Analogies between optical and quantum-mechanical angular momentum

Gerard Nienhuis
Huygens-Kamerlingh Onnes Laboratory, Leiden University,
P.O. Box 9504, 2300 RA Leiden, The Netherlands

The insight that a beam of light can carry orbital angular momentum in its propagation direction came up in 1992 as a surprise. Nevertheless, the existence of momentum and angular momentum of an electromagnetic field has been well-known since the days of Maxwell. We compare the expressions for densities of angular momentum in general three-dimensional modes and in paraxial modes. Despite their classical nature, these expressions have a suggestive quantum-mechanical appearance, in terms of linear operators acting on mode functions. In addition, paraxial wave optics has several analogies with real quantum mechanics, both with the wave function of a free quantum particle, and with a quantum harmonic oscillator. We discuss how these analogies can be applied.

PACS numbers: 42.50.Tx, 03.50.De, 12.20.-m

I. INTRODUCTION

The renewed interest in angular momentum (AM) of radiation fields that started 25 years ago did not arise from the discovery that optical AM exists. In fact, in the very first sentences of Ref. [1] it is stated that this is a well-known result of Maxwell's theory, and explained in textbooks on electrodynamics. The novelty of this work consists in the explicit expressions for the density of orbital AM of Laguerre-Gaussian light beams. It was argued that these beams possess a well-defined component of orbital AM along the beam axis. By using laser beams, the effects of orbital AM transfer were expected to be readily accessible to experimental realization, in particular when cylindrical lenses are used to convert Hermite-Gaussian beams into Laguerre-Gaussian ones. It was commonly known that the circular polarization of photons leads to a spin component in the propagation direction. This may be viewed as an example of the concept of helicity, well-known in particle physics. However the picture of photons carrying orbital AM in their propagation direction turned out to be intriguing, and a new field of research came up. This also stimulated the study of angular momentum in general Maxwell fields, outside the restrictions of the paraxial approximation. In particular, the (im)possibility of a meaningful separation of optical AM into an orbital and a spin part was widely reconsidered [2–7].

In the present paper we compare the expressions for the densities of conserved quantities like energy, momentum, spin and orbital AM in paraxial modes with the case of arbitrary modes in three dimensions. These quantities are expressed in terms of operators acting on mode functions. This gives the classical expressions for the densities a quantummechanical appearance. The expressions simplify when the concept of the optical helicity is included in the discussion [8–10]. We consider eigenmodes of the various operators. In a classical description, the density of these quantities as well as their integral values are always well-determined, whether or not the mode is an eigenfunction of the operators. In the case of paraxial beams, we point out a general correspondence between solutions of the paraxial wave equation and wave functions of the quantum harmonic oscillator in two dimensions. This analogy explains the structure of standard basis sets of Hermite-Gaussian and Laguerre-Gaussian modes, as well as the transformation between these sets. Moreover, due to the simplicity of the dynamics of harmonic oscillators, this correspondence allows simple analytical studies of light beams with special structures, such as vortices.

II. MODES AND MODE OPERATORS

A mode of the radiation field is a monochromatic solution of Maxwell's equation in the absence of charges and currents. Because of the linearity of these equations, any solution can be expanded in terms of modes. As usual we represent a mode in complex notation, so that the physical electric and magnetic fields fields are $\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \mathrm{Re}\ \mathbf{E}(\mathbf{r})\exp(-i\omega t)$ and $\boldsymbol{\mathcal{B}}(\mathbf{r},t) = \mathrm{Re}\ \mathbf{B}(\mathbf{r})\exp(-i\omega t)$, with frequency $\omega = ck$. The fields can be expressed in terms of the transverse (divergence-free) complex vector potential $\mathbf{A}$, so that

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A},\ \mathbf{E} = i\omega\mathbf{A}. \tag{1}$$

Fields and potentials are transverse complex vector solutions of Helmholtz' equation $\nabla^2\mathbf{A} = -k^2\mathbf{A} = -\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{A})$. Alternatively, the fields can be expressed in the electric vector potential $\mathbf{C}$, so that [11]

$$\mathbf{E} = -\boldsymbol{\nabla}\times\mathbf{C},\ c^2\mathbf{B} = i\omega\mathbf{C}. \tag{2}$$

Notice that the vector potentials are related as $\boldsymbol{\nabla}\times\mathbf{C} = -i\omega\mathbf{A}$ or $c^2\boldsymbol{\nabla}\times\mathbf{A} = i\omega\mathbf{C}$. The corresponding real potentials are $\boldsymbol{\mathcal{A}}$ and $\boldsymbol{\mathcal{C}}$.

A. Mode operators for angular momentum and helicity

In this paper we shall make use of Hermitian linear operators acting on mode space. These operators are analogous to quantum operators on wave functions or spinors, although their significance is purely classical. Therefore we avoid the use of $\hbar$, as the quantum of AM. We denote mode operators by a caret. Vector operators analogous to momentum and orbital AM are

$$\hat{\mathbf{P}} = -i\boldsymbol{\nabla},\ \hat{\mathbf{L}} = -i\mathbf{r}\times\boldsymbol{\nabla} = \mathbf{r}\times\hat{\mathbf{P}}. \tag{3}$$

The operator $\hat{\mathbf{P}}$ generates translations of the vector field. The separate operators $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ generate rotations of the $\mathbf{r}$-dependence and the vector nature, which in general does not preserve the transversality of a mode [2, 4]. Since a mode obeys Helmholtz' equation, it is by definition an eigenfunction of $\hat{P}^2$, with eigenvalue k^2. Since modes are vector fields, mode operators can also have a matrix character. An example is the vector operator $\hat{\mathbf{S}}$ for spin. Its three components $\hat{S}_i$ are 3×3 matrices acting on the vector, with matrix elements

$$(\hat{S}_i)_{jk} = -i\epsilon_{ijk}, \tag{4}$$

with ϵ_{ijk} the fully antisymmetric tensor of rank 3. The operator $\hat{\mathbf{S}}$ obeys the compact identity

$$\mathbf{A}^*\cdot\hat{\mathbf{S}}\cdot\mathbf{A} = -i\mathbf{A}^*\times\mathbf{A}. \tag{5}$$

Maxwell's equations in free space are symmetric for interchanging electric and magnetic fields [4, 12]. This duality symmetry can be expressed by the action of a dimensionless mode operator defined by

$$\hat{H}\mathbf{A} = \frac{1}{k}\boldsymbol{\nabla}\times\mathbf{A}. \tag{6}$$

It has the significance of a helicity operator, that generates the corresponding symmetry transformation of the electromagnetic field [10, 13–15]. This symmetry also suggests that fundamental conserved quantities of the electromagnetic field should be invariant under this symmetry transformation [16]. Since $\hat{H}^2 = \hat{I}$ is the unit operator, the eigenvalues of $\hat{H}$ are ± 1. This implies that the two operators

$$\hat{P}_\pm = \frac{1}{2}[\hat{I}\pm\hat{H}] \tag{7}$$

obey the identity $\hat{P}_\pm^2 = \hat{P}_\pm$. They are projection operators on the subspaces of eigenmodes of $\hat{H}$ with eigenvalues ± 1. These two subspaces are spanned by the plane-wave modes with right- or left-hand circular polarization. Application of the helicity operator on the fields gives the results

$$\hat{H}\mathbf{E} = \frac{1}{k}\boldsymbol{\nabla}\times\mathbf{E} = ic\mathbf{B},\ \hat{H}\mathbf{B} = \frac{1}{k}\boldsymbol{\nabla}\times\mathbf{B} = -\frac{i}{c}\mathbf{E}. \tag{8}$$

