Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Chapter 5

Spectral stability analysis

In this chapter we prove the two main spectral approximation results presented in Chapter 3: we show that the zeros of the Evans function $\mathcal{E}_\varepsilon$ are approximated by the ones of the reduced Evans function $\mathcal{E}_0$ and we derive an expansion of the critical spectral curve attached to the origin. Yet, we start with collecting some properties of the reduced Evans function $\mathcal{E}_0$, which are necessary for the proof of these approximation results.

5.1 The reduced Evans function

In this section we study the reduced Evans function $\mathcal{E}_0$, which is defined in terms of the three eigenvalue problems (3.6), (3.8) and (3.9). Thereby, we provide the proofs of Propositions 3.10, 3.11 and 3.12.

5.1.1 The fast Evans function

The homogeneous fast eigenvalue problem (3.6) arises when linearizing $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_\mathrm{h}(x, u_0)$ – see assumption **(E1)**. The homoclinic $\psi_\mathrm{h}(x, u_0) = (v_\mathrm{h}(x, u_0), q_\mathrm{h}(x, u_0))$ to (2.3) at $u = u_0$ converges exponentially to the hyperbolic saddle 0 as $x \to \pm\infty$. Hence, system (3.6) is asymptotically hyperbolic. Consequently, it has exponential dichotomies on both half-lines respecting analyticity in λ. This leads to the construction of the analytic fast Evans function $\mathcal{E}_{f,0}$ which detects the values of λ equation (3.6) has exponentially localized solutions. The above is the content of the following lemma and proposition.

Lemma 5.1. *Let $\mathcal{K} \subset \mathbb{C}^m$ be an open and bounded set containing the orbit segment $\{u_\mathrm{s}(\check{x}) : \check{x} \in [0, 2\ell_0]\}$ such that $\overline{K} \subset U$ – see **(S1)** and **(E2)**. There exists $\Lambda_0 > 0$ such that for $\Lambda \in (-\Lambda_0, 0)$ the spectrum of the matrix,*

$$A(u, \lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, 0, 0) + \lambda & 0 \end{pmatrix}, \tag{5.1}$$

is bounded away from the imaginary axis on $\overline{\mathcal{K}} \times C_\Lambda$ by some constant $\mu_r > 0$.

Proof. For $k \in \mathbb{Z}_{>0}$ and a matrix $A \in \mathrm{Mat}_{k\times k}(\mathbb{C})$ denote by $\mathcal{F}(A) = \{v^*Av : v \in \mathbb{C}^k, \|v\| = 1\}$ its field of values. Since $\partial_v G(u, 0, 0)$ has positive definite real part by **(S2)**, the field of values $\mathcal{F}(\partial_v G(u, 0, 0))$ is for every $u \in \overline{K}$ contained in the positive half-plane by [48, Property 1.2.5a]. In fact, by compactness of $\overline{K}$ there exists $\Lambda_0 > 0$ such that we have $\mathcal{F}(\partial_v G(u, 0, 0)) \subset C_{-\Lambda_0}$ for every $u \in \overline{K}$. Let $\Lambda \in (-\Lambda_0, 0)$. For $u \in \overline{K}$ and $\lambda \in C_\Lambda$ we establish using [48, Property 1.2.3] and [48, Corollary 1.7.7]

$$\begin{aligned}\sigma((\partial_v G(u, 0, 0) + \lambda)D_2^{-1}) &\subset (\mathcal{F}(\partial_v G(u, 0, 0)) + \lambda)\mathcal{F}(D_2^{-1})\\ &\subset \left\{z \in \mathbb{C} : \mathrm{Re}(z) \geq d_{\max}^{-1}(\Lambda_0 + \Lambda)\right\},\end{aligned}$$

where $d_{\max}$ is the largest diagonal value of D_2. The eigenvalues of $A(u, \lambda)$ are given by the square roots of the eigenvalues of $(\partial_v G(u, 0, 0) + \lambda)D_2^{-1}$. Therefore, we obtain for $u \in \overline{K}$ and $\lambda \in C_\Lambda$ that any eigenvalue $z \in \sigma(A(u, \lambda))$ satisfies $|\mathrm{Re}(z)| \geq \cos(\pi/4)\sqrt{(\Lambda_0 + \Lambda)/d_{\max}}$, which concludes the proof. □

Proposition 5.2. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. The homogeneous fast eigenvalue problem* (3.6) *admits for every $\lambda \in C_\Lambda$ exponential dichotomies on $[0, \infty)$ and $(-\infty, 0]$ with constants $C(\lambda), \mu_r > 0$ and rank n projections $Q_{f,\pm}(x, \lambda)$, where $\mu_r > 0$ is as in Lemma 5.1. The projections $Q_{f,\pm}(\pm x, \cdot)$ are analytic on C_Λ for each $x \geq 0$. Morover, the map $\lambda \mapsto C(\lambda)$ is continuous on C_Λ.*

Let $B_f^{u,s} \colon C_\Lambda \to \mathrm{Mat}_{2n\times n}(\mathbb{C})$ be analytic bases such that $Q_{f,+}(0, \lambda)[\mathbb{C}^{2n}] = B_f^u(\lambda)[\mathbb{C}^n]$ and $\ker(Q_{f,-}(0, \lambda)) = B_f^s(\lambda)[\mathbb{C}^n]$ for $\lambda \in C_\Lambda$. The analytic function $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ given by $\mathcal{E}_{f,0}(\lambda) = \det(B_f^u(\lambda), B_f^s(\lambda))$ has the following properties:

1. *$\mathcal{E}_{f,0}(\lambda) = 0$ if and only if* (3.6) *admits a non-trivial, exponentially localized solution;*
2. *$\mathcal{E}_{f,0}(\lambda) \neq 0$ if and only if* (3.6) *has an exponential dichotomy on $\mathbb{R}$;*
3. *The zero set $\mathcal{E}_{f,0}^{-1}(0)$ is discrete and independent of the choice of bases $B_f^{u,s}$;*
4. *The multiplicity of a zero $\lambda \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of λ as an eigenvalue of the operator $\mathcal{L}_f$, defined in* (3.7).

Proof. By Lemma 5.1 the asymptotic matrix $A(u_0, \cdot)$, defined in (5.1), is hyperbolic on C_Λ with spectral gap larger than μ_r, where u_0 is as in **(E2)**. The stable and unstable eigenspaces of $A(u_0, \lambda)$ have dimension n for any $\lambda \in C_\Lambda$. Moreover, estimate (2.6) implies

$$\left\|\mathcal{A}_{22,0}(x, u_0, \lambda) - A(u_0, \lambda)\right\| \leq Ke^{-\mu_h|x|}, \quad x \in \mathbb{R}, \lambda \in C_\Lambda,$$

where $K > 0$ is a λ-independent constant. Therefore, system (3.6) admits by Proposition 4.7 exponential dichotomies on both half-lines with the desired properties.

By [86, Proposition 2.1] we have $\mathcal{E}_{f,0}(\lambda) \neq 0$ if and only if (3.6) has an exponential dichotomy on $\mathbb{R}$. On the other hand, every exponentially localized solution $\varphi(x, \lambda)$ to (3.6) must satisfy $\varphi(0, \lambda) \in B^u_f(\lambda)[\mathbb{C}^n] \cap B^s_f(\lambda)[\mathbb{C}^n]$. This settles the first two properties. The third and fourth property are the content of [1, Section E]. □

Proposition 5.2 provides the fast Evans function and thereby proves Proposition 3.10.

Definition 5.3. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. The map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ given by $\mathcal{E}_{f,0}(\lambda) = \det(B^u_f(\lambda), B^s_f(\lambda))$, obtained in Proposition 5.2, is called the *fast Evans function*.

An important consequence of the exponential dichotomies established in Proposition 5.2 is that the differential operator associated with (3.6) is Fredholm.

Corollary 5.4. *Let* $\Lambda \in (-\Lambda_0, 0)$ *with* $\Lambda_0 > 0$ *as in Lemma 5.1. For each* $\lambda \in C_\Lambda$ *the bounded operator* $\mathcal{L}_\lambda \colon C^1_b(\mathbb{R}, \mathbb{C}^{2n}) \to C_b(\mathbb{R}, \mathbb{C}^{2n})$ *given by*

$$\mathcal{L}_\lambda \varphi = \varphi_x - \mathcal{A}_{22,0}(\cdot, u_0, \lambda)\varphi,$$

is Fredholm of index 0. Moreover, $\mathcal{L}_\lambda$ *is invertible if and only if* $\lambda \in C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$. *The multiplicity of a zero* $\lambda_\diamond \in C_\Lambda$ *of* $\mathcal{E}_{f,0}$ *coincides with the algebraic multiplicity of the operator pencil* $\lambda \mapsto \mathcal{L}_\lambda$ *at* $\lambda = \lambda_\diamond$.

Proof. This follows readily from Proposition 5.2, [86, Lemma 4.2] and [1, Section E]. We also refer to [6, Section 3.2]. □

5.1.2 The slow Evans function

The slow Evans function $\mathcal{E}_{s,0}$ is explictly given by (3.10). The matrix solution $\mathcal{X}_{in}(x, u_0, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8) at $u = u_0$ is one of the key ingredients of $\mathcal{E}_{s,0}$. We prove that $\mathcal{X}_{in}(x, u_0, \cdot)$ is meromorphic for each $x \in \mathbb{R}$. Singularities of $\mathcal{X}_{in}(x, u_0, \cdot)$ only occur at the zeros of the fast Evans function $\mathcal{E}_{f,0}$.

Proposition 5.5. *Let* $\Lambda \in (-\Lambda_0, 0)$ *with* $\Lambda_0 > 0$ *as in Lemma 5.1. There exists a unique solution* $\mathcal{X}_{in} \colon \mathbb{R} \times U_\mathrm{h} \times [C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)] \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ *to the inhomogeneous fast eigenvalue problem (3.8) with the following properties:*

1. $\mathcal{X}_{in}(x, u_0, \cdot)$ *is meromorphic on* C_Λ *and analytic on* $C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$ *for all* $x \in \mathbb{R}$*;*
2. *If* $\lambda \mapsto \mathcal{X}_{in}(x, u_0, \lambda)$ *has a pole at* $\lambda = \lambda_\diamond$*, then its order is at most the multiplicity of* $\lambda_\diamond$ *as a root of* $\mathcal{E}_{f,0}$*;*
3. $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ *is exponentially localized for each* $\lambda \in C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$. *In particular, there exists* λ*-independent constants* $C, \mu_{in} > 0$ *such that*

$$\|\mathcal{X}_{in}(x, u_0, \lambda)\| \leq C e^{-\mu_{in}|x|}, \quad x \in \mathbb{R},$$

for all $\lambda \in C_\Lambda$ *with* $\mathrm{Re}(\sqrt{\lambda}) > C$*;*

4. *Let $\lambda_\diamond \in C_\Lambda$ be a simple zero of $\mathcal{E}_{f,0}$. Denote by $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ exponentially localized solutions to (3.6) and its adjoint equation* (3.14)*, respectively, at $\lambda = \lambda_\diamond$ satisfying*

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \varphi_{\lambda_\diamond}(z) dz = 1. \tag{5.2}$$

There exists a neighborhood $B_{\lambda_\diamond} \subset C_\Lambda$ of $\lambda_\diamond$ and a mapping $\mathcal{X}_{\lambda_\diamond} : \mathbb{R} \times B_{\lambda_\diamond} \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$, such that

$$\mathcal{X}_{in}(x, u_0, \lambda) = \frac{\varphi_{\lambda_\diamond}(x)}{\lambda - \lambda_\diamond} \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \mathcal{A}_{21,0}(z, u_0) dz + \mathcal{X}_{\lambda_\diamond}(x, \lambda), \quad (x, \lambda) \in \mathbb{R} \times B_{\lambda_\diamond}.$$

Here, $\mathcal{X}_{\lambda_\diamond}(x, \cdot)$ is analytic on $B_{\lambda_\diamond}$ for every $x \in \mathbb{R}$. Moreover, $\mathcal{X}_{\lambda_\diamond}(\cdot, \lambda)$ is exponentially localized for every $\lambda \in B_{\lambda_\diamond}$.

Proof. For $\lambda \in C_\Lambda$, the operator $\mathcal{L}_\lambda$, established in Corollary 5.4, is Fredholm of index 0 and $\mathcal{L}_\lambda$ is invertible if and only if $\mathcal{E}_{f,0}(\lambda) \neq 0$. The multiplicity of a zero $\lambda_\diamond \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ at $\lambda = \lambda_\diamond$. Combining the latter with [74, Theorem 1.3.1] settles the first two properties.

We establish the third property. The homogeneous fast eigenvalue problem (3.6) has by Proposition 5.2 an exponential dichotomy on $\mathbb{R}$ for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Thus, since $\mathcal{A}_{21,0}(\cdot, u_0)$ is exponentially localized by **(S1)** and estimate (2.6), the same holds for $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ by Proposition 4.15. The coordinate change $(v, q) \mapsto (v, \sqrt{|\lambda|} w)$ puts system (3.6) into the form,

$$\begin{aligned} D_2 v_x &= \sqrt{|\lambda|} w, \\ w_x &= \left(\frac{\partial_v G(u_0, v_h(x, u_0), 0)}{\sqrt{|\lambda|}} + \frac{\lambda}{\sqrt{|\lambda|}} \right) v, \end{aligned} \qquad (v, w) \in \mathbb{C}^{2n}, \tag{5.3}$$

where we denote by $\sqrt{\cdot}$ the principal square root. By Proposition 4.12 there exists a constant $K > 0$ such that for any

$$\lambda \in \Sigma_K := \left\{ \lambda \in C_\Lambda : \mathrm{Re}\left(\sqrt{\lambda}\right) > K \right\} \subset \left\{ \lambda \in C_\Lambda : |\lambda| > K^2 \right\}, \tag{5.4}$$

system (5.3) admits an exponential dichotomy on $\mathbb{R}$ with constants $K_1, \mu(\lambda) > 0$, where $\mu(\lambda) = \mu_1 \mathrm{Re}(\sqrt{\lambda})$ and $K_1, \mu_1 > 0$ are independent of λ. Therefore, system (3.6) has for each $\lambda \in \Sigma_K$ an exponential dichotomy on $\mathbb{R}$ with constants $K_2(\lambda), \mu(\lambda) > 0$, where $K_2(\lambda) = \sqrt{|\lambda|} K_1$. Note that $\lambda \mapsto \frac{K_2(\lambda)}{\mu(\lambda)}$ is bounded by a λ-independent constant on Σ_K. Combining this fact with Proposition 4.15 yields the third property.

Finally, let $\lambda_\diamond \in C_\Lambda$ be a simple zero of $\mathcal{E}_{f,0}$. By Corollary 5.4 the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = \lambda_\diamond$. Hence, the fourth property follows by an application of Keldysh formula – see [74, Theorem 1.6.5]. □

Remark 5.6. If $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$, then it is always possible to choose exponentially localized solutions $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ to (3.6) and its adjoint equation (3.14) satisfying (5.2). Indeed, the kernels of the operator $\mathcal{L}_{\lambda_\diamond}$ and its adjoint $\mathcal{L}_{\lambda_\diamond}^*$ are one-dimensional by Corollary 5.4. In addition, since equation (3.6) has exponential dichotomies on both half-lines by Proposition 5.2, the same holds for its adjoint (3.14). So, the spaces of exponentially localized solutions to (3.6) and (3.14) are one-dimensional. Now, take non-trivial, exponentially localized solutions $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ to (3.6) and (3.14), respectively. Since the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = \lambda_\diamond$ by Corollary 5.4, the generalized eigenvalue problem,

$$\mathcal{L}_{\lambda_\diamond}\varphi = \partial_\lambda \mathcal{L}_{\lambda_\diamond}\varphi_{\lambda_\diamond},$$

has no bounded solutions. Hence, [86, Lemma 4.2] implies

$$0 \neq \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \partial_\lambda \mathcal{L}_{\lambda_\diamond}\varphi_{\lambda_\diamond}(z)dz = \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \varphi_{\lambda_\diamond}(z)dz. \qquad \blacksquare$$

Remark 5.7. Let $f \in C_b(\mathbb{R}, \mathbb{C}^{2n})$. The Fredholm alternative in [86, Lemma 4.2] states that the inhomogeneous equation,

$$\partial_x \varphi = \mathcal{A}_{22,0}(x, u_0, \lambda)\varphi + f(x), \quad \varphi \in \mathbb{C}^{2n},$$

has a bounded solution if and only if the solvability condition,

$$\int_{-\infty}^{\infty} \psi(x)^* f(x) dx = 0,$$

is satisfied for all bounded solutions ψ to the adjoint equation (3.14). This agrees with the fact that $\mathcal{X}_{in}(x, u_0, \cdot)$ has a removable singularity at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$ if and only if we have

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \mathcal{A}_{21,0}(z, u_0) dz = 0,$$

by the fourth assertion in Proposition 5.5. $\blacksquare$

Remark 5.8. It is possible to obtain expressions for the singular part of the Laurent series of $\mathcal{X}_{in}$ at a zero of $\mathcal{E}_{f,0}$ of higher multiplicity by looking at a canonical system of generalized eigenfunctions. However, for simplicity of exposition we restrict ourselves to the (generic) case of a simple zero of $\mathcal{E}_{f,0}$. The interested reader is referred to [74, Chapter 1] for the general set-up. $\blacksquare$

Using Proposition 5.5, we prove Proposition 3.11.

Proof of Proposition 3.11. Assumption **(S1)**, estimate (2.6) and Proposition 5.5 yield that $\partial_u H_2(u_0, v_\mathrm{h}(\cdot, u_0))$ and $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ are exponentially localized for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Thus, the integral $\mathcal{G}(u_0, \lambda)$ converges for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ and $\mathcal{E}_{s,0}$ is well-defined.

It is well-known [60, Lemma 2.1.4] that, when the coefficient matrix depends analytically on a parameter, then the evolution is analytic in this parameter too. Combining this with Proposition 5.5 yields the first two properties.

Since the solution $\psi_s(\check{x})$ to the slow reduced system (2.4) crosses $\ker(I - R_s)$ at $\check{x} = \ell_0$ by **(E2)**, the slow eigenvalue problem (3.9) is R_s-reversible at $\check{x} = \ell_0$, i.e. the evolution $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ of (3.9) satisfies $R_s\mathcal{T}_s(2\ell_0, 0, \lambda)R_s = \mathcal{T}_s(0, 2\ell_0, \lambda)$ for each $\lambda \in \mathbb{C}$. Moreover, we have $R_s\Upsilon(u_0, \lambda) = \Upsilon(u_0, \lambda)^{-1}R_s$ and the matrices $\Upsilon(u_0, \lambda)$ and $\mathcal{T}_s(2\ell_0, 0, \lambda)$ have determinant 1 for any $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. This yields the third property.

Proving the fourth property is more elaborate. We denote by $C > 0$ a constant, which is independent of λ and γ. Putting $\check{y} = \sqrt{|\lambda|}\check{x}$ and $p = \sqrt{|\lambda| D_1} r$ rescales the slow eigenvalue problem (3.9) into

$$\begin{aligned} \sqrt{D_1}u_{\check{y}} &= r, \\ \sqrt{D_1}r_{\check{y}} &= \left(\frac{\partial_u H_1\left(u_s\left(|\lambda|^{-1/2}\check{y}\right), 0, 0\right)}{|\lambda|} + \frac{\lambda}{|\lambda|}\right)u, \end{aligned} \qquad (u, r) \in \mathbb{C}^{2m}, \lambda \in \mathbb{C} \setminus \{0\}. \tag{5.5}$$

Denote by $\mathcal{T}_{s1}(\check{y}, \check{z}, \lambda)$ the evolution operator of system (5.5). It holds

$$C_\lambda \Upsilon_1(\lambda)\mathcal{T}_{s1}\left(2\sqrt{|\lambda|}\ell_0, 0, \lambda\right)C_\lambda^{-1} = \Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda), \quad \lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0), \tag{5.6}$$

with

$$C_\lambda := \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\lambda| D_1} \end{pmatrix}, \quad \Upsilon_1(\lambda) := \begin{pmatrix} I & 0 \\ (|\lambda| D_1)^{-\frac{1}{2}}\,\mathcal{G}(u_0, \lambda) & I \end{pmatrix}.$$

We regard system (5.5) as a perturbation of

$$\begin{aligned} \sqrt{D_1}u_{\check{y}} &= r, \\ \sqrt{D_1}r_{\check{y}} &= \frac{\lambda}{|\lambda|}u, \end{aligned} \qquad (u, r) \in \mathbb{C}^{2m}, \lambda \in \mathbb{C} \setminus \{0\}. \tag{5.7}$$

Consider the set Σ_K defined in (5.4). Clearly, there exists a $K > 0$ such that (5.7) has for each $\lambda \in \Sigma_K$ an exponential dichotomy on $\mathbb{R}$ with λ-independent constants and rank m projections,

$$P_1(\lambda) = \frac{1}{2}\begin{pmatrix} I & -\frac{|\lambda|}{\lambda} I \\ -\frac{\lambda}{|\lambda|} I & I \end{pmatrix}. \tag{5.8}$$

Taking $K > 0$ larger if necessary, Proposition 4.12 implies that (5.5) admits an exponential dichotomy on $[0, 2\sqrt{|\lambda|}\ell_0]$ with λ-independent constants and corresponding projections $P_2(x, \lambda)$ satisfying

$$\|P_2(x, \lambda) - P_1(\lambda)\| \leq \frac{C}{|\lambda|}, \quad x \in \left[0, 2\sqrt{|\lambda|}\ell_0\right], \lambda \in \Sigma_K. \tag{5.9}$$

One readily observes from (5.8) that there exists bases $B_1^{u,s}(\lambda) \in \mathrm{Mat}_{2m\times m}(\mathbb{C})$ of $P_1(\lambda)[\mathbb{C}^{2m}] = B_1^s(\lambda)[\mathbb{C}^m]$ and $\ker(P_1(\lambda)) = B_1^u(\lambda)[\mathbb{C}^m]$ such that for each $\lambda \in \Sigma_K$ the quantity $\det(B_1^u(\lambda), B_1^s(\lambda))$ is bounded away from 0 by a λ-independent constant. Define $B_2^s(\lambda) = P_2(0,\lambda)B_1^s(\lambda)$ and $B_2^u(\lambda) = (I - P_2(2\sqrt{|\lambda|}\ell_0, \lambda))B_1^u(\lambda)$. By estimate (5.9) it holds

$$\left\|B_2^{u,s}(\lambda) - B_1^{u,s}(\lambda)\right\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_K. \tag{5.10}$$

Consider the invertible matrix,

$$\mathcal{H}(\lambda) := \left(\mathcal{T}_{s1}\left(0, 2\sqrt{|\lambda|}\ell_0, \lambda\right) B_2^u(\lambda), B_2^s(\lambda)\right), \quad \lambda \in \Sigma_K.$$

Taking $K > 0$ larger if necessary, Proposition 5.5 yields that $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ is for each $\lambda \in \Sigma_K$ exponentially localized with λ-independent decay rates. Thus, by (5.10) we have

$$\left\|\left(\Upsilon_1(\lambda)\mathcal{T}_{s1}\left(2\sqrt{\lambda}\ell_0, 0, \lambda\right) - \gamma\right)\mathcal{H}(\lambda) - \left(B_1^u(\lambda), -\gamma B_1^s(\lambda)\right)\right\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \lambda \in \Sigma_K.$$

Taking determinants in the previous expression gives

$$\left\|\mathcal{E}_{s,0}(\lambda,\gamma)\det(\mathcal{H}(\lambda)) - (-\gamma)^m \det\left(B_1^u(\lambda), B_1^s(\lambda)\right)\right\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \gamma \in S^1, \lambda \in \Sigma_K,$$

using (5.6). Since $\lambda \mapsto \det(B_1^u(\lambda), B_1^s(\lambda))$ is bounded away from zero on Σ_K by a λ-independent constant and $\det(\mathcal{H}(\lambda))$ is non-zero on Σ_K, the slow Evans function $\mathcal{E}_{s,0}$ has no roots in $\Sigma_K \times S^1$, provided $K > 0$ is sufficiently large. This proves the third property, because $C_\Lambda \setminus \Sigma_K$ is bounded. □

Finally, we establish the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot,\gamma)$ at a simple zero of $\mathcal{E}_{f,0}$ and thereby prove Proposition 3.12.

