Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Chapter 3

Stability results

3.1 Introduction

In this chapter we present the outcomes of our spectral stability analysis performed in Chapter 5. We assume that conditions **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Then, Theorem 2.3 provides a reversibly symmetric, $2L_\varepsilon$-periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x)$ to (2.1). This yields a stationary, periodic pulse solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$ to system (1.10). We denote by $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ the corresponding solution to the rescaled system (1.9). The stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ is determined by the spectrum of the linearization $\mathcal{L}_\varepsilon$ of (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$. The (critical) spectrum of the periodic differential operator $\mathcal{L}_\varepsilon$ is a union of curves parameterized over the unit circle S^1 by Floquet theory. Due to translational invariance one of these curves is attached to the origin. The spectral curves can be located by tracing the zeros of the analytic Evans function [38].

When the spectrum of $\mathcal{L}_\varepsilon$ is confined to the left half-plane and bounded away from the imaginary axis, except for a quadratic tangency at the origin, it is known [58, 101, 104] that the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinear diffusively stable as solution to (1.9). Verifying such spectral conditions is in general very hard, especially for multi-component systems. However, as mentioned in the introduction in Chapter 1, the presence of the small parameter ε in (1.9) provides a mechanism to reduce complexity. In the singular limit the Evans function corresponding to the full problem decomposes as a product of a slow and a fast Evans function. The analytic fast and meromorphic slow Evans function are defined in terms of simpler, lower-dimensional eigenvalue problems. The spectrum of $\mathcal{L}_\varepsilon$ can be approximated by the roots of the fast and slow Evans functions. This approximation mechanism provides asymptotic control over the spectrum. However, the critical spectral curve attached to origin shrinks to the origin in the singular limit. Thus, our approximation result is unable to determine the spectral geometry about the origin and asymptotic spectral control is insufficient to establish nonlinear stability. Therefore, we complement our analysis with an expansion of this critical spectral curve.

We start this chapter by linearizing (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$ and characterizing the spectrum of the linearization $\mathcal{L}_\varepsilon$ via Floquet-Bloch decomposition. Then, we provide conditions on the spectrum of $\mathcal{L}_\varepsilon$ yielding nonlinear stability. Subsequently, we introduce the analytic Evans function and reformulate the spectral stability conditions in terms of this function. Next, we state our two main spectral approximation results: the slow-fast decomposition of the Evans function in the singular limit and the expansion of the critical spectral curve. These two results then lead to explicit criteria yielding stability and instability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ in terms of simpler, lower-dimensional eigenvalue problems. Finally, we further simplify these criteria in the case $n = 1$ or $m = 1$ and we illustrate our results by explicit calculations in the slowly nonlinear toy problem (2.27).

3.2 Linearizing about the periodic pulse solution

We linearize system (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$ and obtain the periodic differential operator $\mathcal{L}_\varepsilon$ on $C_{\mathrm{ub}}(\mathbb{R}, \mathbb{R}^{m+n})$ with domain $C^2_{\mathrm{ub}}(\mathbb{R}, \mathbb{R}^{m+n})$ given by

$$\mathcal{L}_\varepsilon \psi = D_\varepsilon \psi_{\check{x}\check{x}} - \mathcal{B}_\varepsilon \psi,$$

with

$$D_\varepsilon := \begin{pmatrix} D_1 & 0 \\ 0 & \varepsilon^2 D_2 \end{pmatrix},$$

and

$$\mathcal{B}_\varepsilon(\check{x}) := \begin{pmatrix} \partial_u H_1(\check{\phi}_{\mathrm{p},\varepsilon}, \varepsilon) + \varepsilon^{-1} \partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}) & \partial_v H_1(\check{\phi}_{\mathrm{p},\varepsilon}, \varepsilon) + \varepsilon^{-1} \partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}) \\ \partial_u G(\check{\phi}_{\mathrm{p},\varepsilon}, \varepsilon) & \partial_v G(\check{\phi}_{\mathrm{p},\varepsilon}, \varepsilon) \end{pmatrix}, \tag{3.1}$$

where we suppress the $\check{x}$-dependence of $\check{\phi}_{\mathrm{p},\varepsilon}$. Here, $C^k_{\mathrm{ub}}(\mathbb{R}, \mathbb{R}^{m+n})$ denotes the Banach space of k times continuously differentiable functions, with derivatives up to order k bounded and uniformly continuous. It is endowed with the supremum norm,

$$\|f\| = \sum_{i=0}^{k} \left\| (\partial_{\check{x}})^i f \right\|_\infty .$$

Note that $\mathcal{L}_\varepsilon$ is closed, densely defined and sectorial by [72, Corollary 3.1.9.ii] and [44, Theorem 1.3.2].

3.2.1 Floquet-Bloch decomposition

By Theorem 2.3, $\mathcal{L}_\varepsilon$ is a $2\ell_\varepsilon$-periodic differential operator, where $\ell_\varepsilon := \varepsilon L_\varepsilon \to \ell_0$ as $\varepsilon \to 0$ with $\ell_0 > 0$ defined in **(E2)**. Therefore, Floquet-Bloch decomposition [38] of $\mathcal{L}_\varepsilon$ yields a family of closed and densely defined operators $\mathcal{L}_{\nu,\varepsilon}$ on $L^2_{\mathrm{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ with domain $H^2_{\mathrm{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ given by

$$\mathcal{L}_{\nu,\varepsilon} \psi = D_\varepsilon \left(\partial_{\check{x}} - \frac{i\nu}{2\ell_\varepsilon} \right)^2 \psi - \mathcal{B}_\varepsilon \psi, \quad \nu \in [-\pi, \pi],$$

where $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ is the space of L^2-integrable functions that are $2\ell_\varepsilon$-periodic and $H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ is the subspace of $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ of functions that have weak derivatives up to order 2. By the Rellich compactness theorem the space $H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ is compactly embedded in $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$. Therefore, $\mathcal{L}_{\nu,\varepsilon}$ has compact resolvent. Consequently, its spectrum is discrete and consists entirely of eigenvalues. The spectrum of $\mathcal{L}_\varepsilon$ is given by the union,

$$\sigma(\mathcal{L}_\varepsilon) = \bigcup_{\nu\in[-\pi,\pi]} \sigma(\mathcal{L}_{\nu,\varepsilon}). \tag{3.2}$$

Indeed, if $\lambda \in \sigma(\mathcal{L}_{\nu,\varepsilon})$ is an eigenvalue and $\varphi \in H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ denotes the corresponding eigenfunction, then the natural extension of $\varphi(\check{x})e^{-i\nu\check{x}/(2\ell_\varepsilon)}$ to $\mathbb{R}$ yields an eigenfunction of $\mathcal{L}_\varepsilon$. Conversely, given $\lambda \in \sigma(\mathcal{L}_\varepsilon)$, there exists by Floquet theory a $\gamma \in S^1$ and a corresponding eigenfunction $\psi \in C^2_{ub}(\mathbb{R},\mathbb{C}^{m+n})$ satisfying $\psi(\check{x}) = \gamma\psi(\check{x}+2\ell_\varepsilon)$ for all $\check{x}\in\mathbb{R}$. The restriction of $\psi(\check{x})e^{i\nu\check{x}/(2\ell_\varepsilon)}$ to $[0,2\ell_\varepsilon]$ is the eigenfunction of $\mathcal{L}_{\nu,\varepsilon}$, where $e^{i\nu}=\gamma$. The spectral decomposition (3.2) gives rise to the following definition.

Definition 3.1. Let $\nu\in[-\pi,\pi]$ and $\gamma = e^{i\nu}\in S^1$. A point $\lambda\in\sigma(\mathcal{L}_{\nu,\varepsilon})$ is called a *γ-eigenvalue* of $\mathcal{L}_\varepsilon$. The algebraic multiplicity of λ as an eigenvalue of $\mathcal{L}_{\nu,\varepsilon}$ is the *algebraic γ-multiplicity* of λ.

3.3 Nonlinear stability by linear approximation

In this section we collect nonlinear (in)stability results from the literature. More precisely, we present conditions on the spectrum of the linearization $\mathcal{L}_\varepsilon$ of (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$ yielding some form of nonlinear stability or instability.

3.3.1 Spectral conditions yielding nonlinear stability

By translational invariance, 0 is always a 1-eigenvalue of $\mathcal{L}_\varepsilon$. Indeed, the restriction of the derivative $\check{\phi}'_{\mathrm{p},\varepsilon}(\check{x})$ to $[0,2\ell_\varepsilon]$ is contained in the kernel of $\mathcal{L}_{0,\varepsilon}$. If we assume that 0 has algebraic 1-multiplicity 1, then there exists by the implicit function theorem a spectral curve $\lambda_\varepsilon\colon U_\varepsilon\to\mathbb{C}$, where $U_\varepsilon\subset[-\pi,\pi]$ is a neighborhood of 0, such that $\lambda_\varepsilon(0)=0$ and $\lambda_\varepsilon(\nu)$ is a $e^{i\nu}$-eigenvalue for $\nu\in U_\varepsilon$. By assuming that this critical spectral curve touches the origin in a quadratic tangency and the rest of the spectrum is confined to the left half-plane, bounded away from the imaginary axis, we establish some form of nonlinear stability. This leads to the following definition.

Definition 3.2. The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) is *spectrally stable* if 0 is a simple eigenvalue of $\mathcal{L}_{0,\varepsilon}$ and there exists $\varsigma>0$, possibly dependent on ε, such that

$$\begin{aligned}\mathrm{Re}(\lambda_\varepsilon(\nu)) &\leq -\varsigma\nu^2, \quad \nu\in U_\varepsilon,\\ \sigma(\mathcal{L}_\varepsilon)\setminus\lambda_\varepsilon[U_\varepsilon] &\subset \{\lambda\in\mathbb{C} : \mathrm{Re}(\lambda) < -\varsigma\}\,.\end{aligned}$$

Spectral stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ implies nonlinear diffusive stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ with respect to localized perturbations. In addition, an initial displacement of the periodic pulse can be tracked for large times.

Theorem 3.3. *[101, Theorem 1] Suppose $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable. Take $b \in (0, \frac{1}{2})$. There are $\delta, C > 0$, possibly dependent on ε, such that the following holds. The solution $\check{\phi}(x,t)$ to* (1.9) *with initial condition,*

$$\check{\phi}(\check{x}, 0) = \check{\phi}_{\mathrm{p},\varepsilon}(\check{x} + \theta_0(\check{x})) + v_0(\check{x}),$$

with $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ and $\theta_0 \in H^3(\mathbb{R}, \mathbb{R})$ satisfying $\|\theta_0\rho\|_{H^3}, \|v_0\rho\|_{H^2} \leq \delta$ with $\rho(\check{x}) = (1 + \check{x}^2)^{3/2}$, exists for all times $t \geq 0$ and can be written as

$$\check{\phi}(\check{x}, t) = \check{\phi}_{\mathrm{p},\varepsilon}(\check{x} + \theta(\check{x}, t)) + v(\check{x}, t), \quad t > 0,$$

where $\theta \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ and $v \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}^{m+n}$. There exists a constant $\theta_{\lim} \in \mathbb{R}$ such that

$$\sup_{\check{x} \in \mathbb{R}} [|\theta(\check{x}, t) - \theta_{\lim} G(\check{x}, t)| + \|v(\check{x}, t)\|] \leq C(1 + t)^{-1+b}, \quad t > 0,$$

where G is the Gaussian,

$$G(\check{x}, t) = \frac{1}{\sqrt{4\alpha\pi(1 + t)}} e^{-\check{x}^2/(4\alpha(1+t))},$$

with $\alpha := -\lambda_\varepsilon''(0)$. In particular, we have

$$\sup_{\check{x} \in \mathbb{R}} \left\| \check{\phi}(\check{x}, t) - \check{\phi}_{\mathrm{p},\varepsilon} (\check{x} + \theta_{\lim} G(\check{x}, t)) \right\| \leq C(1 + t)^{-1+b}, \quad t > 0.$$

The above result is to be compared with [104, Theorem 1.1]. Here, the class of allowed perturbations is larger, i.e. one requires $v_0\tilde{\rho} \in H^{1/2+b}(\mathbb{R}, \mathbb{R}^{m+n})$ with $\tilde{\rho}(\check{x}) = 1 + \check{x}^2$. However, in [104] one obtains a weaker decay bound of the form $\sup_{\check{x} \in \mathbb{R}} \|v(\check{x}, t)\| \leq C(1+t)^{-1/2}$. Moreover, in [58] pointwise nonlinear estimates are obtained with respect to perturbations $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ satisfying $\|v_0(\check{x})\| \leq E_0 e^{-\check{x}^2/M}$ or $\|v_0(\check{x})\| \leq E_0(1 + |\check{x}|)^{-r}$ for some $M > 1, r > 2$ and $E_0 > 0$. The decay rates obtained in [58] are comparable to those in Theorem 3.3, yet they are more specific, since they depend pointwise on $\check{x}$. Finally, we emphasize that both in [58] and [104] one does not consider an initial displacement in time in contrast to Theorem 3.3.

As mentioned in the introduction of this chapter, verifying spectral stability is in general very hard. The main outcome of our spectral analysis is explicit conditions in terms of simpler, lower-dimensional eigenvalue problems that yield spectral stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ – see §3.7. This reduction of complexity is achieved by a slow-fast decomposition of the Evans function in the singular limit and an expansion of the critical spectral curve $\lambda_\varepsilon(\nu)$ – see §3.5 and §3.6, respectively.

