hence $\|y\| \leq \alpha\|z\|$ and we conclude that Y is α-normal.

We prove (3). Let $B(X,Y)$ be α-absolutely normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, to conclude that X is approximately α-absolutely conormal, it is sufficient to prove that X' is α-absolutely normal. Let $f, g \in X'$ satisfy $\pm f \leq g$, and let $0 \neq y \in Y_+$. Then $\pm f \otimes y \leq g \otimes y$, and by the α-absolute normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|g \otimes y\| = \alpha\|g\|\|y\|.$$

Therefore $\|f\| \leq \alpha\|g\|$, and hence X' is α-absolutely normal. With $X'_+ \neq \{0\}$, let $0 \neq f \in X'_+$ be arbitrary, and $y, z \in Y$ such that $\pm y \leq z$. Then $\pm f \otimes y \leq f \otimes z$ in $B(X,Y)$, and by the α-absolute normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\|f \otimes z\| = \alpha\|f\|\|z\|$$

Hence $\|y\| \leq \alpha\|z\|$ and we conclude that Y is α-absolutely normal.

We prove (4). Let $B(X,Y)$ is α-sum-normal. With $Y_+ \neq \{0\}$, by Theorem 3.3.7, it is sufficient to prove that X' is α-sum-normal to conclude that X is approximately α-max-conormal. Let $0 \neq y \in Y_+$ and $f, g, h \in X'$ satisfy $g \leq f \leq h$. Then $g \otimes y \leq f \otimes y \leq h \otimes y$ in $B(X,Y)$, and by the α-sum-normality of $B(X,Y)$,

$$\|f\|\|y\| = \|f \otimes y\| \leq \alpha\left(\|g \otimes y\| + \|h \otimes y\|\right) = \alpha(\|g\| + \|h\|)\|y\|.$$

Hence $\|f\| \leq \alpha(\|g\| + \|h\|)$ and X' is α-sum-normal. With $X'_+ \neq \{0\}$, to prove that Y is α-sum-normal, let $u, v, y \in Y$ satisfy $u \leq y \leq v$ and let $0 \neq f \in X'_+$. Then $f \otimes u \leq f \otimes y \leq f \otimes v$ in $B(X,Y)$, and hence, $\|y\| \leq \alpha(\|u\| + \|v\|)$ as before.

The proof of (5) is analogous to that of (4). □

Converse-like implications to the previous result also hold, giving sufficient conditions for $B(X,Y)$ to have a normality property. Part (1) and the case $\alpha = \beta = 1$ of part (3) are due to Batty and Robinson Batty and Robinson [6, Proposition 1.7.3, Corollary 1.7.5]. The special case $X = Y$ and $\alpha = \beta = 1$ of part (3) is due to Yamamuro [48, 1.3]. The case where X is approximately α-sum-conormal and Y is β-max-normal of part (4) is due to Wickstead [45, Theorem 3.1].

Theorem 3.4.2. *Let X and Y be pre-ordered Banach spaces with closed cones and $\alpha, \beta > 0$.*

(1) If X_+ is generating and Y is α-normal, then there exists some $\gamma > 0$ for which $B(X,Y)$ is γ-normal.

(2) If X is approximately α-conormal and Y is β-normal, then $B(X,Y)$ is $(2\alpha + 1)\beta$-normal.

(3) If X is approximately α-absolutely conormal and Y is β-absolutely normal, then $B(X,Y)$ is $\alpha\beta$-absolutely normal.

(4) If X is approximately α-sum-conormal and Y is β-normal (β-absolutely normal, β-max-normal, β-sum-normal respectively), then $B(X,Y)$ is $\alpha\beta$-normal ($\alpha\beta$-absolutely normal, $\alpha\beta$-max-normal, $\alpha\beta$-sum-normal respectively)

Proof. We prove (1). By Andô's Theorem [3, Lemma 1], the fact that X_+ is generating in X implies that there exists some $\beta > 0$ such that X is β-max-conormal. Let $T, S \in B(X,Y)$ be such that $0 \le T \le S$. Then, for any $x \in X$, let $a, b \in X_+$ be such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \le \beta\|x\|$, so that $0 \le Ta \le Sa$ and $0 \le Tb \le Sb$. By α-normality of Y,

$$\|Tx\| \le \|Ta\| + \|Tb\| \le \alpha(\|Sa\| + \|Sb\|) \le \alpha\|S\|(\|a\| + \|b\|) \le 2\alpha\beta\|S\|\|x\|,$$

hence $\|T\| \le 2\alpha\beta\|S\|$.

We prove (2). Let $T, S \in B(X,Y)$ be such that $0 \le T \le S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ such that $\{0, x\} \le a$ and $\|a\| \le \alpha\|x\| + \varepsilon$. Since $x = a - (a - x)$ and $a, a - x \ge 0$, we obtain $0 \le Ta \le Sa$ and $0 \le T(a - x) \le S(a - x)$, and hence

$$\begin{aligned}
\|Tx\| &= \|Ta - T(a-x)\| \\
&\le \|Ta\| + \|T(a-x)\| \\
&\le \beta\|Sa\| + \beta\|S(a-x)\| \\
&\le \beta\|S\|(\alpha\|x\| + \varepsilon) + \beta\|S\|(\alpha\|x\| + \varepsilon + \|x\|) \\
&= (2\alpha + 1)\beta\|S\|\|x\| + 2\varepsilon\beta\|S\|.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\| \le (2\alpha + 1)\beta\|S\|$.

We prove (3). Let $T, S \in B(X,Y)$ satisfy $\pm T \le S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ satisfying $\pm x \le a$ and $\|a\| < \alpha\|x\| + \varepsilon$. Then

$$Tx = T\left(\frac{a+x}{2}\right) - T\left(\frac{a-x}{2}\right),$$

and hence,

$$\pm Tx = \pm T\left(\frac{a+x}{2}\right) \mp T\left(\frac{a-x}{2}\right).$$

Since $(a + x)/2 \ge 0$, $(a - x)/2 \ge 0$ and $\pm T \le S$, we find

$$\pm Tx \le S\left(\frac{a+x}{2}\right) + S\left(\frac{a-x}{2}\right) = Sa.$$

Now, because Y is β-absolutely normal, we obtain

$$\|Tx\| \le \beta\|Sa\| \le \beta\|S\|\|a\| \le \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X,Y)$ is $\alpha\beta$-absolutely normal.

We prove (4). Let X be approximately α-sum-conormal and let Y be β-normal. Let $T, U \in B(X,Y)$ satisfy $0 \le T \le U$ and let $x \in X$ be arbitrary. Then, for every

$\varepsilon > 0$, there exist $x_1, x_2 \in X_+$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| < \alpha\|x\| + \varepsilon$. Also, $0 \le Tx_i \le Ux_i$ implies $\|Tx_i\| \le \beta\|Ux_i\|$ for $i = 1, 2$. Therefore,

$$\begin{aligned}\|Tx\| &\le \|Tx_1\| + \|Tx_2\| \\ &\le \beta\|Ux_1\| + \beta\|Ux_2\| \\ &\le \beta\|U\|(\|x_1\| + \|x_2\|) \\ &\le \alpha\beta\|U\|\|x\| + \varepsilon\beta\|U\|.\end{aligned}$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, we may conclude that $B(X,Y)$ is $\alpha\beta$-normal. The case where X is approximately α-sum-conormal and Y is β-absolutely normal follows similarly.

Let X be approximately α-sum-conormal and let Y be β-max-normal. Let $T, U, V \in B(X,Y)$ satisfy $U \le T \le V$ and let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exist $x_1, x_2 \in X_+$ such that $x = x_1 - x_2$ and $\|x_1\| + \|x_2\| < \alpha\|x\| + \varepsilon$. Also, $Ux_i \le Tx_i \le Vx_i$ implies $\|Tx_i\| \le \beta \max\{\|Ux_i\|, \|Vx_i\|\}$ for $i = 1, 2$. Therefore,

$$\begin{aligned}\|Tx\| &\le \|Tx_1\| + \|Tx_2\| \\ &\le \beta \max\{\|Ux_1\|, \|Vx_1\|\} + \beta \max\{\|Ux_2\|, \|Vx_2\|\} \\ &\le \beta \max\{\|U\|, \|V\|\}(\|x_1\| + \|x_2\|) \\ &\le \alpha\beta \max\{\|U\|, \|V\|\}\|x\| + \varepsilon\beta \max\{\|U\|, \|V\|\}.\end{aligned}$$

Since $x \in X$ and $\varepsilon > 0$ were arbitrary, we may conclude that $B(X,Y)$ is $\alpha\beta$-max-normal. The case where X is approximately α-sum-conormal and Y is β-sum-normal follows similarly. □

If one has further knowledge of the behavior of the positive bounded linear operators, specifically that their norms are determined by their behavior on the cone, then one can improve the constant in (2) of the above theorem. This will be discussed in the rest of this section.

Definition 3.4.3. Let X be a pre-ordered Banach space with a closed cone and Y a Banach space. For $T \in B(X,Y)$, we define $\|T\|_+ := \sup\{\|Tx\| : x \in X_+,\ \|x\| = 1\}$.

If $X = \overline{X_+ - X_+}$, then $\|\cdot\|_+$ is a norm on $B(X,Y)$, called the *Robinson norm* (as named by Yamamuro in [48]). We will say that the operator norm on $B(X,Y)$ is *positively attained* (as named by Batty and Robinson in [6]) if $\|T\| = \|T\|_+$ for all positive operators $T \in B(X,Y)_+$.

If X_+ is closed and generating, $\|\cdot\|_+$ is in fact equivalent to the usual operator norm on $B(X,Y)$. The following result is a slight refinement of a remark by Batty and Robinson [6, p. 248].

Proposition 3.4.4. *If X is a pre-ordered Banach space with a closed generating cone and Y a Banach space, then the Robinson norm is equivalent to the operator norm on $B(X,Y)$.*

Proof. By Andô's Theorem [3, Lemma 1], X is α-max-conormal for some $\alpha > 0$. Let $x \in X$ and $T \in B(X, Y)$ be arbitrary, then there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \alpha\|x\|$. Hence

$$\begin{aligned} \|Tx\| &= \|Ta - Tb\| \\ &\leq \|Ta\| + \|Tb\| \\ &\leq \|T\|_+(\|a\| + \|b\|) \\ &\leq 2\alpha\|T\|_+\|x\|. \end{aligned}$$

Therefore, $\|T\|_+ \leq \|T\| \leq 2\alpha\|T\|_+$. □

Part (2) of Theorem 3.4.2 can be improved if we know that the operator norm is positively attained.

Proposition 3.4.5. *Let X and Y be pre-ordered Banach spaces with closed cones, with Y α-normal for some $\alpha > 0$. If the operator norm on $B(X, Y)$ is positively attained, then $B(X, Y)$ is α-normal.*

Proof. Let $T, S \in B(X, Y)$ satisfy $0 \leq T \leq S$. Then, for any $x \in X_+$, $0 \leq Tx \leq Sx$, and hence $\|Tx\| \leq \alpha\|Sx\|$. We then see that

$$\begin{aligned} \|T\| &= \|T\|_+ \\ &= \sup\{\|Tx\| : x \in X_+,\ \|x\| \leq 1\} \\ &\leq \alpha \sup\{\|Sx\| : x \in X_+,\ \|x\| \leq 1\} \\ &= \alpha\|S\|_+ \\ &= \alpha\|S\|, \end{aligned}$$

and conclude that $B(X, Y)$ is α-normal. □

The following theorem gives one necessary condition and some sufficient conditions to have that an operator norm is positively attained. The sufficiency of (1)[3], (2), and the necessity of approximate 1-conormality in the following theorem are due to Batty and Robinson in [6, Proposition 1.7.8.].

Theorem 3.4.6. *Let X and Y be pre-ordered Banach spaces with closed cones.*

If $Y_+ \neq \{0\}$ and the operator norm on $B(X, Y)$ is positively attained, then X is approximately 1-conormal.

Any of the following conditions is sufficient for the operator norm on $B(X, Y)$ to be positively attained:

(1) The space X is approximately 1-max-conormal and Y is 1-max-normal.

(2) The space X is approximately 1-absolutely conormal and Y is absolutely monotone (i.e., if $X = Y$, X is 1-absolutely Davies-Ng regular).

[3] There is a small error in the statement of (1) in [6, Proposition 1.7.8.]. We give its correct statement and proof.

(3) *The space X is approximately* 1*-sum-conormal (in which case $\|T\| = \|T\|_+$ even holds for all $T \in B(X,Y)$).*

Proof. We prove the necessity of approximate 1-conormality of X when $Y_+ \neq \{0\}$ and the operator norm on $B(X,Y)$ is positively attained. Let $f \in X'_+$ be arbitrary and let $0 \neq y \in Y_+$. Then, since the operator norm on $B(X,Y)$ is positively attained,

$$\|f\|\|y\| = \|f \otimes y\| = \|f \otimes y\|_+ = \|f\|_+\|y\|,$$

so that $\|f\| = \|f\|_+$. For all $f, g \in X'$ satisfying $0 \leq f \leq g$, we obtain $\|f\| = \|f\|_+ \leq \|g\|_+ = \|g\|$. Therefore X' is monotone, and by part (2)(d) of Theorem 3.3.7, X is approximately 1-conormal.

We prove the sufficiency of (1). Let $T \in B(X,Y)_+$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Then, since X is 1-max-conormal, there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} < \|x\| + \varepsilon$. We notice that $-b \leq x \leq a$ and $T \geq 0$ imply that $-Tb \leq Tx \leq Ta$. Then, since Y is 1-max-normal,

$$\|Tx\| \leq \max\{\|Ta\|, \|Tb\|\} \leq \|T\|_+ \max\{\|a\|, \|b\|\} \leq \|T\|_+(\|x\| + \varepsilon).$$

Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\|_+ \leq \|T\| \leq \|T\|_+$.

We prove the sufficiency of (2). Let $T \in B(X,Y)_+$. Let $x \in X$ and $\varepsilon > 0$ be arbitrary, then there exists a $z \in X_+$ such that $\{-x, x\} \leq z$ and $\|z\| < \|x\| + \varepsilon$. Then, since $T \geq 0$, we see that $\{-Tx, Tx\} \leq Tz$, and therefore, since Y is absolutely monotone,

$$\|Tx\| \leq \|Tz\| \leq \|T\|_+\|z\| \leq \|T\|_+(\|x\| + \varepsilon).$$

Because $\varepsilon > 0$ was chosen arbitrarily, we conclude that $\|T\|_+ \leq \|T\| \leq \|T\|_+$.

We prove the sufficiency of (3). Let $x \in X$ be arbitrary. Since X is approximately 1-sum-conormal, for every $\varepsilon > 0$, there exist $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \|x\| + \varepsilon$. For any $T \in B(X,Y)$, we have

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \|T\|_+(\|a\| + \|b\|) \leq \|T\|_+(\|x\| + \varepsilon).$$

Since $\varepsilon > 0$ and $x \in X$ were chosen arbitrarily, we obtain $\|T\|_+ \leq \|T\| \leq \|T\|_+$. ☐

3.5 Quasi-lattices and their basic properties

In this section we will define quasi-lattices, establish their basic properties and provide a number of illustrative (non-)examples.

Let X be a pre-ordered Banach space and A any subset of X. For $x \in X$, by $A \leq x$ we mean that $a \leq x$ for all $a \in A$ and say *x is an upper bound of A*. We will use the Greek letter 'upsilon' to denote the set of all upper bounds of A, written as $\upsilon(A)$. If $x \in X$ is such that $A \leq x$ and, for any $y \in X$, $A \leq y \leq x$ implies $x = y$, we say that x is a *minimal upper bound of A*. We will use the Greek letter 'mu' to denote the set of all minimal upper bounds of A, written as $\mu(A)$. We note that $\upsilon(A)$ and $\mu(A)$ could be empty for some $A \subseteq X$.

For any fixed $x, y \in X$, we define the function $\sigma_{x,y} : X \to \mathbb{R}_{\geq 0}$ by $\sigma_{x,y}(z) := \|z - x\| + \|z - y\|$ for all $z \in X$, and note that $\sigma_{x,y}(z) \geq \|x - y\|$ for all $x, y, z \in X$. We will refer to $\sigma_{x,y}$ as *the distance sum to x and y.*

We introduce the following definitions and notation:

Definition 3.5.1. Let X be a pre-ordered Banach space with a closed cone.

(1) We say that X is an *υ-quasi-lattice* if, for every pair of elements $x, y \in X$, $\upsilon(\{x, y\})$ is non-empty and there exists a unique element $z \in \upsilon(\{x, y\})$ minimizing $\sigma_{x,y}$ on $\upsilon(\{x, y\})$. The element z will be called the *υ-quasi-supremum* of $\{x, y\}$.

(2) We say that X is a *μ-quasi-lattice* if, for every pair of elements $x, y \in X$, $\mu(\{x, y\})$ is non-empty and there exists a unique element $z \in \mu(\{x, y\})$ minimizing $\sigma_{x,y}$ on $\mu(\{x, y\})$. The element z will be called the *μ-quasi-supremum* of $\{x, y\}$.

We immediately note that all Banach lattices are μ-quasi-lattices:

Proposition 3.5.2. *If X is a lattice ordered Banach space with a closed cone* (*in particular, if X is a Banach lattice*), *then X is a μ-quasi-lattice and its lattice structure coincides with its μ-quasi-lattice structure.*

Proof. Since for every $x, y \in X$, $\mu(\{x, y\}) = \{x \vee y\}$ is a singleton, this is clear. □

Remark 3.5.3. If X is a pre-ordered Banach space with a closed cone, then, for $x, y \in X$, the set $\upsilon(\{x, y\})$ is closed and convex, and hence techniques from convex optimization can be used to establish whether a pre-ordered Banach space is an υ-quasi-lattice (cf. Theorem 3.6.1). The set $\mu(\{x, y\})$ need not be convex in general (cf. Example 3.5.9), and hence it is usually more difficult to determine whether or not a space is a μ-quasi-lattice than an υ-quasi-lattice.

Except in the case of monotone υ-quasi-lattices which are also μ-quasi-lattices with coinciding υ– and μ-quasi-lattice structures (cf. Theorem 3.5.12), no further relationship is known between υ– and μ-quasi-lattices. Example 3.5.5 will provide a Banach lattice, and hence μ-quasi-lattice, that is not an υ-quasi-lattice. Furthermore, Example 3.5.13 will provide a non-monotone υ-quasi-lattice, which exhibits υ-quasi-suprema that are not minimal, hence if this space were a μ-quasi-lattice (which is currently not known), then its υ– and μ-quasi-lattice structures will not coincide.

To avoid repetition, we will often use the term quasi-lattice when it is unimportant whether a space is an υ– or μ-quasi-lattice, i.e., a quasi-lattice is either an υ– or μ-quasi-lattice. In such cases we will refer to the relevant υ– or μ-quasi-supremum as just the quasi-supremum. When it is indeed important whether a space is an υ– or μ-quasi-lattice, we will mention it explicitly.

The following notation will be used for both υ– and μ-quasi-lattices. Let X be a quasi-lattice and $x, y \in X$ arbitrary. We will denote the quasi-supremum of $\{x, y\}$ by $x \tilde{\vee} y$. This operation is symmetric, i.e., $x \tilde{\vee} y = y \tilde{\vee} x$. We define the *quasi-infimum* of $\{x, y\}$ by $x \tilde{\wedge} y := -((-x) \tilde{\vee} (-y))$. It is elementary to see that $x \tilde{\wedge} y \leq \{x, y\}$. We define the *quasi-absolute value* of x by $\lceil x \rceil := (-x)\tilde{\vee}(x)$. We will often use the notation $x^+ := 0\tilde{\vee}x$ and $x^- := 0\tilde{\vee}(-x)$.

Before establishing the basic properties of quasi-lattices, we will give a few examples of spaces that are (not) quasi-lattices.

The following is an example of a quasi-lattice that is not a Riesz space, and hence not a Banach lattice:

Example 3.5.4. The space $\{\mathbb{R}^3, \|\cdot\|_2\}$, endowed with the Lorentz cone

$$C := \{(x_1, x_2, x_3) : x_1 \geq (x_2^2 + x_3^2)^{1/2}\}.$$

There are many minimal upper bounds of, e.g., $\{(0,0,0),(0,0,2)\}$ (cf. Example 3.5.9 and Proposition 3.7.5). Hence no supremum exists, and this space is not a Riesz space. Another method to establish this would be to note that C has more than distinct 3 extreme rays, while every lattice cone in $\mathbb{R}^3$ has at most 3 disinct extreme rays [2, Theorem 1.45].

This space is (simultaneously an υ-quasi-lattice and) a μ-quasi-lattice. Intuitively, this can be seen by taking arbitrary elements, $x, y \in \mathbb{R}^3$, and seeing that there exists a unique element in $\mu(\{x,y\})$ with least first coordinate, which is then the quasi-supremum. It is possible give a more explicit proof, but this is not needed in view of the general Theorem 3.7.10 which is applicable to this example.

The following is an example of a Banach lattice, hence a μ-quasi-lattice, that is not an υ-quasi-lattice:

Example 3.5.5. Consider the space $\{\mathbb{R}^3, \|\cdot\|_\infty\}$ with the standard cone. Let $x := (1,-1,0)$. Then

$$\upsilon(\{0,x\}) = \{(z_1,z_2,z_3) \in \mathbb{R}^3 : z_1 \geq 1,\ z_2, z_3 \geq 0\},$$

and hence, for all $z \in \upsilon(\{0,x\})$, we see $\sigma_{0,x}(z) = \|z\|_\infty + \|(z_1-1, z_2+1, z_3)\|_\infty \geq 1+1 = 2$. But, for every $t \in [0,1]$, $\{0,x\} \leq z_t := (1,0,t)$ is such that

$$\sigma_{x,0}(z_t) = \|z_t - x\|_\infty + \|z_t - 0\|_\infty = \|(0,1,t)\|_\infty + \|(1,0,t)\|_\infty = 2,$$

so that there exists no unique upper bound of $\{x,0\}$ minimizing the distance sum to x and 0. We conclude that this space is not an υ-quasi-lattice.

There do exist ordered Banach spaces endowed with closed proper generating cones that are not normed Riesz spaces, nor μ-quasi-lattices or υ-quasi-lattices:

Example 3.5.6. Let $\{\mathbb{R}^3, \|\cdot\|_\infty\}$ be endowed with the cone defined by the four extreme rays $\{(\pm 1, \pm 1, 1)\}$. Let $x := (0,0,0)$ and $y := (2,0,0)$. It can be seen that $\mu(\{x,y\}) = \{(1,t,1) \in \mathbb{R}^3 : t \in [-1,1]\}$. Since this set is not a singleton, this space is not a Riesz space. Moreover, $\sigma_{x,y}$ takes the constant value 2 on $\mu(\{x,y\})$, and hence there does not exist a unique element minimizing $\sigma_{x,y}$ on $\mu(\{x,y\})$. Therefore this space is not a μ-quasi-lattice. Furthermore, if $z \in \upsilon(\{x,y\})$ and $z_3 > 1$, then $\sigma_{x,y}(z) > 2$, and since $\upsilon(\{x,y\}) \cap \{z \in \mathbb{R}^3 : z_3 \leq 1\} = \mu(\{x,y\})$, all minimizers of $\sigma_{x,y}$ in $\upsilon(\{x,y\})$ must be elements of $\mu(\{x,y\})$. Since $\sigma_{x,y}$ is constant on $\mu(\{x,y\})$, this space is not an υ-quasi-lattice.

The following results establish some basic properties of quasi-lattices.

Proposition 3.5.7. *If X is a quasi-lattice, then X_+ is a proper and generating cone.*

Proof. If X_+ is not proper, there exists an $x \in X$ such that $x > 0$ and $-x > 0$. Let $z \geq \{0, x\}$ be arbitrary. Then $z - x > z \geq 0 > x$, so that $z > z - x \geq \{0, x\}$. Hence no upper bound of $\{0, x\}$ is minimal and therefore X cannot be a μ-quasi-lattice.

Moreover, every $z \in \{\lambda x : \lambda \in [-1, 1]\}$ minimizes $\sigma_{-x,x}$ on $\upsilon(\{x, -x\})$, therefore X cannot be an υ-quasi-lattice either.

For all $x \in X$, since $\upsilon(\{x, 0\})$ is non-empty, taking any $z \in \upsilon(\{x, 0\})$ and writing $x = z - (z - x)$ shows that X_+ is generating in X. □

Surprisingly, many elementary Riesz space properties have direct analogues in quasi-lattices. Many of the proofs below follow arguments from [49, Sections 5, 6] nearly verbatim.

Theorem 3.5.8. *Let X be a quasi-lattice, and $x, y, z \in X$ arbitrary. Then:*

(1) $x \tilde{\vee} x = x \tilde{\wedge} x = x$.

(2) For $\alpha \geq 0$, $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x\tilde{\vee}y)$ *and* $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x\tilde{\wedge}y)$.

(3) For $\alpha \leq 0$, $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x\tilde{\wedge}y)$ *and* $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x\tilde{\vee}y)$.

(4) $(x\tilde{\vee}y) + z = (x + z)\tilde{\vee}(y + z)$ *and* $(x\tilde{\wedge}y) + z = (x + z)\tilde{\wedge}(y + z)$.

(5) $x^{\pm} \geq 0$, $x^- = (-x)^+$.

(6) $\lceil x \rceil \geq 0$ *and, for all* $\alpha \in \mathbb{R}$, $\lceil \alpha x \rceil = |\alpha| \lceil x \rceil$. *In particular* $\lceil -x \rceil = \lceil x \rceil$.

(7) $x = x^+ - x^-$; $x^+ \tilde{\wedge} x^- = 0$ *and* $\lceil x \rceil = x^+ + x^-$.

(8) If $x \geq 0$, *then* $x\tilde{\wedge}0 = 0$ *and* $x = x^+ = \lceil x \rceil$.

(9) $\lceil \lceil x \rceil \rceil = \lceil x \rceil$.

(10) $x\tilde{\vee}y + x\tilde{\wedge}y = x + y$ *and* $x\tilde{\vee}y - x\tilde{\wedge}y = \lceil x - y \rceil$.

(11) $x\tilde{\vee}y = \frac{1}{2}(x + y) + \frac{1}{2}\lceil x - y \rceil$ *and* $x\tilde{\wedge}y = \frac{1}{2}(x + y) - \frac{1}{2}\lceil x - y \rceil$.

Proof. Assertion (1) follows from $x \leq x$ and the fact that $\sigma_{x,x}(x) = 0$ and $\sigma_{x,x}(y) > 0$ for all $y \neq x$.

We prove the assertion (2) for μ-quasi-lattices. The case $\alpha = 0$ follows from (1), hence we assume $\alpha > 0$. By definition, $x\tilde{\vee}y$ is a minimal upper bound of $\{x, y\}$. Since $\alpha > 0$, the element $\alpha(x\tilde{\vee}y)$ is then a minimal upper bound of $\{\alpha x, \alpha y\}$. Suppose that $\alpha(x\tilde{\vee}y) \neq (\alpha x)\tilde{\vee}(\alpha y)$, then there exists a minimal upper bound of $\{\alpha x, \alpha y\}$, say z_0, such that

$$\sigma_{\alpha x, \alpha y}(z_0) = \|z_0 - \alpha x\| + \|z_0 - \alpha y\| < \|\alpha(x\tilde{\vee}y) - \alpha x\| + \|\alpha(x\tilde{\vee}y) - \alpha y\|.$$

But then $\alpha^{-1}z_0$ is a minimal upper bound for $\{x, y\}$, and

$$\sigma_{x,y}(\alpha^{-1}z_0) = \|\alpha^{-1}z_0 - x\| + \|\alpha^{-1}z_0 - y\| < \|(x\tilde{\vee}y) - x\| + \|(x\tilde{\vee}y) - y\|,$$

contradicting the definition of $x\tilde{\vee}y \in \mu(\{x, y\})$ as the unique element minimizing $\sigma_{x,y}$ on $\mu(\{x, y\})$. We conclude that $(\alpha x)\tilde{\vee}(\alpha y) = \alpha(x\tilde{\vee}y)$. The same argument holds for υ-quasi-lattices by ignoring the word 'minimal' in the previous argument. By using what was just established, showing that $(\alpha x)\tilde{\wedge}(\alpha y) = \alpha(x\tilde{\wedge}y)$ holds is an elementary calculation.

The assertion (3) follows from applying (2) with $\beta := -\alpha \geq 0$.

The assertion (4) follows from the translation invariance of both the metric defined by the norm and the partial order, and (5) is immediate from the definitions.

To establish (6), we notice that $\{x, -x\} \leq \lceil x \rceil$ implies $0 \leq x - x \leq 2\lceil x \rceil$. The second part follows by noticing that $(-\alpha x)\tilde{\vee}(\alpha x) = (-|\alpha|x)\tilde{\vee}(|\alpha|x)$ and applying (2).

We prove (7). By (4), $x^+ - x = (x\tilde{\vee}0) - x = (x - x)\tilde{\vee}(-x) = 0\tilde{\vee}(-x) = x^-$, so $x = x^+ - x^-$. By this, we then have $0 = -x^- + x^- = x\tilde{\wedge}0 + x^- = (x + x^-)\tilde{\wedge}(x^-) = (x^+)\tilde{\wedge}(x^-)$. By (2) and (4), $\lceil x \rceil = (-x)\tilde{\vee}x = (-x)\tilde{\vee}x + x - x = 0\tilde{\vee}(2x) - x = 2x^+ - x^+ + x^- = x^+ + x^-$.

We prove (8). Let $x \geq 0$, then 0 is an upper bound of $\{0, -x\}$. Moreover, since the cone is proper, 0 is a minimal upper bound for $\{0, -x\}$. But, for any $z \in X$ (and in particular all (minimal) upper bounds of $\{0, -x\}$), we have

$$\sigma_{-x,0}(0) = \|0 - (-x)\| + \|0 - 0\| = \|0 - (-x)\| \leq \sigma_{-x,0}(z).$$

Hence we have $0 = (-x)\tilde{\vee}0 = x^-$, and hence, by (7), $x = x^+ = \lceil x \rceil$.

The assertion (9) follows from (6) and (8).

We prove (10). We observe that $x\tilde{\vee}y = ((x - y)\tilde{\vee}0) + y = (x - y)^+ + y$, and $x\tilde{\wedge}y = x + (0\tilde{\wedge}(y - x)) = x - (x - y)^+$. Adding these two equations yields $x\tilde{\vee}y + x\tilde{\wedge}y = x + y$, and subtracting gives $x\tilde{\vee}y - x\tilde{\wedge}y = 2(x - y)^+ + y - x = (2(x - y)\tilde{\vee}0) - (x - y) = ((x - y)\tilde{\vee}(-(x - y)) = \lceil x - y \rceil$.

The assertion (11) follows by adding and subtracting the equations established in (10). □

In a sense the more interesting results concerning quasi-lattices are ones outlining how they may differ from Riesz spaces and Banach lattices. An important remark, that may at first sight be counterintuitive, is the following: For elements x, y, z in a quasi-lattice, $x \leq z$ and $y \leq z$ does not, in general, imply that $x\tilde{\vee}y \leq z$. The following example shows how this may happen:

Example 3.5.9. We continue with Example 3.5.4. Let $x = (0, 0, 0)$ and $y = (0, 0, 2)$, then $x\tilde{\vee}y = (1, 0, 1)$. The set of minimal upper bounds of $\{x, y\}$ forms a branch of a hyperbola. Choosing z from this hyperbola such that z and $x\tilde{\vee}y$ are not comparable, say any $z = (\sqrt{t^2 + 1}, \pm t, 1)$ with $t > 0$, we see that, although $x \leq z$ and $y \leq z$, it does not hold that $x\tilde{\vee}y \leq z$.

The previous example shows how it may sometimes happen in quasi-lattices that the quasi-supremum operation is not monotone: $x \leq y$ does not necessarily imply

$x^+ \leq y^+$. We can therefore not expect distributive laws, Birkhoff type inequalities or the Riesz decomposition property to hold in general quasi-lattices.

The following example shows how a quasi-supremum operation need not even be associative:

Example 3.5.10. Let $\{\mathbb{R}^3, \|\cdot\|_2\}$ be endowed with a 'half Lorentz cone'

$$C := \{(x_1, x_2, x_3) : x_1 \geq (x_2^2 + x_3^2)^{1/2}, x_2 \geq 0\}.$$

By Corollary 3.6.2, this space is a μ-quasi-lattice.

For any $x, y \in \mathbb{R}^3$, we claim that $(x\tilde{\vee}y)_2 = \max\{x_2, y_2\}$. To this end, let $z \geq \{x, y\}$ be arbitrary and define $z' := (z_1, \max\{x_2, y_2\}, z_3)$. We first show that $z' \geq \{x, y\}$. Firstly, $z_2' - x_2 = \max\{x_2, y_2\} - x_2 \geq 0$ and $z_2' - y_2 = \max\{x_2, y_2\} - y_2 \geq 0$. Since $z_2 - x_2 \geq 0$ and $z_2 - y_2 \geq 0$, we also have $z_2 \geq z_2'$. Also, where we use the fact that $(z_2 - z_2')(z_2 - x_2) \geq 0$ and $(z_2 - z_2')^2 \geq 0$ in the last step,

$$\begin{aligned} z_1' - x_1 &= z_1 - x_1 \\ &\geq \sqrt{(z_2 - x_2)^2 + (z_3 - x_3)^2} \\ &= \sqrt{(z_2 - z_2' + z_2' - x_2)^2 + (z_3 - x_3)^2} \\ &= \sqrt{(z_2 - z_2')^2 + 2(z_2 - z_2')(z_2 - x_2) + (z_2' - x_2)^2 + (z_3' - x_3)^2} \\ &\geq \sqrt{(z_2' - x_2)^2 + (z_3' - x_3)^2}. \end{aligned}$$

Similarly $z_1' - y_1 \geq \sqrt{(z_2' - y_2)^2 + (z_3' - y_3)^2}$, so that $z' \geq \{x, y\}$. We claim that $\sigma_{x,y}(z) \geq \sigma_{x,y}(z')$. Indeed, again since $(z_2 - z_2')(z_2 - x_2) \geq 0$ and $(z_2 - z_2')^2 \geq 0$,

$$\begin{aligned} & \|z - x\|_2 \\ = & \sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2 + (z_3 - x_2)^2} \\ = & \sqrt{(z_1 - x_1)^2 + (z_2 - z_2' + z_2' - x_2)^2 + (z_3 - x_2)^2} \\ = & \sqrt{(z_1 - x_1)^2 + (z_2 - z_2')^2 + 2(z_2 - z_2')(z_2' - x_2) + (z_2' - x_2)^2 + (z_3 - x_2)^2} \\ \geq & \sqrt{(z_1 - x_1)^2 + (z_2' - x_2)^2 + (z_3 - x_2)^2} \\ = & \sqrt{(z_1' - x_1)^2 + (z_2' - x_2)^2 + (z_3' - x_2)^2} \\ = & \|z' - x\|_2. \end{aligned}$$

Similarly we have $\|z - y\|_2 \geq \|z' - y\|_2$. Therefore $\sigma_{x,y}(z) = \|z - x\|_2 + \|z - y\|_2 \geq \sigma_{x,y}(z')$. We conclude that $(x\tilde{\vee}y)_2 = \max\{x_2, y_2\}$, else, by the above construction, there would exist an upper bound of $\{x, y\}$ different from $x\tilde{\vee}y$, but which also minimizes $\sigma_{x,y}$ on $\mu(\{x, y\})$.

Now let $a := (0, 0, 0)$, $b := (0, -1, 1)$ and $c := (0 - 1, -1)$. Using what was just proven and the fact that the space is a μ-quasi-lattice, it can be seen that $a\tilde{\vee}b$ must be an element of the plane $\{x \in \mathbb{R}^3 : x_2 = 0\}$ and must be a minimal upper bound of $\{a,b\}$. The minimal upper bounds of $\{a,b\}$ that are elements of $\{x \in \mathbb{R}^3 : x_2 = 0\}$ can be parameterized by $\gamma : t \mapsto (\sqrt{1+(1-t)^2}, 0, t)$ with $t \in (-\infty, 1]$ and the function $t \mapsto \sigma_{a,b}(\gamma(t))$ attains its minimum at $t = \sqrt{3}-1$. Therefore $a \tilde{\vee} b = (2\sqrt{2-\sqrt{3}}, 0, \sqrt{3}-1)$. Again, using similar reasoning, it can be verified (using a computer algebra system) that $(a\tilde{\vee}b)\tilde{\vee}c = (\sqrt{1+(1+\kappa)^2}, 0, \kappa)$, where $\kappa := 23^{-1}\left(-29 - 8\sqrt{2} + 9\sqrt{3} + 12\sqrt{6}\right)$. Also, since $(1,-1,0)$ is the only minimal upper bound of $\{b,c\}$ that is an element of the plane $\{x \in \mathbb{R}^3 : x_2 = -1\}$, we must have $b\tilde{\vee}c = (1,-1,0)$. It can then be verified that $a\tilde{\vee}(b\tilde{\vee}c) = (2,0,0)$. We conclude that $a\tilde{\vee}(b\tilde{\vee}c) \neq (a\tilde{\vee}b)\tilde{\vee}c$.

The triangle and reverse triangle inequality take the following form in quasi-lattices. They reduce to the familiar ones in lattice-ordered μ-quasi-lattices.

Theorem 3.5.11. (*Triangle and reverse triangle inequality*) *Let X be a quasi-lattice and $x, y \in X$ be arbitrary. Then*

$$\{x+y, -(x+y)\} \leq \lceil x \rceil + \lceil y \rceil,$$

and

$$\{x - \lceil y \rceil, -x - \lceil y \rceil, y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil.$$

Proof. By Theorem 3.5.8 (7), for all $z \in X$, we have $\lceil z \rceil \geq z^{\pm} \geq \pm z$, and hence we obtain $\lceil x \rceil + \lceil y \rceil \geq x^+ + y^+ \geq x + y$ and $\lceil x \rceil + \lceil y \rceil \geq x^- + y^- \geq -x - y$. Therefore $\lceil x \rceil + \lceil y \rceil$ is an upper bound of $\{x+y, -(x+y)\}$.

To establish the second inequality, we use what was just established to see, by Theorem 3.5.8 (6) and (9), that $\{x, -x\} = \{(x \pm y) \mp y, -((x \pm y) \mp y)\} \leq \lceil x \pm y \rceil + \lceil \mp y \rceil = \lceil x \pm y \rceil + \lceil y \rceil$. Hence $\{x - \lceil y \rceil, -x - \lceil y \rceil\} \leq \lceil x \pm y \rceil$. Similarly, $\{y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil$, and finally we conclude that $\{x - \lceil y \rceil, -x - \lceil y \rceil, y - \lceil x \rceil, -y - \lceil x \rceil\} \leq \lceil x \pm y \rceil$. □

The following result allows us to conclude that monotone υ-quasi-lattices are in fact μ-quasi-lattices:

Theorem 3.5.12. *Every monotone υ-quasi-lattice is a μ-quasi-lattice, and its υ– and μ-quasi-lattice structures coincide.*

Proof. We first claim that, if X is a monotone ordered Banach space, then, for $x, y \in X$, if $z_0 \in \upsilon(\{x,y\})$ is such that $\|z - x\| + \|z - y\| > \|z_0 - x\| + \|z_0 - y\|$ for all $z \in \upsilon(\{x,y\})$ with $z \neq z_0$, then z_0 is a minimal upper bound of $\{x,y\}$.

As to this, by translating, we may assume that $y = 0$. Let $z \in X$ be any element satisfying $\{x, 0\} \leq z \leq z_0$. Then $0 \leq z \leq z_0$ and $0 \leq z - x \leq z_0 - x$. By monotonicity, $\|z\| \leq \|z_0\|$ and $\|z - x\| \leq \|z_0 - x\|$, so that $\|z\| + \|z - x\| \leq \|z_0\| + \|z_0 - x\|$. The hypothesis on z_0 then implies that $z = z_0$. Hence z_0 is a minimal upper bound of $\{x,y\}$, establishing the claim.

Let X be a monotone υ-quasi-lattice and $x, y \in X$ arbitrary. By the above claim, the υ-quasi-supremum of $\{x,y\}$ is a minimal upper bound of $\{x,y\}$. Since

$\mu(\{x,y\}) \subseteq \upsilon(\{x,y\})$ we have that the υ-quasi-supremum of $\{x,y\}$ is also the μ-quasi-supremum. We conclude that X is also a μ-quasi-lattice, and that its $\upsilon-$ and μ-quasi-lattice structures coincide. ☐

The following example shows that there exist υ-quasi-lattices in which some υ-quasi-suprema are not minimal upper bounds.

Example 3.5.13. Consider the space $\{\mathbb{R}^3, \|\cdot\|_2\}$, endowed with the cone

$$C := \{(a,b,c) \in \mathbb{R}^3 : ax^2 + bx + c \geq 0 \text{ for all } x \in [0,1]\}.$$

By Theorem 3.6.1, this space is an υ-quasi-lattice. Let $x := (0,1,0)$, $y := (0,-1,1)$. It can be verified (using a computer algebra system) that

$$x \tilde{\vee} y = (2^{-1}(2-\sqrt{3}), -2^{-1}(2-\sqrt{3}), 1),$$

while $\{x,y\} \leq (1,-1,1) < x\tilde{\vee}y$. Therefore $x\tilde{\vee}y \notin \mu(\{x,y\})$.

By comparing the norms of the elements in $0 \leq (0,-1,1) \leq (0,0,1)$, we see that this space is not monotone. We can therefore not draw any conclusion from Theorem 3.5.12 as to whether this space is a μ-quasi-lattice. A valid conclusion we may draw is that, if this space is indeed also a μ-quasi-lattice in addition to being an υ-quasi-lattice, its $\mu-$ and υ-quasi-lattice structures will *not* coincide.

3.6 A concrete class of quasi-lattices

In the previous section we have already noted that lattice ordered Banach spaces are μ-quasi-lattices (cf. Proposition 3.5.2) and gave a number of examples of quasi-lattices. We begin this section by proving that quite a large class of (not necessarily lattice ordered) ordered Banach spaces with closed generating cones are in fact quasi-lattices. Afterwards, we briefly investigate conditions under which a space has a quasi-lattice as a dual, or is the dual of a quasi-lattice.

We recall that a normed space X is *strictly convex* or *rotund* if, for $x, y \in X$, $\|x+y\| = \|x\| + \|y\|$ implies that either x or y is a non-negative multiple of the other [29, Definition 5.1.1, Proposition 5.1.11].

The following theorem shows that there exist relatively many quasi-lattices:

Theorem 3.6.1. *Every strictly convex reflexive ordered Banach space X with a closed proper generating cone is an υ-quasi-lattice.*

Proof. We need to prove that every pair of elements $x_0, y_0 \in X$ has an υ-quasi-supremum in X.

If x_0 and y_0 are comparable, by exchanging the roles of x_0 and y_0 if necessary, we may assume $x_0 \leq y_0$. We may further assume that $x_0 = 0$ by translating over $-x_0$. We will denote the distance sum to 0 and y_0 by σ instead of σ_{0,y_0}.

If $y_0 = 0$, then $\sigma(z) = 0$ if and only if $z = 0$, so that $0\tilde{\vee}0 = 0$. If $0 \neq y_0 \geq 0$, we have that $y_0 \in \upsilon(\{0, y_0\})$ and, for all $z \in \upsilon(\{0, y_0\})$, we have $\sigma(z) = \|y_0 - z\| + \|z\| \geq \|y_0\| = \sigma(y_0)$. Suppose that $z_0 \in \upsilon(\{0, y_0\})$ is such that $\sigma(y_0) = \sigma(z_0)$. We must have $z_0 \neq 0$, else $0 \leq y_0 \leq z_0 = 0$ hence, since X_+ is proper, $y_0 = 0$, while $y_0 \neq 0$. Then, since

$$\|y_0 - z_0 + z_0\| = \|y_0\| = \sigma(y_0) = \sigma(z_0) = \|y_0 - z_0\| + \|z_0\|,$$

by strict convexity we obtain $y_0 - z_0 = \lambda z_0$ for some $\lambda \geq 0$. Hence $z_0 \geq y_0 = (1+\lambda)z_0 \geq z_0$ and then, since X_+ is proper, $y_0 = z_0$. Therefore $0\tilde{\vee}y_0 = y_0$.

