Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Chapter 4

Prerequisites for the spectral stability analysis

In the spectral stability analysis in Chapter 5 we encounter linear ODEs, some of which depend on a small parameter $\varepsilon > 0$ or on a spectral parameter $\lambda \in \mathbb{C}$. In this chapter we collect the necessary techniques to control such systems.

4.1 A Grönwall type estimate for linear systems

In the spectral stability analysis of solutions to singularly perturbed equations one often needs to compare a linear system with its perturbation. Our analysis requires the following approximation result for linear systems, which follows from a direct application of Grönwall's inequality.

Lemma 4.1. *[87, Lemma 1] Let $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ with $a < b$ and $A, B \in C([a,b], \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose there are constants $K, \mu > 0$ such that the evolution operator $T_1(x,y)$ of system,*

$$\varphi_x = A(x)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.1}$$

satisfies

$$\|T_1(x,y)\| \leq K e^{\mu|x-y|}, \quad x, y \in [a,b]. \tag{4.2}$$

Denote by $T_2(x,y)$ the evolution operator of system,

$$\varphi_x = B(x)\varphi, \quad \varphi \in \mathbb{C}^n. \tag{4.3}$$

It holds

$$\|T_1(x,y) - T_2(x,y)\| \leq K \int_a^b \|A(z) - B(z)\| dz \exp\left(\mu(b-a) + K \int_a^b \|A(z) - B(z)\| dz\right),$$

for $x, y \in [a,b]$.

Remark 4.2. If $M > 0$ is such that $M \geq \sup\{\|A(x)\| : x \in [a,b]\}$, then (4.2) is satisfied for $\mu = M$ and $K = 1$ by Grönwall's inequality. ■

4.2 Asymptotically constant systems

The eigenvalue problems arising in our spectral stability analysis are non-autonomous linear systems of the form,

$$\varphi_x = A(x,\lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.4}$$

depending analytically on a spectral parameter λ. Often we are looking for the eigenvalues $\lambda \in \mathbb{C}$ for which (4.4) admits a non-trivial bounded (or exponentially localized) solution. Therefore, we are interested in the asymptotic behavior of solutions to (4.4).

Linearizing about pulse type solutions leads to eigenvalue problems (4.4) that have an asymptotically constant coefficient matrix. In such systems the asymptotics of solutions is dictated by the behavior of the constant coefficient system at $\pm\infty$ – see also Proposition 4.7. The following result concerns the construction of a unique solution with the highest decay rate to an asymptotically constant system.

Proposition 4.3. *[90, Proposition 1.2] Let $n \in \mathbb{Z}_{>0}, \Omega \subset \mathbb{C}$ open and $A \in C([0,\infty) \times \Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ such that $A(x,\cdot)$ is analytic on Ω for each $x \geq 0$. Suppose that there exists $\mu, K > 0$ and $A_\infty : \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ analytic such that*

$$\|A(x,\lambda) - A_\infty(\lambda)\| \leq Ke^{-\mu x}, \quad x \geq 0, \lambda \in \Omega. \tag{4.5}$$

Furthermore, suppose that the eigenvalue $\mu(\lambda)$ of $A_\infty(\lambda)$ of smallest real part is simple for all $\lambda \in \Omega$. Denote by $v(\lambda)$ an analytic eigenvector of A_∞ corresponding to $\mu(\lambda)$. For any compact subset $\Omega_b \subset \Omega$, there exists $C > 0$, independent of λ, and a unique solution $y(x,\lambda)$ to (4.4) *satisfying*

$$\left\| e^{-\mu(\lambda)x} y(x,\lambda) - v(\lambda) \right\| \leq Ce^{-\mu x}, \quad x \geq 0, \lambda \in \Omega_b.$$

The solution $y(x,\cdot)$ is analytic on the interior of Ω_b for each $x \geq 0$.

4.3 Exponential dichotomies

Exponential dichotomies enable us to track solutions in linear systems by separating the solution space in solutions that either decay exponentially in forward time or else in backward time. Moreover, their associated projections inherit analytic dependence of the problem on a spectral parameter λ. Therefore, they provide a natural framework [98] to capture the linear dynamics of eigenvalue problems of the form (4.4) arising in our spectral stability analysis.

Definition 4.4. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x,y)$ the evolution operator of (4.1). Equation (4.1) has *an exponential dichotomy on J with constants $K, \mu > 0$ and projections $P(x)\colon \mathbb{C}^n \to \mathbb{C}^n$* if for all $x, y \in J$ it holds

- $P(x)T(x,y) = T(x,y)P(y)$;
- $\|T(x,y)P(y)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x,y)(I - P(y))\| \leq Ke^{-\mu(y-x)}$ for $y \geq x$.

Let $P(x)$ be the family of projections associated with an exponential dichotomy on J. For each $x, y \in J$, we denote by $T^s(x,y) = T(x,y)P(y)$ and $T^u(x,y) = T(x,y)(I - P(y))$ the *stable and unstable evolution* of system (4.1), leaving the projection $P(y)$ implicit.

Below we give a short overview of the properties of exponential dichotomies that we need for our spectral stability analysis. For an extensive introduction on dichotomies the reader is referred to [14, 96]. A generalization of the concept of exponential dichotomies is the notion of *exponential separation*, which is treated in [85]. In particular, one can define *exponential trichotomies* to capture linear systems that exhibit centre behavior in addition to exponential decay in forward and backward time – see §4.4.

4.3.1 Dichotomy projections

Exponential dichotomies on an interval $J \subset \mathbb{R}$ are in general not unique. If $J = [0, \infty)$, then the range of the dichotomy projection corresponds to the space of solutions decaying in forward time and is therefore uniquely determined, whereas its kernel can be any complement.

Lemma 4.5. *[96, Lemma 1.2(ii)] Let $n \in \mathbb{Z}_{>0}$ and $A \in C([0,\infty), \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation (4.1) admits an exponential dichotomy on $[0,\infty)$ with projections $P(x)$. If $Y \subset \mathbb{C}^n$ satisfies $Y \oplus P(0)[\mathbb{C}^n] = \mathbb{C}^n$, then (4.1) admits an exponential dichotomy on $[0,\infty)$ with projections $Q(x)$, where $Q(0)$ is the projection on $P(0)[\mathbb{C}^n]$ along Y.*

An autonomous linear system $\varphi_x = A_0\varphi$, where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is hyperbolic, admits an exponential dichotomy on $\mathbb{R}$. The associated dichotomy projection is given by the spectral projection onto the stable eigenspace of A_0. If a non-autonomous linear system (4.1), which admits an exponential dichotomy on $[0,\infty)$, converges to a hyperbolic system as $x \to \infty$, then the dichotomy projections converge to the associated spectral projection.

Lemma 4.6. *[86, Lemma 3.4] Let $n \in \mathbb{Z}_{>0}$ and $A \in C([0,\infty), \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation (4.1) admits an exponential dichotomy on $[0,\infty)$ with constants $K, \mu > 0$ and projections $P(x)$. In addition, suppose there exists a hyperbolic matrix $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ with spectral gap larger than μ such that*

$$\|A_0\| \leq K, \quad \|A(x) - A_0\| \leq Ke^{-\mu x}, \quad x \geq 0.$$

Then, there exists a constant $C > 0$, depending on n, μ and K only, such that

$$\|P(x) - P_0\| \leq Ce^{-\mu x}, \quad x \geq 0,$$

where P_0 is the spectral projection onto the stable eigenspace of A_0.

