Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Chapter 2

Existence analysis

2.1 Introduction

In this thesis we are interested in of stationary, spatially periodic pulse solutions to the class of reaction-diffusion systems (1.10), where we assume that the interaction terms satisfy **(S1)**. Such solutions are constant in time and they are periodic and symmetric in space. In addition, the v-components exhibit spatially localized pulses, whereas the u-components are non-localized. We refer to Figure 1 for a plot of the pulse profile in the case $m = n = 1$.

In this chapter we focus on the construction of such solutions. Finding stationary solutions to (1.10) is equivalent to solving the singularly perturbed ordinary differential equation,

$$\begin{aligned} D_1 u_x &= \varepsilon p, \\ p_x &= \varepsilon H_1(u, v, \varepsilon) + H_2(u, v), \\ D_2 v_x &= q, \\ q_x &= G(u, v, \varepsilon), \end{aligned} \qquad (u, p, v, q) \in \mathbb{R}^{2(m+n)}, \tag{2.1}$$

which is R-reversible, where $R\colon \mathbb{R}^{2(m+n)} \to \mathbb{R}^{2(m+n)}$ is the reflection in the space $p = q = 0$. Taking the limit $\varepsilon \to 0$ in properly scaled versions of (2.1) yields slow and fast reduced systems. By piecing together orbit segments of these reduced systems in such a way that they form a closed loop, one obtains a so-called singular periodic orbit. Although this singular periodic orbit is not an actual solution to (2.1), one can prove that (under certain conditions) an actual periodic solution to (2.1) arises from the singular one, provided $\varepsilon > 0$ is sufficiently small.

In this chapter we perform a slow-fast decomposition of (2.1) and construct a singular periodic orbit from the slow and fast reduced systems. Next, we use geometric singular perturbation theory [34, 54, 57] to study the dynamics of system (2.1) in the neighborhood of the singular orbit. Then, we have the ingredients to prove the existence of an actual periodic pulse solution to (2.1) in the vicinity of the singular one. The R-reversibility of system (2.1) plays an essential role in the proof. Therefore, both the periodic pulse solution and its singular limit naturally respect the R-reversibility of system (2.1). Since the stability analysis in Chapter 5 relies crucially on how the periodic pulse solutions are approximated by the singular limit structure, we provide detailed (pointwise) estimates along with the existence result. Finally, we apply the existence result to an explicit slowly nonlinear toy model.

2.2 The singular limit

2.2.1 Slow-fast decomposition

We perform a slow-fast decomposition of the singularly perturbed equation (2.1). Fast and slow reduced systems arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of (2.1). First, if set $\varepsilon = 0$ in (2.1), then the dynamics is given by the *fast reduced system*,

$$\begin{aligned} u_x &= 0, \\ p_x &= H_2(u,v), \\ D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^{2(m+n)}, \tag{2.2}$$

System (2.2) is governed by the family of $2n$-dimensional systems,

$$\begin{aligned} D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (v,q) \in \mathbb{R}^{2n}, \tag{2.3}$$

parameterised over $u \in U$. Note that (2.3) is R_f-reversible, where $R_f \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the reflection in the space $q = 0$. Moreover, we observe that *the slow manifold*,

$$\mathcal{M} := \{(u,p,0,0) : u \in U, p \in \mathbb{R}^m\},$$

consists entirely of equilibria of (2.2) by assumption **(S1)**. When $\varepsilon > 0$, the manifold $\mathcal{M}$ consists no longer of equilibria, but remains invariant for the dynamics of (2.1). The flow restricted to $\mathcal{M}$ is of order $\mathcal{O}(\varepsilon)$. In the spatial scale $\check{x} = \varepsilon x$, the dynamics of (2.1) on $\mathcal{M}$ is to leading order governed by the *slow reduced system*,

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= H_1(u,0,0), \end{aligned} \qquad (u,p) \in \mathbb{R}^{2m}. \tag{2.4}$$

Note that system (2.4) is R_s-reversible, where $R_s \colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is the reflection in the space $p = 0$.

Although the fast and slow reduced systems (2.2) and (2.4) are simpler, lower-dimensional systems, enough information can be obtained from them to determine the leading-order dynamics of the full system (2.1) close to the slow manifold $\mathcal{M}$ for $0 < \varepsilon \ll 1$ – see §2.3.

2.2.2 Construction of the singular periodic pulse

In this section we construct a singular periodic orbit by concatenating solutions of the fast and slow reduced systems (2.2) and (2.4) in such a way that they form a closed loop. The singular orbit consist of a pulse satisfying the fast reduced system (2.2) and a segment on the invariant slow manifold $\mathcal{M}$, satisfying the slow reduced system (2.4). We emphasize that such a singular orbit is not a solution to (2.1). However, when the singular orbit satisfies certain conditions, we will prove that an actual periodic pulse solution lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small – see §2.4.

The first ingredient for constructing the singular periodic orbit is the existence of a pulse solution in the fast reduced system (2.2). This is ensured by the following assumption.

(E1) ***Existence of a pulse solution to the fast reduced system***
There exists $u_\diamond \in U$ such that system (2.3) has for $u = u_\diamond$ a solution $\psi_{\mathrm{h}}(x, u_\diamond) = (v_{\mathrm{h}}(x, u_\diamond), q_{\mathrm{h}}(x, u_\diamond))$ homoclinic to the hyperbolic saddle 0. The stable manifold $W^s_{u_\diamond}(0)$ intersects the space $\ker(I - R_f)$ transversely in the point $\psi_{\mathrm{h}}(0, u_\diamond)$.