B. Density of energy, momentum and helicity

The expressions for the density of energy $w = \epsilon_0(\mathcal{E}^2 + c^2\mathcal{B}^2)/2$ and momentum $\mathbf{p} = \epsilon_0\boldsymbol{\mathcal{E}}\times\boldsymbol{\mathcal{B}}$ of the electromagnetic field are well-known from Maxwell's theory. A useful definition of the helicity density is $h = \epsilon_0(c^2\boldsymbol{\mathcal{A}}\cdot\boldsymbol{\mathcal{B}} - \boldsymbol{\mathcal{C}}\cdot\boldsymbol{\mathcal{E}})/(2c)$ [8, 9, 17, 18]. These quantities are conserved, in the sense that the densities obey continuity equations, even in the presence of sources [15, 19]. Here we restrict ourselves to the case of a single mode, defined by the complex transverse vector potential $\mathbf{A}$. Moreover we consider only the expressions averaged over time, so that rapidly oscillating terms proportional to $\exp(\pm 2i\omega t)$ are omitted. In that case, the densities are conveniently expressed in terms of the helicity operator, with the result

$$\begin{aligned}
w &= \frac{\epsilon_0}{4}\omega^2\left[\mathbf{A}^*\cdot\mathbf{A} + (\hat{H}\mathbf{A})^*\cdot(\hat{H}\mathbf{A})\right],\\
\mathbf{p} &= \frac{\epsilon_0}{2}\mathrm{Re}\,[\mathbf{E}^*\times\mathbf{B}] = \frac{\epsilon_0}{2}\omega k\ \mathrm{Im}\left[\mathbf{A}^*\times(\hat{H}\mathbf{A})\right],\\
h &= \frac{\epsilon_0}{2}\omega\ \mathrm{Re}\ \mathbf{A}^*\cdot(\hat{H}\mathbf{A})
\end{aligned} \tag{9}$$

The density of AM is $\mathbf{j} = \mathbf{r}\times\mathbf{p}$. By using the commutator between the operators $\mathbf{r}$ and $\boldsymbol{\nabla}$ this can be rewritten as

$$\mathbf{j} = \frac{1}{2}\epsilon_0\omega k\ \mathrm{Re}\ \left[\mathbf{A}^*\cdot(\hat{\mathbf{J}})\mathbf{A} + i(\mathbf{A}^*\cdot\boldsymbol{\nabla})(\mathbf{r}\times\mathbf{A})\right]. \tag{10}$$

The vector operator $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$ generates rotations of the full mode.

Expressions for the total energy, momentum, AM and helicity of the mode are obtained by integrating these densities over space, so that $W = \int wd_3\mathbf{r}$, etc. This leads to the results

$$\begin{aligned}
W &= \frac{1}{2}\epsilon_0\omega^2\int d_3\mathbf{r}\ \mathbf{A}^*\cdot\mathbf{A},\\
\mathbf{P} &= \frac{1}{2}\epsilon_0\omega\int d_3\mathbf{r}\ \mathbf{A}^*\cdot(\hat{\mathbf{P}})\mathbf{A},\\
\mathbf{J} &= \frac{1}{2}\epsilon_0\omega\int d_3\mathbf{r}\ \mathbf{A}^*\cdot(\hat{\mathbf{J}})\mathbf{A},\\
H &= \frac{1}{2}\epsilon_0\omega^2\int d_3\mathbf{r}\ \mathbf{A}^*\cdot\hat{H}\mathbf{A},
\end{aligned} \tag{11}$$

in terms of the mode operators. In the case of $\mathbf{P}$ and $\mathbf{J}$ partial integration has been applied. These expressions have the flavor of quantum-mechanical expectation values. This suggests that the classical mode functions $\mathbf{A}$, or the electromagnetic field in general, may be viewed as wave function of photons. In fact, this can be justified in some sense [20, 21].

In a similar spirit, it is natural to distinguish eigenmodes of the helicity operator $\hat{H}$ with opposite eigenvalues, which may be thought of as representing photons with opposite spin in the propagation direction. For a mode $\mathbf{A}_\pm$ with eigenvalue ± 1, we find for the densities (9) and (10) the particularly simple form

$$\begin{aligned}
w &= \frac{\epsilon_0}{2}\omega^2\mathbf{A}_\pm^*\cdot\mathbf{A}_\pm,\\
\mathbf{p} &= \mp\frac{i\epsilon_0}{2}\omega k\mathbf{A}_\pm^*\times\mathbf{A}_\pm,\\
\mathbf{j} &= \mp\frac{i\epsilon_0}{2}\omega k\mathbf{r}\times(\mathbf{A}_\pm^*\times\mathbf{A}_\pm),\\
h &= \pm\frac{\epsilon_0}{2}\omega\mathbf{A}_\pm^*\cdot\mathbf{A}_\pm = \pm w/\omega.
\end{aligned} \tag{12}$$

Note that for helicity eigenmodes the densities of energy and helicity are determined by the local strength of the mode function. The densities of momentum and AM are proportional to the cross product (5), which determines the local polarization, and has the nature of the spin density.

III. EXACT EIGENMODES OF ANGULAR MOMENTUM

A. Spherical modes

Spherical modes arise in a natural way by considering the radiation fields emitted by oscillating multipoles [22]. Here we restrict ourselves to spherical modes without singularities. As our starting point we take the scalar solutions of Helmholtz' equation that arise when a plane wave is expanded in spherical harmonics Y_{lm}. So we introduce the scalar functions

$$f_{klm}(\mathbf{r}) = Cj_l(kr)Y_{lm}(\theta\phi), \tag{13}$$

in terms of standard spherical coordinates r, θ and ϕ, with C an arbitrary constant, and j_l the spherical Bessel functions, always with the argument kr. We denote the unit vectors in the direction of varying spherical coordinates as $\mathbf{e}_r = \mathbf{r}/r$, $\mathbf{e}_\phi = -\mathbf{e}_x \sin\phi + \mathbf{e}_y \cos\phi$, $\mathbf{e}_\theta = \mathbf{e}_r \times \mathbf{e}_\phi$, and the derivatives with respect to these coordinates as ∂_r, ∂_θ and ∂_ϕ. These scalar functions can be transformed into transverse vector fields that solve Helmholtz' equation by applying the vector operator $\hat{\mathbf{L}}$. This produces the set of modes

$$\mathbf{A}_{klm,\mathrm{TE}} \equiv \hat{\mathbf{L}} f_{klm} = -iCj_l(kr)\left(\mathbf{e}_\phi\partial_\theta - \frac{im}{\sin\theta}\mathbf{e}_\theta\right)Y_{lm}(\theta\phi). \tag{14}$$

Here we used that $\partial_\phi Y_{lm} = imY_{lm}$. Since the operator $\hat{\mathbf{L}}$ commutes with the operator $\nabla^2 = -\hat{P}^2$, the complex vector potential (14) solves Helmholtz' equation. Moreover it is obviously transverse, so that it indeed defines a mode. The index TE is an abbreviation for transverse electric, indicating that the electric field $\mathbf{E}_{klm,\mathrm{TE}} = i\omega\mathbf{A}_{klm,\mathrm{TE}}$ of this mode is directed normal to $\mathbf{r}$. The corresponding magnetic field $\boldsymbol{\nabla} \times \mathbf{A}_{klm,\mathrm{TE}}$ is given by the expression

$$\begin{aligned}\mathbf{B}_{klm,\mathrm{TE}} &= iC\left[\left(\frac{1}{r} + \partial_r\right)j_l\right]\left(\mathbf{e}_\theta\partial_\theta + \frac{im}{\sin\theta}\mathbf{e}_\phi\right)Y_{lm} \\ &+ iC\mathbf{e}_r\frac{l(l+1)}{r}j_lY_{lm}.\end{aligned} \tag{15}$$

Another class of modes is generated by application of the helicity operator $\hat{H}$ on the TE modes. If we define

$$\mathbf{A}_{klm,\mathrm{TM}} = \hat{H}\mathbf{A}_{klm,\mathrm{TE}} = \hat{H}\hat{\mathbf{L}}f_{klm}, \tag{16}$$

it is obvious from Eq. (8) that the corresponding electric and magnetic field are

$$\mathbf{E}_{klm,\mathrm{TM}} = ic\mathbf{B}_{klm,\mathrm{TE}},\ c\mathbf{B}_{klm,\mathrm{TM}} = -i\mathbf{E}_{klm,\mathrm{TE}}. \tag{17}$$