4.3.2 Sufficient criteria for exponential dichotomies

As mentioned before, an autonomous linear system $\varphi_x = A_0\varphi$, where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is hyperbolic, admits an exponential dichotomy on $\mathbb{R}$. This result can be extended to non-autonomous systems (4.1) in at least two ways. First, if the coefficient matrix $A(x)$ converges to a hyperbolic matrix $A_{\pm\infty}$ as $x \to \pm\infty$, then exponential dichotomies for (4.1) on the half-lines $[0, \infty)$ and $(-\infty, 0]$ can be constructed from the exponential dichotomies of the asymptotic systems $\varphi_x = A_{\pm\infty}\varphi$. Second, if $A(x)$ is slowly varying and pointwise hyperbolic, then system (4.1) admits an exponential dichotomy.

In our spectral stability analysis, we use these two results to obtain exponential dichotomies for eigenvalue problems of the form (4.4). We emphasize that both constructions respect analyticity in the spectral parameter λ. We start with the first result that focusses on asymptotically hyperbolic systems.

Proposition 4.7. *[86, Lemma 3.4], [99, Theorem 1] Let $n \in \mathbb{Z}_{>0}$, $\Omega \subset \mathbb{C}$ open and $A \in C([0, \infty) \times \Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ such that $A(x, \cdot)$ is analytic on Ω for each $x \geq 0$. Suppose that there exists constants $\mu, K, \alpha > 0$ and an analytic map $A_\infty : \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that*

i. Identity (4.5) is satisfied for each $x \geq 0$ and $\lambda \in \Omega$;

ii. For any $\lambda \in \Omega$ the matrix $A_\infty(\lambda)$ is hyperbolic with spectral gap larger than α.

System (4.4) admits for any $\lambda \in \Omega$ an exponential dichotomy on $[0, \infty)$ with constants $C(\lambda), \alpha > 0$ and projections $P(x, \lambda)$, whose rank equals the dimension of the stable eigenspace of $A_\infty(\lambda)$. The projections $P(x, \cdot)$ are analytic on Ω for each $x \geq 0$. Moreover, the map $\lambda \mapsto C(\lambda)$ is continuous.

For hyperbolic, constant coefficient systems $\varphi_x = A_0(\lambda)\varphi$ the dichotomy projections equal the spectral projections onto the stable eigenspace of $A_0(\lambda)$. Clearly, this spectral projection inherits its analyticity from A_0. This can be extended to non-autonomous systems of the form (4.4): if $A(x, \lambda)$ varies slowly and is pointwise hyperbolic, then (4.1) admits an exponential dichotomy that has analytic projections close to the spectral projections onto the stable eigenspace of $A(x, \lambda)$.

The latter result is proved in [10, Proposition 6.5] in the setting of the FitzHugh-Nagumo system. In addition, in [14, Proposition 6.1] the result is proved for general systems of the form (4.1). However, the result in [14] lacks the desired closeness estimates on the dichotomy projections and analytic dependence on parameters is not shown. Therefore, we provide a proof of these two facts along the lines of [10, Proposition 6.5].

Proposition 4.8. *Let $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ with $b - a > 2$ and $\Omega \subset \mathbb{C}$ open. Denote $X = [a, b] \times \Omega$ and let $A \in C^1(X, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Assume that $A(x, \cdot)$ is analytic on Ω for each $x \in [a, b]$ and that there exists constants $\alpha > 0$ and $M > 1$ such that:*

i. For each $(x, \lambda) \in X$ the matrix $A(x, \lambda)$ is hyperbolic with spectral gap larger than α;

ii. The matrix function A is bounded by M on X.

There exists $\delta > 0$, depending only on α and M, such that, if we have

$$\sup_{(x,\lambda)\in X} \|\partial_x A(x, \lambda)\| \leq \delta,$$

then (4.4) *has an exponential dichotomy on $[a + 1, b - 1]$ for any $\lambda \in \Omega$ with constants $C, \mu > 0$ and projections $P(x, \lambda)$ such that $P(x, \cdot)$ is analytic on Ω for each $x \in [a + 1, b - 1]$. In addition, we have $\mu = \frac{1}{2}\alpha$ and C depends only on M, α and n. Finally, for any $(y, \lambda) \in [a + 1, b - 1] \times \Omega$ we have the estimate,*

$$\|P(y, \lambda) - \mathcal{P}(y, \lambda)\| \leq C \sup_{(x,\lambda)\in X} \|\partial_x A(x, \lambda)\|, \tag{4.6}$$

where $\mathcal{P}(x, \lambda)$ is the spectral projection onto the stable eigenspace of $A(x, \lambda)$.

Proof. In the following, we denote by $C > 0$ a constant depending only on M, n and α.

Our approach is to extend system (4.4) to the whole real line, such that it varies only on the finite interval $[a, b]$. We establish an exponential dichotomy for this extended system using [14, Proposition 6.1]. The range or kernel of the dichotomy projections must be analytic for $x \in \mathbb{R} \setminus [a, b]$ by analyticity of the spectral projections. These analyticity properties can be interpolated to the interval $[a, b]$. Finally, to prove the closeness estimate (4.6), we approximate the stable evolution operator of system (4.4) by $\mathcal{P}(x, \lambda) \exp(A(x, \lambda)(x - y))$, using that the derivative of $A(x, \lambda)$ is small.

We introduce a smooth partition of unity $\chi_i \colon \mathbb{R} \to [0, 1]$, $i = 1, 2, 3$ satisfying

$$\sum_{i=1}^{3} \chi_i(x) = 1, \quad |\chi_2'(x)| \leq 2, \quad x \in \mathbb{R},$$
$$\mathrm{supp}(\chi_1) \subset (-\infty, a + 1), \quad \mathrm{supp}(\chi_2) \subset (a, b), \quad \mathrm{supp}(\chi_3) \subset (b - 1, \infty).$$

The equation,

$$\varphi_x = \mathcal{A}(x, \lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.7}$$

with

$$\mathcal{A}(x, \lambda) := \chi_1(x)A(a, \lambda) + \chi_2(x)A(x, \lambda) + \chi_3(x)A(b, \lambda),$$

coincides with (4.4) on $[a+1, b-1]$. We calculate

$$\partial_x \mathcal{A}(x,\lambda) = \begin{cases} \chi_2(x)\partial_x A(x,\lambda), & x \in (a+1, b-1), \\ \chi_2'(x)(A(x,\lambda) - A(a,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [a, a+1], \\ \chi_2'(x)(A(x,\lambda) - A(b,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [b-1, b], \\ 0, & \text{otherwise.} \end{cases}$$

First, we have $\|\partial_x \mathcal{A}(x,\lambda)\| \leq 3\delta$ for each $(x,\lambda) \in \mathbb{R}\times\Omega$ by the mean value theorem. Second, by the spectral estimates in [83] the Hausdorff distance between the spectra of $A(a,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for each $(x,\lambda) \in (-\infty, a+1] \times \Omega$. Similarly, the Hausdorff distance between the spectra of $A(b,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for every $(x,\lambda) \in [b-1,\infty)\times\Omega$. Hence, for $\delta > 0$ sufficiently small, the matrix $\mathcal{A}(x,\lambda)$ is hyperbolic for each $(x,\lambda) \in \mathbb{R}\times\Omega$ with spectral gap larger than $\frac{1}{2}\alpha$. Third, $\mathcal{A}$ is bounded by M on $\mathbb{R}\times\Omega$. Combining these three items with [14, Proposition 6.1] implies that system (4.7) admits, provided $\delta > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$ with $\mu = \frac{1}{2}\alpha$ and projections $P(x,\lambda)$.