Remark 2.1. In the terminology of [118] homoclinics that lies in the transverse intersection of $W^s_{u_\diamond}(0)$ and $\ker(I - R_f)$ are called *elementary*. In particular, any non-degenerate homoclinic solution is elementary by [118, Lemma 4]. We emphasize that in the case $n = 1$ any homoclinic solution to (2.3) is elementary. ■

Since transverse intersections are robust under perturbations, assumption **(E1)** implies the existence of an open neighborhood $U_{\mathrm{h}} \subset U$ of $u_\diamond$ such that for every $u \in U_{\mathrm{h}}$ there exists a solution $\psi_{\mathrm{h}}(x, u)$ to (2.3), which is homoclinic to the hyperbolic saddle 0, such that $W^s_u(0) \pitchfork \ker(I - R_f) = \{\psi_{\mathrm{h}}(0, u)\}$. The homoclinics $\psi_{\mathrm{h}}(x, u)$ yield solutions,

$$\phi_{\mathrm{h}}(x, u) := \left(u, \int_0^x H_2(u, v_{\mathrm{h}}(z, u))dz, v_{\mathrm{h}}(x, u), q_{\mathrm{h}}(x, u)\right), \quad u \in U_{\mathrm{h}},$$

to the fast reduced system (2.2), which are homoclinic to $\mathcal{M}$. The homoclinics $\phi_{\mathrm{h}}(x, u)$ take off and touch down on the points $\lim_{x\to\pm\infty} \phi_{\mathrm{h}}(x, u) \in \mathcal{M}$. We define the mapping $\mathcal{J} \colon U_{\mathrm{h}} \to \mathbb{R}^m$ by

$$\mathcal{J}(u) = \int_0^\infty H_2(u, v_{\mathrm{h}}(z, u))dz. \tag{2.5}$$

The m-dimensional graphs $\mathcal{T}_\pm := \{(u, \pm\mathcal{J}(u)) : u \in U_{\mathrm{h}}\}$ on $\mathcal{M}$ are the so-called *take-off and touch-down manifolds*. Since 0 is a hyperbolic saddle in (2.3), there exists constants $C, \mu_{\mathrm{h}} > 0$ such that

$$\|\phi_{\mathrm{h}}(\pm x, u) - (u, \pm\mathcal{J}(u), 0, 0)\| \le Ce^{-\mu_{\mathrm{h}} x}, \quad x \ge 0, u \in U_{\mathrm{h}}. \tag{2.6}$$

The manifolds $\mathcal{T}_\pm$ allow us to piece the pulse solutions ϕ_{h} to solutions that lie in $\mathcal{M}$ in order to obtain a singular periodic orbit – see Figure 2.1. Therefore, we shift our attention to the slow reduced system (2.4). Recall that (2.4) is R_s-reversible. In addition, since (2.3) is

R_f-reversible, it holds $R_s[\mathcal{T}_+] = \mathcal{T}_-$. Therefore, to establish a connection between the take-off and touch-down manifolds $\mathcal{T}_\pm$, it is sufficient to find a solution to (2.4) that starts on the touch-down manifold $\mathcal{T}_+$ and crosses $\ker(I - R_s)$ at some point. This is the content of our next assumption.

(E2) ***Existence of connecting orbit in slow reduced system***
There exists a solution $\psi_s(\check{x}) = (u_s(\check{x}), p_s(\check{x}))$ to system (2.4) with initial condition $\psi_s(0) \in \mathcal{T}_+$ and $\psi_s(\ell_0) \in \ker(I - R_s)$ for some $\ell_0 > 0$. Moreover, let $\Phi_s(\check{x}, \check{y})$ be the evolution operator of the associated variational equation,

$$\varphi_{\check{x}} = \mathcal{A}_s(\check{x})\varphi, \quad \varphi \in \mathbb{R}^{2m}, \tag{2.7}$$

with

$$\mathcal{A}_s(\check{x}) := \begin{pmatrix} 0 & D_1^{-1} \\ \partial_u H_1(u_s(\check{x}), 0, 0) & 0 \end{pmatrix}.$$

Denote $u_0 := u_s(0)$, $H_1(u_0, 0, 0) = (h_1, \ldots, h_m)$ and for $i, j \in \{1, \ldots, m\}$ by A_{ij} the $(m \times m)$-submatrix of

$$\Phi_s(\ell_0, 0)\begin{pmatrix} I \\ \partial_u J(u_0) \end{pmatrix},$$

containing rows $\{i, m+1, \ldots, 2m\} \setminus \{m+j\}$. There exists $i_* \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{m} (-1)^j h_j \det(A_{i_* j}) \neq 0. \tag{2.8}$$

By concatenating the orbits of ψ_s and ϕ_h, we obtain the *singular periodic pulse*,

$$\phi_{p,0} := \{(\psi_s(\check{x}), 0) : \check{x} \in (0, 2\ell_0)\} \cup \{\phi_h(x, u_0) : x \in \mathbb{R}\} \subset \mathbb{R}^{2(m+n)}, \tag{2.9}$$

consisting of a pulse satisfying the fast reduced system (2.2) and an orbit segment on the slow manifold. We emphasize that $\phi_{p,0}$ is C^1, except at the two *corners* $(u_0, \pm\mathcal{J}(u_0), 0, 0) = (u_s(0), \pm p_s(0), 0, 0)$. Eventually, our goal is to construct a periodic pulse solution to (2.1) in the vicinity of the singular orbit (2.9), provided $0 < \varepsilon \ll 1$. Therefore, we need some robustness of the structure (2.9) under perturbations. Robustness of the pulse ϕ_h is ensured by the transversality condition in **(E1)**. The orbit ψ_s in the slow system (2.4) persists by regular perturbation arguments on the finite interval $[0, 2\ell_0]$. Lastly, to ensure persistence of the connections between $(\psi_s(\check{x}), 0)$ and $\psi_h(x, u_0)$ at the two corners, we impose the technical condition (2.8) in assumption **(E2)**. For $m = 1$ the condition (2.8) is equivalent to the transversality condition,