In these transverse magnetic modes it is the magnetic field that is normal to $\mathbf{r}$. It is easy to check that the TE and the TM modes are normal to each other, in the sense that

$$\int d_3\mathbf{r}\mathbf{A}^*_{klm,\mathrm{TE}} \cdot \mathbf{A}_{k'l'm',\mathrm{TM}} = 0. \tag{18}$$

The densities of energy, momentum, AM and helicity in each TM mode are obviously the same as in the corresponding TE mode. These densities can be directly evaluated from the general expressions (9). In the modes $\mathbf{A}_{klm,\mathrm{TE}}$ and $\mathbf{A}_{klm,\mathrm{TM}}$ the densities of momentum and AM are

$$\mathbf{p} = p_{lm}\mathbf{e}_\phi,\ \mathbf{j} = -rp_{lm}\mathbf{e}_\theta, \tag{19}$$

in terms of the scalar function

$$p_{lm} = \frac{1}{2}C^2\epsilon_0\omega j_l^2\frac{l(l+1)}{r\sin\theta}m|Y_{lm}|^2. \tag{20}$$

So in geographical language, on each sphere around the origin the momentum density is directed Eastwards, and the density of AM is directed Southwards. The expression for the energy density w is directly found from the fields, but it is not very illuminating. It is relevant to note, however, that the densities of momentum and AM are not simply proportional to the energy density. The helicity density h is found to vanish in each of these modes.

Any vector operator $\hat{\mathbf{V}}$ acting on scalar functions of $\mathbf{r}$ obeys the standard commutation rules $[\hat{L}_i, \hat{V}_j] = \sum_k \epsilon_{ijk}\hat{V}_k$, with the orbital AM operator $\hat{\mathbf{L}}$. This is equivalent to the identity $\hat{J}_i\hat{V}_j = \hat{V}_j\hat{L}_i$. This means that the action of the components $\hat{J}_i$ of the AM operator on the spherical modes is the same as the action of $\hat{L}_i$ on the spherical harmonics Y_{lm}. With $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$ one obtains

$$\begin{aligned}\hat{J}_z\mathbf{A}_{klm,\tau} &= m\mathbf{A}_{klm,\tau}, \\ \hat{J}_\pm\mathbf{A}_{klm,\tau} &= \sqrt{(l \mp m)(l \pm m + 1}\mathbf{A}_{klm\pm1,\tau},\end{aligned} \tag{21}$$

for τ = TE or TM. Therefore, the modes $\mathbf{A}_{klm,\tau}$ are eigenmodes of $\hat{J}_z$ with eigenvalue m, and of $\hat{J}^2$ with eigenvalue $l(l+1)$.

The spherical modes are not eigenmodes of the helicity operator $\hat{H}$. A basis of eigenmodes of the helicity operator is obtained by application of the projection operators $\hat{P}_\pm$ on the TE modes. The resulting modes

$$\mathbf{A}_{klm,\pm} = \frac{1}{2}(\mathbf{A}_{klm,\mathrm{TE}} \pm \mathbf{A}_{klm,\mathrm{TM}}) \tag{22}$$

are orthogonal eigenmodes of $\hat{H}$ with eigenvalues ±1. The modes $\mathbf{A}_{klm,\pm}$ are fully specified by the requirement that they are eigenmodes of the commuting operators $\hat{P}^2$, $\hat{J}^2$, $\hat{J}_z$ and $\hat{H}$, with eigenvalues k^2, $l(l+1)$, m and ±1.

B. Cylindrical modes

Cylindrical modes are obtained in an way analogous to spherical modes, when one starts with scalar solutions of Helmholtz' equation in cylindrical coordinates R, ϕ and z, with $x = R\cos\phi$, $y = R\sin\phi$. As scalar functions we choose

$$g_{\kappa K m}(\mathbf{r}) = G_{\kappa m}(\phi, z) J_m(KR) \tag{23}$$

with $G_{\kappa m} = C\exp(i\kappa z + im\phi)$. J_m is a Bessel function with integer m, always with the argument KR. The functions (23) obey Helmholtz' equation $\nabla^2 g_{\kappa K m} = -k^2 g_{\kappa K m}$, with $k^2 = \kappa^2 + K^2$. Note that K is the wave number in the xy-plane, and κ the component of the wave vector in the z-direction. Then the vector functions

$$\mathbf{A}_{\kappa K m,\mathrm{TE}} \equiv \frac{1}{K}\mathbf{e}_z \times \hat{\mathbf{P}} g_{\kappa K m} \tag{24}$$

have a vanishing divergence and obey Helmholtz' equation, so that they define modes. These are generalizations of the Bessel beams, which have the remarkable property that they are free of diffraction [23, 24]. However, as beams they might be called unphysical, since the total power passing the xy-plane is infinite. Explicitly one finds

$$\mathbf{A}_{\kappa K m,\mathrm{TE}} = -G_{\kappa m}\left(\mathbf{e}_R \frac{m}{KR} J_m + i\mathbf{e}_\phi J_m'\right), \tag{25}$$

with J_m' the derivative of J_m. The electric field $\mathbf{E}_{\kappa K m,\mathrm{TE}} = i\omega \mathbf{A}_{\kappa K m,\mathrm{TE}}$ is then restricted to the xy-plane, which is expressed by the index TE. The magnetic field $\boldsymbol{\nabla} \times \mathbf{A}_{\kappa K m,\mathrm{TE}}$ in this mode is

$$\mathbf{B}_{\kappa K m,\mathrm{TE}} = G_{\kappa m}\left(iK\mathbf{e}_z J_m - \kappa \mathbf{e}_R J_m' - i\kappa \mathbf{e}_\phi \frac{m}{KR} J_m\right) \tag{26}$$

The corresponding TM mode with magnetic field in the xy-plane is introduced by the definition $\mathbf{A}_{\kappa K m,\mathrm{TM}} = \hat{H}\mathbf{A}_{\kappa K m,\mathrm{TE}}$, so that

$$\mathbf{E}_{\kappa K m,\mathrm{TM}} = ic\mathbf{B}_{\kappa K m,\mathrm{TE}},\; c\mathbf{B}_{\kappa K m,\mathrm{TM}} = -i\mathbf{E}_{\kappa K m,\mathrm{TE}}. \tag{27}$$

Again, the TE modes and the TM modes are orthogonal to each other.

When using the identities for Bessel functions

$$\begin{aligned} 2J_m'(\alpha) &= 2\partial_\alpha J_m(\alpha) = J_{m-1}(\alpha) - J_{m+1}(\alpha), \\ 2mJ_m(\alpha) &= \alpha[J_{m-1}(\alpha) + J_{m+1}(\alpha)] \end{aligned} \tag{28}$$

we find for the momentum density in both modes $\mathbf{A}_{\kappa K m,\tau}$

$$\mathbf{p} = \frac{\epsilon_0}{4} C^2 \omega \left(\kappa (J_{m-1}^2 + J_{m+1}^2)\mathbf{e}_z + 2\frac{m}{R} J_m^2 \mathbf{e}_\phi\right). \tag{29}$$

The AM density $\mathbf{j} = (R\mathbf{e}_R + z\mathbf{e}_z) \times \mathbf{p}$ has a z-component

$$j_z = \frac{\epsilon_0}{2} C^2 \omega m J_m^2. \tag{30}$$

The helicity density is found to be

$$h = \frac{\epsilon_0}{4} C^2 c\kappa \left(J_{m-1}^2 - J_{m+1}^2\right). \tag{31}$$

Application of the projection operators $\hat{P}_\pm$ on the modes $\mathbf{A}_{\kappa K m,\mathrm{TE}}$ produces cylindrical modes $\mathbf{A}_{\kappa K m,\pm}$, which are eigenmodes of the operators $\hat{P}^2$, $\hat{P}_z$, $\hat{J}_z$ and $\hat{H}$, with eigenvalues $k^2 = \kappa^2 + K^2$, κ, m and ± 1 [2].