Corollary 5.9. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Suppose $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$. Let $\varphi_{\lambda_\diamond} = (\varphi_{\lambda_\diamond,1}, \varphi_{\lambda_\diamond,2}), \psi_{\lambda_\diamond} = (\psi_{\lambda_\diamond,1}, \psi_{\lambda_\diamond,2}), B_{\lambda_\diamond}$ and $\mathcal{X}_{\lambda_\diamond}$ as in Proposition 5.5. Define for $\lambda \in B_{\lambda_\diamond}$*

$$\begin{aligned}
\varphi &:= \int_{-\infty}^{\infty} \partial_v H_2(u_0, v_{\mathrm{h}}(z))\varphi_{\lambda_\diamond,1}(z)dz \in \mathbb{C}^m,\\
\psi &:= \int_{-\infty}^{\infty} \psi_{\lambda_\diamond,2}(z)^* \partial_u G(u_0, v_{\mathrm{h}}(z), 0)dz \in \mathrm{Mat}_{1\times m}(\mathbb{C}),\\
\mathcal{G}_a(\lambda) &:= \int_{-\infty}^{\infty} \left[\partial_u H_2(u_0, v_{\mathrm{h}}(z)) + \partial_v H_2(u_0, v_{\mathrm{h}}(z))\mathcal{V}_{\lambda_\diamond}(z,\lambda)\right] dz \in \mathrm{Mat}_{m\times m}(\mathbb{C}),
\end{aligned} \tag{5.11}$$

where $\mathcal{V}_{\lambda_\diamond}$ denotes the upper-left $(n\times m)$-block of the $(2n\times 2m)$-matrix $\mathcal{X}_{\lambda_\diamond}$. Moreover, let $(u_i(x,\lambda), p_i(x,\lambda))$, $i = 1,\ldots,2m$ be a fundamental set of solutions to the slow eigenvalue problem (3.9). Finally, let $C(\lambda,\gamma)$ be the cofactor matrix of

$$\mathcal{U}_a(\lambda,\gamma) := \begin{pmatrix} I & 0 \\ \mathcal{G}_a(\lambda) & I \end{pmatrix}\mathcal{T}_s(2\ell_0, 0, \lambda) - \gamma I \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C}).$$

For all $\gamma \in S^1$, the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot, \gamma)$ at $\lambda_\diamond$ is given by

$$\frac{1}{\lambda - \lambda_\diamond} \sum_{i=1}^{2m} (\psi u_i(2\ell_0, \lambda_\diamond)) \left(\varphi^{\mathrm{T}} \left[C_{ji}(\lambda_\diamond, \gamma) \right]_{j=m+1}^{2m} \right).$$

Proof. Assume $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$. Using the Laurent series of $\mathcal{X}_{in}$ provided in Proposition 5.5, we can split off the singular part of $\mathcal{G}(u_0, \lambda)$ at $\lambda_\diamond$. Indeed, we have

$$\mathcal{G}(u_0, \lambda) = \frac{1}{\lambda - \lambda_\diamond} \varphi\psi + \mathcal{G}_a(\lambda), \qquad \lambda \in B_{\lambda_\diamond}.$$

Using the multi-linearity of the determinant, we expand

$$\begin{aligned} \mathcal{E}_{s,0}(\lambda, \gamma) &= \det \left[\mathcal{U}_a(\lambda, \gamma) + \frac{1}{\lambda - \lambda_\diamond} \begin{pmatrix} 0 & 0 \\ \varphi\psi & 0 \end{pmatrix} \mathcal{T}_s(2\ell_0, 0, \lambda) \right] \\ &= \det(\mathcal{U}_a(\lambda, \gamma)) + \frac{1}{\lambda - \lambda_\diamond} \sum_{i=1}^{2m} (\psi u_i(2\ell_0, \lambda_\diamond)) \left(\varphi^{\mathrm{T}} \left[C_{ji}(\lambda_\diamond, \gamma) \right]_{j=m+1}^{2m} \right), \end{aligned}$$

for $\lambda \in B_{\lambda_\diamond}$ and $\gamma \in S^1$. □

Remark 5.10. In the case $m = 1$, Propositions 3.25 and 3.28 imply that γ appears as a factor in the singular part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot, \gamma)$ at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$. Therefore, $\mathcal{E}_{s,0}(\cdot, \gamma)$ has a pole at $\lambda_\diamond$ for *some* $\gamma \in S^1$ if and only if $\mathcal{E}_{s,0}(\cdot, \gamma)$ has a pole at $\lambda_\diamond$ for *all* $\gamma \in S^1$. However, in the general setting of Corollary 5.9, the principal part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot, \gamma)$ is polynomial in γ. So, it could happen that $\mathcal{E}_{s,0}(\cdot, \gamma)$ has a pole at $\lambda_\diamond$ for all but a discrete set of $\gamma \in S^1$. We expect that such a (non-generic) situation occurs precisely when $\lambda_\diamond$ is a limit point of the zero set $\bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_{s,0}(\lambda, \gamma) = 0\}$. ■

5.2 Approximation of the roots of the Evans function

5.2.1 Introduction

In this section we prove Theorem 3.15. Our plan is to factorize the Evans function into a fast and a slow component:

$$\mathcal{E}_\varepsilon(\lambda, \gamma) = \mathcal{E}_{f,\varepsilon}(\lambda, \gamma)\mathcal{E}_{s,\varepsilon}(\lambda, \gamma), \tag{5.12}$$

and to approximate the factors by the fast and slow Evans functions $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$. The factorization (5.12) is induced by diagonalizing the full eigenvalue problem (3.3) via the Riccati transform, which is established in §4.6. Rescaling the p-coordinate in (3.3) by a factor $\sqrt{\varepsilon}$ yields the equivalent system,

$$\varphi_x = \begin{pmatrix} \sqrt{\varepsilon}\tilde{\mathcal{A}}_{11,\varepsilon}(x, \lambda) & \sqrt{\varepsilon}\mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix} \varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{5.13}$$

where $\mathcal{A}_{12,\varepsilon}, \mathcal{A}_{21,\varepsilon}$ and $\mathcal{A}_{22,\varepsilon}$ are as in (3.4) and

$$\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) := \begin{pmatrix} 0 & D_1^{-1} \\ \varepsilon\left(\partial_u H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) + \lambda\right) + \partial_u H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}.$$

System (5.13) has the required slow-fast form (4.19) for an application of the Riccati transform. Moreover, the evolution matrices of systems (3.3) and (5.13) are similar. Therefore, it holds

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \det(\tilde{\mathcal{T}}_\varepsilon(0,-L_\varepsilon,\lambda) - \gamma\tilde{\mathcal{T}}_\varepsilon(0,L_\varepsilon,\lambda)), \tag{5.14}$$

where $\tilde{\mathcal{T}}_\varepsilon(x,y,\lambda)$ is the evolution operator of system (5.13). Yet, an application of the Riccati transformation to (5.13) is only legitimate when system,

$$\psi_x = \mathcal{A}_{22,\varepsilon}(x,\lambda)\psi, \quad \psi \in \mathbb{C}^{2n}, \tag{5.15}$$

has an exponential dichotomy on $\mathbb{R}$. If λ is not a zero of the fast Evans function, then the homogeneous fast eigenvalue problem (3.6) admits an exponential dichotomy on $\mathbb{R}$ by Proposition 5.2. Using roughness techniques the exponential dichotomy of (3.6) carries over to the perturbed problem (5.15), whenever λ is away from $\mathcal{E}_{f,0}^{-1}(0)$. In that case, system (5.13) diagonalizes via the Riccati transform. Consequently, using the periodicity of system (5.13) and identity (5.14), the Evans function $\mathcal{E}_\varepsilon$ factorizes as (5.12) for λ away from the roots of $\mathcal{E}_{f,0}$.

The two blocks, in which (5.13) diagonalizes, can be approximated in terms of the three eigenvalue problems (3.6), (3.8) and (3.9). This corresponds to approximating the factor $\mathcal{E}_{f,\varepsilon}$ by the fast Evans function $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,\varepsilon}$ by the slow Evans function $\mathcal{E}_{s,0}$. Thus, we obtain the desired approximation of the roots of $\mathcal{E}_\varepsilon$ by the zeros of the reduced Evans function $\mathcal{E}_0(\lambda,\gamma) = (-\gamma)^n\mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda,\gamma)$ using Rouché's Theorem.

This section is structured as follows. We start by showing that the spectrum of the linearization $\mathcal{L}_\varepsilon$ is contained in an ε-independent sector. This provides an important a priori bound on the magnitude of the roots of the Evans function $\mathcal{E}_\varepsilon$. Subsequently, we establish an exponential dichotomy for system (5.15) for λ away from the zeros of $\mathcal{E}_{f,0}$. Then, the Riccati transform yields the desired diagonalization of (5.13) and the factorization of $\mathcal{E}_\varepsilon$. Then, we link the factors $\mathcal{E}_{f,\varepsilon}$ and $\mathcal{E}_{s,\varepsilon}$ to $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$. Finally, we apply Rouché's Theorem to conclude the proof of Theorem 3.15.

5.2.2 A priori bounds on the spectrum

In §3.2 we established the linearization $\mathcal{L}_\varepsilon$ of (1.9) about the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$. By [44, Theorem 1.3.2], the differential operator $\mathcal{L}_\varepsilon$ is sectorial as a sum of a sectorial and a bounded operator. The bounded part involves multiplication with the matrix function $\mathcal{B}_\varepsilon(x)$, defined in (3.1), which has a norm of order $\mathcal{O}(\varepsilon^{-1})$. Yet, the spectrum of $\mathcal{L}_\varepsilon$ is confined to an ε-independent sector.

Proposition 5.11. *For $\varepsilon > 0$ sufficiently small, there exists constants $\omega \in \mathbb{R}_{>0}$ and $\varpi \in (\pi/2, \pi)$, both independent of ε, such that the sector $\Sigma := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda-\omega)| \leq \varpi\} \cup \{\omega\}$ is contained in the resolvent set $\rho(\mathcal{L}_\varepsilon)$.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Our approach is to decompose $\mathcal{L}_\varepsilon$ in more elementary building blocks in order to control the ε^{-1}-terms in $\mathcal{B}_\varepsilon$. First, we show that the operator $\mathcal{L}_{1,\varepsilon} \colon C^2_{ub}(\mathbb{R}, \mathbb{R}^m) \subset C_{ub}(\mathbb{R}, \mathbb{R}^m) \to C_{ub}(\mathbb{R}, \mathbb{R}^m)$ given by

$$\mathcal{L}_{1,\varepsilon} u = D_1 u_{\check{x}\check{x}} + \varepsilon^{-1} \partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot)) u,$$

is sectorial with an ε-independent sector. Subsequently, we prove this for $\widehat{\mathcal{L}_\varepsilon} \colon C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \subset C_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \to C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{L}_{1,\varepsilon} u + \varepsilon^{-1} \partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot)) v \\ \varepsilon^2 D_2 v_{\check{x}\check{x}} \end{pmatrix}.$$

Finally, we regard $\mathcal{L}_\varepsilon$ as a perturbation of $\widehat{\mathcal{L}_\varepsilon}$ by a bounded operator with $O(1)$-norm.

Our goal is to show that the spectrum of the periodic differential operator $\mathcal{L}_{1,\varepsilon}$ is contained in an ε-independent sector. By [38, Proposition 2.1] it is sufficient to show that the associated eigenvalue problem,

$$\begin{aligned} \sqrt{D_1} u_{\check{x}} &= \sqrt{\lambda} p, \\ \sqrt{D_1} p_{\check{x}} &= \left(\sqrt{\lambda} + \frac{\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))}{\sqrt{\lambda}\varepsilon} \right) u, \end{aligned} \qquad (u,p) \in \mathbb{C}^{2m}, \tag{5.16}$$

has no non-trivial bounded solutions for λ in some ε-independent sector. Here, $\sqrt{\cdot}$ is the principal square root. Denote by $\mathcal{T}_{1,\varepsilon}(\check{x}, \check{y}, \lambda)$ the evolution operator of system (5.16) and let $\mathcal{T}_1(\check{x}, \check{y}, \lambda)$ be the evolution operator of

$$\begin{aligned} \sqrt{D_1} u_{\check{x}} &= \sqrt{\lambda} p, \\ \sqrt{D_1} p_{\check{x}} &= \sqrt{\lambda} u, \end{aligned} \qquad (u,p) \in \mathbb{C}^{2m}. \tag{5.17}$$

One readily observes that, whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, system (5.17) has an exponential dichotomy on $\mathbb{R}$ with constants $C_0, \mu_\lambda > 0$ with $C_0 = 1$ and $\mu_\lambda := \|D_1\|^{-1/2} \mathrm{Re}(\sqrt{\lambda})$. Since we have $|\varepsilon L_\varepsilon - \ell_0| < C\varepsilon$ by Theorem 2.3, there exists a constant $C_1 > 0$, independent of ε, such that for all $\lambda \in P_1 := \{\mu \in \mathbb{C} : \mathrm{Re}(\sqrt{\mu}) \geq C_1\}$ it holds $h_\lambda := \mu_\lambda^{-1} \sinh^{-1}(4) \leq \ell_\varepsilon := \varepsilon L_\varepsilon$.

Let $\lambda \in P_1$. Using **(S1)** and Theorem 2.3 we estimate

$$\|\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))\| \leq C e^{-\varepsilon^{-1}\mu_0|\check{x}|}, \quad \check{x} \in [-\ell_\varepsilon, \ell_\varepsilon].$$

Let $\check{w}, \check{z} \in \mathbb{R}$ such that $0 \leq \check{z} - \check{w} \leq 2h_\lambda \leq 2\ell_\varepsilon$. Taking into account the $2\ell_\varepsilon$-periodicity of $\check{\phi}_{\mathrm{p},\varepsilon}$, we have

$$\int_{\check{w}}^{\check{z}} \frac{\|\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))\|}{\sqrt{|\lambda|}\varepsilon} d\check{x} \leq \frac{C}{\sqrt{|\lambda|}}.$$

Thus, by Proposition 4.1, we establish

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda) - \mathcal{T}_{1,\varepsilon}(\check{z}, \check{w}, \lambda)\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \check{w}, \check{z} \in \mathbb{R} \text{ with } |\check{w} - \check{z}| \leq 2h_\lambda,$$

where we use that the evolution operator of (5.17) satisfies

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda)\| \leq Ce^{\mathrm{Re}(\sqrt{\lambda})|\check{z}-\check{w}|}, \quad \check{w}, \check{z} \in \mathbb{R}.$$

So, there exists an ε-independent constant $C_2 > 0$ such that, whenever $\lambda \in P_1$ satisfies $|\lambda| > C_2$, then it holds

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda) - \mathcal{T}_{1,\varepsilon}(\check{z}, \check{w}, \lambda)\| < 1, \quad \check{w}, \check{z} \in \mathbb{R} \text{ with } |\check{w} - \check{z}| \leq 2h_\lambda. \tag{5.18}$$

Now let Σ_1 be an ε-independent sector disjoint from $B(0, C_2) \cup [\mathbb{C} \setminus P_1]$ – see Figure 5.1. For all $\lambda \in \Sigma_1$, there are no non-trivial, bounded solutions to (5.16) by combining (5.18) with Proposition 4.14. So, by [38, Proposition 2.1] the resolvent set $\rho(\mathcal{L}_{1,\varepsilon})$ contains the ε-independent sector Σ_1.

Consider the elliptic operator $\mathcal{L}_2 \colon C^2_{ub}(\mathbb{R}, \mathbb{R}^n) \subset C_{ub}(\mathbb{R}, \mathbb{R}^n) \to C_{ub}(\mathbb{R}, \mathbb{R}^n)$ given by $\mathcal{L}_2 v = D_2 v_{\check{x}\check{x}}$. Clearly, we have $\rho(\mathcal{L}_2) = \mathbb{C} \setminus \mathbb{R}_{\leq 0} \supset \Sigma_1$. For $\lambda \in \Sigma_1$ the operator on $C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} (\mathcal{L}_{1,\varepsilon} - \lambda)^{-1}(u - \varepsilon^{-1}\partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot))(\varepsilon^2\mathcal{L}_2 - \lambda)^{-1}(v)) \\ (\varepsilon^2\mathcal{L}_2 - \lambda)^{-1}(v) \end{pmatrix},$$

is an inverse of $\widehat{\mathcal{L}_\varepsilon} - \lambda$. Therefore, the sector Σ_1 is contained in the resolvent set $\rho(\widehat{\mathcal{L}_\varepsilon})$.

Define

$$\mathcal{B}_{b,\varepsilon}(\check{x}) = \begin{pmatrix} \partial_u H_1(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) & \partial_v H_1(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) \\ \partial_u G(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) & \partial_v G(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) \end{pmatrix}.$$

Let $\mathcal{L}_{b,\varepsilon} \colon C_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \to C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ be the multiplication operator $[\mathcal{L}_{b,\varepsilon}\varphi](\check{x}) = \mathcal{B}_{b,\varepsilon}(\check{x})\varphi$. By Theorem 2.3 the norm of $\mathcal{L}_{b,\varepsilon}$ is bounded by an ε-independent constant.

Invoking [44, Theorem 1.3.2] and its proof yields the conclusion: the sum $\mathcal{L}_\varepsilon = \widehat{\mathcal{L}_\varepsilon} + \mathcal{L}_{b,\varepsilon}$ with domain $C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ is sectorial with an ε-independent sector $\Sigma \subset \rho(\mathcal{L}_\varepsilon)$, using that $\|\mathcal{L}_{b,\varepsilon}\|$ is bounded by an ε-independent constant and Σ_1 is independent of ε. □

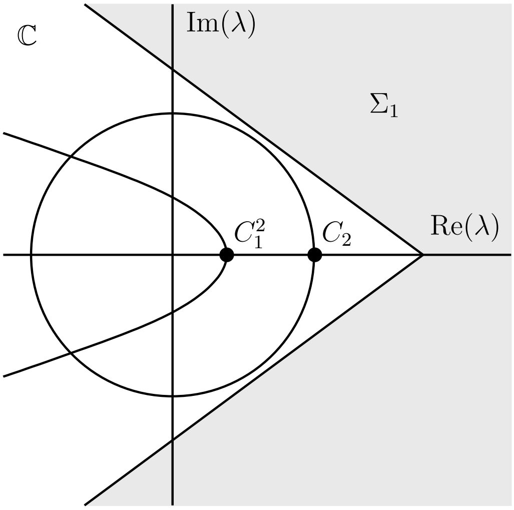

Figure 5.1: Construction of the sector Σ_1 in the proof of Proposition 5.11.

Let $\gamma \in S^1$. By Propositions 3.7 and 5.11 the roots of the Evans function $\mathcal{E}_\varepsilon(\cdot, \gamma)$ in the half-plane C_Λ are confined to an ε- and γ-independent bounded region. In addition, by Propositions 5.2 and 3.11 the same holds for the zeros of the reduced Evans function $\mathcal{E}_0(\cdot, \gamma)$. Thus, we have established the following result.

Corollary 5.12. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. There exists an open and bounded set $\Sigma_{\Lambda,0} \subset C_\Lambda$ such that*

$$\bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda, \gamma) = 0 \text{ or } \mathcal{E}_\varepsilon(\lambda, \gamma) = 0\} \subset \Sigma_{\Lambda,0}.$$

Thus, when proving Theorem 3.15, we may without loss of generality restrict ourselves to the set $\Sigma_{\Lambda,0}$ by the a priori bounds in Corollary 5.12.

5.2.3 An exponential dichotomy capturing the fast dynamics

We wish to apply the Riccati transformation to the rescaled full eigenvalue problem (5.13) in order to factorize $\mathcal{E}_\varepsilon$ into a fast and a slow part as in (5.12). However, according to Theorem 4.19 this is only legitimate, when system (5.15) has an exponential dichotomy on $\mathbb{R}$. By Proposition 5.2 the homogeneous fast eigenvalue problem (3.6) admits an exponential dichotomy on $\mathbb{R}$, whenever $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Using roughness techniques the exponential dichotomy of (3.6) carries over to the perturbed problem (5.15). Therefore, we establish the following result.

Notation 5.13. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1 and $\Sigma_{\Lambda,0} \subset C_\Lambda$ as in Corollary 5.12. For $\delta > 0$, we denote

$$\Sigma_{\Lambda,\delta} := \Sigma_{\Lambda,0} \setminus \bigcup_{\lambda \in \mathcal{E}_{f,0}^{-1}(0)} B(\lambda, \delta).$$

Theorem 5.14. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, systems* (3.6) *and* (5.15) *have for all $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with ε- and λ-independent constants $C, \mu_f > 0$.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ and ε.

Our approach is as follows. First, we establish an exponential dichotomy for (5.15) on an interval $[a, 2L_\varepsilon - a]$ for some $a > 0$, using that the coefficient matrix $\mathcal{A}_{22,\varepsilon}(x, \lambda)$ has slowly varying coefficients and is pointwise hyperbolic along the slow manifold. We extend the exponential dichotomy to $[0, 2L_\varepsilon]$. Similarly, we obtain an exponential dichotomy for (5.15) on $[-2L_\varepsilon, 0]$.

Subsequently, we calculate the minimal opening between the kernels and ranges of the dichotomy projections at 0. By approximating system (5.15) by the fast eigenvalue problem (3.6), we show that, whenever λ is contained in $\Sigma_{\Lambda,\delta}$, this minimal opening is substantial. Therefore, Lemma 4.11 provides exponential dichotomies for (5.15) on $[-2L_\varepsilon, 2L_\varepsilon]$ and for (3.6) on $\mathbb{R}$. Finally, we extend the exponential dichotomy of (5.15) to $\mathbb{R}$.

We start by establishing exponential dichotomies for (5.15) on $[0, 2L_\varepsilon]$ and on $[-2L_\varepsilon, 0]$. Theorem 2.3 yields the following estimates,

$$\begin{aligned} \|v_{\mathrm{p},\varepsilon}(x)\| &\leq C e^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}, \\ \|v'_{\mathrm{p},\varepsilon}(x)\| = \|D_2^{-1} q_{\mathrm{p},\varepsilon}(x)\| &\leq C e^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}, \qquad x \in [0, 2L_\varepsilon], \\ \|u'_{\mathrm{p},\varepsilon}(x)\| = \varepsilon \left\|D_1^{-1} p_{\mathrm{p},\varepsilon}(x)\right\| &\leq C\varepsilon, \end{aligned}$$

which imply

$$\left\|\partial_x \mathcal{A}_{22,\varepsilon}(x, \lambda)\right\|, \left\|\mathcal{A}_{22,\varepsilon}(x, \lambda) - A(u_{\mathrm{p},\varepsilon}(x), \lambda)\right\| \leq C \max\left\{\varepsilon, e^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}\right\}, \tag{5.19}$$

for $x \in [0, 2L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$, where $A(u, \lambda)$ is defined in (5.1). First, by Theorem 2.3 and Lemma 5.1, there exists an ε-independent constant $\alpha > 0$ such that, for $\varepsilon > 0$ sufficiently small, the matrix $A(u_{\mathrm{p},\varepsilon}(x), \lambda)$ is hyperbolic for each $x \in [0, 2L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$ with spectral gap larger than 2α. Thus, by estimate (5.19) there exists $x_0 > 0$, independent of ε, such that $\mathcal{A}_{22,\varepsilon}$ is hyperbolic on $[x_0, 2L_\varepsilon - x_0] \times \Sigma_{\Lambda,\delta}$ with spectral gap larger than α. Second, $\mathcal{A}_{22,\varepsilon}$ is bounded on $[0, 2L_\varepsilon] \times \Sigma_{\Lambda,\delta}$ by an ε-independent constant using Theorem 2.3. Thus, taking $x_0 > 0$ larger if necessary, Proposition 4.8 and (5.19) yield, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy for system (5.15) on $[x_0, 2L_\varepsilon - x_0]$ with ε- and λ-independent constants. Using Lemma 4.9 we extend this to an exponential dichotomy on $[0, 2L_\varepsilon]$ with constants independent of ε and λ. Similarly, we obtain an exponential dichotomy for (5.15) on $[-2L_\varepsilon, 0]$. We conclude that (5.15) has exponential dichotomies on both $[0, 2L_\varepsilon]$ and $[-2L_\varepsilon, 0]$ for every $\lambda \in \Sigma_{\Lambda,\delta}$ with constants $C, \alpha_f > 0$, independent of ε and λ.