$$\partial_u \mathcal{J}(u_0) D_1^{-1} \mathcal{J}(u_0) - H_1(u_0, 0, 0) \neq 0, \tag{2.10}$$

of the touch-down curve $\mathcal{T}_+$ and the solution ψ_s at $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ (and of $\mathcal{T}_-$ and ψ_s at $\psi_s(2\ell_0) = R_s\psi_s(0)$) – see Figure 2.1. In the case $m > 1$, the technical condition (2.8) is employed to generate a 'good' set of initial conditions in $\ker(I - R_s)$. This set becomes under the forward flow of the slow reduced system (2.4) an m-dimensional manifold, which contains the solution ψ_s and intersect $\mathcal{T}_-$ transversally. We emphasize that (2.10) is a necessary condition for (2.8) to hold true for any $m \geq 1$ – see identity (2.19) in the proof of Theorem 2.3.

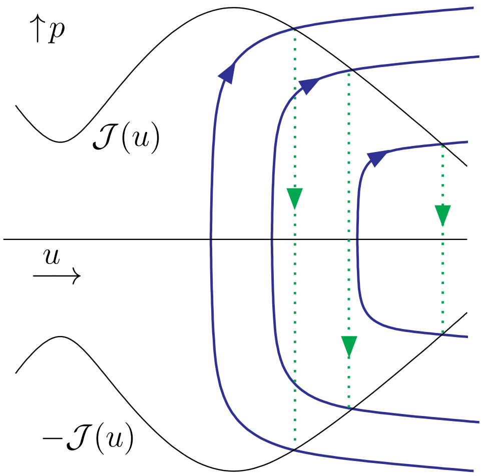

Figure 2.1: Orthogonal projection of three singular periodic orbits on the slow manifold $\mathcal{M}$ in the case $m = n = 1$.

2.3 Dynamics in the vicinity of the slow manifold

Eventually, our goal is prove that close to the singular concatenation (2.9) there exists an actual periodic pulse solution to (2.1), provided $\varepsilon > 0$ is sufficiently small. The singular orbit (2.9) consists of a pulse and an orbit segment on the slow manifold $\mathcal{M}$. Using Grönwall-type arguments it is not difficult to track solutions to (2.1) close to the pulse $\phi_h(x, u_0)$ on an interval $[-X, X]$, where X is ε-independent. However, on the slow manifold $\mathcal{M}$ the dynamics of system (2.1) is of order $\mathcal{O}(\varepsilon)$. Thus, to track solutions to (2.1) close to the orbit segment $\{(\psi_s(\check{x}), 0) : \check{x} \in [0, 2\ell_0]\}$ on $\mathcal{M}$, we need approximations on an interval of length $2\ell_0\varepsilon^{-1}$ and Grönwall type estimates fail. Thus, to capture the dynamics in the vicinity of $\mathcal{M}$, we need additional arguments. First, we require $\mathcal{M}$ to be normally hyperbolic.

(S2) ***Normal hyperbolicity***
For each $u \in U$ the symmetric part $\mathrm{Re}(\mathcal{G}(u)) = \frac{1}{2}\left(\mathcal{G}(u) + \mathcal{G}(u)^T\right)$ of $\mathcal{G}(u) := \partial_v G(u, 0, 0)$ is positive definite.

The behavior of singularly perturbed equations of the form (2.1) close to an invariant, normally hyperbolic manifold is described by *Fenichel geometric singular perturbation theory* [34, 54]. Fenichel theory states that the dynamics close to $\mathcal{M}$ is to leading order governed by the fast and slow reduced systems (2.2) and (2.4), respectively. In this section we collect the facts from Fenichel theory needed to prove our existence result.

2.3.1 Fenichel fibering

Let $\mathcal{M}_0$ be a compact $2m$-dimensional submanifold of the slow manifold $\mathcal{M}$. Fenichel theory [34, Theorem 9.1] states that, the manifold $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a manifold $\mathcal{M}_\varepsilon$, which is diffeomorphic to $\mathcal{M}_0$ and locally invariant for the dynamics of (2.1). Since $\mathcal{M}_0$ is itself locally invariant for the dynamics of (2.1), there exists an ε-independent constant $C > 0$ such that $\mathcal{M}_\varepsilon$ has Hausdorff distance $\mathcal{O}(e^{-C/\varepsilon})$ from $\mathcal{M}_0$ – see also [19, Theorem 2.1] and [121, Theorem 1].