We consider the case where neither $x_0 \leq y_0$ nor $y_0 \leq x_0$. Again, by translating, we may assume without loss of generality that $x_0 = 0$, and that neither $y_0 \leq 0$ nor $0 \leq y_0$. We again denote the distance sum to 0 and y_0 by σ instead of σ_{0,y_0}.

Since X_+ is generating, $\upsilon(\{y_0, 0\}) = X_+ \cap (y_0 + X_+)$ is non-empty, hence let $z_0 \in X_+ \cap (y + X_+)$. Consider the non-empty closed bounded and convex set

$$K := X_+ \cap (y_0 + X_+) \cap \{x \in X : \sigma(x) \leq \sigma(z_0)\}.$$

We note that $0, y_0 \notin K$, since we had assumed that neither $y_0 \leq 0$ nor $0 \leq y_0$ holds.

The function σ is continuous and convex and, since K is bounded closed and convex and X is reflexive, by [5, Theorem 2.11], there exists an element $z_m \in K$ minimizing σ on K. We claim that z_m is the unique minimizer of σ on K. To prove this claim it is sufficient to establish that σ is strictly convex on K, i.e., if $z, z' \in K$ with $z \neq z'$ and $t \in (0,1)$, then $\sigma(tz + (1-t)z') < t\sigma(z) + (1-t)(z')$.

We first claim that the line $\mathbb{R}y_0$ does not intersect K. Indeed, if $\lambda y_0 \in K$ for some $\lambda \in \mathbb{R}$, then we must have $\lambda \neq 0$, since $0 \notin K$. But then $\lambda y_0 \in K \subseteq X_+$ implies that either $y_0 \leq 0$ or $0 \leq y_0$, contrary to our assumption that neither $y_0 \leq 0$ nor $0 \leq y_0$.

We now prove that σ is strictly convex on K. Let $z, z' \in K$ be arbitrary but distinct and $t \in (0,1)$. If $z \neq \lambda z'$ for all $\lambda \geq 0$, then, by strict convexity of X, $\|tz + (1-t)z'\| < t\|z\| + (1-t)\|z'\|$, and hence $\sigma(tz+(1-t)z') < t\sigma(z)+(1-t)\sigma(z')$. On the other hand, if $z' = \lambda z$ for some $\lambda \geq 0$, we must have that $\lambda \neq 1$ (since $z \neq z'$) and $\lambda \neq 0$ (since $0 \notin K$). Therefore, supposing that

$$\|(1-t)(y_0 - z) + t(y_0 - z')\| = (1-t)\|y_0 - z\| + t\|y_0 - z'\|,$$

by strict convexity of X, we obtain $(1-t)(y_0-z) = \rho t(y_0-z')$ for some $\rho > 0$ (if $\rho = 0$, then $y_0 = z \in K$ contradicts $y_0 \notin K$). By rewriting, we obtain $((1-t)-\rho t)y_0 = ((1-t) - \rho t\lambda)z$. If $((1-t) - \rho t\lambda) = 0$, then $((1-t) - \rho t) \neq 0$ since $\lambda \neq 1$ and $\rho t \neq 0$, and hence $y_0 = 0$, contradicting the assumption that neither $y_0 \leq 0$ nor $0 \leq y_0$. Therefore $((1-t)-\rho t\lambda) \neq 0$, and $z \in K \cap \mathbb{R}y_0$, contracting the fact that K and $\mathbb{R}y_0$ are disjoint. Therefore, we must have $\|(1-t)y_0 - (1-t)z + ty_0 - tz'\| < (1-t)\|y_0 - z\| + t\|y_0 - z'\|$, and hence $\sigma(tz + (1-t)z') < t\sigma(z) + (1-t)\sigma(z')$.

We conclude that σ is strictly convex on K, and that $z_m \in K$ is the unique minimizer of σ on K. Then clearly z_m is also the unique minimizer of σ on $\upsilon(\{0, y_0\})$. $\square$

Theorem 3.6.1 and Theorem 3.5.12 together yield the following two corollaries:

Corollary 3.6.2. *Every strictly convex reflexive monotone ordered Banach space with a closed proper generating cone is both an υ-quasi-lattice and a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).*

Corollary 3.6.3. *For $1 < p < \infty$, every L^p-space endowed with a closed proper generating cone is an υ-quasi-lattice. In particular, every ℓ^p-space and every space $\{\mathbb{R}^n, \|\cdot\|_p\}$ that is endowed with a closed proper generating cone is an υ-quasi-lattice. If, in addition, the space is monotone, it is also a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).*

Proof. That an L^p-space is strictly convex for every $1 < p < \infty$ is a consequence of [29, Theorem 5.2.11]. The result then follows from the previous theorem and Theorem 3.5.12. □

The remainder of this section will be devoted to dual considerations, specifically to the question of when the dual of a pre-ordered Banach space is a quasi-lattice. The following result gives necessary conditions for this to be the case.

Proposition 3.6.4. *If a pre-ordered Banach space X with a closed cone has a quasi-lattice as dual, then:*

(1) There exists an $\alpha > 0$ such that X is α-max-normal.

(2) $X_+ - X_+$ is dense in X.

Proof. By Proposition 3.5.7, the dual cone is proper and generating. By part (1)(e) of Theorem 3.3.7, there exists an $\alpha > 0$ such that X is α-max-normal. By [2, Theorem 2.13(2)], $X_+ - X_+$ is weakly dense in X. Since $X_+ - X_+$ is convex, its weak closure and norm closure coincide, and $X_+ - X_+$ is therefore norm dense in X. □

Corollary 3.6.5. *Let X be a pre-ordered Banach space with a closed generating cone. If X has a quasi-lattice as dual, then there exists an $\alpha > 0$ such that X is α-Ellis-Grosberg-Krein regular.*

Proof. By the previous result, there exists a $\beta > 0$ such that X is β-max-normal. The cone X_+ was assumed to be generating, and therefore X is Andô-regular. By Proposition 3.3.10 there exists an $\alpha > 0$ such that X is α-Ellis-Grosberg-Krein regular. □

The following theorem provides sufficient conditions for a pre-ordered Banach space to have a quasi-lattice as dual.

We recall that a normed space X is *smooth* if, for every $x \in X$ with $\|x\| = 1$, there exists a unique element $\phi \in X'$ with $\|\phi\| = 1$ such that $\phi(x) = 1$ [29, Definition 5.4.1, Corollary 5.4.3].

Theorem 3.6.6. *If, for some $\alpha > 0$, X is an α-normal smooth reflexive pre-ordered Banach space with a closed cone such that $X_+ - X_+$ is dense in X, then its dual is an υ-quasi-lattice.*

If, in addition, X is approximately 1-conormal, its dual is a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).

Proof. By [2, Corollary 2.14, Theorem 2.40], the dual cone is proper and generating in X'. That the dual cone is closed is elementary. By [29, Proposition 5.4.7], X' is strictly convex, since X was assumed to be smooth. Therefore X' satisfies the hypotheses of Theorem 3.6.1, and is an υ-quasi-lattice.

If we make the extra assumption that X is approximately 1-conormal, then by part (2)(d) of Theorem 3.3.7, X' is monotone. Then, by Theorem 3.5.12, X' is a μ-quasi-lattice. □

The following example shows that there exist υ-quasi-lattices that are not α-normal for any $\alpha > 0$. It cannot have a quasi-lattice as dual, nor is it the dual of a quasi-lattice. Indeed, by Proposition 3.6.4 its dual is not a quasi-lattice. Moreover, by part (2)(e) of Theorem 3.3.7, since the space is not α-normal for any $\alpha > 0$, its cone is not the dual cone of a pre-ordered Banach space with a closed generating cone, and in particular, it is not the dual of a quasi-lattice.

Example 3.6.7. Consider the following subset of ℓ^2:

$$C := \left\{ x \in \ell^2 : x_1 \geq \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2 \right)^{1/2} \right\}.$$

Clearly, $C \cap (-C) = \{0\}$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. Also, by Minkowski's inequality, $C + C \subseteq C$ so that we may conclude that C is a proper cone. For any $x \in \ell^2$, taking $y := \lambda(1, 0, 0, \ldots) \in C$ with $\lambda \geq |x_1| + \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2\right)^{1/2}$, we see $x = y - (y - x) \in C - C$, so that C is generating in ℓ^2. Since the map $\rho_0 : \ell^2 \to \mathbb{R}$ defined by $\rho_0 : x \mapsto \left(\sum_{m=2}^{\infty} \frac{1}{m} x_m^2\right)^{1/2}$ is a continuous seminorm, the map $\rho : x \mapsto x_1 - \rho_0(x)$ is also continuous. Since $C = \rho^{-1}(\mathbb{R}_{\geq 0})$, we conclude that C is closed. By Theorem 3.6.1, this space is an υ-quasi-lattice.

We claim that this space is not α-normal for any $\alpha > 0$. It is sufficient to show, for every $\alpha \geq 1$, that there exist $x, y \in \ell^2$ with $0 \leq x \leq y$, such that $\|x\| > \alpha\|y\|$. To this end, we set $y := (2, 0, \ldots)$. We define x as follows: let $\mathbb{N} \ni n_\alpha > (2\alpha)^2$ and $x = (1, 0, \ldots, 0, \sqrt{n_\alpha}, 0, \ldots)$ with $\sqrt{n_\alpha}$ occurring at the n_α-th coordinate. We then see that $0 \leq x \leq y$, while

$$\|x\| = (1 + n_\alpha)^{\frac{1}{2}} > n_\alpha^{\frac{1}{2}} > 2\alpha = \alpha\|y\|.$$

We conclude that this space is not α-normal for any $\alpha > 0$.

3.7 A class of quasi-lattices with absolutely monotone spaces of operators

In this section we show that a real Hilbert space $\mathcal{H}$ endowed with a Lorentz cone (defined below) is a 1-absolutely Davies-Ng regular μ-quasi-lattice (that is not a Banach lattice if $\dim \mathcal{H} \geq 3$). Through an application of Theorem 3.4.2, this will resolve the question posed in the introduction of whether there exist non-Banach lattice pre-ordered Banach spaces X and Y for which $B(X, Y)$ is absolutely monotone.

Results established in this section will be collected in Theorem 3.7.10. In particular it will be shown that $\|x\| = \| \lceil x \rceil \|$ for all $x \in \mathcal{H}$ (which is analogous to the identity $\|x\| = \||x|\|$ which holds for all elements x of a Banach lattice). Then, for $\alpha > 0$ and pre-ordered Banach spaces X and Y that are respectively approximately α-absolutely conormal and α-absolutely normal, the spaces of operators $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are shown to be α-absolutely normal. Furthermore, if $\alpha = 1$ (in particular if X and Y are Hilbert spaces endowed with Lorentz cones), then the operator norms of $B(X, \mathcal{H})$ and $B(\mathcal{H}, Y)$ are positively attained.

We begin with the following lemma which outlines sufficient conditions for establishing 1-absolute Davies-Ng regularity of absolutely monotone quasi-lattices:

Lemma 3.7.1. *Let X be an absolutely monotone quasi-lattice satisfying $\|x\| = \| \lceil x \rceil \|$ for all $x \in X$. Then X is 1-absolutely Davies-Ng regular.*

Proof. The fact that $\|x\| = \| \lceil x \rceil \|$ for all $x \in X$ implies that X is 1-absolutely conormal. Therefore X is 1-absolutely Davies-Ng regular. □

Every Banach lattice satisfies the hypotheses of the previous proposition. The rest of this section will be devoted to proving that there exist quasi-lattices that are not Banach lattices, but still satisfy the hypothesis of the previous proposition.

Definition 3.7.2. Let $\mathcal{H}$ be a real Hilbert space. For a norm-one element $v \in \mathcal{H}$, let P be the orthogonal projection onto $\{v\}^{\perp}$. We define the *Lorentz cone*

$$\mathcal{L}_v := \{x \in \mathcal{H} : \langle v|x \rangle \geq \|Px\|\}.$$

As in Example 3.6.7, it is elementary to see that this cone is closed, proper and generating in $\mathcal{H}$.

It is widely known that the Hilbert space $\mathbb{R}^3$ ordered by the Lorentz cone $\mathcal{L}_{e_1} \subseteq \mathbb{R}^3$ is not a Riesz space (cf. Example 3.5.4). This is actually true for arbitrary Hilbert spaces endowed with a Lorentz cone as we will now proceed to show. The following two lemmas will be used in the proof of Proposition 3.7.5 which establishes this fact.

Lemma 3.7.3. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. If $x \in \mathcal{L}_v$ is such that $\langle x|v \rangle = \|Px\|$ and $z_1, z_2 \in \mathcal{L}_v$ are such that $x = z_1 + z_2$, then $z_1, z_2 \in \{\lambda x : \lambda \geq 0\}$.*

Proof. Let P be the orthogonal projection onto $\{v\}^{\perp}$. If $x = 0$, since $\mathcal{L}_v$ is proper, the statement is clear. Let $0 \neq x \in \mathcal{L}_v$ be such that $\langle x|v \rangle = \|Px\|$. Then $\langle x|v \rangle = \|Px\| > 0$, else $x = 0$. Suppose $z_1, z_2 \in \mathcal{L}_v$ are such that $x = z_1 + z_2$. Then

$$\langle x|v \rangle = \langle z_1 + z_2|v \rangle \geq \|Pz_1\| + \|Pz_2\| \geq \|P(z_1 + z_2)\| = \|Px\| = \langle x|v \rangle.$$

Therefore $\|Pz_1\| + \|Pz_2\| = \|Pz_1 + Pz_2\| = \|Px\| > 0$, and hence Pz_1 and Pz_2 cannot both be zero. We assume $Pz_1 \neq 0$, and then, by strict convexity of $\mathcal{H}$, $Pz_2 = \lambda Pz_1$ for some $\lambda \geq 0$. If $\langle z_1|v \rangle > \|Pz_1\|$ or $\langle z_2|v \rangle > \|Pz_2\|$, then $\langle x|v \rangle = \langle z_1|v \rangle + \langle z_2|v \rangle > \|Pz_1\| + \|Pz_2\| = \|Px\|$, contradicting the assumption that $\langle x|v \rangle = \|Px\|$. Hence, since $z_1, z_2 \in \mathcal{L}_v$, we must have $\langle z_1|v \rangle = \|Pz_1\|$ and $\langle z_2|v \rangle = \|Pz_2\|$, and therefore $\langle z_2|v \rangle = \|Pz_2\| = \lambda \|Pz_1\| = \lambda \langle z_1|v \rangle$. Now, since $\langle z_2|v \rangle = \lambda \langle z_1|v \rangle$ and $Pz_2 = \lambda Pz_1$, we obtain $z_2 = \langle z_2|v\rangle v + Pz_2 = \lambda z_1$, and hence $x = z_1 + z_2 = (1+\lambda)z_1$. We conclude that $z_1, z_2 \in \{\lambda x : \lambda \geq 0\}$. □

Lemma 3.7.4. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. If $x \in \mathcal{L}_v$ is such that $\langle x|v\rangle = \|Px\|$ and $0 \leq y \leq x$, then $y \in \{\lambda x : \lambda \in [0,1]\}$.*

Proof. Since $\mathcal{L}_v$ is proper, this is clear if $x = 0$. If $0 \leq y \leq x \neq 0$, then $x = y + (x-y)$ with $y, (x-y) \in \mathcal{L}_v$, so that by the previous lemma $y = \lambda x$ for some $\lambda \geq 0$. If $\lambda > 1$, then $x \leq \lambda x = y \leq x$, since $\mathcal{L}_v$ is proper and $y = \lambda x$, implies $x = y = 0$ contradicting the assumption $x \neq 0$. We conclude that $\lambda \in [0,1]$. □

Proposition 3.7.5. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ such that $\|v\| = 1$. If $\dim(\mathcal{H}) \geq 3$, then $\mathcal{H}$ is not a Riesz space (and hence not a Banach lattice).*

Proof. Let P be the orthogonal projection onto $\{v\}^\perp$ and $\{v, e_1, e_2\} \subseteq \mathcal{H}$ be any orthonormal set. For $t \in \mathbb{R}$, we have

$$\{0, 2e_1\} \leq e_1 + te_2 + \sqrt{t^2+1}v =: z_t.$$

We claim that each z_t is a minimal upper bound of $\{0, 2e_1\}$. We have $\langle z_t|v\rangle = \|Pz_t\|$, and hence by the previous lemma, if $\{0, 2e_1\} \leq y \leq z_t$, we must have $y = \lambda z_t$ for some $\lambda \in [0,1]$. If $\lambda < 1$, then $\lambda\sqrt{t^2+1} = \langle y - 2e_1|v\rangle$ and

$$\begin{aligned} \|P(y-2e_1)\|^2 &= \|P(\lambda e_1 + \lambda te_2 + \lambda\sqrt{t^2+1}v - 2e_1)\|^2 \\ &= \|(\lambda-2)e_1 + \lambda te_2\|^2 \\ &= (\lambda-2)^2 + \lambda^2 t^2 \\ &> 1 + \lambda^2 t^2 \\ &> \lambda^2 + \lambda^2 t^2. \end{aligned}$$

Hence $\langle y - 2e_1|v\rangle = \lambda\sqrt{t^2+1} < \|P(y-2e_1)\|$, contradicting $2e_1 \leq y$. Therefore we must have $\lambda = 1$, and $y = z_t$, and hence z_t is a minimal upper bound of $\{0, 2e_1\}$ for every $t \in \mathbb{R}$. Clearly all z_t are distinct, and therefore there exists no supremum of $\{0, 2e_1\}$. □

Since every Hilbert space is strictly convex, and knowing that Lorentz cones are closed proper and generating, we conclude from Theorem 3.6.1 that every Hilbert space endowed with a Lorentz cone is an v-quasi-lattice. We will now proceed to show that these spaces are absolutely monotone. Once this has been established, Theorem 3.5.12 will imply that they are in fact μ-quasi-lattices.

The following lemma will be applied in Propositions 3.7.7 and 3.7.9, which together will show that Hilbert spaces endowed with a Lorentz cones are in fact 1-absolutely Davies-Ng regular.

Lemma 3.7.6. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v$ where $v \in \mathcal{H}$ is such that $\|v\| = 1$. Let $x \in \mathcal{H}$ and Q be the orthogonal projection onto* $\operatorname{span}\{x, v\}$*. If $\{-x, x\} \leq y$, then $\{-x, x\} \leq Qy$.*

Proof. Let P be the orthogonal projection onto $\{v\}^\perp$ and Q the orthogonal projection onto $\operatorname{span}\{x, v\}$. Let $Q^\perp := \mathrm{id} - Q$. We note that $\operatorname{ran}(\mathrm{Id} - P) = \operatorname{span}\{v\} \subseteq \operatorname{ran}(Q)$, so that $\mathrm{Id}-P$ and Q commute, and hence P and Q also commute. Therefore, from

$$\begin{aligned}
\langle v|Qy \pm x\rangle &= \langle v|Qy + Q^\perp y - Q^\perp y \pm x\rangle \\
&= \langle v|y \pm x\rangle - \langle v|Q^\perp y\rangle \\
&= \langle v|y \pm x\rangle \\
&\geq \|P(y \pm x)\| \\
&\geq \|QP(y \pm x)\| \\
&= \|P(Qy \pm Qx)\| \\
&= \|P(Qy \pm x)\|,
\end{aligned}$$

we conclude that $Qy \geq \{-x, x\}$. □

The following proposition, together with Theorem 3.5.12, will show that every Hilbert space endowed with a Lorentz cone is in fact a μ-quasi-lattice.

Proposition 3.7.7. *A real Hilbert space endowed with a Lorentz cone is absolutely monotone.*

Proof. Let $\mathcal{H}$ be a real Hilbert space ordered by a Lorentz cone $\mathcal{L}_v$, where $v \in \mathcal{H}$ is a norm-one element. Let P be the orthogonal projection onto $\{v\}^\perp$. Let $\{-x, x\} \leq y$ and let Q denote the orthogonal projection onto $V := \operatorname{span}\{x, v\}$. By Lemma 3.7.6, $\{-x, x\} \leq Qy$.

If $x \in \operatorname{span}\{v\}$, then $Px = 0$. Also $V = \operatorname{span}\{v\}$, so that $PQ = 0$. Therefore $\{-x, x\} \leq Qy$ implies $\langle v|Qy \pm x\rangle \geq \|PQ(y \pm x)\| = \|Px\| = 0$, and hence $\langle v|Qy\rangle \geq |\langle v|x\rangle|$. Then $\|Qy\| = |\langle v|Qy\rangle| \geq \langle v|Qy\rangle \geq |\langle v|x\rangle| = \|x\|$, and hence $\|x\| \leq \|Qy\| \leq \|y\|$ as was to be shown.

If $x \notin \operatorname{span}\{v\}$, since $0 \neq Px = x - \langle v|x\rangle v \in V$, we see that

$$e_\pm := (\sqrt{2}\|Px\|)^{-1}(\pm Px + \|Px\|v)$$

are orthonormal elements of $V \cap \mathcal{L}_v$. We claim that $V \cap \mathcal{L}_v = \{\lambda e_+ + \lambda' e_- : \lambda, \lambda' \geq 0\}$. Let $a \in V \cap \mathcal{L}_v$. Since $x \notin \operatorname{span}\{v\}$, $0 \neq Px \in V$ is orthogonal to v, and hence $\{Px, v\}$ is a basis of V. Then, by writing $a = \alpha Px + \beta v$ for some $\alpha, \beta \in \mathbb{R}$, we obtain $\beta = \langle a|v\rangle \geq \|Pa\| = |\alpha|\|Px\|$. Hence, by

$$\begin{aligned}
\langle a|e_\pm\rangle &= (\sqrt{2}\|Px\|)^{-1}\langle \alpha Px + \beta v| \pm Px + \|Px\|v\rangle \\
&= (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\langle Px|Px\rangle + \beta\|Px\|) \\
&= (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\|Px\|^2 + \beta\|Px\|)
\end{aligned}$$

$$\begin{aligned} &\geq (\sqrt{2}\|Px\|)^{-1}(\pm\alpha\|Px\|^2 + |\alpha|\|Px\|^2) \\ &\geq 0, \end{aligned}$$

we conclude that $V \cap \mathcal{L}_v = \{\lambda e_+ + \lambda' e_- : \lambda, \lambda' \geq 0\}$. Now $Qy \pm x \in V \cap \mathcal{L}_v$ implies $\langle Qy \pm x | e_\pm \rangle \geq 0$, so that $\langle Qy | e_\pm \rangle \geq |\langle x | e_\pm \rangle|$, and hence $\|x\| \leq \|Qy\| \leq \|y\|$ as was to be shown. ☐

Remark 3.7.8. If $x \notin \mathrm{span}\{v\}$ we note that $(V, V \cap \mathcal{L}_v)$ in the previous proposition is isometrically order isomorphic to the Banach lattice $\{\mathbb{R}^2, \|\cdot\|_2\}$ with the standard cone through mapping $e_+ \in V$ and $e_- \in V$ to $(1,0) =: e_1 \in \mathbb{R}^2$ and $(0,1) =: e_2 \in \mathbb{R}^2$ respectively.

We can now show that real Hilbert spaces endowed with Lorentz cones satisfy the hypotheses of Lemma 3.7.1:

Proposition 3.7.9. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone. Then $\|x\| = \|\lceil x \rceil\|$ for all $x \in \mathcal{H}$. Hence $\mathcal{H}$ is 1-absolutely conormal.*

Proof. Let $v \in \mathcal{H}$ be a norm one element and order $\mathcal{H}$ with the Lorentz cone $\mathcal{L}_v$. We again denote the projection onto $\{v\}^\perp$ by P. Let $x \in \mathcal{H}$ be arbitrary.

If $x \geq 0$ or $x \leq 0$, then, by Theorem 3.5.8 (6) and (8), $\lceil x \rceil = x$ or $\lceil x \rceil = -x$, respectively, so that $\|x\| = \|\lceil x \rceil\|$.

It remains to show that $\|x\| = \|\lceil x \rceil\|$ when neither $x \geq 0$ nor $x \leq 0$. Then $x \notin \mathrm{span}\{v\}$. We define the two dimensional subspace $V := \mathrm{span}\{x, v\}$, denote the orthogonal projection onto V by Q, and define $Q^\perp := \mathrm{Id} - Q$. By Lemma 3.7.6, if $\{-x, x\} \leq w$, then $\{-x, x\} \leq Qw$.

When $w \notin V$, we see that $Q^\perp w \neq 0$ implies

$$\begin{aligned} &\|w - x\| + \|w + x\| \\ = \;& \sqrt{\|Q(w-x)\|^2 + \|Q^\perp(w-x)\|^2} + \sqrt{\|Q(w+x)\|^2 + \|Q^\perp(w+x)\|^2} \\ = \;& \sqrt{\|Qw - x\|^2 + \|Q^\perp w\|^2} + \sqrt{\|Qw + x\|^2 + \|Q^\perp w\|^2} \\ > \;& \|Qw - x\| + \|Qw + x\|. \end{aligned}$$

We conclude that $\lceil x \rceil$ must be an element of V. Furthermore, by Proposition 3.7.7 and Theorem 3.5.12, $\mathcal{H}$ is a μ-quasi-lattice, and hence $\lceil x \rceil \in V$ is a minimal upper bound of $\{-x, x\}$.

Finally, V endowed with the cone $\mathcal{L}_v \cap V$ is seen to be isometrically order isomorphic to $\{\mathbb{R}^2, \|\cdot\|_2\}$ with the standard cone (cf. Remark 3.7.8). Viewing V as a Banach lattice, we notice that the Banach lattice absolute value $|x|$ in V is the only minimal upper bound for $\{-x, x\}$ in $\mathcal{H}$ that is also an element of V. By the argument in the previous paragraph, we conclude that $\lceil x \rceil = |x|$, and hence that $\|\lceil x \rceil\| = \||x|\| = \|x\|$, by invoking the Banach lattice property $\||x|\| = \|x\|$ in V. ☐

We collect the results established in this section and some of their consequences in the following theorem. We note that (8) below resolves the question alluded to in the introduction of the existence pre-ordered Banach spaces X and Y, which are not Banach lattices, while $B(X, Y)$ is absolutely monotone.

Theorem 3.7.10. *Let $\mathcal{H}$ be a real Hilbert space endowed with a Lorentz cone. Then:*

(1) If $\dim(\mathcal{H}) \geq 3$, then $\mathcal{H}$ is not a Riesz space (and hence not a Banach lattice).

(2) $\mathcal{H}$ is an υ-quasi-lattice.

(3) $\mathcal{H}$ is absolutely monotone.

(4) $\mathcal{H}$ is a μ-quasi-lattice (and its $\upsilon-$ and μ-quasi-lattice structures coincide).

(5) For every $x \in \mathcal{H}$, $\|x\| = \| \lceil x \rceil \|$. Hence $\mathcal{H}$ is 1-absolutely conormal.

(6) $\mathcal{H}$ is 1-absolutely Davies-Ng regular.

(7) If X and Y are pre-ordered Banach spaces with closed cones, with X approximately 1-absolutely conormal and Y absolutely monotone, then the operator norms of both $B(X,\mathcal{H})$ and $B(\mathcal{H},Y)$ are positively attained, i.e., $\|T\| = \sup\{\|Tx\| : x \geq 0,\ \|x\| = 1\}$ for $T \in B(X,\mathcal{H})_+$ or $T \in B(\mathcal{H},Y)_+$. In particular, if $\mathcal{H}_1$ is another real Hilbert space endowed with a Lorentz cone, then the operator norm of $B(\mathcal{H},\mathcal{H}_1)$ is positively attained.

(8) If $\alpha > 0$ and X and Y are pre-ordered Banach spaces with closed cones, with X approximately α-absolutely conormal and Y α-absolutely normal, then both $B(X,\mathcal{H})$ and $B(\mathcal{H},Y)$ are α-absolutely normal. In particular, if $\mathcal{H}_1$ is another real Hilbert space endowed with a Lorentz cone, then $B(\mathcal{H},\mathcal{H}_1)$ is absolutely monotone.

Proof. The assertion (1) is Proposition 3.7.5. The assertion (2) follows from Theorem 3.6.1. The assertion (3) was established in Proposition 3.7.7 and hence (4) follows from Corollary 3.6.2. The assertion (5) was established in Proposition 3.7.9, and hence (6) follows from Lemma 3.7.1. The assertion (7) follows from (6) and part (2) of Theorem 3.4.6. The assertion (8) is then immediate from (6) and part (3) of Theorem 3.4.2. □

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Chapter 4

Crossed products of Banach algebras

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4.1 Introduction and overview

This paper is an analytical continuation of [19] where, motivated by the theory of crossed products of C^*-algebras and its relevance for the theory of unitary group representations, a start was made with the theory of crossed products of Banach algebras. General Banach algebras lack the convenient rigidity of C^*-algebras where, e.g., morphisms are automatically continuous and even contractive, and this makes the task of developing the basics more laborious than it is for crossed products of C^*-algebras. Apart from some first applications, including the usual description of the non-degenerate (involutive) representations of the crossed product associated with a C^*-dynamical system (cf. [19, Theorem 9.3]), [19] is basically concerned with one theorem, the General Correspondence Theorem [19, Theorem 8.1], most of which is formulated as Theorem 4.2.1 below. If $\mathcal{R}$ is a non-empty class of non-degenerate continuous covariant representations of a Banach algebra dynamical system (A, G, α) – all notions will be reviewed in Section 4.2 – then Theorem 4.2.1 gives a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$, provided that A has a bounded approximate left identity. In the current paper, the basic theory is developed further and, in addition, a substantial part is concerned with generalized Beurling algebras $L^1(G, A, \omega; \alpha)$ and their representations. These are weighted Banach spaces of (equivalence classes) of A-valued functions that are also associative algebras with a multiplication that is continuous in both variables, but they are not Banach algebras in general, since the norm need not be submultiplicative. If A equals the scalars, they reduce to the ordinary Beurling algebras $L^1(G,\omega)$ (which are true Banach algebras) for a not necessarily abelian group G. We will describe the non-degenerate bounded representations of generalized Beurling algebras as a consequence of the General Correspondence Theorem, which is thus seen to be a common underlying principle for (at least) both crossed products of C^*-algebras and generalized Beurling algebras.

We will now briefly describe the contents of the paper.

In Section 4.2 we review the relevant definitions and results of [19]. In Section 4.3 it is investigated how the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ depends on $\mathcal{R}$, and it is also shown that there exists an isometric representation of this algebra on a Banach space. The latter result is used in Section 4.4. Loosely speaking, $(A \rtimes_\alpha G)^{\mathcal{R}}$ "generates" all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), and under two mild additional hypotheses it is shown to be the unique such algebra, up to isomorphism (cf. Theorem 4.4.4). This result parallels work of Raeburn's [38]. It is also shown (cf. Proposition 4.4.3) that the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a topological embedding into its left centralizer algebra $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. Since $(A \rtimes_\alpha G)^{\mathcal{R}}$ need not have a bounded approximate right identity, this is not automatic.

Next, in Section 4.5 the generalized Beurling algebras $L^1(A, G, \omega; \alpha)$ make their appearance. These algebras can be defined for any Banach algebra dynamical system (A, G, α) and weight ω on G, provided that α is uniformly bounded. If A has a bounded approximate right identity, then it can be shown that $L^1(A, G, \omega; \alpha)$ is isomorphic to $(A \rtimes_\alpha G)^{\mathcal{R}}$, for a suitably chosen class $\mathcal{R}$ (cf. Theorem 4.5.17). Via this isomorphism the General Correspondence Theorem therefore predicts, if A has a bounded two-sided approximate identity, what the non-degenerate bounded representations of $L^1(A, G, \omega; \alpha)$ are, in terms of the non-degenerate continuous covariant representations of (A, G, α) (cf. Theorem 4.5.20), and some classical results are thus seen to be obtainable from the General Correspondence Theorem. As the easiest example, we retrieve the usual description of the non-degenerate left $L^1(G)$-modules in terms of the uniformly bounded strongly continuous representations of G. Naturally, there is a similar description of the non-degenerate right $L^1(G)$-modules, but an intermediate procedure is needed to obtain such a result from the General Correspondence Theorem, where one always ends up with left modules over the crossed product. This is taken up in Section 4.6, where we investigate all "reasonable" variations on the theme that $\pi : A \to B(X)$ and $U : G \to B(X)$ should be multiplicative, and that $U_r \pi(a) U_r^{-1} = \pi(\alpha_r(a))$ should hold for all $a \in A$ and $r \in G$. We argue that there are only three more "reasonable" requirements (cf. Table 4.1). One of these is, e.g., that π and U are anti-multiplicative and that $U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for all $a \in A$ and $r \in G$; for $A = \mathbb{K}$ and $\alpha = \text{triv}$ this covers the case of right G-modules. Moreover, we show that a pair (π, U) of each of the other three types can be reinterpreted as a covariant representation in the usual sense for a suitable "companion" Banach algebra dynamical system. The example (π, U) given above, where there are three "flaws" in the properties of (π, U), is a covariant representation for the opposite Banach algebra dynamical system (A^o, G^o, α^o). Therefore, if one seeks a Banach algebra of which the non-degenerate bounded (multiplicative) representations "encode" a family of such pairs (π, U), then a crossed product of type

$(A \rtimes_\alpha G)^{\mathcal{R}}$ is not what one should look at, but $(A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$ is to be considered.

Section 4.7 shows, as a particular case of Theorem 4.7.5, how the encoding for various types can be collected in one Banach algebra. For example, the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}} \hat{\otimes} (A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$ correspond to commuting non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}^o}$. These representations can then be respectively related to a usual covariant representation of (A, G, α) and a thrice "flawed" pair (π, U) as above, which again commute.

In the final Section 4.8 we combine the results from Sections 4.5, 4.6 and 4.7. Using the procedure from Section 4.6 and the results from Section 4.5, the relation between thrice "flawed" pairs (π, U) as above and the non-degenerate bounded representations of $L^1(G^o, A^o, \omega^o; \alpha^o)$ is easily established. Since coincidentally the generalized Beurling algebra $L^1(G^o, A^o, \omega^o; \alpha^o)$ turns out to be anti-isomorphic to $L^1(A, G, \omega; \alpha)$, these pairs (π, U) can then also be related to the non-degenerate right $L^1(A, G, \omega; \alpha)$-modules (cf. Theorem 4.8.3). It is then easy to describe the simultaneous left $L^1(A, G, \omega; \alpha)$– and right $L^1(B, H, \eta; \beta)$-modules, where the actions commute (cf. Theorem 4.8.4). In particular this describes the bimodules over a generalized Beurling algebra $L^1(A, G, \omega; \alpha)$. Specializing to the case where A equals the scalars yields a description of the non-degenerate bimodules over an ordinary Beurling algebra $L^1(G, \omega)$ in terms of G-bimodules. Specializing still further to $\omega = 1$ the classical description of the non-degenerate $L^1(G)$-bimodules in terms of a uniformly bounded G-bimodule is retrieved as the simplest case in the general picture.

4.2 Preliminaries and recapitulation

For the sake of self-containment we provide a brief recapitulation of definitions and results from earlier papers [18, 19].

Throughout this paper X and Y will denote Banach spaces. The algebra of bounded linear operators on X will be denoted by $B(X)$. By A and B we will denote Banach algebras, not necessarily unital, and by G and H locally compact groups (which are always assumed to be Hausdorff). We will always use the same symbol λ to denote the left regular representation of various Banach algebras instead of distinguishing between them, as the context will always make precise what is meant. If A is a Banach algebra, X a Banach space, and $\pi : A \to B(X)$ is a Banach algebra representation, when confusion could arise, we will write X_π instead of X to make clear that the Banach space X is related to the representation π. We do not assume that Banach algebra representations of unital Banach algebras are unital. Representations of algebras and groups are always multiplicative (so that we are considering left modules), unless explicitly stated otherwise.

Let A be a Banach algebra, G a locally compact Hausdorff group and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous representation of G on A. Then the triple (A, G, α) is called a *Banach algebra dynamical system.*

Let (A, G, α) be a Banach algebra dynamical system, X a Banach space with $\pi : A \to B(X)$ and $U : G \to B(X)$ representations of the algebra A and group G on

X respectively. If (π, U) satisfies

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^{-1},$$

for all $a \in A$ and $s \in G$, the pair (π, U) is called a *covariant representation* of (A, G, α) on X. The pair (π, U) is said to be *continuous* if π is norm-bounded and U is strongly continuous. The pair (π, U) is called *non-degenerate* if π is non-degenerate (i.e., the span of $\pi(A)X$ lies dense in X).

Integrals of compactly supported continuous Banach space valued functions are, as in [19], defined by duality, following [40, Section 3]. Let $C_c(G, A)$ denote the space of all continuous compactly supported A-valued functions. For any $f, g \in C_c(G, A)$ and $s \in G$ defining the twisted convolution

$$[f * g](s) := \int_G f(r) \alpha_r(g(r^{-1}s))\, dr$$

gives $C_c(G, A)$ the structure of an associative algebra, where integration is with respect to a fixed left Haar measure on G.

If (π, U) is a continuous covariant representation of (A, G, α) on X, then, for $f \in C_c(G, A)$, we define $\pi \rtimes U(f) \in B(X)$, as in [19, Section 3], by

$$\pi \rtimes U(f)x := \int_G \pi(f(s)) U_s x\, ds \quad (x \in X).$$

The map $\pi \rtimes U : C_c(G, A) \to B(X)$ is a representation of the algebra $C_c(G, A)$ on X, and is called the *integrated form* of (π, U).

Let $\mathcal{R}$ be a class of covariant representations of (A, G, α). Then $\mathcal{R}$ is called a *uniformly bounded class of continuous covariant representations* if there exist a constant $C \geq 0$ and function $\nu : G \to [0, \infty)$ which is bounded on compact sets, such that, for any $(\pi, U) \in \mathcal{R}$, we have that $\|\pi\| \leq C$ and $\|U_r\| \leq \nu(r)$ for all $r \in G$. We will always tacitly assume that such a class $\mathcal{R}$ is non-empty. With $\mathcal{R}$ as such, it follows that $\|\pi \rtimes U(f)\| \leq C \left(\sup_{r \in \mathrm{supp}(f)} \nu(r)\right) \|f\|_1$ for all $(\pi, U) \in \mathcal{R}$ and $f \in C_c(G, A)$ [19, Remark 3.3].

We define the algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$ by

$$\sigma^{\mathcal{R}}(f) := \sup_{(\pi, U) \in \mathcal{R}} \|\pi \rtimes U(f)\| \quad (f \in C_c(G, A)),$$

and denote the completion of the quotient $C_c(G, A)/\ker \sigma^{\mathcal{R}}$ by $(A \rtimes_\alpha G)^{\mathcal{R}}$, with $\|\cdot\|^{\mathcal{R}}$ denoting the norm induced by $\sigma^{\mathcal{R}}$. The Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ is called the *crossed product* corresponding to (A, G, α) and $\mathcal{R}$. The quotient homomorphism is denoted by $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$.

A covariant representation of (A, G, α) is called *$\mathcal{R}$-continuous* if it is continuous and its integrated form is bounded with respect to the seminorm $\sigma^{\mathcal{R}}$. For any Banach space X and linear map $T : C_c(G, A) \to X$, if T is bounded with respect to the $\sigma^{\mathcal{R}}$ seminorm, we will denote the canonically induced linear map on $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $T^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to X$, as detailed in [19, Section 3].

If A has a bounded approximate left (right) identity, then it can be shown that $(A \rtimes_\alpha G)^{\mathcal{R}}$ also has a bounded approximate left (right) identity, with estimates for its bound, [19, Theorem 4.4 and Corollary 4.6].

We will denote the left centralizer algebra of a Banach algebra B by $\mathcal{M}_l(B)$. Assuming B has a bounded approximate left identity (u_i), any non-degenerate bounded representation $T : B \to B(X)$ induces a non-degenerate bounded representation $\overline{T} : \mathcal{M}_l(B) \to B(X)$, by defining $\overline{T}(L) := \text{SOT-lim}_i T(Lu_i)$ for all $L \in \mathcal{M}_l(B)$, so that the following diagram commutes (cf. [18, Theorem 4.1]):

$$\begin{array}{ccc} B & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

Moreover, $\overline{T}(L)T(a) = T(La)$ for all $a \in B$ and $L \in \mathcal{M}_l(B)$. We will often use this fact.

With (A, G, α) a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations, we define the homomorphisms $i_A : A \to \text{End}(C_c(G, A))$ and $i_G : G \to \text{End}(C_c(G, A))$ by

$$\begin{aligned} (i_A(a)f)(s) &:= af(s), \\ (i_G(r)f)(s) &:= \alpha_r(f(r^{-1}s)), \end{aligned}$$

for all $a \in A$, $f \in C_c(G, A)$ and $r, s \in G$. For each $a \in A$ and $r \in G$, the maps

$$i_A(a), i_G(r) : (C_c(G, A), \sigma^{\mathcal{R}}) \to (C_c(G, A), \sigma^{\mathcal{R}})$$

are bounded [19, Lemma 6.3], and

$$\begin{aligned} \|i_A(a)\|^{\mathcal{R}} &\leq \sup_{(\pi,U)\in\mathcal{R}} \|\pi(a)\|, \\ \|i_G(r)\|^{\mathcal{R}} &\leq \sup_{(\pi,U)\in\mathcal{R}} \|U_r\|. \end{aligned}$$

Defining $i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f) := q^{\mathcal{R}}(i_A(a)f)$ and $i_G^{\mathcal{R}}(r)q^{\mathcal{R}}(f) := q^{\mathcal{R}}(i_G(r)f)$ for all $a \in A$, $r \in G$ and $\in C_c(G, A)$, we obtain bounded maps

$$i_A^{\mathcal{R}}(a), i_G^{\mathcal{R}}(r) : (A \rtimes_\alpha G)^{\mathcal{R}} \to (A \rtimes_\alpha G)^{\mathcal{R}}.$$

Moreover, the maps $a \mapsto i_A^{\mathcal{R}}(a)$ and $r \mapsto i_G^{\mathcal{R}}(r)$ map A and G into $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. If A has a bounded approximate left identity and $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations, then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Section 6] and the integrated form $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$ equals the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Theorem 7.2].

The main theorem from [19] establishes, amongst others, a bijective relationship between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, by letting (π, U) correspond to $(\pi \rtimes U)^{\mathcal{R}}$. This result will play a fundamental role throughout the rest of this paper, and the relevant part of [19, Theorem 8.1] can be stated as follows:

Theorem 4.2.1. (*General Correspondence Theorem, cf. [19, Theorem 8.1]*) *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α). Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

More precisely:

(1) If (π, U) is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), then $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ is the representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as described above, cf. [19, Section 7].

(2) If T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$, then the pair $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T.$$

4.3 Varying $\mathcal{R}$

For a given Banach algebra dynamical system (A, G, α), one may ask what relationship exists between the crossed products $(A \rtimes_\alpha G)^{\mathcal{R}_1}$ and $(A \rtimes_\alpha G)^{\mathcal{R}_2}$ for two uniformly bounded classes $\mathcal{R}_1$ and $\mathcal{R}_2$ of possibly degenerate continuous covariant representations on Banach spaces. This section investigates this question.

Since uniformly bounded classes of covariant representations might be proper classes, we must take some care in working with them. Nevertheless, we can always choose a set from a uniformly bounded class $\mathcal{R}$ of covariant representations of a Banach algebra dynamical system (A, G, α) so that this set determines $\sigma^{\mathcal{R}}$. Indeed for every $f \in C_c(G, A)$, looking at the subset $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ of $\mathbb{R}$ (subclasses of sets are sets), we may choose a sequence from $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ converging to $\sigma^{\mathcal{R}}(f)$ and regard only those corresponding covariant representations. In this way, we can chose a set S from $\mathcal{R}$ of cardinality at most $|C_c(G, A) \times \mathbb{N}|$ such that $\sigma^S(f) = \sup_{(\pi,U)\in S} \|\pi \rtimes U(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$. Hence the following definition is meaningful; it will be required in Definition 4.3.3 and Proposition 4.3.4.