3.3.2 Spectral conditions yielding nonlinear instability

Spectrum of $\mathcal{L}_\varepsilon$ in the right half-plane yields nonlinear instability of the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ against localized and non-localized perturbations.

Definition 3.4. The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) is *spectrally unstable* if there exists $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda) > 0$.

Theorem 3.5. *[75, Section 4] Let $X = H^2(\mathbb{R}, \mathbb{R}^{m+n})$ or $X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$. Suppose the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) is spectrally unstable. Then, there exists $\delta > 0$ and a sequence of solutions $\check{\phi}_n(\check{x}, t), n \in \mathbb{Z}_{>0}$ to (1.9) satisfying $\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon} \in X$ such that*

$$\left\|\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X \to 0 \text{ as } n \to \infty,$$

but for all $n \in \mathbb{Z}_{>0}$ there exists $t_n > 0$ such that

$$\begin{aligned} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X &\geq \delta, \text{ in the case } X = H^2(\mathbb{R}, \mathbb{R}^{m+n}), \\ \inf_{\theta\in\mathbb{R}} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)\right\|_X &\geq \delta, \text{ in the case } X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n}). \end{aligned}$$

We emphasize that in the case of non-localized perturbations, it is important to measure the distance from the perturbation to the family of all translates of the solution rather than to the solution itself. Indeed, any translate $\check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)$ corresponds to a non-localized perturbation. Yet, such a translate is a solution to (1.9) itself. Thus, $\check{\phi}_{\mathrm{p},\varepsilon}$ is never stable against translation of the profile. We stress that the θ-terms in Theorem 3.3 account for translation of the profile.

Using the outcomes of our spectral analysis, we obtain explicit conditions in terms of simpler, lower-dimensional systems yielding spectral instability – see §3.7. In particular, in the case $n = 1$ or $m = 1$, we can test for instability by calculating the signs of a number of explicit integral expressions – see §3.8.

3.4 The Evans function

In this section we introduce the Evans function as a tool to locate the spectrum of the linearization $\mathcal{L}_\varepsilon$. Recall from §3.2.1 that a point $\lambda \in \mathbb{C}$ is in the spectrum of $\mathcal{L}_\varepsilon$ if and only if there exists $\psi \in C^2_{ub}(\mathbb{R}, \mathbb{C}^{m+n})$ such that $\mathcal{L}_\varepsilon\psi = \lambda\psi$. The latter equation can be rewritten as an ODE in the 'small' spatial scale $x = \varepsilon^{-1}\check{x}$ as follows

$$\varphi_x = \mathcal{A}_\varepsilon(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{3.3}$$

with coefficient matrix,

$$\mathcal{A}_\varepsilon(x, \lambda) := \begin{pmatrix} \mathcal{A}_{11,\varepsilon}(x, \lambda) & \mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix},$$

where the blocks are given by

$$
\begin{aligned}
\mathcal{A}_{11,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & \varepsilon D_1^{-1} \\ \varepsilon\left(\partial_u H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\lambda\right)+\partial_u H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\
\mathcal{A}_{12,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \varepsilon\partial_v H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\partial_v H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\
\mathcal{A}_{21,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) & 0 \end{pmatrix}, \\
\mathcal{A}_{22,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\lambda & 0 \end{pmatrix},
\end{aligned}
\tag{3.4}
$$

and $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$ is the $2L_\varepsilon$-periodic pulse solution to (1.10). We will refer to (3.3) as the *full eigenvalue problem*. By Floquet Theory bounded solutions to (3.3) must satisfy $\varphi(-L_\varepsilon) = \gamma\varphi(L_\varepsilon)$ for some $\gamma \in S^1$. This fact leads to the definition of the Evans function.

Definition 3.6. Denote by $\mathcal{T}_\varepsilon(x,z,\lambda)$ the evolution operator of system (3.3). The *Evans function* $\mathcal{E}_\varepsilon\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is given by

$$\mathcal{E}_\varepsilon(\lambda,\gamma) := \det(\mathcal{T}_\varepsilon(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_\varepsilon(0,L_\varepsilon,\lambda)).$$

Proposition 3.7. *The Evans function has the following properties:*

1. *The Evans function is analytic in both λ and γ;*
2. *We have $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ if and only if there exists $\gamma \in S^1$ such that $\mathcal{E}_\varepsilon(\lambda,\gamma) = 0$. In that case, λ is a γ-eigenvalue and its algebraic γ-multiplicity is equal to the multiplicity of λ as a root of $\mathcal{E}_\varepsilon(\cdot,\gamma)$;*
3. *It holds $\overline{\mathcal{E}_\varepsilon(\lambda,\gamma)} = \mathcal{E}_\varepsilon(\overline{\lambda},\overline{\gamma})$ for $\lambda,\gamma\in\mathbb{C}$. Thus, the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is invariant under complex conjugation;*
4. *We have $\mathcal{E}_\varepsilon(\lambda,\gamma) = \mathcal{E}_\varepsilon(\lambda,\overline{\gamma})\gamma^{2(m+n)}$ for $\lambda \in \mathbb{C}$ and $\gamma \in S^1$. Thus, λ is a γ-eigenvalue if and only if it is a $\overline{\gamma}$-eigenvalue.*

Proof. The first two properties are established in [38]. Since (3.3) is a real-valued problem for $\lambda \in \mathbb{R}$, the third property follows by the reflection principle. Finally, since $\phi_{\mathrm{p},\varepsilon}$ is reversibly symmetric by Theorem 2.3, the eigenvalue problem (3.3) is R-reversible at $x = 0$, i.e. it holds $R\mathcal{T}_\varepsilon(x,y,\lambda)R = \mathcal{T}_\varepsilon(-x,-y,\lambda)$ for $x,y\in\mathbb{R}$. This yields the fourth property. □

Proposition 3.7 shows that the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is an at most countable union of curves, each of which is covered twice by the unit circle S^1. The endpoints of the curves are ±1-eigenvalues. Proposition 3.7 and the implicit function theorem yield the following reformulation of the concept 'spectral stability' introduced in §3.3.

Corollary 3.8. *The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (1.9) *is spectrally stable if and only if*

i. $\mathcal{E}_\varepsilon(\lambda,\gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

ii. $\mathcal{E}_\varepsilon(0,\gamma) \neq 0$ *for all* $\gamma \in S^1 \setminus \{1\}$*;*

iii. $\partial_\lambda \mathcal{E}_\varepsilon(0,1)\partial_{\gamma\gamma}\mathcal{E}_\varepsilon(0,1) < 0$.

3.5 The Evans function in the singular limit

In this section we present one of the main outcomes of our spectral stability analysis. We obtain an explicit *reduced Evans function* $\mathcal{E}_0(\lambda,\gamma)$, whose zeros, for γ restricted to S^1, approximate the zeros of the Evans function $\mathcal{E}_\varepsilon(\lambda,\gamma)$, provided that $\varepsilon > 0$ is sufficiently small, yielding asymptotic control over the spectrum. The reduced Evans function is defined in terms of three simpler, lower-dimensional eigenvalue problems. Therefore, the verification of the first spectral stability condition in Corollary 3.8 simplifies to a calculation of the roots of the reduced Evans function, which does not require understanding of the full eigenvalue problem (3.3), but rather of three simpler, lower-dimensional eigenvalue problems. Thus, when proving spectral stability, understanding of the full eigenvalue problem (3.3) is only necessary for λ close to the origin. The results of this local analysis are presented in §3.6.

This section is structured as follows. First, we define the reduced Evans function in terms of three eigenvalue problems, which are obtained by a slow-fast decomposition of the full eigenvalue problem (3.3). Then, we state our main result concerning the approximation of the zeros of $\mathcal{E}_\varepsilon$ by the ones of $\mathcal{E}_0$. Finally, using this approximation result, we simplify the verification of the first spectral stability condition in Corollary 3.8.

3.5.1 The reduced Evans function

The reduced Evans function $\mathcal{E}_0$ is only defined on half-planes C_Λ of the following form.

Notation 3.9. For every $\Lambda < 0$ we denote by C_Λ the open half-plane,

$$C_\Lambda := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \Lambda\}.$$

The reduced Evans function $\mathcal{E}_0 \colon C_\Lambda \times \mathbb{C} \to \mathbb{C}$ is defined as the product,

$$\mathcal{E}_0(\lambda,\gamma) = (-\gamma)^n \mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda,\gamma). \tag{3.5}$$

Here, the analytic map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ is called the *fast Evans function*. It is associated with the *homogeneous fast eigenvalue problem*,

$$\varphi_x = \mathcal{A}_{22,0}(x,u_0,\lambda)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.6}$$

with

$$\mathcal{A}_{22,0}(x,u,\lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, v_{\rm h}(x,u), 0) + \lambda & 0 \end{pmatrix}, \quad u \in U_{\rm h}.$$

Recall that $U_{\rm h}, v_{\rm h}(x,u), u_0 = u_{\rm s}(0)$ and $u_{\rm s}(\check{x})$ are defined in **(E1)** and **(E2)**. System (3.6) arises as an eigenvalue problem, when linearizing $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_{\rm h}(x, u_0)$. Indeed, equation (3.6) is equivalent to $\mathcal{L}_f\varphi = \lambda\varphi$, where $\mathcal{L}_f \colon L^2(\mathbb{R},\mathbb{R}^n) \to L^2(\mathbb{R},\mathbb{R}^n)$ is the closed, densely defined and sectorial operator – see [72, Theorem 3.1.3] and [44, Theorem 1.3.2] – with domain $H^2(\mathbb{R},\mathbb{R}^n)$ given by

$$\mathcal{L}_f v = D_2 v_{xx} - \partial_v G(u_0, v_{\rm h}(\cdot, u_0), 0)v. \tag{3.7}$$

We establish the existence of the fast Evans function.

Proposition 3.10. *There exists $\Lambda < 0$ and an analytic map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$, which has a zero if and only if* (3.6) *admits a non-trivial, exponentially localized solution. In particular, the multiplicity of a root $\lambda \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of λ as an eigenvalue of the sectorial operator $\mathcal{L}_f$, defined in* (3.7).

The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is determined by two eigenvalue problems. The first is the *inhomogeneous fast eigenvalue problem*,

$$\partial_x \mathcal{X} = \mathcal{A}_{22,0}(x,u,\lambda)\mathcal{X} + \mathcal{A}_{21,0}(x,u), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}), \tag{3.8}$$

with

$$\mathcal{A}_{21,0}(x,u) := \begin{pmatrix} 0 & 0 \\ \partial_u G(u, v_{\rm h}(x,u), 0) & 0 \end{pmatrix}, \quad u \in U_{\rm h}.$$

The matrix system (3.8) describes the dynamics in the limit $\varepsilon \to 0$ of the full eigenvalue problem (3.3). The second is the *slow eigenvalue problem*,

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= (\partial_u H_1(u_{\rm s}(\check{x}), 0, 0) + \lambda)\, u, \end{aligned} \qquad (u,p) \in \mathbb{C}^{2m}. \tag{3.9}$$

which arises as an eigenvalue problem when linearizing system $u_t = D_1 u_{\check{x}\check{x}} - H_1(u,0,0)$ about the stationary solution $u_{\rm s}(\check{x})$ in $L^2_{\rm per}([0,2\ell_0])$. Let $\Lambda < 0$ be as in Proposition 3.10. The *slow Evans function* $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is defined by

$$\mathcal{E}_{s,0}(\lambda,\gamma) = \det\left(\Upsilon(u_0,\lambda)\mathcal{T}_s(2\ell_0, 0, \lambda) - \gamma I\right), \tag{3.10}$$

where $\ell_0 > 0$ is as in **(E2)**, $\mathcal{T}_s(\check{x},\check{y},\lambda)$ is the evolution operator of the slow eigenvalue problem (3.9) and $\Upsilon(u,\lambda)$ is given by

$$\begin{aligned} \Upsilon(u,\lambda) &= \begin{pmatrix} I & 0 \\ \mathcal{G}(u,\lambda) & I \end{pmatrix}, \\ \mathcal{G}(u,\lambda) &= \int_{-\infty}^{\infty} [\partial_u H_2(u, v_{\rm h}(x,u)) + \partial_v H_2(u, v_{\rm h}(x,u))\mathcal{V}_{in}(x,u,\lambda)]\, dx, \end{aligned} \qquad u \in U_{\rm h}, \tag{3.11}$$

where $\mathcal{V}_{in}(x, u, \lambda)$ denotes the upper-left $(n\times m)$-block of the unique matrix solution $\mathcal{X}_{in}(x, u, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8). We collect some properties of the slow Evans functions $\mathcal{E}_{s,0}$.

Proposition 3.11. *The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is well-defined and enjoys the following properties:*

1. *$\mathcal{E}_{s,0}$ is analytic on its domain;*
2. *$\mathcal{E}_{s,0}(\cdot, \gamma)$ is meromorphic on C_Λ for each $\gamma \in \mathbb{C}$ in such a way that the reduced Evans function $\mathcal{E}_0$ is analytic on its domain;*
3. *$\mathcal{E}_{s,0}(\lambda, \cdot)$ is a polynomial of degree $2m$ and it holds $\mathcal{E}_{s,0}(\lambda, \gamma) = \gamma^{2m}\mathcal{E}_{s,0}(\lambda, \overline{\gamma})$ for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ and $\gamma \in S^1$;*
4. *The set of roots,*

$$\bigcup_{\gamma\in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_{s,0}(\lambda, \gamma) = 0\},$$

is bounded.