By assumption **(S2)** any $\psi_0 \in \mathcal{M}_0$ is a saddle-centre equilibrium for system (2.2) having n-dimensional stable and unstable fibers $W_0^{u,s}(\psi_0)$. Fenichel theory [34, Theorem 9.1] states that, for $\varepsilon > 0$ sufficiently small, there exists $\psi_\varepsilon \in \mathcal{M}_\varepsilon$ such that these fibers persist as n-dimensional manifolds $W_\varepsilon^{u,s}(\psi_\varepsilon)$ that have $\mathcal{O}(\varepsilon)$-Hausdorff distance to $W_0^{u,s}(\psi_0)$ within an ε-independent neighborhood $\mathcal{D} \subset \mathbb{R}^{2(m+n)}$ of $\mathcal{M}_0$, i.e. the Hausdorff distance between $W_\varepsilon^{u,s}(\psi_\varepsilon) \cap \mathcal{D}$ and $W_0^{u,s}(\psi_0) \cap \mathcal{D}$ is $\mathcal{O}(\varepsilon)$. Moreover, we have the following invariance principle called *Fenichel fibering*: if $\psi_\varepsilon(x)$ is a solution to (2.1) lying in $\mathcal{M}_\varepsilon$ for $\varepsilon x \in [0, X]$, where $X > 0$ is ε-independent, then the manifolds,

$$\mathcal{P}_\varepsilon^{u,s} = \bigcup_{\varepsilon x \in [0,X]} W_\varepsilon^{u,s}(\psi_\varepsilon(x)),$$

are locally invariant for the dynamics of (2.1). Moreover, solutions in $\mathcal{P}_\varepsilon^s$ or $\mathcal{P}_\varepsilon^u$ converge to $\mathcal{M}_\varepsilon$ exponentially as $x \to \infty$ or $x \to -\infty$, respectively. Finally, $\mathcal{P}_\varepsilon^{u,s}$ have $\mathcal{O}(\varepsilon)$-Hausdorff distance (within $\mathcal{D}$) to the manifolds,

$$\mathcal{P}_0^{u,s} = \bigcup_{\check{x} \in [0,X]} W_0^{u,s}((\psi_0(\check{x}), 0)),$$

where ψ_0 is the solution to the slow reduced system (2.4) governing the leading-order dynamics of ψ_ε. In particular, the stable and unstable manifolds $W_0^s(\mathcal{M}_0)$ and $W_0^u(\mathcal{M}_0)$ defined as the union of the stable and unstable fibers of $\mathcal{M}_0$ in (2.2) persist as locally invariant, stable and unstable manifolds $W_\varepsilon^s(\mathcal{M}_\varepsilon)$ and $W_\varepsilon^u(\mathcal{M}_\varepsilon)$ of $\mathcal{M}_\varepsilon$ in (2.1).

Fenichel fibering gives a detailed description of the behavior of solutions to (2.1) converging to $\mathcal{M}_\varepsilon$. In essence the dynamics is an interplay of the attracting or repelling behavior induced by the fast reduced system (2.2) and the dynamics on $\mathcal{M}_\varepsilon$ described by the slow reduced system (2.4).

2.3.2 Fenichel normal form

Fenichel fibering describes the dynamics of those solutions to (2.1) that converge to $\mathcal{M}_\varepsilon$ as $x \to \pm\infty$. However, to understand the behavior of *any* solution close to $\mathcal{M}_\varepsilon$ it is convenient to put system (2.1) into a canonical form in the neighborhood $\mathcal{D} \subset \mathbb{R}^{2(m+n)}$ of $\mathcal{M}_0$, the so-called *Fenichel normal form* [57, Proposition 1]. For $0 \leq \varepsilon \ll 1$, there exists a C^1-change of coordinates $\Psi_\varepsilon \colon \mathcal{D} \to \mathbb{R}^{2(m+n)}$, depending C^1-smoothly on ε, in which the flow of (2.1) is given by,

$$\begin{aligned} a_x &= A(a,b,c,\varepsilon)a, \\ b_x &= B(a,b,c,\varepsilon)b, \qquad\qquad a,b\in\mathbb{R}^n, c\in\mathbb{R}^{2m}, \\ c_x &= \varepsilon K(c,\varepsilon) + H(a,b,c,\varepsilon)(a\otimes b), \end{aligned} \tag{2.11}$$

where the A, B, K and H are C^1 in their arguments, K maps to $\mathbb{R}^{2m}$, A and B map to the square matrices of order n and H maps to tensors of appropriate rank. Moreover, there exists $\Delta > 0$ and an open and bounded set $U_F \subset \mathbb{R}^{2m}$ such that the image $\Psi_\varepsilon(\mathcal{D})$ contains the compact box,

$$\mathcal{B} := \left\{(a,b,c) : \|a\|, \|b\| \leq \Delta, c \in \overline{U_F}\right\}. \tag{2.12}$$

In addition, there exists $C, \mu > 0$, independent of ε, such that

$$\mathrm{Re}(\sigma(A(a,b,c,\varepsilon))) \leq -\mu, \quad \mathrm{Re}(\sigma(B(a,b,c,\varepsilon))) \geq \mu, \tag{2.13}$$

and

$$\|H(a,b,c,\varepsilon)(a\otimes b)\| \leq C\|a\|\|b\|, \tag{2.14}$$

for all $(a,b,c) \in \mathcal{B}$ and $0 \leq \varepsilon \ll 1$.

In the local *Fenichel coordinates* $\mathcal{M}_\varepsilon$ correspond to the space $a = b = 0$ and the local stable and unstable manifolds $W^{u,s}_\varepsilon(\mathcal{M}_\varepsilon) \cap \mathcal{D}$ of $\mathcal{M}_\varepsilon$ correspond to the spaces $b = 0$ and $a = 0$, respectively. Since system (2.1) is R-reversible, R maps $W^u_\varepsilon(\mathcal{M}_\varepsilon)$ onto $W^s_\varepsilon(\mathcal{M}_\varepsilon)$ and vice versa. Hence, $\ker(I - R) \cap \mathcal{D}$ corresponds to the space $a = b$. Finally, system

$$c_{\check{x}} = K(c, 0), \quad c \in \mathbb{R}^{2m}, \tag{2.15}$$

is equivalent to the slow reduced system (2.4).

In the canonical form (2.11) the dynamics of (2.1) is decomposed in an attracting a-direction, a repelling b-direction and a slowly evolving c-direction.