The next step is to prove that the projections $P(x,\cdot)$ are analytic in Ω for each $x \in \mathbb{R}$. Any solution to the constant coefficient system $\psi_x = A(a,\lambda)\psi$ that converges to 0 as $x \to -\infty$ must be in the kernel of the spectral projection $\mathcal{P}(a,\lambda)$ onto the stable eigenspace of $A(a,\lambda)$. Hence, it holds $\ker(\mathcal{P}(a,\lambda)) = \ker(P(a,\lambda))$ by construction of (4.7). Moreover, the spectral projection $\mathcal{P}(a,\cdot)$ is analytic on Ω, since $A(a,\cdot)$ is analytic on Ω. Thus, $\ker(P(a,\lambda))$ and similarly $P(b,\lambda)[\mathbb{C}^n]$ must be analytic subspaces – see [42, Chapter 18] – in $\lambda \in \Omega$. Denote by $T(x,y,\lambda)$ the evolution operator of (4.7), which is by [60, Lemma 2.1.4] analytic in $\lambda \in \Omega$ for each $x, y \in \mathbb{R}$. We conclude that both $\ker(P(a,\lambda))$ and $P(a,\lambda)[\mathbb{C}^n] = T(a,b,\lambda)P(b,\lambda)[\mathbb{C}^n]$ are analytic subspaces in $\lambda \in \Omega$. Therefore, the projection $P(a,\cdot)$ (and thus any projection $P(x,\cdot)$, $x \in \mathbb{R}$) is analytic in Ω.

Finally, we prove that the projections $P(x,\lambda)$ can be approximated by the spectral projections $\mathcal{P}(x,\lambda)$ onto the stable eigenspace of $\mathcal{A}(x,\lambda)$ for any $(x,\lambda) \in \mathbb{R}\times\Omega$. Define $\delta_* := \sup\{\|\partial_x A(x,\lambda)\| : (x,\lambda) \in X\} > 0$. Take $z \in \mathbb{R}$ and $v \in \mathcal{P}(z,\lambda)[\mathbb{C}^n]$. Observe that

$$\hat{\varphi}(x,\lambda) := \mathcal{P}(x,\lambda)e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}v, \quad (x,\lambda) \in \mathbb{R}\times\Omega,$$

satisfies the inhomogeneous equation,

$$\varphi_x = \mathcal{A}(x,\lambda)\varphi + g(x,\lambda),$$

with

$$g(x,\lambda) := e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}\left[\partial_x\left(\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)\right)(x-z) + \partial_x\mathcal{P}(x,\lambda)\right]v.$$

By uniformity of the bound on the spectral gap of $\mathcal{A}$, there exists a contour $\Gamma \subset \mathbb{C}$, depending only on M, α and n, containing precisely those eigenvalues of $\mathcal{A}(x,\lambda)$ of negative real part for all $(x, \lambda) \in \mathbb{R} \times \Omega$. Thus, we have

$$\mathcal{P}(x, \lambda) = \frac{1}{2\pi i} \oint_\Gamma (w - \mathcal{A}(x, \lambda))^{-1} dw, \quad (x, \lambda) \in \mathbb{R} \times \Omega. \tag{4.8}$$

By [41, Corollary 1.2.4] the norm of the resolvent $(w - \mathcal{A}(x, \lambda))^{-1}$ can be bounded in terms of M, n and the distance $d(w, \sigma(\mathcal{A}(x, \lambda)))$. Hence, choosing the contour Γ appropriately, we observe

$$\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\mathcal{P}(x, \lambda)\| \leq C. \tag{4.9}$$

Since $\mathcal{P}(x, \lambda)$ is the projection onto the stable eigenspace of $\mathcal{A}(x, \lambda)$ and $\mathcal{A}$ is uniformly bounded by M on $\mathbb{R} \times \Omega$ and has a uniform spectral gap larger than $\mu = \frac{1}{2}\alpha$ on $\mathbb{R} \times \Omega$, we have by [41, Theorem 1.2.1] the bound,

$$\sup_{\lambda\in\Omega} \left\| e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)} \right\| \leq Ce^{-\mu(x-z)}, \quad x \geq z. \tag{4.10}$$

Differentiating identity (4.8) yields

$$\partial_x \mathcal{P}(x, \lambda) = \frac{1}{2\pi i} \oint_\Gamma (w - \mathcal{A}(x, \lambda))^{-1} \partial_x \mathcal{A}(x, \lambda)(w - \mathcal{A}(x, \lambda))^{-1} dw,$$

for each $(x, \lambda) \in \mathbb{R} \times \Omega$. Since the norm of the resolvent $(w - \mathcal{A}(x, \lambda))^{-1}$ can be bounded in terms of M, n and $d(w, \sigma(\mathcal{A}(x, \lambda)))$, we observe that $\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\partial_x \mathcal{P}(x, \lambda)\| \leq C\delta_*$. Thus, combining the latter with (4.9) and (4.10) yields

$$\sup_{\lambda\in\Omega} \|g(x, \lambda)\| \leq C\delta_* \|v\|, \quad x \geq z. \tag{4.11}$$

Take $\xi = z - \log(\delta_*)\mu^{-1} \geq z$. By the variation of constants formula there exists $w \in \mathbb{C}^3$ such that

$$\hat{\varphi}(x, \lambda) = T(x, \xi, \lambda)w + \int_z^x T^s(x, y, \lambda)g(y, \lambda)dy + \int_\infty^x T^u(x, y, \lambda)g(y, \lambda)dy, \tag{4.12}$$

for $x \geq z$ and $\lambda \in \Omega$. Evaluating (4.12) at $x = \xi$, while using (4.9), (4.10) and (4.11), we derive $\|w\| \leq C\delta_* \|v\|$. Thus, applying $I - P(z, \lambda)$ to (4.12) at $x = z$, yields the bound $\|(I - P(z, \lambda))v\| \leq C\delta_* \|v\|$ for every $v \in \mathcal{P}(z, \lambda)[\mathbb{C}^n]$ by (4.10) and (4.11). Similarly, one shows that for every $v \in \ker(\mathcal{P}(z, \lambda))$ we have $\|P(z, \lambda)v\| \leq C\delta_* \|v\|$. Thus, we obtain for any $(z, \lambda) \in \mathbb{R} \times \Omega$

$$\|[P - \mathcal{P}](z, \lambda)\| \leq \|[(I - P)\mathcal{P}](z, \lambda)\| + \|[P(I - \mathcal{P})](z, \lambda)\| \leq C\delta_*.$$

Since (4.7) coincides with (4.4) on $[a + 1, b - 1]$, we have established the desired exponential dichotomy of (4.1). $\square$

4.3.3 Extending and pasting exponential dichotomies

Once one puts a linear system in the framework of exponential dichotomies, a great technical toolbox becomes available. First, there are several constructions available to extend the interval of the dichotomy. Second, if an equation admits exponential dichotomies on two neighboring intervals, then these dichotomies can be glued together. Third, exponential dichotomies persist under small perturbations of the equation.