We compare system (5.15) with the homogeneous fast eigenvalue problem (3.6). First, by Theorem 2.3 and estimate (2.6), the corresponding coefficient matrices $\mathcal{A}_{22,\varepsilon}$ and $\mathcal{A}_{22,0}(\cdot, u_0, \cdot)$ are bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ by a constant $M > 1$, which is independent of ε. Second, Theorem 2.3 yields

$$\left\|\mathcal{A}_{22,\varepsilon}(x, \lambda) - \mathcal{A}_{22,0}(x, u_0, \lambda)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [\log(\varepsilon), -\log(\varepsilon)], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.20}$$

Denote by $\mathcal{T}_r(x, y, \lambda)$ and $\mathcal{T}_{f,\varepsilon}(x, y, \lambda)$ the evolution operators of (3.6) and (5.15), respectively. Using Lemma 4.1 and (5.20) we estimate

$$\|\mathcal{T}_r(x, y, \lambda) - \mathcal{T}_{f,\varepsilon}(x, y, \lambda)\| < 1, \quad x, y \in [\log(\varepsilon)/4M, -\log(\varepsilon)/4M], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.21}$$

We recall some facts from Proposition 5.2. First, system (3.6) admits for each $\lambda \in C_\Lambda$ exponential dichotomies on both half-lines with constants that depend continuously on λ. Second, the corresponding projections $Q_{f,\pm}(x, \lambda)$ are analytic in λ. Third, the subspaces $E_0^s(\lambda) := Q_{f,+}(0, \lambda)[\mathbb{C}^{2n}]$ and $E_0^u(\lambda) := \ker(Q_{f,-}(0, \lambda))$ are complementary for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Therefore, Proposition 4.18 implies that the continuous map $\eta_r\colon C_\Lambda \to [0, \infty)$ given by the minimal opening $\eta_r(\lambda) = \eta(E_0^s(\lambda), E_0^u(\lambda))$ is bounded away from 0 on the set $\Sigma_{\Lambda,\delta}$. Hence, the projection on $E_0^s(\lambda)$ along $E_0^u(\lambda)$ is well-defined on $\Sigma_{\Lambda,\delta}$ and bounded by a λ-independent constant by Proposition 4.18. Thus, Lemma 4.11 yields for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy of the homogeneous fast eigenvalue problem (3.6) on $\mathbb{R}$ with λ-independent constants.

Denote by $Q_{\pm,\varepsilon}(x, \lambda)$ the projections corresponding to the exponential dichotomies of (5.15) on $[0, 2L_\varepsilon]$ and on $[-2L_\varepsilon, 0]$. Let $\lambda \in \Sigma_{\Lambda,\delta}$. By combining estimate (5.21) with Lemma 4.13, there exists for each $w \in E_\varepsilon^s(\lambda) := Q_{+,\varepsilon}(0, \lambda)[\mathbb{C}^{2n}]$ an element $v \in E_0^s(\lambda)$ such that

$$\|v - w\| \leq C\varepsilon^{\alpha_f/4M}\|w\|. \tag{5.22}$$

Similarly, there exists for each $w \in E_\varepsilon^u(\lambda) := \ker(Q_{-,\varepsilon}(0, \lambda))$ a vector $v \in E_0^u(\lambda)$ such that (5.22) holds true. Therefore, Proposition 4.18 yields the estimate

$$\left|\eta_r(\lambda) - \eta(E_\varepsilon^s(\lambda), E_\varepsilon^u(\lambda))\right| \leq C\varepsilon^{\alpha_f/4M}, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.23}$$

Finally, we establish the desired exponential dichotomy for (5.15) on $\mathbb{R}$. Recall that the map η_r is bounded away from 0 on $\Sigma_{\Lambda,\delta}$. Thus, by estimate (5.23) and Proposition 4.18 one deduces that, for $\varepsilon > 0$ sufficiently small, $E_\varepsilon^s(\lambda)$ and $E_\varepsilon^u(\lambda)$ are complementary on $\Sigma_{\Lambda,\delta}$. So, the projection $Q_\varepsilon(\lambda)$ onto $E_\varepsilon^s(\lambda)$ along $E_\varepsilon^u(\lambda)$ is well-defined for $\lambda \in \Sigma_{\Lambda,\delta}$. In addition, by Proposition 4.18 and (5.23), the norm of Q_ε is bounded on $\Sigma_{\Lambda,\delta}$ by an ε-independent constant. Therefore, Lemma 4.11 implies that (5.15) admits an exponential dichotomy for each $\lambda \in \Sigma_{\Lambda,\delta}$ on $[-2L_\varepsilon, 2L_\varepsilon]$ with λ- and ε-independent constants. Subsequently, for each

$\lambda \in \Sigma_{\Lambda,\delta}$, Lemma 4.10 yields an exponential dichotomy for system (5.15) on $\mathbb{R}$ with λ- and ε-independent constants, where we use that the coefficient matrix $\mathcal{A}_{22,\varepsilon}$ is ε-uniformly bounded on $\mathbb{R} \times \Sigma_{\Lambda,\delta}$. □

In Theorem 5.14 we established exponential dichotomies on $\mathbb{R}$ for the homogeneous fast eigenvalue problem (3.6) and its perturbation (5.15). This enables us to compare solutions to the inhomogeneous fast eigenvalue problem (3.8) and its perturbation,

$$\Psi_x = \mathcal{A}_{22,\varepsilon}(x,\lambda)\Psi + \mathcal{A}_{21,\varepsilon}(x), \quad \Psi \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}). \tag{5.24}$$

Corollary 5.15. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For each $\lambda \in \Sigma_{\Lambda,\delta}$, there exists a unique bounded solution $\Psi_\varepsilon(x,\lambda)$ to* (5.24) *satisfying*

$$\|\Psi_\varepsilon(x,\lambda) - \mathcal{X}_{in}(x,u_0,\lambda)\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [-L_\varepsilon, L_\varepsilon], \lambda \in \Sigma_{\Lambda,\delta},$$

where $C > 0$ is a λ- and ε-independent constant.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Systems (3.6) and (5.15) have by Theorem 5.14 for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu_f > 0$, which are independent of ε and λ. Let $\mu_0 > 0$ be as in Theorem 2.3 and take $\chi := 2/\min\{\mu_f, \mu_0\}$. Theorem 2.3 yields

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \mathcal{A}_{22,0}(x,u_0,\lambda)\right\|, \left\|\mathcal{A}_{21,\varepsilon}(x) - \mathcal{A}_{21,0}(x,u_0)\right\| \leq C\varepsilon|\log(\varepsilon)|,$$

for $x \in [2\chi\log(\varepsilon), -2\chi\log(\varepsilon)]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Now, we apply Proposition 4.15 to the inhomogeneous equations (3.8) and (5.24): there exists a unique bounded solution $\Psi_\varepsilon(x,\lambda)$ to (5.24) satisfying

$$\|\Psi_\varepsilon(x,\lambda) - \mathcal{X}_{in}(x,u_0,\lambda)\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [\chi\log(\varepsilon), -\chi\log(\varepsilon)], \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.25}$$

where we use that $\mathcal{A}_{22,\varepsilon}, \mathcal{A}_{22,0}(\cdot,u_0,\cdot)$ and $\mathcal{X}_{in}(\cdot,u_0,\cdot)$ are ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ and $\mathcal{A}_{21,\varepsilon}$ and $\mathcal{A}_{21,0}(\cdot,u_0)$ are ε-uniformly bounded on $\mathbb{R}$ by Theorem 2.3 and Proposition 5.5. Furthermore, by Theorem 2.3, estimate (2.6) and **(S1)** we have

$$\begin{aligned} \|\mathcal{A}_{21,\varepsilon}(x)\| &\leq Ce^{-\mu_0|x|}, \quad x \in [-L_\varepsilon, L_\varepsilon], \\ \|\mathcal{A}_{21,0}(x,u_0)\| &\leq Ce^{-\mu_h|x|}, \quad x \in \mathbb{R}, \end{aligned} \quad \lambda \in \Sigma_{\Lambda,\delta}.$$

Combing the latter with Proposition 4.15 implies

$$\begin{aligned} \|\Psi_\varepsilon(x,\lambda)\| &\leq Ce^{-\min\{\mu_f,\mu_0\}|x|/2}, \quad x \in [-L_\varepsilon, L_\varepsilon]. \\ \|\mathcal{X}_{in}(x,u_0,\lambda)\| &\leq Ce^{-\min\{\mu_f,\mu_h\}|x|/2}, \quad x \in \mathbb{R}, \end{aligned} \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.26}$$

which proves that (5.25) actually holds for all $x \in [-L_\varepsilon, L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. □

5.2.4 Factorization of the Evans function via the Riccati transform

We employ the Riccati transform to diagonalize the rescaled full eigenvalue problem (5.13). This yields the factorization (5.12) of the Evans function.

Theorem 5.16. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, there exists a function $U_\varepsilon \colon \mathbb{R} \times \Sigma_{\Lambda,\delta} \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ such that we have the factorization,*

$$\mathcal{E}_\varepsilon(\lambda, \gamma) = \mathcal{E}_{s,\varepsilon}(\lambda, \gamma)\mathcal{E}_{f,\varepsilon}(\lambda, \gamma), \quad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in \mathbb{C},$$

with $\mathcal{E}_{s,\varepsilon}, \mathcal{E}_{f,\varepsilon} \colon \Sigma_{\Lambda,\delta} \times \mathbb{C} \to \mathbb{C}$ given by

$$\begin{aligned}\mathcal{E}_{s,\varepsilon}(\lambda, \gamma) &:= \det(\mathcal{T}_{sd,\varepsilon}(0, -L_\varepsilon, \lambda) - \gamma \mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda)),\\ \mathcal{E}_{f,\varepsilon}(\lambda, \gamma) &:= \det(\mathcal{T}_{fd,\varepsilon}(0, -L_\varepsilon, \lambda) - \gamma \mathcal{T}_{fd,\varepsilon}(0, L_\varepsilon, \lambda)).\end{aligned}$$

where $\mathcal{T}_{sd,\varepsilon}(x, y, \lambda)$ is the evolution operator of system,

$$\chi_x = \sqrt{\varepsilon}\left(\tilde{\mathcal{A}}_{11,\varepsilon}(x, \lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x, \lambda)\right)\chi, \quad \chi \in \mathbb{C}^{2m}, \tag{5.27}$$

and $\mathcal{T}_{fd,\varepsilon}(x, y, \lambda)$ is the evolution operator of system,

$$\omega_x = \left(\mathcal{A}_{22,\varepsilon}(x, \lambda) - \sqrt{\varepsilon}U_\varepsilon(x, \lambda)\mathcal{A}_{12,\varepsilon}(x)\right)\omega, \quad \omega \in \mathbb{C}^{2n}. \tag{5.28}$$

In addition, U_ε enjoys the following properties:

1. *U_ε is bounded by an ε-independent constant on its domain $\mathbb{R} \times \Sigma_{\Lambda,\delta}$;*
2. *$U_\varepsilon(\cdot, \lambda)$ is $2L_\varepsilon$-periodic for each $\lambda \in \Sigma_{\Lambda,\delta}$;*
3. *Take*

$$\Xi_\varepsilon := -\frac{12\log(\varepsilon)}{\min\{\mu_{\mathrm{h}}, \mu_0, \mu_f\}},$$

where $\mu_{\mathrm{h}} > 0$ is as in (2.6), $\mu_0 > 0$ is as in Theorem 2.3 and $\mu_f > 0$ is as in Theorem 5.14. It holds,

$$\left.\begin{aligned}\|U_\varepsilon(x, \lambda) - X_{in}(x, u_0, \lambda)\| &\leq C\sqrt{\varepsilon}|\log(\varepsilon)|, & x &\in [0, 2L_\varepsilon],\\ \|U_\varepsilon(x, \lambda)\| &\leq C\varepsilon^3, & x &\in [\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon],\end{aligned}\right\} \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.29}$$

where $C > 0$ is a λ- and ε-independent constant.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ and ε.

System (5.13) is clearly of the slow-fast form (4.19) with coefficient matrices that are ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ by Theorem 2.3. Furthermore, by Theorem 5.14, system (5.15) admits for every $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with λ- and ε-independent constants $C, \mu_f > 0$. Hence, we can apply the Riccati transform to (5.13). Thus, Theorem 4.19 yields matrix functions $H_\varepsilon(x,\lambda) \in \mathrm{Mat}_{2(m+n)\times 2(m+n)}(\mathbb{C})$ and $U_\varepsilon(x,\lambda) \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ such that the change of variables $\varphi(x) = H_\varepsilon(x,\lambda)\psi(x)$ transforms (5.13) into the diagonal system,

$$\psi_x = \begin{pmatrix} \sqrt{\varepsilon}\left(\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda)\right) & 0 \\ 0 & \mathcal{A}_{22,\varepsilon}(x,\lambda) - \sqrt{\varepsilon}U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x) \end{pmatrix}\psi, \quad (5.30)$$

with $\psi \in \mathbb{C}^{2(m+n)}$. The evolution $\mathcal{T}_{d,\varepsilon}(x,y,\lambda)$ of system (5.30) is a block-diagonal matrix with consecutively $\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)$ and $\mathcal{T}_{fd,\varepsilon}(x,y,\lambda)$ on its diagonal. Furthermore, $H_\varepsilon(\cdot,\lambda)$ and $U_\varepsilon(\cdot,\lambda)$ are $2L_\varepsilon$-periodic by Theorem 4.19 for any $\lambda \in \Sigma_{\Lambda,\delta}$. Finally, since $H_\varepsilon(x,\lambda)$ is a product of two triangular matrices with only ones on the diagonal by (4.23), the determinant of $H_\varepsilon(x,\lambda)$ equals 1 for every $(x,\lambda) \in \mathbb{R}\times\Sigma_{\Lambda,\delta}$. Therefore, we obtain the factorization,

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \det\left(H_\varepsilon(0,\lambda)\left[\mathcal{T}_{d,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{d,\varepsilon}(0,L_\varepsilon,\lambda)\right]H_\varepsilon(L_\varepsilon,\lambda)^{-1}\right) = \mathcal{E}_{s,\varepsilon}(\lambda,\gamma)\mathcal{E}_{f,\varepsilon}(\lambda,\gamma),$$

where we use that the Evans function can be expressed as (5.14).

We establish the above properties of U_ε. The first two properties follow immediately from Theorem 4.19. Furthermore, combining (4.25) with Corollary 5.15, settles the first estimate in (5.29). For the second estimate in (5.29) we use the method of successive approximation. Theorem 2.3 and **(S1)** yield

$$\|\mathcal{A}_{21,\varepsilon}(x)\| \leq Ce^{-\mu_0|x|}, \quad x \in [-L_\varepsilon, L_\varepsilon]. \quad (5.31)$$

Because U_ε is ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$, estimates (4.26) and (5.31) yield $\|U_\varepsilon(x,\lambda)\| \leq C\sqrt{\varepsilon}$ for $x \in [\Xi_\varepsilon/4, 2L_\varepsilon - \Xi_\varepsilon/4]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Thus, employing (4.26) and (5.31) again gives $\|U_\varepsilon(x,\lambda)\| \leq C\varepsilon\sqrt{\varepsilon}$ for $x \in [\Xi_\varepsilon/2, 2L_\varepsilon - \Xi_\varepsilon/2]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Finally, a third application of (4.26) and (5.31) leads to the second estimate in (5.29). □

Theorem 5.16 provides a diagonalization of the rescaled full eigenvalue problem (5.13) into two lower-dimensional problems (5.27) and (5.28). The diagonalization yields a factorization of the Evans function $\mathcal{E}_\varepsilon$ into two factors $\mathcal{E}_{s,\varepsilon}$ and $\mathcal{E}_{f,\varepsilon}$. By relating (5.27) and (5.28) to the three eigenvalue problems (3.6), (3.8) and (3.9), we link $\mathcal{E}_{s,\varepsilon}$ to the slow Evans function $\mathcal{E}_{s,0}$ and $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$.

First, we consider problem (5.27). Along the pulse, the transformation matrix $U_\varepsilon(x,\lambda)$ is approximated by the solution $\mathcal{X}_{in}(x,u_0,\lambda)$ to the inhomogeneous fast eigenvalue problem (3.8). On the other hand, along the slow manifold, U_ε is small and system (5.27) is a perturbation of the slow eigenvalue problem (3.9). Thus, we observe that both (3.8) and (3.9) govern the leading-order dynamics in system (5.27). This leads to the following approximation result.

Lemma 5.17. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, the map $\mathcal{E}_{s,\varepsilon}$, defined in Theorem 5.16, is approximated as*

$$\left|\mathcal{E}_{s,\varepsilon}(\lambda,\gamma) - \mathcal{E}_{s,0}(\lambda,\gamma)\right| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in S^1, \quad (5.32)$$

where $C > 0$ is a constant, which is independent of λ and ε.

Proof. In the following, Ξ_ε is as in Theorem 5.16 and $C > 0$ is a constant, which is independent of ε and λ.

Our approach is as follows. We introduce a splitting of the coefficient matrix of system (5.27) that is consistent with the decay behavior along the slow manifold, i.e. we write

$$\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda) = \mathcal{B}_{1,\varepsilon}(x,\lambda) + \mathcal{B}_{2,\varepsilon}(x,\lambda),$$

with

$$\begin{aligned}
\mathcal{B}_{1,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & D_1^{-1} \\ \varepsilon\left(\partial_u H_1(u_{\mathrm{p},\varepsilon}(x),0,\varepsilon)+\lambda\right) & 0 \end{pmatrix}, \\
\mathcal{B}_{2,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & 0 \\ B_{2,\varepsilon}(x) & 0 \end{pmatrix} + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda), \\
B_{2,\varepsilon}(x) &:= \partial_u H_2(u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x)) + \varepsilon\left(\partial_u H_1(u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x),\varepsilon) - \partial_u H_1(u_{\mathrm{p},\varepsilon}(x),0,\varepsilon)\right).
\end{aligned}$$

Theorems 2.3 and 5.16 and assumption **(S1)** imply that $\mathcal{B}_{1,\varepsilon}$ and $\mathcal{B}_{2,\varepsilon}$ are ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ and it holds

$$\|\mathcal{B}_{2,\varepsilon}(x,\lambda)\| \leq C\varepsilon^3, \quad x\in[\Xi_\varepsilon, 2L_\varepsilon-\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}. \tag{5.33}$$

The splitting gives rise to an intermediate system,

$$\chi_x = \sqrt{\varepsilon}\mathcal{B}_{1,\varepsilon}(x,\lambda)\chi, \quad \chi\in\mathbb{C}^{2m}. \tag{5.34}$$

On the one hand, system (5.34) is a perturbation of (a rescaled version of) the slow eigenvalue problem (3.9). On the other hand, (5.27) and (5.34) are related via the variation of constants formula. This leads to the desired approximation of $\mathcal{E}_{s,\varepsilon}$ by $\mathcal{E}_{s,0}$ on $\Sigma_{\Lambda,\delta}$.

First, we relate systems (5.27) and (5.34) via the variation of constants formula. Denote by $\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)$ and $\mathcal{T}_{is,\varepsilon}(x,y,\lambda)$ the evolution operators of system (5.27) and (5.34), respectively. Lemma 4.1 gives the estimate,

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)-I\|, \|\mathcal{T}_{is,\varepsilon}(x,y,\lambda)-I\| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|, \quad x,y\in[-\Xi_\varepsilon,\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}. \tag{5.35}$$

On the other hand, upon rescaling the p-coordinate in (5.34), one obtains the Grönwall estimates,

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)\|, \|\mathcal{T}_{is,\varepsilon}(x,y,\lambda)\| \leq \frac{C}{\sqrt{\varepsilon}}e^{\varepsilon\mu_s|x-y|}, \quad x,y\in\mathbb{R}, \lambda\in\Sigma_{\Lambda,\delta}, \tag{5.36}$$

where $\mu_s > 0$ is a λ- and ε-independent constant. Thus, combining (5.33) and (5.36) with Lemma 4.1 gives

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)-\mathcal{T}_{is,\varepsilon}(x,y,\lambda)\| \leq C\varepsilon^2, \quad x,y\in[\Xi_\varepsilon, 2L_\varepsilon-\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}, \tag{5.37}$$

where we use that $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ by Theorem 2.3. We apply the variation of constants formula and write

$$\mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda) = \mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda) - \sqrt{\varepsilon} \int_0^{L_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, L_\varepsilon, \lambda)dz, \tag{5.38}$$

for $\lambda \in \Sigma_{\Lambda,\delta}$. Estimates (5.33) and (5.36) yield

$$\left\| \int_{\Xi_\varepsilon}^{L_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, L_\varepsilon, \lambda)dz \right\| \leq C\varepsilon, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.39}$$

Applying (5.36), (5.37) and (5.39) to (5.38) gives

$$\left\| \mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda) - \mathcal{F}_{+,\varepsilon}(\lambda)\mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda) \right\| \leq C\varepsilon \sqrt{\varepsilon} |\log(\varepsilon)|, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.40}$$

where

$$\mathcal{F}_{+,\varepsilon}(\lambda) := I - \sqrt{\varepsilon} \int_0^{\Xi_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz\mathcal{T}_{is,\varepsilon}(\Xi_\varepsilon, 0, \lambda).$$

Using (5.35), we derive

$$\left\| \mathcal{F}_{+,\varepsilon}(\lambda) - I + \sqrt{\varepsilon} \int_0^{\Xi_\varepsilon} \mathcal{B}_{2,\varepsilon}(z, \lambda)dz \right\| \leq C\varepsilon |\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.41}$$

Recall that the $(2n \times 2m)$-matrix $\mathcal{X}_{in}(x, u_0, \lambda)$ is a composition of four block matrices, where $\mathcal{V}_{in}(x, u_0, \lambda)$ is the upper-left $n \times m$-block. Theorems 2.3 and 5.16 and estimates (2.6), (5.26) and (5.41) yield

$$\left\| \mathcal{F}_{+,\varepsilon}(\lambda) - \begin{pmatrix} I & 0 \\ -\sqrt{\varepsilon} \int_0^\infty \left[\partial_u H_2(u_0, v_{\mathrm{h}}(x)) + \partial_v H_2(u_0, v_{\mathrm{h}}(x))\mathcal{V}_{in}(x, u_0, \lambda)\right] dx & I \end{pmatrix} \right\| \leq C\varepsilon |\log(\varepsilon)|^2, \tag{5.42}$$

for any $\lambda \in \Sigma_{\Lambda,\delta}$.

Our next step is to relate systems (3.9) and (5.34). We apply two operations on system (5.34). First, we perform the coordinate change $\chi = C_\varepsilon \tilde{\chi}$, where $C_\varepsilon := \begin{pmatrix} I & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C})$. Second, we switch to the spatial scale $\check{x} = \varepsilon x$. Thus, Lemma 4.1 yields the following estimate,

$$\|C_\varepsilon^{-1}\mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda)C_\varepsilon - \mathcal{T}_s(0, \ell_0, \lambda)\| \leq C\varepsilon, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.43}$$

where $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ is the evolution operator of the slow eigenvalue problem (3.9).

Finally, we approximate $\mathcal{E}_{s,\varepsilon}$ by the slow Evans function $\mathcal{E}_{s,0}$. Applying (5.40), (5.42) and (5.43) to (5.38) yields

$$\left\| C_\varepsilon^{-1}\mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda)C_\varepsilon - \Upsilon_+(\lambda)\mathcal{T}_s(0, \ell_0, \lambda) \right\| \leq C \sqrt{\varepsilon} |\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.44}$$

with

$$\Upsilon_+(\lambda) := \begin{pmatrix} I & 0 \\ -\int_0^\infty [\partial_u H_2(u_0, v_\mathrm{h}(x)) + \partial_v H_2(u_0, v_\mathrm{h}(x))\mathcal{V}_{in}(x, u_0, \lambda)]\, dx & I \end{pmatrix}.$$

Similarly, we derive

$$\left\| C_\varepsilon^{-1} \mathcal{T}_{sd,\varepsilon}(0, -L_\varepsilon, \lambda) C_\varepsilon - \Upsilon_-(\lambda)\mathcal{T}_s(2\ell_0, \ell_0, \lambda) \right\| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.45}$$

with

$$\Upsilon_-(\lambda) := \begin{pmatrix} I & 0 \\ \int_{-\infty}^0 [\partial_u H_2(u_0, v_\mathrm{h}(x)) + \partial_v H_2(u_0, v_\mathrm{h}(x))\mathcal{V}_{in}(x, u_0, \lambda)]\, dx & I \end{pmatrix}.$$

For any $\lambda \in \Sigma_{\Lambda,\delta}$, we have $\det(\mathcal{T}_s(0, \ell_0, \lambda)) = 1$ and $\Upsilon_+(\lambda)^{-1}\Upsilon_-(\lambda) = \Upsilon(u_0, \lambda)$, where $\Upsilon(u, \lambda)$ is defined in (3.11). Combining the latter with estimates (5.44) and (5.45) yields (5.32). □

It remains to link the factor $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$.