2.3.3 The Exchange lemma

Through the Fenichel normal form (2.11) one observes that (2.1) exhibits attracting, repelling and slow dynamics. *Exchange lemmas* [55, 57, 59, 102] provide a way to capture this combination of dynamics.

As mentioned before, we need to track solutions close to the orbit segment on the slow manifold of the singular concatenation (2.9) in order to prove our main existence result. For this reason we need the following exchange lemma, which is (naturally) stated in Fenichel coordinates.

Lemma 2.2. *[102, Theorem 2.3] Let* $a_* \in \mathbb{R}^n$ *with* $\|a_*\| < \Delta$ *and let* $c_0(\check{x})$ *be a solution to* (2.15) *such that* $c_0(\check{x}) \in U_F$ *for* $\check{x} \in [0, X]$ *with* $X > 0$*. Let* $\mathcal{Z}_\varepsilon$ *for* $0 \leq \varepsilon \ll 1$ *be a submanifold of* $\mathbb{R}^{2(m+n)}$ *of dimension* $n + l$*, where* $0 \leq l \leq 2m - 1$*, satisfying the assertions:*

i. $\mathcal{Z} = \{(a, b, c, \varepsilon) : (a, b, c) \in \mathcal{Z}_\varepsilon\}$ *is itself a manifold;*

ii. $\mathcal{Z}_0$ *meets the space* $b = 0$ *transversally at the point* $(a_*, 0, c_0(0))$.

Denote by $\mathcal{P}_\varepsilon, 0 \leq \varepsilon \ll 1$ *the orthogonal projection of the l-dimensional manifold* $\mathcal{Z}_\varepsilon \cap \{(a, 0, c) : a \in \mathbb{R}^n, c \in U_F\}$ *on the space* $a = b = 0$*. We require in addition:*

iii. $\mathcal{P}_0$ *is an l-dimensional manifold and the flow of* (2.15) *is not tangent to* $\mathcal{P}_0$ *at* $c_0(0)$.

Denote by $\mathcal{Z}_\varepsilon^*$ *and* $\mathcal{P}_\varepsilon^*$ *the* $(n + l + 1)$*- and* $(l + 1)$*-dimensional manifolds obtained by flowing initial conditions on* $\mathcal{Z}_\varepsilon$ *and* $\mathcal{P}_\varepsilon$ *forward in* (2.11)*. Then, there exists a* $(n + l + 1)$*-dimensional submanifold* $\mathcal{Z}_{1,\varepsilon}$ *of* $\mathcal{Z}_\varepsilon^*$ *and an ε-independent neighborhood* $U_1 \subset U_F$ *of* $c_0(X)$ *such that the Hausdorff distance between* $\mathcal{Z}_{1,\varepsilon}$ *and the* $(n + l + 1)$*-dimensional manifold,*

$$\{(0, b, c) : b \in \mathbb{R}^n, c \in \mathcal{P}_\varepsilon^* \cap U_1\} \subset W_\varepsilon^u(\mathcal{M}_\varepsilon),$$

is $O(e^{-C/\varepsilon})$*, where* $C > 0$ *is independent of ε. Moreover, trajectories crossing* $\mathcal{Z}_{1,\varepsilon}$ *remain in the box* $\mathcal{B}$ *– see* (2.12) *– during their excursion from* $\mathcal{Z}_\varepsilon$ *to* $\mathcal{Z}_{1,\varepsilon}$*.*

2.4 Main existence result

In this section, we prove that close to the singular concatenation (2.9) there exists an actual periodic pulse solution to (2.1), provided $\varepsilon > 0$ is sufficiently small.

It is a well-known principle that close to a singular periodic orbit, constructed by piecing together orbit segments of the fast and slow reduced systems in such a way that they form a closed loop, one can find an actual periodic orbit, provided $\varepsilon > 0$ is sufficiently small. In [110] this is proved for a large class of slow-fast systems. However, an essential condition for the result in [110] is that the slow components are constant in the fast reduced system. In our case the slow p-components are non-constant along orbits in (2.2). Therefore, the result in [110] is not applicable.

To our knowledge there is no existence result in the literature focusing on periodic solutions in the large class of singularly perturbed systems (2.1) beyond the type of slow-fast systems considered in [110]. However, for the Gierer-Meinhardt equations – see Remark 2.7 – the existence of stationary, spatially periodic pulse solutions is proved in [25]. We emphasize that the framework in [25] differs fundamentally from ours due to a difference in scaling in the p-components and the fact that the u, p, v- and q-components are scalar. In Remark 2.5 we elaborate in more detail on the scaling in the p-components.

We prove the existence of periodic pulse solutions close to (2.9) in the class of systems (2.1) by adapting and extending the techniques in [25, 110] – see Remark 2.4. The proof of our result exploits the fact that every orbit that crosses the space $\ker(I-R)$ twice, must be a closed loop. Therefore, our approach is to start with a 'good' set of initial conditions in $\mathcal{Z} \subset \ker(I-R)$ and track these conditions under the forward flow of (2.1) with the aid of the Exchange Lemma 2.2. We show that the tracked trajectories remain close to the singular orbit (2.9). In particular, we establish that the union of trajectories starting in $\mathcal{Z}$ intersects $\ker(I-R)$ transversally in some point P_ε, which lies close to $\phi_{\mathrm{h}}(0,u_0)$. The desired periodic solution is the one that crosses P_ε.

