Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Chapter 6

Destabilization mechanisms

6.1 Introduction

In this chapter we focus on instabilities of periodic pulse solutions to (1.9) as system parameters are varied. To describe the spectral geometry as the periodic pulse destabilizes, we need as much analytical grip as possible. Therefore, we restrict ourselves to the case $m = n = 1$ – see §3.8. We assume that equation (1.9) depends on a real parameter μ. An generic instability occurs at $\mu = \mu_*$ if one of the spectral stability criteria in Corollary 3.8 fails at $\mu = \mu_*$, while the others are still valid. Depending on which one of these criteria fails, we can identify the type of instability occurring when μ passes through μ_*.

Verification of the three spectral stability criteria in Corollary 3.8 requires explicit knowledge of the Evans function $\mathcal{E}_\varepsilon(\lambda, \gamma)$. In Chapter 3 we approximated the roots of the Evans function $\mathcal{E}_\varepsilon(\lambda, \gamma)$ by the zeros of the reduced Evans function $\mathcal{E}_0(\lambda, \gamma) = -\gamma\mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda, \gamma)$ which is defined in terms of three simpler, lower-dimensional eigenvalue problems. This leads to asymptotic control over the spectrum and simplifies the verification of the first spectral stability criterium in Corollary 3.8. Moreover, we obtained higher-order control over the spectrum about the origin: we derived a leading-order expression $\lambda_0(\nu)$ for the critical spectral curve attached to the origin, which shrinks to the origin as $\varepsilon \to 0$. The latter simplifies the verification of the spectral stability criteria in Corollary 3.8 further, which eventually leads to spectral stability criteria in terms of simpler, lower-dimensional problems – see Corollaries 3.20 and 3.31.

The zeros of the fast Evans function $\mathcal{E}_{f,0}$ will in general depend on the parameter μ. However, by Proposition 3.24 the relative position of these zeros with respect to the origin is fixed, i.e. no root of the fast Evans function can pass through the origin as we vary μ. Thus, by the aforementioned spectral approximation results, generic instabilities occur if either the curve $\lambda_0(\nu)$ or a curve $\lambda_*(\nu)$ satisfying $\mathcal{E}_{s,0}(\lambda_*(\nu), e^{i\nu}) = 0$ transits through the imaginary axis as we vary μ. By Proposition 3.25 and 3.29 this is precisely the case if one of the following two scenarios occurs:

1. One of the quantities $\mathfrak{a}, \mathfrak{b}$ or $\mathfrak{w}$, defined in (3.24) and (3.32), changes sign as we vary μ;

2. For some $\gamma \in S^1$, there is a complex conjugate pair of roots of $\mathcal{E}_{s,0}(\cdot, \gamma)$ moving through the imaginary axis $i\mathbb{R} \setminus \{0\}$ as we vary μ.

By employing Proposition 3.29, we study the spectral configuration about the origin in detail in the first scenario. We establish that the instabilities are of sideband or period doubling type if $\mathfrak{a}$ or $\mathfrak{w}$ changes sign and of Hopf type if $\mathfrak{b}$ changes sign. Moreover, the second destabilization scenario above corresponds to a Hopf instability. We conclude that the only possible primary codimension-one instabilities occurring are of sideband, Hopf or period doubling type.

This second destabilization scenario has been studied in great detail in [27] for the Gierer-Meinhardt equations (2.26) when periodic pulse solutions approach a homoclinic limit. While decreasing the wave number k, the character of destabilization alternates between two kinds of Hopf instabilities. One in which the destabilization is caused by a conjugated pair of 1-eigenvalues crossing the imaginary axis, allowing for perturbations that are exactly in phase with the periodic solution. The other Hopf instability corresponds to a conjugated pair of -1-eigenvalues crossing the imaginary axis, allowing for antiphase perturbations. In (k, μ)-space the curves $\mathcal{H}_{\pm 1}$ corresponding to ± 1-Hopf instabilities intersect infinitely often as they oscillate about each other while both converging to the Hopf destabilization point of the homoclinic limit solution on the line $k = 0$. This phenomenon is called the *Hopf dance*. In the singular limit $\varepsilon \to 0$ the two curves $\mathcal{H}_{\pm 1}$ cover the boundary of the region of stable pulse solutions. The boundary is non-smooth at the (transversal) intersection points of $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$. This corresponds to an associated higher order phenomenon: *the belly dance*. The analysis of these phenomena in the Gierer-Meinhardt system relies crucially on the specific characteristics of the equations; in particular, on the fact that the slow dynamics away from the pulses are driven by linear equations.

We employ our spectral methods to show that both the Hopf and belly dance are persistent mechanisms that occur in the general class (1.9) of *slowly nonlinear* systems – see §1.3. Second, we wish is to identify whether the limiting homoclinic pulse is the last 'periodic' pulse to become unstable as we vary μ. This was conjectured by W.M. Ni in the context of the Gierer-Meinhardt equations [80]. We establish an explicit sign criterion to determine whether the homoclinic pulse solution is the last or the first to destabilize.

This chapter is structured as follows. First, we provide a complete overview of the possible codimension-one instabilities for periodic pulse solutions to (1.9). Then, we study the spectral geometry in the two generic destabilization scenarios above and identify the type of instability occurring. Subsequently, we switch to the regime where the periodic pulse approaches a homoclinic limit. Before we study destabilization mechanisms in the homoclinic limit, we collect results from the literature concerning the existence and spectral properties of homoclinic pulse solutions to (1.9). Next, we provide the leading and next order geometry of the spectral curves crossing the imaginary axis, when periodic pulse solutions undergo a Hopf destabilization in the homoclinic limit. This key result then yields the existence of the Hopf and belly dance destabilization mechanisms and leads to a criterion which determines whether the homoclinic pulse is the last or the first periodic pulse solution to destabilize.

6.2 Classification of codimension-one instabilities

Let $\check{\phi}_{\mathrm{p},\varepsilon}$ be a periodic pulse solution to (1.9), established in Theorem 2.3. We assume that equation (1.9) depends on a real parameter μ. The periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable if the three conditions in Corollary 3.8 are satisfied. A codimension-one instability of $\check{\phi}_{\mathrm{p},\varepsilon}$ occurs if one of these conditions fails as we vary μ, while the others are still valid. Denote by $\mathcal{E}_{\varepsilon,\mu}(\lambda,\gamma)$ the associated Evans function (depending on μ). Suppose one of the conditions in Corollary 3.8 is violated by a pair $(\lambda_*,\nu_*) \in i\mathbb{R} \times [-\pi,\pi]$ at $\mu = \mu_*$. Consequently, it holds $\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*}) = 0$. If we have $\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*}) \neq 0$, the implicit function theorem yields a local expansion of the marginally stable spectral curve $\lambda_c(\nu)$ through λ_*:

$$\lambda_c(\nu) = \lambda_* + \frac{a_2}{2!}(\nu-\nu_*)^2 + \frac{a_4}{4!}(\nu-\nu_*)^4 + \mathcal{O}\left((\nu-\nu_*)^6\right),$$

with $a_2, a_4 \in \mathbb{C}$. Note that Proposition 3.7 implies that the odd coefficients in the expansion of $\lambda_c(\nu)$ must be zero. The leading coefficient a_2 can be computed through implicit differentiation:

$$a_2 = \frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})e^{2i\nu_*}}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})}.$$

In the case $a_2 = 0$, we have

$$a_4 = \frac{-\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})e^{4i\nu_*}}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})}.$$

This gives rise to the following classification of codimension-one instabilities – see [93, Section 3.3].

- γ_*-*Hopf.* The second and third condition in Corollary 3.8 are satisfied and the first condition is violated by a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm 1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1}) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})\gamma_*^{\pm2}}{\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}{\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}\right] \neq 0.$$

- *Spatial period doubling.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at $\gamma = -1$ so that

$$\mathcal{E}_{\varepsilon,\mu_*}(0,-1) = 0, \quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,-1)\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,-1) < 0, \quad \partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(0,-1) \neq 0.$$

- γ_*-*Turing.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at a unique pair $\gamma_*^{\pm} \in S^1 \setminus \{\pm1\}$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(0, \gamma_*^{\pm1}) = 0, \quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0, \gamma_*^{\pm1})\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0, \gamma_*^{\pm1})\gamma_*^{\pm2} < 0, \quad \partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(0, \gamma_*^{\pm1}) \neq 0.$$

- *Sideband.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1)=0,\quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,1)\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1)>0,\quad \partial_{\gamma\gamma\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1)\neq 0.$$

- *Fold/Pitchfork.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,1)=0,\quad \partial_{\lambda\lambda}\mathcal{E}_{\varepsilon,\mu_*}(0,1),\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1),\partial_{\lambda\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1)\neq 0.$$

Using the spectral stability results from Chapter 3 one easily verifies that the only possible primary codimension-one instabilities are of sideband, Hopf or period doubling type.

Proposition 6.1. *Suppose $m=n=1$. The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ to* (1.9) *cannot be destabilized through a Turing or fold instability.*

Proof. In the case of a γ_*-Turing instability, $\mathcal{E}_{\varepsilon,\mu_*}(0,\cdot)$ has double roots $\gamma_*^{\pm 1}$ and 1 with $\gamma_*\in S^1\setminus\{1\}$. However, this is impossible, since $\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma)$ is a quartic polynomial in γ by Proposition 3.11. In the case of a fold instability, 0 is a double root of the reduced Evans function $\mathcal{E}_{0,\mu_*}(\cdot,1)$ by Theorem 3.15. Since 0 is a simple root of the fast Evans function $\mathcal{E}_{f,0,\mu_*}$ by Proposition 3.24, the slow Evans function $\mathcal{E}_{s,0,\mu_*}(\cdot,1)$ also has a root 0. Thus, Proposition 3.25 yields $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*)=-1$. So, by Corollary 3.32 there exists a λ in the spectrum $\sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda)>0$. Hence, the first condition in Corollary 3.8 is not satisfied, which contradicts the occurrence of a fold instability. □

To identify which one of the three remaining instabilities occurs when the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ destabilizes does not require control over the full Evans function $\mathcal{E}_\varepsilon$. In the next section we show that generically it is sufficient to track the quantities $\mathfrak{a},\mathfrak{b}$ and $\mathfrak{w}$ and roots of the slow Evans function $\mathcal{E}_{s,0}$ as we vary μ.

6.3 Generic destabilization mechanisms

Let $\check{\phi}_{\mathrm{p},\varepsilon}$ be a periodic pulse solution to (1.9), established in Theorem 2.3. We assume that equation (1.9) depends on a real parameter μ. In the introduction in §6.1 we observed that generically instabilities occur precisely if either one of the quantities $\mathfrak{a}(\mu)$, $\mathfrak{b}(\mu)$ or $\mathfrak{w}(\mu)$, defined in (3.24) and (3.32), changes sign or, for some $\gamma_*\in S^1$, there is a complex conjugate pair of roots of the slow Evans function $\mathcal{E}_{s,0,\mu}(\cdot,\gamma_*)$ moving through the imaginary axis $i\mathbb{R}\setminus\{0\}$ as μ passes through some value μ_*. Thus, we distinguish between the following generic destabilization scenarios:

(D1) $\mathfrak{w}(\mu_*)=0$, $\partial_\mu\mathfrak{w}(\mu_*)\neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*)>0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda,\gamma)\neq 0$ for all $\gamma\in S^1$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$;

(D2) $\mathfrak{b}(\mu_*)=0$, $\partial_\mu\mathfrak{b}(\mu_*)\neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{w}(\mu_*)>0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda,\gamma)\neq 0$ for all $\gamma\in S^1$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$ and $\lambda\neq 0$;

(D3) $\mathfrak{a}(\mu_*) = 0$, $\partial_\mu \mathfrak{a}(\mu_*) \neq 0$, $\mathfrak{b}(\mu_*)\mathfrak{w}(\mu_*) > 0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $\gamma \in S^1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \neq 0$;

(D4) There is a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm 1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1}) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})\gamma_*^{\pm 2}}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}\right].$$

In addition, $\mathfrak{a}(\mu_*)$, $\mathfrak{b}(\mu_*)$ and $\mathfrak{w}(\mu_*)$ have the same non-zero sign and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $(\lambda, \gamma) \in S^1 \times \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $(\lambda, \gamma) \neq (\pm\lambda_*, \gamma_*^{\pm 1})$.

In this section we identify the type of instability occurring in these four scenarios. Clearly, the following result is an immediate consequence of Theorems 3.15 and 3.17 and Proposition 3.29.

Corollary 6.2. *Assume $m = n = 1$ and **(D4)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (1.9) *destabilizes through a γ_ε-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon - \gamma_*| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

The remainder of this section is devoted to the identification of the type of instability occurring in the three other scenarios, which requires detailed control over the spectral geometry about the origin.

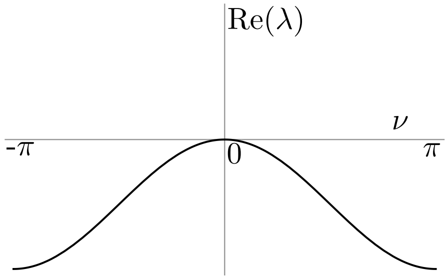
6.3.1 The first destabilization scenario

Let **(D1)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) > 0$. Then, there exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that it holds $\mathcal{E}_{\mathrm{s},0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$. Thus, by Corollary 3.16, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum for any $\mu \in M$. In addition, $\lambda_{\varepsilon,\mu}$ is real-valued and analytic and, by Proposition 3.29, we have the leading-order approximation,

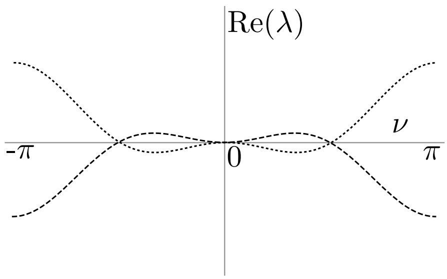
$$\lambda_{\varepsilon,\mu}(\nu) = \varepsilon^2 \mathfrak{a}(\mu)\mathfrak{w}(\mu) \frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}(\mu)\mathfrak{b}(\mu)} + O\left(\varepsilon^3 \left|\log(\varepsilon)\right|^5\right), \tag{6.1}$$

for any $\mu \in M$ and $\nu \in \mathbb{R}$. So, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, for $\mu \in M$ with $|\mu - \mu_*| > \delta$ the approximation (6.1) gives the spectral configuration depicted in Figures 6.1a and 6.1c. Hence, $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and unstable for $\mu > \mu_* + \delta$. For $|\mu - \mu_*| \leq \delta$ our leading-order approximation (6.1) is insufficient to determine the precise position of the critical spectral curve with respect to the imaginary axis. However, since $\lambda_{\varepsilon,\mu}$ is real-valued for any $\mu \in M$ and Turing instabilities do not occur by Proposition 6.1, we have obtained the following result.

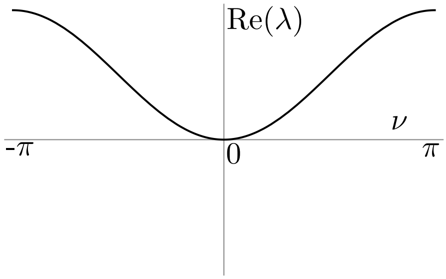
Proposition 6.3. *Assume $m = n = 1$ and **(D1)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (1.9) *destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*



(a) $\mu < \mu_* - \delta$, $\mathfrak{w}(\mu) > 0$



(b) $|\mu - \mu_*| \leq \delta$, $\mathfrak{w}(\mu) \approx 0$



(c) $\mu > \mu_* + \delta$, $\mathfrak{w}(\mu) < 0$

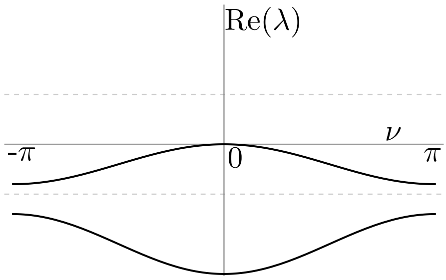
Figure 6.1: The spectral geometry about the origin is depicted in the first generic destabilization scenario **(D1)** with $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) > 0$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

6.3.2 The second destabilization scenario

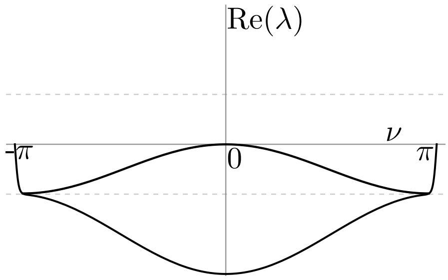
Let **(D2)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{b}(\mu_*) > 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that $\mathfrak{a}(\mu)\mathfrak{w}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0, \delta)$ with $\mathrm{Re}(\lambda) \geq 0$. In addition, it holds $\mathcal{E}_{s,0,\mu}(0, \gamma) \neq 0$ for any $\gamma \in S^1$ and $\mu \in M$ with $\mu < \mu_* - \delta$ by Proposition 3.25. So, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum by Corollary 3.16 for any $\mu \in M$ with $\mu < \mu_* - \delta$. In that situation $\lambda_{\varepsilon,\mu}(\nu)$ is by Proposition 3.29 approximated by (6.1) – see Figure 6.2a. Denote

$$\nu_\diamond(\mu) := \arccos\left(\max\{-1 - 2\mathfrak{a}(\mu)\mathfrak{b}(\mu), -1\}\right), \quad \mu \in M.$$

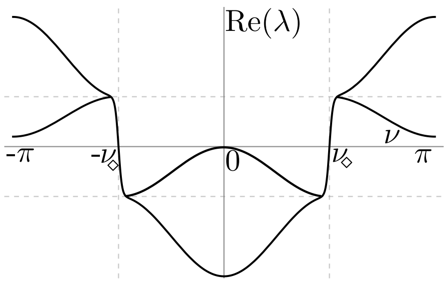
For any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists by Theorem 3.19 and Proposition 3.29 a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0, \delta)$ that is approximated by (6.1) – see Figure 6.2d. Combining this with Proposition 3.25 implies that for any $\mu \in M$ and $\nu \in [-\pi, \pi]$ there are precisely two $e^{i\nu}$-eigenvalues of positive real part if $|\nu| > \nu_\diamond(\mu) + \delta$ and no $e^{i\nu}$-eigenvalues of positive real part if $|\nu| < \nu_\diamond(\mu) - \delta$ – see Figure 6.2c.