Definition 4.3.1. Let $\mathcal{R}$ be a uniformly bounded class of possibly degenerate continuous covariant representations of (A, G, α). We define $[\mathcal{R}]$ to be the collection of all uniformly bounded classes $\mathcal{S}$ that are actually sets and satisfy $\sigma^{\mathcal{R}} = \sigma^{\mathcal{S}}$ on $C_c(G, A)$. Elements of some $[\mathcal{R}]$ will be called *uniformly bounded sets of continuous covariant representations.*

Before addressing the question laid out in the first paragraph, we consider the following aside which will play a key role in Section 4.4.

Definition 4.3.2. Let I be an index set and $\{X_i : i \in I\}$ a family of Banach spaces. For $1 \leq p \leq \infty$, we will denote the ℓ^p-direct sum of $\{X_i : i \in I\}$ by $\ell^p\{X_i : i \in I\}$.

Definition 4.3.3. Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations. For $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, suppressing the p-dependence in the notation, we define the representations $(\oplus_S \pi) : A \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ and $(\oplus_S U) : G \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ by $(\oplus_S \pi)(a) := \bigoplus_{(\pi,U)\in S} \pi(a)$ and $(\oplus_S U)_r := \bigoplus_{(\pi,U)\in S} U_r$ for all $a \in A$ and $r \in G$ respectively.

It is easily seen that $((\oplus_S \pi), (\oplus_S U))$ is a continuous covariant representation, that

$$((\oplus_S \pi) \rtimes (\oplus_S U))(f) = \bigoplus_{(\pi,U)\in S} \pi \rtimes U(f),$$

and that $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^S(f) = \sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G, A)$.

We hence obtain the following (where the statement concerning non-degeneracy is an elementary verification).

Proposition 4.3.4. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations. For any $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, there exists an $\mathcal{R}$-continuous covariant representation of (A, G, α) on $\ell^p\{X_\pi : (\pi, U) \in S\}$, denoted $((\oplus_S \pi), (\oplus_S U))$, such that its integrated form satisfies $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$ and hence induces an isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on $\ell^p\{X_\pi : (\pi, U) \in S\}$.*

If every element of S is non-degenerate, then $((\oplus_S \pi), (\oplus_S U))$ is non-degenerate.

The previous theorem shows, in particular, that crossed products can always be realized isometrically as closed subalgebras of bounded operators on some (rather large) Banach space.

We now return to the original question. The following results examine relations that may exist between crossed products defined by using two different uniformly bounded classes of continuous covariant representations of a Banach algebra dynamical system.

Proposition 4.3.5. *Let (A, G, α) be a Banach algebra dynamical system. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly bounded classes of possibly degenerate continuous covariant representations of (A, G, α) and $M \geq 1$ a constant. Then the following are equivalent:*

(1) There exists a homomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to (A \rtimes_\alpha G)^{\mathcal{R}_1}$ such that $\|h\| \leq M$ and $h \circ q^{\mathcal{R}_2}(f) = q^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$.

(2) The seminorms $\sigma^{\mathcal{R}_1}$ and $\sigma^{\mathcal{R}_2}$ satisfy $\sigma^{\mathcal{R}_1}(f) \leq M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

(3) There exist uniformly bounded sets of continuous covariant representations $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ such that $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$.

(4) If (π, U) is an $\mathcal{R}_1$-continuous covariant representation of (A, G, α) and $M' \geq 0$ is such that $\|\pi \rtimes U(f)\| \leq M'\sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, then (π, U) is an $\mathcal{R}_2$-continuous covariant representation of (A, G, α), and $\|\pi \rtimes U(f)\| \leq M'M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

(5) For any bounded representation $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ such that $T \circ q^{\mathcal{R}_1}(f) = S \circ q^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$ and $\|S\| \leq M\|T\|$.

Proof. We prove that (1) implies (5). Let $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ be a bounded representation. Then $S := T \circ h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ satisfies $T \circ q^{\mathcal{R}_1}(f) = T \circ h \circ q^{\mathcal{R}_2}(f) = S \circ q^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$, and $\|S\| \leq \|T\|\|h\| \leq M\|T\|$.

We prove that (5) implies (4). Let (π, U) be $\mathcal{R}_1$-continuous and $M' \geq 0$ be such that $\|\pi \rtimes U(f)\| \leq M'\sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$. Then, for the bounded representation $(\pi \rtimes U)^{\mathcal{R}_1} : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X_\pi)$, there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X_\pi)$ such that

$$\pi \rtimes U(f) = (\pi \rtimes U)^{\mathcal{R}_1} \circ q^{\mathcal{R}_1}(f) = S \circ q^{\mathcal{R}_2}(f)$$

for all $f \in C_c(G, A)$, and $\|S\| \leq M\|(\pi \rtimes U)^{\mathcal{R}_1}\| \leq MM'$. Hence, (π, U) is $\mathcal{R}_2$-continuous, and $\|\pi \rtimes U(f)\| = \|S \circ q^{\mathcal{R}_2}(f)\| \leq MM'\sigma^{\mathcal{R}_2}(f)$ holds for all $f \in C_c(G, A)$.

We prove that (4) implies (2). Every $(\pi, U) \in \mathcal{R}_1$ is $\mathcal{R}_1$-continuous and satisfies $\|\pi \rtimes U(f)\| \leq \sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$. Then, by hypothesis, (π, U) is $\mathcal{R}_2$-continuous and

$$\|\pi \rtimes U(f)\| \leq M\sigma^{\mathcal{R}_2}(f)$$

for all $f \in C_c(G, A)$. Taking the supremum over all $(\pi, U) \in \mathcal{R}_1$, we obtain $\sigma^{\mathcal{R}_1}(f) \leq M\sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$.

We prove that (2) implies (1). Since $\ker \sigma^{\mathcal{R}_2} \subseteq \ker \sigma^{\mathcal{R}_1}$, a homomorphism

$$h : C_c(G, A)/\ker \sigma^{\mathcal{R}_2} \to C_c(G, A)/\ker \sigma^{\mathcal{R}_1}$$

can be defined by $h(q^{\mathcal{R}_2}(f)) := q^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, and then satisfies $\|h\| \leq M$. The map h therefore extends to a homomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to (A \rtimes_\alpha G)^{\mathcal{R}_1}$ with the same norm.

We prove that (2) implies (3). Let $\mathcal{R}_1' \in [\mathcal{R}_1]$ and $\mathcal{R}_2' \in [\mathcal{R}_2]$ and define $\mathcal{R}_3' := \mathcal{R}_1' \cup \mathcal{R}_2'$. By construction we have that $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f)$ for all $f \in C_c(G, A)$. By hypothesis we have that $\sigma^{\mathcal{R}_1'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$, as well as $M \geq 1$. Therefore,

$$\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) = \max\{\sigma^{\mathcal{R}_1'}(f), \sigma^{\mathcal{R}_2'}(f)\} \leq \max\{M\sigma^{\mathcal{R}_2'}(f), \sigma^{\mathcal{R}_2'}(f)\} = M\sigma^{\mathcal{R}_2'}(f).$$

We prove that (3) implies (2). Let $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ be such that $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and $\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f)$ for all $f \in C_c(G, A)$. Then

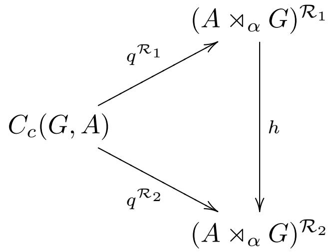
$$\sigma^{\mathcal{R}_1}(f) = \sigma^{\mathcal{R}_1'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M\sigma^{\mathcal{R}_2'}(f) \leq M\sigma^{\mathcal{R}_2}(f).$$

□

We can now describe the relationship between $\mathcal{R}$ and the isomorphism class of the pair $((A \rtimes_\alpha G)^{\mathcal{R}}, q^{\mathcal{R}})$.

Corollary 4.3.6. *Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}_1$ and $\mathcal{R}_2$ be uniformly bounded classes of possibly degenerate continuous covariant representations of (A, G, α). Then the following are equivalent:*

(1) There exists a topological algebra isomorphism $h : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to (A \rtimes_\alpha G)^{\mathcal{R}_2}$ such that the following diagram commutes:



(2) The seminorms $\sigma^{\mathcal{R}_1}$ and $\sigma^{\mathcal{R}_2}$ on $C_c(G, A)$ are equivalent.

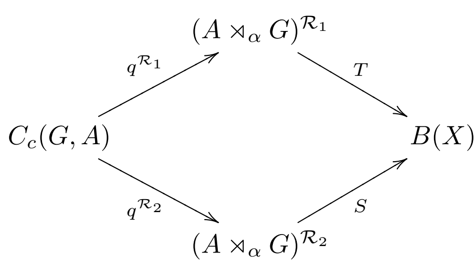
(3) There exist uniformly bounded sets of possibly degenerate continuous covariant representations $\mathcal{R}_1' \in [\mathcal{R}_1]$, $\mathcal{R}_2' \in [\mathcal{R}_2]$ and $\mathcal{R}_3'$ with $\mathcal{R}_1' \cup \mathcal{R}_2' \subseteq \mathcal{R}_3'$ and constants $M_1, M_2 \geq 0$, such that

$$\sigma^{\mathcal{R}_1'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M_1 \sigma^{\mathcal{R}_1'}(f),$$
$$\sigma^{\mathcal{R}_2'}(f) \leq \sigma^{\mathcal{R}_3'}(f) \leq M_2 \sigma^{\mathcal{R}_2'}(f),$$

for all $f \in C_c(G, A)$.

(4) The $\mathcal{R}_1$-continuous covariant representations of (A, G, α) coincide with the $\mathcal{R}_2$-continuous covariant representations of (A, G, α). Moreover, there exist constants $M_1, M_2 \geq 0$, with the property that, if $M' \geq 0$ and (π, U) is $\mathcal{R}_1$-continuous, such that $\|\pi \rtimes U(f)\| \leq M' \sigma^{\mathcal{R}_1}(f)$ for all $f \in C_c(G, A)$, then $\|\pi \rtimes U(f)\| \leq M_1 M' \sigma^{\mathcal{R}_2}(f)$ for all $f \in C_c(G, A)$, and likewise for the indices 1 and 2 interchanged.

(5) There exist constants $M_1, M_2 \geq 0$ with the property that, for every bounded representation $T : (A \rtimes_\alpha G)^{\mathcal{R}_1} \to B(X)$ there exists a bounded representation $S : (A \rtimes_\alpha G)^{\mathcal{R}_2} \to B(X)$ with $\|S\| \leq M_1 \|T\|$, such that the diagram



commutes, and likewise with the indices 1 and 2 interchanged.

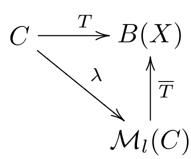
Proof. This follows from Proposition 4.3.5. □

4.4 Uniqueness of the crossed product

Theorem 4.2.1 asserts, amongst others, that all non-degenerate $\mathcal{R}$-continuous covariant representations of a Banach algebra dynamical system (A, G, α) can be generated from the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, with the aid of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ and the pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$. In this section we show that, under mild additional hypotheses, $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra with this generating property. These results are similar in nature as Raeburn's for the crossed product of a C^*-algebra, see [38] or [46, Theorem 2.61].

We start with the general framework of how to generate many non-degenerate $\mathcal{R}$-continuous covariant representations from a suitable basic one, on a Banach space that is a Banach algebra.

Lemma 4.4.1. *Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α). Let C be a Banach algebra with a bounded approximate left identity, and let (k_A, k_G) be a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space C, such that $k_A(A), k_G(G) \subseteq \mathcal{M}_l(C)$. Suppose $T : C \to B(X)$ is a non-degenerate bounded representation of C on a Banach space X. Let $\overline{T} : \mathcal{M}_l(C) \to B(X)$ be the non-degenerate bounded representation of $\mathcal{M}_l(C)$ such that the following diagram commutes:*



Then the pair $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$.

Proof. It is clear that $\overline{T} \circ k_A$ is a continuous representation of A on X. Since $\overline{T}$ is unital [18, Theorem 4.1], $\overline{T} \circ k_G$ is a representation of G on X. Using that $\overline{T}(L)T(c) = T(Lc)$ for $L \in \mathcal{M}_l(C)$ and $c \in C$, (cf. [18, Theorem 4.1]), we find, for $r \in G$, $c \in C$ and $x \in X$, that $(\overline{T} \circ k_G(r))T(c)x = T(k_G(r)c)x$. Since k_G is strongly continuous and T is continuous, we see that

$$\lim_{r \to e} (\overline{T} \circ k_G(r))T(c)x = \lim_{r \to e} T(k_G(r)c)x = T(c)x,$$

for all $c \in C$ and $x \in X$. The non-degeneracy of T, together with [19, Corollary 2.5] then imply that $T \circ k_G$ is strongly continuous. It is a routine verification that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is covariant, so that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a continuous covariant representation of (A, G, α) on C.

We claim that $k_A \rtimes k_G : C_c(G, A) \to B(C)$ has its image in $\mathcal{M}_l(C)$, and that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$. The $\mathcal{R}$-continuity of (k_A, k_G) and the continuity of $\overline{T}$ then show that $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is $\mathcal{R}$-continuous. As to this claim, note that, for $f \in C_c(G, A)$, the integrand in $k_A \rtimes k_G(f) = \int_G k_A(f(r))k_G(r)\, dr$ takes values in the SOT-closed subspace $\mathcal{M}_l(C)$ of $B(C)$, hence the integral is likewise in this subspace. Hence $\overline{T} \circ (k_A \rtimes k_G) : C_c(G, A) \to B(X)$ is a meaningfully defined map. Using that that continuous operators can be pulled through the integral [40, Ch. 3, Exercise 24] and the definition of operator valued integrals [19, Proposition 2.19], we then have for all $x \in X$:

$$\begin{aligned}
\overline{T}\,(k_A \rtimes k_G(f))\,T(c)x &= T\,(k_A \rtimes k_G(f)c)\,x \\
&= T\left(\int_G k_A(f(r))k_G(r)\,dr\ c\right)x \\
&= T\left(\int_G k_A(f(r))k_G(r)c\,dr\right)x \\
&= \int_G T\,(k_A(f(r))k_G(r)c)\,x\,dr \\
&= \int_G \overline{T}\,(k_A(f(r))k_G(r))\,T(c)x\,dr \\
&= \int_G \overline{T} \circ k_A(f(r))\overline{T} \circ k_G(r)T(c)x\,dr \\
&= \left(\int_G \overline{T} \circ k_A(f(r))\overline{T} \circ k_G(r)\,dr\right)T(c)x \\
&= \big((\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G)(f)\big)\,T(c)x.
\end{aligned}$$

Since T is non-degenerate, this establishes the claim.

It remains to show that $\overline{T} \circ k_A$ is non-degenerate. Let $x \in X$ and $\varepsilon > 0$ be arbitrary. Since $\overline{T}$ is non-degenerate [18, Theorem 4.1], there exist finite sets $\{c_i\}_{i=1}^n \subseteq C$ and $\{x_i\}_{i=1}^n \subseteq X$ such that $\| \sum_{i=1}^n T(c_i)x_i - x\| < \varepsilon/2$. Since k_A is non-degenerate, for every $i \in \{1, \ldots, n\}$, there exist finite sets $\{a_{i,j}\}_{j=1}^{m_i} \subseteq A$ and $\{d_{i,j}\}_{j=1}^{m_i} \subseteq C$ such that $\|T\|\|x_i\|\|c_i - \sum_{j=1}^{m_i} k_A(a_{i,j})d_{i,j}\| < \varepsilon/2n$. Then

$$\begin{aligned}
&\left\| x - \sum_{i=1}^n \sum_{j=1}^{m_i} \left(\overline{T} \circ k_A(a_{i,j})\right) T(d_{i,j})x_i \right\| \\
= &\left\| x - \sum_{i=1}^n \sum_{j=1}^{m_i} T\,(k_A(a_{i,j})d_{i,j})\,x_i \right\| \\
\leq &\left\| x - \sum_{i=1}^n T(c_i)x_i \right\| + \left\| \sum_{i=1}^n T(c_i)x_i - \sum_{i=1}^n \sum_{j=1}^{m_i} T\,(k_A(a_{i,j})d_{i,j})\,x_i \right\| \\
\leq &\left\| x - \sum_{i=1}^n T(c_i)x_i \right\| + \sum_{i=1}^n \|T\| \left\| c_i - \sum_{j=1}^{m_i} k_A(a_{i,j})d_{i,j} \right\| \|x_i\| \\
< &\frac{\varepsilon}{2} + \frac{\varepsilon}{2}.
\end{aligned}$$

We conclude that $\overline{T} \circ k_A$ is non-degenerate. □

Naturally any Banach algebra C' isomorphic to C as in the previous lemma has a similar "generating pair" (k'_A, k'_G). The details are in the following result, the routine verification of which is left to the reader.

Lemma 4.4.2. *Let (A, G, α), $\mathcal{R}$, C and (k_A, k_G) be as in Lemma 4.4.1. Suppose C' is a Banach algebra and $\psi : C \to C'$ is a topological isomorphism. Then:*

(1) $\psi_l : \mathcal{M}_l(C) \to \mathcal{M}_l(C')$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l(C)$, is a topological isomorphism.

(2) The pair $(k'_A, k'_G) := (\psi_l \circ k_A, \psi_l \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on C', such that $k'_A(A), k'_G(G) \subseteq \mathcal{M}_l(C')$.

(3) If $T : C \to B(X)$ is a non-degenerate bounded representation, then so is $T' : C' \to B(X)$, where $T' := T \circ \psi^{-1}$.

(4) If $T : C \to B(X)$ is a non-degenerate bounded representation, and $\overline{T'} : \mathcal{M}_l(C') \to B(X)$ is the non-degenerate bounded representation of $\mathcal{M}_l(C')$ such that the diagram

$$\begin{array}{ccc} C' & \xrightarrow{T'} & B(X) \\ & \searrow^{\lambda} & \uparrow \overline{T'} \\ & & \mathcal{M}_l(C') \end{array}$$

commutes, then $\overline{T} \circ k_A = \overline{T'} \circ k'_A$ and $\overline{T} \circ k_G = \overline{T'} \circ k'_G$.

Now let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α), where A has a bounded approximate left identity. Then, according to [19, Theorem 7.2], $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a bounded approximate left identity, and the maps $i_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ form a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space $(A \rtimes_\alpha G)^{\mathcal{R}}$, with images in $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$. According to Lemma 4.4.1, the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ can be used to produce non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$, and, according to Theorem 4.2.1, all non-degenerate $\mathcal{R}$-continuous covariant representations are thus obtained. According to Lemma 4.4.2, any Banach algebra isomorphic to $(A \rtimes_\alpha G)^{\mathcal{R}}$ has the same property. We will now proceed to show the converse: If (B, k_A, k_G) is a triple generating all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), then it can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ as in Lemma 4.4.2.

We start with a preliminary observation that is of some interest in its own right.

Proposition 4.4.3. *Let (A, G, α) be a Banach algebra dynamical system where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations. Then the left regular representation $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a topological embedding.*

Proof. According to Proposition 4.3.4, there exists a non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) such that $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. According to Theorem 4.2.1, $\pi = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}$ and $U = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}$. Furthermore, according to Lemma 4.4.1,

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}) \rtimes (\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}}).$$

We recall [19, Theorem 7.2] that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$. Combining all this, we see, with M denoting an upper bound for an approximate left identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Corollary 4.6], that, for $f \in C_c(G, A)$:

$$\begin{aligned}
\|q^{\mathcal{R}}(f)\|^{\mathcal{R}} &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\
&= \|\pi \rtimes U(f)\| \\
&= \|(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}) \rtimes (\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}})(f)\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(f)\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\
&= \|\overline{(\pi \rtimes U)^{\mathcal{R}}}(\lambda(q^{\mathcal{R}}(f)))\| \\
&\leq M\|(\pi \rtimes U)^{\mathcal{R}}\|\|\lambda(q^{\mathcal{R}}(f))\| \\
&= M\|\lambda(q^{\mathcal{R}}(f))\|.
\end{aligned}$$

Since the inequality $\|\lambda(q^{\mathcal{R}}(f))\| \leq \|q^{\mathcal{R}}(f)\|$ is trivial, the result follows from the density of $q^{\mathcal{R}}(C_c(G, A))$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$. □

Since $q^{\mathcal{R}}(C_c(G, A))$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$, we conclude that $\lambda \circ q^{\mathcal{R}}(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$, i.e., that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$. Together with Proposition 4.4.3 this gives the two additional hypotheses alluded to before, under which the following uniqueness theorem can now be established. As mentioned earlier, this should be compared with Raeburn's result for the crossed product of a C^*-algebra, see [38] or [46, Theorem 2.61].

Theorem 4.4.4. *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α). Let B be a Banach algebra with a bounded approximate left identity, such that $\lambda : B \to \mathcal{M}_l(B)$ is a topological embedding. Let (k_A, k_G) be a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the Banach space B, such that*

(1) $k_A(A), k_G(G) \subseteq \mathcal{M}_l(B)$

(2) $(k_A \rtimes k_G)(C_c(G, A)) \subseteq \lambda(B)$

(3) $(k_A \rtimes k_G)(C_c(G, A))$ *is dense in* $\lambda(B)$.

Suppose that, for each non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a Banach space X, there exists a non-degenerate bounded representation $T : B \to B(X)$ such that the non-degenerate bounded representation

$\overline{T} : \mathcal{M}_l(B) \to B(X)$ in the commuting diagram

$$\begin{array}{ccc} B & \xrightarrow{T} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

generates (π, U) as in Lemma 4.4.1, i.e., is such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$.

Then there exists a unique topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$, such that the induced topological isomorphism $\psi_l : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to \mathcal{M}_l(B)$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, induces (k_A, k_G) as in Lemma 4.4.2 from $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$, i.e., is such that $k_A = \psi_l \circ i_A^{\mathcal{R}}$ and $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

Proof. Proposition 4.3.4 provides a non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) such that $(\pi \rtimes U)^{\mathcal{R}}$ is an isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$. If $T : B \to B(X)$ is a non-degenerate bounded representation such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$, then Lemma 4.4.1 shows that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$, i.e., that $\pi \rtimes U = \overline{T} \circ (k_A \rtimes k_G)$. Hence, for $f \in C_c(G, A)$:

$$\begin{aligned} \|q^{\mathcal{R}}(f)\|^{\mathcal{R}} &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\ &= \|\pi \rtimes U(f)\| \\ &= \|\overline{T} \circ (k_A \rtimes k_G)(f)\| \\ &\leq \|\overline{T}\| \|k_A \rtimes k_G(f)\| \\ &= \|\overline{T}\| \|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\|. \end{aligned}$$

Since (k_A, k_G) is $\mathcal{R}$-continuous, $\|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \leq \|(k_A \rtimes k_G)^{\mathcal{R}}\| \|(q^{\mathcal{R}}(f))\|^{\mathcal{R}}$, for all $f \in C_c(G, A)$, hence we can now conclude, using (2) and (3) and the fact that $\lambda(B)$ is closed, that $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(B)$ is a topological isomorphism of Banach algebras. Since $\lambda : B \to \mathcal{M}_l(B)$ is assumed to be a topological embedding,

$$\psi := \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$$

is a topological isomorphism.

We proceed to show that ψ_l induces k_A and k_G. As a preparation, note that, since $(\pi \rtimes U)^{\mathcal{R}}$ is isometric and $(\pi \rtimes U)^{\mathcal{R}} = \overline{T} \circ (k_A \rtimes k_G)^{\mathcal{R}}$, the map $\overline{T} : \mathcal{M}_l(B) \to B(X)$ is injective on $(k_A \rtimes k_G)^{\mathcal{R}}((A \rtimes_\alpha G)^{\mathcal{R}}) = \lambda(B)$. Now by [19, Proposition 5.3], for $a \in A$ and $f \in C_c(G, A)$, we have

$$\begin{aligned} (\pi \rtimes U)^{\mathcal{R}}(i_A^{\mathcal{R}}(a) q^{\mathcal{R}}(f)) &= \pi \rtimes U(i_A(a) f) \\ &= \pi(a) \pi \rtimes U(f) \\ &= \overline{T} \circ k_A(a) (\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \\ &= \overline{T} \circ k_A(a) \overline{T} \circ (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \\ &= \overline{T}\left(k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right). \end{aligned}$$

Since $\lambda(B)$ is a left ideal in $\mathcal{M}_l(B)$ (as is the case for every Banach algebra), we note that $k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \in \lambda(B)$. On the other hand, we also have

$$(\pi \rtimes U)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) \quad = \quad \overline{T}\left((k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f))\right).$$

Hence the injectivity of $\overline{T}$ on $\lambda(B)$ shows that

$$k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = (k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)).$$

We now apply λ^{-1} to both sides, and use that $\lambda^{-1}(L\circ\lambda(b)) = L(b)$ for all $L \in \mathcal{M}_l(B)$ and $b \in B$, to see that

$$\begin{aligned}
\psi(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) &= \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}}(i_A^{\mathcal{R}}(a)q^{\mathcal{R}}(f)) \\
&= \lambda^{-1}\left(k_A(a)(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right) \\
&= \lambda^{-1}\left(k_A(a) \circ \lambda \circ \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\right) \\
&= \lambda^{-1}\left(k_A(a) \circ \lambda \circ \psi(q^{\mathcal{R}}(f))\right) \\
&= k_A(a)\psi(q^{\mathcal{R}}(f)).
\end{aligned}$$

By density, we conclude that $\psi \circ i_A^{\mathcal{R}}(a) = k_A(a) \circ \psi$, for all $a \in A$, i.e., that $k_A = \psi_l \circ i_A^{\mathcal{R}}$. A similar argument yields $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

As to uniqueness, suppose that $\phi : (A \rtimes_\alpha G)^{\mathcal{R}} \to B$ is a topological isomorphism such that $k_A = \phi_l \circ i_A^{\mathcal{R}}$ and $k_G = \phi_l \circ i_G^{\mathcal{R}}$. Remembering that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$ [19, Theorem 7.2], this readily implies that, for $f \in C_c(G, A)$,

$$\begin{aligned}
\phi \circ (\lambda(q^{\mathcal{R}}(f)) \circ \phi^{-1} &= \phi \circ (i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}(q^{\mathcal{R}}(f)) \circ \phi^{-1} \\
&= (k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)),
\end{aligned}$$

hence

$$(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) \circ \phi = \phi \circ \lambda(q^{\mathcal{R}}(f)).$$

Applying this to $q^{\mathcal{R}}(g)$, for $g \in C_c(G, A)$, we find

$$\begin{aligned}
(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\phi(q^{\mathcal{R}}(g)) &= \phi\left(\lambda(q^{\mathcal{R}}(f))q^{\mathcal{R}}(g)\right) \\
&= \phi(q^{\mathcal{R}}(f)q^{\mathcal{R}}(g)) \\
&= \phi(q^{\mathcal{R}}(f))\phi(q^{\mathcal{R}}(g)) \\
&= \lambda(\phi(q^{\mathcal{R}}(f)))\phi(q^{\mathcal{R}}(g)).
\end{aligned}$$

By density, we conclude that $(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f)) = \lambda(\phi(q^{\mathcal{R}}(f)))$, for all $f \in C_c(G, A)$. Again by density, we conclude that

$$\phi = \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} = \psi.$$

□

4.5 Generalized Beurling algebras as crossed products

In this section we give sufficient conditions for a crossed product of a Banach algebra to be topologically isomorphic to a generalized Beurling algebra (see Definition 4.5.4), cf. Theorem 4.5.13 and Corollary 4.5.14. Since these conditions can always be satisfied, all generalized Beurling algebras are topologically isomorphic to a crossed product (cf. Theorem 4.5.17). Through an application of the General Correspondence Theorem (Theorem 4.2.1) we then obtain a bijection between the non-degenerate bounded representations of such a generalized Beurling algebra and the non-degenerate continuous covariant representations of the Banach algebra dynamical system where the group representation is bounded by a multiple of the weight, cf. Theorem 4.5.20. When the Banach algebra in the Banach algebra dynamical system is taken to be the scalars, and the weight on the group G taken to be the constant 1 function, then Corollary 4.5.14 shows that $L^1(G)$ is isometrically isomorphic to a crossed product, and Theorem 4.5.20 reduces to the classical bijective correspondence between uniformly bounded strongly continuous representations of G and non-degenerate bounded representations of $L^1(G)$, cf. Corollary 4.5.22.

We start with the topological isomorphism between a generalized Beurling algebra and a crossed product.

Definition 4.5.1. For a locally compact group G, let $\omega : G \to [0, \infty)$ be a non-zero submultiplicative Borel measurable function. Then ω is called a *weight* on G.

Note that we do not assume that $\omega \geq 1$, as is done in some parts of the literature. The fact that ω is non-zero readily implies that $\omega(e) \geq 1$. More generally, if $K \subseteq G$ is a compact set, there exist $a, b > 0$ such that $a \leq \omega(s) \leq b$ for all $s \in K$ [26, Lemma 1.3.3].

Let (A, G, α) be a Banach algebra dynamical system, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α). We recall that $C^{\mathcal{R}} := \sup_{(\pi,U)\in\mathcal{R}} \|\pi\|$ and $\nu^{\mathcal{R}} : G \to \mathbb{R}_{\geq 0}$ is defined by $\nu^{\mathcal{R}}(r) := \sup_{(\pi,U)\in\mathcal{R}} \|U_r\|$ as in [19, Definition 3.1]. We note that the map $\nu^{\mathcal{R}}$ is a weight on G. It is clearly submultiplicative, and, being the supremum of a family of continuous maps $\{r \mapsto \|U_r x\| : (\pi, U) \in \mathcal{R},\ x \in X_\pi,\ \|x\| \leq 1\}$, the map $\nu^{\mathcal{R}}$ is lower semicontinuous, hence Borel measurable.

Let ω be a weight on G, such that $\nu^{\mathcal{R}} \leq \omega$. Then, for all $f \in C_c(G, A)$,

$$\begin{aligned}
\sigma^{\mathcal{R}}(f) &= \sup_{(\pi,U)\in\mathcal{R}} \left\| \int_G \pi(f(s))U_s\, ds \right\| \\
&\leq \sup_{(\pi,U)\in\mathcal{R}} \int_G \|\pi(f(s))\| \|U_s\|\, ds \\
&\leq C^{\mathcal{R}} \int_G \|f(s)\| \nu^{\mathcal{R}}(s)\, ds
\end{aligned}$$

$$\begin{aligned} &\leq C^{\mathcal{R}} \int_G \|f(s)\|\omega(s)\, ds \\ &= C^{\mathcal{R}} \|f\|_{1,\omega}, \end{aligned} \tag{4.5.1}$$

where $\|\cdot\|_{1,\omega}$ denotes the ω-weighted L^1-norm. In Theorem 4.5.13, we will give sufficient conditions under which a reverse inequality holds. Then $\sigma^{\mathcal{R}}$ is actually a norm on $C_c(G, A)$ and is equivalent to a weighted L^1-norm, so that $(A \rtimes_\alpha G)^{\mathcal{R}}$ will be isomorphic to a generalized Beurling algebra to be defined shortly.

Definition 4.5.2. Let X be a Banach space, $1 \leq p < \infty$, and $\omega : G \to [0, \infty)$ a weight on G. We define the weighted p-norm on $C_c(G, X)$ by

$$\|h\|_{p,\omega} := \left(\int_G \|h(s)\|^p \omega(s)\, ds \right)^{1/p},$$

and define $L^p(G, X, \omega)$ as the completion of $C_c(G, X)$ with this norm.

Remark 4.5.3. By definition $L^p(G, A, \omega)$ with $1 \leq p < \infty$ has $C_c(G, A)$ as a dense subspace. In view of the central role of $C_c(G, A)$ in our theory of crossed products of Banach algebras, this is clearly desirable, but it would be unsatisfactory not to discuss the relation with spaces of Bochner integrable functions. We will now address this and explain that for $p = 1$ (our main space of interest in the sequel), $L^1(G, A, \omega)$ is (isometrically isomorphic to) a Bochner space.

In most of the literature, the theory of the Bochner integral is developed for finite (or at least σ-finite) measures, and sometimes the Banach space in question is assumed to be separable. Since $\omega d\mu$ (where μ is the left Haar measure on G) need not be σ-finite and A need not be separable, this is not applicable to our situation. In [10, Appendix E], however, the theory is developed for an arbitrary measure μ on a σ-algebra $\mathcal{A}$ of subsets of a set Ω, and functions $f : \Omega \to X$ with values in an arbitrary Banach space X. Such a function f is called Bochner integrable if $f^{-1}(B) \in \mathcal{A}$ for every Borel subset of X, $f(\Omega)$ is separable, and $\int_\Omega \|f(\xi)\| d\mu(\xi) < \infty$ (the measurability of $\xi \mapsto \|f(\xi)\|$ is an automatic consequence of the Borel measurability of f). Identifying Bochner integrable functions that are equal μ-almost everywhere, one obtains a Banach space $L^1(\Omega, \mathcal{A}, \mu, X)$, where the norm is given by $\|[f]\| = \int_\Omega \|f(\xi)\|\, d\mu(\xi)$ with f any representative of the equivalence class $[f] \in L^1(\Omega, \mathcal{A}, \mu, X)$. Although it is not stated as such, it is in fact proved [10, p. 352] that the simple Bochner integrable functions (i.e., all functions of the form $\sum_{i=1}^n \chi_{A_i} \otimes x_i$, where $A_i \in \mathcal{A}$, $\mu(A_i) < \infty$ and $x_i \in X$) form a dense subspace of $L^1(\Omega, \mathcal{A}, \mu, X)$.

We claim that our space $L^1(G, A, \omega)$ is isometrically isomorphic to the Bochner space $L^1(G, \mathcal{B}, \omega d\mu, A)$, where $\mathcal{B}$ is the Borel σ-algebra of G, and μ is the left Haar measure on G again. To start with, if $f \in C_c(G, A)$, then certainly f is Bochner integrable, so that we obtain a (clearly isometric) inclusion map $j : C_c(G, A) \to L^1(G, \mathcal{B}, \omega d\mu, A)$, that can be extended to an isometric embedding of $L^1(G, A, \omega)$ into $L^1(G, \mathcal{B}, \omega d\mu, A)$. To see that the image is dense, it is, in view of the density of the simple Bochner integrable functions in $L^1(G, \mathcal{B}, \omega d\mu, A)$, sufficient to approximate $\chi_S \otimes a$ with elements from $C_c(G, A)$, for arbitrary $a \in A$ and $S \in \mathcal{B}$ with $\int_G \chi_S \omega d\mu < \infty$. Since $C_c(G)$ is dense in $L^1(G, \omega d\mu)$ [26, Lemma 1.3.5], we can choose a sequence

$(f_n) \subseteq C_c(G)$ such that $f_n \to \chi_S$ in $L^1(G, \omega d\mu)$, and then

$$\|f_n \otimes a - \chi_S \otimes a\| = \|a\| \int_G |f_n(r) - \chi_S(r)|\omega(r)\, d\mu(r) \to 0.$$

Hence the image of $j : C_c(G, A) \to L^1(G, \mathcal{B}, \omega d\mu, A)$ is dense and our claim has been established.

For the sake of completeness, we note that one cannot argue that $\omega d\mu$ is "clearly" a regular Borel measure on G, so that $C_c(G)$ is dense in $L^1(G, \omega d\mu)$ by the standard density result [22, Proposition 7.9]. Indeed, although [22, Exercises 7.2.7–9] give sufficient conditions for this to hold (none of which applies in our general case), the regularity is not automatic, see [22, Exercise 7.2.13]. The proof of the density of $C_c(G)$ in $L^1(G, \omega d\mu)$ in [26, Lemma 1.3.5] is direct and from first principles. It uses in an essential manner that ω is bounded away from zero on compact subsets of G, but not that the Haar measure should be σ-finite or that ω should be integrable.

With (A, G, α) a Banach algebra dynamical system and ω a weight on G, if α is uniformly bounded, say $\|\alpha_s\| \leq C_\alpha$ for some $C_\alpha \geq 0$ and all $s \in G$, then, using the submultiplicativity of ω, it is routine to verify that

$$\|f * g\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega} \|g\|_{1,\omega} \quad (f, g \in C_c(G, A)).$$

Hence the Banach space $L^1(G, A, \omega)$ can be supplied with the structure of an associative algebra, such that

$$\|u * v\|_{1,\omega} \leq C_\alpha \|u\|_{1,\omega} \|v\|_{1,\omega} \quad (u, v \in L^1(G, A, \omega)).$$

If $C_\alpha = 1$ (i.e., if α lets G act as isometries on A), then $L^1(G, A, \omega)$ is a Banach algebra, and when $C_\alpha \neq 1$, as is well known, there is an equivalent norm on $L^1(G, A, \omega)$ such that it becomes a Banach algebra. We will show below (cf. Theorem 4.5.17) that such a norm can always be obtained from a topological isomorphism between $L^1(G, A, \omega)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ for a suitable choice of $\mathcal{R}$.

Definition 4.5.4. Let (A, G, α) be a Banach algebra dynamical system with α uniformly bounded and ω a weight on G. The Banach space $L^1(G, A, \omega)$ endowed with the continuous multiplication induced by the twisted convolution on $C_c(G, A)$, given by

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\, dr \quad (f, g \in C_c(G, A),\ s \in G),$$

will be denoted by $L^1(G, A, \omega; \alpha)$ and called a *generalized Beurling algebra.*

As a special case, we note that for $A = \mathbb{K}$, the generalized Beurling algebra reduces to the classical Beurling algebra $L^1(G, \omega)$, which is a true Banach algebra.

Obtaining such a reverse inequality to (4.5.1) rests on inducing a covariant representation of (A, G, α) from the left regular representation $\lambda : A \to B(A)$ of A, analogous to [46, Example 2.14]. The key result is Proposition 4.5.12 and we will now start working towards it.

Definition 4.5.5. Let (A, G, α) be a Banach algebra dynamical system and let $\pi : A \to B(X)$ be a bounded representation of A on a Banach space X. We define the induced algebra representation $\tilde{\pi}$ and left regular group representation Λ on the space of all functions from G to X by the formulae:

$$\begin{aligned} [\tilde{\pi}(a)h](s) &:= \pi(\alpha_s^{-1}(a))h(s), \\ (\Lambda_r h)(s) &:= h(r^{-1}s), \end{aligned}$$

where $h : G \to X$, $r, s \in G$ and $a \in A$.

A routine calculation, left to the reader, shows that $(\tilde{\pi}, \Lambda)$ is covariant.

We need a number of lemmas in preparation for the proof of Proposition 4.5.12.

The following is clear.

Lemma 4.5.6. *If (A, G, α) is a Banach algebra dynamical system with α uniformly bounded by a constant C_α, and $\omega : G \to [0, \infty)$ a weight, then for any bounded representation $\pi : A \to B(X)$ on a Banach space X, both the maps $\tilde{\pi} : A \to B(C_0(G, X))$ (as defined in Definition 4.5.5) and $\tilde{\pi} : A \to B(L^p(G, X, \omega))$ for $1 \leq p < \infty$ (the canonically induced representation $\tilde{\pi}$ of A on $L^p(G, X, \omega)$ as completion of $C_c(G, X)$ with the $\|\cdot\|_{p,\omega}$-norm) are representations with norms bounded by $C_\alpha \|\pi\|$. Moreover, $C_c(G, X)$ is invariant under both A-actions.*

Lemma 4.5.7. *If (A, G, α) is a Banach algebra dynamical system and X a Banach space and ω a weight on G, then both the left regular representations $\Lambda : G \to B(C_0(G, X))$ (as defined in Definition 4.5.5), and $\Lambda : G \to B(L^p(G, X, \omega))$ for $1 \leq p < \infty$ (the canonically induced representation Λ of G on $L^p(G, X, \omega)$ as completion of $C_c(G, X)$ with the $\|\cdot\|_{p,\omega}$-norm) are strongly continuous group representations. The representation $\Lambda : G \to B(C_0(G, X))$ acts as isometries, and $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$. Moreover, $C_c(G, X)$ is invariant under both G-actions.*

Proof. That $\Lambda : G \to B(C_0(G, X))$ acts on $C_0(G, X)$ as isometries is clear.

That $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$ follows from left invariance of the Haar measure and submultiplicativity of ω: For any $h \in C_c(G, X)$ and $s \in G$,

$$\begin{aligned} \|\Lambda_s h\|_{p,\omega}^p &= \int_G \|[\Lambda_s h](t)\|^p \omega(t)\, dt \\ &= \int_G \|h(s^{-1}t)\|^p \omega(t)\, dt \\ &= \int_G \|h(t)\|^p \omega(st)\, dt \\ &\leq \omega(s) \int_G \|h(t)\|^p \omega(t)\, dt \\ &= \omega(s) \|h\|_{p,\omega}^p. \end{aligned}$$

Therefore Λ_s induces a map on $L^p(G, X, \omega)$ with the same norm, denoted by the same symbol, and $\|\Lambda_s\| \leq \omega(s)^{1/p}$.

To prove strong continuity of $\Lambda : G \to B(C_0(G,X))$ and $\Lambda : G \to B(L^p(G,X,\omega))$, it is sufficient to establish strong continuity at $e \in G$ on dense subsets of both $C_0(G,X)$ of $L^p(G,X,\omega)$ respectively [19, Corollary 2.5]. By the uniform continuity of elements in $C_c(G,X)$ [46, Lemma 1.88] and the density of $C_c(G,X)$ in $C_0(G,X)$, the result follows for $\Lambda : G \to B(C_0(G,X))$.