Theorem 2.3. *Assume **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Then, there exists constants* $C,\mu_0,\varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$ *there exists a solution* $\phi_{\mathrm{p},\varepsilon}(x)$ *to* (2.1) *satisfying the following assertions:*

1. ***Periodicity***
   $\phi_{\mathrm{p},\varepsilon}$ *is* $2L_\varepsilon$*-periodic, where* $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$.
2. ***Reversibility***
   We have $\phi_{\mathrm{p},\varepsilon}(x) = R\phi_{\mathrm{p},\varepsilon}(-x)$ *for* $x \in \mathbb{R}$.
3. ***Singular limit***
   Define for $\theta \geq \mu_0^{-1}$ *the quantity* $\Xi_\theta(\varepsilon) := -\theta\log(\varepsilon)$. *The solution* $\phi_{\mathrm{p},\varepsilon}$ *approximates the pulse as,*
   $$\left\|\phi_{\mathrm{p},\varepsilon}(x) - \phi_{\mathrm{h}}(x,u_0)\right\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon),\Xi_\theta(\varepsilon)], \tag{2.16}$$
   and it approximates the orbit segment on the slow manifold as,
   $$\left\|\phi_{\mathrm{p},\varepsilon}(x) - (\psi_{\mathrm{s}}(\varepsilon x),0)\right\| \leq C\varepsilon, \quad x \in [\Xi_\theta(\varepsilon), 2L_\varepsilon - \Xi_\theta(\varepsilon)]. \tag{2.17}$$
4. ***Exponential convergence to slow manifold***
   We have the estimate,
   $$d(\phi_{\mathrm{p},\varepsilon}(x),\mathcal{M}) \leq Ce^{-\mu_0\min\{x,2L_\varepsilon - x\}}. \tag{2.18}$$

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε.

We start with constructing a good manifold of initial conditions in $\ker(I-R)$. Denote by $e_i, i \in 1,\ldots,m$ the unit basis of $\mathbb{R}^m$. Let $\mathcal{U}$ be the $(m \times (m-1))$-matrix with column vectors $e_1,\ldots,e_{i_*-1},e_{i_*+1},\ldots,e_m$, where i^* is as in **(E2)**. Consider the $(m+n-1)$-dimensional manifold,

$$\mathcal{Z} := \{(u_{\mathrm{s}}(\ell_0)+u,0,v,0) : u \in \mathcal{U}[\mathbb{R}^{m-1}], v \in \mathbb{R}^n\} \subset \ker(I-R).$$

The intersection of $\mathcal{Z}$ and $\mathcal{M}$ equals,

$$\mathcal{P}_0 := \{(u_{\mathrm{s}}(\ell_0)+u,0) : u \in \mathcal{U}[\mathbb{R}^{m-1}]\} \subset \ker(I-R_s).$$

By assumption **(E2)** $\mathcal{P}_0$ becomes under the forward flow of the slow reduced system (2.4) an m-dimensional manifold $\mathcal{P}_0^*$, which intersects $\mathcal{T}_- = R_s[\mathcal{T}_+]$ transversely at $\psi_s(2\ell_0) = R_s\psi_s(0) = (u_0, -\mathcal{J}(u_0))$. Indeed, we have by condition (2.8)

$$\begin{aligned}
0 \neq \det&\left(\Phi_s(\ell_0,0)\begin{bmatrix} I \\ \partial_u \mathcal{J}(u_0) \end{bmatrix} \middle| \begin{matrix} 0 & \mathcal{U} \\ H_1(u_s(\ell_0),0,0) & 0 \end{matrix} \right) \\
= \det&\left(\begin{matrix} I & D_1^{-1}\mathcal{J}(u_0) \\ \partial_u \mathcal{J}(u_0) & H_1(u_0,0,0) \end{matrix} \middle| \Phi_s(0,\ell_0)\begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \right) \\
= \det&\left(H_1(u_0,0,0) - \partial_u \mathcal{J}(u_0) D_1^{-1}\mathcal{J}(u_0) \middle| \begin{bmatrix} -\partial_u \mathcal{J}(u_0) & I \end{bmatrix} \Phi_s(0,\ell_0)\begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \right)
\end{aligned} \tag{2.19}$$

where we use that $\Phi_s(0,\ell_0)$ induces an isomorphism between the tangent spaces of $\mathcal{P}_0^*$ at $\psi_s(\ell_0)$ and at $\psi_s(0)$ and that the determinant of $\Phi_s(0,\ell_0)$ equals 1.

Eventually, our goal is to show that the $(m+n)$-dimensional manifold $\mathcal{Z}_\varepsilon^*$ obtained by flowing initial conditions on $\mathcal{Z}$ forward in (2.1) intersects the $(m+n)$-dimensional manifold $\ker(I-R)$ transversally within $\mathbb{R}^{2(m+n)}$ close to the point $\phi_h(0,u_0)$. The unique intersection point then yields a periodic solution.

To describe the dynamics on $\mathcal{Z}_\varepsilon^*$ close to $\mathcal{M}$, we apply Fenichel theory – see §2.3. We choose a compact submanifold $\mathcal{M}_0$ of $\mathcal{M}$ that contains the projection of the singular orbit (2.9) on $\mathcal{M}$, i.e. let $\mathcal{M}_0$ be a compact $2m$-dimensional submanifold of $\mathcal{M}$ such that $\mathcal{M}_0$ serves as a neighborhood of the orbit segment $\{\psi_s(\check{x}) : \check{x} \in [0, 2\ell_0]\}$ and of the projection $\{(u_0, \int_0^x H_2(u_0, v_h(z,u_0))dz) : x \in \mathbb{R}\}$ of the pulse $\phi_h(x,u_0)$ on $\mathcal{M}$.