In our spectral stability analysis we need to establish exponential dichotomies for eigenvalue problems that have a complicated structure. The aforementioned techniques enable us to built exponential dichotomies for these problems from exponential dichotomies of simpler subproblems. In this section, we treat extending and pasting of exponential dichotomies. In the next section, we consider the persistence of exponential dichotomies against small disturbances.

Every exponential dichotomy can be extended for finite time using Grönwall type estimates.

Lemma 4.9. *[14, p. 13] Let $n \in \mathbb{Z}_{>0}$, $J_2 \subset J_1 \subset \mathbb{R}$ intervals and $A \in C(J_1, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation (4.1) admits an exponential dichotomy on J_2 with constants $K, \mu > 0$ and projections $P_2(x)$. In addition, suppose the length of $J_1 \setminus J_2$ is finite. Take $M > 0$ such that $M \geq \sup\{\|A(x)\| : x \in J_1 \setminus J_2\}$.*

Then, system (4.1) has an exponential dichotomy on J_1 with constants $C, \mu > 0$ and projections $P_1(x)$. The constant C depends on K, μ, M and the length of $J_1 \setminus J_2$ only. Moreover, we have $P_1(x) = P_2(x)$ for all $x \in J_2$.

In the case that the equation is periodic, an exponential dichotomy on a sufficiently large interval can be extended to the whole line.

Lemma 4.10. *[87, Theorem 1] Let $n \in \mathbb{Z}_{>0}$, $T > 0$ and $A \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose that A is T-periodic and that equation* (4.1) *has an exponential dichotomy on an interval J of length $2T$ with constants $K, \alpha > 0$. Let $M \geq \sup\{\|A(x)\| : x \in \mathbb{R}\}$ and $h := \alpha^{-1}(\sinh^{-1}(4) + \log(K))$.*

If $T > 0$ is so large that $T \geq 2h$, then equation (4.1) *has an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$. We have $\mu = h^{-1}\log 3$ and C depends only on M, K and α.*

Exponential dichotomies on two neighboring intervals can be pasted together as long as their spaces of exponential decaying solutions in forward and backward time are complementary.

Lemma 4.11. *Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Let J_1, J_2 be two intervals such that their union equals J and $\max J_1 = b = \min J_2$ for some $b \in \mathbb{R}$. Suppose equation* (4.1) *has exponential dichotomies on both J_1 and J_2 with constants $K, \mu > 0$ and projections $P_1(x), x \in J_1$ and $P_2(x), x \in J_2$, respectively.*

If $E^u := \ker(P_1(b))$ *and* $E^s := P_2(b)[\mathbb{C}^n]$ *are complementary, then* (4.1) *has an exponential dichotomy on* J *with constants* $K_1, \mu > 0$. *Here,* K_1 *depends only on* K *and* $\|P\|$, *where* P *is the projection on* E^s *along* E^u.

Proof. Let $X(x)$ be the fundamental matrix of (4.1) satisfying $X(b) = I$. Define $P(x) = X(x)PX(x)^{-1}$ for $x \in J$, where P is the projection on E^s along E^u. Observe that $P = P(b)$ has the same range as $P_2(b)$ and the same kernel as $P_1(b)$. Now, the exposition in [14, pp. 16-17] shows that (4.1) has exponential dichotomies on J_1 and on J_2 with constants $K_1, \mu > 0$ and projections $P(x)$ for $x \in J_1$ and $x \in J_2$, respectively. We have $K_1 = K + K^2\|P\| + K^3$. To conclude the proof we need to show that the dichotomy estimates remain true on the union $J = J_1 \cup J_2$. Indeed, take $x \in J_2$ and $y \in J_1$. We estimate

$$\|T(x,y)P(y)\| \leq \|T(x,b)P_2(b)\|\|P\|\|P_1(b)T(b,y)\| \leq K^2\|P\|e^{-\mu(x-y)},$$

where we use $P_2(b)P = P$ and $PP_1(b) = P$. Similarly, one estimates $\|T(y,x)(I - P(x))\| \leq K^2\|P\|e^{-\mu(x-y)}$ for $x \in J_2$ and $y \in J_1$. □

4.3.4 Roughness of exponential dichotomies

Exponential dichotomies are particularly useful to study the spectral properties of perturbed differential equations, since they are robust against small disturbances. This property is often referred to as *roughness*. The following result concerns roughness of exponential dichotomies on arbitrary intervals.

Proposition 4.12. *[14, Proposition 5.1] Let* $n \in \mathbb{Z}_{>0}$. *Take an interval* $J \subset \mathbb{R}$ *and* $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ *such that* (4.1) *has an exponential dichotomy on* J *with constants* $K, \alpha > 0$ *and projections* $P(x)$. *Then, for any* $0 < \varepsilon < \alpha$, *there exists* $\delta > 0$ *depending only on* K, α *and* ε *such that if* $B \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ *satisfies*

$$\sup_{x \in J} \|A(x) - B(x)\| \leq \delta,$$

then equation (4.3) *has an exponential dichotomy on* J *with constants* $C, \mu > 0$ *and projections* $Q(x)$, *where* $\mu = \alpha - \varepsilon$ *and* C *depends on* K *only. Moreover, for all* $x \in J$ *we have*

$$\|P(x) - Q(x)\| \leq \frac{C\delta}{\alpha}.$$

Proposition 4.12 establishes an exponential dichotomy for any perturbation of an equation admitting an exponential dichotomy. The constructed dichotomy projections are close to the dichotomy projections of the unperturbed equation. The next result shows that the reverse is also true: if two equations that are close to each other admit exponential dichotomies, then the 'gap' between the ranges and the kernels of the dichotomy projections can be estimated.

Lemma 4.13. *Let* $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ *with* $a < b$ *and* $A, B \in C([a,b], \mathrm{Mat}_{n\times n}(\mathbb{C}))$. *Suppose equations* (4.1) *and* (4.3) *have exponential dichotomies on* $[a,b]$ *with constants* $K_{1,2}, \mu_{1,2} > 0$

and projections $P_{1,2}(x)$. Denote by $T_{1,2}(x,y)$ the evolution operators of systems (4.1) *and* (4.3). *Let $\delta \geq 0$ such that*

$$\|T_1(a,b) - T_2(a,b)\| \leq \delta.$$

Then, for every $v \in E_1^s(a) = P_1(a)[\mathbb{C}^n]$, there exists $w \in E_2^s(a) = P_2(a)[\mathbb{C}^n]$ such that

$$\|v - w\| \leq (\delta + K_2 e^{-\mu_2(b-a)}) K_1 e^{-\mu_1(b-a)} \|v\|. \tag{4.13}$$

Similarly, for every $v \in E_1^u(b) = \ker(P_1(b))$, there exists $w \in E_2^u(b) = \ker(P_2(b))$ such that (4.13) *holds true.*

Proof. Let $v \in E_1^s(a)$ and consider $w = T_2(a,b)P_2(b)T_1(b,a)v \in E_2^s(a)$. We estimate

$$\begin{aligned}\|w - v\| &\leq [\|T_2(a,b) - T_1(a,b)\| + \|T_2(a,b)(I - P_2(b))\|] \, \|T_1(b,a)v\| \\ &\leq (\delta + K_2 e^{-\mu_2(b-a)}) K_1 e^{-\mu_1(b-a)} \|v\|.\end{aligned}$$

The other statement is proven in an analogous way. □

In our spectral stability analysis we are interested in non-trivial bounded solutions to eigenvalue problems of the form (4.4). If the eigenvalue problem has an exponential dichotomy on $\mathbb{R}$, then it admits no non-trivial bounded solutions. It is possible to achieve persistence against perturbations of the latter fact under milder conditions than those stated in Proposition 4.12.