To establish the result for $L^p(G,X,\omega)$, let $\varepsilon > 0$ and $h \in C_c(G,X)$ be arbitrary. Let $K := \mathrm{supp}(h)$ and W a precompact neighbourhood of $e \in G$. By uniform continuity of h, there exists a symmetric neighbourhood $V \subseteq W$ of $e \in G$ such that $\|\Lambda_s h - h\|_\infty^p < \varepsilon^p / \left(\sup_{r\in WK} \omega(r)\right) \mu(WK)$ for all $s \in V$. Then, for $s \in V$,

$$\begin{aligned} \|\Lambda_s h - h\|_{p,\omega}^p &= \int_{WK} \|h(s^{-1}r) - h(r)\|^p \omega(r)\, dr \\ &\leq \frac{\varepsilon^p}{\left(\sup_{r\in WK} \omega(r)^p\right) \mu(WK)} \int_{WK} \omega(r)\, dr \\ &\leq \varepsilon^p. \end{aligned}$$

By the density of $C_c(G,X)$ in $L^p(G,X,\omega)$, the result follows. □

Lemma 4.5.8. *Let (A,G,α) be a Banach algebra dynamical system where α is uniformly bounded by a constant C_α and ω a weight on G. Let $\pi : A \to B(X)$ be a non-degenerate bounded representation on a Banach space X. If $f \in C_c(G,X)$, then there exist a compact subset K of G, containing* $\mathrm{supp}(f)$*, and a sequence $(f_n) \subseteq$* $\mathrm{span}\left(\tilde{\pi}(A)(C_c(G) \otimes X)\right)$ *such that* $\mathrm{supp}(f_n) \subseteq K$ *for all n, and (f_n) converges uniformly to f on G. Consequently the representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\tilde{\pi} : A \to B(L^p(G,X,\omega))$ for $1 \leq p < \infty$ (as yielded by Definition 4.5.5) are then non-degenerate.*

Proof. Let $f \in C_c(G,X)$ and $\varepsilon > 0$ be arbitrary. Since $\mathrm{supp}(f)$ is compact, we can fix some precompact open set U_f containing $\mathrm{supp}(f)$. Since π is non-degenerate, for every $s \in G$, there exist finite sets $\{a_{i,s}\}_{i=1}^{n_s}$ and $\{x_{i,s}\}_{i=1}^{n_s}$ such that $\|f(s) - \sum_{i=1}^{n_s} \pi(a_{i,s})x_{i,s}\| < \varepsilon$. Since α is strongly continuous, for each $s \in G$ and $i \in \{1,\ldots,n_s\}$, there exists some precompact neighbourhood $W_{i,s}$ of s, such that $t \in W_{i,s}$ implies $n_s\|\pi\|\|x_{i,s}\|\|a_{i,s} - \alpha_t^{-1} \circ \alpha_s(a_{i,s})\| < \varepsilon$. Furthermore, for any $s \in G$, we can choose a precompact neighbourhood V_s of s such that $t \in V_s$ implies $\|f(s) - f(t)\| < \varepsilon$. Define $W_s := \bigcap_{i=1}^{n_s} W_{i,s} \cap V_s \cap U_f$. Now $\{W_s\}_{s\in G}$ is an open cover of $\mathrm{supp}(f)$, hence let $\{W_{s_j}\}_{j=1}^m$ be a finite subcover. Let $\{u_j\}_{j=1}^m \subseteq C_c(G)$ be a partition of unity such that, for all $j \in \{1,\ldots,m\}$, $0 \leq u_j(t) \leq 1$ for $t \in G$, $\mathrm{supp}(u_j) \subseteq W_{s_j}$, $\sum_{j=1}^m u_j(t) = 1$ for $t \in \mathrm{supp}(f)$, and $\sum_{j=1}^m u_j(t) \leq 1$ for $t \in G$. Then, for $t \in G$,

$$\begin{aligned} &\left\| f(t) - \left(\sum_{j=1}^m \sum_{i=1}^{n_{s_j}} \tilde{\pi}(\alpha_{s_j}(a_{i,s_j}))u_j \otimes x_{i,s_j} \right)(t) \right\| \\ = &\left\| f(t) - \sum_{j=1}^m \sum_{i=1}^{n_{s_j}} u_j(t)\pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j}))x_{i,s_j} \right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \sum_{j=1}^{m} u_j(t) f(t) - \sum_{j=1}^{m} u_j(t) f(s_j) + \sum_{j=1}^{m} u_j(t) f(s_j) - \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} \right. \\
&\qquad \left. + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \sum_{j=1}^{m} u_j(t) \left\| f(t) - f(s_j) \right\| + \sum_{j=1}^{m} u_j(t) \left\| f(s_j) - \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} \right\| \\
&\qquad + \left\| \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \left\| \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \pi(a_{i,s_j}) x_{i,s_j} - \sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} u_j(t) \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \left\| \pi(a_{i,s_j}) x_{i,s_j} - \pi(\alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})) x_{i,s_j} \right\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \|\pi\| \|x_{i,s_j}\| \|a_{i,s_j} - \alpha_t^{-1} \circ \alpha_{s_j}(a_{i,s_j})\| \\
&\leq \varepsilon + \varepsilon + \sum_{j=1}^{m} u_j(t) \sum_{i=1}^{n_{s_j}} \frac{\varepsilon}{n_{s_j}} \\
&\leq \varepsilon + \varepsilon + \varepsilon.
\end{aligned}
$$

Since

$$\sum_{j=1}^{m} \sum_{i=1}^{n_{s_j}} \tilde{\pi}(\alpha_{s_j}(a_{i,s_j})) u_j \otimes x_{i,s_j}$$

is supported in the fixed compact set $\overline{U_f}$, the result follows. □

Combining the previous three lemmas yields:

Corollary 4.5.9. *If (A, G, α) is a Banach algebra dynamical system with α uniformly bounded by a constant C_α, ω a weight on G and $\pi : A \to B(X)$ a bounded representation on a Banach space X, then the pair $(\tilde{\pi}, \Lambda)$ (as yielded by Definition 4.5.5) is a continuous covariant representation of (A, G, α) on $C_0(G, X)$ or $L^p(G, X, \omega)$ for $1 \leq p < \infty$ respectively. Moreover:*

(1) Both representations $\tilde{\pi} : A \to B(C_0(G, X))$ and $\tilde{\pi} : A \to B(L^p(G, X, \omega))$ satisfy $\|\tilde{\pi}\| \leq C_\alpha \|\pi\|$.

(2) The left regular group representation $\Lambda : G \to B(C_0(G, X))$ acts as isometries on $C_0(G, X)$, and the left regular group representation $\Lambda : G \to B(L^p(G, X, \omega))$ is bounded by $\omega^{1/p}$ on G.

(3) The space $C_c(G,X)$, seen as a subspace of $C_0(G,X)$ or $L^p(G,X,\omega)$, is invariant under actions of both A and G on $C_0(G,X)$ or $L^p(G,X,\omega)$ through the representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\Lambda : G \to B(C_0(G,X))$, or $\tilde{\pi} : A \to B(L^p(G,X,\omega))$ and $\Lambda : G \to B(L^p(G,X,\omega))$, respectively.

(4) If $\pi : A \to B(X)$ is non-degenerate, so are both representations $\tilde{\pi} : A \to B(C_0(G,X))$ and $\tilde{\pi} : A \to B(L^p(G,X,\omega))$.

If α is uniformly bounded by $C_\alpha \geq 0$, Corollary 4.5.9 shows that the left regular representation $\lambda : A \to B(A)$ of A is such that the covariant representation $(\tilde{\lambda}, \Lambda)$ of (A,G,α) on $L^1(G,A,\omega)$ (as yielded by Definition 4.5.5) is continuous with $\|\tilde{\lambda}\| \leq C_\alpha$ and $\|\Lambda_s\| \leq \omega(s)$. Moreover, if A has a bounded left or right approximate identity, then λ is non-degenerate, and hence $(\tilde{\lambda}, \Lambda)$ is non-degenerate.

We need two more results before Proposition 4.5.12 can be established.

Lemma 4.5.10. *Let (A,G,α) be a Banach algebra dynamical system with α uniformly bounded. Let ω be a weight on G, and $\lambda : A \to B(A)$ the left regular representation of A. Let $(\tilde{\lambda}, \Lambda)$ be the continuous covariant representation of (A,G,α) on $L^1(G,A,\omega)$ (as yielded by Definition 4.5.5). Then, for all $f \in C_c(G,A)$, $\tilde{\lambda} \rtimes \Lambda(f) \in B(L^1(G,A,\omega))$ leaves the subspace $C_c(G,A)$ of $L^1(G,A,\omega)$ invariant. In fact, if $h \in C_c(G,A) \subseteq L^1(G,A,\omega)$, then $\tilde{\lambda} \rtimes \Lambda(f)h \in L^1(G,A,\omega)$ is given by the pointwise formula*

$$[\tilde{\lambda} \rtimes \Lambda(f)h](s) = \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\, dr \quad (s \in G).$$

Proof. We proceed indirectly, via $C_0(G,A)$, and write $(\tilde{\lambda}_0, \Lambda_0)$ and $(\tilde{\lambda}_1, \Lambda_1)$ for the continuous covariant representations of (A,G,α) on $C_0(G,A)$ and $L^1(G,A,\omega)$, respectively. Let $f, h \in C_c(G,A)$ and consider the integral

$$\tilde{\lambda}_1 \rtimes \Lambda_1(f)h = \int_G \tilde{\lambda}_1(f(r))\Lambda_{1,r}h\, dr \in L^1(G,A,\omega).$$

Let $K := \mathrm{supp}(f) \cdot \mathrm{supp}(h)$, and put $C_0(G,A)_K := \{g \in C_0(G,A) : \mathrm{supp}(g) \subseteq K\}$. Then $C_0(G,A)_K$ is a closed subspace of $C_0(G,A)$ and the inclusion $j_K : C_0(G,A)_K \to L^1(G,A,\omega)$ is bounded, since ω is bounded on compact sets. Define $\psi : G \to C_0(G,A)_K$ by $\psi(r) := \tilde{\lambda}_0(f(r))\Lambda_{0,r}h$ for all $r \in G$. Then ψ is continuous and supported on the compact set $\mathrm{supp}(f) \subseteq G$. Now, by the boundedness of j_K,

$$\int_G \tilde{\lambda}_1(f(r))\Lambda_{1,r}h\, dr = \int_G j_K \circ \psi(r)\, dr = j_K\left(\int_G \psi(r)\, dr\right).$$

Since $\int_G \psi(r)\, dr \in C_0(G,A)_K$, we conclude that $\tilde{\lambda}_1 \rtimes \Lambda_1(f)h \in C_c(G,A)$.

Since the evaluations $\mathrm{ev}_s : C_0(G,A)_K \to A$, sending $g \in C_0(G,A)_K$ to $g(s) \in A$, are bounded for all $s \in G$, we find that, for all $s \in G$,

$$\begin{aligned}\left(\int_G \psi(r)\, dr\right)(s) &= \mathrm{ev}_s\left(\int_G \psi(r)\, dr\right) \\ &= \int_G \mathrm{ev}_s \circ \psi(r)\, dr\end{aligned}$$

$$
\begin{aligned}
&= \int_G \psi(r)(s)\, dr \\
&= \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\, dr.
\end{aligned}
$$

Therefore $[\tilde{\lambda}_1 \rtimes \Lambda_1(f)h](s) = \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\, dr$. □

Lemma 4.5.11. *Let A be a Banach algebra with bounded approximate right identity (u_i) and let $K \subseteq A$ be compact. Then, for any $\varepsilon > 0$, there exists an index i_0 such that $\|au_i\| \geq \|a\| - \varepsilon$ for all $a \in K$ and all $i \geq i_0$.*

Proof. Let $M \geq 1$ be an upper bound for (u_i) and $\varepsilon > 0$ be arbitrary. By compactness of K, there exist $a_1, \ldots, a_n \in K$ such that for all $a \in K$ there exists an index $k \in \{1, \ldots, n\}$ with $\|a - a_k\| < \varepsilon/3M \leq \varepsilon/3$. Let i_0 be such that $\|a_k u_i - a_k\| < \varepsilon/3$ for all $k \in \{1, \ldots, n\}$ and all $i \geq i_0$.

Now, for $a \in K$ arbitrary, let $k_0 \in \{1, \ldots, n\}$ be such that $\|a - a_{k_0}\| < \varepsilon/3$. For any $i \geq i_0$,

$$
\begin{aligned}
\|au_i\| &\geq \|a\| - \|au_i - a_{k_0}u_i\| - \|a_{k_0}u_i - a_{k_0}\| - \|a_{k_0} - a\| \\
&> \|a\| - \frac{\varepsilon}{3M}M - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\
&= \|a\| - \varepsilon.
\end{aligned}
$$

□

Finally, we combine Lemmas 4.5.6–4.5.11 to obtain the following:

Proposition 4.5.12. *Let (A, G, α) be a Banach algebra dynamical system where A has an M-bounded approximate right identity and α is uniformly bounded by a constant C_α. Let ω be a weight on G, and $\lambda : A \to B(A)$ the left regular representation of A. Let $W \subseteq G$ be a precompact neighbourhood of $e \in G$. Then the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) satisfies*

$$
\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq \frac{1}{C_\alpha M \sup_{s \in W} \omega(s)} \|f\|_{1,\omega}
$$

for all $f \in C_c(G, A)$. Consequently $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ is a faithful representation.

Proof. Let (u_i) be an M-bounded approximate right identity of A and $W \subseteq G$ any precompact neighbourhood of $e \in G$. Let $f \in C_c(G, A)$ and $\varepsilon > 0$ be arbitrary. By the uniform continuity of f, there exists a symmetric neighbourhood $V \subseteq W$ of $e \in G$ such that $\|f(r) - f(rs)\| < \varepsilon/2C_\alpha M$ for all $s \in V$ and $r \in G$. By continuity of all maps involved and the assumption that f is compactly supported, the set $\{\alpha_s^{-1}(f(s)) : s \in G\} \subseteq A$ is compact. Lemma 4.5.11 then asserts the existence of an index i_0, such that $\|au_{i_0}\| \geq \|a\| - \varepsilon/2$ for all $a \in \{\alpha_s^{-1}(f(s)) : s \in G\}$.

By Urysohn's Lemma, let $h_0 : G \to [0,1]$ be continuous with $h_0(e) = 1$ and $\mathrm{supp}(h_0) \subseteq V$, so that $h_0 \in C_c(G)$. We may assume $h_0(r) = h_0(r^{-1})$ for all $r \in G$, by replacing h_0 with $r \mapsto \max\{h_0(r), h_0(r^{-1})\}$. Define

$$h := \left(\int_G h_0(t)\,dt\right)^{-1} h_0 \otimes u_{i_0} \in C_c(G, A).$$

Then

$$\begin{aligned}
\|h\|_{1,\omega} &= \left(\int_G h_0(t)\,dt\right)^{-1} \int_G h_0(r)\|u_{i_0}\|\omega(r)\,dr \\
&\leq M \sup_{r\in V} \omega(r) \\
&\leq M \sup_{r\in W} \omega(r).
\end{aligned}$$

For every $s \in G$, we find, using the reverse triangle inequality, noting that $\|f(s)\| = \|\alpha_s \circ \alpha_{s^{-1}}(f(s))\| \leq C_\alpha \|\alpha_{s^{-1}}(f(s))\|$, remembering that h_0 is supported in V, and applying Lemma 4.5.10, that

$$\begin{aligned}
&\|[\tilde{\lambda} \rtimes \Lambda(f)h](s)\| \\
={}& \left\| \int_G \alpha_s^{-1}(f(r))h(r^{-1}s)\,dr \right\| \\
={}& \left\| \int_G \alpha_s^{-1}(f(sr))h(r^{-1})\,dr \right\| \\
={}& \left(\int_G h_0(t)\,dt\right)^{-1} \left\| \int_G h_0(r^{-1})\alpha_s^{-1}(f(sr))u_{i_0}\,dr \right\| \\
\geq{}& \left(\int_G h_0(t)\,dt\right)^{-1} \left\| \int_G h_0(r^{-1})\alpha_s^{-1}(f(s))u_{i_0}\,dr \right\| \\
&\quad - \left(\int_G h_0(t)\,dt\right)^{-1} \left\| \int_G h_0(r^{-1})\alpha_s^{-1}(f(s) - f(sr))u_{i_0}\,dr \right\| \\
\geq{}& \left(\int_G h_0(t)\,dt\right)^{-1} \left(\int_G h_0(r)\,dr\right) \|\alpha_s^{-1}(f(s))u_{i_0}\| \\
&\quad - \left(\int_G h_0(t)\,dt\right)^{-1} \frac{\varepsilon C_\alpha M \left(\int_G h_0(r)\,dr\right)}{2C_\alpha M} \\
\geq{}& \|\alpha_{s^{-1}}(f(s))\| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \\
\geq{}& \frac{1}{C_\alpha}\|f(s)\| - \varepsilon.
\end{aligned}$$

Hence, with $L := \sup_{s\in\mathrm{supp}(f)} \omega(s)$, which is finite since ω is bounded on compact sets,

$$\begin{aligned}\|\tilde{\lambda} \rtimes \Lambda(f)h\|_{1,\omega} &\geq \int_{\mathrm{supp}(f)} \|[\tilde{\lambda} \rtimes \Lambda(f)h](s)\|\omega(s)\,ds \\ &\geq \int_{\mathrm{supp}(f)} \left(\frac{1}{C_\alpha}\|f(s)\| - \varepsilon\right)\omega(s)\,ds \\ &\geq \frac{1}{C_\alpha}\|f\|_{1,\omega} - \varepsilon\mu(\mathrm{supp}(f))L.\end{aligned}$$

Now, since $\|h\|_{1,\omega} \leq M \sup_{r\in W} \omega(r)$, we obtain

$$\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq \frac{1}{C_\alpha M \sup_{r\in W}\omega(r))}\|f\|_{1,\omega} - \frac{\varepsilon L}{M\sup_{r\in W}\omega(r)}\mu(\mathrm{supp}(f)).$$

Because $\varepsilon > 0$ was chosen arbitrarily, $\|\tilde{\lambda} \rtimes \Lambda(f)\| \geq (C_\alpha M \sup_{r\in W}\omega(r))^{-1}\|f\|_{1,\omega}$ now follows. ☐

We now combine our previous results, notably (4.5.1) and Proposition 4.5.12, to obtain sufficient conditions for a crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ to be isomorphic to a generalized Beurling algebra, and also collect a number of direct consequences in the following result. The desired reverse inequality to (4.5.1) is a consequence of Proposition 4.5.12, supplying the first inequality in (4.5.2) and the second inequality in (4.5.2), which follows from the assumption that $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous.

Theorem 4.5.13. *Let (A, G, α) be a Banach algebra dynamical system where A has an M-bounded approximate right identity and α is uniformly bounded by a constant C_α. Let ω be a weight on G. Let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) with $C^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|\pi\| < \infty$ and satisfying $\nu^{\mathcal{R}}(r) = \sup_{(\pi,U)\in\mathcal{R}} \|U_r\| \leq \omega(r)$ for all $r \in G$. Let λ be the left regular representation of A, and suppose that the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) is $\mathcal{R}$-continuous. Then, for all $f \in C_c(G, A)$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set,*

$$\begin{aligned}\left(\frac{1}{C_\alpha M \inf_{W\in\mathcal{Z}} \sup_{r\in W}\omega(r)}\right)\|f\|_{1,\omega} &\leq \|\tilde{\lambda} \rtimes \Lambda(f)\| \qquad (4.5.2)\\ &\leq \|\tilde{\lambda} \rtimes \Lambda\|\sigma^{\mathcal{R}}(f) \leq \|\tilde{\lambda} \rtimes \Lambda\|C^{\mathcal{R}}\|f\|_{1,\omega}.\end{aligned}$$

In particular, $\sigma^{\mathcal{R}}$ is a norm on $C_c(G, A)$, so that $C_c(G, A)$ can be identified with a subspace of $(A \rtimes_\alpha G)^{\mathcal{R}}$. Since the norms $\sigma^{\mathcal{R}}$ and $\|\cdot\|_{1,\omega}$ on $C_c(G, A)$ are equivalent, there exists a topological isomorphism between the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ and the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ that is the identity on $C_c(G, A)$.

The multiplication on the common dense subspace $C_c(G, A)$ of the spaces $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is given by

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \quad (f, g \in C_c(G, A),\ s \in G).$$

The faithful representation $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ of the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ into $B(L^1(G, A, \omega))$.

Using Corollary 4.5.9, we have the following consequence of Theorem 4.5.13, where the isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is isometric.

Corollary 4.5.14. *Let (A, G, α) be a Banach algebra dynamical system where A has a 1-bounded approximate right identity and α lets G act as isometries on A. Let ω be a weight on G, and λ the left regular representation of A. Then the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ on $L^1(G, A, \omega)$ (as yielded by Definition 4.5.5) is such that $\tilde{\lambda}$ is contractive and Λ is bounded by ω.*

Suppose furthermore that $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set, and that $\mathcal{R}$ is a uniformly bounded class of continuous covariant representations with $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, and satisfying

$$C^{\mathcal{R}} = \sup_{(\pi, U) \in \mathcal{R}} \|\pi\| \leq 1,$$

and

$$\nu^{\mathcal{R}}(r) = \sup_{(\pi, U) \in \mathcal{R}} \|U_r\| \leq \omega(r) \quad (r \in G).$$

Then $\sigma^{\mathcal{R}}(f) = \|f\|_{1,\omega}$ for $f \in C_c(G, A)$, and hence $(A \rtimes_\alpha G)^{\mathcal{R}}$ is isometrically isomorphic to the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$.

Moreover, $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ is an isometric embedding as a Banach algebra.

Proof. Since $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, we have $\|\tilde{\lambda} \rtimes \Lambda\| \leq 1$, and by hypothesis $C^{\mathcal{R}} \leq 1$. Therefore, by Theorem 4.5.13, for every $f \in C_c(G, A)$,

$$\|f\|_{1,\omega} \leq \|\tilde{\lambda} \rtimes \Lambda(f)\| \leq \|\tilde{\lambda} \rtimes \Lambda\| \sigma^{\mathcal{R}}(f) \leq C^{\mathcal{R}} \|\tilde{\lambda} \rtimes \Lambda\| \|f\|_{1,\omega} \leq \|f\|_{1,\omega}.$$

We conclude that $C^{\mathcal{R}} = \|\tilde{\lambda} \rtimes \Lambda\| = 1$, and the result now follows. □

Remark 4.5.15. Certainly if the weight $\omega : G \to [0, \infty)$ is continuous in $e \in G$ and $\omega(e) = 1$, then $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, for example if ω is taken to be a continuous positive character of G.

Remark 4.5.16. We note that the representation

$$(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$$

does not equal the left regular representation of $L^1(G, A, \omega; \alpha)$ in general, but they are always conjugate. To see this, define, for $h \in C_c(G, A)$ and $s \in G$, $\check{h}(s) := \alpha_{s^{-1}}(h(s))$, $\hat{h}(s) := \alpha_s(h(s))$. Then $\hat{\ } : C_c(G, A) \to C_c(G, A)$ and $\check{\ } : C_c(G, A) \to C_c(G, A)$ are mutual inverses and, since α is uniformly bounded, extend to mutually inverse Banach space isomorphisms of $L^1(G, A, \omega; \alpha)$ onto itself. Then $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ and the left regular representation λ of $L^1(G, A, \omega; \alpha)$ are conjugate under $\hat{\cdot}$. Indeed by Lemma 4.5.10, for $f, h \in C_c(G, A)$ and $s \in G$,

$$\begin{aligned}
\left(\tilde{\lambda} \rtimes \Lambda(f)\check{h}\right)^{\wedge}(s) &= \alpha_s\left([\tilde{\lambda} \rtimes \Lambda(f)\check{h}](s)\right) \\
&= \alpha_s\left(\int_G \alpha_s^{-1}(f(r))\check{h}(r^{-1}s)\,dr\right) \\
&= \alpha_s\left(\int_G \alpha_s^{-1}(f(r))\alpha_{s^{-1}r}(h(r^{-1}s))\,dr\right) \\
&= \int_G f(r)\alpha_r(h(r^{-1}s))\,dr \\
&= [f * h](s) \\
&= [\lambda(f)h](s).
\end{aligned}$$

Hence $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ and the left regular representation

$$\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$$

of $L^1(G, A, \omega; \alpha)$ are conjugate as claimed. Note that $\hat{\cdot}$ is the identity if $\alpha = \mathrm{triv}$, hence in that case $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} = \lambda$.

We continue the main line with the following trivial but important observation: If $(\tilde{\lambda}, \Lambda) \in \mathcal{R}$, for example, by taking $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$, then certainly $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous, hence the conclusions in Theorem 4.5.13 hold, and in particular the algebras $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ are topologically isomorphic. A similar remark is applicable to Corollary 4.5.14, giving sufficient conditions for the mentioned topological isomorphism to be isometric. Hence we have the following:

Theorem 4.5.17. *Let (A, G, α) be a Banach algebra dynamical system where A has a bounded approximate right identity and α is uniformly bounded. Let ω be a weight on G and let the non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ be as yielded by Definition 4.5.5. Then the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ are topologically isomorphic via an isomorphism that is the identity on $C_c(G, A)$.*

Furthermore, the map $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a topological embedding of $L^1(G, A, \omega; \alpha)$ into $B(L^1(G, A, \omega))$.

If A has a 1-bounded two-sided approximate identity, α lets G act as isometries on A and $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set, then the isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ is an isometry, and the embedding of $L^1(G, A, \omega; \alpha)$ into $B(L^1(G, A, \omega))$ is isometric.

Remark 4.5.18. As noted in Remark 4.5.16, when $\alpha = \mathrm{triv}$, then $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ equals the left regular representation $\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$ of $L^1(G, A, \omega; \alpha)$.

Remark 4.5.19. We note that, for $(A,G,\alpha)=(\mathbb{K},G,\mathrm{triv})$, the second part of Theorem 4.5.17 asserts that $(\mathbb{K}\rtimes_{\mathrm{triv}}G)^{\mathcal{R}}$ is isometrically isomorphic to the classical Beurling algebra $L^1(G,\omega)$, provided that $\inf_{W\in\mathcal{Z}}\sup_{r\in W}\omega(r)=1$ (which is certainly true if ω is continuous at $e\in G$ and $\omega(e)=1$). In particular $L^1(G)$ is isometrically isomorphic to a crossed product. Under the condition $\inf_{W\in\mathcal{Z}}\sup_{r\in W}\omega(r)=1$, combining Remark 4.5.16 and Theorem 4.5.17 also shows that the left regular representation of $L^1(G,\omega)$ is an isometric embedding of $L^1(G,\omega)$ into $B(L^1(G,\omega))$.

Hence, provided that A has a bounded approximate right identity, the generalized Beurling algebras $L^1(G,A,\omega;\alpha)$, and in particular the classical Beurling algebras $L^1(G,\omega)$ for $A=\mathbb{K}$, are isomorphic to a crossed product associated with a Banach algebra dynamical system. Therefore, in the case where the algebra A has a two-sided identity, the General Correspondence Theorem (Theorem 4.2.1) determines the non-degenerate bounded representations of generalized Beurling algebras. This we will elaborate on in the rest of the section. In cases where the algebra is trivial, i.e., $A=\mathbb{K}$, we regain classical results on the representation theory of $L^1(G)$ and other classical Beurling algebras.

Assume, in addition to the hypothesis in Theorem 4.5.13, that A has an M-bounded two-sided approximate identity and that all continuous covariant representations in $\mathcal{R}$ are non-degenerate. In that case, we claim that the non-degenerate $\mathcal{R}$-continuous covariant representations are precisely the non-degenerate continuous covariant representations (π,U) of (A,G,α), with no further restriction on π, but with U such that $\|U_r\|\leq C_U\omega(r)$ for all $r\in G$ and a U-dependent constant C_U. To see this, we start by noting that, for $f\in C_c(G,A)$,

$$\begin{aligned}
\|\pi\rtimes U(f)\| &\leq \int_G \|\pi(f(r))\|\|U_r\|\,dr\\
&\leq \int_G \|\pi\|\|f(r)\|C_U\omega(r)\,dr\\
&\leq C_U\|\pi\|\int_G \|f(r)\|\omega(r)\,dr\\
&= C_U\|\pi\|\|f\|_{1,\omega}\\
&\leq C'_{(\pi,U)}\sigma^{\mathcal{R}}(f)
\end{aligned}$$

for some $C'_{(\pi,U)}\geq 0$, since $\|\cdot\|_{1,\omega}$ and $\sigma^{\mathcal{R}}$ are equivalent.

For the converse, we use that A has a bounded approximate left identity and that $\mathcal{R}$ consists of non-degenerate continuous covariant representations. If (π,U) is a non-degenerate $\mathcal{R}$-continuous representation of (A,G,α), then the General Correspondence Theorem (Theorem 4.2.1) asserts that

$$(\pi,U)=(\overline{(\pi\rtimes U)^{\mathcal{R}}}\circ i_A^{\mathcal{R}},\overline{(\pi\rtimes U)^{\mathcal{R}}}\circ i_G^{\mathcal{R}}),$$

where $\overline{(\pi\rtimes U)^{\mathcal{R}}}:\mathcal{M}_l((A\rtimes_\alpha G)^{\mathcal{R}})\to B(X_\pi)$ is the non-degenerate bounded representation induced by the non-degenerate bounded representation $(\pi\rtimes U)^{\mathcal{R}}:(A\rtimes_\alpha G)^{\mathcal{R}}\to B(X_\pi)$. However if $T:(A\rtimes_\alpha G)^{\mathcal{R}}\to B(X)$ is any non-degenerate bounded representation, then [19, Proposition 7.1] asserts that there exists a constant $C_T := M_l^{\mathcal{R}}\|T\|$, with $M_l^{\mathcal{R}}$ a bound for a bounded approximate left identity in $(A \rtimes_\alpha G)^{\mathcal{R}}$, such that

$$\|\overline{T} \circ i_G^{\mathcal{R}}(r)\| \leq C_T \nu^{\mathcal{R}}(r) \leq C_T \omega(r) \quad (r \in G). \tag{4.5.3}$$

Therefore, $r \mapsto \|U_r\|$ is bounded by a multiple of ω, as claimed.

We now take $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ as in Theorem 4.5.17. Theorem 4.5.17 shows that the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ can be identified with those of $(A \rtimes_\alpha G)^{\mathcal{R}}$. By the General Correspondence Theorem (Theorem 4.2.1) the latter are in natural bijection with the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and these we have just described. Hence the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ are in natural bijection with pairs (π, U) as above. Furthermore, slightly simplified versions of [19, Equations (8.1) and (8.2)] (cf. Remark 4.5.21) give explicit formulas for retrieving (π, U) from a non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}} \simeq L^1(G, A, \omega; \alpha)$. Combining all this, we obtain the following correspondence between the non-degenerate continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$:

Theorem 4.5.20. *Let (A, G, α) be a Banach algebra dynamical system where A has a two-sided approximate identity and α is uniformly bounded by C_α. Let ω be a weight on G. Then the following maps are mutual inverses between the non-degenerate continuous covariant representations (π, U) of (A, G, α) on a Banach space X, satisfying $\|U_r\| \leq C_U \omega(r)$ for some $C_U \geq 0$ and all $r \in G$, and the non-degenerate bounded representations $T : L^1(G, A, \omega; \alpha) \to B(X)$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ on X:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G \pi(f(r)) U_r \, dr \right) =: T^{(\pi,U)} \quad (f \in C_c(G, A)),$$

determining a non-degenerate bounded representation $T^{(\pi,U)}$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, and,

$$T \mapsto \begin{pmatrix} a \mapsto \text{SOT-lim}_{(V,i)} T(z_V \otimes a u_i), \\ s \mapsto \text{SOT-lim}_{(V,i)} T(z_V(s^{-1} \cdot) \otimes u_i) \end{pmatrix} =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G, A)$ is chosen such that $z_V \geq 0$, supported in $V \in \mathcal{Z}$, $\int_G z_V(r) dr = 1$, and (u_i) is any bounded approximate left identity of A.

Furthermore, if A has an M-bounded approximate left identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U \|\pi\|$,

(2) $\|\pi^T\| \leq (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \, \|T\|$,

(3) $\|U_s^T\| \leq M \, (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \, \|T\| \omega(s) \quad (s \in G)$.

Proof. Except for the claimed bounds for $\|T^U\|$, $\|\pi^T\|$ and $\|U^T\|$, all statements have been proven preceding the statement of the theorem. We will now establish these three bounds.

We prove (1). Let (π, U) be a non-degenerate continuous covariant representations of (A, G, α) on a Banach space X, satisfying $\|U_r\| \leq C_U \omega(r)$ for some $C_U \geq 0$ and all $r \in G$. Then, for any $f \in C_c(G, A)$,

$$\begin{aligned}
\|T^{(\pi,U)}(f)\| &= \left\| \int_G \pi(f(r)) U_r \, dr \right\| \\
&\leq \int_G \|\pi\| \|f(r)\| \|U_r\| \, dr \\
&\leq \|\pi\| C_U \int_G \|f(r)\| \omega(r) \, dr \\
&= \|\pi\| C_U \|f\|_{1,\omega}.
\end{aligned}$$

Therefore $\|T^{(\pi,U)}\| \leq \|\pi\| C_U$.

We prove (2). Let $T : L^1(G, A, \omega; \alpha) \to B(X)$ be a non-degenerate bounded representations of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ on X. Choose a bounded two-sided approximate identity (u_i) of A. Then, for any $a \in A$,

$$\begin{aligned}
\|T(z_V \otimes au_i)\| &\leq \|T\| \|z_V \otimes au_i\|_{1,\omega} \\
&\leq \|T\| \int_G z_V(r) \|au_i\| \omega(r) \, dr \\
&= \|T\| \|au_i\| \int_G z_V(r) \omega(r) \, dr \\
&\leq \|T\| \|au_i\| \sup_{r \in V} \omega(r) \int_G z_V(r) \, dr \\
&= \|T\| \|au_i\| \sup_{r \in V} \omega(r).
\end{aligned}$$

Since, in particular, (u_i) is an approximate right identity of A, for any $\varepsilon_1 > 0$, there exists an index i_0 such that $i \geq i_0$ implies $\|au_i\| \leq \|a\| + \varepsilon_1$. Also, for any $\varepsilon_2 > 0$, there exists some $V_0 \in \mathcal{Z}$ such that $\sup_{r \in V_0} \omega(r) \leq \inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2$. Now, if $(V, i) \geq (V_0, i_0)$, then $V_0 \supseteq V$ and $i \geq i_0$, and hence

$$\begin{aligned}
\|T(z_V \otimes au_i)\| &\leq \|T\| \|au_i\| \sup_{r \in V} \omega(r) \\
&\leq \|T\| \|au_i\| \sup_{r \in V_0} \omega(r) \\
&\leq \|T\| (\|a\| + \varepsilon_1) \left(\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2 \right).
\end{aligned}$$

Therefore, if $x \in X$, then

$$\begin{aligned}
\|\pi^T(a)x\| &= \lim_{(V,i)} \|T(z_V \otimes au_i)x\| \\
&= \lim_{(V,i)\geq(V_0,i_0)} \|T(z_V \otimes au_i)x\| \\
&\leq \|T\|(\|a\|+\varepsilon_1)\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon_2\right)\|x\|.
\end{aligned}$$

Since ε_1 and ε_2 we chosen arbitrarily, $\|\pi^T\| \leq \|T\|\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)\right)$ now follows.

We prove (3). Let (u_i) be an M-bounded approximate left identity of A. Fix $s \in G$. Let $\varepsilon > 0$ be arbitrary and let $V_0 \in \mathcal{Z}$ be such that $\sup_{r\in V_0}\omega(r) \leq \inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon$. Fix some index i_0, then, for every $(V,i)\geq(V_0,i_0)$,

$$\begin{aligned}
\|T(z_V(s^{-1}\cdot)\otimes u_i)\| &\leq \|T\|\|z_V(s^{-1}\cdot)\otimes u_i\|_{1,\omega} \\
&= \|T\|\int_G z_V(s^{-1}r)\|u_i\|\omega(r)\,dr \\
&\leq M\|T\|\int_G z_V(r)\omega(sr)\,dr \\
&\leq M\|T\|\int_G z_V(r)\omega(s)\omega(r)\,dr \\
&= M\|T\|\omega(s)\int_V z_V(r)\omega(r)\,dr \\
&\leq M\|T\|\left(\sup_{r\in V_0}\omega(r)\right)\omega(s)\int_V z_V(r)\,dr \\
&\leq M\|T\|\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon\right)\omega(s).
\end{aligned}$$

Therefore, if $x \in X$, then

$$\begin{aligned}
\|U_s^T x\| &= \lim_{(V,i)} \|T(z_V(s^{-1}\cdot)\otimes u_i)x\| \\
&= \lim_{(V,i)\geq(V_0,i_0)} \|T(z_V(s^{-1}\cdot)\otimes u_i)x\| \\
&\leq M\|T\|\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon\right)\omega(s)\|x\|.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily, $\|U_r^T\| \leq M\|T\|\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)\right)\omega(s)$ now follows. □

Remark 4.5.21. Our reconstruction formulas in Theorem 4.5.20 differs slightly from those given in [19, Equations (8.1) and (8.2)], where the reconstruction formula for U^T is given as

$$s \mapsto \text{SOT-lim}_{(V,i)}T(z_V(s^{-1}\cdot)\otimes\alpha_s(u_i)), \tag{4.5.4}$$

with (u_i) any bounded approximate left identity of A. However, if (u_i) is any bounded approximate left identity of A and $s \in G$ is fixed, then $(\alpha_{s^{-1}}(u_i))$ is also a bounded approximate left identity of A, and using this particular choice in (4.5.4) gives the formula in Theorem 4.5.20.

For the Banach algebra dynamical system $(\mathbb{K}, G, \mathrm{triv})$ and weight ω on G, Theorem 4.5.20 simplifies. We collect the statements from Theorem 4.5.20 concerning representations and some material from Remark 4.5.16, Corollary 4.5.14 in the following result, which contains a few classical results as special cases: For one-dimensional representations, the result reduces to the bijection between ω-bounded characters of G and multiplicative functionals of the Beurling algebra $L^1(G,\omega)$, see, e.g., [26, Theorem 2.8.2] (where, contrary to our general groups, G is assumed to be abelian). In the case where ω is the constant 1, the result reduces to the classical bijection between uniformly bounded strongly continuous representations of G and non-degenerate bounded representations of $L^1(G)$, see, e.g., [24, Assertion VI.1.32].

Corollary 4.5.22. *Let ω be a weight on G. With (z_V) as in Theorem 4.5.20, the maps*

$$U \mapsto \left(f \mapsto \int_G f(r) U_r \, dr \right) =: T^U \quad (f \in C_c(G)),$$

determining a non-degenerate bounded representation T^U of the Beurling algebra $L^1(G,\omega)$, and

$$T \mapsto \left(s \mapsto \text{SOT-lim}_V T(z_V(s^{-1}\cdot)) \right) =: U^T$$

are mutual inverses between the strongly continuous group representations U of G on a Banach space X, satisfying $\|U_r\| \leq C_U \omega(r)$, for some $C_U \geq 0$ and all $r \in G$, and the non-degenerate bounded representations $T : L^1(G,\omega) \to B(X)$ of the Beurling algebra $L^1(G,\omega)$ on X).

If the weight satisfies $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, then $\|T^U\| = \sup_{r \in G} \|U_r\|/\omega(r)$ and $\|U^T_r\| \leq \|T\|\omega(r)$ for all $r \in G$.

Proof. The only statement that does not follow directly from Theorem 4.5.20 is that $\|T^U\| = \sup_{r\in G} \|U_r\|/\omega(r)$, when $\sup_{W \in \mathcal{Z}} (\sup_{r \in W} \omega(r))^{-1} = 1$.

To establish this, we note that

$$\|U_r\| = \omega(r) \frac{\|U_r\|}{\omega(r)} \leq \left(\sup_{s \in G} \frac{\|U_s\|}{\omega(s)} \right) \omega(r).$$

Therefore, we can replace C_U with $\sup_{r\in G} \|U_r\|/\omega(r)$, and, by the bound (1) in Theorem 4.5.20, $\|T^U\| \leq \sup_{r \in G} \|U_r\|/\omega(r)$. The reverse inequality follows from (3) in Theorem 4.5.20, when noting that the maps $U \mapsto T^U$ and $T \mapsto U^T$ are mutual inverses. ◻

Remark 4.5.23. For one-dimensional representations, Corollary 4.5.22 implies that continuous characters $\chi : G \to \mathbb{C}^\times$ of G, such that $|\chi(r)| \leq C_\chi \omega(r)$ for some C_χ and all $r \in G$, are in natural bijection with the one-dimensional representations of

$L^1(G,\omega)$. Since this is a Banach algebra, such representations are contractive, and the final part of Corollary 4.5.22 then asserts that one can actually take $C_\chi = 1$ (cf. [26, Lemma 2.8.2] for abelian G). One can also verify this directly by noting that, if there exists some $s \in G$ for which $|\chi(s)| > \omega(s)$, then, for all $n \in \mathbb{N}$, by submultiplicativity of ω,

$$\left(\frac{|\chi(s)|}{\omega(s)}\right)^n = \frac{|\chi(s^n)|}{\omega(s)^n} \leq C_\chi \frac{\omega(s^n)}{\omega(s)^n} \leq C_\chi.$$

Therefore, since $|\chi(s)| > \omega(s)$, we must have that $C_\chi = \infty$, which is absurd. Hence $|\chi(r)| \leq \omega(r)$ for all $r \in G$.

4.6 Other types for (π, U)

For a given Banach algebra dynamical system (A, G, α) we have thus far been concerned with a uniformly bounded class of pairs (π, U), where $\pi : A \to B(X)$ and $U : G \to B(X)$ are multiplicative representations, U is strongly continuous, and satisfy the covariance condition

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_r(a))$$

for all $r \in G$ and $a \in A$. On the other hand, in [19, Proposition 6.5], we have encountered an example of a pair (π, U) where π and U are both anti-multiplicative and satisfy the anti-covariance condition

$$U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$$

for all $r \in G$ and $a \in A$. Suppose one has a uniformly bounded class $\mathcal{R}$ of such pairs (π, U), with U strongly continuous, π non-degenerate and that A has a bounded "appropriately sided" approximate identity, can one then find a Banach algebra of crossed product type again, such that its non-degenerate bounded (perhaps anti-) representations are in natural bijection with the $\mathcal{R}$-continuous pairs (ρ, V), satisfying the aforementioned requirements for elements of $\mathcal{R}$? What about pairs (π, U) where π is multiplicative, U is anti-multiplicative and a covariance condition is satisfied? Can one, to ask a more fundamental question, expect a meaningful theory to exist for such pairs?

In this section we address these matters. We start by determining what appears to be the natural "reasonable" requirements in this vein on (π, U) for a meaningful theory to exist (and which are not met in the second-mentioned example). There turn out to be four cases. For each case we indicate a Banach algebra dynamical system (B, H, β) such that $B = A$ and $H = G$ as sets, and such that the given maps $\pi : B \to B(X)$ and $U : H \to B(X)$ are now multiplicative and satisfy a covariance condition. This brings us back into the realm of the correspondence as in Theorem 4.2.1 or [19, Theorem 8.1], but we leave it to the reader to formulate the resulting correspondence theorem for the other three types of uniformly bounded classes of non-degenerate continuous pairs (π, U).

After this, we turn to actions of A and G on $C_c(G, A)$. While this is not, in general, a Banach space, several Banach spaces are naturally obtained from $C_c(G, A)$ via quotients and/or completions, hence it is for this space that we list sixteen canonical pairs of actions, with each of the four "reasonable" properties occurring four times. We then explain that, even though the formulas look quite different, there is essentially only one pair, and the fifteen others can be derived from it. We conclude with natural pairs (π, U) of commuting actions on $C_c(G, A)$.

This section is, in a sense, elementary and almost entirely algebraic in nature. Nevertheless, we thought it worthwhile to make a systematic inventorization, once and for all, of the "reasonable" properties of pairs (π, U), the natural actions on A-valued function spaces on G, and the interrelations between the various formulas. A particular case of the results in the present section will be instrumental in Section 4.8 where we explain how non-degenerate right– and bimodules over generalized Beurling algebras fit into the general framework of crossed products of Banach algebras.

To start with, let (A, G, α) be a Banach algebra dynamical system. What are the "reasonable" properties of (π, U) that can lead to a meaningful theory? Let us assume that $\pi : A \to B(X)$ is linear and multiplicative or anti-multiplicative, that $U : G \to B(X)$ is a multiplicative or anti-multiplicative map of G into the group of invertible elements of $B(X)$, and that

$$U_r \pi(a) U_r^{-1} = \pi(\delta_r(a)) \tag{4.6.1}$$

for all $a \in A$ and $r \in G$, where δ is a multiplicative or anti-multiplicative map from G into the automorphisms or anti-automorphisms of A. This is "asking for the most general setup". We start by arguing that δ should map G into the automorphisms of A. Indeed, if π is multiplicative, $r \in G$ and $a_1, a_2 \in A$, then

$$\begin{aligned} \pi(\delta_r(a_1 a_2)) &= U_r \pi(a_1 a_2) U_r^{-1} \\ &= U_r \pi(a_1) U_r^{-1} U_r \pi(a_2) U_r^{-1} \\ &= \pi(\delta_r(a_1)) \pi(\delta_r(a_2)) \\ &= \pi(\delta_r(a_1) \delta_r(a_2)). \end{aligned}$$

If π is anti-multiplicative, then again

$$\begin{aligned} \pi(\delta_r(a_1 a_2)) &= U_r \pi(a_1 a_2) U_r^{-1} \\ &= U_r \pi(a_2) U_r^{-1} U_r \pi(a_1) U_r^{-1} \\ &= \pi(\delta_r(a_2)) \pi(\delta_r(a_1)) \\ &= \pi(\delta_r(a_1) \delta_r(a_2)). \end{aligned}$$

Hence one is led to assume that δ maps G into $\mathrm{Aut}(A)$, still leaving open the possible choice of $\delta : G \to \mathrm{Aut}(A)$ being multiplicative or anti-multiplicative.

To continue, if U is anti-multiplicative, then (4.6.1) implies, for $a \in A$ and $r_1, r_2 \in G$,

$$\pi(\delta_{r_1 r_2}(a)) = U_{r_1 r_2} \pi(a) U_{r_1 r_2}^{-1}$$

$$
\begin{aligned}
&= U_{r_2} U_{r_1} \pi(a) U_{r_1}^{-1} U_{r_2}^{-1} \\
&= \pi(\delta_{r_2} \circ \delta_{r_1}(a)).
\end{aligned}
$$

Therefore, unless one imposes a further relation between π and U, it seems that only the possibility that δ is also anti-multiplicative will lead to a meaningful theory. Likewise, the multiplicativity of U "implies" that δ should be multiplicative. Using that δ_r is multiplicative on A for $r \in G$, it is easily seen that the covariance condition yields no implications on the nature of π.