By assumption **(S2)** $\mathcal{M}_0$ is normally hyperbolic. So, according to Fenichel theory, $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a manifold $\mathcal{M}_\varepsilon$, which is diffeomorphic to $\mathcal{M}_0$ and locally invariant for the dynamics of (2.1). In addition, $\mathcal{M}_\varepsilon$ has Hausdorff distance $O(e^{-C/\varepsilon})$ to $\mathcal{M}_0$.

To track solutions on $\mathcal{Z}_\varepsilon^*$ we apply the Exchange Lemma 2.2. By switching to Fenichel coordinates in the neighborhood $\mathcal{D}$ of $\mathcal{M}_0$ – see §2.3.2 – it is readily seen that $\mathcal{Z} \subset \ker(I-R)$ intersects the local stable manifold $W_0^s(\mathcal{M}_0) \cap \mathcal{D}$ of $\mathcal{M}_0$ in the fast reduced system (2.2) transversally at $(\psi_s(\ell_0), 0)$. Moreover, the slow reduced flow (2.4) on $\mathcal{M}_0$ is not tangent to $\mathcal{P}_0$ at $\psi_s(\ell_0)$ by (2.19). We conclude that the conditions for the Exchange Lemma 2.2 are satisfied.

Denote by $\mathcal{P}_\varepsilon \subset \mathcal{M}_\varepsilon$ the $(m-1)$-dimensional manifold, where $\mathcal{Z}$ and the local stable manifold $W_\varepsilon^s(\mathcal{M}_\varepsilon) \cap \mathcal{D}$ meet transversally. Moreover, let $\mathcal{P}_\varepsilon^* \subset \mathcal{M}_\varepsilon$ be the m-dimensional manifold obtained by flowing initial conditions on $\mathcal{P}_\varepsilon$ forward in (2.1). Finally, we denote by

$$\mathcal{Y}_\varepsilon := \bigcup_{\varphi \in \mathcal{P}_\varepsilon^*} W_\varepsilon^u(\varphi) \subset W_\varepsilon^u(\mathcal{M}_\varepsilon),$$

the union of unstable fibers in (2.1) with base points in $\mathcal{P}_\varepsilon^* \subset \mathcal{M}_\varepsilon$. Note that $\mathcal{Y}_\varepsilon$ is locally invariant in (2.1) by Fenichel fibering – see §2.3.1. By the Exchange Lemma, there exists an

$(m+n)$-dimensional submanifold $\mathcal{Z}_{1,\varepsilon}$ of $\mathcal{Z}^*_\varepsilon$ and an ε-independent neighborhood $\mathcal{D}_1 \subset \mathcal{D}$ of $(\psi_s(2\ell_0), 0)$ such that the Hausdorff distance between $\mathcal{D}_1 \cap \mathcal{Y}_\varepsilon$ and $\mathcal{Z}_{1,\varepsilon}$ is $O(e^{-C/\varepsilon})$. Moreover, trajectories crossing $\mathcal{Z}_{1,\varepsilon}$ remain in $\mathcal{D}$ during the excursion from $\mathcal{Z}$ to $\mathcal{Z}_{1,\varepsilon}$.

We aim to show that the $(m+n)$-dimensional manifold $\mathcal{Y}_\varepsilon$ intersects $\ker(I-R)$ transversally. Then, by the above closeness estimate the same holds for the $(m+n)$-dimensional manifold $\mathcal{Z}^*_{1,\varepsilon} \subset \mathcal{Z}^*_\varepsilon$ obtained by flowing $\mathcal{Z}_{1,\varepsilon}$ forward in (2.1). Therefore, we determine the singular limit $\mathcal{Y}_0$ of $\mathcal{Y}_\varepsilon$. First, recall that $\mathcal{P}^*_0$ intersects $\mathcal{T}_-$ transversely at $\psi_s(2\ell_0)$. Second, the unstable manifold $W^u_0(\mathcal{M}_0)$ of $\mathcal{M}_0$ in (2.2) intersects $\ker(I-R)$ transversely in an m-dimensional manifold $\mathcal{S}_0 := \{\phi_h(0,u) : u \in U_h\}$ by assumption **(E1)**. The α-limit set of $\mathcal{S}_0$ equals the touch-down manifold $\mathcal{T}_-$ in $\mathcal{M}$. We now put these two items together and conclude that the $(m+n)$-dimensional union,

$$\mathcal{Y}_0 := \bigcup_{\varphi \in \mathcal{P}^*_0} W^u_0(\varphi) \subset W^u_0(\mathcal{M}_0),$$

of unstable fibers in (2.2) with base points in $\mathcal{P}^*_0$ intersects the $(m+n)$-dimensional manifold $\ker(I-R)$ transversally in the point $\phi_h(0,u_0)$.

By Fenichel fibering – see §2.3.1 – the manifolds $\mathcal{Y}_\varepsilon$ and $\mathcal{Y}_0$ have Hausdorff distance $O(\varepsilon)$ in a neighborhood of the intersection point $\phi_h(0,u_0)$. Therefore, provided $\varepsilon > 0$ is sufficiently small, $\mathcal{Y}_\varepsilon$ intersects $\ker(I-R)$ transversally in some point $P_{h,\varepsilon}$, which lies $O(\varepsilon)$-close to $\phi_h(0,u_0)$. Denote by $\phi_{h,\varepsilon}(x)$ the solution to (2.1) with initial condition $\phi_{h,\varepsilon}(0) = P_{h,\varepsilon}$.