Proposition 4.14. *[88, Theorem 1] Let $n \in \mathbb{Z}_{>0}$ and $A, B \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose A is bounded on $\mathbb{R}$ and system* (4.1) *has an exponential dichotomy on $\mathbb{R}$ with constants $K, \mu > 0$. Denote by $T_{1,2}(x,y)$ the evolution operators of systems* (4.1) *and* (4.3)*, respectively. If there exists $\tau \geq \mu^{-1}(\sinh^{-1}(4) + \log(K))$ such that for all $x, y \in \mathbb{R}$ with $|x - y| \leq 2\tau$ we have*

$$\|T_1(x,y) - T_2(x,y)\| < 1,$$

then (4.3) *admits no non-trivial bounded solutions.*

4.3.5 Inhomogeneous problems

In our spectral stability analysis inhomogeneous problems arise when decomposing complicated eigenvalue problems into a simpler principal part and a remainder. If the associated homogeneous problem admits an exponential dichotomy on $\mathbb{R}$, then the splitting of exponential growth and decay induces a splitting of the integrals in the variation of constants formula leading to a characterisation of the unique bounded solution to the inhomogeneous problem.

This above characterisation allows us to compare bounded solutions to an inhomogeneous problem and its perturbation *on the whole real line*, whereas with Grönwall type arguments, one only obtain sharp estimates on finite intervals. This is the content of the following result.

Proposition 4.15. *Let $n \in \mathbb{Z}_{>0}$, $f, g \in C(\mathbb{R}, \mathbb{C}^n)$ bounded and $A, B \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation* (4.1) *has an exponential dichotomy on $\mathbb{R}$ with constants $K, \mu > 0$. Then the inhomogeneous problem,*

$$\omega_x = A(x)\omega + f(x), \quad \omega \in \mathbb{C}^n, \tag{4.14}$$

has a unique bounded solution $\varphi(x)$. Furthermore, if A and B are bounded and $a, b \in \mathbb{R}$ with $a < b$, then, for any bounded solution $\psi(x)$ to the inhomogeneous problem,

$$\omega_x = B(x)\omega + g(x), \quad \omega \in \mathbb{C}^n, \tag{4.15}$$

we estimate for $x \in [a, b]$,

$$\begin{aligned} \|\varphi(x) - \psi(x)\| \leq \frac{K}{\mu}\left(e^{-\mu(x-a)} + e^{-\mu(b-x)}\right)(\|\psi\|\|A - B\| + \|f - g\|) \\ + \frac{2K}{\mu}\left(\|\psi\| \sup_{z\in[a,b]} \|A(z) - B(z)\| + \sup_{z\in[a,b]} \|f(z) - g(z)\|\right). \end{aligned} \tag{4.16}$$

Proof. Denote by $T(x, y)$ the evolution operator of system (4.1). By [14, Proposition 8.2] system (4.14) has a unique bounded solution given by

$$\varphi(x) = \int_{-\infty}^{x} T^s(x, z)f(z)dz + \int_{\infty}^{x} T^u(x, z)f(z)dz, \quad x \in \mathbb{R}.$$

Now, let A and B be bounded and ψ a bounded solution to (4.15). Note that $w: \mathbb{R} \to \mathbb{C}^n$ defined by $w(x) = \varphi(x) - \psi(x)$ is a bounded solution to the inhomogeneous equation,

$$w_x = A(x)w + h(x),$$

where the inhomogeneity $h: \mathbb{R} \to \mathbb{C}^n$ given by $h(x) = (A(x) - B(x))\psi(x) + f(x) - g(x)$ is bounded on $\mathbb{R}$. By applying [14, Proposition 8.2] once again we deduce that $w(x)$ is given by

$$w(x) = \int_{-\infty}^{x} T^s(x, z)h(z)dz + \int_{\infty}^{x} T^u(x, z)h(z)dz, \quad x \in \mathbb{R}. \tag{4.17}$$

Now, let $a, b \in \mathbb{R}$ with $a < b$. Estimate (4.16) for $x \in [a, b]$ is achieved by splitting both integrals in expression (4.17) into two parts. The first integral is split in integrals over $(-\infty, a)$ and over (a, x). Similarly, the second integral is split in integrals over (x, b) and over (b, ∞). This yields four integrals, which can be estimated separately in order to obtain estimate (4.16). □

4.4 Exponential trichotomies

In Chapter 2 we proved the existence of stationary, spatially periodic pulse solutions to (1.10) by separating attracting, repelling and slowly evolving dynamics in the existence problem (2.1).

Naturally, the linearization of system (1.10) about the periodic pulse has a similar structure. Therefore, we encounter eigenvalue problems in our spectral stability analysis that exhibit fast exponential decay in forward and backward time as well as slow 'centre' behavior. Exponential trichotomies capture the dynamics in such linear systems. We employ the following definition.

Definition 4.16. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x, y)$ the evolution operator of (4.1). Equation (4.1) has an *exponential trichotomy* on J with constants $K, \mu > 0$ and projections $P^u(x), P^s(x), P^c(x)\colon \mathbb{C}^n \to \mathbb{C}^n$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) + P^c(x) = I$;
- $P^{u,s,c}(x)T(x, y) = T(x, y)P^{u,s,c}(y)$;
- $\|T(x, y)P^s(y)\|, \|T(y, x)P^u(x)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x, y)P^c(y)\| \leq K$.

We often use the abbreviations $T^{u,s,c}(x, y) = T(x, y)P^{u,s,c}(y)$ leaving the associated projections of the exponential trichotomy implicit.

If a linear system has a special structure, then exponential trichotomies can be generated explicitly from exponential dichotomies of a subsystem. For instance, consider the upper-triangular block system,

$$\varphi_x = \begin{pmatrix} A(x) & B(x) \\ 0 & C(x) \end{pmatrix} \varphi, \quad \varphi \in \mathbb{C}^{m+n}, \tag{4.18}$$

where A, B, C are bounded matrix functions. If the invariant subsystem $\psi_x = C(x)\psi$ admits an exponential dichotomy on some interval $J \subset \mathbb{R}$ and all solutions to $\omega_x = A(x)\omega$ are bounded on J, then system (4.18) has an exponential trichotomy on J. The latter fact is readily seen using variation of constants formulas. In the spectral stability analysis we use a similar construction to generate exponential trichotomies, see §5.3.2 and §5.3.3.

4.5 The minimal opening between subspaces

The minimal opening [42, Section 13.3] is a quantity measuring the 'gap' between two subspaces.