The analytic reduced Evans function is defined as the product (3.5) of the meromorphic slow Evans function and the analytic fast Evans function. Thus, when determining the zeros of $\mathcal{E}_0(\cdot, \gamma)$ one should be aware of the possibility of zero-pole cancelation at all points in $\mathcal{E}_{f,0}^{-1}(0)$. The next proposition focuses on this issue.

Proposition 3.12. *Let $\lambda_\diamond$ be a simple zero of $\mathcal{E}_{f,0}$. Then, $\lambda_\diamond$ is also a zero of $\mathcal{E}_0(\cdot, \gamma)$ for any $\gamma \in S^1$ if it holds*

$$\int_{-\infty}^{\infty} \partial_v H_2(u_0, v_\mathrm{h}(z, u_0))\varphi_{\lambda_\diamond,1}(z)dz = 0, \tag{3.12}$$

where $\varphi_{\lambda_\diamond}(x) = (\varphi_{\lambda_\diamond,1}(x), \varphi_{\lambda_\diamond,2}(x))$ is a non-trivial, exponentially localized solution to (3.6) *at $\lambda = \lambda_\diamond$, or*

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond,2}(z)^* \partial_u G(u_0, v_\mathrm{h}(z, u_0), 0)dz = 0, \tag{3.13}$$

where $\psi_{\lambda_\diamond}(x) = (\psi_{\lambda_\diamond,1}(x), \psi_{\lambda_\diamond,2}(x))$ denotes a non-trivial, exponentially localized solution to the adjoint equation,

$$\varphi_x = -\mathcal{A}_{22,0}(x, u_0, \lambda)^*\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.14}$$

of (3.6) *at $\lambda = \lambda_\diamond$.*

The orthogonality relations (3.12) and (3.13) imply that there is no zero-pole cancelation. However, the converse is not true as pointed out in §3.8.4. One can show that the integrals in the right hand sides of (3.12) and (3.13) appear as one of multiple factors in the principal part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot,\gamma)$. Although it is possible to write down the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot,\gamma)$ explicitly at a zero $\lambda \in \mathcal{E}_{f,0}^{-1}(0)$, we decide to postpone this to §5.1.2 for the benefit of exposition, since the involved expressions are rather complex (except in the case $m = 1$ – see Proposition 3.28). Eventually, these principal parts provide a tool to determine precisely whether zero-pole cancelation occurs or not. Therefore, Proposition 3.12 is weaker – but better digestible – than the statements in §5.1.2.

Remark 3.13. Note that the fast eigenvalue problem (3.6) at $\lambda = 0$ equals the variational equation,

$$\varphi_x = \mathcal{A}_f(x)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.15}$$

about the homoclinic solution $\psi_{\mathrm{h}}(x,u_0)$ to (2.3) at $u = u_0$ with

$$\mathcal{A}_f(x) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 \end{pmatrix}, \quad u \in U_{\mathrm{h}}.$$

Therefore, the derivative of the homoclinic solution $\partial_x\psi_{\mathrm{h}}(x,u_0)$ is a non-trivial, exponentially localized solution to (3.6) at $\lambda = 0$. Thus, it holds $\mathcal{E}_{f,0}(0) = 0$ by Proposition 3.10. Now assume 0 is a simple root of $\mathcal{E}_{f,0}$. Since, we have

$$\int_{-\infty}^{\infty} \partial_v H_2(u_0, v_{\mathrm{h}}(x,u_0))\partial_x v_{\mathrm{h}}(x,u_0)dx = 0,$$

there occurs no zero-pole cancelation at $\lambda = 0$ by Proposition 3.12. We infer $\mathcal{E}_0(0,\gamma) = 0$ for each $\gamma \in S^1$. The latter corresponds to the existence of the critical spectral curve attached to the origin – see §3.5.3. ■

The proof of Propositions 3.10, 3.11 and 3.12 are provided in §5.1.

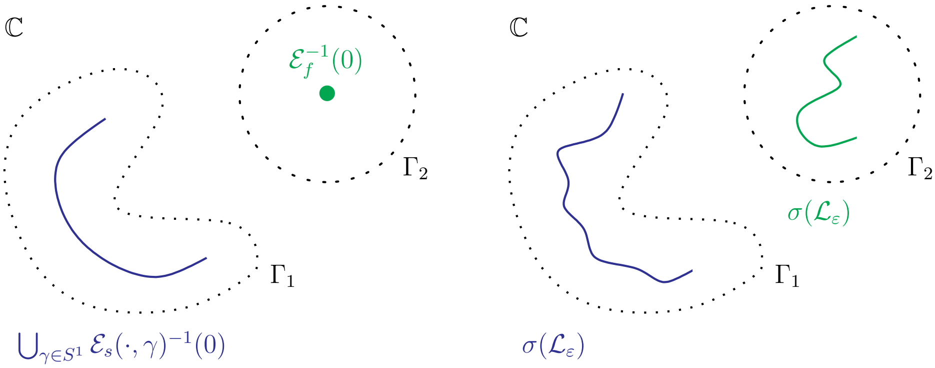
3.5.2 The spectral approximation result

We state our main result concerning the approximation of the zeros of $\mathcal{E}_\varepsilon$ by the ones of $\mathcal{E}_0$.

Theorem 3.14. *Let $\Lambda < 0$ be as in Proposition 3.10. Take a simple closed curve Γ in $C_\Lambda \setminus \mathcal{N}_0$, where*

$$\mathcal{N}_0 := \bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0\}.$$

Then, for $\varepsilon > 0$ sufficiently small, the number of roots (counting multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ and $\mathcal{E}_\varepsilon(\cdot,\gamma)$ interior to Γ coincides for any $\gamma \in S^1$.



(a) Spectrum in the limit $\varepsilon \to 0$ with disjoint contours Γ_1 and Γ_2.

(b) The true spectrum remains in the interior of Γ_1 and Γ_2 for $\varepsilon > 0$ sufficiently small.

Figure 3.1: Approximation of the spectrum $\sigma(\mathcal{L}_\varepsilon)$.

Combining Proposition 3.7 with Theorem 3.14 yields that the number of γ-eigenvalues (counting algebraic γ-multiplicity) of $\mathcal{L}_\varepsilon$ interior to Γ equals the number of roots (counting multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ for any $\gamma \in S^1$. In particular, Theorem 3.14 shows that the spectrum $\sigma(\mathcal{L}_\varepsilon) \cap C_\Lambda$ converges to a subset of $\mathcal{N}_0$ in the limit $\varepsilon \to 0$. Indeed, choose contours close enough to and disjoint from the connected components of $\mathcal{N}_0$, with, say, Hausdorff distance δ. This results in an $\varepsilon_\delta > 0$ such that, if $\varepsilon \in (0, \varepsilon_\delta)$, then $\sigma(\mathcal{L}_\varepsilon) \cap C_\Lambda$ is contained in a δ-neighborhood of $\mathcal{N}_0$.

To see that the singular limit of $\sigma(\mathcal{L}_\varepsilon)$ in fact *equals* $\mathcal{N}_0$, we need the following generalization of Theorem 3.14.

Theorem 3.15. *Let $\Lambda < 0$ be as in Proposition 3.10. Let $S \subset S^1$ be a closed subset. Take a simple closed curve Γ in $C_\Lambda \setminus \mathcal{N}_S$, where*

$$\mathcal{N}_S := \bigcup_{\gamma\in S} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0\}. \tag{3.16}$$

Then, for $\varepsilon > 0$ sufficiently small, the number of roots (including multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ and $\mathcal{E}_\varepsilon(\cdot,\gamma)$ interior to Γ coincides for any $\gamma \in S^1$.

Theorem 3.15 allows us, by taking $S = \{\gamma\}$ for some $\gamma \in S^1$, to follow individual γ-eigenvalues as they converge to the roots of $\mathcal{E}_0(\cdot,\gamma)$ as $\varepsilon \to 0$ or, equivalently, to establish the convergence of the spectrum $\sigma(\mathcal{L}_{\nu,\varepsilon})$ to the discrete set $\{\lambda \in \mathbb{C} : \mathcal{E}_0(\lambda, e^{i\nu}) = 0\}$.

The proof of Theorem 3.15 is provided in §5.2.

3.5.3 Consequences of the spectral approximation result

The results in §3.5.2 imply that we can approximate the roots of the Evans function, which is defined in terms of the $2(m+n)$-dimensional full eigenvalue problem (3.3), by the roots of the reduced Evans function, which is defined in terms of the $2n$- and $2m$-dimensional, ε-independent, eigenvalue problems (3.6), (3.8) and (3.9). Therefore, this approximation result leads to a reduction in complexity, when verifying the first spectral stability condition in Corollary 3.8.

However, asymptotic spectral control through the reduced Evans function is insufficient to establish spectral stability, since the critical spectral curve attached to the origin shrinks to the origin as $\varepsilon \to 0$. Hence, we cannot determine whether the critical curve lies in the left half-plane and touches the origin in a quadratic tangency – see the second and third condition in Corollary 3.8. Yet, using the approximation results from §3.5.2, we can isolate the critical spectral curve from the rest of the spectrum. All in all, we obtain the following result.

Corollary 3.16. *Suppose the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_{s,0}(0,\gamma) \neq 0$ *for each* $\gamma \in S^1$*;*

iii. $\mathcal{E}_0(\lambda,\gamma) \neq 0$ *for each* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$.

Then, there exists $\sigma_0, \varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$ *there exists a* 2π*-periodic, analytic map* $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ *with the following properties:*

1. $\sigma(\mathcal{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} = \lambda_\varepsilon[\mathbb{R}]$*;*
2. $\lambda_\varepsilon(\nu) = \lambda_\varepsilon(-\nu)$ *is a simple zero of* $\mathcal{E}_\varepsilon(\cdot, e^{\pm i\nu})$ *for each* $\nu \in [0,\pi]$*;*
3. $\lambda_\varepsilon(0), \lambda_\varepsilon'(0), \lambda_\varepsilon'(\pi) = 0$*;*
4. $\lambda_\varepsilon(\nu)$ *converges to* 0 *as* $\varepsilon \to 0$ *for each* $\nu \in \mathbb{R}$.

In particular, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable if there exists* $\varsigma > 0$*, possibly dependent on* ε*, such that* $\lambda_\varepsilon(\nu) \leq -\varsigma\nu^2$ *holds for all* $\nu \in [0,\pi]$.

Proof. Since $\mathcal{E}_{s,0}(\cdot,\gamma)$ has no pole at $\lambda = 0$ by Remark 3.13, we deduce that 0 is a simple root of $\mathcal{E}_0(\cdot,\gamma)$ for each $\gamma \in S^1$. In addition, the set $\mathcal{N}_0$, defined in Theorem 3.14, is bounded by Propositions 3.10 and 3.11. So, there exists $\sigma_0 > 0$ such that, if $\mathcal{E}_0(\lambda,\gamma) = 0$ is satisfied for some $\gamma \in S^1$ and $\lambda \in C_\Lambda \setminus \{0\}$, then we have $\mathrm{Re}(\lambda) < -\sigma_0$. Let $\delta \in (0,\sigma_0)$. Theorem 3.14 yields $\varepsilon_\delta > 0$ such that for each $\varepsilon \in (0,\varepsilon_\delta)$ and $\nu \in \mathbb{R}$ there exists precisely one (simple) root $\lambda_\varepsilon(\nu)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ in $B(0,\delta)$. Thus, λ_ε defines a 2π-periodic function from $\mathbb{R}$ to $\mathbb{C}$ satisfying $\lambda_\varepsilon(\nu) \to 0$ as $\varepsilon \to 0$ for each $\nu \in \mathbb{R}$. Since $\mathcal{E}_\varepsilon$ is analytic in both of its arguments by Proposition 3.7 and the root $\lambda_\varepsilon(\nu)$ is simple, it follows by the implicit function theorem that $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{C}$ is analytic.

By Proposition 3.7 it holds

$$0 = \mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{i\nu})e^{-2(m+n)i\nu} = \mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{-i\nu}) = \mathcal{E}_\varepsilon(\overline{\lambda_\varepsilon(\nu)}, e^{i\nu}), \quad \nu \in \mathbb{R}.$$

Thus, by uniqueness of the root of $\mathcal{E}(\cdot, e^{i\nu})$ in $B(0,\delta)$, we conclude that $\lambda_\varepsilon(-\nu) = \lambda_\varepsilon(\nu) = \overline{\lambda_\varepsilon(\nu)}$. Hence, λ_ε must be real-valued and $\lambda_\varepsilon'(0), \lambda_\varepsilon'(\pi) = 0$. Recall from §3.3 that 0 is always a 1-eigenvalue of $\mathcal{L}_\varepsilon$ due to translational invariance, i.e. it holds $\mathcal{E}_\varepsilon(0,1) = 0$. We derive $\lambda_\varepsilon(0) = 0$. The fact that $\mathcal{E}_0(\cdot,\gamma)$ has no roots in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\}$ except 0 yields by Theorem 3.14 that $\mathcal{E}_\varepsilon(\cdot,\gamma)$ has no roots in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} \setminus B(0,\delta)$ for each $\gamma \in S^1$. This proves $\sigma(\mathcal{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} = \lambda_\varepsilon[\mathbb{R}]$. □

3.6 Expansion of the critical spectral curve

We present the second main outcome of our spectral stability analysis; that is, we provide an expansion of the critical spectral curve $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ established in Corollary 3.16.