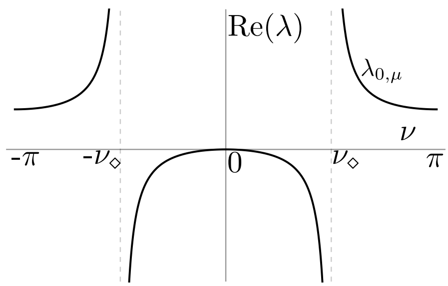
(a) $\mu < \mu_* - \delta$, $\mathfrak{b}(\mu) > 0$



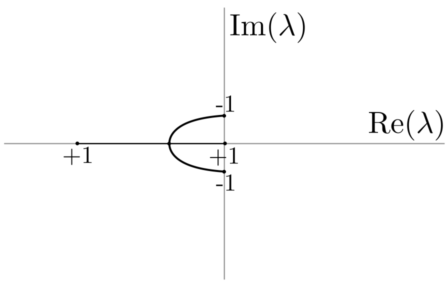
(b) $|\mu - \mu_*| < \delta$, $\mathfrak{b}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



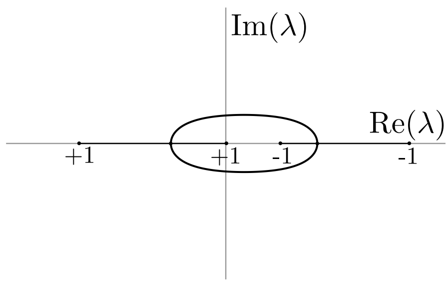
(c) $\mu > \mu_* + \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$ with singularities $\pm\nu_\diamond(\mu)$



(e) Spectrum for $|\mu - \mu_*| < \delta$, $\mathfrak{b}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(f) Spectrum for $\mu > \mu_* - \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$

Figure 6.2: The spectral geometry about the origin is depicted in the second destabilization scenario **(D2)** with $\mathfrak{a}(\mu_*)\partial_\mu\mathfrak{b}(\mu_*) > 0$. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = \mathcal{O}(\varepsilon^2)$.

Therefore, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and there is unstable spectrum for $\mu > \mu_* + \delta$. In particular, we observe that $e^{i\nu}$-eigenvalues with $|\nu \pm \pi| < \delta$ are in the right half-plane strictly before $e^{i\nu}$-eigenvalue with $|\nu| < \delta$ as μ increases. Thus, a sideband instability cannot occur.

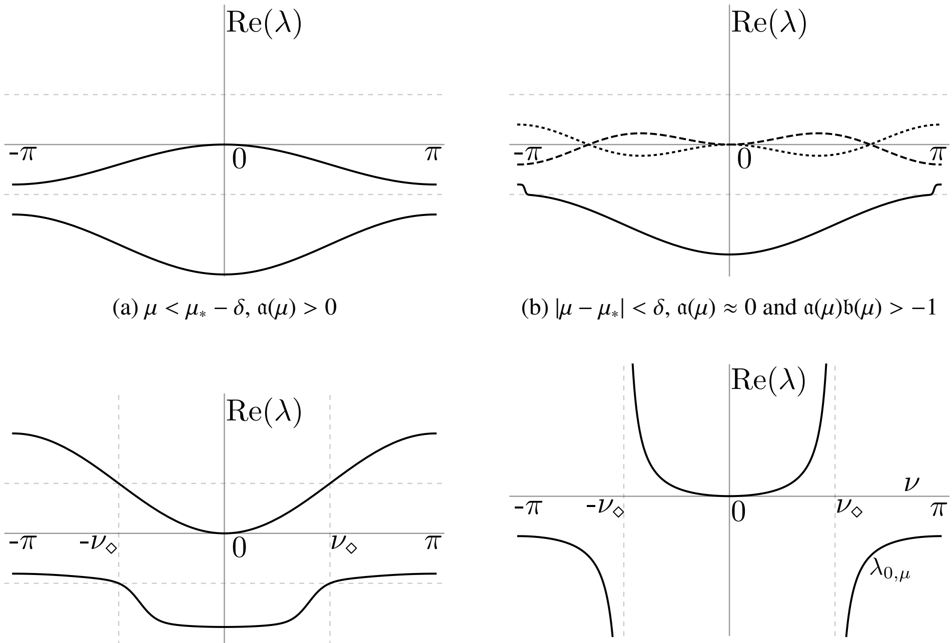
Now suppose a spatial period doubling bifurcation occurs at $\mu = \mu_\varepsilon$. By the previous observations there are precisely two -1-eigenvalues in the right half-plane for $\mu \in M$ with $\mu > \mu_* + \delta \geq \mu_\varepsilon$. By definition of a period doubling bifurcation, the most unstable one of these -1-eigenvalues must have crossed the imaginary axis at the origin. Since the spectrum is symmetric in the real axis – see Proposition 3.7 – the same holds for the other -1-eigenvalue. If the -1-eigenvalues cross simultaneously, then $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ has a root 1 of multiplicity two and a root -1 of multiplicity four, which is impossible, since $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ is a quartic polynomial by Proposition 3.11. If one -1-eigenvalue crosses first, then, by the implicit function theorem and symmetry of the spectrum in the real axis, this -1-eigenvalue is attached to a spectral branch that lies on the real axis. So, if the second -1-eigenvalue crosses at $\mu = \tilde{\mu}_\varepsilon > \mu_\varepsilon$, then $\mathcal{E}_{\varepsilon,\tilde{\mu}_\varepsilon}(0,\cdot)$ has double roots 1 and -1 and simple roots $\gamma^{\pm 1}$ for some $\gamma \in S^1 \setminus \{\pm 1\}$, which is again impossible. We conclude that a period doubling bifurcation cannot occur. So, by Proposition 6.1 a Hopf instability occurs – see Figure 6.2b. Thus, we obtain the following result.

Proposition 6.4. *Assume $m = n = 1$ and **(D2)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) destabilizes through a γ_ε-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon + 1| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

6.3.3 The third destabilization scenario

Let **(D3)** hold true and assume without loss of generality $\mathfrak{w}(\mu_*)\partial_\mu \mathfrak{a}(\mu_*) > 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that $\mathfrak{w}(\mu)\mathfrak{b}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda,\gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0,\delta)$ with $\mathrm{Re}(\lambda) \geq 0$. As in the second destabilization scenario **(D2)**, for any $\mu \in M$ with $\mu < \mu_* - \delta$, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum and it is approximated by (6.1) – see Figure 6.3a. Also similar to scenario **(D2)**, we establish that for any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi,\pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0,\delta)$ that is approximated by (6.1) – see Figure 6.3d. Combining this with Proposition 3.25 implies that for any $\mu \in M$ with $\mu > \mu_* + \delta$ and $\nu \in [-\pi,\pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there is precisely one $e^{i\nu}$-eigenvalue of positive real part. This excludes the possibility of a Hopf destabilization. So, by Proposition 6.1 either a sideband instability or period doubling bifurcation occurs – see Figure 6.3. Thus, we obtain the following result.

Proposition 6.5. *Assume $m = n = 1$ and **(D3)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*



(a) $\mu < \mu_* - \delta$, $\mathfrak{a}(\mu) > 0$

(b) $|\mu - \mu_*| < \delta$, $\mathfrak{a}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$

(c) $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$

(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$ with singularities $\pm\nu_\diamond(\mu)$

Figure 6.3: The spectral geometry about the origin is depicted in the third destabilization scenario **(D3)**. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = \mathcal{O}(\varepsilon^2)$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

6.4 Destabilization mechanisms in the homoclinic limit

In this section we are interested in the destabilization mechanisms of periodic pulse solutions to (1.9) approaching a homoclinic limit. We assume that (1.9) depends on a real parameter μ. It is well-known [39, 99] that the spectral curves corresponding to the periodic pulse shrink to the eigenvalues associated with the limiting homoclinic as the wavelength tends to infinity. This process is of particular interest, when it occurs in the vicinity of a destabilization of the homoclinic pattern.

Generic instabilities of symmetric homoclinic pulse solutions are either of Hopf, saddle-node or pitchfork type [30]. A saddle-node or pitchfork bifurcation occurs if a (simple) real eigenvalue passes through the origin as we vary μ. At a Hopf destabilization a pair of complex conjugate eigenvalues transits through the imaginary axis as we vary μ.

Suppose the homoclinic pulse destabilizes at $\mu = \mu_*$. Since the spectral curves corresponding to a long-wavelength periodic pulse lie close to the eigenvalues associated with the homoclinic, the periodic pulse is also unstable for certain μ-values close to μ_*. However, whether the periodic pulse solution also *destabilizes* at some μ-value close to μ_* depends on the position of the critical spectral curve attached to 0 – see §3.6. We establish that the relative position of the critical curve with respect to the imaginary axis does not change in the homoclinic limit.

If the critical spectral curve is confined to the left half-plane and the homoclinic pulse undergoes a Hopf instability at $\mu = \mu_*$, then the long-wavelength periodic pulse solution also destabilizes at some μ-value close to μ_*. The character of destabilization alternates between two kinds of Hopf instabilities as the wavelength tends to infinity. As explained in the introduction §6.1 the latter is called the 'Hopf dance' and the associated higher order phenomenon the 'belly dance'.

In general it is quite challenging to determine the spectral structure, when a periodic pulse solution approaches a homoclinic limit. However, the spectral reduction mechanisms in Chapter 3 for periodic pulses and in [30] for homoclinic pulses allow us to describe this process in great detail in the singular limit $\varepsilon \to 0$. In this limit it is therefore possible to prove the occurrence of the Hopf and belly dance destabilization mechanisms.

This section is structured as follows. We start by collecting results from the literature concerning the existence and spectral properties of homoclinic pulse solutions to (1.9). Second, we construct a family of periodic pulse solutions to (2.1) that converges to a homoclinic pulse. Third, we study the geometry of the spectral curves associated with the periodic pulses in the long-wavelength limit. Then, using these spectral results, we prove the occurrence of the Hopf and belly dance destabilization mechanisms. In addition, we establish an explicit sign criterion to determine whether the limiting homoclinic pulse solution is the last (or the first) 'periodic' pattern to destabilize in the case of a Hopf destabilization.

6.4.1 Existence of homoclinic pulse solutions

In Chapter 2 we constructed a singular periodic orbit by concatenating a pulse solution to the fast reduced systems (2.2) and an orbit segment on the slow manifold $\mathcal{M}$, satisfying the slow reduced system (2.4), in such a way that they form a closed loop. Then, we proved that an actual periodic pulse solution to (2.1) lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. Similarly, one can construct a singular homoclinic orbit by gluing a pulse solution to the fast reduced system (2.2) to a solution to the slow reduced system (2.4) that converges to a fixed point on $\mathcal{M}$. In the case $m = n = 1$ one proves in [30] the existence of an actual homoclinic solution close to the singular one:

Theorem 6.6. *[30, Theorem 2.1] Let* $m = n = 1$ *and assume* ***(S1)**, **(S2)** and **(E1)** hold true. Suppose there exists a solution* $\check{\psi}_\infty(\check{x}) = (u_\infty(\check{x}), p_\infty(\check{x}))$ *to* (2.4)*, which intersects the touch down curve* $\mathcal{T}_+$ *transversally at* $\check{x} = 0$ *and satisfies* $\lim_{\check{x}\to\infty} p_\infty(\check{x}) = 0$.

Then, for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a homoclinic solution $\phi_{\infty,\varepsilon}(x)$ to (2.1) *satisfying the following assertions:*

1. ***Reversibility***
   We have $\phi_{\infty,\varepsilon}(x) = R\phi_{\infty,\varepsilon}(-x)$ for $x \in \mathbb{R}$, where $R\colon \mathbb{R}^4 \to \mathbb{R}^4$ is the reflection in the space $p = q = 0$.

2. ***Singular limit***
   The Hausdorff distance between the orbit of $\phi_{\infty,\varepsilon}$ in $\mathbb{R}^4$ and the singular concatenation

$$\{(u_\infty(\check{x}), \pm p_\infty(\check{x}), 0, 0) : \check{x} \geq 0\} \cup \{\phi_{\mathrm{h}}(x, u_\infty(0)) : x \in \mathbb{R}\}, \tag{6.2}$$

   is smaller than δ.

6.4.2 Spectral properties of homoclinic pulse solutions

Suppose $\phi_{\infty,\varepsilon}(x)$ is a homoclinic pulse solution established in Theorem 6.6 with singular limit (6.2). Let $\check{\phi}_{\infty,\varepsilon}(\check{x})$ be the corresponding solution to (1.9). To study destabilization mechanisms as periodic pulse solutions to (1.9) approach a homoclinic limit, we need analytical grip on the spectrum of the linearization about $\check{\phi}_{\infty,\varepsilon}$. We linearize system (1.9) about $\check{\phi}_{\infty,\varepsilon}$ and obtain a differential operator $\mathcal{L}_{\infty,\varepsilon}$ on the space $C_{ub}(\mathbb{R}, \mathbb{R}^2)$. By [72, Theorem 3.1.9.ii] and [44, Theorem 1.3.2] $\mathcal{L}_{\infty,\varepsilon}$ is a closed, densely defined and sectorial operator with domain $C^2_{ub}(\mathbb{R}, \mathbb{R}^2)$. The eigenvalue problem $\mathcal{L}_{\infty,\varepsilon}\varphi = \lambda\varphi$ can be written as a first order system,

$$\varphi_x = \mathcal{A}_{\infty,\varepsilon}(x, \lambda)\varphi, \quad \varphi \in \mathbb{R}^4. \tag{6.3}$$

As in Chapter 3, we define an analytic Evans function in terms of (6.3) that locates the (critical) spectrum of $\mathcal{L}_{\infty,\varepsilon}$. Since $\check{\phi}_{\infty,\varepsilon}(x)$ is homoclinic, the limits $\lim_{x\to\pm\infty} \mathcal{A}_{\infty,\varepsilon}(x, \lambda) = \mathcal{A}_{*,\varepsilon}(\lambda)$ exist. Write $u_* = \lim_{\check{x}\to\infty} u_\infty(\check{x})$. Because $(u_*, 0)$ is a hyperbolic saddle in system (2.4), there exists $\Lambda < 0$ such that

$$-\min\{\partial_v G(u_*, 0, 0), \partial_u H_1(u_*, 0, 0)\} < \Lambda < 0.$$

One readily observes that the matrix $\mathcal{A}_{*,\varepsilon}(\lambda)$ is hyperbolic on the half-plane C_Λ. Hence, by Proposition 4.7, system (6.3) admits for $\lambda \in C_\Lambda$ exponential dichotomies on both half-lines $[0, \infty)$ and $(-\infty, 0]$ such that the associated projections are analytic in λ. Note that the dichotomy constants depend on ε and λ. Denote by $\varphi^s_{1,\varepsilon}(x, \lambda)$ and $\varphi^s_{2,\varepsilon}(x, \lambda)$ two solutions that span the space of exponentially decaying solutions to (6.3) as $x \to \infty$. Similarly, let $\varphi^u_{1,\varepsilon}(x, \lambda)$ and $\varphi^u_{2,\varepsilon}(x, \lambda)$ span the space of exponentially decaying solutions as $x \to -\infty$. By [98] the spectrum in C_Λ is located by the analytic Evans function $\mathcal{E}_{\infty,\varepsilon}\colon C_\Lambda \to \mathbb{C}$ given by

$$\mathcal{E}_{\infty,\varepsilon}(\lambda) = \det\left(\varphi^s_{1,\varepsilon}(0, \lambda) \mid \varphi^s_{2,\varepsilon}(0, \lambda) \mid \varphi^u_{1,\varepsilon}(0, \lambda) \mid \varphi^u_{2,\varepsilon}(0, \lambda)\right).$$

More precisely, a point $\lambda \in C_\Lambda$ is in the spectrum $\sigma(\mathcal{L}_{\infty,\varepsilon})$ if and only if we have $\mathcal{E}_{\infty,\varepsilon}(\lambda) = 0$. We emphasize that the spectrum of $\mathcal{L}_{\infty,\varepsilon}$ in C_Λ consists of point spectrum only – see [98].

