Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

Rijk, B. de

Author: Rijk, B. de
Title: Periodic pulse solutions to slowly nonlinear reaction-diffusion systems
Issue Date: 2016-12-22

Chapter 7

Outlook

In this chapter we outline possible future research topics.

7.1 Breaking the symmetry

This thesis focusses on stationary, spatially symmetric, periodic pulse solutions in the singularly perturbed reaction-diffusion system (1.9). Such solutions arise naturally, because the existence problem (2.1) is R-reversible. Therefore, the associated eigenvalue problem (3.3) is also R-reversible. These symmetries can be broken by adding *advection* terms to system (1.9) or by studying *traveling*-wave solutions to (1.9) instead of stationary ones. We emphasize that in some applications advection terms occur naturally, leading to reaction-advection-diffusion models [2, 63]. It is therefore an interesting and relevant question how symmetry breaking affects our analysis.

Our existence analysis in Chapter 2 relies heavily on the R-reversibility of system (2.1): we exploit that any orbit that crosses $\ker(I - R)$ twice must be periodic. In the absence of such a symmetry, additional transversality arguments are required to construct a periodic orbit using geometric singular perturbation theory. If the singular periodic orbit consists of fast heteroclinic connections and orbit segments on slow invariant manifolds, then the required transverse intersections (of the stable and unstable foliations of the slow manifolds) are often already present in the fast reduced systems arising in the limit $\varepsilon \to 0$ – see for instance [110, Section 7]. On the other hand, if the singular periodic orbit is a concatenation of a *homoclinic* connection with an orbit segment on the slow manifold – as is the case in our work – then these transversal intersections exists only for $\varepsilon > 0$ and tools like Melnikov theory for slowly varying systems [94] can be employed to find them – see for instance [26, 108] for constructions of periodic traveling waves in the two-component (Klausmeier-)Gray-Scott model. Moreover, controlling the periodic orbits close to the transverse intersections might be subtile [108]. It remains an open problem whether the techniques in [26, 108] can be extended to the general class of multi-component systems (1.9) with additional advection terms.

In general, the spectrum associated with periodic wave trains to reaction-advection-diffusion systems consists of continuous images of the unit circle S^1 [38]. The presence of symmetry yields degenerate spectrum: the image of S^1 covers each curve of spectrum twice so that any γ-eigenvalue is also a $\overline{\gamma}$-eigenvalue – see Proposition 3.7. Thus, breaking the symmetry changes the structure of the spectrum fundamentally. Yet, we expect that the present spectral techniques extend to the non-symmetric case without any problems. Let us elaborate on this claim. Recall that our spectral analysis is based on two reduction results: the approximation of the roots of the Evans function by the ones of the reduced Evans function and the expansion of the critical spectral curve attached to the origin. First, our Evans-function analysis is based on the results in [17], where we do not assume that the periodic pulse solutions are symmetric. We observe that R-reversibility of the eigenvalue problem is not essential for the decomposition of the Evans function and its reduction. This makes an extension to models with advection terms straightforward – see also [108].

Second, our expansion method of the critical spectral curve is based on the analyses in [10, 100] using Lin's method – see §5.3.6. Both in [10] and in [100] the eigenvalue problem does not admit a (reversible) symmetry, since one considers *traveling* waves. Therefore, we expect that the present expansion method remains valid in the non-symmetric case. Yet, we foresee that the outcome of the analysis will be different: we conjecture that the critical curve is not confined to the real axis and scales with ε instead of ε^2. Indeed, the essential spectrum is no longer degenerate and the $O(\varepsilon)$-terms in the expansion of the critical curve will no longer vanish due to parity arguments. Our hypothesis is further strengthened by the fact that the critical spectral curve associated with periodic traveling waves in the FitzHugh-Nagumo equations scales with ε and is non-real – see [32].

The non-degeneracy of the spectrum in the non-symmetric case affects the destabilization mechanisms discussed in Chapter 6. Numerical investigations in the Klausmeier-Gray-Scott system indicate that the Hopf and belly dance destabilization mechanisms break down in the presence of $O(1)$ advection: the boundary of the Busse ballon consists no longer of curves of ± 1-Hopf instabilities in the limit $\varepsilon \to 0$ and the codimension-two points disappear. Instead, the boundary is smooth in the limit $\varepsilon \to 0$ and consists of oscillating curves of γ-Hopf instabilities, where γ can be *any* Floquet multiplier in S^1. It remains an open problem to confirm this analytically.