Lemma 5.18. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. There exists $\mu_p > 0$ such that, for $\varepsilon > 0$ sufficiently small, there is a map $h_\varepsilon \colon \Sigma_{\Lambda,\delta} \to \mathbb{C}$ satisfying*

$$\begin{aligned} 0 < |h_\varepsilon(\lambda)| &\leq Ce^{-\mu_p L_\varepsilon}, \\ \left|\mathcal{E}_{f,\varepsilon}(\lambda, \gamma)h_\varepsilon(\lambda) - (-\gamma)^n \mathcal{E}_{f,0}(\lambda)\right| &\leq C\varepsilon^{\mu_p}, \end{aligned} \qquad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in S^1,$$

where $\mathcal{E}_{f,\varepsilon}$ is as in Theorem 5.16 and $C > 0$ is a constant, which is independent of λ and ε.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Our approach is as follows. Using roughness techniques we show that system (5.28) has for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with projections $P_{fd,\varepsilon}(x, \lambda)$. Moreover, by Proposition 5.2, the homogeneous fast eigenvalue problem (3.6) admits for every $\lambda \in \Sigma_{\Lambda,\delta}$ exponential dichotomies on $[0, \infty)$ and $(-\infty, 0]$ with λ-independent constants $C, \mu_r > 0$ and projections $Q_{f,\pm}(x, \lambda)$. Recall that the fast Evans function $\mathcal{E}_{f,0}$ is defined in terms of bases $B_f^s(\lambda)$ and $B_f^u(\lambda)$ of $Q_{f,+}(0, \lambda)[\mathbb{C}^{2n}]$ and $\ker(Q_{f,-}(0, \lambda))$, respectively. By comparing system (5.28) to (3.6), we construct bases $B_\varepsilon^{u,s}(\lambda)$ of $P_{fd,\varepsilon}(0, \lambda)[\mathbb{C}^{2n}]$ and $\ker(P_{fd,\varepsilon}(0, \lambda))$, which are close to $B_f^{u,s}(\lambda)$. By tracking the bases $B_\varepsilon^{u,s}(\lambda)$ either forward or backward, we obtain bases of $P_{fd,\varepsilon}(L_\varepsilon, \lambda)[\mathbb{C}^{2n}]$ and $\ker(P_{fd,\varepsilon}(L_\varepsilon, \lambda))$. These bases will form the column vectors of a matrix $\mathcal{H}_\varepsilon(\lambda)$, which connects $\mathcal{E}_{f,\varepsilon}$ to $\mathcal{E}_{f,0}$.

We start by establishing an exponential dichotomy on $\mathbb{R}$ for system (5.28). System (5.15) has by Theorem 5.14 an exponential dichotomy on $\mathbb{R}$ with ε- and λ-independent constants. In addition, by Theorems 2.3 and 5.16, $\mathcal{A}_{12,\varepsilon}$ and U_ε are ε-uniformly bounded on $\mathbb{R}$ and $\mathbb{R} \times \Sigma_{\Lambda,\delta}$, respectively. Therefore, Proposition 4.12 yields that (5.28) has, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with projections $P_{fd,\varepsilon}(x, \lambda)$ and ε- and λ-independent constants $C, \mu_d > 0$. Since the coefficient matrix of (5.28) is $2L_\varepsilon$-periodic by Theorem 5.16, the projections $P_{fd,\varepsilon}(\cdot, \lambda)$ are also $2L_\varepsilon$-periodic – see [14, Proposition 8.4].

Our next step is to compare system (5.28) to the homogeneous fast eigenvalue problem (3.6). Theorems 2.3 and 5.16 yield

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \sqrt{\varepsilon} U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x) - \mathcal{A}_{22,0}(x,u_0,\lambda)\right\| \leq C\sqrt{\varepsilon}, \quad x \in [\log(\varepsilon), -\log(\varepsilon)]. \tag{5.46}$$

By **(E1)** there exists an $M > 1$ such that $\mathcal{A}_{22,0}(\cdot, u_0, \cdot)$ is bounded by M on $\mathbb{R} \times \Sigma_{\Lambda,\delta}$. Denote by $\mathcal{T}_r(x,y,\lambda)$ and $\mathcal{T}_{fd,\varepsilon}(x,y,\lambda)$ the evolution operators of (3.6) and (5.28), respectively. Provided that $\varepsilon > 0$ is sufficiently small, Lemma 4.1 and estimate (5.46) imply

$$\|\mathcal{T}_r(x,y,\lambda) - \mathcal{T}_{fd,\varepsilon}(x,y,\lambda)\| < 1, \quad x,y \in \left[\frac{\log(\varepsilon)}{8M}, -\frac{\chi\log(\varepsilon)}{8M}\right], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.47}$$

Since $\lambda \mapsto B_f^{u,s}(\lambda)$ is analytic on C_Λ by Proposition 5.2, $B_f^{u,s}$ is bounded on $\Sigma_{\Lambda,\delta}$. Now, combine estimate (5.47) and Lemma 4.13: there exists, for $\varepsilon > 0$ sufficiently small, bases $B_\varepsilon^{u,s} : \Sigma_{\Lambda,\delta} \to \mathrm{Mat}_{2n\times n}(\mathbb{C})$ of $P_{fd,\varepsilon}(0,\lambda)[\mathbb{C}^{2n}] = B_\varepsilon^s[\mathbb{C}^n]$ and $\ker(P_{fd,\varepsilon}(0,\lambda)) = B_\varepsilon^u[\mathbb{C}^n]$, such that

$$\|B_\varepsilon^{u,s}(\lambda) - B_f^{u,s}(\lambda)\| \leq C\varepsilon^{\mu_r/(8M)}, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.48}$$

Since $B_f^{u,s}$ is bounded on $\Sigma_{\Lambda,\delta}$, the some holds for $B_\varepsilon^{u,s}$ by (5.48).

Finally, we link $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$. Define

$$\mathcal{H}_\varepsilon(\lambda) := \left(\mathcal{T}_{fd,\varepsilon}(-L_\varepsilon, 0, \lambda)B_\varepsilon^u(\lambda), \mathcal{T}_{fd,\varepsilon}(L_\varepsilon, 0, \lambda)B_\varepsilon^s(\lambda)\right), \quad \lambda \in \Sigma_{\Lambda,\delta}.$$

Since $P_{fd,\varepsilon}(\cdot,\lambda)$ is $2L_\varepsilon$-periodic, the first n columns of $\mathcal{H}_\varepsilon(\lambda)$ form a basis of the space $\ker(P_{fd,\varepsilon}(L_\varepsilon,\lambda))$ and the last n columns form a basis of $P_{fd,\varepsilon}(L_\varepsilon,\lambda)[\mathbb{C}^{2n}]$. Thus, $\mathcal{H}_\varepsilon(\lambda)$ is invertible. By Hadamard's inequality we have $|\det(\mathcal{H}_\varepsilon(\lambda))| \leq Ce^{-2n\mu_d L_\varepsilon}$ for each $\lambda \in \Sigma_{\Lambda,\delta}$. Moreover, using that $P_{fd,\varepsilon}(\cdot,\lambda)$ is $2L_\varepsilon$-periodic, we estimate

$$\begin{aligned} \|\mathcal{T}_{fd,\varepsilon}(0, L_\varepsilon, \lambda)\mathcal{T}_{fd,\varepsilon}(-L_\varepsilon, 0, \lambda)B_\varepsilon^u(\lambda)\| &\leq Ce^{-2\mu_d L_\varepsilon}, \\ \|\mathcal{T}_{fd,\varepsilon}(0, -L_\varepsilon, \lambda)\mathcal{T}_{fd,\varepsilon}(L_\varepsilon, 0, \lambda)B_\varepsilon^s(\lambda)\| &\leq Ce^{-2\mu_d L_\varepsilon}, \end{aligned} \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.49}$$

We combine estimates (5.48) and (5.49) and derive

$$\left\|\left(\mathcal{T}_{fd,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{fd,\varepsilon}(0,L_\varepsilon,\lambda)\right)\mathcal{H}_\varepsilon(\lambda) - \left(B_f^u(\lambda), \gamma B_f^s(\lambda)\right)\right\| \leq C\varepsilon^{\mu_r/(8M)},$$

for $\lambda \in \Sigma_{\Lambda,\delta}$ and $\gamma \in S^1$. Taking determinants and defining $h_\varepsilon(\lambda) := \det(\mathcal{H}_\varepsilon(\lambda))$ concludes the proof. □

Remark 5.19. In the proof of Lemma 5.18, the connection between $\mathcal{E}_{f,\varepsilon}$ and $\mathcal{E}_{f,0}$ is given by the matrix $\mathcal{H}_\varepsilon$. This idea is taken from the proof of [99, Theorem 2]. However, the context in [99] is different: here it is shown that the eigenvalues of a periodic boundary value problem are exponentially close to the eigenvalues of the corresponding unbounded problem. ■

5.2.5 Conclusion

In contrast to the approximation of $\mathcal{E}_{s,\varepsilon}$ by $\mathcal{E}_{s,0}$ in Lemma 5.17, we need to rescale $\mathcal{E}_{f,\varepsilon}$ in Lemma 5.18 by an exponentially small quantity h_ε in order to approximate it by the ε-independent fast Evans function $\mathcal{E}_{f,0}$. This quantity prevents us from directly estimating the Evans function $\mathcal{E}_\varepsilon$ by the reduced Evans function $\mathcal{E}_0(\lambda,\gamma) = (-\gamma)^n\mathcal{E}_{s,0}(\lambda,\gamma)\mathcal{E}_{f,0}(\lambda)$. Nevertheless, it is still possible to compare the zero sets of $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0$ using the classical symmetric version of Rouché's Theorem due to Estermann. This yields the proof of Theorem 3.15.

Proof of Theorem 3.14. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Let $S \subset S^1$ be closed. Take a simple closed curve Γ in $C_\Lambda \setminus [\mathcal{N}_S \cup \mathcal{E}_{f,0}^{-1}(0)]$, where $\mathcal{N}_S$ is as in (3.16). Since $\mathcal{E}_\varepsilon(\cdot,\gamma)$ and $\mathcal{E}_0(\cdot,\gamma)$ have no roots in $C_\Lambda \setminus \Sigma_{\Lambda,0}$ for each $\gamma \in S$ by Corollary 5.12, we may assume $\Gamma \subset \Sigma_{\Lambda,0} \setminus [\mathcal{N}_S \cup \mathcal{E}_{f,0}^{-1}(0)]$. Observe that there exists $\delta > 0$ such that $\Gamma \subset \Sigma_{\Lambda,\delta} \setminus \mathcal{N}_S$, since Γ avoids the set of roots of $\mathcal{E}_{f,0}$, which is discrete by Proposition 5.2.

By Propositions 3.7 and 3.11, $\mathcal{E}_\varepsilon(\cdot,\gamma)$ and $\mathcal{E}_0(\cdot,\gamma)$ are analytic on Γ and its interior for each $\gamma \in S$. Furthermore, $\mathcal{E}_0$ is bounded away from 0 on the compact set $\Gamma \times S$, because Γ is disjoint from $\mathcal{N}_S$ and $\mathcal{E}_0$ is analytic on $C_\Lambda \times \mathbb{C}$ by Proposition 3.11. Hence, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned}|\mathcal{E}_\varepsilon(\lambda,\gamma) - \mathcal{E}_0(\lambda,\gamma)| &\leq |\mathcal{E}_{s,\varepsilon}(\lambda,\gamma)\mathcal{E}_{f,\varepsilon}(\lambda,\gamma)h_\varepsilon(\lambda) - (-\gamma)^n\mathcal{E}_{s,0}(\lambda,\gamma)\mathcal{E}_{f,0}(\lambda)| \\ &\quad + (1-h_\varepsilon(\lambda))|\mathcal{E}_\varepsilon(\lambda,\gamma)| \qquad\qquad \lambda \in \Gamma, \gamma \in S, \\ &< |\mathcal{E}_0(\lambda,\gamma)| + |\mathcal{E}_\varepsilon(\lambda,\gamma)|,\end{aligned}$$

by Theorem 5.16 and Lemmas 5.17 and 5.18. The result follows by an application of the symmetric version of Rouché's Theorem. □

Remark 5.20. The technical Lemmas 5.17 and 5.18 seem to provide a rate at which the spectrum $\sigma(\mathcal{L}_\varepsilon)$ converges to its singular limit. However, the approximations in these lemmas are only valid away from the zeros of the fast Evans function $\mathcal{E}_{f,0}$! So, one can only deduce that spectrum converging to

$$\left\{\lambda \in \mathbb{C} : \mathcal{E}_{s,0}(\lambda,\gamma) = 0 \text{ for some } \gamma \in S^1\right\} \setminus \mathcal{E}_{f,0}^{-1}(0),$$

does this at an algebraic rate of order $O(\sqrt{\varepsilon})$. We expect that this rate is in fact of order $O(\varepsilon)$ and that the square root appears due to the rescaling of the full eigenvalue problem (3.3) in §5.2.1. By making the parameter δ appearing in the proof of Lemma 5.18 dependent on ε, it might be possible to derive an overall rate at which the spectrum $\sigma(\mathcal{L}_\varepsilon)$ converges to its singular limit spectrum. However, this is beyond the scope of this thesis. Yet, we derive in §5.3 that the critical spectral curve, which is attached to the origin, scales with ε^2. This suggest that spectrum converging to the roots of $\mathcal{E}_{f,0}$ does this at an algebraic rate of order $O(\varepsilon^2)$. ■

5.2.6 Discussion

As mentioned in the introduction in Chapter 1, our factorization method via the Riccati transformation of the Evans function offers one unified analytic alternative to both the elephant trunk procedure developed by Alexander, Gardner and Jones [1, 37] and the NLEP approach of [21, 22] – that both have a geometric nature. It is worthwhile to compare and discuss the links between our work and these methods.

Consider a localized pulse solution to a 2-component, singularly perturbed reaction-diffusion system of the form (1.1). When the associated eigenvalue problem has a slow-fast structure (1.5), it is a general phenomenon that it decouples outside the pulse region due to exponential decay of the solution to its asymptotic background state. This yields a decomposition of the solution space into three subspaces $V_{s\pm} \oplus V_{c\pm} \oplus V_{u\pm}$ at both sides ($\pm$) of the pulse region. Here, $V_{s\pm}$ consists of (fast) exponentially decaying solutions, whereas $V_{u\pm}$ consists of exponentially increasing solutions. Lastly, $V_{c\pm}$ consists of solutions that evolve slowly. In the sense of [85], one could say that the eigenvalue problem admits exponential separations with respect to the decompositions $V_{s\pm} \oplus V_{c\pm} \oplus V_{u\pm}$. The difficulty is to 'glue' the subspaces $V_{\cdot+}$ and $V_{\cdot-}$ for $\cdot = u, s, c$ together, yielding an exponential separation of the eigenvalue problem on the whole line. Eventually, this induces a factorization of the Evans function into a fast and slow component.

Gardner and Jones achieved this in [37] by considering the eigenvalue problem in projective space. When the eigenvalue problem is asymptotically of constant coefficient type, one can obtain stable and unstable bundles. These bundles are then split into fast and slow (un)stable subbundles. The elephant trunk lemma is used to track the fast (un)stable bundle through the pulse region. By the control on the fast subbundle, it is possible to approximate the dynamics of the slow (un)stable subbundles. Eventually, this yields a $(1, 2, 1)$-exponential separation (in the sense of [85]) of the eigenvalue problem on $\mathbb{R}$. Note that the 2-dimensional center direction corresponds to the slow (un)stable subbundles. In our stability analysis, the Riccati transformation plays the role of the elephant trunk lemma – see Section 5.2.4. This transformation yields an $(n, 2m, n)$-exponential separation on $\mathbb{R}$ of the eigenvalue problem as long as we are not close to the eigenvalues of the operator $\mathcal{L}_f$, defined in (3.7).

Although the proof of the elephant trunk lemma has been worked out in full detail for some specific 2-component models [22, 32, 37, 95] only, it is widely accepted that the method can be followed for a larger class of systems. However, there are some limitations. For instance, the elephant trunk lemma is only suitable for eigenvalue problems that have an asymptotically constant coefficient matrix. This is neither a restriction for slowly linear systems as the classical Gray-Scott and Gierer-Meinhardt models nor for homoclinic pulses on $\mathbb{R}$. However, the eigenvalue problem associated with spatially periodic patterns in slowly nonlinear systems exhibits non-autonomous behavior in the background state on its domain of periodicity – and thus does not approach a constant coefficient matrix. This prohibits the application of the elephant trunk procedure. Moreover, the elephant trunk lemma is only capable of tracking the 'most unstable' fast solution, which corresponds to the (simple) eigenvalue of largest real part of the asymptotic coefficient matrix. Therefore, it is unclear how to obtain the exponential separation with the elephant trunk method in the multi-component setting $n > 1$.

Furthermore, there is a major difference in the mathematical framework used in [1, 37] and our work. The framework in [1, 37] has a highly geometrical character, whereas our method is of a more analytical nature. Alexander, Gardner and Jones track solutions via vector bundles arising from the projectivized eigenvalue problem. This has the advantage that the generated bundles have a clean and natural characterization as ε tends to zero, whereas the actual solutions of the eigenvalue problem become singular. On the other hand, one could argue that exponential dichotomies provide a natural framework to capture the dynamics of the eigenvalue problem being a non-autonomous linear system, which depends analytically on the spectral parameter λ. The Riccati transformation is naturally formulated in terms of exponential dichotomies and is explicit in terms of the coefficient matrix of the eigenvalue problem. Therefore, the exponential separation of the solution space is much more explicit than in [1, 37], which shortens proofs. Finally, it is interesting to remark that in both the approach initiated by Alexander, Gardner and Jones and our method we need an a-priori ε-independent estimate on the sector containing the spectrum. Our proof of this fact in Proposition 5.11 forms an analytical counterpart to the geometrical proof provided in [1, Proposition 2.2] and [37, Lemma 3.3].

Based on the geometric methods of Alexander, Gardner and Jones [1, 37], the NLEP approach was developed in the context of the stability of homoclinic (multi-)pulse patterns in the Gray-Scott equation [22] and Gierer-Meinhardt-type models [21]. This method established the approximation of the Evans function by the product (1.4) of an analytic fast Evans function and a meromorphic slow Evans function and provided explicit analytic expressions for both factors. The NLEP approach was extended to the spectral analysis of spatially periodic pulse patterns in the generalized Gierer-Meinhardt equations in [114] and to the stability of heteroclinic and homoclinic multi-front patterns in 2- and 3-component bistable systems of FitzHugh-Nagumo-type [23, 116]. Moreover, the method has recently been generalized to the stability of homoclinic pulses in slowly nonlinear systems in [30, 120]. In each of these works, the fast and slow Evans functions are interpreted geometrically in terms of fast and slow transmission functions that encode the passage of specially selected fast and slow basis functions over the fast pulse regions. The expressions for the slow transmission functions include Melnikov-type components. The meromorphic character of the slow Evans function generates the zero-pole cancelation mechanism – also called NLEP paradox – in each of these models. The spectral analysis for periodic pulse solutions developed here shows that these phenomena occur in a broad class of multi-component singularly perturbed reaction-diffusion systems.

Although the present work stands in the tradition of [21, 22, 23, 30, 114, 120], the methods differ fundamentally. Unlike these works, our analysis is based on an intrinsically analytic reduction method. This has the advantage that our spectral analysis allows for non-autonomous behavior of the eigenvalue problem outside the pulse region – a crucial extension in the case of spatially periodic patterns in slowly nonlinear systems. This extended applicability of the present method also plays a role in the spectral analysis of homoclinic patterns as outlined in Remark 1.4. Moreover, in contrast to the present work, the slow and fast eigenvalue problems appearing in [21, 22, 23, 30, 114, 120] are scalar, which significantly simplifies the analysis of these problems. In [21, 22, 114] the slow and fast Evans functions can be explicitly computed in terms of hypergeometric functions, while in [23, 116] the stability of the (multi-)fronts is determined by spectrum near the origin, so that the relevant reductions can be determined in a relatively straightforward manner. An extensive analysis of the multi-component slow and fast eigenvalue problems, as we did in Section 5.1, is thus not necessary in these cases.

5.3 The critical spectral curve

5.3.1 Introduction

In this section we prove Theorem 3.19. Thus, we assume 0 is a simple zero of the fast Evans function $\mathcal{E}_{f,0}$. Moreover, we take $\delta > 0$ and denote

$$\mathcal{N}_\diamond = \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta = \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta).$$

For each $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$ the reduced Evans function $\mathcal{E}_0(\cdot, e^{i\nu})$ has a simple root at 0 by Remark 3.13. Since $\mathcal{E}_0$ is analytic by Proposition 3.11, there exists $\varsigma > 0$ such that there are no other roots of $\mathcal{E}_0(\cdot, e^{i\nu})$ in the closed ball $\overline{B(0, \varsigma)}$ for any $\nu \in \mathcal{S}_\delta$. So, provided $\varepsilon > 0$ is sufficiently small, there exists by Theorem 3.15 a unique (simple) root $\lambda_\varepsilon(\nu)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ in $B(0, \varsigma)$ for each $\nu \in \mathcal{S}_\delta$. By Proposition 3.7 $\lambda_\varepsilon \colon \mathcal{S}_\delta \to B(0, \varsigma)$ is real-valued, 2π-periodic and even. Moreover, since $\mathcal{E}_\varepsilon$ is analytic by Proposition 3.7, $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ is also analytic by the implicit function theorem. By translational invariance, it holds $\lambda_\varepsilon(0) = 0$ if we have $0 \in \mathcal{S}_\delta$. Thus, all that remains to prove Theorem 3.19 is to approximate $\lambda_\varepsilon(\nu)$ for any $\nu \in \mathcal{S}_\delta$ with an error bound that is ν-uniform.

We describe our approach to obtain a leading-order approximation for $\lambda_\varepsilon(\nu)$ for each $\nu \in \mathcal{S}_\delta$. Fix $\nu \in \mathcal{S}_\delta$. On the one hand, since $\mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{i\nu}) = 0$, the full eigenvalue problem (3.3) admits at $\lambda = \lambda_\varepsilon(\nu)$ a solution $\tilde{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$, which satisfies $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(x + 2L_\varepsilon)$ for each $x \in \mathbb{R}$. On the other hand, the derivative $\phi_{\mathrm{p},\varepsilon}'(x)$ of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), p_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x), q_{\mathrm{p},\varepsilon}(x))$ to (2.1) is a solution to (3.3) at $\lambda = 0$. Therefore,

$$\psi_{\nu,\varepsilon}(x) := \begin{pmatrix} \tilde{v}_{\nu,\varepsilon}(x) - v_{\mathrm{p},\varepsilon}'(x) \\ \tilde{q}_{\nu,\varepsilon}(x) - q_{\mathrm{p},\varepsilon}'(x) \end{pmatrix},$$

solves the inhomogeneous problem,

$$\psi_x = \mathcal{A}_f(x)\psi + \begin{pmatrix} 0 \\ \mathcal{B}_{\nu,\varepsilon}(x) + \lambda_\varepsilon(\nu)\tilde{v}_{\nu,\varepsilon}(x) \end{pmatrix}, \quad \psi \in \mathbb{C}^{2n}, \tag{5.50}$$

where $\mathcal{A}_f(x)$ is the coefficient matrix of the fast variational equation (3.15) and $B_{\nu,\varepsilon}(x)$ is given by

$$\mathcal{B}_{\nu,\varepsilon}(x) := \begin{pmatrix} \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x,u_0),0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \tilde{u}_{\nu,\varepsilon}(x) - u'_{p,\varepsilon}(x) \\ \tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x) \end{pmatrix}.$$

By Proposition 5.2 and Corollary 5.4 the fast variational equation (3.15) has exponential dichotomies on both half-lines and the corresponding differential operator $\mathcal{L}_0$ is Fredholm of index 0. Since 0 is a simple root of $\mathcal{E}_{f,0}$, $\mathcal{L}_0$ has a one-dimensional kernel by Corollary 5.4. So, there exists a non-trivial, exponentially localized solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint problem (3.19), which is unique up to scalar multiples. Applying the solvability condition in [86, Lemma 4.2] to equation (5.50) leads to the key identity,

$$\lambda_\varepsilon(\nu) \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x) dx = -\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \mathcal{B}_{\nu,\varepsilon}(x) dx. \tag{5.51}$$

Hence, to obtain a leading-order expression of $\lambda_\varepsilon(\nu)$, it is sufficient to approximate the two integrals in (5.51). Thus, we need leading-order expressions of the solution $\tilde{\varphi}_{\nu,\varepsilon}(x)$ to (3.3), of the solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ to (1.10) and of the difference $\tilde{\varphi}_{\nu,\varepsilon}(x) - \phi'_{\mathrm{p},\varepsilon}(x)$. Clearly, we can approximate $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ by its singular limit – see Theorem 2.3. To obtain leading-order expressions for the other quantities in (5.51), we proceed as follows.