Similarly to the case of periodic pulse solutions – see Chapter 3 – we can define an explicit reduced Evans function $\mathcal{E}_{\infty,0}\colon C_\Lambda \to \mathbb{C}$, whose zeros approximate those of $\mathcal{E}_{\infty,\varepsilon}$, provided that $\varepsilon > 0$ is sufficiently small. Again, the reduced Evans function reflects the slow-fast structure of the eigenvalue problem (6.3). Thus, the analytic map $\mathcal{E}_{\infty,0}$ is given by the product,

$$\mathcal{E}_{\infty,0}(\lambda) = \mathcal{E}_{\infty,f}(\lambda)\mathcal{E}_{\infty,s}(\lambda). \tag{6.4}$$

Here, the analytic fast Evans function $\mathcal{E}_{\infty,f}\colon C_\Lambda \to \mathbb{C}$ locates the eigenvalues $\lambda \in \mathbb{C}$ of the homogeneous fast eigenvalue problem,

$$\varphi_x = \mathcal{A}_{22,0}(x, u_\infty(0), \lambda)\varphi, \quad \varphi \in \mathbb{C}^2. \tag{6.5}$$

The slow Evans function $\mathcal{E}_{\infty,s}\colon C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0) \to \mathbb{C}$ is defined in terms of the inhomogeneous fast eigenvalue problem (3.8) and the slow eigenvalue problem,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(\check{x}, \lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \qquad \mathcal{A}_\infty(\check{x}, \lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\infty(\check{x}), 0, 0) + \lambda & 0 \end{pmatrix}. \tag{6.6}$$

Note that the coefficient matrix of (6.6) converges as $\check{x} \to \infty$ to the asymptotic matrix,

$$\mathcal{A}_*(\lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_*, 0, 0) + \lambda & 0 \end{pmatrix}, \tag{6.7}$$

which is hyperbolic on C_Λ with eigenvalues $\pm\sqrt{\partial_u H_1(u_*, 0, 0) + \lambda}$. An application of Proposition 4.3 yields a unique analytic solution $\varphi_\infty(\check{x}, \lambda) = (\hat{u}_\infty(\check{x}, \lambda), \hat{p}_\infty(\check{x}, \lambda))$ to (6.6) that satisfies

$$\lim_{\check{x}\to\infty} \hat{u}_\infty(\check{x}, \lambda) e^{\check{x}\sqrt{\partial_u H_1(u_*,0,0)+\lambda}} = 1, \quad \lambda \in C_\Lambda. \tag{6.8}$$

Thus, the slow Evans function is explicitly given by

$$\mathcal{E}_{\infty,s}(\lambda) = \det\left(\varphi_\infty(0, \lambda) \mid \Upsilon(u_\infty(0), \lambda) R_s \varphi_\infty(0, \lambda)\right),$$

where the term $\Upsilon(u, \lambda)$ is defined in (3.11). We emphasize that the slow Evans function $\mathcal{E}_{\infty,s}$ is meromorphic on C_Λ such that the product $\mathcal{E}_{\infty,0}$ given in (6.4) is analytic on C_Λ. Having defined the reduced Evans function $\mathcal{E}_{\infty,0}$, we state the approximation result.

Theorem 6.7. *[30, Section 4] Let Γ be a simple closed curve, contained in $C_\Lambda \setminus \mathcal{E}_{\infty,0}^{-1}(0)$. For $\varepsilon > 0$ sufficiently small, the number of zeros of $\mathcal{E}_{\infty,\varepsilon}$ interior to Γ equals the number of zeros of $\mathcal{E}_{\infty,0}$ interior to Γ including multiplicity.*

By [30, Lemma 5.9] the slow Evans function at 0 can be expressed as

$$\mathcal{E}_{\infty,s}(0) = -2\mathfrak{d}_\infty \mathfrak{a}_\infty, \tag{6.9}$$

with

$$\mathfrak{d}_\infty := -\mathcal{J}(u_\infty(0)), \quad \mathfrak{a}_\infty := \mathcal{J}'(u_\infty(0))\mathcal{J}(u_\infty(0)) - H_1(u_\infty(0), 0, 0), \tag{6.10}$$

where $\mathcal{J}\colon U_{\mathrm{h}} \to \mathbb{R}$ is defined in (2.5). This leads to the following result, whose proof is along the lines of the proof of Proposition 3.32.

Proposition 6.8. *[30, Section 5] There exists a positive root of* $\mathcal{E}_{\infty,s}$ *if it holds* $\mathfrak{a}_\infty \mathfrak{d}_\infty < 0$ *or* $(u_\infty(0) - u_*)\mathfrak{d}_\infty < 0$, *where* $\mathfrak{d}_\infty$ *and* $\mathfrak{a}_\infty$ *are defined in* (6.10).

As in Proposition 3.24 one establishes in [30, Section 4] that the roots of the fast Evans function $\mathcal{E}_{\infty,f}$ are real and simple. In particular, 0 is a root of $\mathcal{E}_{\infty,f}$ and there is precisely one positive zero $\lambda_{\infty,1} > 0$. Let $(v_{\infty,1}(x), q_{\infty,1}(x))$ be an eigenfunction of (6.5) corresponding to $\lambda_{\infty,1}$. By [30, Lemma 5.1] the slow Evans function $\mathcal{E}_{\infty,s}$ has a pole at $\lambda_{\infty,1}$ if and only if the generic condition $\mathfrak{i}_\infty \neq 0$ is satisfied, where

$$\mathfrak{i}_\infty := \hat{u}_\infty(0, \lambda_{\infty,1}) \int_{-\infty}^{\infty} v_{\infty,1}(x) \frac{\partial H_2}{\partial v}(u_\star, v_{\mathrm{h}}(x, u_\star))dx \int_{-\infty}^{\infty} v_{\infty,1}(x) \frac{\partial G}{\partial u}(u_\star, v_{\mathrm{h}}(x, u_\star), 0)dx, \tag{6.11}$$

where $u_\star := u_\infty(0)$ and $\hat{u}_\infty(\check{x}, \lambda)$ denotes the u-coordinate of the solution $\varphi_\infty(\check{x}, \lambda)$ to (6.6). Thus, due to zero-pole cancelation, the reduced Evans function $\mathcal{E}_{\infty,0}$ has a zero at $\lambda_{\infty,1}$ if and only if $\mathfrak{i}_\infty = 0$.

6.4.3 Destabilization mechanisms for homoclinic pulse solutions

We study codimension-one instabilities of the homoclinic pulse solution $\check{\phi}_{\infty,\varepsilon}$ to (1.9), which is established in Theorem 6.6. Since the critical spectrum of $\mathcal{L}_{\infty,\varepsilon}$ is given by $\mathcal{E}_{\infty,\varepsilon}^{-1}(0)$, an instability occurs precisely if a root of the Evans function $\mathcal{E}_{\infty,\varepsilon}$ moves through the imaginary axis as we vary a real parameter μ. By Theorem 6.7 the roots of $\mathcal{E}_{\infty,\varepsilon}$ are approximated by the roots of the reduced Evans function $\mathcal{E}_{\infty,0}(\lambda) = \mathcal{E}_{\infty,f}(\lambda)\mathcal{E}_{\infty,s}(\lambda)$. One establishes in [30, Section 4] that the roots of the fast Evans function $\mathcal{E}_{\infty,f}$ are real and simple and that their relative location with respect to the origin is fixed as we vary μ. In addition, 0 is always a root of $\mathcal{E}_{\infty,f}$. Thus, generic instabilities occur precisely if roots of the slow Evans function $\mathcal{E}_{\infty,s}$ transit through the imaginary axis as we vary μ. Thus, by identity (6.9) we distinguish between the following generic destabilization scenarios:

1. One of the quantities $\mathfrak{a}_\infty$ or $\mathfrak{d}_\infty$, defined in (6.10), changes sign as we vary μ;

2. There is a complex conjugate pair of roots of $\mathcal{E}_{\infty,s}$ moving through the imaginary axis $i\mathbb{R} \setminus \{0\}$ as we vary μ.

In [30] one establishes that the homoclinic pulse undergoes a Hopf destabilization in the second scenario. Moreover, a saddle-node or pitchfork bifurcation occurs if $\mathfrak{a}_\infty$ changes sign.

6.4.4 Existence of a family of periodic pulse solutions approaching a homoclinic limit

In this section we establish with the aid of Theorems 2.3 and 6.6 a family of periodic pulse solutions to (2.1) approaching a homoclinic pulse solution in the long-wavelength limit. Key to the construction of such a family is the existence of a saddle in the slow reduced system (2.4).

(E3) ***Existence of saddle in the slow reduced system***
There exists $u_* \in U$ such that $\psi_* := (u_*, 0)$ is a hyperbolic saddle in (2.4). In addition, the touch-down curve $\mathcal{T}_+ = \{(u, \mathcal{J}(u)) : u \in U_\mathrm{h}\}$ intersects the stable manifold $W^s(\psi_*)$ transversally in some point ψ_0.

Theorem 6.9. *Let $m = n = 1$ and assume **(S1), (S2), (E1)** and **(E3)** hold true. Let $\psi_\infty(\check{x})$ be the solution to* (2.4) *in $W^s(\psi_*)$ with initial condition $\psi_\infty(0) = \psi_0$. There exists $\ell_0, \varepsilon_0 > 0$ such that the following assertions hold true:*

1. ***Saddle dynamics in slow reduced system***
For $\ell \in (\ell_0, \infty)$ there exists a solution $\psi_\ell(\check{x}) = (u_\ell(\check{x}), p_\ell(\check{x}))$ to (2.4) *that intersects $\mathcal{T}_+$ transversally at $\check{x} = 0$ and crosses the line $p = 0$ at $\check{x} = \ell$. In addition, $\psi_\ell(\check{x})$ converges as $\ell \to \infty$ to $\psi_\infty(\check{x})$ for each $\check{x} \in [0, \ell]$.*

2. ***Existence of family of periodic pulse solutions***
For $(\ell, \varepsilon) \in (\ell_0, \infty) \times (0, \varepsilon_0)$ there exists a reversibly symmetric, $2L_{\ell,\varepsilon}$-periodic pulse solution $\phi_{\ell,\varepsilon}$ to (2.1)*, whose orbit converges in the Hausdorff distance to the singular concatenation,*

$$\{(u_\ell(\check{x}), p_\ell(\check{x}), 0, 0) : \check{x} \in (0, 2\ell)\} \cup \{\phi_\mathrm{h}(x, u_\ell(0)) : x \in \mathbb{R}\}, \tag{6.12}$$

as $\varepsilon \to 0$ and whose period satisfies $\varepsilon L_{\ell,\varepsilon} \to \ell$ as $\varepsilon \to 0$.

3. ***Long wavelength limit***
For every $\varepsilon \in (0, \varepsilon_0)$ the family of solutions $\phi_{\ell,\varepsilon}$ converges pointwise on $[0, L_{\ell,\varepsilon}]$ to a reversibly symmetric, homoclinic pulse solution $\phi_{\infty,\varepsilon}$ to (2.1) *as $\ell \to \infty$. Moreover, $\phi_{\infty,\varepsilon}$ converges in Hausdorff distance to the singular concatenation,*

$$\{(u_\infty(\check{x}), \pm p_\infty(\check{x}), 0, 0) : \check{x} \in (0, \infty)\} \cup \{\phi_\mathrm{h}(x, u_\infty(0)) : x \in \mathbb{R}\}, \tag{6.13}$$

as $\varepsilon \to 0$.

Proof. The first assertion is immediate by Hamiltonian nature of the planar system (2.4). For any fixed $\ell > \ell_0$ the existence of a periodic pulse solution $\phi_{\ell,\varepsilon}(x)$ for $0 < \varepsilon \ll 1$ follows from Theorem 2.3. Following the proof of Theorem 2.3, one observes that the ε-bound is in fact ℓ-uniform. This establishes the second assertion. The existence of the homoclinic pulse solution $\phi_{\infty,\varepsilon}(x)$ for $0 < \varepsilon \ll 1$ follows from Theorem 6.6. Now fix $\varepsilon \in (0, \varepsilon_0)$. From the proof of Theorem 2.3 we deduce that the pointwise limits $\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x)$ exist for each $x \in \mathbb{R}$ and must lie on the stable manifold $W^s(\phi_{*,\varepsilon})$ in (2.1), where $\phi_{*,\varepsilon} \in \mathcal{M}$ is a saddle converging to $(\psi_*, 0)$ as $\varepsilon \to 0$. Moreover, the limiting orbit $\{\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x) : x \in \mathbb{R}\}$ is reversibly symmetric. On the other hand, the proof of Theorem 6.6 – see [30, Theorem 2.1] – shows that the 2-dimensional manifold $W^s(\phi_{*,\varepsilon})$ intersects the reversible symmetry plane $p = q = 0$ transversely in $\phi_{\infty,\varepsilon}(0)$. This intersection point is locally unique in a small ε- and ℓ-independent neighborhood of $\phi_{\infty,\varepsilon}(0)$. Thus, we conclude that for $x \in [0, L_{\ell,\varepsilon}]$ the pointwise limits $\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x)$ are given by the homoclinic $\phi_{\infty,\varepsilon}(x)$. □

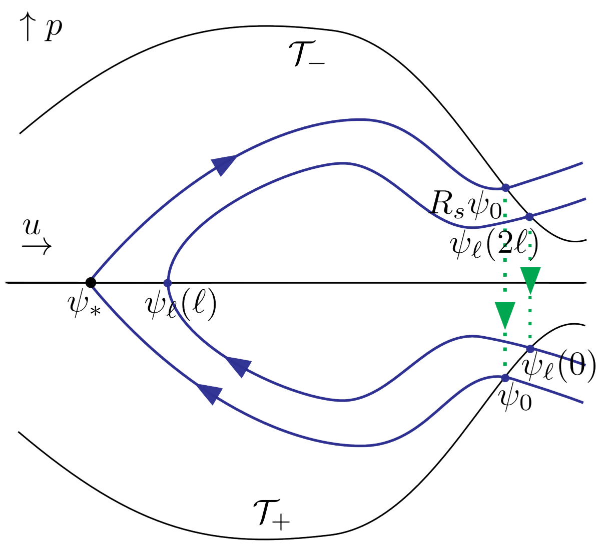


Figure 6.4: Depicted are the orthogonal projections of the singular periodic orbit (6.12) and the singular homoclinic orbit (6.13) onto the slow manifold $\mathcal{M}$ and the take-off and touch-down curves $\mathcal{T}_\pm$.

Remark 6.10. Theorem 6.9 proves that for fixed $\varepsilon \in (0, \varepsilon_0)$ the orbit of the periodic pulse $\phi_{\ell,\varepsilon}$ converges to the orbit of the homoclinic $\phi_{\infty,\varepsilon}$ as $\ell \to \infty$. If we subsequently take the limit $\varepsilon \to 0$, we obtain the singular concatenation (6.13). On the other hand, the orbit of $\phi_{\ell,\varepsilon}$ converges to (6.12) in the limit $\varepsilon \to 0$. Taking subsequently the long-wavelength limit $\ell \to \infty$ yields again (6.13). Thus, we may conclude that the limits $\lim_{\varepsilon\to 0}\lim_{\ell\to\infty}\phi_{\ell,\varepsilon}$ and $\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\phi_{\ell,\varepsilon}$ with respect to Hausdorff metric on $\mathbb{R}^4$ are equal. ■

6.4.5 Spectral geometry of long-wavelength periodic pulse solutions

Let $n = m = 1$ and assume **(S1)**, **(S2)**, **(E1)** and **(E3)** hold true. For fixed $\varepsilon \in (0, \varepsilon_0)$, Theorem 6.9 provides a family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$ to (1.9) converging pointwise to a homoclinic pulse solution $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. For any $\ell \in (\ell_0, \infty)$ we denote by $\mathcal{E}_{\ell,\varepsilon}$ the Evans function associated with the spectrum of the linearization of (1.9) about $\check{\phi}_{\ell,\varepsilon}$ and by

$$\mathcal{E}_{\ell,0}(\lambda, \gamma) = -\gamma\mathcal{E}_{\ell,f}(\lambda)\mathcal{E}_{\ell,s}(\lambda, \gamma),$$

the corresponding reduced Evans function – see §3.4 and §3.5.1.

We are interested in Hopf destabilization of long-wavelength periodic pulses $\check{\phi}_{\ell,\varepsilon}$, $\ell \gg 0$. Such a destabilization is caused by two complex conjugate curves of spectrum moving through the imaginary axis away from the origin – see §6.2. Since these spectral curves converge [39, 99] to the eigenvalues associated with the homoclinic limit as $\ell \to \infty$, Hopf destabilizations of

$\check{\phi}_{\ell,\varepsilon}$ occur in the vicinity of a Hopf instability of $\check{\phi}_{\infty,\varepsilon}$ as long as the critical spectral curve is confined to the left half-plane. Hopf instabilities of the homoclinic pulse occur when a conjugate pair of roots $\lambda_{\infty,\pm}$ of $\mathcal{E}_{\infty,s}$ moves through the imaginary axis.

Thus, to understand the character of the Hopf destabilization of long-wavelength periodic pulses, we need to control three spectral curves. First, we are interested in the position of the critical spectral curve attached to the origin for $\ell \gg 0$. Second, we need to understand the geometry of the spectral curves that shrink to $\lambda_{\infty,\pm}$ as $\ell \to \infty$. The first curve is by Theorem 3.17 and Proposition 3.29 to leading order approximated by the quantity $\lambda_{0,\ell}(\nu)$, defined in (3.31). The other two curves will be embedded in the set $\{\lambda \in \mathbb{C} : \mathcal{E}_{\ell,s}(\lambda,\gamma) = 0, \gamma \in S^1\}$ as $\varepsilon \to 0$ by Theorem 3.14 and Proposition 3.24.