IV. PARAXIAL MODES

A. Paraxial approximation

The paraxial approximation for the description of a radiation field applies when the wave vectors of the field fall within a cone with a small opening angle. This is commonly the case for laser beams. This small angle is of the order of the ratio of the components of the wave vector normal and parallel to the beam axis, which is used as a smallness parameter. To zeroth order, the complex vector potential $\mathbf{A}$ of a monochromatic light beam that freely propagates in the positive z-direction can then be written as the product of a plane wave and a slowly-varying envelope $\mathbf{a}$ as

$$\mathbf{A}(\mathbf{r}) = \mathbf{a}(\mathbf{R}, z)\exp(ikz), \tag{32}$$

where the vector $\mathbf{a}$ is restricted to the transverse (xy) plane. Here $\mathbf{R} = (x, y)$ is the 2D transverse component of the position vector $\mathbf{r}$. In this approximation, the complex electric and magnetic fields can likewise be written as $\mathbf{E} = \mathbf{f}\exp(ikz)$ and $\mathbf{B} = \mathbf{b}\exp(ikz)$. It follows from Eq. (1) that in the paraxial approximation the fields are related to $\mathbf{a}$ as

$$\mathbf{f} = i\omega\mathbf{a},\; \mathbf{b} = ik\mathbf{e}_z \times \mathbf{a}. \tag{33}$$

Just as the envelope potential $\mathbf{a}$, the fields $\mathbf{f}$ and $\mathbf{b}$ lie in the transverse plane. The vector potential is a quarter phase ahead of the electric field, while at each point the magnetic field $c\mathbf{b}$ is equal to the electric field $\mathbf{f}$ rotated over an angle $\pi/2$.

Since $\mathbf{A}$ is divergence-free, to first order it must have a small z-component so that $\boldsymbol{\partial}_\mathbf{R} \cdot \mathbf{a} + ika_z = 0$, where we introduced the symbol $\boldsymbol{\partial}_\mathbf{R} = (\partial_x, \partial_y)$ for the two-dimensional gradient operator in the transverse plane. A similar argument holds for the fields $\mathbf{E}$ and $\mathbf{B}$. So the fields $\mathbf{a}$, $\mathbf{f}$ and $\mathbf{b}$ in each xy-plane determine the z-components

$$\begin{aligned} a_z &= \frac{i}{k}\boldsymbol{\partial}_\mathbf{R} \cdot \mathbf{a}, \\ f_z &= \frac{i}{k}\boldsymbol{\partial}_\mathbf{R} \cdot \mathbf{f} = -c\boldsymbol{\partial}_\mathbf{R} \cdot \mathbf{a}, \\ b_z &= \frac{i}{k}\boldsymbol{\partial}_\mathbf{R} \cdot \mathbf{b} = -\boldsymbol{\partial}_\mathbf{R} \cdot (\mathbf{e}_z \times \mathbf{a}). \end{aligned} \tag{34}$$

B. Helicity, momentum and angular momentum of paraxial beams

The helicity operator $\hat{H}$ for a paraxial mode (32) can be expressed as $\hat{H}\mathbf{a} = i\mathbf{e}_z \times \mathbf{a}$. From the third equality (9) one finds for the helicity density in this case

$$h = \frac{\epsilon_0}{2}\omega \mathrm{Re}\, i\mathbf{a}^* \cdot (\mathbf{e}_z \times \mathbf{a}) = \frac{\epsilon_0}{2}\omega\sigma\mathbf{a}^* \cdot \mathbf{a}. \tag{35}$$

We introduced the polarization parameter σ by the relation $-i\mathbf{e}_z \cdot (\mathbf{a}^* \times \mathbf{a}) = \sigma\mathbf{a}^* \cdot \mathbf{a}$, so that σ measures the local polarization (or the spin per photon). At points of circular polarization, $\sigma = \pm 1$, and σ vanishes for linear polarization. Notice that the polarization may vary over the transverse plane.

Expressions for the densities of energy and momentum in a paraxial beam follow from the other equations (9). The density of energy and the z-component of momentum have a leading zeroth-order term that obeys the relation

$$w = \frac{\epsilon_0}{2}\omega^2 \mathbf{a}^* \cdot \mathbf{a} = cp_z. \tag{36}$$

However, we are not interested in the AM arising from this photon momentum along the axis, but in the component j_z of the AM density in the propagation direction. This component arises from the transverse component $\mathbf{p}_t = (p_x, p_y)$ of the momentum density, since

$$j_z \mathbf{e}_z = \mathbf{R} \times \mathbf{p}_t. \tag{37}$$

According to Eq. (9) $\mathbf{p}_t$ originates from products of the xy-component of $\mathbf{A}$ and the z-component of $\hat{H}\mathbf{A}$, and *vice versa*. The transverse component of the momentum density is

$$\mathbf{p}_t = \frac{\epsilon_0}{2}\mathrm{Re}\left[f_z^*(\mathbf{e}_z \times \mathbf{b}) + (\mathbf{f}^* \times \mathbf{e}_z) b_z\right] \,, \tag{38}$$

which is of first order. This result can be expressed in terms of the vector potential $\mathbf{a}$ and a_z by using Eqs. (33) and (34). After some rewriting one finds a separation in the form $\mathbf{p}_t = \mathbf{p}_{\mathrm{phase}} + \mathbf{p}_{\mathrm{pol}}$, in a term arising from a phase gradient of the field, and a helicity-dependent term. These contributions are

$$\mathbf{p}_{\mathrm{phase}} = -i\frac{\epsilon_0}{2}\omega \mathrm{Re}[\mathbf{a}^* \cdot (\boldsymbol{\partial}_{\mathbf{R}})\mathbf{a}],\ \mathbf{p}_{\mathrm{pol}} = -\frac{1}{2}\mathbf{e}_z \times \boldsymbol{\partial}_{\mathbf{R}} h. \tag{39}$$

Notice that the operator $-i\boldsymbol{\partial}_{\mathbf{R}}$ is the transverse component of the mode operator $\hat{\mathbf{P}}$.

The AM density $j_z = l + s$ of a paraxial beam is likewise separated in an orbital and a spin part. In cylindrical coordinates the density of angular momentum in the z-direction takes the form $j_z = l + s$, with

$$l = \frac{\epsilon_0}{2}\omega \mathrm{Re}[\mathbf{a}^* \cdot \partial_\phi \mathbf{a}/i],\ s = -\frac{1}{2}R\partial_R h. \tag{40}$$

Eqs. (39) and (40) show that both the density of transverse momentum and of AM in the propagation direction can be separated in an orbital part, that is determined by the phase gradient of the mode, and a spin part, that contains the gradient of the helicity density. The spin contribution is significant where the helicity density varies appreciably. This can be roughly understood when one imagines the helicity density as consisting of local momentum circulations. As long as the density of circulations is uniform, the circulation of neighboring points compensate each other. A net momentum density remains only due to a gradient of the density of circulation, in a direction normal to this gradient. This picture is not unlike the net current arising from a non-uniform magnetization density. It is easy to check that $\int s d_2\mathbf{R} = \int h d_2\mathbf{R}$, so that the spin per unit length of the beam is equal to the helicity per unit length.

Modes $\mathbf{a}_\pm$ with a uniform circular polarization $\sigma = \pm 1$ are eigenmodes of $\hat{H}$ as well as of the operator $\hat{S}_z$, both with eigenvalue ± 1. Then the helicity density is equal to $h = \pm w/\omega$, whereas the spin density is $s = \mp R\partial_R w/(2\omega)$.