Theorem 3.17. *Suppose the conditions in Corollary 3.16 are met. Then, provided $\varepsilon > 0$ is sufficiently small, the critical spectral curve $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$, established in Corollary 3.16, is approximated as*

$$\left|\lambda_\varepsilon(\nu) - \varepsilon^2\lambda_0(\nu)\right| \leq C\varepsilon^3|\log(\varepsilon)|^5, \tag{3.17}$$

where $C > 0$ is a constant independent of ε and ν and the analytic function $\lambda_0 \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\lambda_0(\nu) := \frac{\int_{-\infty}^{\infty} \langle \partial_u G(u_0, v_\mathrm{h}(x,u_0), 0)^* \psi_{\mathrm{ad},2}(x) x, B(\nu)\rangle\, dx}{\int_{-\infty}^{\infty} \langle \psi_{\mathrm{ad},2}(x), \partial_x v_\mathrm{h}(x,u_0)\rangle\, dx}, \tag{3.18}$$

with $\psi_\mathrm{ad}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ a non-trivial, exponentially localized solution to the adjoint,

$$\varphi_x = -\mathcal{A}_f(x)^*\varphi, \quad \varphi \in \mathbb{R}^{2n}. \tag{3.19}$$

of the fast variational equation (3.15) *and*

$$\begin{aligned} B(\nu) &:= D_1^{-1}\begin{pmatrix} 0 & I \end{pmatrix}\mathcal{B}(\nu),\\ \mathcal{B}(\nu) &:= \Upsilon_0^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0)\end{pmatrix} - \left(I - e^{-i\nu}\Upsilon_0\Phi_\mathrm{s}(2\ell_0,0)\Upsilon_0\right)^{-1}\begin{pmatrix} 2D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix},\\ \Upsilon_0 &:= \begin{pmatrix} I & 0 \\ \partial_u\mathcal{J}(u_0) & I\end{pmatrix}, \end{aligned} \tag{3.20}$$

where $\mathcal{J} \colon U_\mathrm{h} \to \mathbb{R}^m$ is defined in (2.5) *and $\Phi_\mathrm{s}(\check{x}, \check{y})$ is the evolution operator of the slow variational equation* (2.7).

Remark 3.18. The integral $\int_{-\infty}^{\infty} \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x, u_0) \rangle \, dx$ in the denominator of $\lambda_0(\nu)$ in Theorem 3.17 arises as a solvability condition for the generalized eigenvalue problem at $\lambda = 0$ associated with the linearization of $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_{\mathrm{h}}(x, u_0)$. Since 0 is a simple zero of the fast Evans function $\mathcal{E}_{f,0}$, this integral is non-zero – see also Proposition 5.21. ■

The critical spectral curve $\lambda_\varepsilon(\nu)$ arises as the solution curve to the equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ about $(\lambda, \nu) = (0, 0)$. The equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ is defined in terms of the $2(m+n)$-dimensional full eigenvalue problem (3.3). The leading-order approximation $\lambda_0(\nu)$ of the solution curve $\lambda_\varepsilon(\nu)$, established in Theorem 3.17, is defined in terms of the ε-independent, $2m$-dimensional slow variational equation (2.7) and $2n$-dimensional fast variational equation (3.15). Therefore, Theorem 3.17 yields a reduction of complexity in the local analysis of the full eigenvalue problem (3.3) about $\lambda = 0$ simplifying the verification of the spectral stability conditions in Corollary 3.8. Combining this with Corollary 3.16 leads to a set of spectral stability conditions in terms of simpler, lower-dimensional systems, which we will present in §3.7.

When we have $\mathcal{E}_{s,0}(0, e^{i\nu_\diamond}) = 0$ for some $\nu_\diamond \in \mathbb{R}$, the approximation of $\lambda_\varepsilon(\nu_\diamond)$ in Theorem 3.17 fails. Since it holds $\det\left(I - e^{-i\nu}\Upsilon_0 \Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right) = \mathcal{E}_{s,0}(0, e^{-i\nu}) = e^{2im\nu}\mathcal{E}_{s,0}(0, e^{i\nu}) = 0$ by Proposition 3.11, we observe that λ_0 has a pole at $\nu_\diamond$. Yet, for ν away from $\nu_\diamond$, the approximation (3.18) is still valid. This leads to the following generalization of Theorem 3.17.

Theorem 3.19. *Suppose* 0 *is a simple zero of* $\mathcal{E}_{f,0}$. *Let* $\delta > 0$ *and denote*

$$\mathcal{N}_\diamond := \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta := \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta). \tag{3.21}$$

Then, for $\varepsilon > 0$ *sufficiently small, there exists for any* $\nu \in \mathcal{S}_\delta$ *a unique root* $\lambda_\varepsilon(\nu)$ *of* $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ *converging to* 0 *as* $\varepsilon \to 0$. *The root* $\lambda_\varepsilon(\nu)$ *is real-valued and satisfies* (3.17), *where* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$ *is given by* (3.18) *and* $C > 0$ *is independent of* ε *and* ν. *In addition, the functions* $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ *and* λ_0 *are analytic, even and* 2π*-periodic. Finally, we have* $\lambda_\varepsilon(0) = 0$ *if* $0 \in \mathcal{S}_\delta$.

The proof of Theorem 3.19 is provided in §5.3. The proof of Theorem 3.17 follows by combining Corollary 3.16 with Theorem 3.19.

3.7 Explicit criteria for spectral stability and instability

Using the spectral approximation results in §3.5 and §3.6, we obtain explicit conditions in terms of simpler, lower-dimensional problems yielding spectral stability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9). Indeed, using Theorem 3.14 we can approximate the roots of the Evans function by the zeros of the reduced Evans function, which is defined in terms of the $2n$- and $2m$-dimensional eigenvalue problems (3.6), (3.8) and (3.9). This simplifies verifying the first spectral stability condition in Corollary 3.8. Then, using Corollary 3.16, we can isolate the most critical part of the spectrum: the curve $\lambda_\varepsilon(\nu)$ attached to the origin. Theorem 3.17

provides a leading-order approximation of $\lambda_\varepsilon(\nu)$ in terms of the $2m$- and $2n$-dimensional variational equations (2.7) and (3.15). This simplifies verifying the spectral stability conditions in Corollary 3.8 for λ close to the origin.

Thus, we readily obtain the following result by combining Theorems 3.14 and 3.17 and Corollary 3.16.

Corollary 3.20. *Suppose the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_{s,0}(0,\gamma) \neq 0$ *for each* $\gamma \in S^1$*;*

iii. $\mathcal{E}_0(\lambda,\gamma) \neq 0$ *for each* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

iv. $\lambda_0''(0) < 0$*,* $\lambda_0(\pi) < 0$ *and* $\lambda_0'(\nu) \neq 0$ *for each* $\nu \in (0,\pi)$*, where* $\lambda_0 \colon \mathbb{R} \to \mathbb{R}$ *is defined by* (3.17).

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable.*

Observe that, if the conditions in Corollary 3.20 are satisfied, then we obtain by Theorem 3.3 nonlinear diffusive stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ as a solution to (1.9) with $\alpha = -\varepsilon^2\lambda_0''(0) + O(\varepsilon^3|\log(\varepsilon)|^5)$.

Regarding instability, Theorems 3.15 and 3.19 yield the following result.

Corollary 3.21. *If one of the following is true:*

i. There exists $\gamma_\diamond \in S^1$ *and* $\lambda_\diamond \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda_\diamond) > 0$ *satisfying* $\mathcal{E}_0(\lambda_\diamond,\gamma_\diamond) = 0$*;*

ii. It holds $\lambda_0(\nu) > 0$ *for some* $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$*, where* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$ *is given by* (3.17) *and* $\mathcal{N}_\diamond$ *is defined in* (3.21).

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally unstable.*

Thus, if one of the conditions in Corollary 3.21 is satisfied, then the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinearly unstable against localized and non-localized perturbations by Theorem 3.5.

We emphasize that the conditions in Corollaries 3.20 and 3.21 can be computed with only the singular limit (2.9) of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ as input. More specifically, one needs understanding of the (adjoint) variational equations about the solutions $\psi_{\mathrm{h}}(x,u_0)$ and $\psi_{\mathrm{s}}(\check{x})$ to systems (2.3) at $u = u_0$ and (2.4), respectively, and of the eigenvalue problems arising when linearizing equations $v_t = D_2 v_{xx} - G(u_0,v,0)$ and $u_t = D_1 u_{\check{x}\check{x}} - H_1(u,0,0)$ about the stationary solutions $v_{\mathrm{h}}(x,u_0)$ and $u_{\mathrm{s}}(\check{x})$, respectively.

In the case $n = 1$ or $m = 1$, the conditions for spectral stability and instability in Corollaries 3.20 and 3.21 can be further simplified – see §3.8. In the case $n = 1$, the first condition in Corollary 3.20 is always satisfied and the third condition only has to be checked for the *slow* Evans function. In the case $m = 1$, the second and fourth condition in Corollary 3.20 are satisfied precisely if the signs of three explicit (integral) expressions are equal.

Remark 3.22. As mentioned in §1.4.1, weak coupling $H_2(u, v) \equiv 0$ is allowed in our spectral analysis. In that case, the integral terms $\mathcal{J}(u)$ and $\mathcal{G}(u, \lambda)$ in (2.5) and (3.11) are identically 0, which implies that $\mathcal{E}_{s,0}$ is only determined by the slow eigenvalue problem (3.9). Therefore, $\mathcal{E}_{s,0}$ is analytic on C_Λ and zeros of $\mathcal{E}_{f,0}$ cannot be canceled by poles of $\mathcal{E}_{s,0}$. We conclude that the spectral stability problem fully splits into slow and fast subproblems with no interaction between them. As a consequence, zeros of $\mathcal{E}_{f,0}$ of positive real part yield spectral (and nonlinear) instability by Proposition 3.21. In particular, in the case $n = 1$, the fast Evans function $\mathcal{E}_{f,0}$ always has a zero in the right half-plane, as we will show in Proposition 3.24.

In addition, $\mathcal{E}_{s,0}(\cdot, 1)$ has a root if and only if the slow eigenvalue problem (3.9) admits a $2\ell_0$-periodic solution. Since we have $\mathcal{J}(u_0) = 0$, it holds $\psi_s(0) = (u_0, 0) = \psi_s(2\ell_0)$ by **(E2)**. Thus, the derivative $\psi_s'(\check{x})$ is a $2\ell_0$-periodic solution to (3.9) at $\lambda = 0$. Hence, the reduced Evans function $\mathcal{E}_0(\cdot, 1)$ has a double root at 0. In particular, in the case $m = 1$, Sturm-Liouville theory [7, Theorems 2.4.2 and 2.5.1] implies that there exists a $\lambda_* > 0$ such that (3.9) has a $2\ell_0$-periodic solution at $\lambda = \lambda_*$, because $u_s'(\check{x})$ vanishes at $\check{x} = \ell_0$. Consequently, $\mathcal{E}_{s,0}(\cdot, 1)$ has a zero in the right half-plane.

Consequently, if we have $H_2(u, v) \equiv 0$, then all periodic pulse solutions $\check{\phi}_{p,\varepsilon}$ to (1.9) are spectrally unstable in the case $n = 1$ or $m = 1$. This motivates the scaling in (1.8). ■

Remark 3.23. Suppose the conditions in Theorem 3.17 are met. Observe that the derivative ψ_s' of the solution ψ_s to (2.4) is a solution to the slow variational equation (2.7). By assumption **(E2)** $\psi_s(\check{x})$ intersects the touch-down manifold $\mathcal{T}_+$ at $\check{x} = 0$ in the point $(u_0, \mathcal{J}(u_0))$. Therefore, we have $\psi_s'(0) = (D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$ and by reversible symmetry it holds $\psi_s'(2\ell_0) = R_s\psi_s'(0) = (-D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$. Thus, we deduce

$$\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0 \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} = \begin{pmatrix} -D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} + (\Upsilon_0\Phi_s(2\ell_0, 0) - I)\begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix},$$

where $\mathfrak{a} := \partial_u\mathcal{J}(u_0)D_1^{-1}\mathcal{J}(u_0) - H_1(u_0, 0, 0) \in \mathbb{R}^m$. Rewriting the latter equation gives

$$\begin{aligned}&\left(1 + e^{-i\nu}\right)\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} \\ &\qquad = e^{-i\nu}\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0\right)^{-1}(\Upsilon_0\Phi_s(2\ell_0, 0) - I)\begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix} + \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix},\end{aligned}$$

for $\nu \in [0, \pi]$. Hence, we obtain the following expression for the quantity $B(\nu)$ in Theorem 3.17

$$B(\nu) = D_1^{-1} \left[\frac{\begin{pmatrix} 0 & I \end{pmatrix}}{1 + e^{-i\nu}} \left(I - e^{-i\nu} \Upsilon_0 \Phi_s(2\ell_0, 0) \Upsilon_0 \right)^{-1} (I - \Upsilon_0 \Phi_s(2\ell_0, 0)) \begin{pmatrix} 0 \\ 2I \end{pmatrix} - I \right] \mathfrak{a},$$

for $\nu \in [0, \pi)$. So, $\mathfrak{a} = 0$ implies $\lambda_0(\nu) = 0$ for any $\nu \in [0, \pi)$ by Theorem 3.17. Therefore, $\mathfrak{a}$ passing through zero, suggests a transition of the critical spectral curve through the imaginary axis. This coincides with a loss of transversality: condition (2.8) in assumption **(E2)** fails if $\mathfrak{a} = 0$ – see §2.2.2. For the case $m = n = 1$, we show in §6.3 that the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ destabilizes through a spatial period doubling bifurcation or sideband instability as $\mathfrak{a}$ passes through zero. ■

3.8 Stability results in lower dimensions

In §3.7 we established explicit conditions yielding spectral stability and instability in terms of the eigenvalue problems (3.6), (3.8) and (3.9) and the variational equations (2.7) and (3.15). In this section we interpret these results in the case $n = 1$ or $m = 1$. Then, the aforementioned systems become 2-dimensional and we can employ techniques tailored for 2-dimensional linear systems to further simplify the spectral (in)stability conditions in Corollaries 3.20 and 3.21.