Regarding the first spectral curve, we have the following result.

Theorem 6.11. *Suppose that the quantities* $\mathfrak{a}_\infty$ *and* $\mathfrak{d}_\infty$*, defined in* (6.10)*, are non-zero. Let* $\omega_* := \sqrt{\partial_u H_1(u_*,0,0)}$ *and take* $\varsigma_* \in (0,\omega_*)$*. Then, for* $0 \ll \ell < \infty$*, the analytic curve* $\lambda_{0,\ell}(\nu)$*, given by* (3.31)*, can be expanded in terms of* $e^{-2\omega_*\ell}$ *as*

$$\left|\lambda_{0,\ell}(\nu) - \frac{2\mathfrak{w}_\infty \omega_* e^{-2\omega_*\ell}(\cos(\nu)-1)}{\mathfrak{d}_\infty}\right| \leq Ce^{-(2\omega_*+\varsigma_*)\ell}, \tag{6.14}$$

where $C > 0$ *is independent of* ℓ *and* ν *and*

$$\mathfrak{w}_\infty := -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_\infty(0), v_{\mathrm{h}}(x,u_\infty(0)),0)\partial_x v_{\mathrm{h}}(x,u_\infty(0))x dx}{\int_{-\infty}^{\infty} (\partial_x v_{\mathrm{h}}(x,u_\infty(0)))^2\, dx}. \tag{6.15}$$

Remark 6.12. In [100] one studies the critical spectral curve associated with long-wavelength periodic solutions to reaction-diffusion systems without assuming the presence of a small parameter ε. Thus, the above result could also have been obtained by taking the singular limit $\varepsilon \to 0$ of the expansion in [100, Theorem 5.5]. However, we stress that one should check whether the error estimates in [100] are in fact ε-uniform. ■

The second key result reveals the leading and next order geometry of the other two spectral curves converging to the eigenvalues $\lambda_{\infty,\pm}$ as $\ell \to \infty$.

Theorem 6.13. *Let* $\lambda_\infty \in C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0)$ *be a simple zero of* $\mathcal{E}_{\infty,s}$ *satisfying*

$$-4\mathrm{Re}(\lambda_\infty)\omega_*^2 < \mathrm{Im}(\lambda_\infty)^2, \tag{6.16}$$

where $\omega_* := \sqrt{\partial_u H_1(u_*,0,0)}$*. Define* $\omega_\infty := \sqrt{\partial_u H_1(u_*,0,0)+\lambda_\infty}$*. Take* ς_* *and* ς_∞ *such that*

$$0 < \varsigma_* < \omega_* < \varsigma_\infty < \mathrm{Re}(\omega_\infty).$$

For $0 \ll \ell < \infty$ *there exists an analytic curve* $\lambda_\ell \colon [-1,1] \to \mathbb{C}$ *satisfying the following assertions:*

1. *For each* $\gamma \in S^1$ *the point* $\lambda_\ell(\mathrm{Re}(\gamma))$ *is the unique zero of* $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ *converging to* λ_∞ *as* $\ell \to \infty$*;*

2. *The curve* λ_ℓ *can be expanded in terms of* $e^{-2\omega_*\ell}$ *as*

$$\lambda_\ell(\gamma_r) = \lambda_\infty + L_1 e^{-2\omega_*\ell} + \mathcal{R}_{2,\ell}(\gamma_r),$$
$$L_1 := \frac{2\left(\omega_* \lim_{\check{x}\to\infty} (u_\infty(\check{x}) - u_*)\, e^{\omega_*\check{x}}\right)^2}{\mathfrak{a}_\infty \mathcal{E}'_{\infty,s}(\lambda_\infty)} \Bigg([\hat{u}_\infty(0,\lambda_\infty)]^2\, \partial_u \mathcal{G}(u_\infty(0),\lambda_\infty) \qquad (6.17)$$
$$+\, 2\int_0^\infty \partial_{uu} H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})\,[\hat{u}_\infty(\check{x},\lambda_\infty)]^2\, d\check{x}\Bigg),$$

where $\mathfrak{a}_\infty$ *is defined in* (6.10) *and the remainder* $\mathcal{R}_{2,\ell}(\gamma_r)$ *is bounded as* $|\mathcal{R}_{2,\ell}(\gamma_r)| \leq C\max\left\{e^{-3\varsigma_*\ell}, e^{-2\varsigma_\infty\ell}\right\}$ *with* $C > 0$ *independent of* ℓ *and* γ_r*. Moreover,* $\hat{u}_\infty(\check{x},\lambda)$ *denotes the u-coordinate of the unique solution* $\varphi_\infty(\check{x},\lambda)$ *to* (6.6) *satisfying* (6.8) *and* $\tilde{u}_\infty(\check{x})$ *is the solution to the initial value problem,*

$$\tilde{u}_{\check{x}\check{x}} = \partial_u H_1(u_\infty(\check{x}),0,0)\tilde{u}, \qquad \tilde{u}(0) = 1, \quad \tilde{u}'(0) = \mathcal{J}'(u_\infty(0));$$

3. *The derivatives of* λ_ℓ *at* $\gamma_r \in [-1,1]$ *are approximated by*

$$\left|\lambda'_\ell(\gamma_r) - \frac{4\omega_\infty e^{-2\omega_\infty\ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right| \leq Ce^{-(2\varsigma_\infty+\varsigma_*)\ell},$$
$$\left|\lambda''_\ell(\gamma_r) - \left(\frac{4\omega_\infty e^{-2\omega_\infty\ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right)^2\left(\frac{-2\ell}{\omega_\infty} + \frac{1}{\omega_\infty^2} - \frac{\mathcal{E}''_{\infty,s}(\lambda_\infty)}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right)\right| \leq Ce^{-(4\varsigma_\infty+\varsigma_*)\ell}, \qquad (6.18)$$

with $C > 0$ *independent of* ℓ *and* γ_r*.*

The quantities $\pm\omega_*$ in Theorems 6.11 and 6.13 correspond to the eigenvalues of the linearization about the fixed point $(u_*, 0)$ in the slow reduced system (2.4). Moreover, $\pm\omega_\infty$ are the spatial eigenvalues of the asymptotic system obtained by taking the limit $\check{x} \to \pm\infty$ in the slow eigenvalue problem (6.6) at $\lambda = \lambda_\infty$. Furthermore, the condition (6.16) is equivalent to $\omega_* < \mathrm{Re}(\omega_\infty)$. In particular, any $\lambda_\infty \in i\mathbb{R} \setminus \{0\}$ satisfies (6.16).

The proofs of Theorems 6.11 and 6.13 are provided in §6.4.8.

6.4.6 Spectral stability of long-wavelength periodic pulse solutions

Consider the family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$, established in Theorem 6.9, converging pointwise to the homoclinic limit $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. The fact that the spectral curves corresponding to $\check{\phi}_{\ell,\varepsilon}$ shrink to the eigenvalues associated with the homoclinic $\check{\phi}_{\infty,\varepsilon}$ as $\ell \to \infty$, does not imply that spectral stability properties of the homoclinic are inherited by the periodic pulses – see [100]. This depends on the location of critical spectral curve attached to the origin.

By Theorem 6.11 the relative location of the critical curve with respect to the imaginary axis does not alter as $\ell \to \infty$ under the generic assumption that the quantities $\mathfrak{a}_\infty, \mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, defined in (6.10) and (6.15), are non-zero. Depending on the sign of these quantities, long-wavelength periodic pulses inherit the (spectral) stability properties of the limiting homoclinic.

Corollary 6.14. *Suppose the slow Evans function $\mathcal{E}_{\infty,s}$ has no roots $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) \geq 0$ *and the quantities $\mathfrak{i}_\infty, \mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, defined in* (6.10)*,* (6.11) *and* (6.15)*, are non-zero. Then, there exists $\ell_0 > 0$ such that for each $\ell \in (\ell_0, \infty)$ the following holds true.*

1. *If $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$ have the same sign, then the periodic pulse solution $\check{\phi}_{\ell,\varepsilon}$ to* (1.9) *is spectrally stable, provided $\varepsilon > 0$ is sufficiently small.*
2. *If $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$ have different signs, then $\check{\phi}_{\ell,\varepsilon}$ is spectrally unstable, provided $\varepsilon > 0$ is sufficiently small.*

Proof. Observe that the quantity $\mathfrak{i}_\ell$, defined in (3.36), converges to $\mathfrak{i}_\infty$ as $\ell \to \infty$ by Theorem 6.9. Thus, by Propositions 3.24 and 3.28, $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ has precisely one pole in the right half-plane for any $\gamma \in S^1$ and $\ell > 0$ sufficiently large. In addition, all roots of $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ in the right half-plane converge to roots of $\mathcal{E}_{\infty,s}$ as $\ell \to \infty$ by Theorem 6.13. Therefore, using Proposition 3.24, we conclude that $\mathcal{E}_{\ell,0}(\cdot,\gamma)$ has no roots $\lambda \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(\lambda) \geq 0$ for any $\gamma \in S^1$ and $\ell > 0$ sufficiently large. In addition, 0 is a simple root of $\mathcal{E}_{\ell,f}$ and $\mathcal{E}_{\ell,s}(0,\gamma) \neq 0$ for each $\gamma \in S^1$ and $\ell > 0$ sufficiently large.

Hence, spectral stability is determined by the position of the critical spectral curve $\lambda_{\varepsilon,\ell}(\nu)$ attached to the origin by Proposition 3.16, which is approximated by the curve $\lambda_{0,\ell}(\nu)$, defined in (3.31), by Proposition 3.29. By Theorem 6.11 the sign of $\lambda_{0,\ell}(\nu)$ and its derivatives is determined by the signs of $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, provided $\ell > 0$ is sufficiently large. This proves the result. □

We stress that the conditions in Corollary 6.14 comprise some form of nonlinear stability for the homoclinic $\check{\phi}_{\infty,\varepsilon}$ to (1.9). Indeed, these conditions imply that $\mathcal{E}_{\infty,0}$ has no zeros $\lambda \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(\lambda) \geq 0$ and 0 is a simple root of $\mathcal{E}_{\infty,0}$ – see §6.4.2. Hence, the same holds for $\mathcal{E}_{\infty,\varepsilon}$, provided $\varepsilon > 0$ is sufficiently small, by Theorem 6.7. So, there exists $\beta > 0$ such that all $\lambda \in \sigma(\mathcal{L}_{\infty,\varepsilon}) \setminus \{0\}$ satisfy $\mathrm{Re}(\lambda) < -\beta$ and $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_{\infty,\varepsilon}$. The latter implies by [44, Section 5.1] nonlinear stability with asymptotic phase. On the other hand, spectral (in)stability implies nonlinear (in)stability for the *periodic* pulse solution $\check{\phi}_{\ell,\varepsilon}$ by the analysis in §3.3. Thus, Corollary 6.14 can be employed to test whether or not nonlinear stability of the homoclinic $\check{\phi}_{\infty,\varepsilon}$ implies nonlinear stability of the nearby periodics $\check{\phi}_{\ell,\varepsilon}$, $\ell \gg 0$.

6.4.7 Hopf destabilization in the homoclinic limit

Consider the family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$, established in Theorem 6.9, converging pointwise to the homoclinic limit $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. In this section we study the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$, when the homoclinic $\check{\phi}_{\infty,\varepsilon}$ undergoes a Hopf destabilization. In §6.4.5 we reasoned that the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$ is determined by the geometry of three spectral curves: the critical spectral curve attached to the origin and the two spectral curves converging to the critical eigenvalues associated with the homoclinic. We employ Theorems 6.11 and 6.13 to control these spectral curves.

Thus, let $\lambda_\infty \in C_\Lambda$ be a simple zero of $\mathcal{E}_{\infty,s}$ in the vicinity of the imaginary axis $i\mathbb{R} \setminus \{0\}$ such that $\lambda_\infty \notin \mathcal{E}_{\infty,f}^{-1}(0)$ and the condition (6.16) is satisfied. We infer from Theorem 6.13 that there is a unique curve $\lambda_\ell \colon [-1,1] \to \mathbb{C}$ of zeros of $\mathcal{E}_{\ell,s}$ shrinking to λ_∞ as $\ell \to \infty$ exponentially with rate $-2\omega_*\ell$. By (6.18) the curve λ_ℓ is to leading order a straight line that rotates with frequency $\mathrm{Im}(\omega_\infty)/\pi$ and whose length decays exponentially with rate $-2\mathrm{Re}(\omega_\infty)\ell$ as $\ell \to \infty$. Therefore, the point on the curve with largest real part will generically be one of the endpoints $\lambda_\ell(\pm 1)$. The following result shows that this is actually always the case – see Figure 6.5.

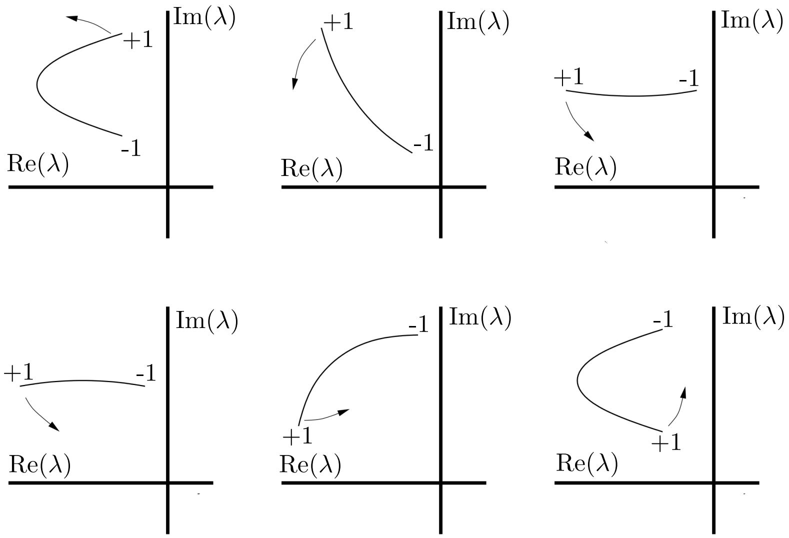


Figure 6.5: Depicted is a series of snapshots of the spectral curve λ_ℓ as ℓ increases. The pictures are corrected for exponential shrinking of the curve. Note that the spectral curve is to leading order a straight line that rotates and its 'belly' always points to the left. The point on the curve with largest real part is always one of the endpoints $\lambda_\ell(\pm 1)$.

Corollary 6.15. *Let* $\lambda_\infty \in C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0)$ *be a simple zero of* $\mathcal{E}_{\infty,s}$ *satisfying* (6.16). *For* $0 \ll \ell < \infty$ *the point of largest real part on* $\lambda_\ell([-1,1])$*, where* $\lambda_\ell \colon [-1,1] \to \mathbb{C}$ *is established in Theorem 6.13, is always one of the endpoints* $\lambda_\ell(\pm 1)$*. In particular, consider the quantity*

$$\chi_\ell := \frac{4\omega_\infty e^{-2\omega_\infty \ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}, \tag{6.19}$$

with $\omega_\infty := \sqrt{\partial_u H_1(u_*,0,0) + \lambda_\infty}$*. If* $\mathrm{Re}(\chi_\ell) \neq 0$*, then* $\lambda_\ell(\mathrm{sgn}(\mathrm{Re}(\chi_\ell)))$ *is the point of largest real part on* $\lambda_\ell([-1,1])$*.*

Proof. By (6.18) the curve $\lambda_\ell(\gamma_r)$ is to leading order a straight line. Its orientation is determined by the argument of the quantity χ_ℓ. Thus, in the case $\chi_\ell \notin i\mathbb{R}$, it is clear that $\lambda_\ell(\mathrm{sgn}(\mathrm{Re}(\chi_\ell)))$ must be the endpoint of largest real part. Now suppose $\chi_\ell \in i\mathbb{R}$. Since λ_∞ is a simple zero of $\mathcal{E}_{\infty,s}$, χ_ℓ is non-zero. Thus, we have $\chi_\ell^2 < 0$. By (6.18) the quadratic deformation of the curve λ_ℓ is to leading order determined by the quantity $-2\chi_\ell^2 \ell \omega_\infty^{-1}$, which has strictly positive real part. Hence, we derive $\mathrm{Re}(\lambda_\ell(\pm 1)) \geq \mathrm{Re}(\lambda_{\ell_v}(\gamma_r))$ for all $\gamma_r \in [-1,1]$. This concludes the proof. □

Now suppose equation (1.9) depends on a real parameter μ. We make the following assumption:

(HO) There is $\mu_* \in \mathbb{R}$ and a unique pair $\pm\lambda_\infty$ with $\lambda_\infty \in i\mathbb{R} \setminus \{0\}$ satisfying $\mathcal{E}_{\infty,s,\mu_*}(\pm\lambda_\infty) = 0$ and

$$\mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{\infty,s,\mu_*}(\lambda_\infty)}{\partial_\lambda \mathcal{E}_{\infty,s,\mu_*}(\lambda_\infty)}\right] < 0.$$

In addition, we have $\mathfrak{i}_\infty(\mu_*) \neq 0$, $\mathfrak{d}_\infty(\mu_*)\mathfrak{w}_\infty(\mu_*) > 0$ and $\mathcal{E}_{\infty,s,\mu_*}(\lambda) \neq 0$ for all $\lambda \in \mathbb{C} \setminus \{\pm\lambda_\infty\}$ with $\mathrm{Re}(\lambda) \geq 0$.