In Ref. [1] expressions similar to Eqs. (39) and (58) were derived for the special case of a Laguerre-Gaussian beam with uniform polarization. Our results are valid for an arbitrary paraxial beam, also for a position-dependent polarization. They are derived from the general equations (9) in the special case of paraxial modes. The momentum and AM of paraxial beams that arise as superpositions of beams with different polarizations and different phase distributions have also been discussed in Ref. [25].

C. Paraxial wave equation

The paraxial approximation can be viewed as a lowest-order term of an expansion in the small paraxial parameter $\delta = 1/(k\gamma_0)$, with γ_0 the beam waist [26]. The zeroth order is given by Eq. (32) in terms of the envelope $\mathbf{a}$ that is restricted to the transverse plane. The two-dimensional vector $\mathbf{a}$ can be decomposed in two polarization components, which propagate independently, as described by a scalar wave equation. This equation follows from Helmholtz' equation with the assumption that the transverse variation of the envelope function is small over a wavelength. In that case $\mathbf{a}$ varies slowly with z, so that its second derivative with respect to z can be ignored. For a freely propagating paraxial beam this leads to the scalar paraxial wave equation for each polarization component

$$\partial_z u = \frac{i}{2k}\partial_{\mathbf{R}}^2 u \tag{41}$$

A paraxial beam with uniform polarization can be expressed by the two-dimensional vector $\mathbf{a} = \mathbf{e}u$, where u is a solution of the paraxial wave equation, and $\mathbf{e}$ a complex unit vector in the xy-plane. An arbitrary paraxial beam can always be decomposed into a superposition of two such beams with opposite uniform polarization.

It is easy to demonstrate that the energy, the orbital and spin AM and the helicity, all calculated per unit length of the beam, are invariant under free propagation [27]. The paraxial wave equation is exactly equivalent to the Schrödinger equation for the wave function of a freely moving quantum particle in a two-dimensional space. The propagation of the mode in the z-direction is equivalent to the time dependence of the wave function.

V. PARAXIAL BEAMS AND QUANTUM HARMONIC OSCILLATORS

A. Hermite-Gaussian modes

An orthonormal basis of exact normalized solutions of the scalar paraxial wave equation (41) is given by the Hermite-Gaussian (HG) mode functions [28, 29]. They are separable in the Cartesian coordinates. For our purposes it is convenient to express them in the form [30]

$$u_{n_x n_y}(\mathbf{R}, z) = \frac{1}{\gamma}\psi_{n_x}(\frac{x}{\gamma})\psi_{n_y}(\frac{y}{\gamma}) \exp\left(\frac{ikR^2}{2q}\right) e^{-i\chi(n_x+n_y+1)}. \tag{42}$$

The width γ, the Gouy phase χ and the radius of curvature q of the wave fronts are functions of z, and their z dependence is determined by the equalities

$$\frac{1}{\gamma^2} - \frac{ik}{q} = \frac{k}{b+iz}, \; \tan\chi = \frac{z}{b}. \tag{43}$$

The length b is the diffraction length (or the Rayleigh range, indicated as z_R in [28, 29]). The Gouy phase increases by an amount π from $z = -\infty$ to ∞. At the focal plane $z = 0$ the width γ takes the value $\gamma(0) = \gamma_0 = \sqrt{b/k}$. The functions $\psi_n(\xi)$ for $n = 0$, 1, ... are the real normalized eigenfunctions with eigenvalue $n + 1/2$ of the dimensionless Hamiltonian

$$\hat{H}_\xi = \frac{1}{2}\left(-\partial_\xi^2 + \xi^2\right), \tag{44}$$

for the quantum harmonic oscillator (HO) in one dimension. As is well-known from basic quantum mechanics, their explicit expressions are

$$\psi_n(\xi) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \exp(-\xi^2/2) H_n(\xi), \tag{45}$$

with H_n the Hermite polynomials.

For a single HG mode with uniform polarization $\mathbf{e}$, the transverse component of the momentum density can be evaluated from Eq. (39), where we substitute $\mathbf{a} = \mathbf{e}u_{n_x n_y}$. The only phase variation over the beam profile arises from the curvature radius q in Eq. (42), and we find for the momentum density arising from the phase gradient

$$\mathbf{p}_{\text{phase}} = \mathbf{e}_R \frac{w}{\omega}\frac{kzR}{z^2+b^2}. \tag{46}$$

This is just the momentum density in the transverse direction which results form the curvature of the wave fronts. Before focus, for $z < 0$, the momentum density points towards the axis, and after focus the momentum points outwards. The total transverse momentum integrated over the beam profile vanishes. Since this part of the momentum density is directed in the radial direction, the density of orbital AM vanishes in a HG mode.

B. Gaussian modes and ladder operators

The Gouy phase term in the HG modes (42) is proportional to the energy eigenvalue $n_x + n_y + 1$ of the two-dimensional quantum HO. This allows us to express an arbitrary solution of the paraxial wave equation in terms of an arbitrary time-dependent solution of the Schrödinger equation

$$\partial_\chi \Psi(\xi, \eta, \chi) = -i\left(\hat{H}_\xi + \hat{H}_\eta\right)\Psi(\xi, \eta, \chi) \tag{47}$$

of the two-dimensional HO, where we replace time by the Gouy phase χ. The wave functions $\psi_{n_x}(\xi)\psi_{n_y}(\eta)\exp(-i(n_x+n_y+1)\chi)$ are stationary solutions of this Schrödinger equation, and by taking linear combination of these one obtains the most general solution $\Psi(\xi, \eta, \chi)$. We conclude that an arbitrary solution $\Psi(\xi, \eta, \chi)$ of the Schrödinger equation for the HO gives an arbitrary solution $u(\mathbf{R}, z)$ of (41), by the identification [31]

$$u(\mathbf{R}, z) = \frac{1}{\gamma}\Psi(\xi, \eta, \chi) \exp\left(\frac{ikR^2}{2q}\right), \tag{48}$$

with $\xi = x/\gamma$, $\eta = y/\gamma$, and where the parameters γ, q and χ are specified by Eq. (43) as functions of z.

The correspondence (48) is exact, and it works both ways. For a given HO wave function we find a paraxial mode, after choosing a value for the Rayleigh range b, which is a measure of the focal region. The overlap of two modes is the same as the overlap of the two corresponding wave functions. In particular, a normalized mode u corresponds to a normalized wave function Ψ. Since the Gouy phase increases by an amount π, any mode function u from $z = -\infty$ to ∞ can be mapped on half a cycle of the oscillator. The HG-mode (42) corresponds to the stationary state $\Psi_{n_x n_y} \exp(-i(N+1)\chi)$ of the HO, with $N = n_x + n_y$.

It is well-known from elementary quantum mechanics that the eigenfunctions $\psi_n(\xi)$ of the HO are connected by (lowering and raising) ladder operators $\hat{a}_\xi$ of the form

$$\hat{a}_\xi = \frac{1}{\sqrt{2}}(\xi + \partial_\xi), \; \hat{a}_\xi^\dagger = \frac{1}{\sqrt{2}}(\xi - \partial_\xi). \tag{49}$$

They obey the bosonic commutation rules $[\hat{a}_\xi, \hat{a}_\xi^\dagger] = 1$, and the Hamiltonian (44) is equal to $\hat{a}_\xi^\dagger\hat{a}_\xi + 1/2$. From these rules it can be shown that neighboring eigenstates are connected by

$$\hat{a}_\xi \psi_n = \sqrt{n}\psi_{n-1},\ \hat{a}_\xi^\dagger \psi_n = \sqrt{n+1}\psi_{n+1}. \tag{50}$$

For the two-dimensional HO, expressions similar to (49) define the ladder operators $\hat{a}_\eta$ and $\hat{a}_\eta^\dagger$. The eigenstates $\Psi_{n_x n_y}$ of the HO can be reached from the ground state by repeated application of the raising operators $\hat{a}_\xi^\dagger$ and $\hat{a}_\eta^\dagger$, so that

$$\Psi_{n_x n_y} = \frac{1}{\sqrt{n_x! n_y!}} (\hat{a}_\xi^\dagger)^{n_x} (\hat{a}_\eta^\dagger)^{n_y} \Psi_{00}. \tag{51}$$

The correspondence (48) between paraxial modes and HO wave functions implies the existence of similar ladder operators $\hat{b}_x$ and $\hat{b}_x^\dagger$ for the modes. As a function of z the lowering operators take the form [30]

$$\hat{b}_x(z) = \frac{1}{\sqrt{2bk}} \left(kx + (b + iz)\partial_x \right), \tag{52}$$

and a similar expression holds for $\hat{b}_y$. For each value of z, the HG modes (42) are eigenmodes of the operators $\hat{b}_x^\dagger(z)\hat{b}_x(z)$ and $\hat{b}_y^\dagger(z)\hat{b}_y(z)$, with eigenvalues n_x and n_y. It is important to notice that the ladder operator do not describe the annihilation or creation of photons. They just link classical modes of different order.