We proceed as follows. First, we study the slow and fast Evans function and the (leading-order) critical spectral curve $\lambda_0(\nu)$ in the lower-dimensional setting. Subsequently, we interpret the spectral stability conditions in Corollary 3.20 in the case $n = 1$ or $m = 1$. Finally, we present an instability test using parity-type arguments in the regime $n = m = 1$.

Throughout this section we assume without loss of generality $D_1 = 1$ in the case $m = 1$ and $D_2 = 1$ in the case $n = 1$ – see Remark 1.5.

3.8.1 The reduced Evans function

In the case $n = 1$, the homogeneous fast eigenvalue problem (3.6) becomes 2-dimensional. The ordering of the eigenvalues of (3.6), i.e. the roots of the fast Evans function, can be understood with Sturm-Liouville theory. Thus, we obtain the following result.

Proposition 3.24. *Suppose $n = 1$. All zeros of the fast Evans function $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ are real and simple. Moreover, there is precisely one positive zero λ_* of $\mathcal{E}_{f,0}$. Finally, 0 is a root of $\mathcal{E}_{f,0}$.*

Proof. By [60, Theorem 2.3.3] all eigenvalues of the operator $\mathcal{L}_f$, defined in (3.7), are real and simple. In addition, the eigenvalues can be enumerated in strictly decreasing order as $\lambda_N < \ldots < \lambda_0$. The eigenfunction corresponding to $\lambda_i, i = 0, \ldots, N$ has precisely i zeros. Hence, all zeros of $\mathcal{E}_{f,0}$ are real and simple by Proposition 3.10. Furthermore, the derivative

$\partial_x v_{\mathrm{h}}(x, u_0)$ lies in the kernel of $\mathcal{L}_f$. The function $\partial_x v_{\mathrm{h}}(x, u_0)$ has precisely one zero by **(E1)**. So, we derive $\lambda_1 = 0$ and $\lambda_0 > 0$. □

In the case $m = 1$, the slow eigenvalue problem (3.9) becomes 2-dimensional. Since the solution $\psi_{\mathrm{s}}(\check{x})$ to (2.4) crosses the reversible symmetry line $\ker(I - R_s)$ at $\check{x} = \ell_0$ by assumption **(E2)**, it holds $\psi_{\mathrm{s}}(\check{x}) = \psi_{\mathrm{s}}(2\ell_0 - \check{x})$ for each $\check{x} \in [0, 2\ell_0]$. Thus, system (3.9) is R_s-reversible at $\check{x} = \ell_0$, i.e. if $\varphi(\check{x}, \lambda)$ is a solution to (3.9), then so is $\check{x} \mapsto R_s\varphi(2\ell_0 - \check{x}, \lambda)$. Hence, there exists non-trivial solutions $u_+(\check{x}, \lambda)$ and $u_-(\check{x}, \lambda)$ to

$$u_{\check{x}\check{x}} = \left(\frac{\partial H_1}{\partial u}(u_{\mathrm{s}}(\check{x}), 0, 0) + \lambda\right) u, \qquad u \in \mathbb{R}, \tag{3.22}$$

which are symmetric and antisymmetric about ℓ_0, respectively. In particular, at $\lambda = 0$ the derivative $u_{\mathrm{s}}'(\check{x})$ is an antisymmetric solution about ℓ_0 to (3.22). A symmetric solution to (3.22) at $\lambda = 0$ can now be found using Rofe-Beketov's formula [7, Chapter 1.9]. This leads to the following result.

Proposition 3.25. *Suppose $m = 1$. Let $u_+(\check{x}, \lambda)$ and $u_-(\check{x}, \lambda)$ be solutions to* (3.22)*, which are symmetric and antisymmetric about ℓ_0, respectively, and have Wronskian 1. The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is given by*

$$\mathcal{E}_{s,0}(\lambda, \gamma) = \gamma^2 - \mathfrak{t}(\lambda)\gamma + 1,$$

where $\mathfrak{t} \colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$ is the analytic function given by

$$\begin{aligned} \mathfrak{t}(\lambda) &:= \mathrm{Tr}(\Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda)) \\ &= 2\left[\frac{d}{d\check{x}}[u_+(\check{x}, \lambda)u_-(\check{x}, \lambda)](0) - \mathcal{G}(u_0, \lambda)u_+(0, \lambda)u_-(0, \lambda)\right], \end{aligned} \tag{3.23}$$

with $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$, $\Upsilon(u, \lambda)$ and $\mathcal{G}(u, \lambda)$ defined in §3.5.1. In particular, we find

$$\mathfrak{t}(0) = -2\left(1 + 2\mathfrak{a}\mathfrak{b}\right),$$

where

$$\begin{aligned} \mathfrak{a} &:= \mathcal{J}'(u_0)\mathcal{J}(u_0) - H_1(u_0, 0, 0), \\ \mathfrak{b} &:= \mathcal{J}(u_0)\int_0^{\ell_0} \frac{(\partial_u H_1(u_{\mathrm{s}}(\check{x}), 0, 0) + 1)[(u_{\mathrm{s}}'(\check{x}))^2 - (H_1(u_{\mathrm{s}}(\check{x}), 0, 0))^2]}{[(u_{\mathrm{s}}'(\check{x}))^2 + (H_1(u_{\mathrm{s}}(\check{x}), 0, 0))^2]^2} d\check{x} \\ &\quad + \frac{H_1(u_0, 0, 0)}{(\mathcal{J}(u_0))^2 + (H_1(u_0, 0, 0))^2}. \end{aligned} \tag{3.24}$$

Finally, $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ is a γ-eigenvalue for some $\gamma \in S^1$ if and only if it holds $\mathfrak{t}(\lambda) \in [-2, 2]$. In that case, we have $2\mathrm{Re}(\gamma) = \mathfrak{t}(\lambda)$.

Proof. The formula (3.23) follows readily by expanding $\mathcal{E}_{s,0}(\lambda, \gamma)$ as a quadratic polynomial in γ and expressing $\mathcal{T}_s(2\ell_0, 0, \lambda)$ in terms of symmetric and antisymmetric solutions.

Calculating $\mathfrak{t}(0)$ is more elaborate. First, note that the derivative $u_s'(x)$ is a solution to (3.22) at $\lambda = 0$, which is antisymmetric about ℓ_0. By Rofe-Beketov's formula [7, Chapter 1.9] a symmetric solution about ℓ_0 to (3.22) at $\lambda = 0$ is given by

$$\begin{aligned} z(\check{x}) := u_s'(\check{x}) \int_{\check{x}}^{\ell_0} \frac{(\partial_u H_1(u_s(\check{y}), 0, 0) + 1)[(u_s'(\check{y}))^2 - (H_1(u_s(\check{y}), 0, 0))^2]}{[(u_s'(\check{y}))^2 + (H_1(u_s(\check{y}), 0, 0))^2]^2} d\check{y} \\ + \frac{H_1(u_s(\check{x}), 0, 0)}{(u_s'(\check{x}))^2 + (H_1(u_s(\check{x}), 0, 0))^2}. \end{aligned} \tag{3.25}$$

Note that the Wronskian of z and u_s' has value 1. Second, the matrix function $(\partial_u \psi_h(x, u_0) \mid 0)$ is a solution to the fast inhomogeneous problem (3.8) at $\lambda = 0$ and $u = u_0$. This implies $2\mathcal{J}'(u_0) = \mathcal{G}(u_0, 0)$. Putting these two items into (3.23), yields $\mathfrak{t}(0) = -2\,(1 + 2\mathfrak{ab})$. □

In §3.8.3 we present an instability test using parity-type arguments. Therefore, we are interested in the asymptotic behavior of the trace map $\mathfrak{t}(\lambda)$.

Lemma 3.26. *Let $m = 1$. Consider the map $\mathfrak{t}\colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$, defined in (3.23). We have* $\lim_{\lambda\to\infty} \mathfrak{t}(\lambda) = \infty$.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ. Consider system,

$$\begin{aligned} u_{\check{x}} &= \sqrt{\lambda} p, \\ p_{\check{x}} &= \left(\frac{1}{\sqrt{\lambda}} \frac{\partial H_1}{\partial u}(u_s(\check{x}), 0, 0) + \sqrt{\lambda} \right) u, \end{aligned} \qquad (u, p) \in \mathbb{C}^2, \tag{3.26}$$

with evolution $\mathcal{T}_{s1}(\check{x}, \check{y}, \lambda)$. Denote by $\mathcal{T}_{s2}(\check{x}, \check{y}, \lambda)$ the evolution operator of the autonomous system,

$$\begin{aligned} u_{\check{x}} &= \sqrt{\lambda} p, \\ p_{\check{x}} &= \sqrt{\lambda} u, \end{aligned} \qquad (u, p) \in \mathbb{C}^2. \tag{3.27}$$

Proposition 4.1 yields

$$\|\mathcal{T}_{s1}(2\ell_0, 0, \lambda) - \mathcal{T}_{s2}(2\ell_0, 0, \lambda)\| \le \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0}, \quad \lambda > 0. \tag{3.28}$$

On the other hand, the slow eigenvalue problem (3.9) is equivalent to system (3.26) upon performing a coordinate change. Indeed, it holds

$$C_\lambda \mathcal{T}_{s1}(2\ell_0, 0, \lambda) C_\lambda^{-1} = \mathcal{T}_s(2\ell_0, 0, \lambda), \quad C_\lambda := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \quad \lambda > 0. \tag{3.29}$$

We refer to Proposition 5.5 for the fact that, for $\lambda > 0$ sufficiently large, the solution $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8) at $u = u_0$ is exponentially localized with λ-independent decay rates. Hence, $\mathcal{G}(u_0, \lambda)$ remains bounded as $\lambda \to \infty$. Thus, $\mathfrak{t}(\lambda)$ is for $\lambda > 0$ sufficiently large approximated as

$$\left\| \mathfrak{t}(\lambda) - \operatorname{tr}\left(\begin{pmatrix} 1 & 0 \\ \frac{\mathcal{G}(u_0,\lambda)}{\sqrt{\lambda}} & 1 \end{pmatrix} \mathcal{T}_{s2}(2\ell_0, 0, \lambda) \right) \right\| \leq \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0},$$

by (3.28) and (3.29). The latter yields

$$\left\| \mathfrak{t}(\lambda) - e^{2\sqrt{\lambda}\ell_0} \right\| \leq \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0},$$

for $\lambda > 0$ sufficiently large, where we use explicit expressions for the evolution $\mathcal{T}_{s2}(\check{x}, \check{y}, \lambda)$ of system (3.27). We conclude $\mathfrak{t}(\lambda) \to \infty$ as $\lambda \to \infty$. □

Example 3.27. In [114] the spectral stability of spatially periodic pulse patterns is studied, where (1.10) is the generalized Gierer-Meinhardt equation (2.26). Thus, the slow eigenvalue problem (3.9) corresponds to the autonomous system $u_{\check{x}\check{x}} = (\mu + \lambda)u$. The condition $\mathfrak{t}(\lambda) \in [-2, 2]$ in Proposition 3.25 simplifies in that case to

$$2\cosh(2\ell_0\sqrt{\mu + \lambda}) + \frac{\mathcal{G}(u_0, \lambda)\sinh(2\ell_0\sqrt{\mu + \lambda})}{\sqrt{\mu + \lambda}} \in [-2, 2],$$

where $\mathcal{G}(u, \lambda)$ is defined in (3.11) and $\sqrt{\cdot}$ denotes the principal square root. Although derived with a different method, this result agrees with [114, Theorem 1.1.I]. ■

As mentioned in §3.5.1, it is possible to obtain explicit expressions of the principal part of the Laurent series of $\mathcal{E}_{s,0}(\cdot, \gamma)$ at a zero $\lambda \in \mathcal{E}_{f,0}^{-1}(0)$. Because of their complexity the expansions are treated separately in §5.1.2. However, in the case $m = 1$, the expressions simplify significantly. Therefore, it is worthwhile to devote a separate proposition to this case.