The condition **(HO)** implies that the homoclinic $\check{\phi}_{\infty,\varepsilon}$ undergoes a Hopf destabilization at a μ-value close to μ_* – see §6.4.2 and §6.4.3. The assumption $\mathfrak{d}_\infty(\mu_*)\mathfrak{w}_\infty(\mu_*) > 0$ in **(HO)** yields that the critical spectral curve associated with $\check{\phi}_{\ell,\varepsilon}$ is confined to the left half-plane by Corollary 6.14 for $\ell > 0$ sufficiently large. Hence, the long-wavelength periodic pulse $\check{\phi}_{\ell,\varepsilon}$ also undergoes a Hopf destabilization at a μ-value close to μ_*, since two spectral curves corresponding to $\check{\phi}_{\ell,\varepsilon}$ converge to the critical eigenvalues of the homoclinic $\check{\phi}_{\infty,\varepsilon}$ by Theorems 3.15, 6.7 and 6.13 as $\ell \to \infty$. The (leading-order) geometry of these spectral curves given in Theorem 6.13 and Corollary 6.15 determines the type of Hopf instability and whether the homoclinic pulse solution is the last (or first) periodic pulse to destabilize – see Figure 6.6. Thus, Theorems 3.15, 6.7, 6.11 and 6.13 and Corollary 6.15 yield the following result.

Corollary 6.16. *Assume* ***(HO)*** *and fix* $\delta > 0$*. Then, there exists* $\ell_0 > 0$ *such that for each* $\ell \in (\ell_0, \infty)$ *the following holds true for* $\varepsilon > 0$ *sufficiently small:*

1. *The homoclinic pulse solution* $\check{\phi}_{\infty,\varepsilon}$ *to* (1.9) *undergoes a Hopf destabilization at* $\mu = \mu_{\infty,\varepsilon}$ *with* $|\mu_{\infty,\varepsilon} - \mu_*| < \delta$*;*

2. *The periodic pulse solution* $\check{\phi}_{\ell,\varepsilon}$ *to* (1.9) *undergoes a* γ_ℓ*-Hopf destabilization at* $\mu = \mu_{\ell,\varepsilon}$ *with* $|\mu_{\ell,\varepsilon} - \mu_*| < \delta$*. It holds either* $|\gamma_\ell - 1| < \delta$ *or* $|\gamma_\ell + 1| < \delta$*;*

3. *If the real part of* $\chi_\ell = \chi_\ell(\mu_*)$*, defined in* (6.19)*, is non-zero, then we have* $|\gamma_\ell - \mathrm{sgn}(\mathrm{Re}(\chi_\ell))| < \delta$*;*

4. *If the quantity* $L_1 = L_1(\mu_*)$*, defined in* (6.17)*, is non-zero, then it holds* $\mathrm{sgn}(\mu_{\infty,\varepsilon} - \mu_{\ell,\varepsilon}) = \mathrm{sgn}(L_1)$*, i.e. the homoclinic pulse solution is the last to destabilize if* $L_1 > 0$*.*

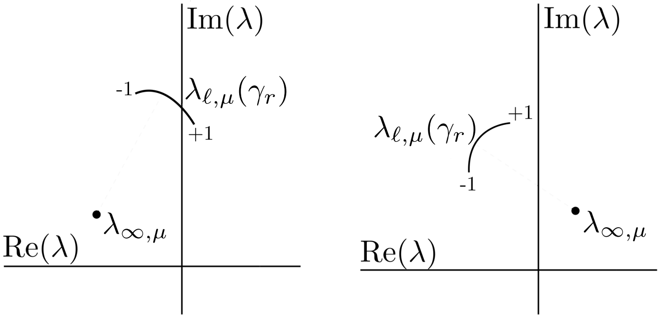


Figure 6.6: The homoclinic pulse solution undergoes a Hopf destabilization at $\mu = \mu_{\infty,\varepsilon}$. We denote by $\lambda_{\ell,\mu}$ the unique spectral curve converging to one of the critical eigenvalues $\lambda_{\infty,\mu}$ as $\ell \to \infty$. The left panel shows the spectral configuration in the case $L_1(\mu_*) > 0$ for $\mu < \mu_{\infty,\varepsilon}$: any (long-wavelength) periodic pulse solution destabilizes at some μ-value $\mu_{\ell,\varepsilon}$ smaller than $\mu_{\infty,\varepsilon}$. The right panel shows the spectral configuration in the case $L_1(\mu_*) < 0$ for $\mu > \mu_{\infty,\varepsilon}$: the homoclinic pulse is unstable, while there are still long-wavelength periodic pulse solutions that are spectrally stable.

Corollary 6.16 implies that, as the wave number $k = \ell^{-1}$ decreases, the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$ alternates between ±1-Hopf instabilities in the limit $\varepsilon \to 0$. This has the following implications for the region of stable pulse solutions in (k, μ)-space, which is also known as the *Busse balloon* [8, 27, 115]. By Corollary 6.16 the boundary $\{(\ell^{-1}, \mu_{\ell,\varepsilon}) : \ell \in (\ell_0, \infty)\}$ of the Busse balloon is in the limit $\varepsilon \to 0$ covered by two curves $\mathcal{H}_{\pm1}$ corresponding to ±1-Hopf instabilities of $\check{\phi}_{\ell,\varepsilon}$. The curves $\mathcal{H}_{\pm,1}$ intersect infinitely often as they oscillate about each other while both converging to the point $\lim_{\varepsilon\to0}(0, \mu_{\infty,\varepsilon}) = (0, \mu_*)$ on the line $k = 0$. Moreover, Corollary 6.15 implies that in the limit $\varepsilon \to 0$ the boundary of the Busse balloon is non-smooth at the intersection points of $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$. Thus, we have established the occurrence of the Hopf and belly dance destabilization mechanisms – see §6.1 – for the general class (1.10) of slowly nonlinear systems.

It was conjectured by W.M. Ni in the context of the Gierer-Meinhardt equations [80] that the homoclinic pulse solution is the last 'periodic' pulse to become unstable as we vary μ – see also [27, Remark 5.4]. Preliminary numerical simulations in the slowly nonlinear toy model (2.27) indicate that there exists parameter regimes, where the quantity L_1, defined in (6.17), has negative sign upon destabilization. This suggests that Ni's conjecture does not hold beyond the slowly linear Gierer-Meinhardt equations. We stress that a structural difference can be readily observed between both cases: the derivative $\partial_{uu}H_1(u_\infty(\check{x}), 0, 0)$ in (6.17) vanishes in the slowly linear case.

6.4.8 Proofs of key results

In this section we prove Theorems 6.11 and 6.13. Our approach is as follows. Let λ_∞ be a simple root of $\mathcal{E}_{\infty,s}$ satisfying (6.16). We want to understand the geometry of the critical curve $\lambda_{0,\ell}(\nu)$, defined in (3.31), and of the unique solution curve $\lambda_\ell(\gamma)$, satisfying $\mathcal{E}_{\ell,s}(\lambda_\ell(\gamma), \gamma) = 0$ for each $\gamma \in S^1$, which converges to λ_∞ as $\ell \to \infty$. By Propositions 3.25 and 3.29 we have

$$\lambda_{0,\ell}(\nu) = \mathfrak{a}_\ell \mathfrak{w}_\ell \frac{\cos(\nu) - 1}{2e^{-i\nu}\mathcal{E}_{\ell,s}(0, e^{i\nu})}, \tag{6.20}$$

where

$$\begin{aligned} \mathfrak{a}_\ell &:= \mathcal{J}'(u_\ell(0))\mathcal{J}(u_\ell(0)) - H_1(u_\ell(0), 0, 0), \\ \mathfrak{w}_\ell &:= -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_\ell(0), v_{\mathrm{h}}(x, u_\ell(0)), 0)\partial_x v_{\mathrm{h}}(x, u_\ell(0)) x dx}{\int_{-\infty}^{\infty} [\partial_x v_{\mathrm{h}}(x, u_\ell(0))]^2 \, dx}. \end{aligned}$$

One readily observes $\mathfrak{a}_\ell \to \mathfrak{a}_\infty$ and $\mathfrak{w}_\ell \to \mathfrak{w}_\infty$ as $\ell \to \infty$ by Theorem 6.9. Thus, to prove Theorems 6.11 and 6.13, we need to relate the periodic slow Evans function $\mathcal{E}_{\ell,s}$ to the homoclinic slow Evans function $\mathcal{E}_{\infty,s}$. The homoclinic slow Evans function $\mathcal{E}_{\infty,s}$ is defined in terms of the unique solution $\varphi_\infty(\check{x}, \lambda)$ to the *homoclinic slow eigenvalue problem* (6.6) that satisfies (6.8). Our approach is to find an analytic solution $\varphi_\ell(\check{x}, \lambda)$ to the *periodic slow eigenvalue problem*,

$$\varphi_{\check{x}} = \mathcal{A}_\ell(\check{x}, \lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \qquad \mathcal{A}_\ell(\check{x}, \lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\ell(\check{x}), 0, 0) + \lambda & 0 \end{pmatrix}, \tag{6.21}$$

which is (pointwise) close to $\varphi_\infty(\check{x}, \lambda)$ and decays exponentially on $[0, 2\ell]$. Recall from §3.8.1 that system (6.21) is R_s-reversible at $\check{x} = \ell$, i.e. the evolution $\mathcal{T}_\ell(\check{x}, \check{y}, \lambda)$ of (6.21) satisfies $R_s\mathcal{T}_\ell(\check{x}, \check{y}, \lambda)R_s = \mathcal{T}_\ell(2\ell - \check{x}, 2\ell - \check{y}, \lambda)$ for $\check{x}, \check{y} \in [0, 2\ell]$. In particular, $\varphi_\ell^r(\check{x}, \lambda) := R_s\varphi_\ell(2\ell - \check{x}, \lambda)$ is also a solution to (6.21). Now, to relate the periodic slow Evans function $\mathcal{E}_{\ell,s}$ to $\mathcal{E}_{\infty,s}$, we multiply $\mathcal{E}_{\ell,s}(\lambda, \gamma)$ with the ($\check{x}$-independent) Wronskian $\mathcal{W}_\ell(\lambda) := \det(\varphi_\ell(\check{x}, \lambda) \mid \varphi_\ell^r(\check{x}, \lambda))$. Using the 2-linearity of the determinant and $\det(\Upsilon(u, \lambda)), \det(\mathcal{T}_\ell(\check{x}, \check{y}, \lambda)) = 1$ for all $\check{x}, \check{y} \in [0, 2\ell]$, $\lambda \in C_\Lambda$ and $u \in U_{\mathrm{h}}$, we derive the key identity,

$$\gamma^{-1}\mathcal{E}_{\ell,s}(\lambda, \gamma)\mathcal{W}_\ell(\lambda) := 2\mathrm{Re}(\gamma)\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda), \tag{6.22}$$

where $\mathcal{K}_\ell \colon C_\Lambda \to \mathbb{C}$ is defined by

$$\begin{aligned} \mathcal{K}_\ell(\lambda) = {} & \det(\varphi_\ell(0, \lambda) \mid \Upsilon(u_\ell(0), \lambda)R_s\varphi_\ell(0, \lambda)) \\ & + \det(\Upsilon(u_\ell(0), \lambda)\varphi_\ell(2\ell, \lambda) \mid R_s\varphi_\ell(2\ell, \lambda)). \end{aligned} \tag{6.23}$$

Since $\varphi_\ell(\check{x},\lambda)$ decays exponentially as $\check{x}\to\infty$, one observes that the right hand side of (6.22) converges to the homoclinic slow Evans function $\mathcal{E}_{\infty,s}(\lambda)$ as $\ell\to\infty$. This leads to the desired approximation (6.14) of $\lambda_{0,\ell}(\nu)$ in Theorem 6.11.

To prove Theorem 6.13, we apply the implicit function theorem on (6.22). This yields the existence of a curve $\lambda_\ell\colon[-1,1]\to\mathbb{C}$ such that for each $\gamma\in S^1$ the point $\lambda_\ell(\mathrm{Re}(\gamma))$ is the unique zero of $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ converging to λ_∞ as $\ell\to\infty$. To calculate the leading-order difference $\lambda_\ell(\mathrm{Re}(\gamma))-\lambda_\infty$ in order to prove (6.17), we need the leading order of the differences $\varphi_\ell(\check{x},\lambda)-\varphi_\infty(\check{x},\lambda)$ and $\psi_\ell(\check{x})-\psi_\infty(\check{x})$ of the solutions to the slow eigenvalue problems and the slow reduced system, respectively. Finally, identity (6.18) is proved by implicit differentiation of identity (6.22).

Thus, the set-up of this section is as follows. First, we will establish a leading-order expression for the difference $\psi_\ell(\check{x})-\psi_\infty(\check{x})$ of the solutions to the slow reduced system (2.4). This allows us to approximate $u_\ell(0)$ by $u_\infty(0)$ in (6.23). Second, we construct the desired solution $\varphi_\infty(\check{x},\lambda)$ to (6.21) that is close to the solution $\varphi_\infty(\check{x},\lambda)$ to (6.6) and decays exponentially on $[0,2\ell]$. At the same time, we establish a leading-order expression for the difference $\varphi_\ell(\check{x},\lambda)-\varphi_\infty(\check{x},\lambda)$. Finally, we provide the proofs of Theorems 6.11 and 6.13 using the approach described above.

Approximations in the slow reduced subsystem

We start by collecting some basic facts for the situation described in §6.4.4. Recall the definition of ς_* and ω_* provided in Theorems 6.11 and 6.13. Since $\psi_*=(u_*,0)$ is a hyperbolic saddle in (2.4) by **(E3)**, we have

$$\|\psi_\infty(\check{x})-\psi_*\|\leq Ce^{-\varsigma_*\check{x}},\quad \check{x}\geq 0,\tag{6.24}$$

where $C>0$ is a constant. The eigenvectors of the linearization of (2.4) about ψ_* are given by $w_\pm:=(1,\pm\omega_*)$. We obtain by the stable manifold theorem:

$$\left\|e^{\omega_*\check{x}}(\psi_\infty(\check{x})-\psi_*)-\alpha_*w_-\right\|,\left\|e^{\omega_*\check{x}}\psi_\infty'(\check{x})+\alpha_*\omega_*w_-\right\|\leq Ce^{-\varsigma_*\check{x}},\quad \check{x}\geq 0,\tag{6.25}$$

where $\alpha_*\in\mathbb{R}\setminus\{0\}$ is given by

$$\alpha_*:=\lim_{\check{x}\to\infty}e^{\omega_*\check{x}}(u_\infty(\check{x})-u_*).$$

It is well-known that in a neighborhood of the point ψ_* one can give growth and decay rates of solutions to the (un)stable manifolds, see for example [56, Proposition 3.1]. Using these bounds one can estimate the distance between ψ_ℓ and ψ_∞ in terms of the 'time of flight' ℓ. Indeed, it holds for $0\ll\ell<\infty$

$$\|\psi_\ell(\check{x})-\psi_\infty(\check{x})\|\leq Ce^{-\varsigma_*(2\ell-\check{x})},\quad \check{x}\in[0,2\ell],\tag{6.26}$$

with $C>0$ a constant independent of ℓ.

We need a leading-order expression for the difference $\psi_\ell(\check{x})-\psi_\infty(\check{x})$. Identity (6.26) gives an a priori estimate for this quantity, which is used in the proof of the next proposition.

Proposition 6.17. *For $0 \ll \ell < \infty$ we have the following expansion,*

$$\psi_\ell(\check{x}) = \psi_\infty(\check{x}) - \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix} 1 \\ \mathcal{J}'(u_\infty(0)) \end{pmatrix} + \mathcal{R}_{1,\ell}(\check{x}), \quad \check{x} \in [0,\ell], \tag{6.27}$$

where $\mathfrak{a}_\infty$ is defined in (6.10) *and the remainder $\mathcal{R}_\ell \colon [0,\ell] \to \mathbb{C}^2$ is bounded by $\|\mathcal{R}_\ell(\check{x})\| \leq Ce^{-\varsigma_*(3\ell-\check{x})}$ with $C > 0$ independent of ℓ, and $\Phi_\infty(\check{x},\check{y})$ denotes the evolution operator of the variational equation of* (2.4) *about ψ_∞,*

$$\theta_{\check{x}} = \mathcal{A}_\infty(\check{x})\theta, \quad \theta \in \mathbb{R}^2, \qquad \mathcal{A}_\infty(\check{x}) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\infty(\check{x}),0,0) & 0 \end{pmatrix}. \tag{6.28}$$

Proof. In the following, we denote by $C > 0$ a constant independent of ℓ.