Definition 4.17. Let $n \in \mathbb{Z}_{>0}$. The *minimal opening* between two non-trivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ is given by

$$\eta(\mathcal{M}, \mathcal{N}) = \inf\{\|x - y\| : x \in \mathcal{M}, y \in \mathcal{N}, \max(\|x\|, \|y\|) = 1\}.$$

The minimal opening has the useful property that the norm of the projection on $\mathcal{M}$ along $\mathcal{N}$ can be bounded in terms of $\eta(\mathcal{M}, \mathcal{N})$. This norm estimate is essential for the application of the "pasting" Lemma 4.11 in our spectral stability analysis.

Proposition 4.18. *Let $n \in \mathbb{Z}_{>0}$. The following assertions hold true.*

1. *If P is a non-trivial projection on $\mathbb{C}^n$, then it holds*

$$\|P\| \leq \frac{1}{\eta(P[\mathbb{C}^n], \ker(P))}.$$

2. *For non-trivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ it holds $\eta(\mathcal{M}, \mathcal{N}) \neq 0$ if and only if $\mathcal{M} \cap \mathcal{N} = \{0\}$.*

3. *Let $\mathcal{M}_{1,2}$ and $\mathcal{N}_{1,2}$ be non-trivial subspaces of $\mathbb{C}^n$. Suppose that there exists $0 < \delta < 1$ such that for each $v \in \mathcal{M}_i$ there exists a $w \in \mathcal{N}_i$ such that $\|v - w\| \leq \delta\|v\|$ for $i = 1, 2$. Then, we have the estimate*

$$\eta(\mathcal{N}_1, \mathcal{N}_2) \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + 4\delta.$$

4. *Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ are continuous families of subspaces on Ω, i.e. there exist continuous families of projections $P_\mathcal{M}, P_\mathcal{N} \colon \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $P_\mathcal{M}(\lambda)[\mathbb{C}^n] = \mathcal{M}(\lambda)$ and $P_\mathcal{N}(\lambda)[\mathbb{C}^n] = \mathcal{N}(\lambda)$ for $\lambda \in \Omega$. Then, the map $\lambda \mapsto \eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda))$ is also continuous on Ω.*

Proof. The first two assertions are derived in [42, p. 396] and [42, Proposition 13.2.1], respectively. For the third assertion take $\epsilon > 0$. There exists $v_1 \in \mathcal{M}_1$ and $v_2 \in \mathcal{M}_2$ with $\max(\|v_1\|, \|v_2\|) = 1$ such that $\|v_1 - v_2\| \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + \epsilon$. Without loss of generality we may assume $\|v_1\| = 1$. By hypothesis there exists $w_1 \in \mathcal{N}_1$ such that $\|v_1 - w_1\| \leq \delta$. Because we have $\delta < 1$, we can normalize w_1 and define $z_1 := \frac{w_1}{\|w_1\|}$. One readily estimates $\|v_1 - z_1\| \leq 2\delta$. Similarly, there exists $w_2 \in \mathcal{N}_2$ such that $\|v_2 - w_2\| \leq \delta$. In the case $\|w_2\| > 1$, take $z_2 := \frac{w_2}{\|w_2\|}$. One easily verifies $\|v_2 - z_2\| \leq 2\delta$. In the case $\|w_2\| \leq 1$, we just take $z_2 := w_2$. Finally, we estimate

$$\eta(\mathcal{N}_1, \mathcal{N}_2) \leq \|z_1 - z_2\| \leq \|v_1 - v_2\| + \|v_1 - z_1\| + \|v_2 - z_2\| \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + 4\delta + \epsilon.$$

Since ϵ is arbitrarily chosen, the second assertion follows. Finally, for the fourth assertion let $P_\mathcal{M}(\lambda)$ and $P_\mathcal{N}(\lambda)$ be continuous families of projections on Ω with ranges $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$, respectively. With the aid of identities (13.1.4), (13.2.5) and (13.2.7) in [42] we derive for $\lambda_0 \in \Omega$

$$|\eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) - \eta(\mathcal{M}(\lambda_0), \mathcal{N}(\lambda_0))| \leq \sqrt{2}\left(\|P_\mathcal{M}(\lambda) - P_\mathcal{M}(\lambda_0)\| + \|P_\mathcal{N}(\lambda) - P_\mathcal{N}(\lambda_0)\|\right).$$

This shows that $\lambda \to \eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda))$ is continuous on Ω. □

4.6 The Riccati transformation

As mentioned in the introduction in Chapter 1, the eigenvalue problem associated with the linearization of system (1.10) about the periodic pulse solution can be put in the following slow-fast block structure,

$$\begin{aligned} \varphi_x &= \epsilon(A_{11}(x,\epsilon)\varphi + A_{12}(x,\epsilon)\psi), \\ \psi_x &= A_{21}(x,\epsilon)\varphi + A_{22}(x,\epsilon)\psi, \end{aligned} \qquad (\varphi,\psi) \in \mathbb{C}^{n_1+n_2}, \tag{4.19}$$

where $0 < \epsilon \ll 1$, $n_1, n_2 \in \mathbb{Z}_{>0}$ and A_{ij} are bounded and continuous matrix functions. The *Riccati transformation* is a tool for diagonalizing linear systems of the form (4.19). This linear non-autonomous transformation, decouples (4.19) into

$$\begin{aligned} \chi_x &= \epsilon\left[A_{11}(x,\epsilon) + A_{12}(x,\epsilon)U_\epsilon(x)\right]\chi, \\ \omega_x &= \left[A_{22}(x,\epsilon) - \epsilon U_\epsilon(x)A_{12}(x,\epsilon)\right]\omega, \end{aligned} \qquad (\chi,\omega) \in \mathbb{C}^{n_1+n_2}, \tag{4.20}$$

where $U_\epsilon(x)$ is a family of matrix functions satisfying a certain matrix Riccati equation as detailed below. Decoupling the full eigenvalue problem associated with the linearization about the periodic pulse into lower-dimensional, fast and a slow eigenvalue problems leads to a reduction of complexity in the spectral stability analysis. Eventually, the decoupling yields the factorization (1.3) of the Evans function.

Although the construction of the transformation is based on two results of Chang [12, Theorem 1] and [13, Lemma 1], the assumptions on the coefficient matrices in [13] are too restrictive. Therefore, we need a refinement of his statements. For this reason and the fact that the Riccati transformation lies at the core of our analytic factorization method, we present the full construction of the transformation. Moreover, we prove that periodicity of the coefficient matrix implies periodicity of the Riccati transform, which appears to be a new result – see Remark 4.20.