Proposition 3.28. *Suppose $m = 1$. Let $\lambda_\diamond$ be a simple zero of $\mathcal{E}_{f,0}$. The singular part of the Laurent expansion at $\lambda = \lambda_\diamond$ of the map $\mathfrak{t}\colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$, defined in Proposition 3.25, is given by*

$$\frac{u(2\ell_0, \lambda_\diamond)}{\lambda - \lambda_\diamond} \int_{-\infty}^{\infty} \partial_v H_2(u_0, v_{\mathrm{h}}(x, u_0)) v_{\lambda_\diamond}(x) dx \int_{-\infty}^{\infty} \tilde{v}_{\lambda_\diamond}(x)^* \frac{\partial G}{\partial u}(u_0, v_{\mathrm{h}}(x, u_0), 0) dx,$$

where $u(\check{x}, \lambda_\diamond)$ is the solution to (3.22) at $\lambda = \lambda_\diamond$ having initial values $u(0) = 0, u'(0) = 1$. Moreover, $v_{\lambda_\diamond}$ is an exponentially localized solution to

$$D_2 v_{xx} = (\partial_v G(u_0, v_{\mathrm{h}}(x, u_0), 0) + \lambda_\diamond)\, v, \quad v \in \mathbb{C}^n, \tag{3.30}$$

and $\tilde{v}_{\lambda_\diamond}(x)$ is an exponentially localized solution to the adjoint problem,

$$D_2 v_{xx} = \left(\partial_v G(u_0, v_{\mathrm{h}}(x, u_0), 0)^* + \overline{\lambda_\diamond}\right) v, \quad v \in \mathbb{C}^n,$$

such that

$$\int_{-\infty}^{\infty} \tilde{v}_{\lambda,\diamond}(x)^* v_{\lambda,\diamond}(x)dx = 1.$$

Proof. The statement is proven in a more general setting in Proposition 5.9. □

3.8.2 The critical spectral curve

In the case $m = 1$, the leading-order approximation (3.18) of the critical spectral curve simplifies. Indeed, the slow variational equation (2.7) becomes 2-dimensional. So, besides the derivative $\psi_s'(\check{x})$, a second, linearly independent solution to (2.7) can be found using Rofe-Beketov's formula. This leads to the following result.

Proposition 3.29. *Let* $m = 1$. *Suppose that* 0 *is a simple zero of* $\mathcal{E}_{f,0}$. *Then, the analytic map* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$, *defined in Theorem 3.19, is given by*

$$\lambda_0(\nu) = \mathfrak{a}\mathfrak{w}\frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}\mathfrak{b}}, \tag{3.31}$$

where $\mathfrak{a}, \mathfrak{b}$ *are defined in* (3.24) *and* $\mathfrak{w}$ *is given by*

$$\mathfrak{w} := -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_\mathrm{h}(x, u_0), 0)^* \psi_{\mathrm{ad},2}(x) x dx}{\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_\mathrm{h}(x, u_0) dx}, \tag{3.32}$$

with $\psi_\mathrm{ad}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ *a non-trivial, exponentially localized solution to* (3.19). *In the case* $n = 1$, *the expression for* $\mathfrak{w}$ *simplifies to*

$$\mathfrak{w} = -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_\mathrm{h}(x, u_0), 0) \partial_x v_\mathrm{h}(x, u_0) x dx}{\int_{-\infty}^{\infty} (\partial_x v_\mathrm{h}(x, u_0))^2 \, dx}. \tag{3.33}$$

Proof. As in the proof of Proposition 3.25 we observe that, at $\lambda = 0$, the derivative $u_s'(\check{x})$ is a solution to (3.22), which is antisymmetric about ℓ_0, and $z(\check{x})$, given by (3.25), is a solution to (3.22), which is symmetric about ℓ_0. In addition, the Wronskian of $z(\check{x})$ and $u_\mathrm{s}'(\check{x})$ equals 1. Expressing the evolution $\Phi_\mathrm{s}(2\ell_0, 0)$ of (2.7) in terms of $\psi_\mathrm{s}'(0)$ and $(z(0), z'(0))$, simplifies the expression for $B(\nu)$ in (3.20) to

$$\begin{aligned} B(\nu) &= -\left[\mathfrak{a} - \begin{pmatrix} 0 & 1 \end{pmatrix} \left(I - e^{-i\nu} \Upsilon_0 \Phi_\mathrm{s}(2\ell_0, 0) \Upsilon_0\right)^{-1} \begin{pmatrix} 2\mathcal{J}(u_0) \\ 0 \end{pmatrix}\right], \\ &= -\left[\mathfrak{a} - \frac{4\mathfrak{a}(1 + \mathfrak{a}\mathfrak{b})e^{-i\nu}}{\mathcal{E}_{s,0}(0, e^{-i\nu})}\right] \end{aligned}$$

where we use $\mathfrak{b} = z(0)$, $\det\left(I - e^{-i\nu}\Upsilon_0\Phi_\mathrm{s}(2\ell_0, 0)\Upsilon_0\right) = e^{2i\nu}\mathcal{E}_{s,0}(0, e^{i\nu}) \neq 0$ and $\psi_\mathrm{s}(0) = (u_0, \mathcal{J}(u_0))$ by **(E2)**. By Proposition 3.25 it holds $e^{i\nu}\mathcal{E}_{s,0}(0, e^{-i\nu}) = 2(\cos(\nu) + 1 + 2\mathfrak{a}\mathfrak{b})$.

Substituting this into the above expression for $B(\nu)$ leads to the desired formula (3.31) for $\lambda_0(\nu) = -\mathfrak{w}B(\nu)$. Finally, in the case $n = 1$, we observe that $(-\partial_x q_h(x, u_0), \partial_x v_h(x, u_0))$ is a solution to equation (3.19) yielding (3.33). This concludes the proof. □

Remark 3.30. Let $m = 1$. Proposition 3.29 indicates that the geometry of the critical spectral curve attached to the origin is to leading order determined by the expressions $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$. In addition, the value of the slow Evans function $\mathcal{E}_{s,0}(\lambda, \gamma)$ at $\lambda = 0$ is fixed by $\mathfrak{a}$ and $\mathfrak{b}$ using Proposition 3.25. Thus, $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ determine the spectral configuration about the origin and play an important role in destabilization processes – see §6.3. We elaborate on the geometric interpretation of these quantities.

As mentioned in §2.2.2, the quantity $\mathfrak{a}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution ψ_s to (2.4) at $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ and, by symmetry, between the take-off curve $\mathcal{T}_-$ and ψ_s at $\psi_s(2\ell_0) = R_s\psi_s(0)$ – see Figure 3.2. If $\mathfrak{a} = 0$, then $\psi_s(\check{x})$ is tangent to the touch-down curve at $\check{x} = 0$.

The quantity $\mathfrak{b}$ depends on the dynamics in the slow reduced system (2.4) only. Since $\psi_s(\ell_0)$ is contained in $\ker(I - R_s)$ by assumption **(E2)**, the vector $\psi_\diamond = (H_1(u_s(\ell_0), 0, 0)^{-1}, 0)$ is a normal to the tangent space of the curve $\psi_s(\check{x})$ at $\check{x} = \ell_0$ such that $\det(\psi_\diamond \mid \psi_s'(\ell_0)) = 1$. Tracking the tangent space along the flow of (2.4) to $\check{x} = \check{x}_0$, the vector $\psi_\diamond$ becomes $\Phi_s(\check{x}_0, \ell_0)\psi_\diamond$. Since system (2.7) is R_s-reversible at $\check{x} = \ell_0$, the first component $z(\check{x})$ of the solution $\Phi_s(\check{x}, \ell_0)\psi_\diamond$ to (2.7) is symmetric at $\check{x} = \ell_0$. Hence, $z(0)$ equals the quantity $\mathfrak{b}$.

Observe that $z(\check{x})$ has precisely one root between two consecutive zeros of $u_s'(\check{x})$, since the derivative of $u_s'(\check{x})/z(\check{x})$ never vanishes between these two zeros of u_s'. Therefore, given that the orbit of ψ_s in the slow reduced system (2.4) crosses the line $p = 0$ at $u = u_\pm$ with $u_- < u_+$, there is precisely one initial value $u_0 = u_s(0) \in (u_-, u_+)$ for which $\mathfrak{b} = 0$ – see Figure 3.2.

The quantity $\mathfrak{w}$ occurs in [92], where one derives asymptotic interaction laws for quasi-stationary pulse solutions to models of the form (1.9). More precisely, one establishes in [92] an ODE, which describes the (leading-order) evolution of the pulse locations over time, assuming existence and smoothness of the quasi-stationary pulse pattern. The pulse locations of our *stationary*, periodic pulse $\check{\phi}_{p,\varepsilon}(\check{x})$ to (1.9) correspond naturally to an equilibrium of this ODE. The quantity $\mathfrak{w}$ occurs as a factor in the linearization of the ODE about this equilibrium – see [92, Section 6.2.1]. Thus, the sign of $\mathfrak{w}$ corresponds to the character of the equilibrium. Loosely speaking, $\mathfrak{w}$ measures the stability of $\check{\phi}_{p,\varepsilon}(\check{x})$ against perturbations of the pulse locations. This relates to the fact that vanishing of $\mathfrak{w}$ corresponds to a transition of the critical spectral curve through the imaginary axis – see §6.3. ■

3.8.3 Criteria for spectral stability and instability

The results in §3.8.1 and §3.8.2 lead to the following simplification of the spectral stability conditions in Corollary 3.20 in the lower-dimensional setting.

Corollary 3.31. *Suppose $m = 1$ and the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_0(\lambda, \gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

iii. The quantities $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{w}$*, defined in* (3.24) *and* (3.32)*, have the same (non-zero) sign.*

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable.*

Moreover, in the case $n = 1$*, conditions i. and ii. above are satisfied if and only* $\mathfrak{c}_+ = 0$*, with*

$$\begin{aligned} \mathfrak{c}_\pm &:= \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{1}{2\pi i} \oint_{\Gamma_R^\pm} \frac{\partial_\lambda \mathcal{E}_{s,0}(\lambda, e^{i\nu})}{\mathcal{E}_{s,0}(\lambda, e^{i\nu})} d\lambda + 1 \right| d\nu \\ &= \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{1}{2\pi i} \oint_{\Gamma_R^\pm} \frac{\mathfrak{t}'(\lambda)}{\mathfrak{t}(\lambda) + 2\cos(\nu)} d\lambda + 1 \right| d\nu, \end{aligned} \tag{3.34}$$

where $\Gamma_R^\pm$ *is the (counter-clockwise) contour in the complex plane consisting of the circle segment* $\{z \in \mathbb{C} : |z \pm R^{-1}| = R, \mathrm{Re}(z) \geq \mp R^{-1}\}$ *and the line joining the points* $iR \mp R^{-1}$ *and* $-iR \mp R^{-1}$.

Proof. Since we have $\mathfrak{a}\mathfrak{b} > 0$, it holds $\mathcal{E}_{s,0}(0, \gamma) \neq 0$ for each $\gamma \in S^1$ by Proposition 3.25. Thus, the first three conditions in Corollary 3.20 are satisfied. Moreover, by Proposition 3.29 we have

$$\lambda_0(\pi) = -\frac{\mathfrak{w}}{\mathfrak{b}}, \qquad \lambda_0''(0) = -\frac{\mathfrak{a}\mathfrak{w}}{2 + 2\mathfrak{a}\mathfrak{b}}, \qquad \lambda_0'(\nu) = -\frac{2\mathfrak{a}(1 + \mathfrak{a}\mathfrak{b})\mathfrak{w}\sin(\nu)}{(1 + 2\mathfrak{a}\mathfrak{b} + \cos(\nu))^2}, \tag{3.35}$$

with $\nu \in \mathbb{R}$. Since $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ are non-zero and have the same sign, the fourth condition in Corollary 3.20 is also satisfied. We conclude that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable.

In the case $n = 1$, 0 is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero λ_* of positive real part by Proposition 3.24. Thus, by Proposition 3.11 the conditions i. and ii. are satisfied if and only if $\mathcal{E}_{s,0}(\lambda, \gamma)$ has precisely one pole of order 1 at $\lambda = \lambda_*$ and no zeros in the closed right half-plane for each $\gamma \in S^1$. Using the argument principle and Proposition 3.25 the latter is the case if and only if $\mathfrak{c}_+ = 0$. □

Thus, in the case $m = n = 1$, we can establish spectral stability by evaluating four expressions $\mathfrak{a}, \mathfrak{b}, \mathfrak{w}$ and $\mathfrak{c}_+$. Even if these four expressions cannot be determined exactly, one can prove spectral stability using rigorously verified computing. To estimate the errors one needs explicit bounds on the solutions $\psi_{\mathrm{h}}(x, u)$ and $\psi_{\mathrm{s}}(\check{x})$ to (2.3) at $u = u_0$ and (2.4) that constitute the singular limit (2.9) and on the functions H_1, H_2, G.

On the other hand, the lower-dimensional setting allows us to test for *instability* using parity-type arguments.

Corollary 3.32. *Let $m = n = 1$. If one of the following is true:*

i. We have $\mathfrak{c}_- \neq 0$, where $\mathfrak{c}_-$ is defined in (3.34)*;*

ii. The quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$, defined in (3.24) *and* (3.32)*, are non-zero and have different signs;*

iii. We have $\mathcal{J}(u_0) = 0$;

iv. It holds $\mathfrak{i} \leq 0$ with

$$\mathfrak{i} := u(2\ell_0, \lambda_*) \int_{-\infty}^{\infty} \frac{\partial H_2}{\partial v}(u_0, v_{\rm h}(x, u_0)) v_{\lambda_*}(x) dx \int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_{\rm h}(x, u_0), 0) v_{\lambda_*}(x) dx, \tag{3.36}$$

where $\lambda_* > 0$ is as in Proposition 3.24, $u(\check{x}, \lambda)$ is the solution to (3.22) *with initial values $u(0) = 0, u'(0) = 1$ and v_{λ_*} is a normalized, exponentially localized solution (having L^2-norm 1) to* (3.30) *at $\lambda_\diamond = \lambda_*$.*

Then, provided $\varepsilon > 0$ is sufficiently small, the periodic pulse solution $\check{\phi}_{\rm p,\varepsilon}$ to (1.9) *is spectrally unstable.*

Proof. First, 0 is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero λ_* of positive real part by Proposition 3.24. Thus, if $\mathfrak{c}_- \neq 0$, then there exists by the argument principle a $\gamma \in S^1$, such that either $\mathcal{E}_{s,0}(\cdot, \gamma)$ has no pole at λ_* or it has a zero $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$. Thus, it holds either $\mathcal{E}_0(\lambda_*, \gamma) = 0$ or $\mathcal{E}_0(\lambda_0, \gamma) = 0$, which implies by Corollary 3.21 that $\check{\phi}_{\rm p,\varepsilon}$ is spectrally unstable.