C. Laguerre-Gaussian modes

HO eigenstates with a circular nature can be obtained from the ground state by application of the circular raising operators

$$\hat{a}_\pm^\dagger = \frac{1}{\sqrt{2}} (\hat{a}_\xi^\dagger \pm i\hat{a}_\eta^\dagger). \tag{53}$$

Expressed in polar coordinates ρ and ϕ, these operators have an azimuthal dependence $\exp(\pm i\phi)$. The HO eigenstate

$$\Psi_{n_+ n_-} = \frac{1}{\sqrt{n_+! n_-!}} (\hat{a}_+^\dagger)^{n_+} (\hat{a}_-^\dagger)^{n_-} \Psi_{00} \tag{54}$$

is a linear combination of the eigenstates (51) with $n_x + n_y = N = n_+ + n_-$. It has an angular momentum $m = n_+ - n_-$, expressed by the azimuthal dependence $\exp(im\phi)$. The operator identity

$$-i\partial_\phi = \hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-, \tag{55}$$

confirms this azimuthal dependence. The basis of stationary states (54) of the HO corresponds to the basis of Laguerre-Gaussian (LG) paraxial modes. Commonly these modes are labeled by the azimuthal mode index m, and the index $p = \min(n_+, n_-)$ for the radial dependence, so we indicate the LG modes as u_{pm}. When we use the same labels for the circular HO states, Eq. (54) leads to their explicit expression

$$\Psi_{pm}(\rho, \phi) = \sqrt{\frac{p!}{\pi(p+m)!}} \rho^{|m|} e^{im\phi - \rho^2/2} L_p^{|m|}(\rho^2), \tag{56}$$

with $L_p^{|m|}$ the generalized Laguerre polynomial of order p and degree $|m|$. The energy eigenvalue of the state (54) is $n_+ + n_- + 1 = 2p + |m| + 1$. When applying the correspondence (48), this leads to the Gouy phase term $\exp(-i(2p + |m| + 1)\chi)$ for the LG modes u_{pm}. This is an eigenmode of the operator $\hat{L}_z$, with eigenvalue m.

For a single LG mode with uniform polarization $\mathbf{e}$, the only phase variation over the beam profile arises both from from the curvature radius q and the term $\exp(im\phi)$ in Eq. (56), and the momentum density arising from the phase gradient is

$$\mathbf{p}_{\text{phase}} = \frac{w}{\omega} \left(\mathbf{e}_\phi \frac{m}{R} + \mathbf{e}_R \frac{kzR}{z^2 + b^2} \right). \tag{57}$$

The orbital part l of the AM density is found from Eq. (40), in the form

$$l = \frac{\epsilon_0}{2} \omega m \mathbf{a}^* \cdot \mathbf{a} = mw/\omega. \tag{58}$$

In this case the ratio of the density l of orbital AM and the density w of energy has the uniform value m/ω. One should notice that the density j_z of AM as found in (30) has no uniform ratio with the density of energy or energy flux.

D. General basis sets of Gaussian modes

It is now rather obvious to find basis sets of generalized Gaussian modes. Basis sets of stationary states of the HO arise from the ground state by application of other pairs of bosonic raising operators

$$\begin{aligned} \hat{a}^\dagger(\theta, \phi) &= \hat{a}_+^\dagger e^{-i\phi/2} \cos\frac{\theta}{2} + \hat{a}_-^\dagger e^{i\phi/2} \sin\frac{\theta}{2}, \\ \hat{b}^\dagger(\theta, \phi) &= -\hat{a}_+^\dagger e^{-i\phi/2} \sin\frac{\theta}{2} + \hat{a}_-^\dagger e^{i\phi/2} \cos\frac{\theta}{2}. \end{aligned} \tag{59}$$

The basis consists of the states

$$\Psi_{n_a n_b} = \frac{1}{\sqrt{n_a! n_b!}} \left(\hat{a}^\dagger\right)^{n_a} \left(\hat{b}^\dagger\right)^{n_b} \Psi_{00}, \tag{60}$$

and the corresponding paraxial modes are shape-invariant. The value of (θ, ϕ) determines a point on a sphere, and each point defines a different basis. These basis sets form a continuous transition between the HG modes for $\theta = \pi/2$ (on the Equator), and the LG modes for $\theta = 0$ or π (on the poles). The sphere is analogous to the Bloch sphere for spins 1/2, and has been termed the Hermite-Laguerre sphere [32].

The notation in terms of the ladder operators demonstrates the perfect analogy between the various sets of the energy eigenfunctions of the two-dimensional HO, and the various basis sets of Gaussian paraxial modes. The Hermite-Gaussian modes are analogous to the Cartesian set of HO eigenfunctions, which are just products of eigenfunctions of the one-dimensional HO in the x and the y-direction. They have vanishing orbital AM. The Laguerre-Gaussian modes are analogous to the HO eigenfunctions of energy and orbital AM. These functions factorize in cylindrical coordinates, with azimuthal dependence $\exp(im\phi)$. The basis sets (60) of the HO provide a continuous transition between the HG and the LG sets. The transformation between these basis sets is identical for paraxial modes and for HO eigenfunctions. It is based on the relations (53) and (59) between the ladder operators. Substitution of these relations in the expressions (54) and (60) for the basis sets leads to explicit expressions for the transformations between basis sets. These transformations are identical for the HO states and the corresponding Gaussian modes. These transformations only couple modes where the sums of the mode indices are equal, so that $n_x+n_y = n_+ + n_- = n_a+n_b = N$. For a given value of N, the basis sets contain $N+1$ states. Any linear combination of these HO states is stationary. For the corresponding paraxial mode this means that the beam profile is invariant under propagation, apart from a scaling factor that is equal to $\gamma(z)$. This algebraic technique of mode transformation is much simpler and more transparent than analytical methods [33, 34].

The use of ladder operators is also useful in the case of astigmatic modes, as they arise in resonators with two astigmatic mirrors [35]. In that case, the corresponding HO states are no longer stationary, and the paraxial modes are not shape-invariant anymore. In the case of general astigmatism, the ellipses of constant intensity and of constant phase are not parallel [32, 36]. These modes can carry orbital angular momentum even in the fundamental mode, which contains no vortex.

The correspondence rule (48) combined with the algebra of ladder operators is quite convenient in many cases. As an example we mention the dynamics of vortices in paraxial beams. For example, a vortex with charge 1 imprinted on a Gaussian beam at the point $\mathbf{R}_0 = (x_0, y_0)$ is expressed by multiplying the mode function in a transverse plane by $x-x_0+i(y-y_0)$, or, in the HO-language by $\xi-\xi_0+i(\eta-\eta_0)$. This factor is equivalent to application of the operator $\hat{a}_- + \hat{a}_+^\dagger - \xi_0 - i\eta_0$, which, when acting on the ground state, gives a simple combination of circular states (54). The propagation of this imprinted vector in the picture of the HO can be expressed in analytical terms [31].