With (A, G, α) given, the relevant non-trivial choice for a multiplicative δ is α, and for an anti-multiplicative δ it is α^o where $\alpha_r^o := \alpha_{r^{-1}}$ for all $r \in G$; the reason for this notation will become clear in a moment. We will consider these non-trivial choices for δ first, and return to $\delta = \mathrm{triv}$ later.

Hence we have to consider four meaningful possibilities for a pair (π, U) and the relation between π and U. If we let, e.g., (a, m) denote the case where π is anti-multiplicative and U is multiplicative, then, for (m, m) and (a, m), one should require

$$
U_r \pi(a) U_r^{-1} = \pi(\alpha_r(a)),
$$

and for (m, a) and (a, a), one should require

$$
U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a)) = \pi(\alpha_r^o(a))
$$

for all $a \in A$ and $r \in G$.

Now note that, with G^o denoting the opposite group, $\alpha^o : G^o \to \mathrm{Aut}(A)$ is a multiplicative strongly continuous map if α is. Therefore, if (A, G, α) is a Banach algebra dynamical system, then so is (A, G^o, α^o). Furthermore, if A^o is the opposite algebra, then $\mathrm{Aut}(A) = \mathrm{Aut}(A^o)$. Therefore, if (A, G, α) is a Banach algebra dynamical system, so is (A^o, G, α). Combining these two, a Banach algebra dynamical system has a third natural companion Banach algebra dynamical system, namely (A^o, G^o, α^o). In each of these three cases, the Banach algebra is A as a set, and the group is G as a set. Hence the given maps $\pi : A \to B(X)$ and $U : G \to B(X)$ can be viewed unaltered as maps for the new system, denoted by $\tilde{\pi}$ and $\tilde{U}$. The crux is, then, that anti-multiplicative representations of A correspond to multiplicative representations of A^o, and likewise for G and G^o. Hence, regardless of the type of (π, U), one can always pass to a suitable companion Banach algebra dynamical system to ensure that the same pair of maps is a pair of type (m, m) for the companion Banach algebra dynamical system. For example, if (π, U) is of type (a, a) for (A, G, α) and satisfies $U_r \pi(a) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for $a \in A$ and $r \in G$, then $\tilde{\pi} : A^o \to B(X)$ and $\tilde{U} : G^o \to B(X)$ form a pair of type (m, m) for (A^o, G^o, α^o), satisfying $\tilde{U}_r \tilde{\pi}(a) \tilde{U}_r^{-1} = \pi(\alpha_r^o(a))$ for $a \in A^o$ and $r \in G^o$. Hence, $(\tilde{\pi}, \tilde{U})$ is a covariant pair of type (m, m) for (A^o, G^o, α^o), and we are back at our original type of objects. One can argue similarly for the types (a, m) and (m, a), and this leads to Table 4.1.

We can now point out how classes of pairs (π, U) of other types than (m, m) can be related to representations of a crossed product of a Banach algebra. For

Type of (π, U) for (A, G, α)	Should require that $U_r\pi(a)U_r^{-1} =$	$(\tilde{\pi}, \tilde{U})$ is type (m, m) for	$\tilde{U}_r\tilde{\pi}(a)\tilde{U}_r^{-1} =$
(m, m)	$\pi(\alpha_r(a))$	(A, G, α)	$\tilde{\pi}(\alpha_r(a))$
(m, a)	$\pi(\alpha_{r^{-1}}(a))$	(A, G^o, α^o)	$\tilde{\pi}(\alpha_r^o(a))$
(a, m)	$\pi(\alpha_r(a))$	(A^o, G, α)	$\tilde{\pi}(\alpha_r(a))$
(a, a)	$\pi(\alpha_{r^{-1}}(a))$	(A^o, G^o, α^o)	$\tilde{\pi}(\alpha_r^o(a))$

Table 4.1

example, suppose that $\mathcal{R}$ is a uniformly bounded class (as in Section 4.2) of non-degenerate continuous pairs (π, U) where $\pi : A \to B(X)$ and $U : G \to B(X)$ are both anti-multiplicative satisfying $U_r\pi(a)U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$. We pass to the system (A^o, G^o, α^o) and consider the class $\tilde{\mathcal{R}}$ consisting of all pairs $(\tilde{\pi}, \tilde{U}) = (\pi, U)$, for $(\pi, U) \in \mathcal{R}$. Then $\tilde{\mathcal{R}}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A^o, G^o, α^o), and the general correspondence theorem, Theorem 4.2.1 or [19, Theorem 8.1] furnishes a bijection between the non-degenerate bounded (multiplicative) representations of $(A^o \rtimes_{\alpha^o} G^o)^{\tilde{\mathcal{R}}}$ and the non-degenerate $\tilde{\mathcal{R}}$-continuous covariant representations of (A^o, G^o, α^o). It is then a matter of routine, left to the reader, to reformulate the latter class as pairs (π, U) of type (a, a) for (A, G, α) again, being aware that the Haar measure for G differs from that of G^o by the modular function. The remaining types (m, a) and (a, m) can be treated similarly and bring the non-degenerate bounded (always multiplicative) representations of $(A \rtimes_{\alpha^o} G^o)^{\tilde{\mathcal{R}}}$ and $(A^o \rtimes_{\alpha} G)^{\tilde{\mathcal{R}}}$, respectively, into play.

We now turn to what can perhaps be regarded as the sixteen canonical types of actions of A and G on the linear space $C_c(G, A)$ (and hence on many natural Banach spaces). They are listed in Table 4.2 and were originally obtained by judiciously experimenting with various candidate expressions. In this table $a \in A$, $r, s \in G$, $f \in C_c(G, A)$ and $\chi : G \to \mathbb{C}^\times$ is a continuous character. The possibility of inserting χ enables one to arrange, by choosing the modular function, that the group actions as in the lines 3, 8, 11 and 16 are isometric on L^p-type spaces for $1 \leq p < \infty$.

We will now explain why, essentially, there is only one canonical type of action from which all others can be derived. To start with, note that the spaces $C_c(G, A)$, $C_c(G^o, A)$, $C_c(G, A^o)$ and $C_c(G^o, A^o)$ can all be identified. This can be put to good use as follows: Suppose one has verified that the formulas in line 1 yield a pair (π, U) of type (m, m) for any Banach algebra dynamical system. Then one can apply this to (A, G^o, α^o) and view the resulting actions of A and G^o on $C_c(G^o, A)$, which are of type (m, m), as actions of A and G on $C_c(G, A)$. It is immediate that the resulting pair (π, U) will be of type (m, a) for (A, G, α). In fact, it is line 5 in the table. Likewise, line 1 for (A^o, G, α) and for (A^o, G^o, α^o) yields line 9 and 13 for (A, G, α), respectively. Similarly line 2 yields the lines 6, 10 and 14, line 3 yields the lines 7, 11 and 15, and line 4 yields the lines 8, 12 and 16. Thus the actions in lines 1 through 4 generate all others via passing to companion Banach algebra dynamical systems. These four actions of (A, G, α) of type (m, m) are, in turn, also essentially

No.	$(\pi(a)f)(s)$	$(U_r f)(s)$	Type (π, U)	$U_r\pi(a)U_r^{-1}$
1	$af(s)$	$\chi_r\alpha_r(f(r^{-1}s))$	(m,m)	$\pi(\alpha_r(a))$
2	$af(s)$	$\chi_r\alpha_r(f(sr))$	(m,m)	$\pi(\alpha_r(a))$
3	$\alpha_s(a)f(s)$	$\chi_r f(sr)$	(m,m)	$\pi(\alpha_r(a))$
4	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r f(r^{-1}s)$	(m,m)	$\pi(\alpha_r(a))$
5	$af(s)$	$\chi_r\alpha_{r^{-1}}(f(sr^{-1}))$	(m,a)	$\pi(\alpha_{r^{-1}}(a))$
6	$af(s)$	$\chi_r\alpha_{r^{-1}}(f(rs))$	(m,a)	$\pi(\alpha_{r^{-1}}(a))$
7	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r f(rs)$	(m,a)	$\pi(\alpha_{r^{-1}}(a))$
8	$\alpha_s(a)f(s)$	$\chi_r f(sr^{-1})$	(m,a)	$\pi(\alpha_{r^{-1}}(a))$
9	$f(s)a$	$\chi_r\alpha_r(f(r^{-1}s))$	(a,m)	$\pi(\alpha_r(a))$
10	$f(s)a$	$\chi_r\alpha_r(f(sr))$	(a,m)	$\pi(\alpha_r(a))$
11	$f(s)\alpha_s(a)$	$\chi_r f(sr)$	(a,m)	$\pi(\alpha_r(a))$
12	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r f(r^{-1}s)$	(a,m)	$\pi(\alpha_r(a))$
13	$f(s)a$	$\chi_r\alpha_{r^{-1}}(f(sr^{-1}))$	(a,a)	$\pi(\alpha_{r^{-1}}(a))$
14	$f(s)a$	$\chi_r\alpha_{r^{-1}}(f(rs))$	(a,a)	$\pi(\alpha_{r^{-1}}(a))$
15	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r f(rs)$	(a,a)	$\pi(\alpha_{r^{-1}}(a))$
16	$f(s)\alpha_s(a)$	$\chi_r f(sr^{-1})$	(a,a)	$\pi(\alpha_{r^{-1}}(a))$

Table 4.2

the same: They are, in fact, equivalent under linear automorphisms of $C_c(G, A)$. In order to see this, define, for a continuous character $\chi : G \to \mathbb{C}^\times$, the linear order 2 automorphism $T_\chi : C_c(G, A) \to C_c(G, A)$ by

$$(T_\chi f)(s) := \chi_s f(s^{-1})$$

for all $s \in G$ and $f \in C_c(G, A)$. Adding line numbers in brackets in the obvious way, one then verifies that

$$\pi_{(2)}(a) = T_{\chi(1)\chi(2)^{-1}}\pi_{(1)}(a)T^{-1}_{\chi(1)\chi(2)^{-1}}$$

for all $a \in A$, and

$$U_{(2),r} = T_{\chi(1)\chi(2)^{-1}}U_{(1),r}T^{-1}_{\chi(1)\chi(2)^{-1}}$$

for all $r \in G$. Thus the actions in the lines 1 and 2 are equivalent. Likewise,

$$\pi_{(4)}(a) = T_{\chi(4)\chi(3)^{-1}}\pi_{(3)}(a)T^{-1}_{\chi(4)\chi(3)^{-1}}$$

for all $a \in A$, and

$$U_{(4),r} = T_{\chi(4)\chi(3)^{-1}}U_{(3),r}T^{-1}_{\chi(4)\chi(3)^{-1}}$$

for all $r \in G$. Hence the actions in the lines 3 and 4 are equivalent. Furthermore, with $\chi : G \to \mathbb{C}^\times$ a continuous character as before, we let $S_\chi : C_c(G, A) \to C_c(G, A)$ be defined by

$$(S_\chi f)(s) := \chi_{s^{-1}}\alpha_{s^{-1}}(f(s))$$

for all $s \in G$ and $f \in C_c(G,A)$. Then S_χ is a linear automorphism of $C_c(G,A)$ and its inverse is given by

$$(S_\chi^{-1} f)(s) = \chi_s \alpha_s(f(s)).$$

It is then straightforward to check that

$$\pi_{(4)}(a) = S_{\chi(1)\chi(4)^{-1}} \pi_{(1)}(a) S_{\chi(1)\chi(4)^{-1}}^{-1}$$

for all $a \in A$, and

$$U_{(4),r} = S_{\chi(1)\chi(4)^{-1}} U_{(1),r} S_{\chi(1)\chi(4)^{-1}}^{-1}$$

for all $r \in G$. Thus the actions in the lines 1 and 4 are equivalent, and hence all actions of type (m,m) in the lines 1 through 4 are equivalent. Therefore, in spite of the different appearances, there is essentially only one type of canonical action in Table 4.2.

We conclude this section with a discussion of the remaining case $\delta = \mathrm{triv}$ in (4.6.1), i.e., commuting actions of A and G. It is interesting to note that, given a Banach algebra dynamical system (A,G,α), we have eight canonical commuting actions of A and G on $C_c(G,A)$. They are listed in Table 4.3, with the same notational conventions as in Table 4.2.

No.	$(\pi(a)f)(s)$	$(U_r f)(s)$	Type (π, U)	$U_r \pi(a) U_r^{-1}$
1	$\alpha_s(a)f(s)$	$\chi_r \alpha_r(f(r^{-1}s))$	(m,m)	$\pi(a)$
2	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r \alpha_r(f(sr))$	(m,m)	$\pi(a)$
3	$\alpha_{s^{-1}}(a)f(s)$	$\chi_r \alpha_{r^{-1}}(f(sr^{-1}))$	(m,a)	$\pi(a)$
4	$\alpha_s(a)f(s)$	$\chi_r \alpha_{r^{-1}}(f(rs))$	(m,a)	$\pi(a)$
5	$f(s)\alpha_s(a)$	$\chi_r \alpha_r(f(r^{-1}s))$	(a,m)	$\pi(a)$
6	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r \alpha_r(f(sr))$	(a,m)	$\pi(a)$
7	$f(s)\alpha_{s^{-1}}(a)$	$\chi_r \alpha_{r^{-1}}(f(sr^{-1}))$	(a,a)	$\pi(a)$
8	$f(s)\alpha_s(a)$	$\chi_r \alpha_{r^{-1}}(f(rs))$	(a,a)	$\pi(a)$

Table 4.3

We employ a similar mechanism as before. Indeed, suppose we have verified that, for any Banach algebra dynamical system, the formulas in line 1 yield commuting actions of type (m,m). Applying this to (A, G^o, α^o) one obtains a commuting pair of type (m,a): line 3 in Table 4.3. Likewise, line 1 for (A^o, G, α) and for (A^o, G^o, α^o) yield line 5 and line 7, respectively. Similarly line 2 yields the lines 4, 6 and 8. Furthermore, with $\mathbf{1} : G \to \mathbb{C}^\times$ denoting the trivial character, one checks that

$$\pi_{(2)}(a) = T_{\mathbf{1}} \pi_{(1)}(a) T_{\mathbf{1}}^{-1}$$

for all $a \in A$, and

$$U_{(2),r} = T_{\mathbf{1}} U_{(1),r} T_{\mathbf{1}}^{-1}$$

for all $r \in G$. Thus the actions in lines 1 and 2 are equivalent, and again there is essentially only one pair of actions in Table 4.3. In this case, one can even go a bit further: Define

$$\begin{aligned}(\tilde{\pi}(a)f)(s) &:= af(s)\\ (\tilde{U}_r f)(s) &:= f(r^{-1}s)\end{aligned}$$

for all $a \in A$, $r \in G$ and $f \in C_c(G, A)$. Then $(\tilde{\pi}, \tilde{U})$ is "the" canonical covariant pair of type (m, m) for (A, G, triv), and one verifies that

$$\pi_{(1)}(a) = S_{\chi(1)}^{-1}\tilde{\pi}(a)S_{\chi(1)}$$

for all $a \in A$, and

$$U_{(1),r} = S_{\chi(1)}^{-1}\tilde{U}_r S_{\chi(1)}$$

for all $r \in G$. Hence all the commuting actions for A and G in Table 4.3 essentially originate from the canonical covariant pair $(\tilde{\pi}, \tilde{U})$ for (A, G, triv).

4.7 Several Banach algebra dynamical systems and classes

Suppose (A_i, G_i, α_i), with $i \in \{1, \ldots, n\}$, are finitely many Banach algebra dynamical systems, and that $\mathcal{R}_i$ is a non-empty uniformly bounded class of non-degenerate continuous covariant representations of (A_i, G_i, α_i). We will show (cf. Theorem 4.7.5) that, for a Banach space X, there is a natural bijection between the non-degenerate bounded representations of the projective tensor product $\widehat{\bigotimes}_{i=1}^n (A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ on X and the n-tuples $((\pi_1, U_1), \ldots, (\pi_n, U_n))$, where, for each $i \in \{1, \ldots, n\}$, (π_i, U_i) is a non-degenerate $\mathcal{R}_i$-continuous covariant representation of (A_i, G_i, α_i) on X, and (π_i, U_i) commutes (to be defined below) with (π_j, U_j) for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$,. Such situations are quite common. For example if X is a G-bimodule (i.e., X is supplied with a left action U of G and a right action V of G that commute), then this can be interpreted as commuting non-degenerate continuous covariant representations (id, U) and (id, V) of $(\mathbb{K}, G, \mathrm{triv})$ and $(\mathbb{K}, G^o, \mathrm{triv})$, respectively (where G^o denotes the opposite group of G). In a similar vein, if (π, U) is a non-degenerate continuous covariant representation of (A, G, α) on X, and (ρ, V) is a non-degenerate continuous pair of type (a, a) (in the terminology of Section 4.6) and (π, U) and (ρ, V) commute, then (π, U) and (ρ, V) can be interpreted as a pair of commuting non-degenerate continuous covariant representations of (A, G, α) and (A^o, G^o, α^o), respectively (where A^o and G^o are, respectively, the opposite Banach algebra and group of A and G, with $\alpha^o_r := \alpha_{r^{-1}}$ for all $r \in G$ as in Section 4.6). Theorem 4.7.5 explains, as a special case, how such a pair of commuting non-degenerate covariant representations (π, U) and (ρ, V) can be related to a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}_1} \hat{\otimes} (A^o \rtimes_{\alpha^o} G^o)^{\mathcal{R}_2}$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are uniformly bounded classes of non-degenerate continuous covariant representations of (A, G, α) and (A^o, G^o, α^o) respectively, and (π, U) and (ρ, V) are respectively $\mathcal{R}_1$-continuous and $\mathcal{R}_2$-continuous.

We will now proceed to establish Theorem 4.7.5, and start with a rather obvious definition.

Definition 4.7.1. Let X be a Banach space and let $\varphi_i : S_i \to B(X)$ be maps from sets S_i into $B(X)$ for $i \in \{1,2\}$. Then φ_1 and φ_2 are said to *commute* if $\varphi_1(s_1)\varphi_2(s_2) = \varphi_2(s_2)\varphi_1(s_1)$ for all $s_1 \in S_1$ and $s_2 \in S_2$.

Let (A_1, G_1, α_1) and (A_2, G_2, α_2) be Banach algebra dynamical systems with (π_1, U_1) and (π_2, U_2) pairs of maps $\pi_1 : A_1 \to B(X)$, $U_1 : G_1 \to B(X)$ and $\pi_2 : A_2 \to B(X)$, $U_2 : G_2 \to B(X)$. Then the pairs (π_1, U_1) and (π_2, U_2) are said to *commute* if each of π_1 and U_1 commutes with both π_2 and U_2.

We then have the following:

Lemma 4.7.2. *Let X be a Banach space. For $i \in \{1,2\}$, let (A_i, G_i, α_i) be a Banach algebra dynamical system and let (π_i, U_i) be a non-degenerate continuous covariant representation of (A_i, G_i, α_i) on X. Then the following are equivalent:*

(1) (π_1, U_1) *and* (π_2, U_2) *commute.*

(2) $\pi_1 \rtimes U_1 : C_c(G_1, A_1) \to B(X)$ *and* $\pi_2 \rtimes U_2 : C_c(G_2, A_2) \to B(X)$ *commute.*

If, for $i \in \{1,2\}$, $\mathcal{R}_i$ is a non-empty class of continuous covariant representations of (A_i, G_i, α_i), such that (π_i, U_i) is $\mathcal{R}_i$-continuous, then (1) *and* (2) *are also equivalent to*

(3) $(\pi_1 \rtimes U_1)^{\mathcal{R}_1} : (A_1 \rtimes_{\alpha_1} G_1)^{\mathcal{R}_1} \to B(X)$ *and* $(\pi_2 \rtimes U_2)^{\mathcal{R}_2} : (A_2 \rtimes_{\alpha_2} G_2)^{\mathcal{R}_2} \to B(X)$ *commute.*

Proof. That (1) implies (2) can be seen through repeated application of [19, Proposition 5.5.iii]. We note that non-degeneracy is not required in this step.

That (2) implies (1) follows again by repeated applications of [19, Propositions 5.5.iii], and relies on the non-degeneracy of (π_i, U_i) for $i \in \{1,2\}$.

That (2) is equivalent to (3) follows from the density of $q^{\mathcal{R}_i}(C_c(G_i, A_i))$ in $(A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ and the fact that $(\pi_i \rtimes U_i)^{\mathcal{R}_i}(q^{\mathcal{R}_i}(f)) = \pi_i \rtimes U_i(f)$ for all $f \in C_c(G_i, A_i)$, for $i \in \{1,2\}$. We again note that non-degeneracy is not required in this step. □

The next step is to investigate the bounded representations of the projective tensor product $B_1 \hat{\otimes} B_2$ of two Banach algebras B_1 and B_2 (which will later be taken to be crossed products). We refer to [26, Section 1.5] for the details concerning the (canonical) algebra structure on the underlying projective tensor product $B_1 \hat{\otimes} B_2$ of the Banach spaces B_1 and B_2, and start with a lemma.

Lemma 4.7.3. *Let B_1 and B_2 be Banach algebras with commuting bounded representations $\pi_1 : B_1 \to B(X)$ and $\pi_2 : B_2 \to B(X)$ on the same Banach space X. Then the map $\pi_1 \odot \pi_2 : B_1 \otimes B_2 \to B(X)$ given by*

$$\pi_1 \odot \pi_2 \left(\sum_{i=1}^{n} b_1^{(i)} \otimes b_2^{(i)} \right) := \sum_{i=1}^{n} \pi_1(b_1^{(i)}) \pi_2(b_2^{(i)}),$$

where $b_j^{(i)} \in B_j$ for $j \in \{1,2\}$ and $i \in \{1, \ldots, n\}$, is well defined and extends uniquely to a bounded representation $\pi_1 \hat{\odot} \pi_2 : B_1 \hat{\otimes} B_2 \to B(X)$.

Furthermore,

(1) $\|\pi_1\hat{\odot}\pi_2\| \leq \|\pi_1\|\|\pi_2\|$,

(2) $\pi_1\hat{\odot}\pi_2 : B_1\hat{\otimes}B_2 \to B(X)$ *is non-degenerate if and only if* $\pi_1 : B_1 \to B(X)$ *and* $\pi_2 : B_2 \to B(X)$ *are non-degenerate.*

Proof. It is routine to verify that $\pi_1 \odot \pi_2$ is well defined and that $\|\pi_1 \odot \pi_2\| \leq \|\pi_1\|\|\pi_2\|$. The fact that π_1 and π_2 commute implies that $\pi_1 \odot \pi_2$ is a representation of $B_1 \otimes B_2$, and then the existence of $\pi_1\hat{\odot}\pi_2$ as a bounded representation of $B_1\hat{\otimes}B_2$ is clear, as is (1).

Since obviously $\overline{\mathrm{span}(\pi_1\hat{\odot}\pi_2(B_1\hat{\otimes}B_2)X)} \subseteq \overline{\mathrm{span}(\pi_i(B_i)X)}$ for $i \in \{1, 2\}$, the non-degeneracy of $\pi_1\hat{\odot}\pi_2$ implies the non-degeneracy of both π_1 and π_2.

Conversely, assume that both π_1 and π_2 are non-degenerate, and let $x \in X$ and $\varepsilon > 0$ be arbitrary. Choose $b_1^{(j)} \in B_1$ and $x^{(j)} \in X$ with $j \in \{1, \ldots, n\}$ such that $\left\|x - \sum_{j=1}^n \pi_1(b_1^{(j)})x^{(j)}\right\| \leq \varepsilon/2$. Next, choose $b_2^{(j,k)} \in B_2$ and $x^{(j,k)} \in X$ with $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m_j\}$ such that $\|\pi_1(b_1^{(j)})\| \left\|x^{(j)} - \sum_{k=1}^{m_j} \pi_2(b_2^{(j,k)})x^{(j,k)}\right\| \leq \varepsilon/2n$ for all $j \in \{1, \ldots, n\}$. Then

$$\begin{aligned}
&\left\|x - \sum_{j=1}^n\sum_{k=1}^{m_j} \pi_1 \odot \pi_2(b_1^{(j)} \otimes b_2^{(j,k)})x^{(j,k)}\right\| \\
\leq\ & \left\|x - \sum_{j=1}^n \pi_1(b_1^{(j)})x^{(j)}\right\| + \left\|\sum_{j=1}^n \pi_1(b_1^{(j)})x^{(j)} - \sum_{j=1}^n\sum_{k=1}^{m_j} \pi_1 \odot \pi_2(b_1^{(j)} \otimes b_2^{(j,k)})x^{(j,k)}\right\| \\
\leq\ & \frac{\varepsilon}{2} + \sum_{j=1}^n \|\pi_1(b_1^{(j)})\| \left\|x^{(j)} - \sum_{k=1}^{m_j} \pi_2(b_2^{(j,k)})x^{(j,k)}\right\| \\
<\ & \varepsilon.
\end{aligned}$$

Hence $\pi_1\hat{\odot}\pi_2$ is non-degenerate. □

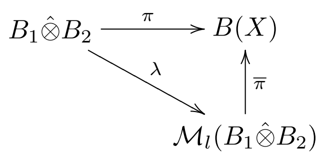
If both B_1 and B_2 have a bounded approximate left identity, then all non-degenerate bounded representations of $B_1\hat{\otimes}B_2$ arise in this fashion for unique (necessarily non-degenerate, in view of Lemma 4.7.3) bounded π_1 and π_2. More precisely, we have the following result, for which we have not been able to find a reference.

Proposition 4.7.4. *Let* B_1 *and* B_2 *be Banach algebras both having a bounded approximate left identity, and let* X *be a Banach space. If* $\pi_1 : B_1 \to B(X)$ *and* $\pi_2 : B_2 \to B(X)$ *are commuting non-degenerate bounded representations, then* $\pi_1\hat{\odot}\pi_2 : B_1\hat{\otimes}B_2 \to B(X)$ *is a non-degenerate bounded representation, and all non-degenerate bounded representations of* $B_1\hat{\otimes}B_2$ *are obtained in this fashion, for unique non-degenerate bounded representations* π_1 *and* π_2*. Then*

(1) $\|\pi_1\hat{\odot}\pi_2\| \leq \|\pi_1\|\|\pi_2\|$

(2) If, for $i \in \{1, 2\}$, B_i *has an* M_i*-bounded approximate left identity, then* $\|\pi_i\| \leq M_1M_2\|\lambda_{B_i}\|\|\pi_1\hat{\odot}\pi_2\|$, *with* $\lambda_{B_i} : B_i \to B(B_i)$ *denoting the left regular representation of* B_i.

Proof. Part of the proposition, including (1), has already been established in Lemma 4.7.3. We start from a given non-degenerate bounded representation $\pi : B_1\hat{\otimes}B_2 \to B(X)$ and construct the non-degenerate bounded representations π_1 and π_2 such that $\pi = \pi_1\hat{\odot}\pi_2$. First, we note that $B_1\hat{\otimes}B_2$ has an approximate left identity bounded by M_1M_2 [26, Lemma 1.5.3]. Therefore, if we let $\overline{\pi} : \mathcal{M}_l(B_1\hat{\otimes}B_2) \to B(X)$ denote the non-degenerate bounded representations of $\mathcal{M}_l(B_1\hat{\otimes}B_2)$ such that the diagram



commutes, then $\|\overline{\pi}\| \leq M_1M_2\|\pi\|$ [18, Theorem 4.1]. We will now compose $\overline{\pi}$ with bounded homomorphisms of B_1 and B_2 into $\mathcal{M}_l(B_1\hat{\otimes}B_2)$ to obtain the sought representations π_1 and π_2. For $b_1 \in B_1$ consider $\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2} \in B(B_1\hat{\otimes}B_2)$, where $\lambda_{B_1}(b_1)$ is the image under the left regular representation $\lambda_{B_1} : B_1 \to B(B_1)$ of B_1. Clearly, $\|\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}\| = \|\lambda_{B_1}(b_1)\| \leq \|\lambda_{B_1}\|\|b_1\|$, and one readily verifies that $\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2} \in \mathcal{M}_l(B_1\hat{\otimes}B_2)$. If we define $l_1 : B_1 \to \mathcal{M}_l(B_1\hat{\otimes}B_2)$ by $l_1(b_1) := \lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}$ for $b_1 \in B_1$, then l_1 is a bounded homomorphism, and $\|l_1\| \leq \|\lambda_{B_1}\|$. Likewise, $l_2 : B_2 \to \mathcal{M}_l(B_1\hat{\otimes}B_2)$, defined by $l_2(b_2) := \mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2)$ for $b_2 \in B_2$, is a bounded homomorphism, and $\|l_2\| \leq \|\lambda_{B_2}\|$. Now, for $i \in \{1,2\}$, define $\pi_i : B_i \to B(X)$ as $\pi_i := \overline{\pi} \circ l_i$. We note that $\|\pi_i\| \leq \|\overline{\pi}\|\|l_i\| \leq M_1M_2\|\lambda_{B_i}\|\|\pi\|$. Since l_1 and l_2 obviously commute, the same holds true for π_1 and π_2. Therefore $\pi_1\hat{\odot}\pi_2 : B_1\hat{\otimes}B_2 \to B(X)$ is a bounded representation.

We will proceed to show that $\pi_1\hat{\odot}\pi_2 = \pi$, and that π_1 and π_2 are uniquely determined. We compute, for $x \in X$, $b_1^{(1)}, b_1^{(2)} \in B_1$ and $b_2^{(1)}, b_2^{(2)} \in B_2$:

$$\begin{aligned}
& \pi_1\hat{\odot}\pi_2(b_1^{(1)} \otimes b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \pi_1(b_1^{(1)})\pi_2(b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \overline{\pi}(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2})\overline{\pi}(\mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2^{(1)}))\pi(b_1^{(2)} \otimes b_2^{(2)})x \\
= \; & \overline{\pi}(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2})\pi(\mathrm{id}_{B_1}\hat{\otimes}\lambda_{B_2}(b_2^{(1)})(b_1^{(2)} \otimes b_2^{(2)}))x \\
= \; & \pi(\lambda_{B_1}(b_1^{(1)})\hat{\otimes}\mathrm{id}_{B_2}(b_1^{(2)} \otimes b_2^{(1)}b_2^{(2)}))x \\
= \; & \pi(b_1^{(1)}b_1^{(2)} \otimes b_2^{(1)}b_2^{(2)})x \\
= \; & \pi(b_1^{(1)} \otimes b_2^{(1)})\pi(b_1^{(2)} \otimes b_2^{(2)})x.
\end{aligned}$$

Since π is non-degenerate and $B_1 \otimes B_2$ is dense in $B_1\hat{\otimes}B_2$, the restriction of π to $B_1 \otimes B_2$ is also non-degenerate. Hence we conclude from the above that $\pi_1\hat{\odot}\pi_2(b_1 \otimes b_2) = \pi(b_1 \otimes b_2)$ for all $b_1 \in B_1$ and $b_2 \in B_2$, i.e., that $\pi_1\hat{\odot}\pi_2 = \pi$. It is now clear that $\|\pi_i\| \leq M_1M_2\|\lambda_{B_i}\|\|\pi_1\hat{\odot}\pi_2\|$. As already mentioned preceding the proposition, π_1 and π_2 are necessarily non-degenerate.

As to uniqueness, assume that $\rho_1 : B_1 \to B(X)$ and $\rho_2 : B_2 \to B(X)$ are commuting bounded representations such that $\rho_1\hat{\odot}\rho_2 = \pi$. Then, for $x \in X$, $b_1, b_1' \in$

B and $b_2' \in B_2$,

$$\begin{aligned}\rho_1(b_1)\pi(b_1' \otimes b_2')x &= \rho_1(b_1)\rho_1 \hat{\odot} \rho_2(b_1' \otimes b_2')x \\ &= \rho_1(b_1)\rho_1(b_1')\rho_2(b_2')x \\ &= \rho_1(b_1 b_1')\rho_2(b_2')x \\ &= \rho_1 \hat{\odot} \rho_2(b_1 b_1' \otimes b_2')x \\ &= \pi(\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2}(b_1' \otimes b_2'))x \\ &= \overline{\pi}(\lambda_{B_1}(b_1)\hat{\otimes}\mathrm{id}_{B_2})\pi(b_1' \otimes b_2')x \\ &= \pi_1(b_1)\pi(b_1' \otimes b_2')x.\end{aligned}$$

The non-degeneracy of π then implies that necessarily $\rho_1 = \pi_1$ and likewise that $\rho_2 = \pi_2$. □

The following is now simply a matter of combining the General Correspondence Theorem (Theorem 4.2.1), Lemma 4.7.3, Proposition 4.7.4, and an induction argument.

Theorem 4.7.5. *For $i \in \{1, \ldots, n\}$, let (A_i, G_i, α_i) be a Banach algebra dynamical system, where A_i has a bounded approximate left identity, and $\mathcal{R}_i$ is a non-empty uniformly bounded class of non-degenerate continuous covariant representations of (A_i, G_i, α_i). Let X be a Banach space. Let $((\pi_1, U_1), \ldots, (\pi_n, U_n))$ be an n-tuple where, for each $i \in \{1, \ldots, n\}$, the pair (π_i, U_i) is a non-degenerate $\mathcal{R}_i$-continuous covariant representation of (A_i, G_i, α_i) on X, and all (π_i, U_i) and (π_j, U_j) commute for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then the map sending $((\pi_1, U_1), \ldots, (\pi_n, U_n))$ to the representation*

$$\widehat{\bigodot}_{i=1}^{n} (\pi_i \rtimes U_i)^{\mathcal{R}_i} : \widehat{\bigotimes}_{i=1}^{n} (A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i} \to B(X),$$

is a bijection between the set of all such n-tuples and the set of all non-degenerate bounded representations of $\widehat{\bigotimes}_{i=1}^{n} (A_i \rtimes_{\alpha_i} G_i)^{\mathcal{R}_i}$ on X.

For the sake of completeness, we mention that the commutativity assumption applies only to the non-degenerate $\mathcal{R}_i$-continuous covariant representations (π_i, U_i), not to the elements of $\mathcal{R}_i$.

In Remark 4.8.5 we will apply Theorem 4.7.5 to relate bimodules over generalized Beurling algebras to left modules over a projective tensor product of the algebra acting on the left and the opposite algebra of the one acting on the right.

4.8 Right and bimodules over generalized Beurling algebras

Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded two-sided approximate identity and α is uniformly bounded, and let ω be a weight on G. In Section 4.5 we have seen that the Banach space $L^1(G, A, \omega)$ has the structure of an associative algebra, denoted $L^1(G,A,\omega;\alpha)$, with multiplication continuous in both variables, determined by

$$[f *_\alpha g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,d\mu(r) \quad (f,g \in C_c(G,A),\ s \in G).$$

Here we have written $*_\alpha$ rather than $*$ to indicate the α-dependence of the multiplication (twisted convolution) on $C_c(G,A)$, as another multiplication will also appear. For the same reason we have now also written $d\mu$ for the chosen left Haar measure on G. Furthermore, we have seen in Section 4.5 that $L^1(G,A,\omega;\alpha)$ is isomorphic to the Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$, when $\mathcal{R}$ is chosen suitably. As a consequence of the General Correspondence Theorem (Theorem 4.2.1), it was then shown that if (π,U) is a non-degenerate continuous covariant representation of (A,G,α), such that $\|U_r\| \leq C_U\omega(r)$ for all $r \in G$, then $\pi \rtimes U(f) = \int \pi(f)U_r\,d\mu(r)$, for $f \in C_c(G,A)$, determines a non-degenerate bounded representation of $L^1(G,A,\omega;\alpha)$, and that all non-degenerate bounded representations of $L^1(G,A,\omega;\alpha)$ are uniquely determined in this way by such pairs (π,U).

In the current section we will explain how the non-degenerate bounded anti-representations of $L^1(G,A,\omega;\alpha)$ (i.e., non-degenerate right $L^1(G,A,\omega;\alpha)$-modules) are in natural bijection with the pairs (π,U), where $\pi : A \to B(X)$ is non-degenerate, bounded and anti-multiplicative, $U : G \to B(X)$ is strongly continuous and anti-multiplicative, satisfy

$$U_r\pi(a)U_r^{-1} = \pi(\alpha_{r^{-1}}(a)) \quad (a \in A,\ r \in G),$$

(i.e., with the non-degenerate continuous pairs (π,U) of type (a,a) as in Section 4.6, called thrice "flawed" in the introduction) and are such that $\|U_r\| \leq C_U\omega(r)$, for some $C_U \geq 0$ and all $r \in G$. This may look counterintuitive to the idea of Section 4.6, where it was argued that one can "always" reinterpret given data so as to end up with pairs of type (m,m) for a (companion) Banach algebra dynamical system, and then formulate a General Correspondence Theorem involving the non-degenerate bounded representations of a companion crossed product: anti-representations of the resulting crossed product never enter the picture. Yet this is precisely what we will do, but it is only the first step.

In this first step the relevant crossed product will, as in Section 4.5, turn out to be topologically isomorphic to $L^1(G^o,A^o,\omega^o;\alpha^o)$ (where ω^o equals ω, seen as a weight on G^o). As it happens, $L^1(G^o,A^o,\omega^o;\alpha^o)$ is topologically anti-isomorphic to $L^1(G,A,\omega;\alpha)$. Hence, in the second step, the non-degenerate bounded representations of $L^1(G^o,A^o,\omega^o;\alpha^o)$ are viewed as the non-degenerate bounded anti-representations of $L^1(G,A,\omega;\alpha)$, which are thus, in the end, related to pairs (π,U) of type (a,a) as above. For this result, therefore, one should not think of $L^1(G,A,\omega;\alpha)$ as being topologically isomorphic to a crossed product as in Section 4.5. Although this is also the case, its main feature here is that it is anti-isomorphic to the algebra $L^1(G^o,A^o,\omega^o;\alpha^o)$ which, in turn, is topologically isomorphic to the crossed product that "actually" explains the situation.

Once this has been completed, we remind ourselves again that $L^1(G,A,\omega;\alpha)$ itself is topologically isomorphic to a crossed product, and combine the results in the first part of this section with those in Sections 4.5 and 4.7 in Theorem 4.8.4, to describe for two Banach algebra dynamical systems (A, G, α) and (B, H, β) the non-degenerate simultaneously left $L^1(G, A, \omega; \alpha)$– and right $L^1(H, B, \eta; \beta)$-modules, and, in the special case where $(A, G, \alpha) = (B, H, \beta)$, the non-degenerate $L^1(G, A, \omega; \alpha)$-bimodules.

To start, recall that the canonical left invariant measure μ on the opposite group G^o of G is given by $\mu^o(E) := \mu(E^{-1})$, for E a Borel subset of G. Then, recalling that $\int_G f\,d\mu = \int_G f(r^{-1})\Delta(r^{-1})\,d\mu(r)$ [46, Lemma 1.67], for $f \in C_c(G)$, we have

$$\int_{G^o} f(r)\,d\mu^o(r) = \int_G f(r^{-1})\,d\mu(r) = \int_G f(r)\Delta(r^{-1})\,d\mu(r).$$

We recall from Section 4.6 if (A, G, α) is a Banach algebra dynamical system, then so is (A^o, G^o, α^o), where A^o is the opposite algebra of A, G^o is the opposite group of G, and $\alpha^o : G^o \to \mathrm{Aut}(A^o) = \mathrm{Aut}(A)$ is given by $\alpha^o_s = \alpha_{s^{-1}}$ for all $s \in G^o$. The vector spaces $C_c(G, A)$ and $C_c(G^o, A^o)$ can be identified, but there are two convolution structures on it. If $\circledcirc$ denotes the multiplication in A^o and G^o, then

$$[f *_\alpha g](s) = \int_G f(r)\alpha_r(g(r^{-1}s))\,d\mu(r) \quad (f, g \in C_c(G, A),\ s \in G),$$

and

$$[f *_{\alpha^o} g](s) = \int_G f(r) \circledcirc \alpha^o_r(g(r^{-1} \circledcirc s))\,d\mu^o(r) \quad (f, g \in C_c(G^o, A^o),\ s \in G^o).$$

Hence we have two associative algebras: $C_c(G, A)$ with multiplication $*_\alpha$, and $C_c(G^o, A^o)$ with multiplication $*_{\alpha^o}$, having the same underlying vector space. The first observation we need is then the following:

Lemma 4.8.1. *Let (A, G, α) be a Banach algebra dynamical system with companion opposite system (A^o, G^o, α^o), and let $\chi : G \to \mathbb{C}^\times$ be a continuous character of G. For $f \in C_c(G, A)$, define $\hat{f} \in C_c(G^o, A^o)$ by $\hat{f}(s) := \chi(s^{-1})\alpha_{s^{-1}}(f(s))$ for $s \in G^o$. Then the map $f \mapsto \hat{f}$ is an anti-isomorphism of the associative algebras $C_c(G, A)$ with multiplication $*_\alpha$, and $C_c(G^o, A^o)$ with multiplication $*_{\alpha^o}$. The inverse is given by $g \mapsto \check{g}$, where $\check{g}(s) := \chi(s)\alpha_s(g(s))$ for $g \in C_c(G^o, A^o)$ and $s \in G$.*

Proof. It is clear that $\hat{\cdot}$ and $\check{\cdot}$ are mutually inverse linear bijections. As to the multiplicative structures, we compute, for $f, g \in C_c(G, A)$ and $s \in G^o$,

$$\begin{aligned}
[\hat{f} *_{\alpha^o} \hat{g}](s) &= \int_{G^o} \hat{f}(r) \circledcirc \alpha^o_{r^{-1}}(\hat{g}(r^{-1} \circledcirc s))\,d\mu^o(r) \\
&= \int_G \hat{f}(r^{-1}) \circledcirc \alpha^o_{r^{-1}}(\hat{g}(r \circledcirc s))\,d\mu(r) \\
&= \int_G \alpha_r(\hat{g}(sr))\hat{f}(r^{-1})\,d\mu(r) \\
&= \int_G \alpha_r(\chi((sr)^{-1})\alpha_{(sr)^{-1}}g(sr))\chi(r)\alpha_r(f(r^{-1}))\,d\mu(r)
\end{aligned}$$

$$\begin{aligned}
&= \chi(s^{-1}) \int_G \alpha_{s^{-1}}(g(sr))\alpha_r(f(r^{-1}))\, d\mu(r) \\
&= \chi(s^{-1})\alpha_{s^{-1}} \left(\int_G g(sr))\alpha_{sr}(f(r^{-1}))\, d\mu(r) \right) \\
&= \chi(s^{-1})\alpha_{s^{-1}} \left(\int_G g(r))\alpha_r(f(r^{-1}s))\, d\mu(r) \right) \\
&= (g *_\alpha f)^\wedge(s).
\end{aligned}$$

□

Choosing χ suitably, we obtain a topological isomorphism in the next result.

Proposition 4.8.2. *Let (A, G, α) be a Banach algebra dynamical system, where α is uniformly bounded. Let ω be a weight on G and view $\omega^o := \omega$ also as a weight on G^o. Then the map $f \mapsto \hat{f}$, where $\hat{f}(s) := \Delta(s)\alpha_{s^{-1}}(f(s))$ for $f \in C_c(G, A)$ and $s \in G^o$ defines a topological anti-isomorphism between $L^1(G, A, \omega; \alpha)$ and $L^1(G^o, A^o, \omega^o; \alpha^o)$. The inverse map is determined by $g \mapsto \check{g}$ where $\check{g}(s) := \Delta(s^{-1})\alpha_s(g(s))$ for $g \in C_c(G^o, A^o)$ and $s \in G$.*

Proof. In view of Lemma 4.8.1, we need only show that $\hat{}$ and $\check{}$ are isomorphisms between the normed spaces $(C_c(G, A), \|\cdot\|_{1,\omega})$ and $(C_c(G^o, A^o), \|\cdot\|_{1,\omega^o})$. Let α be uniformly bounded by C_α. If $f \in C_c(G, A)$, then

$$\begin{aligned}
\|\hat{f}\|_{1,\omega^o} &= \int_{G^o} \|\hat{f}(r)\|\omega^o(r)\, d\mu^o(r) \\
&= \int_{G^o} \|\Delta(r)\alpha_{r^{-1}}(f(r))\|\omega(r)\, d\mu^o(r) \\
&\leq C_\alpha \int_{G^o} \|f(r)\|\omega(r)\Delta(r)\, d\mu^o(r) \\
&= C_\alpha \int_G \|f(r^{-1})\|\omega(r^{-1})\Delta(r^{-1})\, d\mu(r) \\
&= C_\alpha \int_G \|f(r)\|\omega(r)\, d\mu(r) \\
&= C_\alpha \|f\|_{1,\omega}.
\end{aligned}$$

Similarly $\|\check{f}\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega^o}$ for all $f \in C_c(G^o, A^o)$. □

It is now an easy matter to combine the ideas of Sections 4.5 and 4.6 with the above Proposition 4.8.2.