Define $\theta_\ell(\check{x}) = \psi_\ell(\check{x}) - \psi_\infty(\check{x})$ for $\check{x} \in [0,\ell]$. Our approach is to obtain a leading-order expression for $\theta_\ell(\check{x})$ using Lin's method [70, 118]. Note that θ_ℓ solves the boundary value problem,

$$\theta_{\check{x}} = \mathcal{A}_\infty(\check{x})\theta + g_0(\theta,\check{x}),$$

$$\theta(0) + \psi_\infty(0) \in T_+, \tag{6.29}$$

$$\theta(\ell) + \psi_\infty(\ell) \in \ker(I - R_s), \tag{6.30}$$

where $g_0 \colon \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$g_0(\theta,\check{x}) := f(\psi_\infty(\check{x}) + \theta) - f(\psi_\infty(\check{x})) - \mathcal{A}_\infty(\check{x})\theta.$$

Our plan is to study the inhomogeneous equation,

$$\theta_{\check{x}} = \mathcal{A}_\infty(\check{x})\theta + g(\check{x}), \quad \theta \in \mathbb{R}^2. \tag{6.31}$$

with $g \in C([0,\ell],\mathbb{R}^2)$ first. Using the exponential dichotomy of the variational equation, we construct a solution operator to (6.31). Subsequently, we substitute $g_0(\theta,\check{x})$ for $g(\check{x})$ and formulate an integral formulation for $\theta_\ell(\check{x})$ that is of fixed point type. This enables us to obtain a leading-order expression for $\theta_\ell(\check{x})$.

We establish an exponential dichotomy for the variational equation (6.28). First, the matrix function $\mathcal{A}_\infty(\check{x})$ converges as $\check{x} \to \infty$ to the asymptotic matrix $\mathcal{A}_*$. More precisely, by (6.24) it holds for $\check{x} \geq 0$

$$\|\mathcal{A}_\infty(\check{x}) - \mathcal{A}_*\| \leq Ce^{-\varsigma_*\check{x}}.$$

Second, the derivative $\psi'_\infty(\check{x})$ is a solution to (6.28), which is bounded as $\check{x} \to \infty$. Combining these items with Proposition 4.7 yields an exponential dichotomy of (6.28) on $[0,\infty)$ with constants $C, \varsigma_* > 0$ and projections $P_\infty(\check{x})$. By Lemma 4.5 we may without loss of generality assume that $P_\infty(0)$ is the projection on $\mathrm{Sp}(\psi'_\infty(0))$ along $\mathrm{Sp}(1, \mathcal{J}'(u_\infty(0)))$, since the stable manifold $W^s(\psi_*)$ intersects the touch-down curve $\mathcal{T}_+$ transversally in $\psi_\infty(0)$ by **(E3)**. In addition, Lemma 4.6 yields the estimate,

$$\|P_\infty(\check{x}) - P_*\| \leq Ce^{-\varsigma_*\check{x}}, \quad \check{x} \geq 0, \tag{6.32}$$

where P_* denotes the spectral projection of $\mathcal{A}_*$ on $\mathrm{Sp}(w_-)$ along $\mathrm{Sp}(w_+)$.

We proceed by constructing a solution operator to the boundary value problem (6.29)-(6.30). Denote by $\Phi_\infty^{u,s}(\check{x},\check{y})$ the (un)stable evolution operator of (6.28) under the exponential dichotomy. The bounded, linear solution operator $W_\ell\colon \ker(P_*) \times P_\infty(0)[\mathbb{R}^2] \times C([0,\ell],\mathbb{R}^2) \to C([0,\ell],\mathbb{R}^2)$ given by

$$W_\ell(a,b,g)[\check{x}] = \Phi_\infty^u(\check{x},\ell)a + \Phi_\infty^s(\check{x},0)b + \int_0^{\check{x}} \Phi_\infty^s(\check{x},z)g(z)dz - \int_{\check{x}}^{\ell} \Phi_\infty^u(\check{x},z)g(z)dz,$$

solves (6.31). Since G is C^3 on its domain by **(S1)**, the homoclinic solution $\kappa_\mathrm{h}(x,u) = (v_\mathrm{h}(x,u), q_\mathrm{h}(x,u))$ to (2.3) is C^3 on its domain $\mathbb{R} \times U_\mathrm{h}$. Therefore, $\mathcal{J}$ is C^3 on U_h. We expand $\mathcal{J}(u)$ in the neighborhood U_h of $u_\infty(0)$ with Taylor's Theorem as

$$\mathcal{J}(u) = \mathcal{J}(u_\infty(0)) + \mathcal{J}'(u_\infty(0))(u - u_\infty(0)) + h(u - u_\infty(0)), \quad u \in U_\mathrm{h},$$

where $h(u - u_\infty(0)) \leq C|u - u_\infty(0)|^2$. Since $\psi_\infty(0)$ equals $(u_\infty(0), \mathcal{J}(u_\infty(0))) \in T_+$, $\theta(\check{x}) = W_\ell(a,b,g)[\check{x}]$ satisfies condition (6.29) if and only if there exists $\rho \in U_\mathrm{h} - u_\infty(0)$ such that

$$\Phi_\infty^u(0,\ell)a + b - \int_0^\ell \Phi_\infty^u(0,z)g(z)dz = \rho\begin{pmatrix} 1 \\ \mathcal{J}'(u_\infty(0)) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\rho) \end{pmatrix}. \tag{6.33}$$

For a vector $w := (w_1, w_2) \in \mathbb{R}^2$ we denote by $w^\perp$ the vector $(-w_2, w_1)$, which is perpendicular to w. Taking the inner product on both sides of (6.33) with $\psi_\infty'(0)^\perp$ yields

$$\left\langle \Phi_\infty^u(0,\ell)a - \int_0^\ell \Phi_\infty^u(0,z)g(z)dz, \psi_\infty'(0)^\perp \right\rangle = \rho\mathfrak{a}_\infty + h(\rho)u_\infty'(0). \tag{6.34}$$

Since $\mathcal{T}_+$ intersects the stable manifold $W^s(\psi_*)$ transversally by **(E3)**, the quantity $\mathfrak{a}_\infty$ is non-zero. Therefore, the right hand side of (6.34) defines an invertible function in ρ on a neighborhood of 0. Hence, there exists an ℓ-independent neighborhood A_0 of $0 \in \ker(P_0) \times C([0,\ell],\mathbb{R}^2)$ and a Lipschitz continuous map $\rho\colon A_0 \to \mathbb{R}$ such that $\rho(a,g)$ satisfies (6.34) and is bounded by

$$|\rho(a,g)| \leq C(e^{-\varsigma_*\ell}\|a\| + \|g\|). \tag{6.35}$$

Now substitute $\rho(a,g)$ in (6.33) and apply $P_\infty(0)$ on both sides. This gives rise to Lipschitz continuous map $b\colon A_0 \to P_\infty(0)[\mathbb{R}^2]$ satisfying

$$b(a,g) = \frac{-h(\rho(a,g))}{\mathfrak{a}_\infty}\psi_\infty'(0), \quad \|b(a,g)\| \leq C(e^{-\varsigma_*\ell}\|a\| + \|g\|)^2, \tag{6.36}$$

using that $P_\infty(0)$ projects on $\mathrm{Sp}(\psi_\infty'(0))$ along $\mathrm{Sp}(1,\mathcal{J}'(u_\infty(0)))$. By construction $\theta[\check{x}] = W_\ell(a,b(a,g),g)[\check{x}]$ satisfies (6.33) and thus (6.29). Similarly, $\theta[\check{x}] = W_\ell(a,b(a,g),g)[\check{x}]$ satisfies condition (6.30) if there exists $\beta \in \mathbb{R}$ such that

$$(I - P_\infty(\ell))a + \Phi_\infty^s(\ell,0)b(a,g) + \int_0^\ell \Phi_\infty^s(\ell,z)g(z)dz + \psi_\infty(\ell) - \psi_* = \beta\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6.37}$$

By estimate (6.32) it holds

$$\|(I - P_\infty(\ell))w_+ - w_+\| \leq Ce^{-\varsigma_*\ell}, \tag{6.38}$$

Estimate (6.38) shows that the inner product $\langle\begin{pmatrix} 1 \\ 0 \end{pmatrix}, [(I - P_\infty(\ell))w_+]^\perp\rangle$ is to leading order given by the non-zero quantity $-\omega_*$. Thus, taking the inner product on both sides of (6.37) with $[(I - P_\infty(\ell))w_+]^\perp$ yields a Lipschitz continuous map $\beta\colon A_0 \to \mathbb{R}$ given by

$$\beta(a,g) = \frac{\left\langle \Phi_\infty^s(\ell,0)b(a,g) + \int_0^\ell \Phi_\infty^s(\ell,z)g(z)dz + \psi_\infty(\ell) - \psi_*, [(I - P_\infty(\ell))w_+]^\perp \right\rangle}{\left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, [(I - P_\infty(\ell))w_+]^\perp \right\rangle},$$

satisfying for $(a,g),(a_1,g) \in A_0$

$$|\beta(a,g)| \leq C(e^{-\varsigma_*\ell} + \|g\| + e^{-2\varsigma_*\ell}\|a\|), \quad |\beta(a,g) - \beta(a_1,g)| \leq Ce^{-\varsigma_*\ell}\|a - a_1\|, \tag{6.39}$$

by estimate (6.24). Now substitute $\beta(a,g)$ in (6.37) and apply $I - P_\infty(\ell)$ on both sides. This yields

$$a = (P_\infty(\ell) - P_*)a - (I - P_\infty(\ell))\left[\psi_\infty(\ell) - \psi_* - \beta(a,g)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] \tag{6.40}$$

One readily verifies that the right hand side of (6.40) defines a contraction mapping in a for $\ell > 0$ sufficiently large, using estimates (6.32) and (6.39). Therefore, there exists by the Banach fixed point theorem an ℓ-independent neighborhood A_b of $0 \in C([0,\ell],\mathbb{R}^2)$ and a Lipschitz continuous map $a\colon A_b \to \ker(P_*)$ such that $a(g)$ satisfies equation (6.40) for each $g \in A_b$. The map a enjoys the bound

$$\|a(g)\| \leq C(e^{-\varsigma_*\ell} + \|g\|) \tag{6.41}$$

We conclude that the Lipschitz continuous map $W_{1,\ell}\colon A_b \to C([0,\ell],\mathbb{R}^2)$ given by $W_{1,\ell}(g) = W_\ell(a(g),b(a(g),g),g)$ satisfies (6.29)-(6.31). Therefore, θ_ℓ is the unique solution to the fixed point problem

$$\theta = W_{1,\ell}(g_0(\theta,\cdot)). \tag{6.42}$$

By shrinking A_b if necessary, it is not difficult to verify that the right hand side of (6.42) defines indeed a contraction mapping in $\theta \in C([0,\ell],\mathbb{R}^2)$.

Finally, the above fixed point arguments provide a mechanism to expand θ_ℓ in terms of $\ell \gg 0$. The first observation is that a priori the norm of $\theta_\ell(\check{x})$ is bounded by $Ce^{-\varsigma_*(2\ell-\check{x})}$ by estimate (6.26). Thus, the map $\hat{g}\colon [0,\ell] \to \mathbb{R}^2$ defined by $\hat{g}(\check{x}) = g_0(\theta_\ell(\check{x}),\check{x})$ is bounded by $Ce^{-2\varsigma_*(2\ell-\check{x})}$. We invoke the bounds (6.35), (6.36), (6.39) and (6.41) on the maps ρ, b, β and a to obtain the estimates

$$\|a(\hat{g})\| \leq Ce^{-\varsigma_*\ell}, \quad |\rho(a(\hat{g}),\hat{g})| \leq Ce^{-2\varsigma_*\ell},$$
$$\|b(a(\hat{g}),\hat{g})\| \leq Ce^{-4\varsigma_*\ell}, \quad |\beta(a(\hat{g}),\hat{g})| \leq Ce^{-\varsigma_*\ell}.$$

Combining the latter estimates with (6.25), (6.32) and (6.38) results in the expansions

$$\beta(a(\hat{g}),\hat{g}) = \frac{\alpha_* \left\langle w_-, w_+^\perp \right\rangle e^{-\omega_*\ell}}{\left\langle \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), w_+^\perp \right\rangle} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right) = 2\alpha_* e^{-\omega_*\ell} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right),$$
$$a(\hat{g}) = (I - P_*)\left[\beta(a(\hat{g}),\hat{g})\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right] + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right) = \alpha_* w_+ e^{-\omega_*\ell} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right).$$

Substituting these expansions in $\theta_\ell = W_\ell(a(\hat{g}), b(a(\hat{g}),\hat{g}),\hat{g})$ yields

$$\psi_\ell(\check{x}) = \psi_\infty(\check{x}) + \alpha_*\Phi_\infty^u(\check{x},\ell)w_+e^{-\omega_*\ell} + \mathcal{O}\left(e^{-\varsigma_*(3\ell-\check{x})}\right), \quad \check{x} \in [0,\ell]. \tag{6.43}$$

Note that $P_\infty(\check{x})$ is the projection on $\mathrm{Sp}(\psi_\infty'(\check{x}))$ along $\mathrm{Sp}\left(\Phi_\infty(\check{x},0)\left(\begin{smallmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{smallmatrix}\right)\right)$. Thus, we estimate with the aid of (6.25)

$$\begin{aligned}\Phi_\infty^u(\check{x},\ell)w_+ &= \frac{\langle w_+, \psi_\infty'(\ell)^\perp\rangle}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{pmatrix}\\ &= \frac{-2\omega_*^2\alpha_*e^{-\omega_*\ell}}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{pmatrix} + \mathcal{O}\left(e^{-\varsigma_*(2\ell-\check{x})}\right),\end{aligned} \tag{6.44}$$

for $\check{x} \in [0,\ell]$. Combining (6.43) and (6.44) yields (6.27). □

Remark 6.18. The proof of Proposition 6.17 is based on [118, Theorem 6]. The fundamental difference with [118] is that it is not the existence of θ_ℓ that is of our interest, but the leading-order behavior. Moreover, we have nonlinear boundary conditions in contrast to [118]. ■

Approximation in slow eigenvalue problems

We proceed by constructing an analytic solution $\varphi_\ell(\check{x},\lambda)$ to (6.21) that is close to the solution $\varphi_\infty(\check{x},\lambda)$ to (6.6) and decays exponentially on $[0,2\ell]$. At the same time, we establish a leading-order expression for the difference $\varphi_\ell(\check{x},\lambda) - \varphi_\infty(\check{x},\lambda)$. We start by collecting some facts about the solution $\varphi_\infty(\check{x},\lambda)$ to (6.6). Recall that the coefficient matrix of (6.6) converges as $\check{x} \to \infty$ to the asymptotic matrix $\mathcal{A}_*(\lambda)$, defined in (6.7), which is hyperbolic on C_Λ. The eigenvalues of $\mathcal{A}_*(\lambda)$ are given by $\pm\omega(\lambda)$ and corresponding eigenvectors are $v_\pm(\lambda) := (1, \pm\omega(\lambda))$, where

$$\omega(\lambda) := \sqrt{\partial_u H_1(u_*,0,0) + \lambda},$$

denotes the principal square root. Note that both $\omega(\lambda)$ and $v_\pm(\lambda)$ are analytic on C_Λ. Choose an open and bounded subset $C_{b,\Lambda} \subset C_\Lambda$. An application of Proposition 4.3 yields the following estimate,

$$\|e^{\omega(\lambda)\check{x}}\varphi_\infty(\check{x},\lambda) - v_-(\lambda)\| \leq Ce^{-\varsigma_*\check{x}}, \quad \check{x} \geq 0, \lambda \in C_{b,\Lambda}, \tag{6.45}$$

where $C > 0$ is a constant independent of λ.

We are now ready to prove the existence of the desired solution $\varphi_\ell(\check{x}, \lambda)$ to (6.21). To state the result, we take $\delta > 0$ such that we have

$$\mu(\lambda) := \mathrm{Re}(\omega(\lambda)) - \delta > 0,$$

for all λ in the bounded set $C_{b,\Lambda}$.

Proposition 6.19. *For $0 \ll \ell < \infty$, there exists a solution $\varphi_\ell \colon [0, 2\ell] \times C_{b,\Lambda} \to \mathbb{C}^2$ to the periodic slow eigenvalue problem* (6.21)*, satisfying the bounds*

$$\begin{aligned} \|\varphi_\ell(\check{x}, \lambda)\| &\leq C e^{-\mu(\lambda)\check{x}}, \\ \|\varphi_\ell(0, \lambda) - \varphi_\infty(0, \lambda)\| &\leq C e^{-2\min\{\varsigma_*, \mu(\lambda)\}\ell}, \\ \|\varphi_\ell(\ell, \lambda) - \varphi_\infty(\ell, \lambda)\| &\leq C e^{-(\varsigma_* + \mu(\lambda))\ell}, \end{aligned} \qquad \begin{aligned} \check{x} &\in [0, 2\ell], \\ \lambda &\in C_{b,\Lambda}, \end{aligned} \tag{6.46}$$

where $C > 0$ is a constant independent of ℓ and λ. Moreover, $\varphi_\ell(\check{x}, \cdot)$ is analytic on $C_{b,\Lambda}$ for each $\check{x} \in [0, 2\ell]$. Finally, we have the expansion for $\lambda \in C_{b,\Lambda}$

$$\begin{aligned} &\varphi_\ell(0, \lambda) - \varphi_\infty(0, \lambda) = \\ &\qquad \int_0^\ell Q_\infty(\lambda) \mathcal{T}_\infty(0, \check{y}, \lambda) \left[\mathcal{A}_\ell(\check{x}, \lambda) - \mathcal{A}_\infty(\check{x}, \lambda)\right] \varphi_\infty(\check{y}, \lambda) d\check{y} + \mathcal{R}_{1,\ell}(\lambda), \end{aligned} \tag{6.47}$$

where $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda)$ denotes the evolution operator of system (6.6)*, $Q_\infty(\lambda)$ is an analytic projection along* $\mathrm{Sp}(\varphi_\infty(0, \lambda))$ *and the remainder $\mathcal{R}_{1,\ell} \colon C_{b,\Lambda} \to \mathbb{C}^2$ is bounded as $\|\mathcal{R}_{1,\ell}(\lambda)\| \leq C \max\left\{e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right\}$.*

Proof. In the following, we denote by $C > 0$ a constant independent of ℓ and λ.