Next, suppose the non-zero quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ have different signs and $1 + \mathfrak{a}\mathfrak{b} > 0$. Then, the calculations (3.35) show that there exists $\nu \in \mathbb{R}$ such that $\lambda_0(\nu) > 0$. By Corollary 3.21 $\check{\phi}_{\rm p,\varepsilon}$ is spectrally unstable.

Now suppose $1 + \mathfrak{a}\mathfrak{b} \leq 0$ and $\mathfrak{i} > 0$. Then, we have $\mathfrak{t}(0) = -2\,(1 + 2\mathfrak{a}\mathfrak{b}) \geq 2$ by Proposition 3.25. On the other hand, the quantity $\mathfrak{i}$ corresponds to the singular part of the Laurent expansion of $\mathfrak{t}(\lambda)$ at $\lambda = \lambda_*$ by Proposition 3.28. Thus, if we have $\mathfrak{i} > 0$, there exists by the intermediate value theorem a $\lambda_0 \in (0, \lambda_*)$ such that $\mathfrak{t}(\lambda_0) = 2$. Hence, Proposition 3.25 yields $\mathcal{E}_0(\lambda_0, 1) = 0$. Therefore, $\check{\phi}_{\rm p,\varepsilon}$ is spectrally unstable by Corollary 3.21.

Suppose $\mathfrak{i} \leq 0$. In the case $\mathfrak{i} = 0$ we have $\mathcal{E}_0(\lambda_*, \gamma) = 0$ for any $\gamma \in S^1$ by Proposition 3.28. On the other hand, it is shown in Lemma 3.26 that $\mathfrak{t}(\lambda)$ tends to infinity as $\lambda \to \infty$. Therefore, the intermediate value theorem implies that, if $\mathfrak{i} < 0$, then there exists $\lambda_0 \in (\lambda_*, \infty)$ such that $\mathfrak{t}(\lambda_0) = 2$. Hence, by Propositions 3.25 and Corollary 3.21 $\check{\phi}_{\rm p,\varepsilon}$ is spectrally unstable if $\mathfrak{i} \leq 0$.

Finally, in the case $\mathcal{J}(u_0) = 0$, it follows $\mathfrak{a}\mathfrak{b} = -1$ by a direct calculation. Hence, $\check{\phi}_{\rm p,\varepsilon}$ is spectrally unstable by the analysis in the previous three paragraphs. □

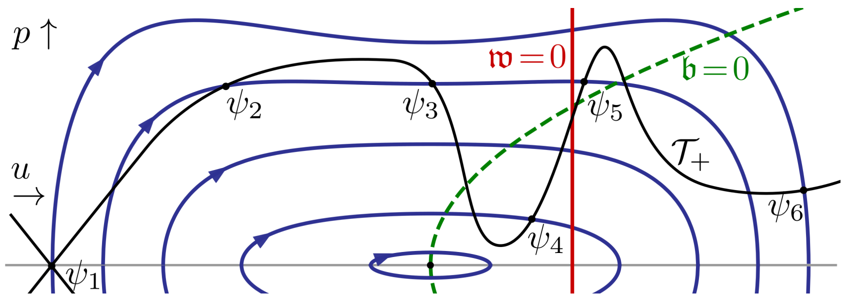


Figure 3.2: Depicted are five orbits of the slow reduced system (2.4) (in purple). The touch-down curve $\mathcal{T}_+$ intersects these orbits transversally at ψ_i, $i = 1, \ldots, 6$. The green dashed line corresponds to the initial values such that $\mathfrak{b} = 0$. We have $\mathfrak{b} > 0$ at ψ_1, ψ_2, ψ_3 and ψ_5 and $\mathfrak{b} < 0$ at ψ_4 and ψ_6. The red line corresponds to initial values with $\mathfrak{w} = 0$. We have $\mathfrak{w} > 0$ at ψ_1, ψ_2, ψ_3 and ψ_4 and $\mathfrak{w} < 0$ at ψ_5 and ψ_6. Finally, we have $\mathfrak{a} < 0$ at ψ_1, ψ_3 and ψ_6 and $\mathfrak{a} > 0$ at ψ_2, ψ_4 and ψ_5. The periodic pulse solutions touching-down at ψ_1, ψ_3, ψ_4 and ψ_5 are spectrally unstable by Corollary 3.32. The solutions touching down at ψ_2 and ψ_6 are potentially spectrally stable.

We stress that the value of $\mathfrak{a}, \mathfrak{b}$, $\mathcal{J}(u_0)$ and $\mathfrak{w}$ depends only on the initial value $u_0 = u_s(0)$ of the solution ψ_s to the slow reduced system (2.4) and can directly be read off from the phase plane of (2.4) – see Figure 3.2.

Remark 3.33. If the periodic pulse $\check{\phi}_{p,\varepsilon}$ approaches a homoclinic limit, then the verification of the conditions in Corollaries 3.31 and 3.32 simplifies significantly – see §6.4.6. In particular, we can test for spectral (in)stability by approximating the quantities $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_-, \mathfrak{i}$ and $\mathfrak{w}$ in the long-wavelength limit. ■

Example 3.34. In [114] the spectral stability of stationary, spatially periodic pulse solutions is studied in the generalized Gierer-Meinhardt equation (2.26). The slow variational equation (2.7) corresponds in this setting to the autonomous equation $u_{\check{x}\check{x}} = \mu u$. The v-component of the homoclinic solution $\psi_h(x, u_0)$ to system (2.3) at $u = u_0$ is given by

$$v_h(x, u_0) = u_0^{-\frac{\alpha_2}{\beta_2 - 1}} w_h(x), \quad w_h(x) := \left(\frac{\beta_2 + 1}{2} \operatorname{sech}^2\left(\frac{(\beta_2 - 1)\, x}{2}\right)\right)^{\frac{1}{\beta_2 - 1}}.$$

Thus, using integration by parts, we calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ in Proposition 3.29,

$$\mathfrak{a} = \mathcal{J}(u_0)\mathcal{J}'(u_0) - \mu u_0, \quad \mathfrak{b} = \frac{\cosh^2\left(\ell_0 \sqrt{\mu}\right)}{4\mu u_0}, \quad \mathfrak{w} = -\frac{\alpha_2 \int_{-\infty}^{\infty} w_h(x)^{\beta_2 + 1} dx}{u_0\, (\beta_2 + 1) \int_{-\infty}^{\infty} \left(w_h'(x)\right)^2 dx},$$

where $\mathcal{J}\colon (0,\infty) \to \mathbb{R}$ is given by

$$\mathcal{J}(u) = \frac{u^{\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2-1}}}{2} \int_{-\infty}^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx. \tag{3.37}$$

It holds $\mathfrak{bw} > 0$, since we have $\beta_{1,2} > 1, \alpha_2 < 0$ and $\mu > 0$. In addition, the signs of $\mathfrak{aw}$ and $\mathfrak{ab}$ are equal to the sign of

$$\frac{\mathfrak{a}}{u_0} = \left(\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2 - 1}\right)\left(u_0^{\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2-1} - 1} \int_{-\infty}^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx\right)^2 - \mu.$$

Thus, the sign of $\frac{\mathfrak{a}}{u_0}$ determines whether condition iii. in Corollary 3.31 is satisfied. The quantity $\frac{\mathfrak{a}}{u_0}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution ψ_{s} – see Remark 3.30.

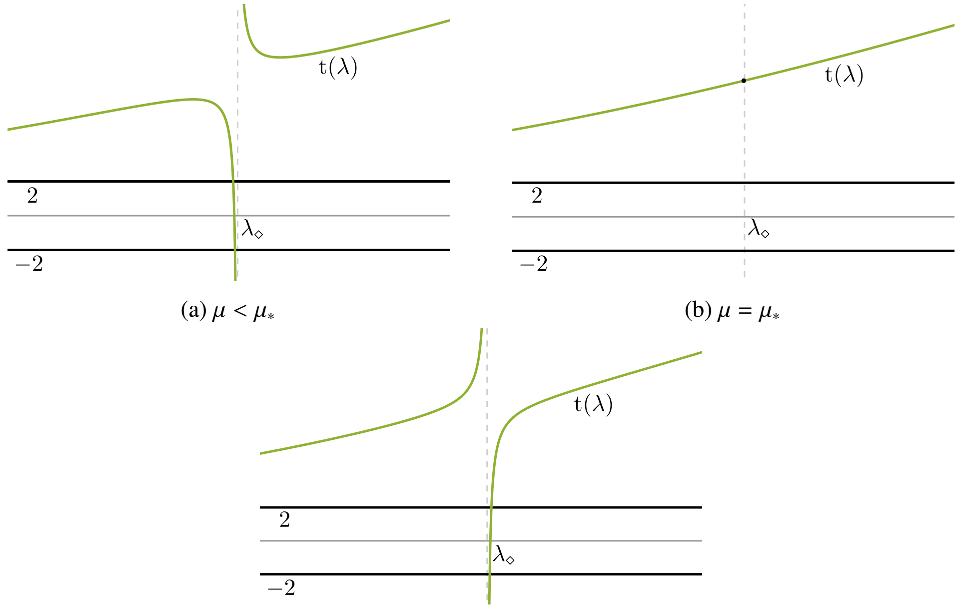
One can verify that the leading-order expression (3.31) of the critical spectral curve coincides with the one in [114] derived with a different method – see §1.2. ■

3.8.4 A closer look at zero-pole cancelation

Proposition 3.28 shows that for $m = 1$ the slow Evans function $\mathcal{E}_{s,0}(\cdot, \gamma)$ has a removable singularity at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$ if and only if one of the identities (3.12), (3.13) holds true or there exists a non-trivial solution to (3.22) at $\lambda = \lambda_\diamond$ with boundary values $u(0) = 0 = u(2\ell_0)$. The set of $\lambda_\diamond \in \mathbb{C}$ for which (3.12) or (3.13) holds true will in general be discrete, since the involved expressions are analytic in $\lambda_\diamond$. Moreover, [128, Theorem 4.3.1-6] shows that this is also the case for the set of $\lambda_\diamond \in C_\Lambda$ for which the boundary value problem (3.22), $u(0) = 0 = u(2\ell_0)$ admits a non-trivial solution. Hence, zero-pole cancelation is a robust phenomenon in the absence of additional structure (such as the translational invariance at $\lambda = 0$ mentioned in Remark 3.13).

Being robust, zero-pole cancelation can still fail in one-parameter families. Suppose equation (1.9) depends on a real parameter μ and $\mathcal{E}_{f,0}$ has a simple zero $\lambda_\diamond$ with $\mathrm{Re}(\lambda_\diamond) > 0$, independent of μ. Denote by $\mathfrak{i}(\mu)$ the singular part of the Laurent expansion at $\lambda = \lambda_\diamond$ of $\mathfrak{t}_\mu(\lambda)$ – see Propositions 3.25 and 3.28. Assume there is a value $\mu_* \in \mathbb{R}$ such that $\mathfrak{i}(\mu_*) = 0$ and $\partial_\mu \mathfrak{i}(\mu_*) \neq 0$. Then, for any $\gamma \in S^1$, $\mathcal{E}_{0,\mu_*}(\lambda_\diamond, \gamma) = 0$ and $\mathcal{E}_{0,\mu}(\lambda_\diamond, \gamma) \neq 0$ for any $\mu \neq \mu_*$ close to μ_*.

The transition of μ through a point μ_* may seem like a blue sky catastrophe, which makes the pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ 'suddenly' spectrally unstable. However, such a transition from cancelation to non-cancelation is caused by unstable spectrum moving through the point $\lambda_\diamond$. This can be seen by noting that the there exists a neighborhood $N \subset \mathbb{R}$ of $\lambda_\diamond$ such that $\mathfrak{t}_\mu(N)$ covers the whole real line as μ approaches μ_* – see Figure 3.3. In particular, $\mathfrak{t}_\mu(N)$ covers the interval $[-2, 2]$ as $\mu \to \mu_*$. Thus, by Proposition 3.25 there is a branch of unstable spectrum moving through the point $\lambda_\diamond$. We remark that the orientation of the spectral curve changes in this proces – see Figure 3.3.



(a) $\mu < \mu_*$ (b) $\mu = \mu_*$

(c) $\mu > \mu_*$

Figure 3.3: The trace function $\mathrm{t}_\mu(\lambda)$ about $\lambda_\diamond$.

Example 3.35. We provide an example where zero-pole cancelation fails. Consider the Gierer-Meinhardt equation (2.26), where $\alpha_2 \neq 0$ and $\mu < 0$. We emphasize that in this case the slow reduced system (2.4) is linear of center type. This differs from the 'standard' Gierer-Meinhardt setting considered in [21, 25, 50, 114, 123], where $\mu > 0$ and the slow reduced system is linear of saddle type.