Let X be a Banach space and let (A, G, α) be a Banach algebra dynamical system, where A has a bounded two-sided approximate identity and α is uniformly bounded. As in Section 4.6, the pairs (π, U), where $\pi : A \to B(X)$ is non-degenerate, bounded and anti-multiplicative, $U : G \to B(X)$ is strongly continuous and anti-multiplicative, and $U_r^{-1}\pi(a)U_r = \pi(\alpha_{r^{-1}}(a))$ for $a \in A$ and $r \in G$, can be identified with the pairs (π^o, U^o), where $\pi^o : A^o \to B(X)$, with $\pi^o(a) := \pi(a)$ for $a \in A$, is non-degenerate, bounded and multiplicative, $U^o : G^o \to B(X)$, with $U^o_r = U_r$ for all $r \in G^o$, is strongly continuous and multiplicative, and $U^o_r \pi^o(a) U^{o-1}_r = \pi^o(\alpha^o_r(a))$ for $a \in A^o$ and $r \in G^o$. Furthermore, if ω is a weight on G, also viewed as a weight $\omega^o := \omega$ on G^o, then there exists a constant C_U such that $\|U_r\| \leq C_U \omega(r)$ for all $r \in G$ if and only if there exists a constant C_{U^o} such that $\|U^o_r\| \leq C_{U^o} \omega^o(r)$ for all $r \in G^o$: take the same constant. Now the collection of all such pairs (π^o, U^o) is, in view of Theorem 4.5.20, in natural bijection with the collection of all non-degenerate bounded representations of $L^1(G^o, A^o, \omega^o; \alpha^o)$ on X. As a consequence of Proposition 4.8.2, this can in turn be viewed as the collection of all non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on X. Combining these three bijections, we can let pairs (π, U) as described above correspond bijectively to the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on X: If (π, U) is such a pair, we associate with it the non-degenerate bounded anti-representation of $L^1(G, A, \omega; \alpha)$ determined by sending $f \in C_c(G, A)$ to $\pi^o \rtimes U^o(\hat{f})$. Explicitly, for $f \in C_c(G, A)$,

$$
\begin{aligned}
\pi^o \rtimes U^o(\hat{f}) &= \int_{G^o} \pi^o(\hat{f}(r)) U^o_r \, d\mu^o(r) \\
&= \int_{G^o} \pi(\Delta(r) \alpha_{r^{-1}}(f(r))) U_r \, d\mu^o(r) \\
&= \int_G \pi(\alpha_r(f(r^{-1}))) U_{r^{-1}} \Delta(r^{-1}) \, d\mu(r) \\
&= \int_G \pi(\alpha_{r^{-1}}(f(r))) U_r \, d\mu(r) \\
&= \int_G U_r U_r^{-1} \pi(\alpha_{r^{-1}}(f(r))) U_r \, d\mu(r) \\
&= \int_G U_r \pi(\alpha_r \circ \alpha_{r^{-1}}(f(r))) \, d\mu(r) \\
&= \int_G U_r \pi(f(r)) \, d\mu(r).
\end{aligned}
$$

To retrieve the pair (π, U) from a non-degenerate bounded anti-representation T of $L^1(G, A, \omega; \alpha)$, we note that, by Proposition 4.8.2, $T \circ \check{\ }$ is a non-degenerate bounded representation of $L^1(G^o, A^o, \omega^o; \alpha^o)$, and hence, we can apply [19, Equations (8.1) and (8.2)] to $T \circ \check{\ }$. A bounded approximate left identity of A^o is then needed, and for this we take a bounded approximate right identity (u_i) of A. Furthermore, if V runs through a neighbourhood base $\mathcal{Z}$ of $e \in G$, of which all elements are contained in a fixed compact set of G, and $z_V \in C_c(G)$ is positive, supported in V, and $\int_G z_V(r^{-1}) \, d\mu(r) = \int_{G^o} z_V(r) \, d\mu^o(r) = 1$, then the $z_V \in C_c(G)$ are as required for [19, Equations (8.1) and (8.2)]. Hence, again taking Remark 4.5.21 into account, we have, for $a \in A$,

$$
\begin{aligned}
\pi(a) = \pi^o(a) &= \text{SOT-lim}_{(V,i)} T((z_V \otimes a \circledcirc u_i)^\vee) \\
&= \text{SOT-lim}_{(V,i)} T((z_V \otimes u_i a)^\vee),
\end{aligned}
$$

where $(z_V \otimes u_i a)^\vee(r) = \Delta(r^{-1}) z_V(r) \alpha_r(a u_i)$ for $r \in G$, and, for $s \in G$,

$$\begin{aligned} U_s = U_s^o &= \text{SOT-lim}_{(V,i)} T((z_V(s^{-1} \circledcirc \cdot) \otimes u_i)^\vee) \\ &= \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee), \end{aligned}$$

where $(z_V(\cdot s^{-1}) \otimes u_i)^\vee(r) = \Delta(r^{-1}) z_V(r s^{-1}) \alpha_r(u_i)$ for $r \in G$.

All in all, we have the following result in analogy to Theorem 4.5.20:

Theorem 4.8.3. *Let (A, G, α) be a Banach algebra dynamical system where A has a two-sided approximate identity and α is uniformly bounded by a constant C_α, and let ω be a weight on G. Let X be a Banach space. Let the pair (π, U) be such that $\pi : A \to B(X)$ is a non-degenerate bounded anti-representation, $U : G \to B(X)$ is a strongly continuous anti-representation satisfying $U_r \pi(\alpha) U_r^{-1} = \pi(\alpha_{r^{-1}}(a))$ for all $a \in A$ and $r \in G$, and with C_U a constant such that $\|U_r\| \leq C_U \omega(r)$ for all $r \in G$. Let $T : L^1(G, A, \omega; \alpha) \to B(X)$ be a non-degenerate bounded anti-representation of $L^1(G, A, \omega; \alpha)$ on X. Then the following maps are mutual inverses between all such pairs (π, U) and the non-degenerate bounded anti-representations T of $L^1(G, A, \omega; \alpha)$:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G U_r \pi(f(r))\, dr \right) =: T^{(\pi, U)} \quad (f \in C_c(G, A)),$$

determining a non-degenerate bounded anti-representation $T^{(\pi,U)}$ of the generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, and,

$$T \mapsto \left(\begin{array}{l} a \mapsto \text{SOT-lim}_{(V,i)} T((z_V \otimes u_i a)^\vee), \\ s \mapsto \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee) \end{array} \right) =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G)$ is chosen such that $z_V \geq 0$, supported in $V \in \mathcal{Z}$, $\int_G z_V(r^{-1})\, dr = 1$, and (u_i) is any bounded approximate right identity of A.

Furthermore, if A has an M-bounded approximate right identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U \|\pi\|$,

(2) $\|\pi^T\| \leq (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \|T\|$,

(3) $\|U_s^T\| \leq M (\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r)) \|T\| \omega(s) \quad (s \in G)$.

Proof. Except for the bounds, all statements were proven in the discussion preceding the statement of the theorem. Establishing the bound (1) proceeds as in Theorem 4.5.20.

To establish (2), we choose a bounded two-sided approximate identity (u_i) of A. Let $a \in A$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ be arbitrary. There exists an index i_0 such that $\|u_i a\| \leq \|a\| + \varepsilon_1$ for all $i \geq i_0$. There exists some $W_1 \in \mathcal{Z}$ such that $\sup_{r \in W_1} \omega(r) \leq \inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2$. Since $r \mapsto \|\alpha_r\|$ is lower semicontinuous and $\|\alpha_e\| = 1$, there exists some $W_2 \in \mathcal{Z}$ such that $\|\alpha_r\| \leq 1 + \varepsilon_3$ for all $r \in W_2$. Let $V_0 \in \mathcal{Z}$ be such that $V_0 \subseteq W_1 \cap W_2$. If $(V, i) \geq (V_0, i_0)$, then $V \subseteq V_0$ and $i \geq i_0$, hence

$$\begin{aligned}
\|T((z_V \otimes u_i a)^\vee)\| &\leq \|T\| \, \|(z_V \otimes u_i a)^\vee\|_{1,\omega} \\
&= \|T\| \int_G \|(z_V \otimes u_i a)^\vee(r)\| \omega(r) \, dr \\
&= \|T\| \int_G \Delta(r^{-1}) z_V(r) \|\alpha_r(a u_i)\| \omega(r) \, dr \\
&\leq \|T\| \|a u_i\| (1 + \varepsilon_3) \left(\sup_{r \in V} \omega(r) \right) \int_G \Delta(r^{-1}) z_V(r) \, dr \\
&\leq \|T\| (\|a\| + \varepsilon_1)(1 + \varepsilon_3) \left(\sup_{r \in V_0} \omega(r) \right) \int_G z_V(r^{-1}) \, dr \\
&\leq \|T\| (\|a\| + \varepsilon_1)(1 + \varepsilon_3) \left(\inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2 \right).
\end{aligned}$$

From this, the bound in (2) now follows as in the proof of Theorem 4.5.20.

As to (3), we fix $s \in G$. The operator $U_s^T = \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes u_i)^\vee)$ does not depend on the particular choice of the bounded approximate right identity (u_i) (see Remark 4.5.21). If (u_i) is an M-bounded approximate right identity of A, then $(\alpha_{s^{-1}}(u_i))$ is also a bounded approximate right identity of A, and hence $U_s^T = \text{SOT-lim}_{(V,i)} T((z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee)$. Let $\varepsilon_1, \varepsilon_2 > 0$ be arbitrary. Choose $W_1 \in \mathcal{Z}$ such that $\|\alpha_r\| \leq 1 + \varepsilon_1$ for all $r \in W_1$, and $W_2 \in \mathcal{Z}$ such that $\sup_{r \in W_2} \omega(r) \leq \inf_{V \in \mathcal{Z}} \sup_{r \in V} \omega(r) + \varepsilon_2$. Let $V_0 \in \mathcal{Z}$ be such that $V_0 \subseteq W_1 \cap W_2$. If $(V, i) \geq (V_0, i_0)$, then $V \subseteq V_0$ and $i \geq i_0$, hence

$$\begin{aligned}
& \|T((z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee)\| \\
\leq \; & \|T\| \left\| (z_V(\cdot s^{-1}) \otimes \alpha_{s^{-1}}(u_i))^\vee \right\|_{1,\omega} \\
= \; & \|T\| \int_G \|(z_V \otimes \alpha_{s^{-1}}(u_i))^\vee(r)\| \omega(r) \, dr \\
= \; & \|T\| \int_G \Delta(r^{-1}) z_V(r s^{-1}) \|\alpha_{r s^{-1}}(u_i)\| \omega(r) \, dr \\
= \; & \|T\| \int_G z_V(r^{-1} s^{-1}) \|\alpha_{r^{-1} s^{-1}}(u_i)\| \omega(r^{-1}) \, dr \\
= \; & \|T\| \int_G z_V(r^{-1}) \|\alpha_{r^{-1}}(u_i)\| \omega(r^{-1} s) \, dr \\
\leq \; & \|T\| \int_G z_V(r^{-1}) \|\alpha_{r^{-1}}(u_i)\| \omega(r^{-1}) \omega(s) \, dr \\
\leq \; & \|T\| \int_G z_V(r^{-1}) (1 + \varepsilon_1) \|u_i\| \left(\sup_{r \in V^{-1}} \omega(r^{-1}) \right) \omega(s) \, dr \\
\leq \; & \|T\| (1 + \varepsilon_1) M \left(\sup_{r \in V^{-1}} \omega(r^{-1}) \right) \omega(s) \int_G z_V(r^{-1}) \, dr
\end{aligned}$$

$$\leq \|T\|(1+\varepsilon_1)M\left(\sup_{r\in V_0}\omega(r)\right)\omega(s)$$
$$\leq \|T\|(1+\varepsilon_1)M\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)+\varepsilon_2\right)\omega(s).$$

Once again, the bound in (3) now follows as in the proof of Theorem 4.5.20. □

We will now describe the non-degenerate bimodules over generalized Beurling algebras as a special case of a more general result. Let (A,G,α) and (B,H,β) be Banach algebra dynamical systems, where A and B have bounded two-sided approximate identities, and both α and β are uniformly bounded. Let ω be a weight on G, and η a weight on H. Remembering that $L^1(G,A,\omega;\alpha)$ and $L^1(H,B,\eta;\beta)$ are themselves also (isomorphic to) a crossed product of a Banach algebra dynamical system, Theorem 4.5.13, it is now easy to describe the non-degenerate simultaneously left $L^1(G,A,\omega;\alpha)$– and right $L^1(H,B,\eta;\beta)$-modules, as follows: Let X be a Banach space. Suppose that $T^m : L^1(G,A,\omega;\alpha)\to B(X)$ is a non-degenerate bounded representation of $L^1(G,A,\omega;\alpha)$ on X, and $T^a : L^1(H,B,\eta;\beta)\to B(X)$ is a non-degenerate bounded anti-representation, such that T^m and T^a commute. We know from Theorem 4.5.20 and Theorem 4.8.3 that T^m and T^a correspond to pairs (π^m,U^m) and (π^a,U^a), respectively, each with the appropriate properties. But then (π^m,U^m) and (π^a,U^a) must also commute in the sense of Definition 4.7.1. Indeed, (π^a,U^a) corresponds to T^a as being the pair such that the integrated form of $(\pi^{a,o},U^{a,o})$ gives rise to the non-degenerate bounded representation T^a of $L^1(H^o,B^o,\eta^o;\beta^o)$ on X. But since $L^1(H^o,B^o,\eta^o;\beta^o)$ is (isomorphic to) a crossed product, and likewise for $L^1(G,A,\omega;\alpha)$, the fact that (π^m,U^m) and $(\pi^{a,o},U^{a,o})$ commute then follows from Lemma 4.7.2 and the fact that T^m and T^a commute. Since $\pi^{a,o}=\pi^a$ and $U^{a,o}=U^a$ as set-theoretic maps, (π^m,U^m) and (π^a,U^a) also commute. The same kind of arguments show that the converse is equally true.

Combining these results, we obtain the following following description of the non-degenerate simultaneously left $L^1(G,A,\omega;\alpha)$– and right $L^1(H,B,\eta;\beta)$-modules. If $(A,G,\alpha)=(B,G,\beta)$ and $\omega=\eta$ it describes the non-degenerate $L^1(G,A,\omega;\alpha)$-bimodules.

Theorem 4.8.4. *Let (A,G,α) and (B,H,β) be a Banach algebra dynamical systems, where A and B have bounded two-sided approximate identities, and both α and β are uniformly bounded. Let ω be a weight on G, and η a weight on H. Let X be a Banach space.*

Suppose that (π^m,U^m) is a non-degenerate continuous covariant representation of (A,G,α) on X such that $\|U^m_r\|\leq C_{U^m}\omega(r)$ for some constant C_{U^m} and all $r\in G$. Suppose that the pair (π^a,U^a) is such that $\pi^a : B\to B(X)$ is a non-degenerate bounded anti-representation, that $U^a : H\to B(X)$ is a strongly continuous anti-representation, such that $U^a_s\pi^a(b)U^{a-1}_s=\pi^a(\alpha_{s^{-1}}(b))$ for all $b\in B$ and $s\in H$, and $\|U^a_s\|\leq C_{U^a}\eta(s)$ for some constant C_{U^a} and all $s\in H$. Furthermore, let (π^m,U^m) and (π^a,U^a) commute.

Then the map

$$T^m(f) := \int_G \pi^m(f(r))U^m_r \, d\mu_G(r) \quad (f \in C_c(G, A))$$

determines a non-degenerate bounded representation of $L^1(G, A, \omega; \alpha)$ *on* X*, and the map*

$$T^a(g) := \int_H U^a_s \pi^a(g(s)) \, d\mu_H(s) \quad (g \in C_c(H, B))$$

determines a non-degenerate bounded anti-representation of $L^1(H, B, \eta; \beta)$ *on* X*. Moreover,* $T^m : L^1(G, A, \omega; \alpha) \to B(X)$ *and* $T^a : L^1(H, B, \eta; \beta) \to B(X)$ *commute.*

All pairs (T^m, T^a)*, where* T^m *and* T^a *commute, are non-degenerate, bounded,* T^m *is a representation of* $L^1(G, A, \omega; \alpha)$ *on* X*, and* T^a *is an anti-representation of* $L^1(H, B, \eta; \beta)$ *on* X*, are obtained in this fashion from unique (necessarily commuting) pairs* (π^m, U^m) *and* (π^a, U^a) *with the above properties.*

For reasons of space, we do not repeat the formulas in Theorem 4.5.20 and Theorem 4.8.3 retrieving (π^m, U^m) from T^m and (π^a, U^a) from T^a, or the upper bounds therein.

Remark 4.8.5. The results of Section 4.6 make it possible to establish a bijection between the commuting pairs (π^m, U^m) and (π^a, U^a) as in Theorem 4.8.4 and the non-degenerate bounded representations of one single algebra (rather than two). To see this, note that, though $L^1(G, A, \omega; \alpha)$ and $L^1(H^o, B^o, \eta^o; \beta^o)$ are not Banach algebras in general, the continuity of the multiplication still implies that $L^1(G, A, \omega; \alpha)\hat{\otimes}L^1(H^o, B^o, \eta^o; \beta^o)$ can be supplied with the structure of an associative algebra such that multiplication is continuous. If $L^1(G, A, \omega; \alpha) \simeq C_1$ and $L^1(H^o, B^o, \eta^o; \beta^o) \simeq C_2$ as topological algebras, where C_1 and C_2 are crossed products of the relevant Banach algebra dynamical systems as in Section 4.5, then clearly

$$L^1(G, A, \omega; \alpha)\hat{\otimes}L^1(H, B, \eta; \beta)^o \simeq L^1(G, A, \omega; \alpha)\hat{\otimes}L^1(H^o, B^o, \eta^o; \beta^o) \simeq C_1\hat{\otimes}C_2$$

where Proposition 4.8.3 was used in the first step. From Theorem 4.7.5 we know what the non-degenerate bounded representations of $C_1\hat{\otimes}C_2$ are. Hence, combining all information, we see that the commuting pairs (π^m, U^m) and (π^a, U^a) as in Theorem 4.8.4 are in bijection with the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)\hat{\otimes}L^1(H, B, \eta; \beta)^o$, by letting (π^m, U^m) and (π^a, U^a) correspond to the non-degenerate bounded representation $T^m \odot T^a$, where T^m and T^a are as in Theorem 4.8.4 (the latter now viewed as a non-degenerate bounded representation of $L^1(H, B, \eta; \beta)^o$). Our notation is slightly imprecise here, since $L^1(G, A, \omega; \alpha)$ and $L^1(H, B, \eta; \beta)^o$ are not Banach algebras in general, but it is easily seen that Lemma 4.7.4 is equally valid when the norm need not be submultiplicative, but multiplication is still continuous.

Finally, we note that the special case where $(A, G, \alpha) = (B, H, \beta) = (\mathbb{K}, G, \mathrm{triv})$ in Theorem 4.8.4 states that the non-degenerate bimodules over $L^1(G, \omega)$ correspond naturally to the G-bimodules determined by a pair (U^m, U^a) of commuting maps U^m and U^a, where $U^m : G \to B(X)$ is a strongly continuous representation, U^a :

$G \to B(X)$ is a strongly continuous anti-representation, and $\|U_r^m\| \leq C_{U^m}\omega(r)$ and $\|U_r^a\| \leq C_{U^a}\omega(r)$ for some constants C_{U^m} and C_{U^a} and all $r \in G$. Specializing further by taking $\omega = 1$, we see that the non-degenerate bimodules over $L^1(G)$ correspond naturally to the G-bimodules determined by a commuting pair (U^m, U^a) as above, with now each of U^m and U^a uniformly bounded. This is a classical result, cf. [25, Proposition 2.1].

Chapter 5

Crossed products of pre-ordered Banach algebras

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5.1 Introduction

This paper is a continuation of [19] and [14], where, inspired by the theory of crossed products of C^*-algebras, the theory of crossed products of Banach algebras is developed. The lack of the convenient rigidity that C^*-algebras provide, where, e.g., morphisms are automatically continuous and even contractive, makes the task of developing the basics more laborious than it is for crossed products of C^*-algebras.

The paper [19] is for a large part concerned with one result: the General Correspondence Theorem [19, Theorem 8.1], most of which is formulated as Theorem 5.2.22 below. With (A, G, α) a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces – all notions will be reviewed in Section 5.2 – the General Correspondence Theorem, in the presence of a bounded approximate left identity of A, yields a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), and the non-degenerate bounded representations of the crossed product Banach algebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ associated with (A, G, α) and $\mathcal{R}$.

In [14] the theory established in [19] is developed further. Amongst others, there it is shown that (under mild conditions) the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra, up to topological isomorphism, which "generates" all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) [14, Theorem 4.4]. Furthermore, given a weight ω on G and assuming α is uniformly bounded, for a particular choice of $\mathcal{R}$ it is shown that the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is topologically isomorphic to a generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ [14, Section 5]. These algebras, as introduced in [14], are weighted Banach spaces of (equivalence classes) of A-valued functions that are also associative algebras with a multiplication that is continuous in both variables, but they are not Banach algebras in general, since the norm need not be submultiplicative. The General Correspondence Theorem then provides a bijection between the non-degenerate continuous covariant representations of (A, G, α), of which the representation of G is bounded by a multiple of ω, and the non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ [14, Theorem 5.20]. When A is taken to be the scalars, generalized Beurling algebras reduce to classical Beurling algebras, which are true Banach algebras, and then [14, Corollary 5.22] describes their non-degenerate bounded representations. In the case where $\omega = 1$ as well, this specializes to the classical bijection between uniformly bounded representations of G on Banach spaces and non-degenerate bounded representations of $L^1(G)$ (cf. [24, Assertion VI.1.32]).

In the current paper we adapt the theory developed in [19] and [14] to the ordered context: that of pre-ordered Banach spaces and algebras. Apart from its intrinsic interest, this is also motivated by the proven relevance of crossed products of C^*-algebras for unitary group representations. As is well known, a decomposition of a general unitary group representation into a direct integral of irreducible unitary representations is obtained via the group C^*-algebra (a particularly simple crossed product), and Mackey's Imprimitivity Theorem can, by Rieffel's work, now be conceptually interpreted in terms of (strong) Morita equivalence of a crossed product of a C^*-algebra and a group C^*-algebra. We hope that the results in the present paper will contribute to similar developments in the theory of positive representations of groups on pre-ordered Banach spaces (and Banach lattices in particular), which exist in abundance.

We are mainly concerned with four topics: Firstly, an adaptation of the construction of crossed products of Banach algebras from [19] to the ordered context (cf. Section 5.3). Secondly, proving a version of the General Correspondence Theorem in this context (cf. Theorem 5.3.13). Thirdly, for a pre-ordered Banach algebra dynamical system (A, G, α) and uniformly bounded class of positive continuous covariant representations $\mathcal{R}$, we establish (under mild conditions) the uniqueness, up to bipositive topological isomorphism, of the associated pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ as the unique pre-ordered Banach algebra which "generates" all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) (cf. Theorem 5.4.7). And fourthly, we describe the positive non-degenerate bounded representations of a pre-ordered Beurling algebra $L^1(G, A, \omega; \alpha)$ in terms of positive non-degenerate continuous covariant representations of the pre-ordered Banach algebra dynamical system (A, G, α) to which $L^1(G, A, \omega; \alpha)$ is associated (cf. Section 5.5).

We now briefly describe the structure of the paper.

Section 5.2 contains all preliminary definitions and results concerning pre-ordered vector spaces and crossed products. Specifically, Sections 5.2.1–5.2.3 provide preliminary definitions and results concerning pre-ordered vector spaces and algebras and pre-ordered normed spaces and algebras. Some of the material is completely standard and/or elementary, but since the fields of representation theory and positivity seem to be somewhat disjoint, we have included it in an attempt to enhance the accessibility of this paper, which draws on both disciplines. Sections 5.2.4 and 5.2.5

provide a brief recapitulation of all relevant notions from [19] relating to Banach algebra dynamical systems and crossed products.

In Section 5.3 we define pre-ordered Banach algebra dynamical systems and provide the construction of pre-ordered crossed products associated with such systems. The construction is largely the same as in the general unordered case, but differs in keeping track of how order structures of (A, G, α) and $\mathcal{R}$ induce a natural cone, denoted $(A \rtimes_\alpha G)^{\mathcal{R}}_+$, which defines a pre-order on the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. Theorem 5.3.8 collects properties of the cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ (and thereby the order structure) of a pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ in terms of the order properties of the pre-ordered Banach algebra dynamical system (A, G, α) and uniformly bounded class $\mathcal{R}$ of continuous covariant representations. Finally, we adapt the General Correspondence Theorem to the ordered context. In the presence of a positive bounded approximate left identity of A, it gives a canonical bijection between the positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, and positive non-degenerate bounded representations of the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces (cf. Theorem 5.3.13).

Paralleling work of Raeburn's [38], in Section 5.4 we show that (under mild additional hypotheses) the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ associated with (A, G, α) and $\mathcal{R}$ is the unique pre-ordered Banach algebra, up to bipositive topological isomorphism, which "generates" all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) (cf. Theorem 5.4.7).

Lastly, in Section 5.5, we study pre-ordered generalized Beurling algebras (denoted $L^1(G, A, \omega; \alpha)$). These algebras can be defined for any pre-ordered Banach algebra dynamical system (A, G, α) and weight ω on G, provided that α is uniformly bounded. If A has a bounded approximate right identity, for a specific choice of $\mathcal{R}$ the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is shown to be bipositively topologically isomorphic to a pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ (cf. Theorem 5.5.7). In the presence of a positive bounded approximate left identity of A, our ordered version of the General Correspondence Theorem, Theorem 5.3.13, then provides a bijection between the positive non-degenerate bounded representations of $L^1(G, A, \omega; \alpha)$ and the positive non-degenerate continuous covariant representations of the pre-ordered Banach algebra dynamical system (A, G, α), where the representation of the group G is bounded by a multiple of ω (cf. Theorem 5.5.9). In the case where A is a Banach lattice algebra, it is shown that $L^1(G, A, \omega; \alpha)$ also becomes a Banach lattice (although it is not generally a Banach algebra), and, under further conditions, becomes a Banach lattice algebra (cf. Theorem 5.5.8). In the simplest case, where A is taken to be the real numbers and $\omega = 1$, our results reduce to a bijection between the positive strongly continuous uniformly bounded representations of G on pre-ordered Banach spaces with closed cones on the one hand, and the positive non-degenerate bounded representations of $L^1(G)$ on such spaces on the other hand; this also follows from [24, Assertion VI.1.32].

5.2 Preliminaries and recapitulation

In this section we will introduce the terminology and notation used in the rest of the paper and give a brief recapitulation of Banach algebra dynamical systems and their crossed products. Sections 5.2.1–5.2.3 will introduce general notions concerning pre-ordered (normed) vector spaces and algebras. Sections 5.2.4 and 5.2.5 will give a brief overview of results from [19] on Banach algebra dynamical systems and their crossed products.

Throughout this paper all vector spaces are assumed to be over the reals, and all locally compact topologies are assumed to be Hausdorff.

Let X and Y be normed spaces. The normed space of bounded linear operators from X to Y will be denoted by $B(X,Y)$, and by $B(X)$ if $X=Y$. The group of invertible elements in $B(X)$ will be denoted by $\mathrm{Inv}(X)$. If A is a normed algebra, by $\mathrm{Aut}(A)$ we will denote its group of bounded automorphisms. We do not assume algebras to be unital.

For a locally compact topological space Ω and topological vector space V, we will denote the space of all continuous compactly supported functions on Ω taking values in V by $C_c(\Omega,V)$. If $V=\mathbb{R}$, we write $C_c(\Omega)$ for $C_c(\Omega,\mathbb{R})$.

If G is a locally compact group, we will denote its identity element by $e\in G$. For $f\in C_c(G)$, we will write $\int_G f(s)\,ds$ for the integral of f with respect to a fixed left Haar measure μ on G.

5.2.1 Pre-ordered vector spaces and algebras

We introduce the following terminology.

Let V be a vector space. A subset $C\subseteq V$ will be called a *cone* if $C+C\subseteq C$ and $\lambda C\subseteq C$ for all $\lambda\geq 0$. The pair (V,C) will be called a *pre-ordered vector space* and, for $x,y\in V$, by $y\leq x$ we mean $x-y\in C$. Elements of C will be called *positive.* We will often suppress mention of the cone C, and merely say that V is a pre-ordered vector space. In this case, we will denote the implicit cone by V_+ and refer to it as *the cone of* V. A cone $C\subseteq V$ will be said to be a *proper cone* if $C\cap(-C)=\{0\}$, in which case $\leq$ is a partial order, and then (V,C) will be called an *ordered vector space.* A cone $C\subseteq V$ will be said to be *generating* (*in* V) if $V=C-C$. If (V,C) is a pre-ordered vector space and V is also an associative algebra such that $C\cdot C\subseteq C$, we will say (V,C) is a *pre-ordered algebra.*

If (V_1,C_1) and (V_2,C_2) are pre-ordered vector spaces, we will say a linear map $T:V_1\to V_2$ is *positive* if $TC_1\subseteq C_2$. If T is injective and both $TC_1\subseteq C_2$ and $T^{-1}C_2\subseteq C_1$ hold, we will say T is *bipositive.*

With $W\subseteq V$ a subspace and $q:V\to V/W$ the quotient map, $q(C)\subseteq V/W$ will be called the *quotient cone.* Then $(V/W,q(C))$ is a pre-ordered vector space. Clearly $q:V\to V/W$ is positive and $q(C)$ is generating in V/W if C is generating in V.

5.2.2 Pre-ordered normed spaces and algebras

We give a brief description of pre-ordered normed vector spaces and algebras. In Section 5.3 we will apply the results from this section to describe the order structure of crossed products associated with pre-ordered Banach algebra dynamical systems.

If A is a pre-ordered algebra that is also a normed algebra, then we will call A a *pre-ordered normed algebra,* and a *pre-ordered Banach algebra* if A is complete. The *positive automorphism group* (*of* A) is defined by $\mathrm{Aut}_+(A) := \{\alpha \in \mathrm{Aut}(A) : \alpha^{\pm 1}(A_+) \subseteq A_+\} \subseteq B(A)$.

Let X and Y be pre-ordered normed spaces. We will always assume that $B(X,Y)$ is endowed with the *natural operator cone* $B(X,Y)_+ := \{T \in B(X,Y) : TX_+ \subseteq Y_+\}$, so that $B(X,Y)$ is a pre-ordered normed space, and $B(X)$ is a pre-ordered normed algebra. We define the *group of bipositive invertible operators* on X by $\mathrm{Inv}_+(X) := \{T \in \mathrm{Inv}(X) : T^{\pm 1}X_+ \subseteq X_+\}$. We will say X_+ is *topologically generating* (*in* X) if $X = \overline{X_+ - X_+}$. If the ordering defined by X_+ is a lattice-ordering (i.e., if every pair of elements from X has a supremum, denoted by $\vee$) we will call X a *normed vector lattice* if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in X$, where $|x| := x \vee (-x)$. A complete normed vector lattice will be called a *Banach lattice.* A pre-ordered Banach algebra that is also a Banach lattice will be called a *Banach lattice algebra.* A subspace $Y \subseteq X$ in a vector lattice X is called an *order ideal* if, for $g \in Y$ and $f \in X$, $|f| \leq |g|$ implies $f \in Y$.

We will need completions of pre-ordered normed spaces in Section 5.3, to be able to describe pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems.

Definition 5.2.1. Let V be a pre-ordered normed space. We define *the completion of* V by $(\overline{V}, \overline{V_+})$, where $\overline{V}$ denotes the usual metric completion of the normed space V, and $\overline{V_+}$ the closure of V_+ in $\overline{V}$.

The following two elementary observations are included for later reference.

Lemma 5.2.2. *Let V be a pre-ordered normed space, X a pre-ordered Banach space with a closed cone, and $T : V \to X$ a positive bounded linear operator. Then the bounded extension of T to the completion of V is a positive operator.*

Lemma 5.2.3. *If V is a pre-ordered normed algebra, then its completion is a pre-ordered Banach algebra with a closed cone.*

Together with Corollary 5.2.8, the following two elementary results will be used in Theorem 5.3.8 to give sufficient conditions for the cone of a crossed product of a pre-ordered Banach algebra dynamical system to be (topologically) generating.

Lemma 5.2.4. *Let V be a pre-ordered normed space. If V_+ is topologically generating in V, then V_+, and hence the cone $\overline{V_+}$, is topologically generating in the completion $\overline{V}$.*

Proof. Let $w \in \overline{V}$ be arbitrary, and let $(v_n) \subseteq V$ be such that $v_n \to w$. For every $n \in \mathbb{N}$, let $a_n, b_n \in V_+$ be such that $\|v_n - (a_n - b_n)\| < 2^{-n}$. Since

$$\|w - (a_n - b_n)\| \leq \|w - v_n\| + \|v_n - (a_n - b_n)\| < \|w - v_n\| + 2^{-n},$$

$(a_n - b_n) \subseteq V_+ - V_+$ converges to w. $\square$

In certain cases the conclusion of the previous lemma for $\overline{V_+}$ may be strengthened.

Lemma 5.2.5. *Let V be a pre-ordered normed space and $(\cdot)^+ : V \to V_+$ a function such that $v \leq v^+$ for all $v \in V$. Then V_+ is generating in V, and if $(\cdot)^+ : V \to V_+$ maps Cauchy sequences to Cauchy sequences, then the cone $\overline{V_+}$ is generating in the completion $\overline{V}$.*

Proof. It is obvious that the fact that V_+ is generating in V is equivalent with the existence of a function $(\cdot)^+ : V \to V_+$ such that $v \leq v^+$ for all $v \in V$.

Assuming that $(\cdot)^+ : V \to V_+$ maps Cauchy sequences to Cauchy sequences, let $w \in \overline{V}$ be arbitrary and let $(v_n) \subseteq V$ be such that $v_n \to w$. The sequence $(v_n) \subseteq V$ is Cauchy, hence, by hypothesis, so is $(v_n^+) \subseteq V_+ \subseteq \overline{V_+}$. Since $\overline{V_+}$ is closed in $\overline{V}$, (v_n^+) converges to some $w' \in \overline{V_+}$. Since $v_n^+ - v_n \in V_+ \subseteq \overline{V_+}$, we have $w' - w = \lim_{n\to\infty}(v_n^+ - v_n) \in \overline{V_+}$. Writing $w = w' - (w' - w)$ yields the result. $\square$

Remark 5.2.6. If V is a normed vector lattice, then the map $v \mapsto v \vee 0$ is uniformly continuous and hence maps Cauchy sequences to Cauchy sequences [41, Proposition II.5.2]. Hence $\overline{V_+}$ is generating in the completion $\overline{V}$. Since, in this case, $\overline{V}$ is actually a Banach lattice [41, Corollary 2, p. 84], this is not unexpected.

The following refinement of Andô's Theorem [3, Lemma 1] is a special case of [13, Theorem 4.1], of which the essence is that the decomposition of elements as the difference of positive elements can be chosen in a bounded, continuous and positively homogeneous manner. Its proof proceeds through applications of a generalization of the usual Open Mapping Theorem [13, Theorem 3.2] and the Michael Selection Theorem [1, Theorem 17.66]. It will be applied in Theorem 5.3.8 to prove that the cones of certain crossed products associated with pre-ordered Banach algebras are topologically generating.

Theorem 5.2.7. *Let X be a pre-ordered Banach space with closed generating cone. Then there exist a constant $\alpha > 0$ and continuous positively homogeneous maps $(\cdot)^\pm : X \to X_+$ such that $x = x^+ - x^-$ and $\|x^+\| + \|x^-\| \leq \alpha\|x\|$ for all $x \in X$.*

Simply through composition with the functions $(\cdot)^\pm : X \to X_+$, cones of continuous X_+-valued functions are then immediately seen to be generating in spaces of continuous X-valued functions. For example:

Corollary 5.2.8. *Let Ω be a locally compact Hausdorff space and X be a pre-ordered Banach space with closed generating cone. Then the cone $C_c(\Omega, X_+)$ is generating in $C_c(\Omega, X)$. In fact, there exists a constant $\alpha > 0$ with the property that, for every $f \in C_c(G, X)$, there exist $f^\pm \in C_c(G, X)$ such that $\|f^+(\omega)\| + \|f^-(\omega)\| \leq \alpha\|f(\omega)\|$ for all $\omega \in \Omega$. In particular, $\|f^\pm\|_\infty \leq \alpha\|f\|_\infty$ and* $\operatorname{supp}(f^\pm) \subseteq \operatorname{supp}(f)$.

Remark 5.2.9. The earliest results of this type known to the authors are [4, Theorem 2.3] and [45, Theorem 4.4]. Both results proceed through an application of Lazar's affine selection theorem to show that canonical cones of certain spaces of continuous affine functions are generating. The result [4, Theorem 2.3] shows, with K a Choquet simplex and X a pre-ordered Banach space with a closed cone, that the space $A(K, X)$ of continuous affine functions from K to X has $A(K, X_+)$ as a generating cone. By taking K to be the regular Borel probability measures on a compact Hausdorff space Ω, this result includes the case that $C(\Omega, X_+)$ is generating in $C(\Omega, X)$, which is part of the statement of Corollary 5.2.8.

We will now define normality and conormality properties for a pre-ordered Banach space X with a closed cone, and subsequently show in Theorem 5.2.12 how these properties imply normality properties of the pre-ordered normed space $B(X, Y)$. In Theorem 5.3.8 this will be used to conclude (conditional) normality properties of a pre-ordered crossed product.

Definition 5.2.10. Let X be a pre-ordered Banach space with closed cone and $\alpha > 0$. We define the following ***normality properties****:*

(1) We will say that X is *α-normal* if, for any $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq \alpha\|y\|$.

(2) We will that X is *α-absolutely normal* if, for any $x, y \in X$, $\pm x \leq y$ implies $\|x\| \leq \alpha\|y\|$.

We define the following ***conormality properties****:*

(1) We will say that X is *approximately α-absolutely conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exists some $a \in X_+$ such that $\pm x \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$.

(2) We will say that X is *approximately α-sum-conormal* if, for any $x \in X$ and $\varepsilon > 0$, there exist some $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| < \alpha\|x\| + \varepsilon$.

Remark 5.2.11. Normality (terminology due to Krein [28]) and (approximate) conormality (terminology due to Walsh [44]) are dual properties for pre-ordered Banach spaces with closed cones. Roughly speaking, a pre-ordered Banach space with a closed cone has some normality property precisely if its dual has a corresponding conormality property, and vice versa. The most complete reference for such duality relationships seems to be [6].

For a pre-ordered Banach space X with a closed cone, elementary arguments will show that α-absolutely normality of X implies that X is α-normal, which in turn implies that X_+ is a proper cone. Also, approximate α-sum-conormality of X implies that X is approximately α-absolute conormal, which in turn implies that X_+ is generating in X. An application of Andô's Theorem [3, Lemma 1] shows conversely that, if X_+ is generating in X, then there exists some $\beta > 0$ such that, for every $x \in X$, there exists $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$ (another form of conormality, clearly implying approximate 2β-sum-conormality). We note that Banach lattices are always 1-absolutely normal and approximately 1-absolutely conormal.

The following results relate normality properties of spaces of operators to the normality and conormality properties of the underlying spaces. Part (2) is due to Wickstead [45, Theorem 3.1]. Part (3) is a slight refinement of a result due to Yamamuro [48, Theorem 1.3], where it is proven for the case $X = Y$ and $\alpha = \beta = 1$. No reference for part (4) is known to the authors. We include proofs for convenience of the reader.

Theorem 5.2.12. *Let X and Y be pre-ordered Banach spaces with closed cones and $\alpha, \beta > 0$.*

(1) If X_+ is generating and Y_+ is a proper cone, then $B(X,Y)_+$ is a proper cone.

(2) If X_+ is generating and Y is α-normal, then there exists some $\gamma > 0$ for which $B(X,Y)$ is γ-normal.

(3) If X is approximately α-absolutely conormal and Y is β-absolutely normal, then $B(X,Y)$ is $\alpha\beta$-absolutely normal.

(4) If X is approximately α-sum-conormal and Y is β-normal, then $B(X,Y)$ is $\alpha\beta$-normal.

Proof. We prove (1). Let $T \in B(X,Y)_+ \cap (-B(X,Y)_+)$. If $x \in X_+$, then $Tx \geq 0$ and $(-T)x \geq 0$. Hence $Tx = 0$, since Y_+ is proper. Since X_+ is generating, we have $T = 0$ as required.

We prove (2). By Andô's Theorem [3, Lemma 1], the fact that X_+ is generating in X implies that there exists some $\beta > 0$ such that, for every $x \in X$, there exist $a, b \in X_+$ such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$. Let $T, S \in B(X,Y)$ be such that $0 \leq T \leq S$. Then, for any $x \in X$, let $a, b \in X_+$ be such that $x = a - b$ and $\max\{\|a\|, \|b\|\} \leq \beta\|x\|$, so that $0 \leq Ta \leq Sa$ and $0 \leq Tb \leq Sb$. By α-normality of Y,

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \alpha(\|Sa\| + \|Sb\|) \leq \alpha\|S\|(\|a\| + \|b\|) \leq 2\alpha\beta\|S\|\|x\|,$$

hence $\|T\| \leq 2\alpha\beta\|S\|$.

We prove (3). Let $T, S \in B(X,Y)$ satisfy $\pm T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exists some $a \in X_+$ satisfying $\pm x \leq a$ and $\|a\| < \alpha\|x\| + \varepsilon$. Then

$$Tx = T\left(\frac{a+x}{2}\right) - T\left(\frac{a-x}{2}\right),$$

and hence

$$\pm Tx = \pm T\left(\frac{a+x}{2}\right) \mp T\left(\frac{a-x}{2}\right).$$

Since $(a+x)/2 \geq 0$, $(a-x)/2 \geq 0$ and $\pm T \leq S$, we find

$$\pm Tx \leq S\left(\frac{a+x}{2}\right) + S\left(\frac{a-x}{2}\right) = Sa.$$

Now, because Y is β-absolutely normal, we obtain

$$\|Tx\| \leq \beta\|Sa\| \leq \beta\|S\|\|a\| \leq \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X,Y)$ is $\alpha\beta$-absolutely normal.

We prove (4). Let $T, S \in B(X,Y)$ satisfy $0 \leq T \leq S$. Let $x \in X$ be arbitrary. Then, for every $\varepsilon > 0$, there exist $a, b \in X_+$ such that $x = a - b$ and $\|a\| + \|b\| \leq$

$\alpha\|x\| + \varepsilon$. Because $0 \leq T \leq S$, we have $0 \leq Ta \leq Sa$ and $0 \leq Tb \leq Sb$. Since Y is β-normal, we obtain

$$\|Tx\| \leq \|Ta\| + \|Tb\| \leq \beta\|Sa\| + \beta\|Sb\| \leq \beta\|S\| \left(\|a\| + \|b\|\right) \leq \alpha\beta\|S\|\|x\| + \varepsilon\beta\|S\|.$$

Since $\varepsilon > 0$ was chosen arbitrarily, we conclude that $B(X, Y)$ is $\alpha\beta$-normal. □

Remark 5.2.13. We note some specific cases of the above theorem. Any Banach lattice X is both approximately 1-absolutely conormal and 1-absolutely normal, therefore (3) in the previous result implies that $B(X)$ is always 1-absolutely normal in this case. Also, if X is a Banach lattice and Y and Z are pre-ordered Banach spaces with closed cones that are respectively α-absolutely normal and approximately α-absolutely conormal for some $\alpha > 0$, then $B(X, Y)$ and $B(Z, X)$ are α-absolutely normal, again by (3) above. If a Banach lattice X is an AL-space (i.e., $\|x + y\| = \|x\| + \|y\|$ for all $x, y \in X_+$), then it is approximately 1-sum-conormal, and hence, for any α-normal pre-ordered Banach space Y with a closed cone, $B(X, Y)$ is α-normal by (4) in the previous result.