Define

$$D_{\eta,\varepsilon} := \{\lambda \in \mathbb{C} : |\lambda||\log(\varepsilon)| < \eta\}, \tag{5.52}$$

with $\eta > 0$ an ε-independent constant. Moreover, consider the intervals,

$$I_{f,\varepsilon} := [-\Xi_\varepsilon, \Xi_\varepsilon], \quad I_{s,\varepsilon} := [\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \quad \Xi_\varepsilon := -\frac{8\log(\varepsilon)}{\min\{\mu_0, \mu_r, \mu_{\mathrm{h}}\}}, \tag{5.53}$$

with $\mu_{\mathrm{h}} > 0$ as in (2.6), $\mu_0 > 0$ as in Theorem 2.3 and $\mu_r > 0$ as in Lemma 5.1. For any $\nu \in \mathcal{S}_\delta$ and $\lambda \in D_{\eta,\varepsilon}$ we establish a *piecewise continuous* solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$ – see Figure 5.2. We explicitly construct $\varphi_{\nu,\varepsilon}$ via Lin's method [10, 70, 118] using the singular limit structure (2.9) the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}$ as our framework.

By Theorem 2.3, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{f,\varepsilon}$ approximated by the pulse solution $\phi_{\mathrm{h}}(x,u_0)$ to the fast reduced system (2.2). Moreover, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{s,\varepsilon}$ approximated by the solution $(\psi_{\mathrm{s}}(\varepsilon x), 0)$ on the slow manifold, where ψ_{s} solves the slow reduced system (2.4). The endpoints of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}$ correspond to the x-values for which $\phi_{\mathrm{p},\varepsilon}(x)$ converges to one of the two non-smooth corners $(u_0, \pm\mathcal{J}(u_0))$ of the singular concatenation (2.9) as $\varepsilon \to 0$.

For $x \in I_{f,\varepsilon}$, we establish a reduced eigenvalue problem by setting ε and λ to 0 in (3.3), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x,u_0)$. The reduced eigenvalue problem admits

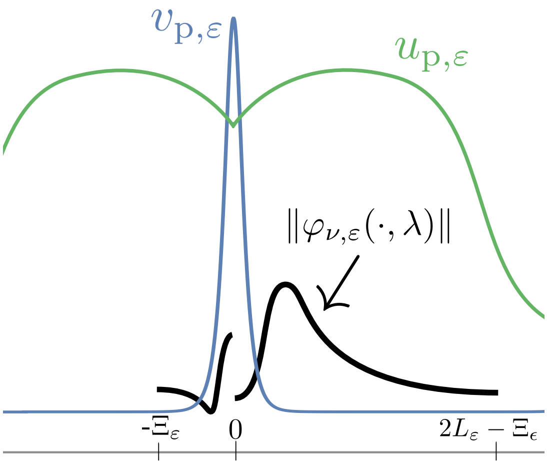

Figure 5.2: A sketch of the piecewise continuous eigenfunction $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ on its domain of definition $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$. Also depicted are the u- and v-component of the periodic pulse solution $\hat{\phi}_{\mathrm{p},\varepsilon}$ (in the case $n = m = 1$).

exponential trichotomies on both half-lines. Hence, one can construct solutions to (3.3) for $\lambda \in D_{\eta,\varepsilon}$ using variation of constants formulas on the intervals,

$$I_{f,\varepsilon}^- := [-\Xi_\varepsilon, 0], \quad I_{f,\varepsilon}^+ := [0, \Xi_\varepsilon]. \tag{5.54}$$

We can control the perturbation terms in these formulas by taking $\eta, \varepsilon > 0$ sufficiently small.

For $x \in I_{s,\varepsilon}$, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in (3.3) is exponentially small by assumption **(S1)** and Theorem 2.3. Thus, we obtain a reduced eigenvalue problem by setting $\mathcal{A}_{21,\varepsilon}(x)$ to 0 in (3.3), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by $(\psi_{\mathrm{s}}(\varepsilon x), 0)$. The reduced eigenvalue problem is upper-triangular and the spectrum of the lower-right block has a consistent splitting into n unstable and n stable eigenvalues. This splitting yields the existence of an exponential trichotomy on the interval $I_{s,\varepsilon}$. Thus, one can construct solutions to (3.3) on $I_{s,\varepsilon}$ using the variation of constants formula again.

In summary, we obtain variation of constants formulas for solutions to (3.3) on the three intervals $I_{f,\varepsilon}^\pm$ and $I_{s,\varepsilon}$. Matching of these expressions yields for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$ which has a jump at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$. We show that for any $\nu \in \mathcal{S}_\delta$ the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes at a unique λ-value $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$. Thus, since (3.3) is $2L_\varepsilon$-periodic, there exists a continuous solution $\check{\varphi}_{\nu,\varepsilon}$ to (3.3) at $\lambda = \tilde{\lambda}_\varepsilon(\nu)$ satisfying

$$\left.\begin{aligned} &\check{\varphi}_{\nu,\varepsilon}(x) = \varphi_{\nu,\varepsilon}(x, \tilde{\lambda}_\varepsilon(\nu)), && x \in I_{f,\varepsilon} \cup I_{s,\varepsilon}, \\ &\check{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\check{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x), && x \in \mathbb{R}, \end{aligned}\right\} \quad \nu \in \mathcal{S}_\delta. \tag{5.55}$$

Consequently, $\tilde{\lambda}_\varepsilon(\nu)$ must be a zero of the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$. Since the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ has a unique root $\lambda_\varepsilon(\nu)$ in $B(0,\varsigma)$, we must have $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ for each $\nu \in \mathcal{S}_\delta$. Since the key identity (5.51) is satisfied for any solution $\tilde{\varphi}_{\nu,\varepsilon}$ to (3.3) at $\lambda = \lambda_\varepsilon(\nu)$ satisfying $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x)$ for any $x \in \mathbb{R}$, it holds in particular for $\tilde{\varphi}_{\nu,\varepsilon} = \check{\varphi}_{\nu,\varepsilon}$.

The variation of constants formulas provide leading-order control over $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the intervals $I^\pm_{f,\varepsilon}$ and $I_{s,\varepsilon}$. Consequently, we obtain approximations for $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi'_{\mathrm{p},\varepsilon}$ for each $\nu \in \mathcal{S}_\delta$. Substituting these into (5.51) yields the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

This section is structured as follows. First, we establish the aforementioned reduced eigenvalue problems along the pulse (i.e. for $x \in I_{f,\varepsilon}$) and along the slow manifold (i.e. for $x \in I_{s,\varepsilon}$) and we generate exponential trichotomies for these problems. Then, we construct solutions to (3.3) on $I_{s,\varepsilon}$ and $I^\pm_{f,\varepsilon}$ using variation of constants formulas. By matching these solutions at the endpoints of the intervals $I^\pm_{f,\varepsilon}$ and $I_{s,\varepsilon}$ we obtain the desired piecewise continuous solution $\varphi_{\nu,\varepsilon}$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$. We show that there is a unique λ-value for which the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes. Finally, we substitute leading-order approximations of $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi'_{\mathrm{p},\varepsilon}$ into the key identity (5.51) and obtain the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

5.3.2 A reduced eigenvalue problem along the pulse

We establish a reduced eigenvalue problem along the pulse by setting ε and λ to 0 in (3.3), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x,u_0)$. Thus, the reduced eigenvalue problem reads

$$\varphi_x = \mathcal{A}_0(x)\varphi, \quad \varphi = (u,p,v,q) \in \mathbb{C}^{2(m+n)}, \tag{5.56}$$

with

$$\mathcal{A}_0(x) := \left(\begin{array}{c|c} \mathcal{A}_1(x) & \mathcal{A}_2(x) \\ \hline \mathcal{A}_3(x) & \mathcal{A}_f(x) \end{array}\right)$$
$$:= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \partial_u H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 & \partial_v H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 \\ \hline 0 & 0 & 0 & D_2^{-1} \\ \partial_u G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 & \partial_v G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 \end{array}\right).$$

Note that (5.56) coincides with the variational equation about the pulse solution $\phi_{\mathrm{h}}(x,u_0)$ to the fast reduced system (2.2).

The u-components of any solution to (5.56) are constant, whereas the p-components are slaved to the other components. Moreover, given the values of the u-components, the dynamics in the v- and q-components is determined by (3.15) via the variation of constants formula. Therefore, the reduced eigenvalue problem (5.56) is governed by the variational equation (3.15) about the homoclinic $\psi_{\mathrm{h}}(x,u_0)$ to (2.3) at $u = u_0$.

Thus, before studying problem (5.56), we study the dynamics of the fast variational equation (3.15). Naturally, the derivative $\partial_x \psi_{\mathrm{h}}(x, u_0)$ is a non-trivial, exponentially localized solution to (3.15). Moreover, since $\psi_{\mathrm{h}}(0, u_0)$ is contained in the space $\ker(I - R_f)$ by **(E1)**, system (3.15) is R_f-reversible at $x = 0$. We establish exponential dichotomies for (3.15) on both half-lines that respect the reversible symmetry.

Proposition 5.21. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Then, the fast variational equation* (3.15) *admits exponential dichotomies on* $[0, \infty)$ *and* $(-\infty, 0]$ *with constants* $C, \mu_r > 0$ *and rank* n *projections* $P_{f,\pm}(x)$ *satisfying*

$$\|P_{f,\pm}(\pm x) - \mathcal{P}_f\| \leq C e^{-\min\{\mu_r, \mu_{\mathrm{h}}\} x}, \quad x \geq 0, \tag{5.57}$$

where $\mu_{\mathrm{h}} > 0$ *is as in* (2.6), $\mu_r > 0$ *is as in Lemma 5.1 and* $\mathcal{P}_f$ *denotes the spectral projection onto the stable eigenspace of the asymptotic matrix,*

$$\mathcal{A}_{f,\infty} := \lim_{x \to \pm\infty} \mathcal{A}_f(x) = \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0, 0, 0) & 0 \end{pmatrix}. \tag{5.58}$$

The space of exponentially localized solutions to (3.15) *is spanned by* $\kappa_{\mathrm{h}}(x) = \partial_x \psi_{\mathrm{h}}(x, u_0) = (\partial_x v_{\mathrm{h}}(x, u_0), \partial_x q_{\mathrm{h}}(x, u_0))$. *Similarly, the adjoint* (3.19) *has a non-trivial, exponentially localized solution* $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$, *which is unique up to scalar multiples and satisfies*

$$\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x, u_0) dx \neq 0, \quad \|\psi_{\mathrm{ad}}(y)\| \leq C e^{-\mu_r |y|}, \quad y \in \mathbb{R}.$$

Moreover, we have the decomposition,

$$\mathbb{C}^{2n} = Y^u \oplus Y^s \oplus Y^c \oplus Y^\perp, \tag{5.59}$$

with $Y^c = \mathrm{Sp}(\kappa_{\mathrm{h}}(0))$, $Y^\perp = \mathrm{Sp}(\psi_{\mathrm{ad}}(0))$ *and*

$$\begin{aligned} P_{f,+}(0)[\mathbb{C}^{2n}] &= Y^s \oplus Y^c, & P_{f,-}(0)[\mathbb{C}^{2n}] &= Y^s \oplus Y^\perp, \\ \ker(P_{f,+}(0)) &= Y^u \oplus Y^\perp, & \ker(P_{f,-}(0)) &= Y^u \oplus Y^c. \end{aligned} \tag{5.60}$$

The spaces $Y^u \oplus Y^s$, $Y^\perp$ *and* Y^c *are pairwise orthogonal and the decomposition* (5.59) *respects the reversible symmetry:*

$$R_f \kappa_{\mathrm{h}}(0) = -\kappa_{\mathrm{h}}(0), \quad R_f \psi_{\mathrm{ad}}(0) = \psi_{\mathrm{ad}}(0), \quad R_f[Y^s] = Y^u. \tag{5.61}$$

Proof. Since (3.15) coincides with the fast eigenvalue problem (3.6) at $\lambda = 0$, Proposition 5.2 provides exponential dichotomies for (3.15) on $[0, \infty)$ and $(-\infty, 0]$ with constants $C_r, \mu_r > 0$ and rank n projections $P_{f,\pm}(x)$. By (2.6) it holds

$$\left\| \mathcal{A}_f(x) - \mathcal{A}_{f,\infty} \right\| \leq K e^{-\mu_{\mathrm{h}} |x|}, \quad x \in \mathbb{R},$$

for some $K > 0$. Hence, Lemma 4.6 yields estimate (5.57). In addition, by Proposition 5.2, the space of exponentially localized solutions to (3.15) is one-dimensional, because 0 is a simple root of $\mathcal{E}_{f,0}$. Since $\kappa_\mathrm{h}(x)$ is a non-trivial, exponentially localized solution to (3.15) by **(E1)**, we deduce $Y^c := \mathrm{Sp}(\kappa_\mathrm{h}(0)) = P_{f,+}(0)[\mathbb{C}^{2n}] \cap \ker(P_{f,-}(0))$.

Define Y^s to be the $(n-1)$-dimensional orthogonal complement of Y^c in $P_{f,+}(0)[\mathbb{C}^{2n}]$. Any solution $\varphi(x)$ to (3.15) with initial condition $\varphi(0) \in Y^s$ decays exponentially to 0 as $x \to \infty$. In addition, since system (3.15) is R_f-reversible at $x = 0$, the solution $R_f\varphi(-x)$ to (3.15) decays exponentially to 0 as $x \to -\infty$. Therefore, $Y^u := R_f[Y^s]$ is contained in $\ker(P_{f,-}(0))$. Since R_f is self-adjoint and $R_f[\kappa_\mathrm{h}(0)] = -\kappa_\mathrm{h}(0)$, the n-dimensional space $\ker(P_{f,-}(0))$ arises as the orthogonal sum of Y^c and Y^u.

Because the kernel of the operator $\mathcal{L}_0$ of Fredholm index 0 is one-dimensional by Corollary 5.4, the adjoint $\mathcal{L}_0^*$ has a one-dimensional kernel too. In addition, since equation (3.15) has exponential dichotomies on both half-lines, the same holds for its adjoint (3.19). So, there exists a non-trivial, exponentially localized solution $\psi_\mathrm{ad}(x)$ to (3.19), which is unique up to scalar multiples. The pointwise inner product of $\psi_\mathrm{ad}(x)$ with any solution $\varphi(x)$ to (3.15) is constant in x. Thus, the pointwise inner product of $\psi_\mathrm{ad}(x)$ with solutions $\varphi(x)$ to (3.15) that are decaying to 0 as $x \to \pm\infty$ must equal 0. Hence, the spaces $Y^s \oplus Y^u$, Y^c and $Y^\perp := \mathrm{Sp}(\psi_\mathrm{ad}(0))$ must be pairwise orthogonal. Since we have the decomposition (5.59), we may without loss of generality assume by Lemma 4.5 that $P_{f,-}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^\perp$ and $\ker(P_{f,+}(0)) = Y^u \oplus Y^\perp$.

Finally, $R_f\psi_\mathrm{ad}(-x)$ is also an exponentially localized solution to (3.19). This implies $R_f\psi_\mathrm{ad}(0) = \alpha\psi_\mathrm{ad}(0)$ for some $\alpha \in \sigma(R_f) = \{\pm 1\}$. On the other hand, since the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = 0$ by Corollary 5.4, the generalized eigenvalue problem,

$$\mathcal{L}_0\varphi = \partial_\lambda \mathcal{L}_0\kappa_\mathrm{h},$$

has no bounded solutions. Hence, the Fredholm alternative in [86, Lemma 4.2] implies

$$0 \neq \int_{-\infty}^{\infty} \psi_\mathrm{ad}(x)^* \partial_\lambda \mathcal{L}_0 \kappa_\mathrm{h}(x) dx = \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_\mathrm{h}(x, u_0) dx.$$

Therefore, $\psi_{\mathrm{ad},2}(x)$ cannot be even, because $\partial_x v_\mathrm{h}(x, u_0)$ is an odd function of x. Hence, $\psi_{\mathrm{ad},2}(x)$ is odd and we establish $R_f\psi_\mathrm{ad}(0) = \psi_\mathrm{ad}(0)$. □

The reduced eigenvalue problem (5.56) is governed by the fast variational equation (3.15). More precisely, the evolution operator of (5.56) can be expressed in terms of the evolution operator of (3.15) via variation of constants formulas. Thus, the solution $\kappa_\mathrm{h}(x) = \partial_x\psi_\mathrm{h}(x, u_0)$ to (3.15) yields the non-trivial, exponentially localized solution,

$$\varphi_\mathrm{h}(x) := \begin{pmatrix} \int_\infty^x \mathcal{A}_2(z)\kappa_\mathrm{h}(z)dz \\ \kappa_\mathrm{h}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ H_2(u_0, v_\mathrm{h}(x, u_0)) \\ \partial_x v_\mathrm{h}(x, u_0) \\ \partial_x q_\mathrm{h}(x, u_0) \end{pmatrix} = \partial_x \phi_\mathrm{h}(x, u_0), \tag{5.62}$$

to (5.56). Moreover, since the matrix function $\mathcal{K}_{in}(x) := (\partial_u\psi_\mathrm{h}(x, u_0) \mid 0)$ solves the inhomogeneous problem,

$$\mathcal{X}_x = \mathcal{A}_f(x)\mathcal{X} + \mathcal{A}_3(x), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}),$$

we obtain a family of solutions,

$$\Phi_{in}(x) := \begin{pmatrix} I + \int_0^x [\mathcal{A}_2(z)\mathcal{K}_{in}(z) + \mathcal{A}_1(z)]\, dz \\ \mathcal{K}_{in}(x) \end{pmatrix}. \tag{5.63}$$

to (5.56). By **(S1)** and (2.6) there exists a constant $C > 0$ such that

$$\left\| \Phi_{in}(\pm x) - \begin{pmatrix} \Upsilon_{\pm\infty} \\ 0 \end{pmatrix} \right\| \leq Ce^{-\mu_{\mathrm{h}} x}, \quad x \geq 0, \tag{5.64}$$

with

$$\Upsilon_{\pm\infty} := \begin{pmatrix} I & 0 \\ \pm\partial_u \mathcal{J}(u_0) & I \end{pmatrix} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C}),$$

where $\mathcal{J} \colon U_{\mathrm{h}} \to \mathbb{R}$ is defined in (2.5).

We show that the exponential dichotomies of (3.15), established in Proposition 5.21, yield exponential trichotomies for (5.56) with projections converging to the spectral projections of the asymptotic matrix,

$$\mathcal{A}_\infty := \lim_{x\to\pm\infty} \mathcal{A}_0(x) = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty} \\ 0 & \mathcal{A}_{f,\infty} \end{pmatrix}, \quad \mathcal{A}_{2,\infty} = \begin{pmatrix} 0 & 0 \\ \partial_v H_2(u_0, 0, 0) & 0 \end{pmatrix}, \tag{5.65}$$

where $\mathcal{A}_{f,\infty}$ is defined in (5.58).

Proposition 5.22. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *System* (5.56) *admits exponential trichotomies on* $[0, \infty)$ *and* $(-\infty, 0]$ *with constants* $C, \mu_r > 0$ *and projections* $P_\pm^{u,s,c}(x)$ *satisfying*

$$\left\| P_\pm^{u,s,c}(\pm x) - \mathcal{P}^{u,s,c} \right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x/2}, \quad x \geq 0, \tag{5.66}$$

where $\mu_{\mathrm{h}} > 0$ *is as in (2.6),* $\mu_r > 0$ *is as in Lemma 5.1 and* $\mathcal{P}^u, \mathcal{P}^s$ *and* $\mathcal{P}^c$ *are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix* $\mathcal{A}_\infty$*, respectively. Moreover, it holds*

$$\begin{gathered} P_-^u(0) = \begin{pmatrix} 0 & \int_{-\infty}^0 \mathcal{A}_2(x)\Phi_{f,-}^u(x,0)dx \\ 0 & I - P_{f,-}(0) \end{pmatrix}, \ P_+^u(0) = \begin{pmatrix} 0 & 0 \\ \int_0^\infty \Phi_{f,+}^u(0,x)\mathcal{A}_3(x)dx & I - P_{f,+}(0) \end{pmatrix}, \\ P_+^s(0) = \begin{pmatrix} 0 & \int_\infty^0 \mathcal{A}_2(x)\Phi_{f,+}^s(x,0)dx \\ 0 & P_{f,+}(0) \end{pmatrix}, \quad P_-^s(0) = \begin{pmatrix} 0 & 0 \\ \int_0^{-\infty} \Phi_{f,-}^s(0,x)\mathcal{A}_3(x)dx & P_{f,-}(0) \end{pmatrix}, \end{gathered} \tag{5.67}$$

where $\Phi_{f,\pm}^{u,s}(x,y)$ *denotes the (un)stable evolution operator of the fast variational equation* (3.15) *under the exponential dichotomies, established in Proposition 5.21, with projections* $P_{f,\pm}(x)$*. Finally, we have the decompositions,*

$$\begin{aligned} \ker(P_+^u(0)) &= P_+^s(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \\ \ker(P_-^s(0)) &= P_-^u(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \end{aligned} \tag{5.68}$$

where Φ_{in} *is defined in* (5.63)*, and*

$$\begin{aligned} P_+^s(0)[\mathbb{C}^{2(m+n)}] &= P_+^s(0)[Z^s] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad Z^s := \{(0,b)\colon b \in Y^s\}, \\ P_-^u(0)[\mathbb{C}^{2(m+n)}] &= P_-^u(0)[Z^u] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad Z^u := \{(0,b)\colon b \in Y^u\}. \end{aligned} \tag{5.69}$$

where $Y^{u,s}$ *are as in Proposition 5.21 and* φ_{h} *is defined in* (5.62).

Proof. In the following, we denote by $C > 0$ a constant.