Let $u_0 > 0$. Note that (2.26) satisfies **(S1)**, **(S2)** and **(E1)** with $v_\mathrm{h}(x, u_0) > 0$ for all $x \in \mathbb{R}$. Take $u_1 < 0$ such that $\mu u_1^2 = \mathcal{J}(u_0)^2 + \mu u_0^2$, where $\mathcal{J}\colon (0, \infty) \to \mathbb{R}$ is as in (3.37). Then, assumption **(E2)** is satisfied with $\psi_\mathrm{s}(\check{x})$ the solution to the Hamiltonian system (2.4) with initial condition $\psi_\mathrm{s}(0) = (u_0, \mathcal{J}(u_0))$. Hence, Theorem 2.3 implies that, for $\varepsilon > 0$ sufficiently small, there exists a $2\ell_\varepsilon$-periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(x)$ to (2.26). Moreover, it holds by the Hamiltonian nature of system (2.4)

$$\ell_\varepsilon \to \ell_0 = \ell_0(\mu) := \frac{\pi}{2} + \sin^{-1} \frac{u_0}{\sqrt{\frac{\mathcal{J}(u_0)^2}{\mu} + u_0^2}}, \qquad \text{as } \varepsilon \to 0.$$

In [114, Lemma 3.3] it is shown that $\lambda_\diamond = 1/4(\beta_2 + 1)^2 - 1 > 0$ is the positive zero of the fast Evans function $\mathcal{E}_{f,0}$. Note that both $\partial_v H_2(u_0, v_\mathrm{h}(x, u_0))$ and $\partial_u G(u_0, v_\mathrm{h}(x, u_0), 0)$ are strictly negative for all $x \in \mathbb{R}$. Moreover, the v-component of any non-trivial solution to (3.6) at $\lambda = \lambda_\diamond$ has no zeros by the proof of Proposition 3.24. Therefore, identities (3.12) and (3.13) are not satisfied. Now assume $\mu > \lambda_\diamond$. The solution to (3.22) at $\lambda = \lambda_\diamond$ with initial values $u(0) = 0, u'(0) = 1$ is given by,

$$u(\check{x}, \lambda_\diamond) = \frac{\sin(\sqrt{\mu - \lambda_\diamond}\,\check{x})}{\sqrt{\mu - \lambda_\diamond}},$$

where $\sqrt{\cdot}$ denotes the principal square root. Clearly, it holds $u(2\ell_0, \lambda_\diamond) = 0$ if and only if

$$\mu = \lambda_\diamond + \left(\frac{k\pi}{2\ell_0(\mu)}\right)^2, \tag{3.38}$$

for some $k \in \mathbb{Z}_{\geq 1}$. Since $\ell_0(\mu) \in (\pi/2, \pi)$ for every $\mu > 0$, equation (3.38) will have a solution $\mu = \mu_k > \lambda_\diamond$ for every $k \in \mathbb{Z}_{\geq 1}$. We conclude with the aid of Proposition 3.28 that, if $\mu = \mu_k$ for some $k \in \mathbb{Z}_{\geq 1}$, then $\mathcal{E}_{s,0}(\lambda, \gamma)$ has a removable singularity at $\lambda = \lambda_\diamond$ and it holds $\mathcal{E}_0(\lambda_\diamond, \gamma) = 0$ for any $\gamma \in S^1$. ∎

3.9 Stability in the slowly nonlinear toy problem

In this section, we derive explicit expressions for the reduced Evans function $\mathcal{E}_0(\lambda, \gamma)$ and the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$, defined in (3.24) and (3.32), in the toy problem (2.27). Then, Corollaries 3.31 and 3.32 can be employed to prove spectral stability or instability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ constructed in §2.5.

For the toy problem (2.27), the homogeneous fast eigenvalue problem reads,

$$\begin{aligned} v_x &= q, \\ q_x &= \left(1 - 3\operatorname{sech}^2\left(\tfrac{1}{2}x\right) + \lambda\right) v, \end{aligned} \qquad (v, q) \in \mathbb{R}^2, \tag{3.39}$$

where we used the expressions for $v_{\mathrm{h}}(x, u)$ derived in §2.5. Let $\Lambda = -1$. The solutions to (3.39) can be found using Legendre functions – see [120, Section 3.3]. Thus, we establish two non-trivial solutions $\varphi_\pm(x, \lambda)$ to (3.39), whose v-components are given by

$$\begin{aligned} v_\pm(x, \lambda) = e^{\mp\sqrt{\lambda+1}x} \Bigg(& 4\lambda \frac{\left(\sqrt{\lambda+1} + 3\right) e^{\pm x} + \sqrt{\lambda+1} - 3}{e^{\pm x} + 1} \\ & + 15\left(e^{\pm x} - 1\right) \frac{\left(\sqrt{\lambda+1} + 1\right) e^{\pm 2x} - \sqrt{\lambda+1} + 1}{\left(e^{\pm x} + 1\right)^3} \Bigg), \end{aligned} \qquad \lambda \in C_\Lambda,$$

where $\sqrt{\cdot}$ denotes the principal square root. One readily observes $\lim_{x \to \pm\infty} \varphi_\pm(x, \lambda) = 0$. Hence, the fast Evans function $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ is given by the Wronskian of $\varphi_+(x, \lambda)$ and $\varphi_-(x, \lambda)$:

$$\mathcal{E}_{f,0}(\lambda) := 2\lambda\sqrt{\lambda+1}\left(16\lambda^2 - 8\lambda - 15\right),$$

and we find $\mathcal{E}_{f,0}$ has simple roots $-\frac{3}{4}, 0$ and $\frac{5}{4}$. The inhomogeneous fast eigenvalue problem reads

$$\begin{aligned} v_x &= q, \\ q_x &= \left(1 - 3\mathrm{sech}^2\left(\tfrac{1}{2}x\right) + \lambda\right)v + \tfrac{9}{4}\mathrm{sech}^4\left(\tfrac{1}{2}x\right)f'(u), \end{aligned} \qquad (v,q) \in \mathbb{R}^2,$$

where $u > 0$ and $\lambda \in C_\Lambda$. For any $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ its unique solution $\mathcal{X}_{in}(x,u,\lambda)$ can be found using variation of constants. So, the v-component $\mathcal{V}_{in}(x,u,\lambda)$ of $\mathcal{X}_{in}(x,u,\lambda)$ reads,

$$\begin{aligned} \mathcal{V}_{in}(x,u,\lambda) &= f'(u)\frac{v_+(x,\lambda)I(x,\lambda) + v_-(x,\lambda)I(-x,\lambda)}{\mathcal{E}_{f,0}(\lambda)}, \\ I(x,\lambda) &:= -\frac{9}{4}\int_{-\infty}^{x} v_-(y,\lambda)\mathrm{sech}^4\left(\tfrac{1}{2}y\right)dy. \end{aligned}$$

We emphasize that the integral $I(x,\lambda)$ can be evaluated using hypergeometric functions. Yet, the resulting expressions are quite lengthy, so we decide not to provide these. Using the formula for ψ_s in §2.5, we state the slow eigenvalue problem,

$$\begin{aligned} u_{\check{x}} &= p, \\ p_{\check{x}} &= \left(\lambda + \mu\cos\left[2\mathrm{Am}\left(-k\sqrt{\mu}(\check{x}-c), k^{-2}\right) + \pi\right]\right)u \qquad (u,p) \in \mathbb{R}^2, \check{x} \in [0, 2\ell_0], \\ &= \left(\lambda + 2\mu k^2\mathrm{sn}^2\left[\sqrt{\mu}(\check{x}-c), k^2\right]\right)u, \end{aligned} \tag{3.40}$$

where $\lambda \in C_\Lambda$, $k \in (0,1)$, $c \in \mathbb{R}$ with $|c| < K(k)\mu^{-1/2}$ and $\ell_0 = \ell_0(k,l,c,\mu)$ is defined in (2.31). Here, $K(k)$ is the Jacobi complete integral of the first kind, $\mathrm{Am}(\check{x},k)$ denotes the Jacobi amplitude function and $\mathrm{sn}(\check{x},k)$ is one of the Jacobi elliptic functions. Equation (3.40) is known as Lamé's equation and can be solved explicitly. For $\lambda \in \mathbb{C} \setminus \{0\}$, this is done by first substituting $z = \mathrm{sn}(\sqrt{\mu}(\check{x}-c), k^2)$ and then applying differential Galois theory [65]. This yields two solutions $\psi_\pm(\check{x},\lambda;k,c,\mu)$ to (3.40), which have u-components,

$$u_\pm(\check{x},\lambda;k,c,\mu) := \frac{\sqrt{\frac{\lambda}{k^2\mu} + \mathrm{cn}^2\left(\sqrt{\mu}(\check{x}-c), k^2\right)}}{\exp\left[\pm\sqrt{\frac{\lambda(\lambda-\mu+\mu k^2)}{\mu(\lambda+k^2\mu)}}\Pi\left(\frac{k^2\mu}{\lambda+k^2\mu}, -\mathrm{am}\left(\sqrt{\mu}(\check{x}-c), k^2\right), k^2\right)\right]},$$

where $\Pi(\check{x},\phi,k)$ denotes Legendre's incomplete elliptic integral of the third kind and $\mathrm{cn}(\check{x},k)$ is one of the Jacobi elliptic functions. Thus, we have obtained all ingredients to explicitly calculate the slow Evans function $\mathcal{E}_{s,0}\colon \left[\mathbb{C} \setminus \mathcal{E}_{f,0}^{-1}(0)\right] \times \mathbb{C} \to \mathbb{C}$. Indeed, we obtain

$$\mathcal{E}_{s,0}(\lambda,\gamma) = \det\left(\Upsilon(u_0,\lambda)X(2\ell_0,\lambda)X(0,\lambda)^{-1} - \gamma I\right),$$

with $u_0 = u_0(k,c,\mu) := u_s(0;k,c,\mu)$ – see equation (2.30) – and

$$\begin{aligned} X(\check{x},\lambda) = X(\check{x},\lambda;k,c,\mu) &:= \left(\ \psi_+(\check{x},\lambda;k,c,\mu)\ \middle|\ \psi_-(\check{x},\lambda;k,c,\mu)\ \right), \\ \Upsilon(u,\lambda) = \Upsilon(u,\lambda;\nu_2,\nu_3) &:= \begin{pmatrix} I & 0 \\ \mathcal{G}(u,\lambda;\nu_2,\nu_3) & I \end{pmatrix}, \\ \mathcal{G}(u,\lambda;\nu_2,\nu_3) &:= f(u)\int_{-\infty}^{\infty}\left(3\nu_2\mathrm{sech}^2\left(\tfrac{1}{2}x\right) + \tfrac{27}{4}\nu_3 f(u)\mathrm{sech}^4(\tfrac{1}{2}x)\right)\mathcal{V}_{in}(x,u,\lambda)dx. \end{aligned}$$

So, by multiplying the expressions for the fast and slow Evans functions above, one obtains an explicit reduced Evans function $\mathcal{E}_0(\lambda, \gamma) = -\gamma \mathcal{E}_{s,0}(\lambda, \gamma)\mathcal{E}_{f,0}(\lambda)$.

Our next step is to calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ in the toy problem (2.27). The slow variational equation about $\psi_s(\check{x}; k, c, \mu)$ – see equation (2.30) – equals the slow eigenvalue problem (3.40) at $\lambda = 0$. Naturally, one of the solutions to (3.40) at $\lambda = 0$ is given by the derivative $\psi_1(\check{x}; k, c, \mu) = \psi_s'(\check{x}; k, c, \mu)$, whose u-component reads

$$u_1(\check{x}; k, c, \mu) = -2k\sqrt{\mu}\mathrm{dn}\left(k\sqrt{\mu}(\check{x} - c), k^{-2}\right),$$

where $\mathrm{dn}(\check{x}, k)$ is one of the Jacobi elliptic functions. Note that $u_1(\check{x})$ is antisymmetric about ℓ_0. A second solution $\psi_2(\check{x}; k, l, c, \mu)$, having a symmetric u-component about $\ell_0 = \ell_0(k, l, c, \mu)$, is established via Rofe-Beketov's formula [7, Chapter 1.9]. We gauge ψ_2 such that the Wronskian of ψ_2 and ψ_1 equals 1. The u-component of ψ_2 is given by

$$\begin{aligned}
u_2(\check{x}; k, l, c, \mu) = &\Big[\mathrm{cn}\left(\sqrt{\mu}(\check{x} - c), k^2\right)\Big[(k^2 - 1)\sqrt{\mu}(\check{x} - c) + E\left(\mathrm{am}\left(\sqrt{\mu}(\check{x} - c), k^2\right), k^2\right)\Big] \\
&- \mathrm{dn}\left(\sqrt{\mu}(\check{x} - c), k^2\right)\mathrm{sn}\left(\sqrt{\mu}(\check{x} - c), k^2\right) - \alpha(k, l, \mu)u_1(\check{x}; k, c, \mu)\Big]\frac{1}{2k(k^2 - 1)\mu}, \\
\alpha(k, l, \mu) := &\ \frac{(1 - k^2)(2l + 1)K(k^2) - E\left((l + \frac{1}{2})\pi, k^2\right)}{2k\sqrt{\mu}},
\end{aligned}$$

where $E(\check{x}, k)$ is the Jacobi complete integral of the second kind. Having established the solutions to the slow variational problem, we calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$:

$$\begin{aligned}
\mathfrak{a} &= \frac{72}{25} f(u_0)^3\left(5\nu_2 + 6\nu_3 f(u_0)\right)\left(5\nu_2 + 9\nu_3 f(u_0)\right) f'(u_0) - \mu\sin(u_0), \\
\mathfrak{b} &= u_2(0; k, l, c, \mu), \qquad \mathfrak{w} = \frac{2f'(u_0)}{f(u_0)}.
\end{aligned}$$