Our approach is to regard the periodic slow eigenvalue problem (6.21) as the perturbation,

$$\varphi_{\check{x}} = (\mathcal{A}_\infty(\check{x}, \lambda) + \mathcal{H}_\ell(\check{x}))\, \varphi, \quad \varphi \in \mathbb{C}^2,$$

of system (6.6) on $[0, \ell]$ and as the perturbation,

$$\varphi_{\check{x}} = (\mathcal{A}_\infty(-\check{x}, \lambda) + \mathcal{H}_\ell(\check{x}))\, \varphi, \quad \varphi \in \mathbb{C}^2,$$

of system,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(-\check{x}, \lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \tag{6.48}$$

on $[-\ell, 0)$, where $\mathcal{H}_\ell \colon [-\ell, \ell] \to \mathrm{Mat}_2(\mathbb{C})$ is given by,

$$\mathcal{H}_\ell(\check{x}) := \begin{cases} \mathcal{A}_\ell(\check{x}, \lambda) - \mathcal{A}_\infty(\check{x}, \lambda), & \check{x} \in [0, \ell] \\ \mathcal{A}_\ell(2\ell + \check{x}, \lambda) - \mathcal{A}_\infty(-\check{x}, \lambda), & \check{x} \in [-\ell, 0) \end{cases},$$

By estimate (6.26) the norm of $\mathcal{H}_\ell$ satisfies

$$\|\mathcal{H}_\ell\| \leq Ce^{-\varsigma_* \ell}. \tag{6.49}$$

Let X_b be the space of bounded functions $[-\ell, \ell] \to \mathbb{C}^2$ that are continuous, except for a possible discontinuity at 0. Our plan is to obtain exponential dichotomies for equations (6.6) and (6.48) first. The exponential dichotomies yield a solution operator to the inhomogeneous problem,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(|\check{x}|, \lambda)\varphi + G(\check{x}), \quad \varphi \in \mathbb{C}^2, \tag{6.50}$$

with $G \in X_b$ using the variation of constants formula. Then, using Lin's method [70, 100], we construct a solution operator to (6.50) that satisfies a matching condition at the endpoints $\check{x} = \ell$ and $\check{x} = -\ell$. Finally, we substitute $\mathcal{H}_\ell(\check{x})\varphi$ for $G(\check{x})$ in (6.50) and obtain a solution operator to (6.21). We apply the latter solution operator to the initial condition $\varphi_\infty(0, \lambda)$ to establish the existence of the desired solution $\varphi_\ell(\check{x}, \lambda)$.

We establish exponential dichotomies for the homoclinic slow eigenvalue problems (6.6) and (6.48). By Proposition 4.7 and estimate (6.26), system (6.6) has for $\lambda \in C_{b,\Lambda}$ an exponential dichotomy on $[0, \infty)$ with constants $C, \mu(\lambda) > 0$. The corresponding projections $\mathcal{P}_\infty(\check{x}, \lambda)$ can be chosen analytic on $C_{b,\Lambda}$. Moreover, since $\mathcal{A}_*(\lambda)$ is hyperbolic with spectral gap larger than $\mu(\lambda) \geq \varsigma_*$ and $\mathcal{A}_*$ is bounded on $C_{b,\Lambda}$, Lemma 4.6 and (6.26) yield

$$\|\mathcal{P}_\infty(\check{x}, \lambda) - \mathcal{P}_*(\lambda)\| \leq Ce^{-\varsigma_* \check{x}}, \quad \check{x} \geq 0, \lambda \in C_{b,\Lambda}, \tag{6.51}$$

where $\mathcal{P}_*(\lambda)$ denotes the analytic spectral projection of $\mathcal{A}_*(\lambda)$ on $\mathrm{Sp}(v_-(\lambda))$ along $\mathrm{Sp}(v_+(\lambda))$. Moreover, since we have $R_s v_-(\lambda) = v_+(\lambda)$, the identity,

$$R_s \mathcal{P}_*(\lambda) R_s = I - \mathcal{P}_*(\lambda), \tag{6.52}$$

holds for each $\lambda \in C_\Lambda$. Denote by $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda)$ the evolution operator of system (6.6). By [60, Lemma 2.1.4] $\mathcal{T}_\infty(\check{x}, \check{y}, \cdot)$ is analytic on C_Λ, since $\mathcal{A}_\infty(\check{x}, \cdot)$ is analytic on C_Λ.

Using the reversible symmetry R_s, system (6.48) can be fully described in terms of system (6.6). Indeed, for the evolution $\mathcal{T}_{\infty,r}(\check{x}, \check{y}, \lambda)$ of system (6.48) it holds $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda) = R_s \mathcal{T}_{\infty,r}(-\check{x}, -\check{y}, \lambda) R_s$. Consequently, system (6.48) has for any $\lambda \in C_{b,\Lambda}$ an exponential dichotomy on $(-\infty, 0]$ with constants $C, \mu(\lambda) > 0$. The corresponding projections $\mathcal{P}_{\infty,r}(\check{x}, \lambda)$ satisfy $\mathcal{P}_{\infty,r}(\check{x}, \lambda) = I - R_s \mathcal{P}_\infty(-\check{x}, \lambda) R_s$ for $\check{x} \leq 0$. Moreover, by (6.52) it holds

$$\|\mathcal{P}_{\infty,r}(\check{x}, \lambda) - \mathcal{P}_*(\lambda)\| \leq Ce^{\varsigma_* \check{x}}, \quad \check{x} \leq 0, \lambda \in C_{b,\Lambda}, \tag{6.53}$$

We proceed by constructing a solution operator to the periodic slow eigenvalue problem (6.21).

Consider $W_\ell(\lambda)\colon \mathbb{C}^2 \times \mathbb{C}^2 \times X_b \to X_b$ to (6.50) given by

$$W_\ell(\lambda)(a,b,G)[\check{x}] = \mathcal{T}^u_\infty(\check{x},\ell,\lambda)a + \mathcal{T}^s_\infty(\check{x},0,\lambda)b + \int_0^{\check{x}} \mathcal{T}^s_\infty(\check{x},\check{y},\lambda)G(\check{y})d\check{y} - \int_{\check{x}}^{\ell} \mathcal{T}^u_\infty(\check{x},\check{y},\lambda)G(\check{y})d\check{y}, \qquad \check{x} \in [0,\ell],$$

$$W_\ell(\lambda)(a,b,G)[\check{x}] = -\mathcal{T}^s_{\infty,r}(\check{x},-\ell,\lambda)a - \int_{\check{x}}^{0} \mathcal{T}^u_{\infty,r}(\check{x},\check{y},\lambda)G(\check{y})d\check{y} + \int_{-\ell}^{\check{x}} \mathcal{T}^s_{\infty,r}(\check{x},\check{y},\lambda)G(\check{y})d\check{y}, \qquad \check{x} \in [-\ell,0),$$

where $\mathcal{T}^{u,s}_\infty(\check{x},\check{y},\lambda)$ and $\mathcal{T}^{u,s}_{\infty,r}(\check{x},\check{y},\lambda)$ denote the (un)stable evolution operator of systems (6.6) and (6.48) under the exponential dichotomies established above. Note that W_ℓ is an analytic operator on $C_{b,\Lambda}$, since the evolutions $\mathcal{T}_\infty(\check{x},\check{y},\cdot)$ and the projections $\mathcal{P}_\infty(\check{x},\cdot)$ are analytic. By (6.51) and (6.53) it holds

$$\|\mathcal{P}_\infty(\ell,\lambda) - \mathcal{P}_{\infty,r}(-\ell,\lambda)\| \leq Ce^{-\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}. \tag{6.54}$$

We conclude that the analytic linear operator $A_{1,\ell}(\lambda) := I - \mathcal{P}_\infty(\ell,\lambda) + \mathcal{P}_{\infty,r}(-\ell,\lambda)$ is invertible for $\ell > 0$ sufficiently large. Now define the analytic linear operator $A_{2,\ell}(\lambda)\colon \mathbb{C}^2 \times X_b \to \mathbb{C}^2$ by

$$A_{2,\ell}(\lambda)(b,G) = A_{1,\ell}(\lambda)^{-1}\left(W_\ell(\lambda)(0,b,G)[-\ell] - W_\ell(\lambda)(0,b,G)[\ell]\right).$$

One readily verifies that the analytic linear operator $W_{2,\ell}(\lambda)\colon \mathbb{C}^2 \times X_b \to X_b$ defined by $W_{2,\ell}(\lambda)(b,G) = W_\ell(\lambda)(A_{2,\ell}(\lambda)(b,G),b,G)$ is linear and satisfies

$$W_{2,\ell}(\lambda)(b,G)[-\ell] = W_{2,\ell}(\lambda)(b,G)[\ell], \quad b \in \mathbb{C}^2, G \in X_b, \lambda \in C_{b,\Lambda}. \tag{6.55}$$

Moreover, we have the estimates

$$\begin{aligned} \|A_{2,\ell}(\lambda)(b,G)\| &\leq C(e^{-\mu(\lambda)\ell}\|b\| + \|G\|), \\ \|W_{2,\ell}(\lambda)(b,G)[\check{x}]\| &\leq \begin{cases} C(e^{-\mu(\lambda)\check{x}}\|b\| + \|G\|), & \check{x} \in [0,\ell], \\ C(e^{-\mu(\lambda)(2\ell+\check{x})}\|b\| + \|G\|), & x \in [-\ell,0), \end{cases} \end{aligned} \tag{6.56}$$

for $b \in \mathbb{C}^2, G \in X_b, \lambda \in C_{b,\Lambda}$. Denote by $W_{3,\ell}(\lambda)\colon X_b \to X_b$ the analytic linear map $W_{3,\ell}(\lambda)(w) = W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot w)$, where $\cdot$ denotes pointwise multiplication, i.e. $(\mathcal{H}_\ell \cdot w)[\check{x}] = \mathcal{H}_\ell(\check{x})w(\check{x})$. By (6.49) we have the estimate,

$$\|W_{3,\ell}(\lambda)\| \leq Ce^{-\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}.$$

Hence for $\ell > 0$ sufficiently large, the map $I - W_{3,\ell}(\lambda)$ is invertible. Finally, consider the analytic linear map $W_{4,\ell}(\lambda)\colon \mathbb{C}^2 \to X_b$ given by $W_{4,\ell}(\lambda)(b) = (I - W_{3,\ell}(\lambda))^{-1}(W_{2,\ell}(\lambda)(b,0))$. One readily checks that

$$W_{4,\ell}(\lambda)(b) = W_{2,\ell}(\lambda)(b,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b)), \quad b \in \mathbb{C}^2, \lambda \in C_{b,\Lambda}, \tag{6.57}$$

is satisfied. Define the map $\zeta\colon [0,2\ell) \to [-\ell,\ell]$ by

$$\zeta(\check{x}) = \begin{cases} \check{x}, & \check{x} \in [0,\ell] \\ \check{x} - 2\ell, & \check{x} \in (\ell, 2\ell) \end{cases}.$$

By identities (6.55) and (6.57) we have $W_{4,\ell}(\lambda)(b)[\ell] = W_{4,\ell}(\lambda)(b)[-\ell]$. We conclude for every $\lambda \in C_{b,\Lambda}, b \in \mathbb{C}^2$ and $\ell > 0$ sufficiently large, that $W_{4,\ell}(\lambda)(b)[\zeta(\check{x})]$ is a solution to (6.21) on $[0,2\ell)$ that can be extended to $[0,2\ell]$.

Next, we apply the solution operator $W_{4,\ell}$ to initial condition $b_\lambda := \varphi_\infty(0,\lambda) \in \mathbb{C}^2$ and consider the solution

$$\varphi_\ell(\check{x},\lambda) := W_{4,\ell}(\lambda)(b_\lambda)[\zeta(\check{x})],$$

to (6.21). Note that $\varphi_\ell(\check{x},\cdot)$ is analytic on $C_{b,\Lambda}$, since both $W_{4,\ell}$ and $\varphi_\infty(0,\lambda)$ are analytic on $C_{b,\Lambda}$. Using (6.49), (6.56) and identity (6.57) we estimate

$$\begin{aligned} \|\varphi_\ell(\check{x},\lambda)\| &\leq \|W_{2,\ell}(\lambda)(b_\lambda,0)[\zeta(\check{x})]\| + \|W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\zeta(\check{x})]\| \\ &\leq C\left[e^{-\mu(\lambda)\check{x}} + e^{-\varsigma_*\ell}\int_0^{2\ell}\left(e^{-\mu(\lambda)|\check{x}-\check{y}|} + e^{-\mu(\lambda)(|\ell-\check{x}|+|\ell-\check{y}|)}\right)\|\varphi_\ell(\check{y},\lambda)\|d\check{y}\right], \end{aligned} \tag{6.58}$$

for $\check{x} \in [0,2\ell], \lambda \in C_{b,\Lambda}$. Applying [15, Lemma III.2.1] on the integral inequality (6.58) yields

$$\|\varphi_\ell(\check{x},\lambda)\| \leq Ce^{-\mu(\lambda)\check{x}}, \quad \check{x} \in [0,2\ell], \lambda \in C_{b,\Lambda}, \tag{6.59}$$

provided $\ell > 0$ is sufficiently large. Moreover, we approximate with the aid of (6.54)

$$\begin{aligned} &\|A_{2,\ell}(\lambda)(b_\lambda,0) - T_\infty^s(\ell,0,\lambda)b_\lambda\| \\ &\quad = \|(\mathcal{P}_\infty(\ell,\lambda) - \mathcal{P}_{\infty,r}(-\ell,\lambda))A_{1,\ell}(\lambda)^{-1}T_\infty^s(\ell,0,\lambda)b_\lambda\| \leq Ce^{-(\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \tag{6.60}$$

for $\lambda \in C_{b,\Lambda}$. On the other hand, using (6.49) and (6.59) we estimate

$$\begin{aligned} \|W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\ell]\| &\leq Ce^{-\varsigma_*\ell}\int_0^{2\ell} e^{-\mu(\lambda)|\ell-\check{y}|}\|\varphi_\ell(\check{y},\lambda)\|d\check{y} \\ &\leq Ce^{-(\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \tag{6.61}$$

for $\lambda \in C_{b,\Lambda}$. Using identity (6.57) and estimates (6.60) and (6.61) we expand $\varphi_\ell(\check{x},\lambda)$ at $\check{x} = \ell$ as follows

$$\begin{aligned} \varphi_\ell(\ell,\lambda) &= W_{2,\ell}(\lambda)(b_\lambda,0)[\ell] + W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\ell] \\ &= T_\infty^s(\ell,0,\lambda)b_\lambda + \mathcal{O}\left(e^{-(\mu(\lambda)+\varsigma_*)\ell}\right) \\ &= \varphi_\infty(\ell,\lambda) + \mathcal{O}\left(e^{-(\mu(\lambda)+\varsigma_*)\ell}\right), \end{aligned}$$

for $\lambda \in C_{b,\Lambda}$. Similarly, using identity (6.57) and estimates (6.26), (6.49) and (6.60) we expand $\varphi_\ell(\check{x}, \lambda)$ at $\check{x} = 0$ as follows for $\lambda \in C_{b,\Lambda}$

$$\begin{aligned}\varphi_\ell(0,\lambda) &= W_{2,\ell}(\lambda)(b_\lambda, \mathcal{H}_\ell \cdot W_{2,\ell}(\lambda)(b_\lambda, 0))[0] \\ &\qquad + W_{2,\ell}(\lambda)(0, \mathcal{H}_\ell \cdot W_{2,\ell}(\lambda)(0, \mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda)))[0] \\ &= \mathcal{P}_\infty(0,\lambda)b_\lambda - \int_0^\ell \mathcal{T}^u_\infty(0,\check{y},\lambda)\mathcal{H}_\ell(\check{y})\mathcal{T}^s_\infty(\check{y},0,\lambda)b_\lambda d\check{y} + O\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right) \\ &= \varphi_\infty(0,\lambda) - \int_0^\ell \mathcal{T}^u_\infty(0,\check{y},\lambda)\mathcal{H}_\ell(\check{y})\varphi_\infty(\check{y},\lambda)d\check{y} + O\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right) \\ &= \varphi_\infty(0,\lambda) + O\left(e^{-2\varsigma_*\ell}\right),\end{aligned}$$

where we used that $\mu(\lambda) > \varsigma_*$. □

Since system (6.21) is R_s-reversible at $\check{x} = \ell$, $\varphi^r_\ell(\check{x}, \lambda) = R_s\varphi_\ell(2\ell - \check{x}, \lambda)$ is a also solution to (6.21). The next proposition shows that $\varphi_\ell(\check{x}, \lambda)$ and $\varphi^r_\ell(\check{x}, \lambda)$ are linearly independent and approximates their Wronskian $\mathcal{W}_\ell(\lambda)$.