In the non-symmetric case there are three types of robust instabilities: Hopf, sideband or spatial-temporal period doubling – see [93]. As far as the author knows, there is no numerical evidence that periodic pulse solutions can destabilize through a fold or Turing instability in the absence of symmetry (provided $\varepsilon > 0$ is sufficiently small). This suggests that, as in the symmetric case, fold and Turing instabilities cannot occur. However, analytical grip on the spectrum in the non-symmetric case is needed to confirm this hypothesis.

7.2 Dynamics of periodic pulses upon destabilization

The explicit insights in the spectral geometry in Chapters 3 and 6 is a key to understanding the weakly nonlinear dynamics of periodic pulse solutions to (1.9) as they become spectrally unstable. A first step in this direction has been taken in [119], in which a normal form approach associated with a Hopf destabilization of *homoclinic* pulses in 2-component, slowly nonlinear models of the form (1.9) is developed. Unlike known classical slowly linear examples such as the Gray-Scott and Gierer-Meinhardt models, the Hopf bifurcation for homoclinic pulses can be supercritical. It can even be the first step in a sequence of further bifurcations that leads to complex (amplitude) dynamics of a standing solitary pulse – as observed in the simulations of [120]. We expect that the weakly nonlinear dynamics of periodic pulse solutions to (1.9) beyond their destabilization is also very rich – and thus an interesting direction of future research – as indicated by the Hopf and belly dance destabilization mechanisms described in Chapter 6 and the fact that the pulses that together form the periodic pattern are in semi-strong interaction [24, 92]. We stress that it is still unknown whether the Hopf and belly dance destabilization mechanisms generalize to systems (1.9) with multiple components (the regime $n > 1$ or $m > 1$).

7.3 Multiple spatial dimensions

In this thesis we focus on solutions to singularly perturbed reaction-diffusion systems *on the line*. However, in some applications [46, 73, 89, 109], the associated reaction-diffusion models are naturally formulated on the plane or an unbounded cylinder. This give rise to the following class,

$$\begin{aligned} u_t &= D_1 \Delta u - H(u, v, \varepsilon), \\ v_t &= \varepsilon^2 D_2 \Delta v - G(u, v, \varepsilon), \end{aligned} \qquad u(\check{x}, t) \in \mathbb{R}^m, v(\check{x}, t) \in \mathbb{R}^n, \quad \check{x} \in \mathbb{R} \times \Omega, \tag{7.1}$$

of reaction-diffusion systems, where $0 < \varepsilon \ll 1$ and $\Omega \subset \mathbb{R}^k$ can be a bounded or unbounded domain. Spatially multi-dimensional systems of the form (7.1) are far less well understood than their one-dimensional counterparts (1.1). Obviously, solutions to (1.1) give rise to *striped* solutions to (7.1) by trivially extending them into a transverse spatial direction. Similarly, by switching to polar coordinates one can construct *spots*, which are constant along concentric circles. Using singular perturbation techniques, one can construct more elaborate solutions to (7.1) like spiral waves and target patterns – see [113] and references therein.

In the stability analysis of stripes and spots one proceeds by applying the Fourier transform in the transverse or radial direction – see for instance [36, 77, 105, 111]. Consequently, the associated eigenvalue problem depends on one additional parameter, but is still a singularly perturbed *ordinary* differential equation. Therefore, we expect that many of the spectral reduction techniques presented in this thesis can be employed to determine the stability of spot and stripe solutions to (7.1). Pioneering work in that direction can be found in [29, 109, 117] in the context spots and stripes in FitzHugh-Nagumo and Gray-Scott type models.

If a solution to (7.1) has a more elaborate structure, then such a reduction via the Fourier transform is impossible and the associated eigenvalue problem is a *partial* differential equation. In some specific cases, the eigenvalue problem can be reduced to a *scalar*, non-local PDE. Via this non-local problem one can prove spectral stability of the underlying pattern – see for instance [124, 125], where spectral stability is established for (asymmetric) spotty patterns in the Gray-Scott and Gierer-Meinhardt models on the plane.

However, it is still an open problem whether the spectral reduction results presented in this thesis have infinite-dimensional counterparts. We emphasize that many of the employed ODE-techniques carry over to the PDEs: exponential dichotomies [44], Lin's method [96] and the Evans function [18, 69] can be utilized by rewriting the eigenvalue problem as an evolution equation in the spatial variables corresponding to the unbounded directions. This so-called *spatial dynamics approach* was introduced in [62] – see also [16, 91, 103] and references therein. All in all, this could provide the desired framework that facilitates a reduction induced by the small parameter ε in (7.1), possibly through a factorization of the Evans function. Eventually, this might lead to a systematic approach for studying the (spectral) stability of patterns in singularly perturbed systems of the form (7.1) that allow for multiple components, multiple spatial dimensions and slow nonlinearities.