E. Gaussian modes with non-uniform polarization

When a beam with non-uniform polarization passes a polarizer, the mode profile of the outgoing beam depends

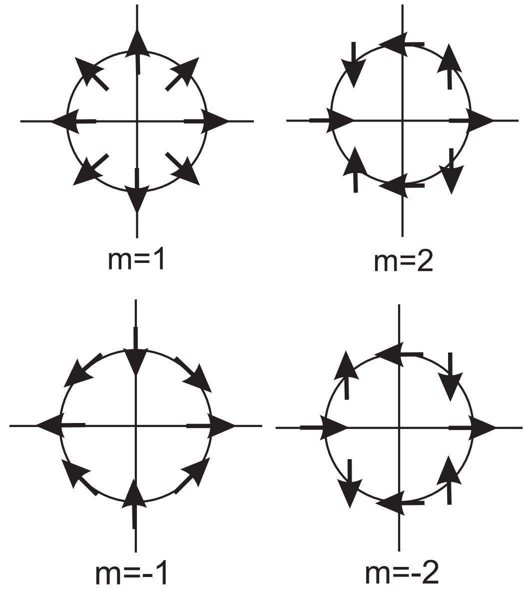

FIG. 1: Sketch of the position-dependent linear polarization for a mode as described by Eq. (61). The arrows indicate the direction of the linear polarization.

on the setting of the polarizer. This means that the mode function $\mathbf{a}$ does not factorize into the form $\mathbf{e}u(\mathbf{R}, z)$, with a fixed polarization vector $\mathbf{e}$. On the quantum level, this means that for each photon in the beam its polarization and its spatial degrees of freedom are entangled. Light beams with a non-uniform linear polarization and axial symmetry are widely studied. They can be generated by using liquid-crystal converters [37]. Another technique is based on spatially varying dielectric gratings [38, 39]. As an example, we consider the superposition of two paraxial Laguerre-Gaussian light beams with the same radial mode number p and opposite azimuthal mode numbers $\pm m$, and with opposite circular polarizations $\mathbf{e}_\pm = (\mathbf{e}_x \pm i\mathbf{e}_y)/\sqrt{2}$. From the analogy (56) between LG modes and the HO states it follows that such a mode pattern can be expressed as

$$\mathbf{a}(R, \phi, z) = F_{mp}(R, z)\left[\mathbf{e}_+ e^{-im\phi} + \mathbf{e}_- e^{im\phi}\right], \qquad (61)$$

where the real function F_{mp}, which contains the Laguerre polynomial $L_p^{|m|}(R^2/\gamma^2)$, does not depend on ϕ. The mode function (61) is the superposition of two components with $\sigma = \pm 1$ and eigenvalue $\mp m$ of the operator $\hat{L}_z$. The local polarization of the mode (61) is linear, in the direction $\mathbf{e}(\phi) = \mathbf{e}_x \cos(m\phi) + \mathbf{e}_y \sin(m\phi)$. Since the polarization is linear, the density h of helicity and the density s of spin are zero. Along a circle around the beam axis the polarization direction makes m full rotations in the positive direction. The mode function is real, apart from the phase variation due to the curvature radius. The orbital part of the momentum density is therefore the same as in Eq. (46), while the density of orbital AM is zero. The directions of linear polarization as a function of ϕ are indicated by the black arrows in Figure 1. For negative values of m, the polarization direction rotates in the negative direction along the circle. In the special case that $m = 1$, the number of rotations is 1, and the pattern is invariant under rotation about the axis. Then the polarization direction is always in the radial direction.

An interesting generalization is the case of a similar superposition of modes with opposite circular polarization, and ϕ-dependent phase terms with two arbitrary m-values. This gives a transverse mode function

$$\mathbf{a}(R,\phi) = F(R)\left[\mathbf{e}_+ e^{im'\phi} + \mathbf{e}_- e^{im\phi}\right], \tag{62}$$

prepared in a single transverse plane, where now the azimuthal mode numbers m and m' are arbitrary integer numbers. We omitted the z-dependence of the mode, since in the general case, the two terms will undergo different diffraction, so that for different transverse planes the radial mode functions will no longer be identical, and the combined mode will not be shape-invariant. When we extract a phase factor $\exp(i(m+m')\phi/2)$, the remaining real polarization vector is $\mathbf{e}(\phi) = \mathbf{e}_x \cos((m-m')\phi/2) + \mathbf{e}_y \sin((m-m')\phi/2)$. The number of rotations of the polarization vector along a circle around the beam axis is now $(m-m')/2$. This is a half-integer value when $m-m'$ is odd. The polarization pattern is illustrated in Figure 2 for the cases that $m - m' = \pm 1$. The overall phase factor $\exp(i(m+m')\phi/2)$ indicates that the phase of the polarized field varies along the circle. The energy density for the beam profile (62) is found to be

$$w = \epsilon_0 \omega^2 F^2. \tag{63}$$

The densities of momentum and AM can be directly evaluated from Eqs. (39) and (40). Since the density of helicity is zero for linear polarization, also $\mathbf{p}_{\mathrm{pol}}$ and s vanish. The remaining densities of orbital momentum and AM are

$$\mathbf{p}_{\mathrm{phase}} = \frac{w}{\omega}\frac{m'+m}{2R}\mathbf{e}_\phi,\; l = \frac{w}{\omega}\frac{m'+m}{2}. \tag{64}$$

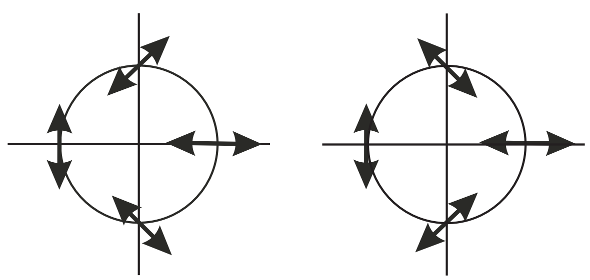

FIG. 2: Sketch of the position-dependent linear polarization for a mode as described by Eq. (62). The arrows indicate the direction of the linear polarization.

In these examples, the local polarization is linear everywhere, with a direction that varies with the angular coordinate ϕ. There is also interest in beams where the polarization varies with the angular as well as the radial coordinate. When the full range of polarization states is covered, as represented by points on the Poincaré sphere, these beams have been termed Poincaré beams [40, 41].

VI. CONCLUSIONS

We have expressed the densities of momentum and angular momentum of a mode of the electromagnetic field in terms of operators acting on mode functions, in particular the operator of helicity. We consider spherical and cylindrical modes, which are defined as eigenmodes of a set of mode operators, and we analyze and discuss explicit expressions for these densities. In the special case of paraxial modes, analogies with the quantum-mechanical harmonic oscillator in two dimensions are helpful to identify various basis sets of Gaussian modes. Laguerre-Gaussian modes are eigenmodes of the operator for the z-component of orbital angular momentum. In this case, the density of orbital angular momentum is proportional to the energy density of the mode. This suggests the picture of a local density of photons, each with a well-defined value $\hbar m$ of the z-component of orbital angular momentum. In this sense, this quantity could be termed well-defined [1]. However, this picture should be handled with care. The cylindrical modes are also eigenmodes of the z-component of the total angular momentum (orbital plus spin), but the density of this angular-momentum component is not proportional to the energy density or to the energy flux density.

Finally, we recall that this paper is entirely based on classical Maxwell theory, even though the description uses mode operators that are reminiscent of quantum mechanics. There is no quantum indeterminacy here, and each physical quantity has a well-defined value for any mode. This is also true for the densities of the three components of spin and orbital AM and for their integral values, in spite of the fact that the corresponding operators are non-commuting. The suggestive expressions (11) for the integral values of conserved quantities are not taken as a starting point, but they originate from well-known expressions for the densities in terms of the Maxwell fields. In contrast, a quantum-mechanical wave function in general does not define a local density of physical quantities. The ladder operators introduced in Sec. V do not refer to photons, but merely change the order of classical modes.