Corollary 6.20. *For $0 \ll \ell < \infty$ the ($\check{x}$-independent) Wronskian $\mathcal{W}_\ell(\lambda) = \det(\varphi_\ell(\check{x}, \lambda) \mid \varphi^r_\ell(\check{x}, \lambda))$ is approximated by*

$$\|\mathcal{W}_\ell(\lambda) - E_\ell(\lambda)\| \leq Ce^{-(2\mu(\lambda)+\varsigma_*)\ell}, \quad \lambda \in C_{b,\Lambda}, \tag{6.62}$$

where $C > 0$ is a constant independent of ℓ and λ and $E_\ell \colon C_{b,\Lambda} \to \mathbb{C}$ is the non-zero analytic map given by $E_\ell(\lambda) = 2\omega(\lambda)e^{-2\omega(\lambda)\ell}$.

Proof. Combining estimates (6.45) and (6.46) yields

$$\left|\det\left(\varphi_\ell(\ell,\lambda) \mid R_s\varphi_\ell(\ell,\lambda)\right) - e^{-2\omega(\lambda)\ell}\det\left(v_-(\lambda) \mid R_s v_-(\lambda)\right)\right| \leq Ce^{-(2\mu(\lambda)+\varsigma_*)\ell},$$

which concludes the proof. □

Conclusion

With the preparatory work done in the previous sections, we are able to prove Theorems 6.11 and 6.13 using the aforementioned approach.

Proof of Theorem 6.11. In the following, we denote by $C > 0$ a constant independent of ℓ. First, using (6.26) and (6.46) we approximate

$$\left|\mathcal{K}_\ell(0) - \mathcal{E}_{\infty,s}(0)\right| \leq Ce^{-2\varsigma_*\ell},$$

where $\mathcal{K}_\ell(\lambda)$ is defined in (6.23). Combining the latter with (6.22) and (6.62) yields

$$\left|e^{-i\nu}\mathcal{E}_{\ell,s}(0, e^{i\nu})\mathcal{W}_\ell(0) - \mathcal{E}_{\infty,s}(0)\right| \leq Ce^{-2\varsigma_*\ell}, \quad \nu \in \mathbb{R}. \tag{6.63}$$

On the other hand, by (6.26) it holds

$$|\mathfrak{a}_\ell - \mathfrak{a}_\infty|, |\mathfrak{w}_\ell - \mathfrak{w}_\infty| \leq Ce^{-2\varsigma_*\ell}. \tag{6.64}$$

Finally, applying (6.9), (6.62), (6.63) and (6.64) on identity (6.20) establishes the desired approximation (6.14). □

Proof of Theorem 6.13. In the following, we denote by $C > 0$ a constant independent of ℓ and λ. Let $\lambda_\infty \in C_\Lambda$ be a simple zero of $\mathcal{E}_{\infty,s}$ satisfying (6.16). Then, we take $C_{b,\Lambda} \subset C_\Lambda$ an open and bounded neighborhood of λ_∞ of $\mathcal{E}_{\infty,s}$ such that it holds $\mathrm{Re}(\omega(\lambda)) > \omega_*$ for all $\lambda \in C_{b,\Lambda}$. We chose $\delta > 0$ such that

$$2\delta < \varsigma_*, \quad \mu(\lambda) := \mathrm{Re}(\omega(\lambda)) - \delta > \omega_*,$$

for all λ in $C_{b,\Lambda}$.

We are looking for zeros of $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ close to λ_∞ for $0 \ll \ell < \infty$ and $\gamma \in S^1$. In other words, we are looking for solutions $\lambda \in C_{b,\Lambda}$ in a neighborhood of λ_∞ to the equation

$$0 = \mathcal{E}_{\ell,s}(\lambda,\gamma). \tag{6.65}$$

By multiplying (6.65) with the non-zero (see Corollary 6.20) quantity $\gamma^{-1}\mathcal{W}_\ell(\lambda)$ on both sides, we obtain the equivalent equation,

$$0 = 2\mathrm{Re}(\gamma)\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda), \quad \lambda \in C_{b,\Lambda}, \quad \gamma \in S^1, \tag{6.66}$$

see also (6.22). Using (6.26) and (6.46) we approximate

$$\left|\mathcal{K}_\ell(\lambda) - \mathcal{E}_{\infty,s}(\lambda)\right| \leq Ce^{-2\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}. \tag{6.67}$$

Note that both $\mathcal{W}_\ell$ and $\mathcal{K}_\ell$ are analytic on $C_{b,\Lambda}$, since $\varphi_\ell(\check{x},\cdot)$ and $\Upsilon(u,\cdot)$ are. By shrinking $C_{b,\Lambda}$ if necessary, the approximations (6.62) and (6.67) provide bounds for the derivatives of the analytic maps $\mathcal{W}_\ell$ and $\mathcal{K}_\ell$ via the estimates,

$$\begin{aligned} \left|\frac{\partial^i}{\partial\lambda^i}\left(\mathcal{K}_\ell(\lambda) - \mathcal{E}_{\infty,s}(\lambda)\right)\right| &\leq Ce^{-2\varsigma_*\ell}, \\ \left|\frac{\partial^i}{\partial\lambda^i}\left(\mathcal{W}_\ell(\lambda) - E_\ell(\lambda)\right)\right| &\leq Ce^{-(2\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \qquad i = 0,1,2, \quad \lambda \in C_{b,\Lambda}. \tag{6.68}$$

Consider the analytic function $\eta_\ell \colon C_{b,\Lambda} \times \mathbb{C} \to \mathbb{C}$ given by $\eta_\ell(\lambda,\gamma_r) = 2\gamma_r\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda)$. Let $\mathcal{D} \subset \mathbb{C}$ be open and bounded such that it contains the closed unit circle. Provided $\ell > 0$ is sufficiently large, we have by (6.62) and (6.67)

$$|\eta_\ell(\lambda,\gamma_r) + \mathcal{E}_{\infty,s}(\lambda)| < |\mathcal{E}_{\infty,s}(\lambda)|,$$

for each $\gamma_r \in \mathcal{D}$ and λ on the boundary of some sufficiently small disk $\mathcal{B} \subset C_{b,\Lambda}$ around λ_∞. Thus, by Rouché's Theorem there exists for each $\gamma_r \in \mathcal{D}$ a unique zero $\lambda_\ell(\gamma_r) \in \mathcal{B}$ of $\eta_\ell(\cdot,\gamma_r)$, which satisfies

$$|\lambda_\ell(\gamma_r) - \lambda_\infty| \leq Ce^{-2\varsigma_*\ell}. \tag{6.69}$$

By estimate (6.68) it holds

$$\left|\partial_\lambda \eta_\ell(\lambda,\gamma_r) - \mathcal{E}'_{\infty,s}(\lambda)\right| \leq C e^{-2\varsigma_*\ell}, \quad \lambda \in \mathcal{B}, \gamma_r \in \mathcal{D}.$$

Hence, using the (analytic) Implicit Function Theorem and the fact that $\mathcal{E}'_{\infty,s}(\lambda_\infty) \neq 0$, we conclude that the map $\lambda_\ell \colon \mathcal{D} \to \mathbb{C}$ is analytic. Implicit differentiation of identity (6.66) yields the derivatives

$$\left.\begin{aligned} \lambda'_\ell(\gamma_r) &= \frac{2\mathcal{W}_\ell(\lambda_\ell(\gamma_r))}{\mathcal{K}'_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}'_\ell(\lambda(\gamma_r))}, \\ \lambda''_\ell(\gamma_r) &= \lambda'_\ell(\gamma_r) \frac{4\mathcal{W}'_\ell(\lambda_\ell(\gamma_r)) - \left[\mathcal{K}''_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}''_\ell(\lambda(\gamma_r))\right]\lambda'_\ell(\gamma_r)}{\mathcal{K}'_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}'_\ell(\lambda(\gamma_r))}, \end{aligned}\right\} \quad \gamma_r \in \mathcal{D}.$$

Approximating these derivatives with (6.69) and (6.68) leads to (6.18). Next, we expand $\mathcal{K}_\ell$ in an ℓ-independent neighborhood V_∞ of λ_∞ with Taylor's Theorem as

$$\mathcal{K}_\ell(\lambda) = \mathcal{K}_\ell(\lambda_\infty) + (\lambda - \lambda_\infty)\mathcal{K}'_\ell(\lambda_\infty) + \hat{\mathcal{K}}_\ell(\lambda - \lambda_\infty), \quad \lambda \in V_\infty, \tag{6.70}$$

with $\|\hat{\mathcal{K}}_\ell(\lambda - \lambda_\infty)\| \leq C|\lambda - \lambda_\infty|^2$. By (6.69) and the ℓ-independence of V_∞ we can substitute $\lambda_\ell(\gamma_r)$ for λ in (6.70) for $\ell > 0$ sufficiently large. Thus, using estimates (6.62), (6.69) and (6.68) we arrive at

$$\begin{aligned} 0 &= 2\gamma_r \mathcal{W}_\ell(\lambda_\ell(\gamma_r)) - \mathcal{K}_\ell(\lambda_\ell(\gamma_r)) \\ &= -\mathcal{K}_\ell(\lambda_\infty) - (\lambda_\ell(\gamma_r) - \lambda_\infty)\mathcal{E}'_{\infty,s}(\lambda_\infty) + \mathcal{O}\left(e^{-4\varsigma_*\ell}, e^{-2\omega(\lambda_\infty)\ell}\right). \end{aligned} \tag{6.71}$$

Hence, we obtain the desired leading-order expression for $\lambda_\ell(\gamma_r) - \lambda_\infty$ by calculating the leading order of $\mathcal{K}_\ell(\lambda_\infty)$. First, since G is C^3 on its domain by **(S1)**, the solutions $\kappa_{\mathrm{h}}(x,u)$ and $\mathcal{X}_{in}(x,u,\lambda)$ to (2.3) and to (3.8) are C^2 on their domains $\mathbb{R} \times U_{\mathrm{h}}$ and $\mathbb{R} \times U_{\mathrm{h}} \times C_{b,\Lambda}$. Therefore, Υ is C^2 on $U_{\mathrm{h}} \times C_{b,\Lambda}$. Thus, by shrinking the ℓ– and λ-independent neighborhood U_∞ of $u_\infty(0)$ if necessary, we expand

$$\Upsilon(u,\lambda) = \Upsilon(u_\infty(0),\lambda) + \partial_u \Upsilon(u_\infty(0),\lambda)(u - u_\infty(0)) + \tilde{\Upsilon}(u,\lambda), \quad u \in U_\infty, \tag{6.72}$$

where $\|\tilde{\Upsilon}(u,\lambda)\| \leq C|u - u_\infty(0)|^2$. With the aid of identities (6.27), (6.46) and (6.72) we expand

$$\begin{aligned} \mathcal{K}_\ell(\lambda) &= \det\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) \\ &\quad + \det\left(\varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda)\right)\right) \\ &\quad + (u_\ell(0) - u_\infty(0))\det\left(\varphi_\infty(0,\lambda) \mid \partial_u\Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) \\ &\quad + \mathcal{E}_{\infty,s}(\lambda) + \mathcal{O}\left(e^{-4\varsigma_*\ell}\right) \\ &= 2\det\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) + \mathcal{E}_{\infty,s}(\lambda) \\ &\quad - \frac{2\omega_*^2\alpha^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\det\left(\varphi_\infty(0,\lambda) \mid \partial_u\Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) + \mathcal{O}\left(e^{-3\varsigma_*\ell}\right), \end{aligned} \tag{6.73}$$

where we used $[\Upsilon(u_\infty(0),\lambda)]^{-1} = \Upsilon(u_\infty(0),\lambda)R_s$, $\det(\Upsilon(u_\infty(0),\lambda)) = 1$, $\det(R_s) = -1$ and the 2-linearity of the determinant. Our aim is to approximate $\varphi_\ell(0,\lambda_\infty) - \varphi_\infty(0,\lambda_\infty)$ in (6.73). First, recall that H_1 is C^3 on its domain. Fix $\check{x} \in [0, \ell]$ Using Taylor's Theorem and estimate (6.26) we approximate

$$|\partial_u H_1(u_\ell(\check{x}),0,0) - \partial_u H_1(u_\infty(\check{x}),0,0) - \partial_{uu} H_1(u_\infty(\check{x}),0,0)(u_\ell(\check{x}) - u_\infty(\check{x}))| \leq Ce^{-2\varsigma_*(2\ell-\check{x})}. \tag{6.74}$$

By estimate (6.27) and (6.74) we obtain

$$\begin{aligned}&\mathcal{A}_\ell(\check{x},\lambda) - \mathcal{A}_\infty(\check{x},\lambda)\\ &\quad= -\frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}\partial_{uu}H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})}{\mathfrak{a}_\infty}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} + \mathcal{O}\left(e^{-\varsigma_*(3\ell-\check{x})}\right),\end{aligned} \tag{6.75}$$

for $\check{x} \in [0, \ell]$. Subsequently, we combine (6.47) and (6.75) to obtain a leading-order approximation of $\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda)$ for $\lambda \in C_{b,\Lambda}$

$$\begin{aligned}\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) &= -\int_0^\ell \mathcal{Q}_\infty(\lambda)\mathcal{T}_\infty(0,\check{y},\lambda)\left(\mathcal{A}_\ell(\check{x},\lambda) - \mathcal{A}_\infty(\check{x},\lambda)\right)\varphi_\infty(\check{y},\lambda)d\check{y}\\ &\quad+ \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right)\\ &= \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\int_0^\infty \mathcal{Q}_\infty(\lambda)\mathcal{T}_\infty(0,\check{y},\lambda)\mathcal{Z}(\check{y},\lambda)d\check{y} + \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right),\end{aligned} \tag{6.76}$$

where we denote

$$\mathcal{Z}(\check{x},\lambda) := \begin{pmatrix}0\\ \partial_{uu}H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})\hat{u}_\infty(\check{x},\lambda)\end{pmatrix}, \quad \check{x} \geq 0.$$

Since the determinant $\mathcal{E}_{\infty,s}(\lambda_\infty) = \det(\varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda))$ equals 0, the vectors $\Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)$ and $\varphi_\infty(0,\lambda_\infty)$ are scalar multiples of each other. As the u-coordinate of both vectors are equal, we have in fact $\varphi_\infty(0,\lambda_\infty) = \Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)$. Moreover, $\mathcal{Q}_\infty(\lambda)$ is a projection along $\mathrm{Sp}(\varphi_\ell(0,\lambda))$. Therefore, the determinant $\det(\mathcal{Q}_\infty(\lambda)w \mid \varphi_\ell(0,\lambda))$ equals $\det(w \mid \varphi_\ell(0,\lambda))$ for any vector $w \in \mathbb{C}^2$ and $\lambda \in C_{b,\Lambda}$. Using the latter two observations and $\det(\mathcal{T}_\infty(0,\check{y},\lambda)) = 1$, we simplify the determinant

$$\begin{aligned}&\det\left(\mathcal{Q}_\infty(\lambda_\infty)\mathcal{T}_\infty(0,\check{y},\lambda_\infty)\mathcal{Z}(\check{y},\lambda_\infty) \mid \Upsilon(u_\ell(0),\lambda_\infty)R_s\varphi_\ell(0,\lambda_\infty)\right)\\ &\quad= \det\left(\mathcal{T}_\infty(0,\check{y},\lambda_\infty)\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(0,\lambda_\infty)\right) = \det\left(\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(\check{y},\lambda_\infty)\right).\end{aligned} \tag{6.77}$$

Finally, using (6.73), (6.76) and (6.77), we rewrite (6.71) as

$$\begin{aligned}\lambda_\ell(\gamma_r) - \lambda_\infty &= -\frac{\mathcal{K}_\ell(\lambda_\infty)}{\mathcal{E}'_{\infty,s}(\lambda_\infty)} + \mathcal{O}\left(e^{-4\varsigma_*\ell}\right)\\ &= \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty\mathcal{E}'_{\infty,s}(\lambda_\infty)}\Bigg(\det\left(\varphi_\infty(0,\lambda_\infty) \mid \partial_u\Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)\right)\\ &\qquad -2\int_0^\infty \det\left(\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(\check{y},\lambda_\infty)\right)d\check{y}\Bigg) + \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda_\infty)\ell}\right)\end{aligned}$$

$$= \frac{2\omega_*^2 \alpha^2 e^{-2\omega_* \ell}}{\mathfrak{a}_\infty \mathcal{E}'_{\infty,s}(\lambda_\infty)} \left(2 \int_0^\infty \partial_{uu} H_1(u_\infty(\check{x}), 0, 0) \tilde{u}_\infty(\check{x}) \, [\hat{u}_\infty(\check{x}, \lambda_\infty)]^2 \, d\check{x} \right.$$
$$\left. + \, [\hat{u}_\infty(0, \lambda_\infty)]^2 \, \partial_u \mathcal{G}(u_\infty(0), \lambda_\infty) \right) + \mathcal{O}\left(e^{-3\varsigma_* \ell}, e^{-2\mu(\lambda_\infty)\ell} \right),$$

which concludes the proof of identity (6.17). □