Let $\mathcal{H}$ be real Hilbert space endowed with a Lorentz cone $\mathcal{L}_v := \{x \in \mathcal{H} : \langle v|x\rangle \geq \|Px\|\}$, where $v \in \mathcal{H}$ is any norm 1 element, and P the projection onto $\{v\}^\perp$. Although not a Banach lattice if $\dim \mathcal{H} \geq 3$, the ordered Banach space $(\mathcal{H}, \mathcal{L}_v)$ is 1-absolutely normal and approximately 1-absolutely conormal [30]. Hence, again by (3) above, $B(\mathcal{H})$ is 1-absolutely normal, and if Y and Z are pre-ordered Banach spaces with closed cones that are respectively α-absolutely normal and approximately α-absolutely conormal for some $\alpha > 0$, then $B(\mathcal{H}, Y)$ and $B(Z, \mathcal{H})$ are α-absolutely normal.

5.2.3 Representations on pre-ordered normed spaces

We will now introduce positive representations of groups and pre-ordered normed algebras on pre-ordered normed spaces. In Section 5.3 we will use the notions in this section to describe a bijection between the positive non-degenerate bounded representations of a crossed product associated with a pre-ordered Banach algebra dynamical system on the one hand, and positive non-degenerate covariant representations of the certain dynamical system on the other hand (cf. Theorem 5.3.13).

Definition 5.2.14. Let A be a normed algebra and X a normed space. An algebra homomorphism $\pi : A \to B(X)$ will be called a *representation of A on X*. We will write X_π for X, if the connection between X_π and π requires emphasis. We will say that π is *non-degenerate* if $\mathrm{span}\{\pi(a)x : a \in A, x \in X\}$ is dense in X.

If A is a pre-ordered normed algebra and X is a pre-ordered normed space, we will say that a representation π of A on X is *positive* if $\pi(A_+) \subseteq B(X)_+$.

Definition 5.2.15. Let G be a locally compact group and X a normed space. A group homomorphism $U : G \to \mathrm{Inv}(X)$ will be called a *representation* (*of G on X*).

If X is a pre-ordered normed space, a group homomorphism $U : G \to \mathrm{Inv}_+(X) \subseteq B(X)$ (cf. Section 5.2.2) will be called a *positive representation* of G on X.

For typographical reasons we will write U_s instead of $U(s)$ for $s \in G$.

Note that continuity is not included in the definition of representations of normed algebras and locally compact groups, and that representations of a unital algebra are not required to be unital.

The left centralizer algebra of a normed algebra, to be introduced next, plays a crucial role in the construction of the bijection mentioned in the first paragraph of this section.

Definition 5.2.16. Let A be a normed algebra. A bounded linear operator $L : A \to A$ will be called a *left centralizer of* A if $L(ab) = (La)b$ for all $a, b \in A$. The unital normed algebra of all left centralizers, with the operator norm inherited from $B(A)$, will be denoted $\mathcal{M}_l(A)$ and called the *left centralizer algebra of* A. The *left regular representation of* A, $\lambda : A \to \mathcal{M}_l(A)$, is defined by $\lambda(a)b := ab$ for $a, b \in A$.

If A is a pre-ordered normed algebra, we will always assume that $\mathcal{M}_l(A)$ is endowed with the cone $\mathcal{M}_l(A) \cap B(A)_+$. Then $\lambda : A \to B(A)$ is a positive contractive representation of A on itself.

Definition 5.2.17. If A is a pre-ordered normed algebra, we will say an approximate left (right) identity (u_i) of A is *positive* if $(u_i) \subseteq A_+$.

The result [18, Theorem 4.1] plays a key role in the proof of the General Correspondence Theorem (Theorem 5.2.22). We collect the relevant parts in Theorem 5.2.18, including how it can be applied to representations of pre-ordered normed algebras on pre-ordered Banach spaces with closed cones. This will be used to adapt the General Correspondence Theorem to the ordered context (cf. Theorem 5.3.13).

Theorem 5.2.18. *Let B be a normed algebra with an M-bounded approximate left identity (u_i) and X a Banach space. If $T : B \to B(X)$ is a non-degenerate bounded representation, then the map $\overline{T} : \mathcal{M}_l(B) \to B(X)$ defined by $\overline{T}(L) :=$ SOT-$\lim_i T(Lu_i)$ is the unique representation of $\mathcal{M}_l(B)$ on X such that the diagram*

$$\begin{array}{ccc} B & \xrightarrow{\;T\;} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T}} \\ & & \mathcal{M}_l(B) \end{array}$$

commutes. Moreover, $\overline{T}$ is non-degenerate and bounded, with $\|\overline{T}\| \leq M\|T\|$, and satisfies $\overline{T}(L)T(b) = T(Lb)$ for all $b \in B$ and $L \in \mathcal{M}_l(B)$.

If, in addition, B is a pre-ordered normed algebra, (u_i) is positive, and X is a pre-ordered Banach space with a closed cone, then $\overline{T} : \mathcal{M}_l(B) \to B(X)$ is a positive non-degenerate bounded representation of $\mathcal{M}_l(B)$ on X.

5.2.4 Banach algebra dynamical systems and crossed products

We recall some basic definitions and results from [19].

Definition 5.2.19. Let A be a Banach algebra, G a locally compact group, and $\alpha : G \to \mathrm{Aut}(A)$ a strongly continuous representation of G on A. Then we will call the triple (A, G, α) a *Banach algebra dynamical system.*

If (A, G, α) is a Banach algebra dynamical system, $C_c(G, A)$ can be made into an associative algebra by defining the twisted convolution

$$(f * g)(s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \quad (f, g \in C_c(G, A),\ s \in G).$$

Here, as in [19], integrals of compactly supported continuous Banach space valued functions are defined by duality, following [40, Section 3].

Let (A, G, α) be a Banach algebra dynamical system and X a normed space. If $\pi : A \to B(X)$ and $U : G \to \mathrm{Inv}(X)$ are representations satisfying

$$\pi(\alpha_s(a)) = U_s\pi(a)U_s^{-1} \quad (a \in A,\ s \in G),$$

we will say that (π, U) is a *covariant representation* of (A, G, α) on X. We will say (π, U) is *continuous* if π is bounded and U is strongly continuous. We will say that (π, U) is *non-degenerate* if π is non-degenerate.

If (π, U) is a continuous covariant representation of (A, G, α) on a Banach space X, then, as in [19, Section 3], for every $f \in C_c(G, A)$, $\pi \rtimes U(f) \in B(X)$ is defined by

$$\pi \rtimes U(f)x := \int_G \pi(f(r))U_r x\,dr \quad (x \in X).$$

The map $\pi \rtimes U : C_c(G, A) \to B(X)$ is then a representation of the algebra $C_c(G, A)$ on X, and is called the *integrated form* of (π, U).

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a class of continuous covariant representations of (A, G, α) on Banach spaces. We will always tacitly assume that $\mathcal{R}$ is non-empty. We will say $\mathcal{R}$ is a *uniformly bounded class* of continuous covariant representations if there exists a constant $C \geq 0$ and a function $\nu : G \to \mathbb{R}_{\geq 0}$, which is bounded on compact subsets of G, such that, for all $(\pi, U) \in \mathcal{R}$, $\|\pi\| \leq C$ and $\|U_r\| \leq \nu(r)$ for all $r \in G$.

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. It follows that $\|\pi \rtimes U(f)\| \leq C\left(\sup_{r\in\mathrm{supp}(f)} \nu(r)\right)\|f\|_1$ for all $(\pi, U) \in \mathcal{R}$ and $f \in C_c(G, A)$ [19, Remark 3.3]. We introduce the (hence finite) algebra seminorm $\sigma^{\mathcal{R}}$ on $C_c(G, A)$, defined by

$$\sigma^{\mathcal{R}}(f) := \sup_{(\pi,U)\in\mathcal{R}} \|\pi \rtimes U(f)\| \quad (f \in C_c(G, A)).$$

The kernel of $\sigma^{\mathcal{R}}$ is a two-sided ideal of $C_c(G, A)$, hence $C_c(G, A)/\ker\sigma^{\mathcal{R}}$ is a normed algebra with norm $\|\cdot\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$. Its completion is called the *crossed product* (*associated with* (A, G, α) *and* $\mathcal{R}$), and denoted by $(A \rtimes_\alpha G)^{\mathcal{R}}$. Multiplication in $(A \rtimes_\alpha G)^{\mathcal{R}}$ will be denoted by $*$. We denote the quotient map from $C_c(G, A)$ to $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$. For any Banach space X and linear map $T : C_c(G, A) \to X$, if T is bounded with respect to the $\sigma^{\mathcal{R}}$ seminorm, we will denote the canonically induced bounded linear map on $(A \rtimes_\alpha G)^{\mathcal{R}}$ by $T^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to X$, as detailed in [19, Section 3].

5.2.5 The correspondence between representations of (A, G, α) and $(A \rtimes_\alpha G)^{\mathcal{R}}$

We briefly describe the General Correspondence Theorem [19, Theorem 8.1], most of which is formulated as Theorem 5.2.22 below. In the presence of a bounded approximate left identity of A, the General Correspondence Theorem describes a bijection between the non-degenerate $\mathcal{R}$-continuous (to be defined below) covariant representations of a Banach algebra dynamical system (A, G, α) and the non-degenerate bounded representations of the associated crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. In Section 5.3 we will adapt the results from this section to pre-ordered Banach algebra dynamical systems and the associated pre-ordered crossed products.

Definition 5.2.20. Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. If (π, U) is a continuous covariant representation of (A, G, α) on a Banach space X, and $\pi \rtimes U : C_c(G, A) \to B(X)$ is bounded with respect to $\sigma^{\mathcal{R}}$, we will say (π, U) is *$\mathcal{R}$-continuous.*

The proof of the General Correspondence Theorem proceeds through an application of Theorem 5.2.18, which requires the existence of a bounded approximate left identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$. The following theorem makes precise how this (and its right-sided counterpart) is inherited from A.

Theorem 5.2.21. *[19, Theorem 4.4] Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let A have a bounded approximate left (right) identity (u_i). Let $\mathcal{Z}$ be a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact subset of G. For each $V \in \mathcal{Z}$, let $z_V \in C_c(G)$ be positive, supported in V, and have integral equal to one. Then the net*

$$\left(q^{\mathcal{R}}(z_V \otimes u_i)\right),$$

where $(V, i) \leq (W, j)$ is defined to mean $i \leq j$ and $W \subseteq V$, is a bounded approximate left (right) identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Let (A, G, α) be a Banach algebra dynamical system. We define the maps $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ by

$$\begin{aligned} (i_A(a)f)(s) &:= af(s), \\ (i_G(r)f)(s) &:= \alpha_r(f(r^{-1}s)), \end{aligned}$$

for $f \in C_c(G, A)$, $a \in A$ and $r, s \in G$. The maps $i_A(a)$ and $i_G(r)$ are bounded on $C_c(G, A)$ with respect to $\sigma^{\mathcal{R}}$ [19, Lemma 6.3], hence we can define the maps

$i_A^{\mathcal{R}} : A \to B((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to B((A \rtimes_\alpha G)^{\mathcal{R}})$, by $i_A^{\mathcal{R}}(a) := i_A(a)^{\mathcal{R}}$ and $i_G^{\mathcal{R}}(r) := i_G(r)^{\mathcal{R}}$ in the notation of Section 5.2.4, for $a \in A$ and $r \in G$. Moreover, the maps $a \mapsto i_A^{\mathcal{R}}(a)$ and $r \mapsto i_G^{\mathcal{R}}(r)$ map A and G into $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ [19, Proposition 6.4]. If A has a bounded approximate left identity and $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, then $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$, and the integrated form $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}}$ equals the left regular representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ [19, Theorem 7.2].

This pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ can be used to "generate" non-degenerate continuous covariant representations of (A, G, α) from non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$. We will investigate this further in Section 5.4, but its key role becomes already apparent in the following result, giving an explicit bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Theorem 5.2.22. (*General Correspondence Theorem, cf. [19, Theorem 8.1]*) *Let (A, G, α) be a Banach algebra dynamical system, where A has a bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on Banach spaces and the non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces.*

More precisely:

(1) If (π, U) is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on a Banach space X_π, then $(\pi \rtimes U)^{\mathcal{R}}$ is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on X_π, and

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ is the non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 5.2.18.

(2) If T is a non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a Banach space X_T, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on X_T, and

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T$$

where $\overline{T}$ is the non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 5.2.18.

5.3 Pre-ordered Banach algebra dynamical systems and crossed products

In this section we study pre-ordered Banach algebra dynamical systems and their associated crossed products. After the preliminary Section 5.3.1, in Section 5.3.2 we will define pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems, and describe properties of the cone of such pre-ordered crossed products (cf. Theorem 5.3.8). Finally, Theorem 5.3.13 in Section 5.3.3 will give an adaptation of the General Correspondence Theorem (Theorem 5.2.22) to the ordered context.

5.3.1 Pre-ordered Banach algebra dynamical systems

We introduce pre-ordered Banach algebra dynamical systems (A, G, α), and verify that the twisted convolution as defined in Section 5.2.4 gives $C_c(G, A)$ a pre-ordered algebra structure. Furthermore, Lemma 5.3.4 shows that positive continuous covariant representations of a pre-ordered Banach algebra dynamical systems (A, G, α) have positive integrated forms.

Definition 5.3.1. Let A be a pre-ordered Banach algebra with closed cone, G a locally compact group, and $\alpha : G \to \mathrm{Aut}_+(A)$ a strongly continuous positive representation of G on A. Then we will call the triple (A, G, α) a *pre-ordered Banach algebra dynamical system.*

Lemma 5.3.2. *If (A, G, α) is a pre-ordered Banach algebra dynamical system, with A having a closed cone, then $(C_c(G, A), C_c(G, A_+))$, with twisted convolution as defined in Section 5.2.4, is a pre-ordered algebra.*

Proof. Let $f, g \in C_c(G, A_+)$ with $f \neq 0$. By [40, Theorem 3.27], for every $s \in G$,

$$\frac{(f * g)(s)}{\mu(\mathrm{supp}(f))} = \int_{\mathrm{supp}(f)} f(r)\alpha_r(g(r^{-1}s)) \,\frac{dr}{\mu(\mathrm{supp}(f))},$$

where μ denotes the chosen left Haar measure on G, lies in the closed convex hull of $\{f(r)\alpha_r(g(r^{-1}s)) : r \in \mathrm{supp}(f)\} \subseteq A_+$. Since A_+ is itself closed and convex, the result follows. $\square$

Definition 5.3.3. Let (A, G, α) be a pre-ordered Banach algebra dynamical system. Let (π, U) be a covariant representation of (A, G, α) on a pre-ordered normed space X. If both π and U are positive representations, we will say that the covariant representation (π, U) is *positive.*

Lemma 5.3.4. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and (π, U) a positive continuous covariant representation of (A, G, α) on a pre-ordered Banach space X with a closed cone. Then the integrated form $\pi \rtimes U : C_c(G, A) \to B(X)$ is a positive algebra representation.*

Proof. Let $f \in C_c(G, A_+)$. Since (π, U) is positive, we have $\pi(f(r))U_r x \in X_+$ for all $x \in X_+$ and $r \in G$. Since X_+ is closed and convex, we obtain $\int_G \pi(f(r))U_r x \, dr \in X_+$ as in the proof of Lemma 5.3.2. $\square$

5.3.2 Crossed products associated with pre-ordered Banach algebra dynamical systems

In this section we will describe the construction of pre-ordered crossed products associated with pre-ordered Banach algebra dynamical systems. The construction as a Banach algebra is as described in Section 5.2.4, so we will focus mainly on the properties of the order structure.

Lemma 5.3.5. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Then the space $C_c(G, A)/\ker\sigma^{\mathcal{R}}$, with norm $\|\cdot\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$ and pre-ordered by the quotient cone $q^{\mathcal{R}}(C_c(G, A_+))$, is a pre-ordered normed algebra.*

Proof. As explained in Section 5.2.4, $C_c(G, A)/\ker\sigma^{\mathcal{R}}$ is a normed algebra with norm induced by $\sigma^{\mathcal{R}}$. That it is a pre-ordered algebra follows from the definition of the quotient cone and the fact that the twisted convolution of positive elements of $C_c(G, A)$ is again positive by Lemma 5.3.2. □

We can now describe $(A \rtimes_\alpha G)^{\mathcal{R}}$ as a pre-ordered Banach algebra:

Definition 5.3.6. Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. The completion (as in Definition 5.2.1) of the pre-ordered normed algebra $(C_c(G, A)/\ker\sigma^{\mathcal{R}}, q^{\mathcal{R}}(C_c(G, A_+)))$, with norm $\|\cdot\|^{\mathcal{R}}$ induced by $\sigma^{\mathcal{R}}$, will be denoted by $(A \rtimes_\alpha G)^{\mathcal{R}}$, the pre-ordering being tacitly understood, and will be called the *pre-ordered crossed product* (*associated with* (A, G, α) *and* $\mathcal{R}$).

We recall the following result from [19], which will be used twice in the proof of Theorem 5.3.8:

Proposition 5.3.7. *[19, Proposition 3.4] Let (A, G, α) be a Banach algebra dynamical system, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Then, for every $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$,*

$$\|d\|^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|(\pi \rtimes U)^{\mathcal{R}}(d)\|.$$

The following theorem describes properties of the closed cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ in a pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Theorem 5.3.8. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone. Let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Then:*

(1) The pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a pre-ordered Banach algebra with a closed cone.

(2) If A_+ is generating in A, then $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is topologically generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.

(3) *If A_+ is generating in A and $(\cdot)^+ : q^{\mathcal{R}}(C_c(G,A)) \to q^{\mathcal{R}}(C_c(G,A_+))$ is a function such that $q^{\mathcal{R}}(f) \leq q^{\mathcal{R}}(f)^+$ for all $f \in C_c(G,A)$, and which maps $\|\cdot\|^{\mathcal{R}}$-Cauchy sequences to $\|\cdot\|^{\mathcal{R}}$-Cauchy sequences, then the cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

(4) *If $\mathcal{R}$ is a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones such that, for every $(\pi, U) \in \mathcal{R}$, the cone $B(X_\pi)_+$ is proper, then the cone $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is a proper cone.*

(5) *If $\beta > 0$ and $\mathcal{R}$ is a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones such that, for every $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ is β-(absolutely) normal, then $(A \rtimes_\alpha G)^{\mathcal{R}}$ is β-(absolutely) normal.*

Proof. As to (1): That $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a pre-ordered Banach algebra with closed cone is immediate from Lemmas 5.3.5 and 5.2.3.

We prove (2). Let A_+ be generating in A. By Corollary 5.2.8, the cone $C_c(G, A_+)$ is generating in $C_c(G, A)$. Therefore the quotient cone is generating in the space $C_c(G,A)/\ker \sigma^{\mathcal{R}}$, and by Lemma 5.2.4, $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is topologically generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.

The statement in (3) follows from Lemma 5.2.5.

We prove (4). Let $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$ be such that $0 \leq d \leq 0$ in $(A \rtimes_\alpha G)^{\mathcal{R}}$. Let $(\pi, U) \in \mathcal{R}$ be arbitrary. By Lemma 5.3.4, $\pi \rtimes U : C_c(G,A) \to B(X_\pi)$ is a positive algebra representation. Therefore the induced map $(\pi \rtimes U)^{\mathcal{R}} : C_c(G,A)/\ker \sigma^{\mathcal{R}} \to B(X_\pi)$ is a positive bounded algebra representation. Since the cone of X_π is closed, so is the cone of $B(X_\pi)$, and therefore, by Lemma 5.2.2, $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X_\pi)$ is a positive algebra representation. Hence $0 \leq d \leq 0$ implies

$$0 \leq (\pi \rtimes U)^{\mathcal{R}}(d) \leq 0,$$

and since $B(X_\pi)_+$ is a proper cone, we obtain $(\pi \rtimes U)^{\mathcal{R}}(d) = 0$. Therefore, by Proposition 5.3.7, $\|d\|^{\mathcal{R}} = \sup_{(\pi,U)\in\mathcal{R}} \|(\pi \rtimes U)^{\mathcal{R}}(d)\| = 0$, and hence $d = 0$. We conclude that $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is a proper cone.

We prove (5). Let $\beta > 0$ be such that, for every $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ is β-absolutely normal. For any $(\pi, U) \in \mathcal{R}$, as previously, $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X_\pi)$ is a positive algebra representation. Hence, if $\pm d_1 \leq d_2$ for $d_1, d_2 \in (A \rtimes_\alpha G)^{\mathcal{R}}$, we have

$$\pm(\pi \rtimes U)^{\mathcal{R}}(d_1) \leq (\pi \rtimes U)^{\mathcal{R}}(d_2).$$

Since $B(X_\pi)$ is β-absolutely normal, we obtain $\|(\pi \rtimes U)^{\mathcal{R}}(d_1)\| \leq \beta \|(\pi \rtimes U)^{\mathcal{R}}(d_2)\|$. By Proposition 5.3.7, taking the supremum over $\mathcal{R}$ on both sides yields $\|d_1\|^{\mathcal{R}} \leq \beta \|d_2\|^{\mathcal{R}}$.

That $(A \rtimes_\alpha G)^{\mathcal{R}}$ is β-normal for some $\beta > 0$ under the assumption that, for every $(\pi, U) \in \mathcal{R}$, $B(X_\pi)$ is β-normal follows similarly. □

Under the hypotheses of Theorem 5.3.8, we see that it is relatively easy to have a topologically generating cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$: It is sufficient that A_+ is generating in A. The condition in (3) under which the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is generating is less easily verified if $\sigma^{\mathcal{R}}$ is not a norm, but we will nevertheless see an example (where $\sigma^{\mathcal{R}}$ is a norm) in Section 5.5 where we conclude that the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is generating through a different method than provided by (3) in the theorem above. Furthermore, according to part (4) and Theorem 5.2.12, if every continuous covariant representation from $\mathcal{R}$ is positive and acts on a pre-ordered Banach space with a closed proper generating cone, then the cone of $(A \rtimes_\alpha G)^{\mathcal{R}}$ is proper. As to (5), an appeal to Remark 5.2.13 shows that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is 1-absolutely normal if every continuous covariant representation from $\mathcal{R}$ is positive and acts on a Banach lattice. We collect the features of the latter case in the following result:

Corollary 5.3.9. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone. Let $\mathcal{R}$ be a uniformly bounded class of positive continuous covariant representations on Banach lattices. Then $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is a closed proper cone and $(A \rtimes_\alpha G)^{\mathcal{R}}$ is a 1-absolutely normal ordered Banach algebra, i.e., for $d_1, d_2 \in (A \rtimes_\alpha G)^{\mathcal{R}}$, if $\pm d_1 \leq d_2$, then $\|d_1\|^{\mathcal{R}} \leq \|d_2\|^{\mathcal{R}}$. If A_+ is generating in A, then $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ is topologically generating in $(A \rtimes_\alpha G)^{\mathcal{R}}$.*

The following example shows that even with A a Banach lattice algebra and the positive representations from $\mathcal{R}$ acting on Banach lattices, $\ker \sigma^{\mathcal{R}}$ need not be an order ideal in the vector lattice $C_c(G, A)$ in general.

Example 5.3.10. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$. We consider the pre-ordered Banach algebra dynamical system $(\mathbb{R}, \mathbb{Z}_2, \mathrm{triv})$ and $\mathcal{R} = \{(\mathrm{id}, \mathrm{triv})\}$ with $(\mathrm{id}, \mathrm{triv})$ the trivial positive non-degenerate continuous covariant representation of $(\mathbb{R}, \mathbb{Z}_2, \mathrm{triv})$ on $\mathbb{R}$. Then, for $f \in C_c(\mathbb{Z}_2)$, $\mathrm{id} \rtimes \mathrm{triv}(f) = f(0) + f(1)$, hence $f \in \ker \sigma^{\mathcal{R}}$ if and only if $f(0) = -f(1)$. In particular, since $f \in \ker \sigma^{\mathcal{R}}$ does not imply $|f| \in \ker \sigma^{\mathcal{R}}$, $\ker \sigma^{\mathcal{R}}$ is not an order ideal in the vector lattice $C_c(\mathbb{Z}_2)$.

5.3.3 The correspondence between positive representations of (A, G, α) and $(A \rtimes_\alpha G)^{\mathcal{R}}$

In this section we give an adaptation of the General Correspondence Theorem (Theorem 5.2.22) to the ordered context. As in the unordered context, Theorem 5.2.18 will be a crucial ingredient, which here will rely on the existence of a positive bounded approximate left identity of the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$. The following result shows that this is inherited from A.

Proposition 5.3.11. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let A have a positive bounded approximate left (right) identity (u_i). Then the net*

$$\left(q^{\mathcal{R}}(z_V \otimes u_i)\right),$$

as described in Theorem 5.2.21, is a positive bounded approximate left (right) identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$.

Proof. Since the quotient map $q^{\mathcal{R}} : C_c(G, A) \to (A \rtimes_\alpha G)^{\mathcal{R}}$ is positive and $z_V \otimes u_i \in C_c(G, A_+)$ for all i and $V \in \mathcal{Z}$, we have $q^{\mathcal{R}}(z_V \otimes u_i) \in (A \rtimes_\alpha G)^{\mathcal{R}}_+$. That $\big(q^{\mathcal{R}}(z_V \otimes u_i)\big)$ is a bounded left (right) identity is the statement of Theorem 5.2.21. □

The following will be used in the proof of Theorem 5.3.13 and in Section 5.4.

Lemma 5.3.12. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone. With $C_c(G, A)$ pre-ordered by the cone $C_c(G, A_+)$, the representations $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ as in defined in Section 5.2.5 are positive.*

If $\mathcal{R}$ is a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, and A has a bounded approximate left identity, then the pair $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ as defined in Section 5.2.5 is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ such that $i_A^{\mathcal{R}}(A) \subseteq \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}}(G) \subseteq \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$.

Proof. That the maps $i_A : A \to \mathrm{End}(C_c(G, A))$ and $i_G : G \to \mathrm{End}(C_c(G, A))$ are positive is clear. By [19, Lemma 6.3] and Lemma 5.2.2 the operators $i_A^{\mathcal{R}}(a)$ and $i_G^{\mathcal{R}}(r)$ ($a \in A_+$, $r \in G$) on $(A \rtimes_\alpha G)^{\mathcal{R}}$ are positive. The remaining statement is contained in [19, Theorem 7.2]. □

We finally establish the following adaptation of the General Correspondence Theorem to the ordered context. Note that the class $\mathcal{R}$ is not required to consist of positive continuous covariant representations. Conditions in that vein affect the properties of the cone in $(A \rtimes_\alpha G)^{\mathcal{R}}$ (cf. Theorem 5.3.8, Corollary 5.3.9), but are not necessary for the correspondence.

Theorem 5.3.13. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a positive bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. Then the map $(\pi, U) \mapsto (\pi \rtimes U)^{\mathcal{R}}$ is a bijection between the positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones and the positive non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on such spaces.*

More precisely:

(1) If (π, U) is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on a pre-ordered Banach space X_π with a closed cone, then $(\pi \rtimes U)^{\mathcal{R}}$ is a positive non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on X_π, and

$$(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U),$$

where $\overline{(\pi \rtimes U)^{\mathcal{R}}}$ denotes the positive non-degenerate bounded representation of $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ as in Theorem 5.2.18.

(2) If T is a positive non-degenerate bounded representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a pre-ordered Banach space X_T with a closed cone, then $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a

positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) *on* X_T*, and*

$$(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T,$$

where $\overline{T}$ *is the positive non-degenerate bounded representation of* $\mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ *as in Theorem 5.2.18.*

Proof. We prove part (1). If (π, U) is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), by Lemma 5.3.4 and Lemma 5.2.2 we obtain that $(\pi \rtimes U)^{\mathcal{R}}$ is a positive bounded representation bounded of $(A \rtimes_\alpha G)^{\mathcal{R}}$. That it is non-degenerate and that $(\overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_A^{\mathcal{R}}, \overline{(\pi \rtimes U)^{\mathcal{R}}} \circ i_G^{\mathcal{R}}) = (\pi, U)$ follows by applying the General Correspondence Theorem (Theorem 5.2.22).

We prove part (2). Since it is assumed that A has a positive bounded approximate left identity, by Proposition 5.3.11, $(A \rtimes_\alpha G)^{\mathcal{R}}$ has a positive bounded approximate left identity. By Theorem 5.2.18, $\overline{T} : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to B(X_T)$ is a positive representation. By Lemma 5.3.12, the maps $i_A^{\mathcal{R}} : A \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ and $i_G^{\mathcal{R}} : G \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ are both positive, and therefore $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a pair of positive representations of respectively A and G on X. The General Correspondence Theorem asserts that $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is also a non-degenerate $\mathcal{R}$-continuous covariant representation, and that $(\overline{T} \circ i_A^{\mathcal{R}} \rtimes \overline{T} \circ i_G^{\mathcal{R}})^{\mathcal{R}} = T$. □

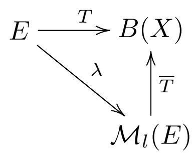
5.4 Uniqueness of the pre-ordered crossed product

Let (A, G, α) be a Banach algebra dynamical system and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces. In [14, Theorem 4.4] it was shown (under mild further hypotheses) that the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique Banach algebra (up to topological isomorphism) such that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ generates all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α), in the sense that, for every non-degenerate bounded representation T of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on a Banach space X, $(\overline{T} \circ i_A^{\mathcal{R}}, \overline{T} \circ i_G^{\mathcal{R}})$ is a non-degenerate $\mathcal{R}$-continuous representation of (A, G, α) on X, and that, moreover, all non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) are obtained in this way.

In this section we will adapt this to pre-ordered Banach algebra dynamical systems. If (A, G, α) is a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones, then we will show that (under similar mild hypotheses as in the unordered case) the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ is the unique pre-ordered Banach algebra (up to bipositive topological isomorphism) such that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ generates all positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) as described above.

We begin with the general framework for generating positive non-degenerate $\mathcal{R}$-continuous representations from a suitable basic one as in [14, Lemma 4.1].

Lemma 5.4.1. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone, and let $\mathcal{R}$ be a uniformly bounded class of continuous covariant representations of (A, G, α) on Banach spaces. Let E be a pre-ordered Banach algebra (with a not necessarily closed cone) and positive bounded approximate left identity, and let (k_A, k_G) be a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space E such that $k_A(A), k_G(G) \subseteq \mathcal{M}_l(E)$. Suppose $T : E \to B(X)$ is a positive non-degenerate bounded representation of E on a pre-ordered Banach space X with a closed cone. Let $\overline{T} : \mathcal{M}_l(E) \to B(X)$ be the positive non-degenerate bounded representation of $\mathcal{M}_l(E)$ such that the following diagram commutes:*



Then the pair $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α), and $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$.

Proof. That $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is a non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) and that $(\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$ follows from [14, Lemma 4.1]. That $(\overline{T} \circ k_A, \overline{T} \circ k_G)$ is positive follows from (k_A, k_G) being positive and $\overline{T} : \mathcal{M}_l(E) \to B(X)$ being positive by Theorem 5.2.18. ☐

Therefore, given a pre-ordered Banach algebra E with such a positive non-degenerate $\mathcal{R}$-continuous covariant representation (k_A, k_G) of (A, G, α) on E, positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) can be generated from positive non-degenerate bounded representations of E.

Clearly, any pre-ordered Banach algebra E' that is bipositively topologically isomorphic to E must also have a similar positive non-degenerate $\mathcal{R}$-continuous covariant generating pair (k'_A, k'_G). This is outlined in the following straightforward adaptation of [14, Lemma 4.2] to the ordered context.

Lemma 5.4.2. *Let (A, G, α), $\mathcal{R}$, E and (k_A, k_G) be as in Lemma 5.4.1. Suppose E' is a pre-ordered Banach algebra and $\psi : E \to E'$ is a bipositive topological isomorphism. Then:*

(1) $\psi_l : \mathcal{M}_l(E) \to \mathcal{M}_l(E')$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l(E)$, is a bipositive topological isomorphism.

(2) The pair $(k'_A, k'_G) := (\psi_l \circ k_A, \psi_l \circ k_G)$ is a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on E' such that $k'_A(A), k'_G(G) \subseteq \mathcal{M}_l(E')$.

(3) If $T : E \to B(X)$ is a positive non-degenerate bounded representation, then so is $T' : E' \to B(X)$, where $T' := T \circ \psi^{-1}$.

(4) *If* $T : E \to B(X)$ *is a positive non-degenerate bounded representation on a pre-ordered Banach space with a closed cone, and* $\overline{T'} : \mathcal{M}_l(E') \to B(X)$ *is the positive non-degenerate bounded representation of* $\mathcal{M}_l(E')$ *such that the diagram*

$$\begin{array}{ccc} E' & \xrightarrow{T'} & B(X) \\ & \searrow^{\lambda} & \uparrow{\scriptstyle \overline{T'}} \\ & & \mathcal{M}_l(E') \end{array}$$

commutes, then $\overline{T} \circ k_A = \overline{T'} \circ k'_A$ *and* $\overline{T} \circ k_G = \overline{T'} \circ k'_G$.

If A has a positive bounded approximate left identity, then, according to Proposition 5.3.11 and Lemma 5.3.12, the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ satisfies the hypotheses of Lemma 5.4.1, and by Theorem 5.3.13 *all* positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) can be "generated" from positive non-degenerate bounded representations of $(A \rtimes_\alpha G)^{\mathcal{R}}$ as in Lemma 5.4.1. By Lemma 5.4.2, a bipositive topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and another pre-ordered Banach algebra yields a triple with the same properties. Our aim in the rest of this section is to establish the converse: If (E, k_A, k_G) (where now E has a closed cone) is a "generating triple" for *all* positive non-degenerate $\mathcal{R}$-continuous covariant representations of (A, G, α) as in Lemma 5.4.1, then, under mild additional hypotheses, this triple can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ via a bipositive topological isomorphism as in Lemma 5.4.2 (cf. Corollary 5.4.8).

In order to do this, we will need the existence of a positive isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on some pre-ordered Banach space with a closed cone. As in [14, Proposition 3.4], this is achieved through combining sufficiently many members of $\mathcal{R}$ into one suitable positive continuous covariant representation.

Definition 5.4.3. Let (A, G, α) be a Banach algebra dynamical system. Let $\mathcal{R}$ be a uniformly bounded class of possibly degenerate continuous covariant representations of (A, G, α) on Banach spaces. We define $[\mathcal{R}]$ to be the collection of all uniformly bounded classes S that are actually sets and satisfy $\sigma^{\mathcal{R}} = \sigma^S$ on $C_c(G, A)$. Elements of some $[\mathcal{R}]$ will be called *uniformly bounded sets of continuous covariant representations.*

We note that $[\mathcal{R}]$ is always non-empty: For every $f \in C_c(G, A)$, considering the set $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\} \subseteq \mathbb{R}$ (subclasses of sets are sets), we may choose a sequence from $\{\|\pi \rtimes U(f)\| : (\pi, U) \in \mathcal{R}\}$ converging to $\sigma^{\mathcal{R}}(f)$, and consider only those corresponding covariant representations from $\mathcal{R}$. In this way we may choose a set S from $\mathcal{R}$ of cardinality at most $|C_c(G, A) \times \mathbb{N}|$ such that $\sigma^S(f) = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$. Therefore the previous definition is non-void.

Definition 5.4.4. Let I be an index set and $\{X_i : i \in I\}$ a family of pre-ordered Banach spaces with closed cones. For $1 \leq p \leq \infty$, we will denote the ℓ^p-direct sum of $\{X_i : i \in I\}$ by $\ell^p\{X_i : i \in I\}$ and endow it with the cone $\ell^p\{(X_i)_+ : i \in I\}$, so that it is a pre-ordered Banach space with a closed cone.

Definition 5.4.5. Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. For $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, suppressing the p-dependence in the notation, we define the positive representations $(\oplus_S \pi) : A \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ and $(\oplus_S U) : G \to B(\ell^p\{X_\pi : (\pi, U) \in S\})$ by $(\oplus_S \pi)(a) := \bigoplus_{(\pi,U)\in S} \pi(a)$ and $(\oplus_S U)_r := \bigoplus_{(\pi,U)\in S} U_r$, for all $a \in A$ and $r \in G$ respectively.

It is easily seen that $((\oplus_S \pi), (\oplus_S U))$ is a positive continuous covariant representation, that

$$((\oplus_S \pi) \rtimes (\oplus_S U))(f) = \bigoplus_{(\pi,U)\in S} \pi \rtimes U(f),$$

and that $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^S(f) = \sigma^{\mathcal{R}}(f)$, for all $f \in C_c(G, A)$.

We hence obtain the following (where the statement concerning non-degeneracy is an elementary verification).

Proposition 5.4.6. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, where A has a closed cone, and $\mathcal{R}$ a uniformly bounded class of positive continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. For any $S \in [\mathcal{R}]$ and $1 \leq p < \infty$, there exists a positive $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space $\ell^p\{X_\pi : (\pi, U) \in S\}$ with a closed cone, denoted $((\oplus_S \pi), (\oplus_S U))$, such that its positive integrated form satisfies $\|((\oplus_S \pi) \rtimes (\oplus_S U))(f)\| = \sigma^{\mathcal{R}}(f)$ for all $f \in C_c(G, A)$ and hence induces a positive isometric representation of $(A \rtimes_\alpha G)^{\mathcal{R}}$ on $\ell^p\{X_\pi : (\pi, U) \in S\}$.*

If every element of S is non-degenerate, then $((\oplus_S \pi), (\oplus_S U))$ is non-degenerate.

In the following theorem we will give sufficient conditions under which a triple (E, k_A, k_G), generating all positive non-degenerate $\mathcal{R}$-continuous covariant representations of a pre-ordered Banach algebra dynamical system (A, G, α) as in Lemma 5.4.1, can be obtained from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ through a bipositive topological isomorphism as in Lemma 5.4.2.

Theorem 5.4.7. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system with A having a closed cone and a positive bounded approximate left identity. Let $\mathcal{R}$ be a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Let E be a pre-ordered Banach algebra, with closed cone and positive bounded approximate left identity and such that $\lambda : E \to \lambda(E) \subseteq \mathcal{M}_l(E)$ is a bipositive topological embedding. Let (k_A, k_G) be a positive non-degenerate $\mathcal{R}$-continuous covariant representation of (A, G, α) on the pre-ordered Banach space E such that:*

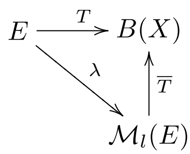
(1) $k_A(A), k_G(G) \subseteq \mathcal{M}_l(E)$,

(2) $(k_A \rtimes k_G)(C_c(G, A)) \subseteq \lambda(E)$,

(3) $(k_A \rtimes k_G)(C_c(G, A))$ *is dense in* $\lambda(E)$,

(4) $(k_A \rtimes k_G)(C_c(G, A_+))$ *is dense in* $\lambda(E) \cap \mathcal{M}_l(E)_+$.

Suppose that, for every positive non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a pre-ordered Banach space X with a closed cone, there exists a positive non-degenerate bounded representation $T : E \to B(X)$ such that the positive non-degenerate bounded representation $\overline{T} : \mathcal{M}_l(E) \to B(X)$ in the commuting diagram



generates (π, U) as in Lemma 5.4.1, i.e., is such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$.

Then there exists a unique topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$ such that the induced topological isomorphism $\psi_l : \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}}) \to \mathcal{M}_l(E)$, defined by $\psi_l(L) := \psi L \psi^{-1}$ for $L \in \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$, induces (k_A, k_G) from $(i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ as in Lemma 5.4.2, i.e., is such that $k_A = \psi_l \circ i_A^{\mathcal{R}}$ and $k_G = \psi_l \circ i_G^{\mathcal{R}}$.

Moreover, ψ is bipositive.

The proof follows largely as in [14, Proposition 4.3], but with some modifications in the first part of the proof, which we now give.

Proof. By hypothesis $\mathcal{R}$ consists of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Hence Proposition 5.4.6 provides a positive non-degenerate $\mathcal{R}$-continuous covariant representation (π, U) of (A, G, α) on a pre-ordered Banach space X with a closed cone such that $(\pi \rtimes U)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(X)$ is a positive non-degenerate isometric representation. By hypothesis, there exists a positive non-degenerate representation $T : E \to B(X)$ such that $\overline{T} \circ k_A = \pi$ and $\overline{T} \circ k_G = U$. By Lemma 5.4.1, we obtain $\pi \rtimes U = (\overline{T} \circ k_A) \rtimes (\overline{T} \circ k_G) = \overline{T} \circ (k_A \rtimes k_G)$. Then, for any $f \in C_c(G, A)$,

$$\begin{aligned}
\|q^{\mathcal{R}}(f)\| &= \|(\pi \rtimes U)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \\
&= \|\pi \rtimes U(f)\| \\
&= \|\overline{T} \circ (k_A \rtimes k_G)(f)\| \\
&\leq \|\overline{T}\| \|k_A \rtimes k_G(f)\| \\
&= \|\overline{T}\| \|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\|.
\end{aligned}$$

Since (k_A, k_G) was assumed to be $\mathcal{R}$-continuous, we obtain $\|(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f))\| \leq \|(k_A \rtimes k_G)^{\mathcal{R}}\| \|q^{\mathcal{R}}(f)\|$. Using (2), (3) and the fact that $\lambda(E)$ is closed, it now follows that $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is a topological isomorphism.

Since (k_A, k_G) is positive and $\mathcal{R}$-continuous, and the cone of E is closed, by Lemmas 5.3.4 and 5.2.2, $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is positive. We claim that $(k_A \rtimes k_G)^{\mathcal{R}}$ is bipositive. Let $b \in E$ be such that $\lambda(b) \in \lambda(E) \cap \mathcal{M}_l(E)_+$, hence by (4) there exists a sequence $(f_n) \subseteq C_c(G, A_+)$ such that $(k_A \rtimes k_G)^{\mathcal{R}}(q^{\mathcal{R}}(f_n)) = (k_A \rtimes k_G)(f_n) \to \lambda(b)$. Since $(k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to \lambda(E)$ is a topological isomorphism, the sequence $(q^{\mathcal{R}}(f_n)) \subseteq (A \rtimes_\alpha G)^{\mathcal{R}}_+$ converges to some $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$

and $(k_A \rtimes k_G)^{\mathcal{R}}(d) = \lambda(b)$. Moreover, since $(A \rtimes_\alpha G)_+^{\mathcal{R}}$ is closed and $(q^{\mathcal{R}}(f_n)) \subseteq (A \rtimes_\alpha G)_+^{\mathcal{R}}$, we have $d \in (A \rtimes_\alpha G)_+^{\mathcal{R}}$. We conclude that $(k_A \rtimes k_G)^{\mathcal{R}}$ is bipositive.

Since $\lambda : E \to \mathcal{M}_l(E)$ is assumed to be a bipositive topological embedding,

$$\psi := \lambda^{-1} \circ (k_A \rtimes k_G)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$$

is a bipositive topological isomorphism.