The evolution $\Phi_0(x,y)$ of (5.56) can be expressed in terms of the evolution $\Phi_f(x,y)$ of (3.15) as follows

$$\Phi_0(x,y) = \begin{pmatrix} I + \int_y^x \left[\mathcal{A}_2(z)\int_y^z \Phi_f(z,w)\mathcal{A}_3(w)dw + \mathcal{A}_1(z)\right]dz & \int_y^x \mathcal{A}_2(z)\Phi_f(z,y)dz \\ \int_y^x \Phi_f(x,z)\mathcal{A}_3(z)dz & \Phi_f(x,y) \end{pmatrix}. \tag{5.70}$$

By Proposition 5.21 equation (3.15) admits exponential dichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C,\mu_r > 0$ and rank n projections $P_{f,\pm}(x)$ satisfying

$$\left\|P_{f,\pm}(\pm x) - \mathcal{P}_f\right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x}, \quad x \geq 0, \tag{5.71}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$, defined in (5.58). We construct an explicit exponential trichotomy for (5.56) on $(-\infty,0]$ using the matrix functions,

$$\begin{aligned} A(x) &:= \int_{-\infty}^x \mathcal{A}_2(z)\Phi_{f,-}^u(z,x)dx, & B(x) &:= \int_x^0 \Phi_{f,-}^u(x,z)\mathcal{A}_3(z)dz, \\ E(x) &:= \int_0^x \mathcal{A}_2(z)\Phi_{f,-}^s(z,x)dx, & D(x) &:= \int_x^{-\infty} \Phi_{f,-}^s(x,z)\mathcal{A}_3(z)dz. \end{aligned}$$

Clearly, A, B, D and E are bounded on $(-\infty,0]$. We consider their asymptotic behavior. By (2.6) and **(S1)**, it holds

$$\|\mathcal{A}_1(x)\|, \|\mathcal{A}_2(x) - \mathcal{A}_{2,\infty}\|, \|\mathcal{A}_3(x)\|, \|\mathcal{A}_f(x) - \mathcal{A}_{f,\infty}\| \leq Ce^{-\mu_{\mathrm{h}}|x|}. \quad x \in \mathbb{R}, \tag{5.72}$$

By writing $B(x)$ as a sum of two integrals over the intervals $(x, x/2)$ and $(x/2, 0)$ and estimating both integrals independently using (5.72) and the exponential dichotomy of (3.15), we deduce that $B(x)$ converges exponentially to 0 as $x \to -\infty$ with rate $\min\{\mu_r,\mu_{\mathrm{h}}\}/2$. Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 5.1, the matrix $\mathcal{A}_f(x)$ is by (5.72) invertible for $x < 0$ sufficiently small. Thus, for $x \ll 0$ we may write

$$A(x) = \int_{-\infty}^x \mathcal{A}_2(z)\mathcal{A}_f(z)^{-1}\partial_z\Phi_{f,-}^u(z,x)dz.$$

Combining the latter with (5.71) and (5.72), leads, via integration by parts, to the approximations,

$$\|B(x)\|, \left\|A(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I - \mathcal{P}_f)\right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x/2}, \quad x \leq 0. \tag{5.73}$$

Similarly, we derive

$$\|D(x)\|, \left\|E(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f\right\| \le Ce^{-\min\{\mu_r,\mu_h\}x/2}, \quad x \le 0. \tag{5.74}$$

We define candidate trichotomy projections,

$$P_-^u(x) := \begin{pmatrix} A(x)B(x) & A(x) \\ B(x) & I - P_{f,-}(x) \end{pmatrix}, \quad P_-^s(x) := \begin{pmatrix} E(x)D(x) & E(x) \\ D(x) & P_{f,-}(x) \end{pmatrix}, \quad x \le 0,$$

and we calculate using (5.70)

$$\begin{aligned} P_-^u(x)\Phi_0(x,y) &= \begin{pmatrix} A(x)\Phi_{f,-}^u(x,y)B(y) & A(x)\Phi_{f,-}^u(x,y) \\ \Phi_{f,-}^u(x,y)B(y) & \Phi_{f,-}^u(x,y) \end{pmatrix} = \Phi_0(x,y)P_-^u(y), \\ P_-^s(y)\Phi_0(y,x) &= \begin{pmatrix} E(y)\Phi_{f,-}^s(y,x)D(x) & E(y)\Phi_{f,-}^s(y,x) \\ \Phi_{f,-}^s(y,x)D(x) & \Phi_{f,-}^s(y,x) \end{pmatrix} = \Phi_0(y,x)P_-^s(x), \end{aligned} \quad x \le y \le 0.$$

Since A, B, D and E are bounded on $(-\infty, 0]$, the above calculations imply

$$\left\|P_-^u(x)\Phi_0(x,y)\right\|, \left\|P_-^s(y)\Phi_0(y,x)\right\| \le Ce^{-\mu_r(y-x)}, \quad x \le y \le 0.$$

Define $P_-^c(x) := I - P_-^s(x) - P_-^u(x)$ for $x \le 0$. Observe that

$$P_-^c(x)\Phi_0(x,y) = \begin{pmatrix} E_1(x,y) & E_2(x,y) \\ E_3(x,y) & 0 \end{pmatrix} = \Phi_0(x,y)P_-^c(y), \quad x, y \le 0,$$

where the matrices,

$$\begin{aligned} E_1(x,y) &:= I + \int_y^x \mathcal{A}_1(z)dz + \int_y^{-\infty} \mathcal{A}_2(z) \int_y^z \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz \\ &\quad + \int_x^{-\infty} \mathcal{A}_2(z) \int_z^0 \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz + \int_y^0 \mathcal{A}_2(z) \int_y^z \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz \\ &\quad + \int_x^0 \mathcal{A}_2(z) \int_z^{-\infty} \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz, \\ E_2(x,y) &:= \int_y^{-\infty} \mathcal{A}_2(z)\Phi_{f,-}^u(z,y)dz + \int_y^0 \mathcal{A}_2(z)\Phi_{f,-}^s(z,y)dz, \\ E_3(x,y) &:= \int_0^x \Phi_{f,-}^u(x,z)\mathcal{A}_3(z)dz + \int_{-\infty}^x \Phi_{f,-}^s(x,z)\mathcal{A}_3(z)dz. \end{aligned}$$

are bounded on $(-\infty, 0] \times (-\infty, 0]$ by (5.72). Therefore, the projections $P_-^{u,s,c}(x)$ define an exponential trichotomy for equation (5.56) on $(-\infty, 0]$. The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by

$$\mathcal{P}^u = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I - \mathcal{P}_f) \\ 0 & I - \mathcal{P}_f \end{pmatrix}, \; \mathcal{P}^s = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f \\ 0 & \mathcal{P}_f \end{pmatrix}, \; \mathcal{P}^c = \begin{pmatrix} I & -\mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1} \\ 0 & 0 \end{pmatrix}, \tag{5.75}$$

Thus, the approximations (5.71), (5.73) and (5.74) yield $\|P_-^{u,s,c}(x) - \mathcal{P}^{u,s,c}\| \leq Ce^{\min\{\mu_r,\mu_h\}x/2}$ for $x \leq 0$. Thus, we have obtained the desired exponential trichotomy for (5.56) on $(-\infty, 0]$. The construction of the exponential trichotomy for (5.56) on $[0, \infty)$ is analogous.

Finally, we establish the decompositions (5.68) and (5.69). The upper $(2m \times 2m)$-block of $\Phi_{in}(0)$ is lower-triangular and has determinant 1. Therefore, the columns of $\Phi_{in}(x)$ constitute $2m$ linearly independent solutions to (5.56), which are bounded, but not exponentially localized by (5.64). On the other hand, $P_\pm^{u,s}(0)$ has rank n, since $P_{f,\pm}(0)$ is a rank n projection. This yields the decomposition (5.68). Furthermore, it holds $P_+^s(0)[\mathbb{C}^{2(m+n)}] = P_+^s(0)[\{(0,b) : b \in P_{f,+}(0)[\mathbb{C}^{2n}]\}]$. Since we have $P_{f,+}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^c$ with $Y^c = \mathrm{Sp}(\kappa_h(0))$ by Proposition 5.21, the decomposition of $P_+^s(0)[\mathbb{C}^{2(m+n)}]$ in (5.69) follows. Analogously, we obtain the decomposition of $P_-^u(0)[\mathbb{C}^{2(m+n)}]$ in (5.69). □

As mentioned in §5.3.1, our goal is to construct a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3), which has a jump at $x = 0$ only. The solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ arises by matching solutions to (3.3), which are defined on the three intervals $I_{f,\varepsilon}^\pm$ and $I_{s,\varepsilon}$, given by (5.53) and (5.54). We match these solutions in such a way that the jump at 0 is confined to the one-dimensional space spannend by $(0, \psi_{\mathrm{ad}}(0))$, where $\psi_{\mathrm{ad}}(x)$ is the solution to the adjoint variational equation (3.15), established in Proposition 5.21. Thus, we need the following lemma.

Lemma 5.23. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Let* Y^s, Y^u, Y^c *and* $Y^\perp$ *be as in Proposition 5.21. Denote by* Q^c *the projection on* Y^c *along* $Y^s \oplus Y^u \oplus Y^\perp$, *by* Q^s *the projection on* Y^s *along* $Y^u \oplus Y^c \oplus Y^\perp$ *and by* Q^u *the projection on* Y^u *along* $Y^s \oplus Y^c \oplus Y^\perp$. *The projections,*

$$
\begin{gathered}
\mathcal{Q}^c := \begin{pmatrix} 0 & 0 \\ 0 & Q^c \end{pmatrix}, \ \hat{\mathcal{Q}}^c := \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dxQ^u - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^s+Q^c) \\ 0 & 0 \end{pmatrix}, \\
\mathcal{Q}^s := \begin{pmatrix} 0 & 0 \\ Q^s \int_0^{-\infty} \Phi_f(0,x)\mathcal{A}_3(x)dx & Q^s \end{pmatrix}, \quad \mathcal{Q}^u := \begin{pmatrix} 0 & 0 \\ Q^u \int_0^{\infty} \Phi_f(0,x)\mathcal{A}_3(x)dx & Q^u \end{pmatrix},
\end{gathered} \tag{5.76}
$$

are well-defined and it holds

$$
Z^\perp = \ker(\mathcal{Q}^c) \cap \ker(\hat{\mathcal{Q}}^c) \cap \ker(\mathcal{Q}^s) \cap \ker(\mathcal{Q}^u), \quad Z^\perp := \{(0,b) \colon b \in Y^\perp\}. \tag{5.77}
$$

Moreover, we have

$$
\begin{gathered}
\mathcal{Q}^c\Phi_{in}(0) = 0, \quad \mathcal{Q}^c\varphi_h(0) = \begin{pmatrix} 0 \\ \kappa_h(0) \end{pmatrix}, \quad \hat{\mathcal{Q}}^c \begin{pmatrix} 0 & 0 \\ 0 & I - R_f \end{pmatrix} = 0, \quad \hat{\mathcal{Q}}^c\Phi_{in}(0) = \begin{pmatrix} I \\ 0 \end{pmatrix}, \\
\hat{\mathcal{Q}}^c = \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^u+Q^c) - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dxQ^s \\ 0 & 0 \end{pmatrix},
\end{gathered} \tag{5.78}
$$

where φ_h *and* Φ_{in} *are defined in* (5.62) *and* (5.63), *respectively, and* $\kappa_h(x) = \partial_x\psi_h(x, u_0)$.

Proof. The integrals in (5.76) converge by (5.60). Thus, the projections in (5.76) are well-defined. Furthermore, the homoclinic solution $\psi_{\mathrm{h}}(x, u_0)$ to (2.3) at $u = u_0$ satisfies $R_f \psi_{\mathrm{h}}(x, u_0) = \psi_{\mathrm{h}}(-x, u_0)$ for any $x \in \mathbb{R}$ by **(E1)**. Taking derivatives yields

$$R_f \kappa_{\mathrm{h}}(0) = -\kappa_{\mathrm{h}}(0), \quad R_f \kappa_{in}(0) = \kappa_{in}(0), \tag{5.79}$$

where $\kappa_{in}(x) = \partial_u \psi_{\mathrm{h}}(x, u_0)$. Consequently, any column of $\kappa_{in}(0)$ lies in the orthogonal complement of $Y^c = \mathrm{Sp}(\kappa_{\mathrm{h}}(0))$, which is given by $Y^s \oplus Y^u \oplus Y^\perp$ by Proposition 5.21. Hence, we have $Q^c \kappa_{in}(0) = 0$ and the first two identities in (5.78) follow.

The fast variational equation (3.15) is R_f-reversible at $x = 0$ by **(E1)**. Thus, by (5.61) it holds $\Phi_f(-x, 0)Q^u = R_f \Phi_f(x, 0)Q^s R_f$ and $\Phi_f(-x, 0)Q^c = R_f \Phi_f(x, 0)Q^c R_f$ for any $x \geq 0$. Combining the latter with (5.79) leads to the other three identities in (5.78), where we use that $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds for any $x \in \mathbb{R}$ by **(E1)**.

Using (5.60) we immediately establish $Z^\perp \subset \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$. Conversely, assume $(a, b) \in \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$ with $a \in \mathbb{C}^{2m}$ and $b \in \mathbb{C}^{2n}$. Then, it holds

$$Q^c b = 0, \quad a = \int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x, 0)dx Q^u b + \int_{\infty}^0 \mathcal{A}_2(x)\Phi_f(x, 0)dx(Q^s + Q^c)b,$$
$$Q^s b = -Q^s \int_0^{-\infty} \Phi_f(0, x)\mathcal{A}_3(x)a dx, \quad Q^u b = -Q^u \int_0^{\infty} \Phi_f(0, x)\mathcal{A}_3(x)a dx.$$

We derive that a is strictly lower-triangular implying $\mathcal{A}_3(x)a = 0$ for any $x \in \mathbb{R}$. Hence, it holds $Q^{u,s,c} b = 0$ yielding $b \in Y^\perp$ and $a = 0$. We conclude $(a, b) \in Z^\perp$. □

5.3.3 A reduced eigenvalue problem along the slow manifold

Along the slow manifold, the v-components of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x)$ are exponentially small and the u-components are approximated by $u_{\mathrm{s}}(\varepsilon x)$ – see Theorem 2.3. Hence, by assumption **(S1)**, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in the full eigenvalue problem (3.3) is exponentially small, whereas the upper-left block $\mathcal{A}_{11,\varepsilon}(x, \lambda)$ is approximated by $\varepsilon \mathcal{A}_s(\varepsilon x)$, where $\mathcal{A}_s$ is the coefficient matrix of the slow variational equation (2.7). Thus, along the slow manifold, we arrive at the reduced eigenvalue problem,

$$\varphi_x = \mathcal{A}_{*,\varepsilon}(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{5.80}$$

with

$$\mathcal{A}_{*,\varepsilon}(x, \lambda) := \begin{pmatrix} \varepsilon \mathcal{A}_s(\varepsilon x) & \mathcal{A}_{12,\varepsilon}(x) \\ 0 & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix}.$$

Due to its upper-triangular block structure, the dynamics in system (5.80) is governed by the blocks on the diagonal via the variation of constants formula. The lower-right block $\mathcal{A}_{22,\varepsilon}(x, \lambda)$ has slowly varying coefficients and is pointwise hyperbolic along the slow manifold. Hence, on the interval $I_{s,\varepsilon}$, defined in (5.53), system (5.15) admits an exponential dichotomy, which yields an exponential trichotomy for the reduced eigenvalue problem (5.80).

Proposition 5.24. *Provided $\varsigma, \varepsilon > 0$ are sufficiently small, system* (5.80) *has for every $\lambda \in B(0,\varsigma)$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of ε and λ, and projections $P_{*,\varepsilon}^{u,s,c}(x,\lambda)$. We have $\mu_s = \frac{1}{2}\mu_r$, where $\mu_r > 0$ is as in Lemma 5.1. The projections $P_{*,\varepsilon}^{u,s,c}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in I_{s,\varepsilon}$ and satisfy*

$$\left\|P_{*,\varepsilon}^{u,s,c}(\Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\|, \left\|P_{*,\varepsilon}^{u,s,c}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.81}$$

where Ξ_ε is as in (5.53) *and $\mathcal{P}^u, \mathcal{P}^s$ and $\mathcal{P}^c$ are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$, defined in* (5.65).

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

We start by establishing an exponential dichotomy for the subsystem (5.15) of the reduced eigenvalue problem (5.80). We define

$$J_{\alpha,\varepsilon} := [\Xi_\varepsilon/\alpha, 2L_\varepsilon - \Xi_\varepsilon/\alpha], \quad \alpha \geq 0.$$

First, by Theorem 2.3 it holds

$$\|u_{\mathrm{p},\varepsilon}'(x)\| = \varepsilon\left\|D_1^{-1}p_{\mathrm{p},\varepsilon}(x)\right\| \leq C\varepsilon, \quad \|v_{\mathrm{p},\varepsilon}'(x)\| = \|D_2^{-1}q_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\partial_x\mathcal{A}_{22,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma).$$

Second, by Theorem 2.3 we have

$$\left\|\hat{\phi}_{\mathrm{p},\varepsilon}(x) - (u_{\mathrm{p},\varepsilon}(x), 0)\right\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - A(u_{\mathrm{p},\varepsilon}(x),\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.82}$$

where $A(u,\lambda)$ is defined in (5.1). By Theorem 2.3 and Lemma 5.1, the matrix $A(u_{\mathrm{p},\varepsilon}(x),\lambda)$ is, provided $\varepsilon > 0$ is sufficiently small, hyperbolic for each $x \in J_{4,\varepsilon}$ and $\lambda \in \mathcal{B}(0,\varsigma)$ with spectral gap larger than $\mu_r = 2\mu_s$. So, by (5.82), the same holds for $\mathcal{A}_{22,\varepsilon}(x,\lambda)$, provided $\varepsilon > 0$ is sufficiently small. Third, $\mathcal{A}_{22,\varepsilon}$ is bounded on $J_{4,\varepsilon} \times B(0,\varsigma)$ by an ε-independent constant using Theorem 2.3. Combining these three items with Proposition 4.8 yields, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy for system (5.15) on $J_{2,\varepsilon}$ with constants $C, \mu_s > 0$ and projections $\Pi_{f,\varepsilon}(x,\lambda)$. The projections $\Pi_{f,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in J_{2,\varepsilon}$ and satisfy

$$\left\|\Pi_{f,\varepsilon}(x,\lambda) - Q_\varepsilon(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{2,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.83}$$

where $Q_\varepsilon(x,\lambda)$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{22,\varepsilon}(x,\lambda)$. On the other hand, by Theorem 2.3 and estimate (2.6) we have

$$\left\|\hat{\phi}_{\mathrm{p},\varepsilon}(\Xi_\varepsilon) - (u_0, 0)\right\| \leq C\varepsilon|\log(\varepsilon)|,$$

yielding

$$\left\|\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{A}_{f,\infty}\right\|\leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right),\quad \lambda\in B(0,\varsigma),$$

where $\mathcal{A}_{f,\infty}$ is given by (5.58). Thus, the same bound holds true for the spectral projections associated with $\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)$ and $\mathcal{A}_{f,\infty}$. Combining the latter with (5.83) yields

$$\left\|\Pi_{f,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{P}_f\right\|\leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right),\quad \lambda\in B(0,\varsigma), \tag{5.84}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$.

The next step is to express the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ of the upper-triangular block system (5.80) in terms of the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ of (5.15) and the evolution $\Phi_{\mathrm{s}}(\check{x},\check{y})$ of the slow variational equation (2.7). Thus, via the variation of constants formula we obtain

$$\mathcal{T}_{*,\varepsilon}(x,y,\lambda)=\begin{pmatrix}\Phi_{\mathrm{s}}(\varepsilon x,\varepsilon y) & \int_y^x \Phi_{\mathrm{s}}(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}_{f,\varepsilon}(z,y,\lambda)dz\\ 0 & \mathcal{T}_{f,\varepsilon}(x,y,\lambda)\end{pmatrix}. \tag{5.85}$$

We define candidate trichotomy projections,

$$\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix}0 & \int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^x \Phi_{\mathrm{s}}(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & \Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix},\\
P^u_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix}0 & \int_{\frac{1}{2}\Xi_\varepsilon}^x \Phi_{\mathrm{s}}(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & I-\Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix}, \qquad x\in I_{s,\varepsilon},\lambda\in B(0,\varsigma),\\
P^c_{*,\varepsilon}(x,\lambda) &:= I-P^s_{*,\varepsilon}(x,\lambda)-P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}$$

where $\mathcal{T}^{u,s}_{f,\varepsilon}(x,y,\lambda)$ denotes the (un)stable evolution under the exponential dichotomy of (5.15) on $J_{2,\varepsilon}$. The projections $P^{u,s,c}_{*,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x\in I_{s,\varepsilon}$, because the projections $\Pi_{f,\varepsilon}(x,\lambda)$ and the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ are analytic in λ using [60, Lemma 2.1.4]. On the other hand, lemma 4.1 it yields

$$\|\Phi_{\mathrm{s}}(\varepsilon x,\varepsilon z)\|\leq C,\quad x,y\in J_{2,\varepsilon}, \tag{5.86}$$

because it holds $|\varepsilon L_\varepsilon-\ell_0|\leq C\varepsilon$ by Theorem 2.3. Using (5.85) we calculate for $x,y\in I_{s,\varepsilon}$ and $\lambda\in B(0,\varsigma)$

$$\left.\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) &:= \begin{pmatrix}0 & \int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^x \Phi_{\mathrm{s}}(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz\\ 0 & \mathcal{T}^s_{f,\varepsilon}(x,y,\lambda)\end{pmatrix}\\
&= \mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^s_{*,\varepsilon}(y,\lambda),\\
P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda) &:= \begin{pmatrix}0 & \int_{\frac{1}{2}\Xi_\varepsilon}^y \Phi_{\mathrm{s}}(\varepsilon y,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz\\ 0 & \mathcal{T}^u_{f,\varepsilon}(y,x,\lambda)\end{pmatrix}\\
&= \mathcal{T}_{*,\varepsilon}(y,x,\lambda)P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}\right\}\quad x\geq y,$$

and

$$P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) := \begin{pmatrix} \Phi_s(\varepsilon x,\varepsilon y) & E_\varepsilon(x,y,\lambda) \\ 0 & 0 \end{pmatrix} = \mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^c_{*,\varepsilon}(y,\lambda), \tag{5.87}$$

$$\begin{aligned} E_\varepsilon(x,y,\lambda) := &-\int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^{y} \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz \\ &-\int_{\frac{1}{2}\Xi_\varepsilon}^{y} \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,y,\lambda)dz. \end{aligned}$$

Using estimate (5.86) and the fact that $\mathcal{A}_{12,\varepsilon}$ is ε-uniformly bounded on $J_{2,\varepsilon}$ by Theorem 2.3, the above calculations imply for $x,y \in I_{s,\varepsilon}$ and $\lambda \in B(0,\varsigma)$

$$\left\|P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\|, \left\|P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda)\right\| \leq Ce^{-\mu_s(x-y)}, \quad x \geq y,$$

and

$$\left\|P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\| \leq C.$$

Therefore, the projections $P^{u,s,c}_{*,\varepsilon}(x)$ define an exponential trichotomy for equation (5.80) on $I_{s,\varepsilon}$.

Finally, we establish the approximations (5.81). Define $\tilde{J}_\varepsilon := \left[\frac{1}{2}\Xi_\varepsilon, \frac{3}{2}\Xi_\varepsilon\right]$. First, by Lemma 4.1 it holds

$$\|\Phi_s(\varepsilon x,\varepsilon y) - I\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in \tilde{J}_\varepsilon, \tag{5.88}$$

Second, by Theorem 2.3 and estimate (2.6) we have

$$\left\|\hat{\phi}_{p,\varepsilon}(x) - (u_0,0)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x,y \in \tilde{J}_\varepsilon,$$

yielding for $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0,\varsigma)$

$$\left\|\mathcal{A}_{12,\varepsilon}(x) - \mathcal{A}_{2,\infty}\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad \left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \mathcal{A}_{f,\infty}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.89}$$

where $\mathcal{A}_{2,\infty}$ is defined in (5.65). Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 5.1, the matrix $\mathcal{A}_{22,\varepsilon}(x,\lambda)$ is by (5.89) invertible for each $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0,\varsigma)$, provided $\varepsilon,\varsigma > 0$ are sufficiently small. Thus, for $\lambda \in B(0,\varsigma)$ we may write

$$\begin{aligned} &\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz \\ &\quad = \int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{A}_{22,\varepsilon}(z,\lambda)^{-1}\partial_z\mathcal{T}_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz\left(I - \Pi_{f,\varepsilon}(\Xi_\varepsilon,\lambda)\right). \end{aligned}$$

Combining the latter with (5.84), (5.88) and (5.89), leads, via integration by parts, to the approximation,

$$\left\|\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}(I - \mathcal{P}_f)\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.90}$$

for $\lambda \in B(0,\varsigma)$, where we use $\mu_r = 2\mu_s$. Similarly, we derive

$$\left\|\int_{\frac{3}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}\mathcal{P}_f\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad (5.91)$$

for $\lambda \in B(0,\varsigma)$. On the other hand, (5.86) yields

$$\left\|\int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^{\frac{3}{2}\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz\right\| \leq C\varepsilon, \quad \lambda \in B(0,\varsigma). \quad (5.92)$$

The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by (5.75). Thus, the approximations (5.84), (5.90), (5.91) and (5.92) yield $\|P^{u,s,c}_{*,\varepsilon}(\Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\| \leq C(\varepsilon|\log(\varepsilon)| + |\lambda|)$ for $\lambda \in B(0,\varsigma)$. The other estimate in (5.81) follows analogously. □

5.3.4 Construction of a piecewise continuous eigenfunction

Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and Ξ_ε be as in (3.21), (5.52) and (5.53), respectively. In this section we establish for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3) on the interval $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$. The construction of $\varphi_{\nu,\varepsilon}$ is based on Lin's method [10, 70, 100].

Theorem 5.25. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Take* $\delta > 0$. *Provided* $\eta,\varepsilon > 0$ *are sufficiently small, there exists for every* $\lambda \in D_{\eta,\varepsilon}$ *and* $\nu \in \mathcal{S}_\delta$ *a piecewise continuous solution* $\varphi_{\nu,\varepsilon}(x,\lambda)$ *to the full eigenvalue problem* (3.3) *on* $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, *which has a jump only at* $x = 0$, *satisfies*

$$\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda),$$

and enjoys the estimates,

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \varphi_h(x)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad x \in [-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \quad (5.93)$$

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{p,\varepsilon}(x) + \varepsilon\Phi_{in}(x)\mathcal{B}(\nu)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^3 + |\lambda|\right), \quad x \in \left[-\tfrac{\Xi_\varepsilon}{2}, \tfrac{\Xi_\varepsilon}{2}\right], \quad (5.94)$$

where $\mathcal{B}(\nu)$, φ_h *and* Φ_{in} *are defined in* (3.20), (5.62) *and* (5.63), *respectively, and* $C > 0$ *is a constant independent of* ε, λ *and* ν. *In addition, for any* $\nu \in \mathcal{S}_\delta$ *there exists a unique* λ*-value* $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$ *for which the jump of* $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ *vanishes.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε, λ and ν.