Theorem 4.19. *Let $n_1, n_2 \in \mathbb{Z}_{>0}$, $\epsilon_0 \in \mathbb{R}_{>0}$ and $A_{ij} \in C(\mathbb{R} \times (0,\epsilon_0), \mathrm{Mat}_{n_i \times n_j}(\mathbb{C}))$ such that A_{ij} are bounded by some constant $K > 0$ on $\mathbb{R} \times (0,\epsilon_0)$ for $i, j = 1, 2$. Suppose that*

$$\psi_x = A_{22}(x,\epsilon)\psi, \quad \psi \in \mathbb{C}^{n_2}, \tag{4.21}$$

admits an exponential dichotomy on $\mathbb{R}$ with constants $K, \mu > 0$, independent of ϵ. Then, for $\epsilon > 0$ sufficiently small, there exists continuously differentiable matrix functions $U_\epsilon(x)$ and $S_\epsilon(x)$ satisfying the matrix Riccati equations,

$$\begin{aligned} U &= A_{22}U - \epsilon U A_{11} - \epsilon U A_{12} U + A_{21}, & U &\in \mathrm{Mat}_{n_2 \times n_1}(\mathbb{C}), \\ S &= \epsilon(A_{11} + A_{12}U)S - S(A_{22} - \epsilon U A_{12}) - A_{12}, & S &\in \mathrm{Mat}_{n_1 \times n_2}(\mathbb{C}), \end{aligned} \tag{4.22}$$

with the following properties:

1. *U_ϵ and S_ϵ are bounded on $\mathbb{R}$ by some constant, which depends on K and μ only.*
2. *The coordinate transform,*

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = H_\epsilon(x)\begin{pmatrix} \chi \\ \omega \end{pmatrix}, \quad H_\epsilon(x) := \begin{pmatrix} I & -\epsilon S_\epsilon(x) \\ U_\epsilon(x) & I - \epsilon U_\epsilon(x) S_\epsilon(x) \end{pmatrix}, \tag{4.23}$$

diagonalizes system (4.19) *into* (4.20).

3. *The unique bounded solution* Ω_ϵ *to the inhomogeneous matrix problem,*

$$\Omega_x = A_{22}(x,\epsilon)\Omega + A_{21}(x,\epsilon), \quad \Omega \in \mathrm{Mat}_{n_2\times n_1}(\mathbb{R},\mathbb{C}), \tag{4.24}$$

satisfies

$$\|U_\epsilon(x) - \Omega_\epsilon(x)\| \leq C\epsilon|\log(\epsilon)|, \quad x \in \mathbb{R}, \tag{4.25}$$

where $C > 0$ *is a constant depending on* K *and* μ *only.*

4. *Let* $a > 0$*. We have the approximation,*

$$\|U_\epsilon(x)\| \leq C\left[\sup_{y\in[x-a,x+a]}\left(\epsilon\|U_\epsilon(y)\|^2 + \|A_{21}(y,\epsilon)\|\right) + e^{-a\mu/2}\right], \quad x \in \mathbb{R}, \tag{4.26}$$

where $C > 0$ *is a constant depending on* K *and* μ *only.*

5. *If the matrices* $A_{ij}(\cdot,\epsilon)$ *are L-periodic for* $1 \leq i,j \leq 2$*, then the coordinate transform* H_ϵ *is also L-periodic.*

Proof. In the following, we denote by $C > 0$ a constant, which depends on K and μ only.

First, we set up an integral equation for U_ϵ and prove global existence via a contraction argument. Since U_ϵ triangulizes the system, an integral equation for S_ϵ can be derived from the variation of constants formula. The first four properties of U_ϵ and S_ϵ follow readily from the integral equations they satisfy. Finally, periodicity of the transform is proven by exponential separation.

Since A_{11} is bounded by K on $\mathbb{R}\times(0,\epsilon_0)$, the evolution $T_{1,\epsilon}(x,y)$ of system,

$$\varphi_x = \epsilon A_{11}(x,\epsilon)\varphi, \quad \varphi \in \mathbb{C}^{n_1},$$

satisfies

$$\|T_{1,\epsilon}(x,y)\| \leq e^{K\epsilon|x-y|}, \quad x,y \in \mathbb{R}. \tag{4.27}$$

Denote by $T_{2,\epsilon}(x,y)$ the evolution operator of system (4.21). Take $\rho = 8K\mu^{-1}\|A_{21}\|$. The ball $B(0,\rho) \subset C_b(\mathbb{R}, \mathrm{Mat}_{n_2\times n_1}(\mathbb{C}))$ is a metric space endowed with the supremum norm. We want to show that the map $\mathcal{A}_\epsilon\colon B(0,\rho) \to B(0,\rho)$ given by

$$\begin{aligned}(\mathcal{A}_\epsilon U)(x) = &\int_{-\infty}^{x} T^s_{2,\epsilon}(x,y)\left[-\epsilon U(y)A_{12}(y,\epsilon)U(y) + A_{21}(y,\epsilon)\right]T_{1,\epsilon}(y,x)dy\\ &- \int_x^\infty T^u_{2,\epsilon}(x,y)\left[-\epsilon U(y)A_{12}(y,\epsilon)U(y) + A_{21}(y,\epsilon)\right]T_{1,\epsilon}(y,x)dy,\end{aligned}$$

is a well-defined contraction. If $\epsilon > 0$ is sufficiently small, it holds for all $U \in B(0,\rho)$

$$\|\mathcal{A}_\epsilon U\| \leq \frac{2K}{\mu - \epsilon K}\left[\epsilon\rho^2\|A_{12}\| + \|A_{21}\|\right] < \rho,$$

using (4.27) and the exponential dichotomy of (4.21). Therefore, $\mathcal{A}_\epsilon$ is well-defined. Similarly, provided $\epsilon > 0$ is sufficiently small, we estimate for $U_1, U_2 \in B(0,\rho)$

$$\|\mathcal{A}_\epsilon U_1 - \mathcal{A}_\epsilon U_2\| \leq \frac{4\epsilon K\rho\|A_{12}\|}{\mu - \epsilon K}\|U_1 - U_2\| < \|U_1 - U_2\|.$$

Hence, $\mathcal{A}_\epsilon$ is a contraction mapping. By the Banach fixed point Theorem the integral equation $\mathcal{A}_\epsilon U = U$ has a unique solution $U_\epsilon(x)$ in $B(0,\rho)$. It is readily seen by differentiating this integral equation that U_ϵ satisfies the matrix Riccati equation (4.22). Moreover, U_ϵ is bounded on $\mathbb{R}$ by $\rho \leq 8K^2\mu^{-1}$. Since (4.21) has an exponential dichotomy on $\mathbb{R}$, (4.24) admits a unique bounded solution Ω_ϵ by Proposition 4.15. Since A_{11} is bounded by K, it holds by Proposition 4.1 for $|x-y| \leq \mu^{-1}|\log(\epsilon)|$

$$\|T_{1,\epsilon}(x,y) - I\| \leq C\epsilon|\log(\epsilon)|. \tag{4.28}$$

Using $U_\epsilon(x) = (\mathcal{A}_\epsilon U_\epsilon)(x)$ we write

$$\begin{aligned} U_\epsilon(x) - \Omega_\epsilon(x) = &\int_{-\infty}^{x} T^s_{2,\epsilon}(x,y)A_{21}(y,\epsilon)(T_{1,\epsilon}(x,y) - I)dy \\ &- \int_{x}^{\infty} T^u_{2,\epsilon}(x,y)A_{21}(y,\epsilon)(T_{1,\epsilon}(x,y) - I)dy \\ &- \int_{-\infty}^{x} \epsilon T^s_{2,\epsilon}(x,y)U(y)A_{12}(y,\epsilon)U(y)T_{1,\epsilon}(y,x)dy \\ &+ \int_{x}^{\infty} \epsilon T^u_{2,\epsilon}(x,y)U(y)A_{12}(y,\epsilon)U(y)T_{1,\epsilon}(y,x)dy, \end{aligned}$$

We split the interval of integration of the first two integrals in the right hand side of the latter equation. This leads to four integrals over $(-\infty, x-b_\epsilon)$, $(x-b_\epsilon, x)$, $(x, x+b_\epsilon)$ and $(x+b_\epsilon, \infty)$, where $b_\epsilon := \mu^{-1}|\log(\epsilon)|$. Thus, we obtain six integrals, which we estimate separately using (4.27), (4.28) and the bound on U_ϵ. This yields the third property.