[1] Allen, L., Beijersbergen, M.W., Spreeuw, R.J.C. and Woerdman, J.P. 1992 Orbital angular momentum and the transformation of Laguerre-Gaussian laser modes. *Phys.*

Rev. A **45**, 8185-8189.

[2] van Enk, S.J. and Nienhuis, G. 1994 Commutation rules and eigenvalues of spin and orbital angular momentum of radiation fields. *J. Mod. Opt.* **41**, 963-977.

[3] Bliokh, K.Y., Alonso, M.A., Ostrovskaya, E.A. and Aiello, A. 2010 Angular momentum and spin-orbit interaction of nonparaxial light in free space. *Phys. Rev. A* **82**, 063825-1-7.

[4] Barnett, S.M. 2010 Rotation of electromagnetic fields and the nature of optical angular momentum. *J. Mod. Opt.* **57**, 1339-1343.

[5] Białynicki-Birula, I. and Białynicka-Birula, Z. 2011 Canonical separation of orbital angular momentum of light into its orbital and spin parts. *J. Opt.* **13**, 064014-1-5.

[6] Bliokh, K.Y., Dressel, J. and Nori, F. 2014 Conservation of the spin and orbital angular momentua in electromagnetism. *New J. Phys.* **16**, 093037-1-21.

[7] Barnett, S.M., Allen, L., Cameron, R.P., Gilson, C.R., Padgett, M.J., Speirits, F.C. and Yao, A.M. 2016 On the nature of the spin and orbital parts of optical angular momentum. *J. Opt.* **18**, 064004-1-11.

[8] Trueba, J.L. and Rañada, A.F. 1996 The electromagnetic helicity. *Eur. J. Phys.* **17**, 141-144.

[9] Afanasiev, G.N. and Stepanovsky, Yu. 1996 The helicity of the free electromagnetic field and its physical meaning. *Il Nuovo Cimento A* **109**, 271-279.

[10] Fernandez-Corbaton, I., Zambrana-Puyalto, X. and Molina-Terriza, G. 2012 Helicity and angular momentum: A symmetry-based framework for the study of light-matter interactions. *Phys. Rev. A* **86**, 042103-1-14.

[11] Bateman, H. 1915 *The mathematical analysis of electrical and optical wave-motion on the basis of Maxwell's equations*, Cambridge: Cambridge University Press.

[12] Calkin, M.G. 1965 An invariance property of the free electromagnetic field. *Am. J. Phys.* **33**, 958-960.

[13] Bliokh, K.Y., Bekshaev A.Y. and Nori, F. 2013 Dual electromagnetism: helicity, spin, momentum and angular momentum. *New J. Phys.* **15**, 033026-1-28.

[14] Fernandez-Corbaton, I., Zambrana-Puyalto, X., Tischler, N., Vidal, X., Juan, M.L. and Molina-Terriza, G. 2013 Electromagnetic duality symmetry and helicity conservation for the macroscopic Maxwell's equations. *Phys. Rev. Lett.* **111**, 060401-1-5.

[15] Nienhuis, G. 2016 Conservation laws and symmetry transformations of the electromagnetic field with sources. *Phys. Rev. A* **93**, 023840-1-9.

[16] Cameron, R.P., Barnett, S.M. and Yao, A.M. 2012 Optical helicity, optical spin and related quantities. *New J. Phys.* **14**, 053050-1-16.

[17] Barnett, S.M., Cameron, R.P. and Yao, A.M. 2012 Duplex symmetry and its relation to the conservation of optical helicity. *Phys. Rev. A* **86**, 013845-1-4.

[18] Cameron, R.P. and Barnett, S.M. 2012 Electric-magnetic symmetry and Noether's theorem. *New. J. Phys.* **14** 123019-1-27.

[19] Nieto-Vesperinas, M. 2015 Optical theorem for the conservation of electromagnetic helicity: Significance for molecular energy transfer and enantiomeric discrimination by circular dichroism. *Phys. Rev. A* **92**, 023813-1-8.

[20] Białynicki-Birula, I. 1996 Photon wave function. *Progress in Optics* **36**, 245-294.

[21] Białynicki-Birula, I. and Białynicka-Birula, Z. 2011 Beams of electromagnetic radiation carrying angular momentum: The Riemann-Silberstein vector and the classical-quantum correspondence. *Opt. Comm.* **264**, 342-351.

[22] Jackson, J.D. 1989 *Classical Electrodynamics*, New York: Wiley.

[23] Durnin, J. 1987 Exact solutions for nondiffracting beams. I. The scalar theory. *J. Opt. Soc. Am. A* **4**, 651-654.

[24] Durnin, J., Micelli, J.J. and Eberly, J.H. 1987 Diffraction-free beams. *Phys. Rev. Lett.* **58**, 1499-1501.

[25] Zambrini, R. and Barnett, S.M. 2007 Angular momentum of multimode and polarization patterns. *Opt. Express* **15**, 15214-15227.

[26] Lax, M., Louisell, W.H. and McKnight, W.B. 1975 From Maxwell to paraxial wave optics. *Phys. Rev. A* **11**, 1365-1370.

[27] van Enk, S.J. and Nienhuis, G. 1992 Eigenfunction description of laser beams and orbital angular momentum of light. *Opt. Comm.* **94**, 147-158.

[28] Kogelnik, H. and Li, T. 1966 Laser beams and resonators. *Appl. Opt.* **5**, 1550-1567.

[29] Siegman, A.E. 1986 *Lasers*, Mill Valley, CA: University Science Books.

[30] Nienhuis, G. and Allen, L. 1993 Paraxial wave optics and harmonic operators. *Phys. Rev. A* **48**, 656-665.

[31] Nienhuis, G. and Visser, J. 2004 Angular momentum and vortices in paraxial beams. *J. Opt. A: Pure Appl. Opt.* **6**, S248-S250.

[32] Visser, J. and Nienhuis, G. 2004 Orbital angular momentum of general astigmatic modes. *Phys. Rev. A* **70**, 013809-1-12.

[33] Abramochkin, E. and Volostnikov, V. 1991 Beam transformations and nontransformed beams. *Opt. Comm.* **83**, 123-135.

[34] Abramochkin, E.G. and Volostnikov, V.G. 2004 Generalized Gaussian beams. *J. Opt. A: Pure Appl. Opt.* **6**, S157-S161.

[35] Habraken, S.J.M. and Nienhuis, G. 2007 Modes of a twisted optical cavity. *Phys. Rev. A* **75**, 033819-1-11.

[36] Arnaud, J.A. and Kogelnik, H. 1969 Gaussian light beams with general astigmatism. *Appl. Opt.* **8**, 1687-1693.

[37] Stalder, M. and Schadt, M. 1996 Linearly polarized light with axial symmetry generated by liquid-crystal polarization converters. *Opt. Lett.* **21**, 1948-1950.

[38] Bomzon, Z., Biener, G., Kleiner, V. and Hasman, E. 2002 Radially and azimuthally polarized beams generated by space-variant dielectric subwavelength gratings. *Opt. Lett.* **27**, 285-287.

[39] Niv, A., Biener, G., Kleiner, V. and Hasman, E. 2003 Formation of linearly polarized light with axial symmetry by use of space-variant subwavelength gratings. *Opt. Lett.* **28**, 510-512.

[40] Beckley, A.M., Brown, T.G. and Alonso, M. 2010 Full Poincaré beams. *Opt. Express* **18**, 10777-10785.

[41] Galvez, E.J., Khadka, S., Schubert, W.H. and Nomoto, S. 2012 Poincaré-beam patterns produced by nonseparable superpositions of Laguerre-Gauss and polarization modes of light. *Appl. Opt.* **51**, 2925-2934.