Let $I_{f,\varepsilon} = I^+_{f,\varepsilon} \cup I^-_{f,\varepsilon}$ and $I_{s,\varepsilon}$ be as in (5.53) and (5.54). Our approach is to regard the full eigenvalue problem (3.3) as a perturbation of the reduced eigenvalue problems (5.56) and (5.80) on the intervals $I_{f,\varepsilon}$ and $I_{s,\varepsilon}$, respectively. Propositions 5.22 and 5.24 yield exponential trichotomies for (5.56) and (5.80). For $\lambda \in D_{\eta,\varepsilon}$, this leads to variation of constants formulas for solutions to (3.3) on the three intervals $I_{s,\varepsilon}$ and $I^{\pm}_{f,\varepsilon}$. We match these solutions at the endpoints $0, \pm\Xi_\varepsilon$ and $2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I^{\pm}_{f,\varepsilon}$ using the estimates (5.66) and (5.81) on the trichotomy projections and identity (5.77). Thus, we obtain for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$. For each $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ the jump

$$J_{\nu,\varepsilon}(\lambda) := \lim_{x\downarrow 0} \varphi_{\nu,\varepsilon}(x,\lambda) - \lim_{x\uparrow 0} \varphi_{\nu,\varepsilon}(x,\lambda), \tag{5.95}$$

is contained in the one-dimensional space $Z^\perp$, defined in (5.77). Pairing the jump with the solution $\psi_{\mathrm{ad}}(x)$ to the adjoint (3.19), established in Proposition 5.21, leads to an (analytic) equation in λ and ν, which has a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

The variation of constants formulas provide leading-order expressions for $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the three intervals $I^{\pm}_{f,\varepsilon}$ and $I_{s,\varepsilon}$. Finally, since the derivative $\phi'_{\mathrm{p},\varepsilon}(x)$ is a solution to (3.3) at $\lambda = 0$, we can write $\phi'_{\mathrm{p},\varepsilon}(x)$ in terms of similar variation of constants formulas on $I^{\pm}_{f,\varepsilon}$ yielding leading-order approximation for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)$.

Thus, we start by establishing expressions for solutions to the full eigenvalue problem (3.3) along the pulse. We regard (3.3) as the perturbation,

$$\varphi_x = \left(\mathcal{A}_0(x) + \mathcal{B}_{0,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.56). By Theorem 2.3, the perturbation matrix $\mathcal{B}_{0,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_0(x)$ is bounded by

$$\|\mathcal{B}_{0,\varepsilon}(x,\lambda)\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.96}$$

By Proposition 5.22, system (5.56) has exponential trichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C, \mu_r > 0$ and projections $P^{u,s,c}_{\pm}(x)$ satisfying (5.66). We denote by $\Phi^{s,u,c}_{0,\pm}(x,y)$ the stable, unstable and neutral evolution operator of system (5.56) under the exponential trichotomies. For convenience, we abbreviate $\Phi^{sc}_{0,\pm}(x,y) = (I - P^u_\pm(x))\Phi_0(x,y)$ and $\Phi^{uc}_{0,\pm}(x,y) = (I - P^s_\pm(x))\Phi_0(x,y)$.

We apply the variation of constants formula. Thus, by the decompositions (5.68) and (5.69), any solution $\varphi^+_{f,\varepsilon}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I^+_{f,\varepsilon}$:

$$\begin{aligned} \varphi^+_{f,\varepsilon}(x,\lambda) = \Phi^u_{0,+}(x,\Xi_\varepsilon)a_+ + \Phi_{in}(x)b_+ + \int_0^x \Phi^s_{0,+}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^+_{f,\varepsilon}(y,\lambda)dy \\ + \varphi_{\mathrm{h}}(x)c_+ + \Phi^s_{0,+}(x,0)d_+ + \int_{\Xi_\varepsilon}^x \Phi^{uc}_{0,+}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^+_{f,\varepsilon}(y,\lambda)dy, \end{aligned} \tag{5.97}$$

for some $a_+ \in P^u_+(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_+ \in \mathbb{C}^{2m}$, $c_+ \in \mathbb{C}$ and $d_+ \in Z^s$, where Z^s is defined in (5.69). Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.96) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi^+_{f,\varepsilon}(x,\lambda)$ to (5.97) on $I^+_{f,\varepsilon}$ using the contraction mapping principle. Note that $\varphi^+_{f,\varepsilon}(x,\lambda)$ is linear in (a_+, b_+, c_+, d_+) and satisfies the bound,

$$\sup_{x\in I^+_{f,\varepsilon}} \|\varphi^+_{f,\varepsilon}(x,\lambda)\| \leq C\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.98}$$

by estimate (5.96), taking $\eta, \varepsilon > 0$ smaller if necessary. Similarly, by (5.68) and (5.69), any solution $\varphi^-_{f,\varepsilon}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I^-_{f,\varepsilon}$:

$$\begin{aligned} \varphi^-_{f,\varepsilon}(x,\lambda) = \Phi^s_{0,-}(x,-\Xi_\varepsilon)a_- + \Phi_{in}(x)b_- + \int_0^x \Phi^u_{0,-}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^-_{f,\varepsilon}(y,\lambda)dy \\ + \varphi_{\mathrm{h}}(x)c_- + \Phi^u_{0,-}(x,0)d_- + \int_{-\Xi_\varepsilon}^x \Phi^{sc}_{0,-}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^-_{f,\varepsilon}(y,\lambda)dy, \end{aligned} \tag{5.99}$$

for some $a_- \in P^s_-(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_- \in \mathbb{C}^{2m}$, $c_- \in \mathbb{C}$ and $d_- \in Z^u$, where Z^u is defined in (5.69). There exists for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi^-_{f,\varepsilon}(x,\lambda)$ to (5.99) on $I^-_{f,\varepsilon}$, which is linear in (a_-, b_-, c_-, d_-) and satisfies the bound,

$$\sup_{x\in I^-_{f,\varepsilon}} \|\varphi^-_{f,\varepsilon}(x,\lambda)\| \leq C\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.100}$$

taking $\eta, \varepsilon > 0$ smaller if necessary.

Our next step is to obtain expressions for solutions to the full eigenvalue problem (3.3) along the slow manifold. We regard (3.3) as the perturbation,

$$\varphi_x = \left(\mathcal{A}_{*,\varepsilon}(x,\lambda) + \mathcal{B}_{*,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.80). By Theorem 2.3 it holds

$$\|u_{\mathrm{p},\varepsilon}(x) - u_{\mathrm{s}}(\varepsilon x)\| \leq C\varepsilon, \quad \|v_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in I_{s,\varepsilon}.$$

Therefore, by **(S1)** the perturbation matrix $\mathcal{B}_{*,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_{*,\varepsilon}(x,\lambda)$ is bounded by

$$\left\|\mathcal{B}_{*,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon\left(\varepsilon + |\lambda|\right), \quad x \in I_{s,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.101}$$

By Proposition 5.24 system (5.80) admits for every $\lambda \in D_{\eta,\varepsilon}$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of ε and λ, and projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ satisfying (5.81). We denote by $\mathcal{T}^{s,u,c}_{*,\varepsilon}(x,y,\lambda)$ the stable, unstable and neutral evolution operator of system (5.80) under the exponential trichotomy.

We apply the variation of constants formula. Thus, any solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I_{s,\varepsilon}$:

$$\begin{aligned} \varphi_{s,\varepsilon}(x,\lambda) = \mathcal{T}^s_{*,\varepsilon}(x,\Xi_\varepsilon,\lambda)f + \mathcal{T}^c_{*,\varepsilon}(x,\Xi_\varepsilon,\lambda)h + \int_{\Xi_\varepsilon}^x \mathcal{T}^{sc}_{*,\varepsilon}(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy \\ + \mathcal{T}^u_{*,\varepsilon}(x,2L_\varepsilon - \Xi_\varepsilon,\lambda)g + \int_{2L_\varepsilon-\Xi_\varepsilon}^x \mathcal{T}^u_{*,\varepsilon}(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy, \end{aligned} \tag{5.102}$$

for some $f \in P^s_{*,\varepsilon}(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $g \in P^u_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$ and $h \in P^c_{*,\varepsilon}(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$. Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.101) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (5.102) on $I_{s,\varepsilon}$ using the contraction mapping principle. The solution $\varphi_{s,\varepsilon}(x,\lambda)$ is linear in (f, g, h) and enjoys the bound,

$$\sup_{x \in I_{s,\varepsilon}} \|\varphi_{s,\varepsilon}(x,\lambda)\| \leq C\left(\|f\| + \|g\| + \|h\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.103}$$

using estimate (5.101) and the fact that $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ by Theorem 2.3.

Now, we match the solutions $\varphi^{\pm}_{f,\varepsilon}(x,\lambda)$ and $\varphi_{s,\varepsilon}(x,\lambda)$, given by (5.97), (5.99) and (5.102), at the endpoints $x = \pm\Xi_\varepsilon$ and $x = 2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I^{\pm}_{f,\varepsilon}$. Applying the projection $P^s_{*,\varepsilon}(\Xi_\varepsilon, \lambda)$ to the difference $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition,

$$\begin{aligned} f &= \mathcal{H}^1_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+), \\ \|\mathcal{H}^1_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.104}$$

where we use (5.66), (5.68), (5.81), (5.96) and (5.98) to obtain the bound on the linear map $\mathcal{H}^1_{\varepsilon,\lambda}$. Similarly, applying $P^u_{*,\varepsilon}(\Xi_\varepsilon,\lambda)$ to $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition,

$$\begin{aligned} a_+ &= \mathcal{H}^2_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+, f, g, h), \\ \|\mathcal{H}^2_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+, f, g, h)\| &\leq C\left[\varepsilon\left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right. \\ &\qquad \left.+\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right)\right], \end{aligned} \tag{5.105}$$

where we use (5.66), (5.68), (5.81), (5.96), (5.98), (5.101), (5.103) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bound on the linear map $\mathcal{H}^2_{\varepsilon,\lambda}$. Finally, applying $P^c_{*,\varepsilon}(\Xi_\varepsilon,\lambda)$ to $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition,

$$\begin{aligned} h &= \begin{pmatrix} \Upsilon_\infty b_+ \\ 0 \end{pmatrix} + \mathcal{H}^3_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+), \\ \|\mathcal{H}^3_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+)\| &\leq \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.106}$$

where we use (5.64), (5.66), (5.81), (5.96) and (5.98) to obtain the bound on the linear map $\mathcal{H}^3_{\varepsilon,\lambda}$. Note that $\mathcal{H}^{1,2,3}_{\varepsilon,\lambda}$ are analytic in λ, because the perturbations matrices $\mathcal{B}_{0,\varepsilon}(x,\lambda)$ and $\mathcal{B}_{*,\varepsilon}(x,\lambda)$, the projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ and the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ are analytic in λ by Proposition 5.24 and [60, Lemma 2.1.4].

Take $\nu \in \mathcal{S}_\delta$. We obtain the following matching conditions for any $\lambda \in D_{\eta,\varepsilon}$ by applying the projections $P^{u,s,c}_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$ to the difference $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon,\lambda)$:

$$\begin{aligned} g &= \mathcal{H}^4_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-), \\ \|\mathcal{H}^4_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \end{aligned} \tag{5.107}$$

$$
\begin{aligned}
a_- &= \mathcal{H}^5_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-, f, g, h),\\
\|\mathcal{H}^5_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-, f, g, h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|)\right.\\
&\quad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right],
\end{aligned} \tag{5.108}
$$

$$
\begin{aligned}
\mathcal{T}^c_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \Xi_\varepsilon, \lambda)h &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty}b_- \\ 0 \end{pmatrix} + \mathcal{H}^6_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-, f, g, h),\\
\|\mathcal{H}^6_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-, f, g, h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|)\right.\\
&\quad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right],
\end{aligned} \tag{5.109}
$$

where we use (5.64), (5.66), (5.68), (5.81), (5.96), (5.100), (5.101), (5.103) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bounds on the linear maps $\mathcal{H}^{4,5,6}_{\varepsilon,\lambda}$, which are analytic in λ. We introduce the shorthand notation $a = (a_+, a_-)$, $b = (b_+, b_-)$, $c = (c_+, c_-)$ and $d = (d_+, d_-)$. Substituting (5.106) into (5.109) yields a linear map $\mathcal{H}^7_{\varepsilon,\lambda}$, which is analytic in λ, satisfying

$$
\begin{aligned}
\begin{pmatrix} \Phi_s(2\ell_0, 0)\Upsilon_\infty b_+ \\ 0 \end{pmatrix} &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty}b_- \\ 0 \end{pmatrix} + \mathcal{H}^7_{\varepsilon,\lambda}(a, b, c, d, f, g, h)\\
\|\mathcal{H}^7_{\varepsilon,\lambda}(a, b, c, d, f, g, h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a\| + \|b\| + \|c\| + \|d\|)\right. \qquad \lambda \in D_{\eta,\varepsilon},\\
&\quad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right],
\end{aligned} \tag{5.110}
$$

where we use (5.87), $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ and the bound,

$$
\|\Phi_s(\varepsilon x, \varepsilon y) - I\| \leq C\varepsilon|\log(\varepsilon)|, \quad |x - y| \leq 2\Xi_\varepsilon, \tag{5.111}
$$

which follows from Proposition 4.1. The matching conditions (5.104), (5.105), (5.106), (5.107), (5.108) and (5.110) constitute a system of 6 linear equations in 11 variables. One readily observes that, provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for $a_\pm, f, g, h$ and b_- yielding linear maps $\mathcal{H}^{8,9}_{\varepsilon,\lambda}$, which are analytic in λ and satisfy

$$
\begin{aligned}
(f, g, a) &= \mathcal{H}^8_{\varepsilon,\lambda}(b_+, c, d),\\
(h, b_-) &= \left(\Upsilon_\infty b_+, e^{-i\nu}\Upsilon_\infty \Phi_s(2\ell_0, 0)\Upsilon_\infty b_+\right) + \mathcal{H}^9_{\varepsilon,\lambda}(b_+, c, d), \qquad \lambda \in D_{\eta,\varepsilon},\\
\|\mathcal{H}^{8,9}_{\varepsilon,\lambda}(b_-, c, d)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)(\|b_-\| + \|c\| + \|d\|).
\end{aligned} \tag{5.112}
$$

Thus, since the projections $P^{u,s,c}_{*,\varepsilon}(x, \lambda)$ are complementary, we observe that (f, g, h, a, b_-) satisfies (5.112) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon, \lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon, \lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon, \lambda)$ hold true.

Our next step is to match the solutions $\varphi^\pm_{f,\varepsilon}(x, \lambda)$, given by (5.97) and (5.99), at $x = 0$ such that the jump $\varphi^+_{f,\varepsilon}(0, \lambda) - \varphi^-_{f,\varepsilon}(0, \lambda)$ is confined to the one-dimensional space $Z^\perp$, which is defined in (5.77). First, we apply the projections $Q^{u,s}$, given by (5.76). By (5.60) and (5.67) it holds

$$
Q^s P^s_-(0) = Q^s, \quad Q^s P^u_+(0) = 0, \quad (I - Q^s P^s_+(0))[Z^s] = 0. \tag{5.113}
$$

Applying the projection Q^s to the difference $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition,

$$\begin{aligned} d_+ &= \mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d), \\ \|\mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left(\|a\|+\|b\|+\|c\|+\|d\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.114}$$

where we use (5.68), (5.69), (5.96), (5.98), (5.100) and (5.113) to obtain the bound on the linear map $\mathcal{H}^{10}_{\varepsilon,\lambda}$, which is analytic in λ. Similarly, applying Q^u to $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$, we establish a linear map $\mathcal{H}^{11}_{\varepsilon,\lambda}$, which is analytic in λ, satisfying

$$\begin{aligned} d_- &= \mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d), \\ \|\mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left(\|a\|+\|b\|+\|c\|+\|d\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \tag{5.115}$$

Next, we apply the projections Q^c and $\hat{Q}^c$, given by (5.76). By (5.60) and (5.67) it holds

$$Q^c P^u_-(0) = Q^c = Q^c P^s_+(0), \quad \hat{Q}^c P^{sc}_-(0) = \hat{Q}^c = \hat{Q}^c P^{uc}_+(0). \tag{5.116}$$

Applying Q^c to the difference $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition,

$$c_+ = c_-, \tag{5.117}$$

where we use (5.78) and (5.116). Finally, applying $\hat{Q}^c$ to $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition,

$$\begin{aligned} \begin{pmatrix} b_+ - b_- \\ 0 \end{pmatrix} &= \int_{-\Xi_\varepsilon}^0 \hat{Q}^c \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\mathrm{h}}(y)c_- dy + \mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d) \\ &\quad - \int_{\Xi_\varepsilon}^0 \hat{Q}^c \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\mathrm{h}}(y)c_+ dy, \qquad \lambda \in D_{\eta,\varepsilon}, \\ \|\mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left[\left(\|a\|+\|b\|+\|d\|\right)\right. \\ &\quad \left. +|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\|c\|\right], \end{aligned} \tag{5.118}$$

where we use (5.78), (5.96), (5.98), (5.100) and (5.116) to obtain the bound on the linear map $\mathcal{H}^{12}_{\varepsilon,\lambda}$, which is analytic in λ.

We wish to approximate the integral expressions in (5.118). Therefore, we split the perturbation $\mathcal{B}_{0,\varepsilon}(y,\lambda)$ in an ε-dependent and λ-dependent part, i.e. it holds

$$\left\|\mathcal{B}_{0,\varepsilon}(y,\lambda)-\mathcal{B}_{0,\varepsilon}(y,0)-\lambda\mathcal{B}_*\right\| \leq C\varepsilon|\lambda|, \quad y \in I_{f,\varepsilon}, \lambda \in \mathbb{C}, \tag{5.119}$$

with

$$\mathcal{B}_* := \begin{pmatrix} 0 & 0 \\ 0 & B_* \end{pmatrix} \in \mathrm{Mat}_{2(n+m)\times 2(n+m)}(\mathbb{C}), \quad B_* := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}).$$

First, we approximate the λ-dependent part of the integrals in (5.118). Recall that system (3.15) is R_f-reversible at $x = 0$ by **(E1)**. Thus, the evolution $\Phi_f(x, y)$ of (3.15) satisfies $R_f \Phi_f(x, y) R_f = \Phi_f(-x, -y)$ for any $x, y \in \mathbb{R}$. Hence, using (5.70) we calculate

$$\begin{aligned}\Phi_0(0, x)\mathcal{B}_*\varphi_{\mathrm{h}}(x) &= \begin{pmatrix} \int_x^0 \mathcal{A}_2(y)\Phi_f(y, x)B_*\kappa_{\mathrm{h}}(x)dy \\ \Phi_f(0, x)B_*\kappa_{\mathrm{h}}(x) \end{pmatrix} = \begin{pmatrix} -\int_{-x}^0 \mathcal{A}_2(y)\Phi_f(y, -x)B_*\kappa_{\mathrm{h}}(-x)dy \\ R_f\Phi_f(0, -x)B_*\kappa_{\mathrm{h}}(-x) \end{pmatrix} \\ &= \begin{pmatrix} -I & 0 \\ 0 & R_f \end{pmatrix}\Phi_0(0, -x)\mathcal{B}_*\varphi_{\mathrm{h}}(-x),\end{aligned}$$

where we use that $R_f B_* = -B_* R_f$, $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$, $R_f\kappa_{\mathrm{h}}(x) = -\kappa_{\mathrm{h}}(-x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds true for any $x \in \mathbb{R}$ by **(E1)**. Combining the latter identity with (5.78) yields

$$\hat{Q}^c\left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy\right] = 0. \tag{5.120}$$

Next, we approximate the ε-dependent part of the integrals in (5.118). This can be done by using that the derivative $\phi'_{\mathrm{p},\varepsilon}(x)$ is a solution to (3.3) at $\lambda = 0$. Thus, $\phi'_{\mathrm{p},\varepsilon}(x)$ satisfies the integral equation (5.97) on $I^+_{f,\varepsilon}$ at $\lambda = 0$, i.e. we have for $x \in I^+_{f,\varepsilon}$

$$\begin{aligned}\phi'_{\mathrm{p},\varepsilon}(x) = \Phi^u_{0,+}(x, \Xi_\varepsilon)a_{\mathrm{p},+} + \Phi_{in}(x)b_{\mathrm{p},+} + \int_0^x \Phi^s_{0,+}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_{\mathrm{h}}(x)c_{\mathrm{p},+} + \Phi^s_{0,+}(x, 0)d_{\mathrm{p},+} + \int_{\Xi_\varepsilon}^x \Phi^{uc}_{0,+}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy,\end{aligned} \tag{5.121}$$

for some constants $a_{\mathrm{p},+} \in P^u_+(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},+} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},+} \in \mathbb{C}$ and $d_{\mathrm{p},+} \in Z^s$, where we suppress their ε-dependence for notational convenience. Similarly, it holds for $x \in I^-_{f,\varepsilon}$

$$\begin{aligned}\phi'_{\mathrm{p},\varepsilon}(x) = \Phi^s_{0,-}(x, -\Xi_\varepsilon)a_{\mathrm{p},-} + \Phi_{in}(x)b_{\mathrm{p},-} + \int_0^x \Phi^u_{0,-}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_{\mathrm{h}}(x)c_{\mathrm{p},-} + \Phi^u_{0,-}(x, 0)d_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^x \Phi^{sc}_{0,-}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy,\end{aligned} \tag{5.122}$$

for some $a_{\mathrm{p},-} \in P^s_-(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},-} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},-} \in \mathbb{C}$ and $d_{\mathrm{p},-} \in Z^u$. By applying suitable projections, we obtain leading-order approximations for the constants $a_{\mathrm{p},\pm}, b_{\mathrm{p},\pm}, c_{\mathrm{p},\pm}$ and $d_{\mathrm{p},\pm}$. This leads to the desired approximations for the integrals in (5.118).

First, Theorem 2.3 and **(S1)** yield

$$\left\|\phi'_{\mathrm{p},\varepsilon}(\pm\Xi_\varepsilon) - \varepsilon\begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0, 0, 0) \\ 0 \\ 0 \end{pmatrix}\right\| \leq C\varepsilon^2|\log(\varepsilon)|, \tag{5.123}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ solves the differential equation (2.1). By applying the projections $P^u_+(\Xi_\varepsilon)$ and $P^c_+(\Xi_\varepsilon)$ to (5.121) at $x = \Xi_\varepsilon$, we derive via (5.68) and (5.69)

$$a_{\mathrm{p},+} = P^u_+(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon), \quad P^c_+(\Xi_\varepsilon)\Phi_{in}(\Xi_\varepsilon)b_{\mathrm{p},+} = P^c_+(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon).$$

Similarly, we apply $P^s_-(-\Xi_\varepsilon)$ and $P^c_-(-\Xi_\varepsilon)$ to (5.122) at $x = -\Xi_\varepsilon$ yielding

$$a_{\mathrm{p},-} = P^s_-(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon), \quad P^c_-(-\Xi_\varepsilon)\Phi_{in}(-\Xi_\varepsilon)b_{\mathrm{p},-} = P^c_-(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon).$$

Combining the latter two identities with (5.64), (5.66), (5.75) and (5.123) gives

$$\|a_{\mathrm{p},\pm}\| \leq C\varepsilon, \quad \left\| b_{\mathrm{p},\pm} - \varepsilon\Upsilon_{\mp\infty}\begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0)\end{pmatrix}\right\| \leq C\varepsilon^2|\log(\varepsilon)|. \tag{5.124}$$

Recall that we have $\varphi_\mathrm{h}(x) = \partial_x\phi_\mathrm{h}(x,u_0)$. Thus, by Theorem 2.3 it holds

$$\left\|\phi'_{\mathrm{p},\varepsilon}(x) - \varphi_\mathrm{h}(x)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in I_{f,\varepsilon}, \tag{5.125}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ and ϕ_h solve (2.1) and (2.2), respectively. Next, we apply Q^c to (5.121) and (5.122) at $x = 0$, yielding

$$c_{\mathrm{p},+} = \frac{\left\langle \left(\begin{smallmatrix} 0 \\ \kappa_\mathrm{h}(0)\end{smallmatrix}\right), \phi'_{\mathrm{p},\varepsilon}(0)\right\rangle}{\|\kappa_\mathrm{h}(0)\|^2} = c_{\mathrm{p},-}, \quad \left|c_{\mathrm{p},\pm} - 1\right| \leq C\varepsilon, \tag{5.126}$$

by (5.78), (5.116) and (5.125). Finally, applying $\hat{Q}^c$ to (5.121) and (5.122) at $x = 0$, gives the identity,

$$\begin{aligned}\begin{pmatrix} b_{\mathrm{p},+} - b_{\mathrm{p},-} \\ 0 \end{pmatrix} = \hat{Q}^c \Bigg[&\Phi^s_{0,-}(0,-\Xi_\varepsilon)a_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ &- \Phi^u_{0,+}(0,\Xi_\varepsilon)a_{\mathrm{p},+} - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy\Bigg],\end{aligned}$$

by (5.78) and (5.116). Using (5.96), (5.124) and (5.125), we approximate both sides of the latter identity, yielding

$$\begin{aligned}&\left\|\hat{Q}^c\left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_\mathrm{h}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_\mathrm{h}(y)dy\right] - \varepsilon\begin{pmatrix} 2D_1^{-1}\mathcal{J}(u_0) \\ 0 \\ 0 \\ 0\end{pmatrix}\right\| \\ &\quad \leq C\varepsilon^2|\log(\varepsilon)|^2,\end{aligned} \tag{5.127}$$

which gives together with (5.119) and (5.120) the desired leading-order expressions of the integrals in (5.118).