The fourth property follows by splitting the interval of integration of the two integrals in the right hand side of the identity $U_\epsilon(x) = (\mathcal{A}_\epsilon U_\epsilon)(x)$. We obtain four integrals over $(-\infty, x-a)$, $(x-a, x)$, $(x, x+a)$ and $(x+a, \infty)$, respectively. We estimate each integral separately using (4.27) and the exponential dichotomy of (4.21). This leads to approximation (4.26).

Since A_{11}, A_{12} and U_ϵ are bounded on $\mathbb{R} \times (0, \epsilon_0)$, the evolution $T_{3,\epsilon}(x,y)$ of system,

$$\chi_x = \epsilon\,[A_{11}(x,\epsilon) + A_{12}(x,\epsilon)U_\epsilon(x)]\,\chi, \quad \chi \in \mathbb{C}^{n_1},$$

is bounded as

$$\|T_{3,\epsilon}(x,y)\| \leq e^{\epsilon C|x-y|}, \quad x,y \in \mathbb{R}. \tag{4.29}$$

On the other hand, equation,

$$\omega_x = (A_{22}(x,\epsilon) - \epsilon U_\epsilon(x)A_{12}(x,\epsilon))\omega, \quad \omega \in \mathbb{C}^{n_2}, \tag{4.30}$$

can be seen as a perturbation of (4.21). By Proposition 4.12 it therefore possesses an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu_1 > 0$. Denote by $T_{4,\epsilon}(x,y)$ the evolution operator of system (4.30). We define $S_\epsilon(x)$ via the variation of constants formula,

$$S_\epsilon(x) = -\int_{-\infty}^{x} T_{3,\epsilon}(x,y)A_{12}(y,\epsilon)T^u_{4,\epsilon}(y,x)dy + \int_x^\infty T_{3,\epsilon}(x,y)A_{12}(y,\epsilon)T^s_{4,\epsilon}(y,x)dy,$$

Using (4.29) and the exponential dichotomy of (4.30) we derive that S_ϵ is bounded on $\mathbb{R}$ by some constant depending on μ and K only. This proves the first property. It is easily verified by differentiation that S_ϵ satisfies the matrix Riccati equation (4.22). Finally, using S_ϵ and U_ϵ satisfy equations (4.22), it is a straightforward calculation to see the change of variables (4.23) transforms system (4.19) into (4.20). This proves the second property.

Only the fifth property remains to be proven. Our plan is to show that system (4.19) is exponentially separated in the sense of [85]. Subsequently, we make use of the fact that exponential separation preserves periodicity. Therefore, denote by $P_\epsilon(x), x \in \mathbb{R}$ the projections corresponding to the exponential dichotomy of system (4.30) on $\mathbb{R}$, established in the latter paragraph. We define the following projections,

$$P_{1,\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & P_\epsilon(0) \end{pmatrix}, \quad P_{2,\epsilon} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{3,\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & I - P_\epsilon(0) \end{pmatrix}.$$

Denote by $V_{i,\epsilon} \subset \mathbb{C}^{n_1+n_2}$ the range of the projection $P_{i,\epsilon}$ for $i = 1,2,3$. Let m_2 be the rank of $P_\epsilon(0)$. Using (4.29) and the exponential dichotomy of (4.30), we conclude system that (4.20) is, for $\epsilon > 0$ sufficiently small, $(m_2, n_1, n_2 - m_2)$-exponentially separated with respect to the decomposition $V_{1,\epsilon} \oplus V_{2,\epsilon} \oplus V_{3,\epsilon}$. As a result, system (4.19) is also $(m_2, n_1, n_2 - m_2)$-exponentially separated with respect to the decomposition $W_{1,\epsilon} \oplus W_{2,\epsilon} \oplus W_{3,\epsilon}$, where $W_{i,\epsilon}$ is the range of the projection $Q_{i,\epsilon} := H_\epsilon(0)P_{i,\epsilon}H_\epsilon(0)^{-1}$ for $i = 1,2,3$.

Now, suppose $A_{ij}(\cdot,\epsilon)$ are L-periodic for $1 \leq i, j \leq 2$. Let $X_\epsilon(x)$ be the fundamental matrix of system (4.19) with $X_\epsilon(0) = I$. Invoking [9, Corollary 4] gives that $X_\epsilon(\cdot)Q_{2,\epsilon}X_\epsilon(\cdot)^{-1}$ is L-periodic. Denote by $T_\epsilon(x,y)$ the evolution operator of the diagonal system (4.20). We calculate for $x \in \mathbb{R}$

$$\begin{aligned} X_\epsilon(x)Q_{2,\epsilon}X_\epsilon(x)^{-1} &= H_\epsilon(x)T_\epsilon(x,0)P_{2,\epsilon}T_\epsilon(0,x)H_\epsilon(x)^{-1} = H_\epsilon(x)P_{2,\epsilon}H_\epsilon(x)^{-1} \\ &= \begin{pmatrix} I - \epsilon S_\epsilon(x)U_\epsilon(x) & \epsilon S_\epsilon(x) \\ U_\epsilon(x) + \epsilon U_\epsilon(x)S_\epsilon(x)U_\epsilon(x) & \epsilon U_\epsilon(x)S_\epsilon(x) \end{pmatrix}. \end{aligned}$$

Hence, $S_\epsilon, U_\epsilon S_\epsilon, S_\epsilon U_\epsilon$ and $U_\epsilon + \epsilon U_\epsilon S_\epsilon U_\epsilon$ are L-periodic. So, $U_\epsilon S_\epsilon U_\epsilon$ is also L-periodic. Combining this with the L-periodicity of $U_\epsilon + \epsilon U_\epsilon S_\epsilon U_\epsilon$, we conclude that U_ϵ is L-periodic. This implies that H_ϵ is L-periodic, which concludes the proof of the fifth statement. □

Remark 4.20. The periodicity of the transform in Theorem 4.19 is a new discovery to the author's knowledge. It is natural to ask whether there always exists a periodic choice for a coordinate change, which transforms a periodic system into diagonal form. However, it is shown in [84, Chapter 5] that this is not the case. It seems that the periodicity of the coordinate change H_ϵ is due to the special (slow-fast) structure of system (4.19). ■

Remark 4.21. The $(m_2, n_1, n_2 - m_2)$-exponential separation of (4.19) obtained in Theorem 4.19 shows that the solution space of systems of the form (4.19) can be decomposed in fast exponentially decaying solutions in forward and backward time and solutions that vary slowly. This type of decomposition is very similar to the one induced by an exponential trichotomy – see §4.4. Yet, in our definition of exponential trichotomies we do not allow for exponential growth in the centre direction. However, we emphasize that some authors do include this in their definition of exponential trichotomies – see for instance [106]. ■

Remark 4.22. The Riccati transform can be employed to diagonalize general linear equations as pointed out in [4, Remark 4.7]. However, the Riccati solutions can become singular in finite time. We use both the slow-fast structure of (4.19) and the exponential dichotomy of (4.21) to achieve global boundedness of the transformation functions U_ϵ and S_ϵ. ■