The remainder of the argument proceeds as in the proof of [14, Theorem 4.4]. □

Under the conditions on (A, G, α) and $\mathcal{R}$ as stated in the previous theorem, one would of course hope that the triple $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ automatically satisfies the hypotheses on (E, k_A, k_G), as happens in the unordered context [14, Theorem 4.4]. Here, as there, the left regular representation $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a topological embedding [14, Proposition 4.3], and, since $q^{\mathcal{R}}(C_c(G, A))$ is dense in $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$ [19, Theorem 7.2], we have that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}})$. Hence (1), (2) and (3) in Theorem 5.4.7 are satisfied by $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$. We claim that the additional assumption that A has a positive bounded approximate right identity gives the remaining conditions that $\lambda : (A \rtimes_\alpha G)^{\mathcal{R}} \to \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})$ is a bipositive topological embedding, and that (4) holds. Indeed, if this is the case, let $(u_i) \subseteq (A \rtimes_\alpha G)_+^{\mathcal{R}}$ be a positive approximate right identity of $(A \rtimes_\alpha G)^{\mathcal{R}}$ (which exists by Theorem 5.3.11), and let $d \in (A \rtimes_\alpha G)^{\mathcal{R}}$ be such that $\lambda(d) \geq 0$. Then $0 \leq \lambda(d)u_i = d * u_i \to d$, so that $d \in (A \rtimes_\alpha G)_+^{\mathcal{R}}$. We conclude that $\lambda^{-1} : \lambda((A \rtimes_\alpha G)^{\mathcal{R}}) \to (A \rtimes_\alpha G)^{\mathcal{R}}$ is also positive. Since $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})^{\mathcal{R}} = \lambda$, this also gives that $(i_A^{\mathcal{R}} \rtimes i_G^{\mathcal{R}})(C_c(G, A_+))$ is dense in $\lambda((A \rtimes_\alpha G)^{\mathcal{R}}) \cap \mathcal{M}_l((A \rtimes_\alpha G)^{\mathcal{R}})_+$. Hence we have the following uniqueness result:

Corollary 5.4.8. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and both a positive bounded approximate left identity and a positive bounded approximate right identity. Let $\mathcal{R}$ be a uniformly bounded class of positive non-degenerate continuous covariant representations of (A, G, α) on pre-ordered Banach spaces with closed cones. Then $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ satisfies all hypotheses on the triple (E, k_A, k_G) in Theorem 5.4.7. Hence triples (E, k_A, k_G) as in Theorem 5.4.7 exist, and every such "generating triple" for all positive non-degenerate $\mathcal{R}$-continuous representations of (A, G, α) originates from $((A \rtimes_\alpha G)^{\mathcal{R}}, i_A^{\mathcal{R}}, i_G^{\mathcal{R}})$ through a bipositive topological isomorphism $\psi : (A \rtimes_\alpha G)^{\mathcal{R}} \to E$ as in Theorem 5.4.7 (so that E necessarily has a positive bounded approximate right identity as well).*

5.5 Pre-ordered generalized Beurling algebras

In [14, Section 5] it was shown that a generalized Beurling algebra (to be defined below) is topologically isomorphic to a crossed product associated with a Banach algebra dynamical system, and the non-degenerate bounded representations of these algebras were described in terms of non-degenerate continuous covariant representations of the underlying Banach algebra dynamical system. We refer the reader to

[14, Section 5] for a more complete treatment of generalized Beurling algebras and how they are constructed from Banach algebra dynamical systems.

In this section we will adapt the main results from [14, Section 5] to the case of pre-ordered Banach algebra dynamical systems and pre-ordered generalized Beurling algebras. Theorem 5.5.7 is the analogue of [14, Theorem 5.17] in the ordered context, and shows that a pre-ordered generalized Beurling algebra is bipositively topologically isomorphic to a crossed product associated with a pre-ordered Banach algebra dynamical system. In Theorem 5.5.9 we modify [14, Theorem 5.20] to explicitly describe a bijection between the positive non-degenerate continuous covariant representations of a pre-ordered Banach algebra dynamical system, where the group representation is bounded by a multiple of a fixed weight on the underlying group, and the positive non-degenerate bounded representations of the associated pre-ordered generalized Beurling algebra.

We begin with a brief description of pre-ordered generalized Beurling algebras and related spaces.

Definition 5.5.1. For a locally compact group G, let $\omega : G \to [0,\infty)$ be a non-zero submultiplicative Borel measurable function. Then ω is called a *weight* on G.

Note that we do not assume that $\omega \geq 1$, as is done in some parts of the literature. The fact that ω is non-zero readily implies that $\omega(e) \geq 1$. More generally, if $K \subseteq G$ is a compact set, there exist $a, b > 0$ such that $a \leq \omega(s) \leq b$ for all $s \in K$ [26, Lemma 1.3.3].

Definition 5.5.2. Let X be a pre-ordered Banach space with a closed cone, and $\omega : G \to [0,\infty)$ a weight on G. We define the weighted 1-norm on $C_c(G, X)$ by

$$\|h\|_{1,\omega} := \int_G \|h(s)\|\omega(s)\, ds \quad (h \in C_c(G, X)),$$

and define the pre-ordered Banach space $L^1(G, X, \omega)$ as the completion (in the sense of Definition 5.2.1) of the pre-ordered vector space $(C_c(G, X), C_c(G, X_+))$ with the $\|\cdot\|_{1,\omega}$-norm.

Given the prominent role of continuous compactly supported functions in the theory, the definition of $L^1(G, X, \omega)$ as the completion of the space $C_c(G, X)$ is clearly convenient. A drawback, however, is that it is then not clear that $L^1(G, X, \omega)_+$, which is, by definition, the closure of $C_c(G, X_+)$, is generating in $L^1(G, X, \omega)$ if X_+ is generating in X. From Corollary 5.2.8 we know that $C_c(G, X_+)$ is generating in $C_c(G, X)$, and then Lemma 5.2.4 yields that $L^1(G, X, \omega)_+$ is topologically generating in $L^1(G, X, \omega)$, but generalities do not seem to help us beyond this point. Similarly, it is not clear that $L^1(G, X, \omega)_+$ is a proper cone if X_+ is proper. To establish these results, we use the fact that, as already observed in [14, Remark 5.3], $L^1(G, X, \omega)$ is isometrically isomorphic to a Bochner space (also if the left Haar measure μ is not σ-finite, or X is not separable). We recall the relevant facts. A function $f : G \to X$ is Bochner integrable (with respect to $\omega d\mu$) if $f^{-1}(B)$ is a Borel subset of G for every Borel subset B of X, $f(G)$ is separable, and $\int_G \|f(s)\|\omega(s)\, d\mu(s) < \infty$ (the measurability of $s \mapsto \|f(s)\|$ is an automatic consequence of the Borel measurability of f). On identifying Bochner integrable functions that are equal $\omega d\mu$-almost everywhere, one obtains a Banach space $L^1(G,\mathcal{B},\omega d\mu,X)$, where $\mathcal{B}$ is the Borel σ-algebra of G, and the norm is given by $\|[f]\| = \int_G \|f(s)\|\omega(s)\,d\mu(s)$, with f any representative of $[f]\in L^1(G,\mathcal{B},\omega d\mu,X)$. Clearly the inclusion map of $(C_c(G,X),\|\cdot\|_{1,\omega})$ into $L^1(G,\mathcal{B},\omega d\mu,X)$ is isometric, and the existence of the aforementioned isometric isomorphism between $L^1(G,X,\omega)$ and $L^1(G,\mathcal{B},\omega d\mu,X)$ is then established by showing that $C_c(G,X)$ is dense in $L^1(G,\mathcal{B},\omega d\mu,X)$. In the present context, if X is a pre-ordered Banach space, then $L^1(G,\mathcal{B},\omega d\mu,X)$ has a natural cone

$$L^1(G,\mathcal{B},\omega d\mu,X_+) := \{f\in L^1(G,\mathcal{B},\omega d\mu,X) : f(s)\in X_+ \text{ for } \omega d\mu\text{-a.a. } s\in G\},$$

where, as usual we have ignored the distinction between equivalence classes and functions. As in the scalar case, a convergent sequence in $L^1(G,\mathcal{B},\omega d\mu,X)$ has a subsequence that converges $\omega d\mu$-almost everywhere to the limit function. Hence if X_+ is closed, then so is $L^1(G,\mathcal{B},\omega d\mu,X_+)$. We then have the following natural result.

Proposition 5.5.3. *Let X be a pre-ordered Banach space with a closed cone. Let G be a locally compact group and ω a weight on G. Then:*

(1) The cone $C_c(G,X_+)$ is dense in the closed cone $L^1(G,\mathcal{B},\omega d\mu,X_+)$.

(2) The pre-ordered Banach spaces

$$(L^1(G,X,\omega),L^1(G,X,\omega)_+) \quad \textit{and} \quad (L^1(G,\mathcal{B},\omega d\mu,X),L^1(G,\mathcal{B},\omega d\mu,X_+))$$

have closed cones and are bipositively isometrically isomorphic through an isomorphism that is the identity on $C_c(G,X)$.

Proof. For the first part we need, in view of the remarks preceding the proposition, only show that $C_c(G,X_+)$ is dense in $L^1(G,\mathcal{B},\omega d\mu,X_+)$. If $f\in L^1(G,\mathcal{B},\omega d\mu,X)$, then the proof of [10, Proposition E.2] shows that there exists a subset S of $\mathbb{Q}f(G)$ and a sequence of of simple functions (f_n), with values in S, such that $f_n(s)\to f(s)$ and $\|f_n(s)\|\leq\|f(s)\|$ for $\omega d\mu$-almost every $s\in G$. Hence by the dominated convergence theorem (see the argument on [10, p. 352]) $f_n\to f$. An inspection of the proof of [10, Proposition E.2] shows that, in fact, S can be chosen to be a subset of $\mathbb{Q}_{\geq 0}f(G)$. It is then clear that the (equivalence classes of) X_+-valued simple functions are dense in $L^1(G,\mathcal{B},\omega d\mu,X_+)$. Therefore, it is sufficient to show that the functions of the form $\chi_B\otimes x\in L^1(G,\mathcal{B},\omega d\mu,X_+)$, where $B\in\mathcal{B}$ and $x\in X_+$, can be approximated arbitrarily closely by elements of $C_c(G,X_+)$. As to this, since $C_c(G)$ is dense in the Beurling algebra $L^1(G,\omega)$ [26, Lemma 1.3.5], there exists a sequence $(g_n)\subseteq C_c(G)$ such that $g_n\to\chi_B$ in $L^1(G,\omega)$. Since $\chi_B\geq 0$ we clearly have $\|g_n^+\otimes x-\chi_B\otimes x\|_{1,\omega}\leq\|g_n-\chi_B\|_{1,\omega}\|x\|\to 0$. Hence $g_n^+\otimes x\to\chi_B\otimes x$, and the proof is complete.

The second part is immediate from the first. ∎

We can now settle the matters mentioned above.

Theorem 5.5.4. *Let X be a pre-ordered Banach space with a closed cone. Let G be a locally compact group and ω a weight on G.*

(1) If X_+ is generating in X, then the closed cone $L^1(G,X,\omega)_+$ is generating in $L^1(G,X,\omega)$.

(2) If X_+ is a proper cone, then the closed cone $L^1(G,X,\omega)_+$ is proper.

Proof. In view of Proposition 5.5.3, it is equivalent to prove the statements for the closed cone $L^1(G,\mathcal{B},\omega d\mu,X_+)$ of $L^1(G,\mathcal{B},\omega d\mu,X)$. Part (2) is then immediate. As to part (1), by Theorem 5.2.7, if X_+ is generating in X there exist continuous positively homogeneous functions $(\cdot)^\pm : X \to X_+$ and a constant $\alpha > 0$ such that $x = x^+ - x^-$ and $\|x^\pm\| \leq \alpha\|x\|$ for all $x \in X$. If $f \in L^1(G,\mathcal{B},\omega d\mu,X)$, we define $f^\pm(s) := (f(s))^\pm$ for all $s \in G$. Since the functions $(\cdot)^\pm : X \to X_+$ are continuous, the measurability of f implies the measurability of $f^\pm$, and the separability of $f(G)$ implies the separability of $f^\pm(G)$. The inequalities $\|x^\pm\| \leq \alpha\|x\|$ $(x \in X)$ imply $\|f^\pm\|_{1,\omega} \leq \alpha\|f\|_{1,\omega} < \infty$. We conclude that $f^\pm \in L^1(G,\mathcal{B},\omega d\mu,X_+)$. Since $f = f^+ - f^-$, the cone $L^1(G,\mathcal{B},\omega d\mu,X_+)$ is generating in $L^1(G,\mathcal{B},\omega d\mu,X)$. □

Thus, in particular, if (A,G,α) is a pre-ordered Banach algebra dynamical system with A having a closed cone, then $L^1(G,A,\omega)_+$ is generating (proper) in $L^1(G,A,\omega)$ if A_+ is generating (proper) in A.

We now turn to the definition of the multiplicative structure on $L^1(G,A,\omega)$ if α is uniformly bounded. Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and ω a weight on G. If α is uniformly bounded, say $\|\alpha_s\| \leq C_\alpha$ for some $C_\alpha \geq 0$ and all $s \in G$, then, using the submultiplicativity of ω, it is routine to verify that

$$\|f * g\|_{1,\omega} \leq C_\alpha \|f\|_{1,\omega}\|g\|_{1,\omega} \quad (f,g \in C_c(G,A)).$$

Since $C_c(G,A)$ is a pre-ordered algebra by Lemma 5.3.2, it is now clear that the pre-ordered Banach space $L^1(G,A,\omega)$ can be supplied with the structure of a pre-ordered algebra with continuous multiplication. If $C_\alpha = 1$ (i.e., if α lets G act as bipositive isometries on A), then $L^1(G,A,\omega)$ is a pre-ordered Banach algebra. When $C_\alpha \neq 1$, as is well known, there is an equivalent norm on $L^1(G,A,\omega)$ such that it becomes a Banach algebra, which is a pre-ordered Banach algebra when endowed with the same cone $L^1(G,A,\omega)_+$. In [14, Theorem 5.17] it was shown that such a Banach algebra norm can be obtained from a topological isomorphism between $L^1(G,A,\omega)$ and the crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ for a suitable choice of $\mathcal{R}$. In Theorem 5.5.7 below, we show that in the ordered context this topological isomorphism is bipositive.

Definition 5.5.5. Let (A,G,α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and α uniformly bounded. Let ω be a weight on G. The pre-ordered Banach space $L^1(G,A,\omega)$ endowed with the continuous multiplication induced by the twisted convolution on $C_c(G,A)$, given by

$$[f * g](s) := \int_G f(r)\alpha_r(g(r^{-1}s))\,dr \quad (f,g \in C_c(G,A),\ s \in G),$$

will be denoted by $L^1(G, A, \omega; \alpha)$ and called a *pre-ordered generalized Beurling algebra.*

We note that if $A = \mathbb{R}$, the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ reduces to a classical Beurling algebra, which is a true Banach algebra.

Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone. The following definition shows how to induce a continuous covariant representation of (A, G, α) from a positive bounded representation of A. Applying this construction to the left regular representation of A, and choosing (for instance) $\mathcal{R}$ to be the singleton containing this continuous covariant representation, yields the desired topological isomorphism (cf. [14, Theorem 5.13]). We keep track of possible order structures in order to show later that this topological isomorphism is bipositive.

Definition 5.5.6. Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, and let $\pi : A \to B(X)$ be a positive bounded representation of A on a pre-ordered Banach space X with a closed cone. We define the induced algebra representation $\tilde{\pi}$ and left regular group representation Λ on the space X^G of all functions from G to X by the formulae:

$$\begin{aligned} [\tilde{\pi}(a)h](s) &:= \pi(\alpha_s^{-1}(a))h(s), \\ (\Lambda_r h)(s) &:= h(r^{-1}s), \end{aligned}$$

where $h : G \to X$, $r, s \in G$ and $a \in A$.

It is easy to see that $(\tilde{\pi}, \Lambda)$ is covariant, and positive if X^G is endowed with the cone X_+^G. If α is uniformly bounded, then $(\tilde{\pi}, \Lambda)$ yields a continuous covariant representation of A on $L^1(G, X, \omega)$ such that $\|\Lambda_r\| \leq \omega(r)$ for all $r \in G$, and if π is non-degenerate, so is $(\tilde{\pi}, \Lambda)$ [14, Corollary 5.9]. Hence, if A has a bounded approximate left or right identity, then, with $\lambda : A \to B(A)$ denoting the left regular representation of A, $(\tilde{\lambda}, \Lambda)$ is a positive non-degenerate continuous covariant representation of (A, G, α) on $L^1(G, A, \omega)$.

Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a (not necessarily positive) bounded approximate right identity. Let ω be a weight on G and $\mathcal{R}$ a uniformly bounded class of non-degenerate continuous covariant representations of (A, G, α) on Banach spaces, such that $\sup_{(\pi,U)\in\mathcal{R}} \|U_r\| \leq \omega(r)$ for all $r \in G$. If $(\tilde{\lambda}, \Lambda)$ is $\mathcal{R}$-continuous, for instance if $\mathcal{R} = \{(\tilde{\lambda}, \Lambda)\}$, then the integrated form $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ is faithful, and hence the seminorm $\sigma^{\mathcal{R}}$ is actually a norm on $C_c(G, A)$ and is equivalent to $\|\cdot\|_{1,\omega}$ [14, Theorem 5.13]. Furthermore, $\tilde{\lambda} \rtimes \Lambda$ extends to a topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ [14, Theorem 5.13]. Since the norms $\sigma^{\mathcal{R}}$ and $\|\cdot\|_{1,\omega}$ are equivalent, the topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ which is the identity on the mutual dense subspace $C_c(G, A)$ is bipositive by construction, as the cones of both spaces are the closure of $C_c(G, A)$. Since the non-degenerate $\mathcal{R}$-continuous covariant representation $(\tilde{\lambda}, \Lambda)$ is positive, so is the topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ by Lemmas 5.3.4 and 5.2.2.

Assuming that, in fact, A has a positive bounded approximate right identity, we claim that the positive topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$

is bipositive. Identifying $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $L^1(G, A, \omega; \alpha)$ through the above bipositive topological isomorphism, the topological embedding $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}$ is conjugate to the left regular representation $\lambda : L^1(G, A, \omega; \alpha) \to B(L^1(G, A, \omega; \alpha))$ through the bipositive map $\hat{\ } : L^1(G, A, \omega; \alpha) \to L^1(G, A, \omega; \alpha)$, determined by $\hat{h}(s) := \alpha_s(h(s))$ for $h \in C_c(G, A)$ and $s \in G$ [14, Remark 5.16]. We denote the inverse of $\hat{\ }$ by $\check{\ }$. With $(u_i) \subseteq L^1(G, A, \omega; \alpha)_+$ a positive approximate right identity (which exists by Proposition 5.3.11 and the fact that $(A \rtimes_\alpha G)^{\mathcal{R}}$ is bipositively topologically isomorphic to $L^1(G, A, \omega; \alpha)$, as described above), if $f \in L^1(G, A, \omega; \alpha)$ is such that $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}(f) \geq 0$, then

$$0 \leq \left((\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}}(f)\check{u_i}\right)^{\wedge} = \lambda(f)u_i = f * u_i \to f.$$

Since $L^1(G, A, \omega; \alpha)_+$ is closed by construction, we obtain $f \in L^1(G, A, \omega; \alpha)_+$, and therefore the claim that $(\tilde{\lambda} \rtimes \Lambda)^{\mathcal{R}} : (A \rtimes_\alpha G)^{\mathcal{R}} \to B(L^1(G, A, \omega))$ is a bipositive topological embedding follows.

We hence obtain the following ordered version of [14, Theorem 5.17]:

Theorem 5.5.7. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a (not necessarily positive) bounded approximate right identity. Let α be uniformly bounded and ω be a weight on G. Let the positive non-degenerate continuous covariant representation $(\tilde{\lambda}, \Lambda)$ of (A, G, α) on $L^1(G, A, \omega)$ be as yielded by Definition 5.5.6. Then the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ and the pre-ordered crossed product $(A \rtimes_\alpha G)^{\mathcal{R}}$ with $\mathcal{R} := \{(\tilde{\lambda}, \Lambda)\}$ are bipositively topologically isomorphic via an isomorphism that is the identity on $C_c(G, A)$.*

Furthermore, the map $\tilde{\lambda} \rtimes \Lambda : C_c(G, A) \to B(L^1(G, A, \omega))$ extends to a positive topological embedding of $(A \rtimes_\alpha G)^{\mathcal{R}}$ into $B(L^1(G, A, \omega))$, and this extension is bipositive if A has a positive bounded approximate right identity.

If A has a 1-bounded right approximate identity, α lets G act as isometries on A and $\inf_{W \in \mathcal{Z}} \sup_{r \in W} \omega(r) = 1$, with $\mathcal{Z}$ denoting a neighbourhood base of $e \in G$ of which all elements are contained in a fixed compact set, then the bipositive topological isomorphism between $(A \rtimes_\alpha G)^{\mathcal{R}}$ and $L^1(G, A, \omega; \alpha)$ and the above positive embedding of $(A \rtimes_\alpha G)^{\mathcal{R}}$ into $B(L^1(G, A, \omega))$ are both isometric.

The following result gives some properties of the cones of pre-ordered generalized Beurling algebras. Here application of Theorem 5.5.4 yields stronger conclusions on the structure of the cone $L^1(G, A, \omega; \alpha)_+$ than can be concluded from the more generally applicable Theorem 5.3.8.

Theorem 5.5.8. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone and a (not necessarily positive) bounded approximate right identity. Let α be uniformly bounded and ω be a weight on G.*

If the cone A_+ is generating (proper) in A, then the cone $L^1(G, A, \omega; \alpha)_+$ is generating (proper) in $L^1(G, A, \omega; \alpha)$.

Furthermore, if A is a Banach lattice algebra, then the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, viewed as pre-ordered Banach space, is a Banach *lattice. If, in addition, α lets G act as bipositive isometries on A, the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ is a Banach lattice algebra.*

Proof. The conclusions on $L^1(G, A, \omega; \alpha)_+$ being generating or proper follow immediately from Theorem 5.5.4.

If A is a Banach lattice algebra, then $(C_c(G, A), C_c(G, A_+))$ with the norm $\|\cdot\|_{1,\omega}$ is a normed vector lattice. Therefore, by [41, Corollary 2, p. 84], $L^1(G, A, \omega; \alpha)$ is a Banach lattice. If α lets G act as bipositive isometries on A, then $L^1(G, A, \omega; \alpha)$ is also a pre-ordered Banach algebra as a consequence of Lemma 5.2.3 and the discussion preceding Definition 5.5.5. Therefore $L^1(G, A, \omega; \alpha)$ is a Banach lattice algebra. □

Through an application of Theorem 5.3.13, we can now adapt [14, Theorem 5.20] to the ordered context, and give an explicit description of the positive non-degenerate bounded representations of pre-ordered generalized Beurling algebras $L^1(G, A, \omega; \alpha)$ on pre-ordered Banach spaces with closed cones in terms of the positive non-degenerate continuous covariant representations of (A, G, α) on such spaces, where the group representation is bounded by a multiple of ω. The result is as follows:

Theorem 5.5.9. *Let (A, G, α) be a pre-ordered Banach algebra dynamical system, with A having a closed cone, a (not necessarily positive) bounded approximate right identity and a positive bounded approximate left identity. Let α be uniformly bounded and ω a weight on G. Then the following maps are mutual inverses between the positive non-degenerate continuous covariant representations (π, U) of (A, G, α) on a pre-ordered Banach space X with closed cone, satisfying $\|U_r\| \leq C_U\omega(r)$ for some $C_U \geq 0$ and all $r \in G$, and the positive non-degenerate bounded representations $T : L^1(G, A, \omega; \alpha) \to B(X)$ of the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$ on X:*

$$(\pi, U) \mapsto \left(f \mapsto \int_G \pi(f(r))U_r \, dr \right) =: T^{(\pi,U)} \quad (f \in C_c(G, A)),$$

determining a positive non-degenerate bounded representation $T^{(\pi,U)}$ of the pre-ordered generalized Beurling algebra $L^1(G, A, \omega; \alpha)$, and,

$$T \mapsto \left(\begin{array}{l} a \mapsto \text{SOT-lim}_{(V,i)} T(z_V \otimes au_i), \\ s \mapsto \text{SOT-lim}_{(V,i)} T(z_V(s^{-1}\cdot) \otimes u_i) \end{array} \right) =: (\pi^T, U^T),$$

where $\mathcal{Z}$ is a neighbourhood base of $e \in G$, of which all elements are contained in a fixed compact subset of G, $z_V \in C_c(G, A)$ is chosen such that $z_V \geq 0$ is supported in $V \in \mathcal{Z}$, $\int_G z_V(r)dr = 1$, and (u_i) is any positive bounded approximate left identity of A.

Furthermore, if A has an M-bounded (not necessarily positive) approximate left identity, then the following bounds for $T^{(\pi,U)}$ and (π^T, U^T) hold:

(1) $\|T^{(\pi,U)}\| \leq C_U\|\pi\|$,

(2) $\|\pi^T\| \leq (\inf_{V\in\mathcal{Z}} \sup_{r\in V} \omega(r)) \|T\|$,

(3) $\|U_s^T\| \leq M\left(\inf_{V\in\mathcal{Z}}\sup_{r\in V}\omega(r)\right)\|T\|\omega(s) \quad (s\in G)$.

In the case where $(A, G, \alpha) = (\mathbb{R}, G, \mathrm{triv})$ with a weight ω on G, by Theorem 5.5.8 we obtain the (here rather obvious fact) fact that the classical Beurling algebra $L^1(G,\omega)$ is a Banach lattice algebra. Furthermore, Theorem 5.5.9 gives a bijection between the positive strongly continuous group representations of G on pre-ordered Banach spaces with closed cones that are bounded by a multiple of ω, and the positive non-degenerate bounded representations of $L^1(G,\omega)$ on such spaces. We hence obtain the following adaptation of [14, Corollary 5.22] to the ordered context:

Corollary 5.5.10. *Let ω be a weight on G. Let (z_V) be as in Theorem 5.5.9. The maps*

$$U \mapsto \left(f \mapsto \int_G f(r)U_r\,dr\right) =: T^U \quad (f\in C_c(G)),$$

determining a positive non-degenerate bounded representation T^U of the ordered Beurling algebra $L^1(G,\omega)$, and

$$T \mapsto \left(s \mapsto \text{SOT-lim}_V T(z_V(s^{-1}\cdot))\right) =: U^T$$

are mutual inverses between the positive strongly continuous group representations U of G on a pre-ordered Banach space X with closed cone, satisfying $\|U_r\| \leq C_U\omega(r)$, for some $C_U \geq 0$ and all $r\in G$, and the positive non-degenerate bounded representations $T : L^1(G,\omega) \to B(X)$ of the ordered Beurling algebra $L^1(G,\omega)$ on X.

If the weight satisfies $\inf_{W\in\mathcal{Z}}\sup_{r\in W}\omega(r) = 1$, where $\mathcal{Z}$ is a neighbourhood base of $e\in G$, of which all elements are contained in a fixed compact subset of G, then $\|T^U\| = \sup_{r\in G}\|U_r\|/\omega(r)$ and $\|U_r^T\| \leq \|T\|\omega(r)$ for all $r\in G$.

As a particular case, the uniformly bounded positive strongly continuous representations of G on a pre-ordered Banach space X with a closed cone are in natural bijection with the positive non-degenerate bounded representations of $L^1(G)$ on X; this also follows from [24, Assertion VI.1.32].

Finally, we note that [14, Theorem 8.3] gives a bijection between the non-degenerate bounded anti-representations of $L^1(G, A, \omega; \alpha)$ on Banach spaces, for a Banach algebra dynamical system (A, G, α) where A has a bounded two-sided approximate identity and α is uniformly bounded, and suitable (not covariant!) pairs (π, U) of anti-representations of A and G. As done above for [14, Theorem 5.20], an ordered version can be derived from this, but this is left to the reader for reasons of space.

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Samenvatting

Veel dynamische systemen in de natuur voldoen aan een 'positiviteits-eis'. In de bevolkingsdynamica bijvoorbeeld hebben negatieve bevolkingen geen betekenis, en net zo zijn negatieve waarden van het concentratieprofiel van materiaal dat diffundeert in een vloeistof niet zinvol. De beschrijving van zulke systemen in wiskundige taal levert vaak vectorruimten op met een (pre)ordening die gedefinieerd wordt door een kegel van positieve elementen. De dynamica van een dergelijk systeem wordt dan gegeven door een groeps– of semigroepsactie op deze vectorruimte die de kegel invariant laat. Vaak is er bij een natuurlijk systeem een symmetriegroep van de onderliggende ruimte (denk aan rotaties van het aardoppervlak) die op een voor de hand liggende manier op een dergelijke vectorruimte werkt, en dan eveneens de kegel invariant laat. Positieve groepsacties komen vaak voor.

Met dit in gedachten, en ook gemotiveerd vanuit de theorie van unitaire representaties van groepen en representaties van C^*-algebra's op Hilbertruimten, is dit proefschrift is een bijdrage aan de theorie van positieve groeps– en algebrarepresentaties op geordende Banachruimten, en meer algemeen van positieve representaties op Banachruimten met een preordening.

Gemotiveerd door onder andere de quantummechanica is er sinds de eerste decennia van de twintigste eeuw veel onderzoek gedaan naar unitaire representaties van lokaal compacte groepen op Hilbertruimten. Voortbouwend op werk van onder andere Peter, Weyl en von Neumann is er onder meer een decompositietheorie voor unitaire representaties van dergelijke groepen op Hilbertruimten ontwikkeld, waarbij aangetoond wordt dat dergelijke representaties ontbonden kunnen worden als een directe som of, meer algemeen, een directe integraal van irreducibele unitaire representaties. Dit heeft tot gevolg dat het bestuderen van zulke unitaire representaties voor een deel gereduceerd kan worden tot het bestuderen van de irreducibele unitaire representaties.

Van belang in deze theorie is de zogeheten groeps C^*-algebra $C^*(G)$ van een lokaal compacte groep G. Deze C^*-algebra heeft de eigenschap dat er een natuurlijke bijectie bestaat tussen de niet ontaarde *-representaties van $C^*(G)$ op Hilbertruimten enerzijds, en de unitaire representaties van G op Hilbertruimten anderzijds. Op die manier kunnen vragen over unitaire representaties van een groep vertaald worden naar vragen over *-representaties van een C^*-algebra. Omdat een C^*-algebra een functionaalanalytische structuur heeft en een groep niet, is dit een essentiële stap om een algemene decompositiestelling voor unitaire representaties van lokaal compacte groepen te kunnen bewijzen.

Met dit en de natuurlijke rol van niet-Hilbertrepresentaties van groepen en algebra's in gedachten, hebben Dirksen, de Jeu en Wortel een begin gemaakt met de theorie van gekruiste producten van Banachalgebra's [19]. De groeps C^*-algebra $C^*(G)$, gebruikt voor ontbindingsstellingen, is een eenvoudig geval van een gekruist product van een C^*-algebra, maar de theorie van algemene gekruiste producten blijkt daarenboven een bevredigend conceptueel kader te bieden voor inductie van unitaire groepsrepresentaties. Omdat de verwachting en hoop is dat dit laatste ook voor niet-Hilbertrepresentaties van groepen het geval zal blijken te zijn, is de theorie van gekruiste producten buiten de context van C^*-algebra's eveneens in zijn algemeenheid opgepakt. In dit proefschrift ontwikkelen we deze theorie van algemene gekruiste producten verder en plaatsen deze ook in de context van positieve representaties van groepen en geordende Banachalgebra's op geordende Banachruimten, en meer algemeen van positieve representaties op Banachruimten met een preordening. Tijdens het onderzoek trad een aantal fundamentele vragen over Banachruimten met een preordening naar voren, die op zichzelf al interessant zijn. Hieronder beschrijven we daarom ook deze fundamentele vragen in samenhang met de theorie van gekruiste producten van Banachalgebra's.

In Hoofdstuk 2 bekijken we de volgende vraag. Laat Ω een compacte Hausdorff ruimte zijn en X een Banachruimte met een preordening, zodanig dat de positieve kegel X_+ een gesloten voortbrengende kegel in X is. Is dan de kegel $C(\Omega, X_+)$ van alle continue X_+-waardige functies voortbrengend in de ruimte van alle continue X-waardige functies $C(\Omega, X)$? Het antwoord hierop is nodig in Hoofdstuk 5.

Als X een Banachrooster is, dan is deze vraag makkelijk te beantwoorden: omdat de roosterafbeeldingen continu zijn, lost de natuurlijke puntsgewijze decompositie in het rooster het problem onmiddellijk op. In het algemene geval is het antwoord niet zo duidelijk. Omdat de kegel X_+ voortbrengend is in X bestaan er, door gebruik te maken van het keuze axioma, functies $(\cdot)^\pm : X \to X_+$, zodanig dat $x = x^+ - x^-$ voor alle $x \in X$. Het probleem is dat deze functies $(\cdot)^\pm : X \to X_+$ nu niet automatisch continu zijn. Door een generalisatie van de Open Afbeeldingsstelling te combineren met de Michael Selectie Stelling kan echter worden aangetoond dat de functies $(\cdot)^\pm : X \to X_+$ toch altijd continu gekozen kunnen worden, zelfs ook nog positief homogeen en begrensd. Het antwoord op onze vraag over functies is daarmee bevestigend. In feite bewijzen we meer: als een Banachruimte voortgebracht wordt door (niet per se aftelbaar veel) gesloten kegels, dan kan de decompositie van een element van deze ruimte als convergente reeks met als termen elementen van de kegels op een continue, positief homogene en begrensde wijze gekozen worden. Het geval van een geordende Banachruimte is simpelweg de situatie waarin de Banachruimte de som is van twee kegels die een minteken schelen.

Veronderstel dat X en Y Banachroosters zijn. Een elementaire berekening laat zien dat $B(X, Y)$ absoluut monotoon is, d.w.z. dat de ongelijkheden $\pm T \leq S$, voor operatoren $T, S \in B(X,Y)$, impliceren dat $\|T\| \leq \|S\|$. Deze eigenschap is een zogeheten normaliteitseigenschap. Wanneer X en Y geen Banachroosters zijn, maar Banachruimten met een preordening gegeven door gesloten kegels, is de situatie alweer minder duidelijk. In Hoofdstuk 3 onderzoeken we de vraag naar nodige en voldoende eigenschappen van X en Y opdat $B(X,Y)$ een dergelijke normaliteitseigenschap heeft. Verder bekijken we ook of er überhaupt voorbeelden bestaan van Banachruimten X en Y met een preordening, die geen Banachroosters zijn, maar waarvoor $B(X,Y)$ toch absoluut monotoon is. Meer in het algemeen definiëren we in dit hoofdstuk een aantal mogelijke normaliteits en conormaliteitseigenschappen van Banachruimten X en Y met een preordening gegeven door gesloten kegels, en we laten zien dat een vorm van conormaliteit van X en normaliteit van Y nodig en voldoende zijn opdat $B(X,Y)$ een normaliteitseigenschap heeft. Het hebben van een dergelijke eigenschap is, op zijn beurt, weer van belang voor de relatie tussen norm en preordening in het gekruiste product in Hoofdstuk 5.

We introduceren een klasse van Banachruimten met een preordening gegeven door een gesloten kegel die *quasi-roosters* heten. In het algemeen zijn dit geen Banachroosters. In de ruimte $(\mathbb{R}^n, \|\cdot\|_2)$ $(n \geq 3)$ zijn er veel kegels die geen roosterkegels zijn; de Lorentzkegel $\{x \in \mathbb{R}^n : x_1 \geq \sqrt{\sum_{i=2}^n x_i^2}\}$ is een bekend voorbeeld. Dit voorbeeld geeft een meetkundig intuïtief beeld van quasi-roosters en motiveert hun definitie, als volgt. In deze driedimensionale ruimte heeft elk paar elementen x en y oneindig veel bovengrenzen, maar er bestaat geen supremum van x en y. Wel bestaat er echter een unieke bovengrens u van x en y die de grootheid $\|x-u\|_2 + \|y-u\|_2$ minimaliseert als functie op de verzameling van bovengrenzen. Dit element $x \tilde{\vee} y := u$ noemen we het *quasi-supremum* van x en y. In het algemeen noemen we een Banachruimte met een preordening, gegeven door een gesloten voortbrengende kegel, een *quasi-rooster* als elk paar elementen een quasi-supremum heeft, d.w.z. wanneer er een unieke bovengrens (of: een unieke minimale bovengrens) is waar de afstandssom een minimum aanneemt op de verzameling van bovengrenzen (of: minimale bovengrenzen). We laten zien dat elke strikt convexe reflexieve Banachruimte X met gesloten voortbrengende kegel X_+, zodanig dat $X_+ \cap (-X_+) = \{0\}$, een quasi-rooster is. Verder blijkt, enigszins verrassend, dat veel van de elementaire identiteiten voor vectorroosters directe analoga hebben voor quasi-roosters. Tenslotte tonen we aan dat niet noodzakelijkerwijs separabele Hilbertruimten H_1 en H_2, geordend door Lorentzkegels, quasi-roosters zijn, zodanig dat $B(H_1, H_2)$ absoluut monotoon is, terwijl H_1 en H_2 toch geen Banachroosters zijn als hun dimensie 3 of hoger is. Zodoende wordt de hierboven gestelde vraag over het bestaan van absoluut monotone ruimten van operatoren positief beantwoord.

Na deze (vanuit de primaire vragen gezien voorbereidende) eerste twee hoofdstukken, komen we in de Hoofdstukken 4 en 5 tot de gekruiste producten van Banachalgebra's en van Banachalgebra's met een preordening.

In [19] werd het gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ gedefinieerd, uitgaande van een gegeven Banachalgebra dynamisch systeem (A, G, α) en een uniform begrensde klasse $\mathcal{R}$ van niet ontaarde continue covariante representaties van het dynamische systeem op Banachruimten. Onder de milde aanname dat A een begrensde linksapproximatie van de identiteit heeft, werd daar aangetoond dat er een natuurlijke bijectie bestaat tussen de niet ontaarde $\mathcal{R}$-continue covariante representaties van het dynamische systeem (A, G, α) op Banachruimten enerzijds, en de niet ontaarde begrensde representaties van de Banachalgebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ anderzijds. In Hoofdstuk 4, laten we onder andere zien dat (onder milde voorwaarden) deze Banachalgebra $(A \rtimes_\alpha G)^{\mathcal{R}}$ de unieke Banachalgebra is waarvoor er een dergelijke bijectie bestaat. We introduceren verder, gegeven een Banachalgebra dynamisch systeem (A, G, α) en een gewicht op G, in dit hoofdstuk ook gegeneraliseerde Beurlingalgebra's. Dit zijn algebra's van (equivalentieklassen van) A-waardige functies op G. Door een specifieke keuze voor de klasse $\mathcal{R}$ laten we zien dat deze gegeneraliseerde Beurling algebra's topologisch isomorf (en soms zelfs isometrisch isomorf) met gekruiste producten zijn. Daardoor kunnen we de representaties van gegeneraliseerde Beurling algebra's beschrijven in termen van de niet ontaarde continue covariante representaties van het Banachalgebra dynamische systeem (A, G, α). Wanneer de Banachalgebra A in het Banachalgebra dynamische systeem gelijk is aan het grondlichaam reduceren deze gegeneraliseerde Beurling algebra's tot klassieke Beurlingalgebra's, waarvan $L^1(G)$ het eenvoudigste geval is. We hervinden dan de klassieke beschrijving van de representaties van deze algebra's als speciaal geval van onze resultaten. Dit laat zien dat de algemene theorie van gekruiste producten van Banach algebra's niet alleen die van C^*-algebra's als speciaal geval omvat (zie [19]), maar ook de representaties van klassieke klassen Banach algebra's beschrijft als een eenvoudig speciaal geval.

In Hoofdstuk 5 komen de voorgaande hoofdstukken samen en ontwikkelen we de theorie van gekruiste producten van Banachalgebra's in een geordende context. Gegeven een Banachalgebra dynamisch system (A, G, α) met een preordening (d.w.z. dat A een Banachalgebra met een preordening is en G als orde-automorfismen op A werkt) en een uniform begrensde klasse $\mathcal{R}$ van niet ontaarde continue covariante representaties van het dynamische systeem op Banachruimten, laten we zien dat het gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ op een natuurlijke manier een Banachalgebra met een preordening is. Wanneer A een positieve begrensde linksapproximatie van de identiteit heeft, bestaat er een natuurlijke bijectie tussen de positieve niet ontaarde $\mathcal{R}$-continue covariante representaties van het dynamische systeem (A, G, α) op Banachruimten met een preordening gegeven door een gesloten kegel enerzijds, en de positieve niet ontaarde begrensde representaties van het Banachalgebra gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ op dergelijke ruimten anderzijds. Als we nu aannemen dat alle covariante representaties in $\mathcal{R}$ *positieve* representaties op Banachruimten met een preordening zijn, dan is (onder milde voorwaarden) het gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ de unieke Banachalgebra met een preordening waarvoor een dergelijke bijectie bestaat.

Voor de ordestruktuur van het gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ maken we gebruik van de resultaten uit de Hoofdstukken 2 en 3. De kegel $(A \rtimes_\alpha G)^{\mathcal{R}}_+$ van het gekruiste product is (topologisch) voortbrengend wanneer de kegel $C_c(G, A_+)$ van A_+-waardige functies op G voortbrengend is in de ruimte $C_c(G, A)$ van A-waardige compact gedragen continue functies op G, en uit Hoofdstuk 2 weten we dat dit het geval is als A_+gesloten en voortbrengend in A is. Verder wordt de normaliteit van het gekruiste product $(A \rtimes_\alpha G)^{\mathcal{R}}$ bepaald door de normaliteit van alle operatorenalgebra's $B(X)$ van de Banachruimten X met een preordening waar de covariante representaties van $\mathcal{R}$ op werken. Hierdoor is de koppeling gelegd met het onderzoek in Hoofdstuk 3.

Als toepassing bestuderen we gegeneraliseerde Beurling algebra's gedefinieerd uitgaande van een Banachalgebra dynamisch systeem (A, G, α) met een preordening en een gewicht op G. Deze algebra's hebben een natuurlijke preordening. Wederom gebruikmakend van een specifieke keuze voor $\mathcal{R}$ gebruiken we de ontwikkelde theorie om de positieve niet ontaarde continue representaties van deze gegeneraliseerde Beurling algebra's te beschrijven in termen van de positieve niet ontaarde continue covariante representaties van (A, G, α). Wanneer $A = \mathbb{R}$ reduceren deze gegeneraliseerde Beurling algebra's tot klassieke reële Beurlingalgebra's (waaronder $L^1(G)$) met de natuurlijke ordening, en specialiseren onze uitspraken over hun positieve representaties tot resultaten die ook eenvoudig zijn af te leiden uit de klassieke resultaten over hun representaties in algemene Banachruimten.

Curriculum Vitae

Miek Messerschmidt was born in Vanderbijlpark, South Africa on July 15, 1984. He matriculated from Vanderbijlpark's Hoërskool Transvalia in 2002. In 2003 he enrolled at the University of Pretoria and graduated in 2005 with a bachelor's degree in Physics (cum laude) with elective subjects chosen from Mathematics, Computer Science and Chemistry. In 2006 he completed his honours degree in Applied Mathematics (cum laude), also at the University of Pretoria. In 2007 he spent six months attempting to teach mathematics to high school students at Carpe Diem Academy in Pretoria, before commencing his master's studies at Utrecht University in the fall of 2007. In 2009 he received his MSc (cum laude) in Mathematical Sciences from Utrecht University, with master's thesis titled *Induced transformations generalized to non-commutative ergodic theory* under the supervision of dr. Karma Dajani. In 2009 he commenced work toward gaining his PhD at Leiden University under supervision of dr. Marcel de Jeu, which culminated in this thesis. During this time he also assisted in the teaching of a number of bachelor's and master's level courses, including the national master course on functional analysis. He attended and spoke at numerous functional analysis seminars in Leiden and Delft and organized the seminar *General ordered vector spaces* in 2012. He gave contributed talks at international conferences in Tianjin, China; Athens, Greece; and Leiden, The Netherlands. During the final stages of writing this thesis in 2012–2013, he also served on the organizing committee of the international conference *Positivity VII* held in Leiden in July 2013.

In 2014 he will begin post-doctoral research at the North West University in Potchefstroom, South Africa under guidance of prof. dr. Sanne ter Horst.