Thus, the matching conditions (5.112), (5.114), (5.115), (5.117) and (5.118) constitute a system of 10 linear equations in 11 variables. Provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for a, b, c_-, d, f, g, h yielding analytic linear maps $\mathcal{H}^{13}_{\varepsilon,\lambda}, \mathcal{H}^{14}_{\varepsilon,\lambda}$ and $\mathcal{H}^{15}_{\varepsilon,\lambda}$

for $\lambda \in D_{\eta,\varepsilon}$ satisfying

$$
\begin{aligned}
(a,d,f,g,h) &= \mathcal{H}^{13}_{\varepsilon,\lambda}(c_+),\\
c_- &= c_+,\\
b_+ &= 2\varepsilon\left(I - e^{-i\nu}\Upsilon_\infty \Phi_{\mathrm{s}}(2\ell_0,0)\Upsilon_\infty\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0)\\ 0\end{pmatrix} c_+ + \mathcal{H}^{14}_{\varepsilon,\lambda}(c_+),\\
b_- &= 2\varepsilon\left(e^{i\nu}\Upsilon_{-\infty}\Phi_{\mathrm{s}}(0,2\ell_0)\Upsilon_{-\infty} - I\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0)\\ 0\end{pmatrix} c_+ + \mathcal{H}^{15}_{\varepsilon,\lambda}(c_+),\\
\|\mathcal{H}^{13}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)|c_+|,\\
\|\mathcal{H}^{14,15}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2|c_+|,
\end{aligned}
\tag{5.128}
$$

where we use (5.119), (5.120), (5.127) and the fact that $\det(I - e^{-i\nu}\Upsilon_\infty\Phi_{\mathrm{s}}(2\ell_0,0)\Upsilon_\infty) = e^{2im\nu}\mathcal{E}_{s,0}(0,e^{i\nu})$ and $\det(e^{i\nu}\Upsilon_{-\infty}\Phi_{\mathrm{s}}(0,2\ell_0)\Upsilon_{-\infty} - I) = \mathcal{E}_{s,0}(0,e^{i\nu})$ are bounded away from 0 by a ν-independent constant.

Recall that (f,g,h,a,b_-) satisfy (5.112) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon,\lambda)$ hold true. Moreover, by identity (5.77), (a,b,c,d) satisfy (5.114), (5.115), (5.117) and (5.118) if and only if the jump $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ lies in $Z^\perp$. Thus, take $c_+ := c_{\mathrm{p},+}$ and define quantities $a_\pm, b_\pm, c_-, d_\pm, f, g$ and h through (5.128), where we suppress their ε-, λ- and ν-dependence for notational convenience. Then, (5.97), (5.99) and (5.102) define for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ in the space $Z^\perp$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$.

Now, estimate (5.93) follows readily by approximating the coefficients (a,b,c,d,f,g,h) in the variation of constants formulations (5.97), (5.99) and (5.102) of the solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ using (5.96), (5.98), (5.100), (5.101), (5.103), (5.126) and (5.128).

Next, we show that for any $\nu \in \mathcal{S}_\delta$ the jump $J_{\nu,\varepsilon}(\lambda)$, defined in (5.95), of $\varphi_{\nu,\varepsilon}(x,\lambda)$ at $x = 0$ vanishes for a unique λ-value in $D_{\eta,\varepsilon}$. Fix $\nu \in \mathcal{S}_\delta$. The jump $J_{\nu,\varepsilon}(\lambda)$ can be expressed as the difference of the two variation of constants formulas (5.97) and (5.99) at $x = 0$ with coefficients $a_\pm, b_\pm, c_\pm$ and $d_\pm$ defined through (5.128) and $c_+ = c_{\mathrm{p},+}$. We observe that $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, because the perturbation $\mathcal{B}_{0,\varepsilon}(x,\lambda)$ and the linear maps $\mathcal{H}^{13}_{\varepsilon,\lambda}, \mathcal{H}^{14}_{\varepsilon,\lambda}$ and $\mathcal{H}^{15}_{\varepsilon,\lambda}$ are analytic in λ. For any $\lambda \in D_{\eta,\varepsilon}$ the jump is approximated by

$$
\begin{aligned}
&\left\| J_{\nu,\varepsilon}(\lambda) - d_+ + d_- - \lambda\int_\infty^0 \Phi^{uc}_{0,+}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \lambda\int_{-\infty}^0 \Phi^{sc}_{0,-}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy\right\|\\
&\quad \leq C|\log(\varepsilon)|^2\left(\varepsilon + |\lambda|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right),
\end{aligned}
\tag{5.129}
$$

using (5.93), (5.96), (5.119) and (5.128). By Proposition 5.21 we have $\psi_{\mathrm{ad}}(0) \in \ker(P_{f,+}(0)^*) \cap P_{f,-}(0)^*[\mathbb{C}^{2n}]$. Therefore, it holds

$$
Z^\perp \subset \ker(P^s_+(0)^*) \cap \ker(P^u_-(0)^*),
\tag{5.130}
$$

by (5.67). The jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ of $\varphi_{\nu,\varepsilon}(x,\lambda)$ at $x = 0$ vanishes if and only if

$$\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle = 0. \tag{5.131}$$

With the aid of (5.130) we calculate

$$\begin{aligned} &\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, \int_\infty^0 \Phi_{0,+}^{uc}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \int_{-\infty}^0 \Phi_{0,-}^{sc}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy \right\rangle \\ &\quad = -\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx. \end{aligned}$$

Combining the latter with (5.129) yields

$$\begin{aligned} &\left\| \left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle + \lambda \int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx \right\| \qquad \lambda \in D_{\eta,\varepsilon}, \\ &\quad \leq C|\log(\varepsilon)|^2 \left(\varepsilon + |\lambda|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right), \end{aligned}$$

since $d_+ \in Z^s$ and $d_- \in Z^u$ are in the orthogonal complement of $Z^\perp$ by Proposition 5.21. Hence, because the λ- and ε-independent integral $\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx$ is non-zero by Proposition 5.21 and the jump $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, Rouché's Theorem implies that equation (5.131) has, provided $\eta, \varepsilon > 0$ are sufficiently small, a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

Our last step is to prove estimate (5.94). Fix $\nu \in \mathcal{S}_\delta$. First, we establish the a priori bound,

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in D_{\eta,\varepsilon}, \tag{5.132}$$

using (5.93) and (5.125). By subtracting (5.121) from (5.97) and (5.122) from (5.99), we obtain variation of constants formulas for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)$ on $I_{f,\varepsilon}^+$ and $I_{f,\varepsilon}^-$, respectively. Our approach is to obtain leading-order expressions for the coefficients $a_\pm - a_{\mathrm{p},\pm}$, $b_\pm - b_{\mathrm{p},\pm}$, $c_\pm - c_{\mathrm{p},\pm}$ and $d_\pm - d_{\mathrm{p},\pm}$ in these variation of constants formulas. By (5.124), (5.126) and (5.128) it holds

$$\begin{aligned} c_\pm - c_{\mathrm{p},\pm} &= 0, \\ \left\|a_\pm - a_{\mathrm{p},\pm}\right\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad \lambda \in D_{\eta,\varepsilon}, \\ \left\|b_\pm - b_{\mathrm{p},\pm} + \mathcal{B}(\nu)\right\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2, \end{aligned} \tag{5.133}$$

where $\mathcal{B}(\nu)$ is defined in (3.20). Estimating $d_\pm - d_{\mathrm{p},\pm}$ is more elaborate. Note that the jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ lies in the kernels of Q^u and Q^s by (5.77). Thus, to estimate $d_+ - d_{\mathrm{p},+}$, we apply the projection Q^s to

$$J_{\nu,\varepsilon}(\lambda) = \lim_{x\downarrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right) - \lim_{x\uparrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right), \quad \lambda \in D_{\eta,\varepsilon},$$

yielding

$$\begin{aligned} d_+ - d_{\mathrm{p},+} = & \int_{-\Xi_\varepsilon}^{0} \Phi^s_{0,-}(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right] dy \\ & - Q^s \int_{\Xi_\varepsilon}^{0} \Phi^c_{0,+}(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right] dy \\ & + \Phi^s_{0,-}(0,-\Xi_\varepsilon)(a_- - a_{\mathrm{p},-}), \end{aligned}$$

by (5.68), (5.113) and (5.133). Therefore, (5.93), (5.96), (5.119), (5.132) and (5.133) imply

$$\left\|d_+ - d_{\mathrm{p},+}\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^2 + |\lambda|\right), \quad \lambda \in D_{\eta,\varepsilon}. \tag{5.134}$$

Subtracting (5.121) from (5.97) gives for each $\lambda \in D_{\eta,\varepsilon}$ a variation of constants formula for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)$ on $I^+_{f,\varepsilon}$:

$$\begin{aligned} \varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x) = & \Phi^u_{0,+}(x,\Xi_\varepsilon)(a_+ - a_{\mathrm{p},+}) + \Phi_{in}(x)(b_+ - b_{\mathrm{p},+}) + \Phi^s_{0,+}(x,0)(d_+ - d_{\mathrm{p},+}) \\ & + \int_0^x \Phi^s_{0,+}(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right] dy \\ & + \int_{\Xi_\varepsilon}^x \Phi^{uc}_{0,+}(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)\right] dy, \end{aligned}$$

where we use $c_+ = c_{\mathrm{p},+}$. Applying (5.93), (5.96), (5.119), (5.132), (5.133) and (5.134) to the latter identity yields the approximation (5.94) on $[0, \Xi_\varepsilon/2]$. The proof of (5.94) on $[-\Xi_\varepsilon/2, 0]$ is analogous. □

Remark 5.26. The proof of Theorem 5.25 provides a Lyapunov-Schmidt type reduction procedure. Finding a bounded solution to the full eigenvalue problem (3.3) amounts to inverting the operator $\mathcal{L}_\varepsilon - \lambda$ defined in §3.2. By constructing the piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) via Lin's method, we invert a certain part of $\mathcal{L}_\varepsilon - \lambda$ and we obtain a one-dimensional reduced equation (5.131) describing the remaining unsolved part.

Thus, solving (5.131) for λ yields the desired simple eigenvalue $\lambda_\varepsilon(\nu)$ of $\mathcal{L}_\varepsilon$ about the origin. A leading-order expression of $\lambda_\varepsilon(\nu)$ can be obtained by calculating the leading order of the ε- and λ-dependent parts of (5.131). Alternatively, we use the key identity (5.51) to derive a leading-order expression for $\lambda_\varepsilon(\nu)$ – see §5.3.5. ■

5.3.5 Conclusion

In this section we provide the proof of Theorem 3.19. Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and Ξ_ε be as in (3.21), (5.52) and (5.53), respectively. In Theorem 5.25 we constructed for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3) on the interval $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$ which has a jump only at $x = 0$. In addition, we obtained leading-order expressions for $\varphi_{\nu,\varepsilon}(x,\lambda)$ and $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi'_{\mathrm{p},\varepsilon}(x)$.

Moreover, we proved in Theorem 5.25 that for any $\nu \in \mathcal{S}_\delta$ there is a unique λ-value $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$ for which the jump of $\varphi_{\nu,\varepsilon}(x,\lambda)$ vanishes. As mentioned in §5.3.1 this λ-value coincides with the unique root $\lambda_\varepsilon(\nu)$ of the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ about the origin. We extend the *continuous* solution $\varphi_{\nu,\varepsilon}(x, \tilde{\lambda}_\varepsilon(\nu))$ to the whole real line via (5.55). In §5.3.1 we derived an identity (5.51) for $\lambda_\varepsilon(\nu)$ in terms of this extended solution $\check{\varphi}_{\nu,\varepsilon}$ to (3.3). Plugging the leading-order expressions for $\check{\varphi}_{\nu,\varepsilon}(x)$ and $\check{\varphi}_{\nu,\varepsilon}(x) - \phi'_{\mathrm{p},\varepsilon}(x)$ into (5.51) yields the desired approximation (3.17) of $\lambda_\varepsilon(\nu)$.

Proof of Theorem 3.19. In the following, we denote by $C > 0$ a constant, which is independent of ε and ν.

In §5.3.1 we established a $\varsigma > 0$ such that, provided $\varepsilon > 0$ is sufficiently small, there exists for any $\nu \in \mathcal{S}_\delta$ a unique (real) root $\lambda_\varepsilon(\nu) \in B(0,\varsigma)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$. We showed that the function $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ is analytic, even and 2π-periodic and satisfies $\lambda_\varepsilon(0) = 0$ whenever $0 \in \mathcal{S}_\delta$.

Fix $\nu \in \mathcal{S}_\delta$. Consider the solution $\varphi_{\nu,\varepsilon}(x, \tilde{\lambda}_\varepsilon(\nu))$ to the full eigenvalue problem (3.3), established in Theorem 5.25, and define $\check{\varphi}_{\nu,\varepsilon}$ by (5.55). Clearly, $\check{\varphi}_{\nu,\varepsilon}$ is a solution to (3.3) on the whole real line. In §5.3.1 we showed that it holds $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ and that the key identity (5.51) is satisfied for $\check{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$. To obtain a leading-order expression for $\lambda_\varepsilon(\nu)$ we approximate the integrals in (5.51) using Theorem 5.25.

First, Theorem 2.3 and estimate (5.93) imply that $\check{\varphi}_{\nu,\varepsilon}$ and $\phi_{\mathrm{p},\varepsilon}$ are bounded on $\mathbb{R}$ by a constant independent of ε and ν. On the other hand, the solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint equation (3.19) satisfies

$$\|\psi_{\mathrm{ad}}(x)\| \leq C e^{-\mu_r |x|}, \quad x \in \mathbb{R},$$

by Proposition 5.21. Thus, using estimate (5.93) we approximate

$$\left\| \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x) dx - \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x, u_0) dx \right\| \leq C |\log(\varepsilon)| \left(\varepsilon |\log(\varepsilon)| + |\lambda_\nu(\varepsilon)|\right), \tag{5.135}$$

In addition, by estimate (5.94) and Theorem 2.3 we have

$$\begin{aligned} &\left\| \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \left(\partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x, u_0), 0)\right) \left(\tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x)\right) dx \right\| \\ &\qquad \leq C\varepsilon |\log(\varepsilon)|^2 \left(\varepsilon^2 |\log(\varepsilon)|^3 + |\lambda_\varepsilon(\nu)|\right) \end{aligned} \tag{5.136}$$

where we use that $\psi_{\mathrm{ad},2}(x)$ is odd by Proposition 5.21, $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even by Theorem 2.3, $v_{\mathrm{h}}(x, u_0)$ is even by **(E1)** and the v-components of $\Phi_{in}(x)\mathcal{B}(\nu)$ are even by **(E1)**. Integration by parts gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) \left(\tilde{u}_{\nu,\varepsilon}(x) - u'_{\mathrm{p},\varepsilon}(x)\right) dx \\ &\qquad = -\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(y), \varepsilon) D_1^{-1} \left(\tilde{p}_{\nu,\varepsilon}(x) - p'_{\mathrm{p},\varepsilon}(x)\right) dy dx, \end{aligned}$$

since $\psi_{\mathrm{ad},2}(x)$ is odd and $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even. Applying estimate (5.94) and Theorem 2.3 to the latter yields

$$\begin{aligned}\Bigg\| \int_{-\infty}^{\infty} & \psi_{\mathrm{ad},2}(x)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) \left(\tilde{u}_{\nu,\varepsilon}(x) - u'_{\mathrm{p},\varepsilon}(x)\right) dx \\ & -\varepsilon^2 \int_{-\infty}^{\infty} \int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(u_0, v_{\mathrm{h}}(y, u_0), 0) dy dx B(\nu) \Bigg\| \\ \leq C\varepsilon & |\log(\varepsilon)| \left(\varepsilon^2 |\log(\varepsilon)|^3 + |\lambda_\nu(\varepsilon)|\right),\end{aligned} \tag{5.137}$$

with $B(\nu)$ defined in (3.20), where we use $\psi_{\mathrm{ad},2}(x)$ is odd, $v_{\mathrm{h}}(x, u_0)$ is even and the p-component of $(I - \Phi_{in}(x))\mathcal{B}(\nu)$ is odd by **(E1)**. Finally, since the integral $\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x, u_0) dx$ is non-zero by Proposition 5.21, the key identity (5.51) in combination with the estimates (5.135), (5.136) and (5.137) gives

$$\begin{aligned}&\left\| \lambda_\varepsilon(\nu) + \varepsilon^2 \frac{\int_{-\infty}^{\infty} \int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(u_0, v_{\mathrm{h}}(y, u_0), 0) dy dx B(\nu)}{\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x, u_0) dx} \right\| \\ &\quad \leq C\varepsilon^3 |\log(\varepsilon)|^5.\end{aligned}$$

The latter yields the leading-order expression (3.17) of $\lambda_\varepsilon(\nu)$ by switching the order of integration in the numerator using that $\psi_{\mathrm{ad},2}$ is odd and $v_{\mathrm{h}}(x, u_0)$ is even. □

Remark 5.27. In the proof of Theorem 3.19 we have obtained for any $\nu \in \mathcal{S}_\delta$ an eigenfunction,

$$\psi_{\nu,\varepsilon}(\check{x}) := \begin{pmatrix} \tilde{u}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x}) \\ \tilde{v}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x}) \end{pmatrix} e^{i\nu\check{x}/2\ell_\varepsilon} \in H^2_{\mathrm{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n}),$$

corresponding to the eigenvalue $\lambda_\varepsilon(\nu)$ of the operator $\mathcal{L}_{\nu,\varepsilon}$ defined in §3.2.1. The approximations in Theorem 5.25 and its proof provide leading-order control over this eigenfunction. We observe that $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by $(0, \partial_x v_{\mathrm{h}}(\varepsilon^{-1}\check{x}, u_0))$ along the pulse. The derivative $\partial_x v_{\mathrm{h}}(x, u_0)$ corresponds to the translational eigenfunction at $\lambda = 0$ of the linearization of $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_{\mathrm{h}}(x, u_0)$. Thus, along the pulse, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is independent of ν. On the other hand, along the slow manifold, i.e. for $\varepsilon\check{x} \in I_{s,\varepsilon}$, $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by the u-components of

$$2\varepsilon e^{i\nu\check{x}/2\ell_\varepsilon} \Phi_{\mathrm{s}}(\check{x}, 0) \Upsilon_0 \left(I - e^{-i\nu} \Upsilon_0 \Phi_{\mathrm{s}}(2\ell_0, 0) \Upsilon_0\right)^{-1} \begin{pmatrix} D_1^{-1} \mathcal{J}(u_0) \\ 0 \end{pmatrix},$$

by (5.87), (5.101), (5.102), (5.103), (5.111), (5.112) and (5.128), where $\mathcal{J}$ is given by (2.5), $\Phi_{\mathrm{s}}(\check{x}, \check{y})$ is the evolution (2.7) and Υ_0 is defined in (3.20). Thus, along the slow manifold, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is dictated by the slow variational equation (2.7) and the value of ν. ■

5.3.6 Discussion

Our approach to expanding the critical spectral curve relies on Lin's method. As mentioned in the introduction in Chapter 1 a similar approach is employed in [10, 100] to determine the spectral geometry about the origin. In this section we compare the analyses in [10, 100] with ours.

In [100] one considers $2L$-periodic wave trains to general reaction-diffusion systems that converge to a homoclinic pulse solution in the long-wavelength limit $L \to \infty$. An expansion of the critical spectral curve is obtained in terms of the period L. It is assumed that the translational eigenvalue at the origin corresponding to the limiting homoclinic pulse is simple. Therefore, the variational equation about the homoclinic pulse has exponential dichotomies on both half-lines such that the spaces of solutions decaying as $x \to \infty$ and $x \to -\infty$ have a one-dimensional intersection. Thus, one obtains a decomposition (5.59) of the solution space as exhibited by our fast variational equation (3.15).

The variational equation about the limiting homoclinic serves as the backbone for the construction of solutions to the eigenvalue problem associated with the periodic wave train. Using Lin's method a piecewise continuous eigenfunction $\varphi_\nu(x)$ is constructed on $[-L, L]$ for any $\nu \in \mathbb{R}$ that has a jump at 0 and satisfies $\varphi(L) = e^{i\nu}\varphi(-L)$. The exponential dichotomies of the variational equation about the homoclinic control the dynamics of the eigenvalue problem on the growing interval $[-L, L]$. The jump at 0 depends on the spectral parameter λ, the period L and the Floquet exponent ν, because the eigenvalue problem is a (λ, L^{-1})-perturbation of the homoclinic variational equation. Using Melnikov theory the jump can be equated to 0 yielding an expansion of the critical spectral curve in terms of e^{-L}.

In our work there are *two* systems that serve as the backbone for the construction of solutions to the full eigenvalue problem (3.3): the reduced eigenvalue problems (5.56) and (5.80) which describe the leading-order dynamics along the fast pulse and along the slow manifold. In contrast to [100], the reduced eigenvalue problems admit exponential *tri*chotomies in accordance with the slow-fast structure of the eigenvalue problem (3.3). Moreover, the full eigenvalue problem (3.3) is a (λ, ε)-perturbation of the reduced eigenvalue problems. As a result, the jump of the obtained piecewise continuous eigenfunction in our work depends on ε, λ and ν. The center dynamics captured by the exponential trichotomies prevents the critical curve from being exponentially small in terms of the period as in [100]; instead the curve scales with ε^2.

In [10] the location of a critical eigenvalue near the origin is determined in the context of fast traveling pulses (with oscillatory tails) in the FitzHugh-Nagumo equations. Again, Lin's method is employed to obtain a leading-order expression for this critical eigenvalue in terms of the small parameter ε. Similar to our work, the slow-fast structure yields a framework for the construction of a piecewise continuous eigenfunction to the associated eigenvalue problem. This framework consists of *four* (reduced) eigenvalue problems arising along the fast front and back and along the orbit segments on the slow manifolds which together constitute the pulse profile in the limit $\varepsilon \to 0$. However, in contrast to our work, it is sufficient to distinguish between center-stable dynamics and unstable dynamics in the eigenvalue problem. Thus, the introduction of an exponential weight yields exponential *di*chotomies for the reduced eigenvalue problems.

Lin's method then yields a piecewise continuous eigenfunction that has *two* ε- and λ-dependent jumps in the middle of the front and the back. Thus, Lyapunov-Schmidt reduction leads to a quadratic equation in λ rather than a linear one as in [100] and our work. One root of the quadratic corresponds to the translational eigenvalue sitting at the origin. The second root corresponds to the critical, non-trivial eigenvalue that scales with ε in the monotone case, while the scaling in the oscillatory case is $\varepsilon^{2/3}$.

In the aforementioned spectral analyses, the fine structure of the spectrum about the origin is decisive for stability, but not detectable in the relevant asymptotic limit. In these cases Lin's method proves to be a powerful tool to determine how the spectrum locally perturbs from the asymptotic limit. Therefore, we expect that Lin's method can be applied to a wide range of spectral perturbation problems – see also Remark 1.3.