Since $\phi_h(x,u_0)$ converges to $(\psi_s(2\ell_0), 0) \in \mathcal{M}$ as $x \to -\infty$, there exists $x_0 > 0$ such that $\phi_h(-x_0,u_0)$ is contained in the neighborhood $\mathcal{D}_1 \subset \mathcal{D}$ of $(\psi_s(2\ell_0), 0)$. Hence, since $\phi_{h,\varepsilon}(0)$ is $O(\varepsilon)$-close to $\phi_h(0,u_0)$ and x_0 is independent of ε, one derives via Grönwall type estimates that $\phi_{h,\varepsilon}(-x_0)$ is contained in $\mathcal{D}_1 \cap \mathcal{Y}_\varepsilon$. Recall that the outcome of the Exchange Lemma is that $\mathcal{Y}_\varepsilon$ has Hausdorff distance $O(e^{-C/\varepsilon})$ from $\mathcal{Z}_{1,\varepsilon}$ in the neighborhood $\mathcal{D}_1$ of $\phi_{h,\varepsilon}(-x_0)$. Thus, using x_0 is ε-independent, we infer, again via Grönwall type estimates, that the Hausdorff distance between $\mathcal{Y}_\varepsilon$ and $\mathcal{Z}^*_{1,\varepsilon}$ is $O(e^{-C/\varepsilon})$ in a neighborhood of $\phi_{h,\varepsilon}(0)$. Therefore, $\mathcal{Z}^*_{1,\varepsilon}$ intersects $\ker(I-R)$ transversally in some point $P_{p,\varepsilon}$, which is $O(\varepsilon)$-close to $\phi_h(0,u_0)$. The solution $\phi_{p,\varepsilon}(x)$ to (2.1) with initial condition $\phi_{p,\varepsilon}(0) = P_{p,\varepsilon}$ is the desired periodic orbit. Indeed, $\phi_{p,\varepsilon}(x)$ crosses $\ker(I-R)$ at $x = 0$ and at some point $x = -L_\varepsilon < 0$, since $\phi_{p,\varepsilon}$ is contained in $\mathcal{Z}^*_\varepsilon$.

All that remains to show is the four assertions in the theorem statement. The second assertion is immediate, since $\phi_{p,\varepsilon}(0) \in \ker(I-R)$. The other assertions require more work.

We start by estimating $\phi_{p,\varepsilon}$ with the pulse solution ϕ_h to the fast reduced system (2.2). Since $\phi_{p,\varepsilon}(0)$ is $O(\varepsilon)$-close to $\phi_h(0,u_0)$, we approximate

$$\left\|\phi_{p,\varepsilon}(x) - \phi_h(x,u_0)\right\| \leq C\varepsilon, \quad x \in [-x_0, 0]. \tag{2.20}$$

Next, we obtain decay estimates of $\phi_{p,\varepsilon}(x)$ to the slow manifold. Without loss of generality we may assume $\phi_{p,\varepsilon}(x)$ is in $\mathcal{D}$ for $x \in [-L_\varepsilon, -x_0]$. Thus, we may express $\phi_{p,\varepsilon}(x)$ in Fenichel coordinates as $\tilde{\phi}_{p,\varepsilon}(x) = (a_{p,\varepsilon}(x), b_{p,\varepsilon}(x), c_{p,\varepsilon}(x)) = \Psi_\varepsilon(\phi_{p,\varepsilon}(x))$ for $x \in [-L_\varepsilon, x_0]$ – see §2.3.2. By [57, Corollary 1] the estimates (2.13) yield a $\mu_0 > 0$, independent of ε, such that

$$\|a_{p,\varepsilon}(x)\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad \|b_{p,\varepsilon}(x)\| \leq Ce^{\mu_0(x+x_0)}, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.21}$$

We prove the fourth assertion. First, $\mathcal{M}_\varepsilon$ corresponds to the space $a = b = 0$ in (2.11). Second, $\mathcal{M}_\varepsilon$ has Hausdorff distance $\mathcal{O}(e^{-C/\varepsilon})$ to $\mathcal{M}_0 \subset \mathcal{M}$. Third, the coordinate transform Ψ_ε is C^1. Combining these items with estimate (2.21) yields the fourth assertion.

We prove the third assertion. We express the pulse solution ϕ_h to the fast reduced system (2.2) in Fenichel coordinates as $\tilde{\phi}_h(x) = \Psi_0(\phi_h(x, u_0))$ for $x \leq -x_0$. Observe that $\tilde{\phi}_h(x)$ satisfies (2.11) for $\varepsilon = 0$ and lies in the unstable space $a = 0$. Consequently, we can write $\tilde{\phi}_h(x) = (0, b_h(x), c_0)$, where c_0 is a constant in U_F and $b_h(x)$ satisfies the equation $b_x = B(0, b, c_0, 0)b$, where B is as in (2.11). Clearly, $b_h(x)$ converges exponentially to 0 as $x \to -\infty$. By estimate (2.20) and C^1-smoothness of Ψ_ε in ε, it holds

$$\left\|\tilde{\phi}_{p,\varepsilon}(-x_0) - \tilde{\phi}_h(-x_0)\right\| \leq C\varepsilon. \tag{2.22}$$

Using estimates (2.14), (2.21) and (2.22) we obtain,

$$\begin{aligned} \|c_{p,\varepsilon}(x) - c_0\| &\leq \int_x^{-x_0} \left(\varepsilon\|K(c_{p,\varepsilon}(y), \varepsilon)\| + \left\|H(\tilde{\phi}_{p,\varepsilon}(y), \varepsilon)(a_{p,\varepsilon}(y) \otimes b_{p,\varepsilon}(y))\right\|\right) dy \\ &\quad + \|c_{p,\varepsilon}(-x_0) - c_0\| \\ &\leq C\varepsilon\Xi_\theta(\varepsilon), \end{aligned} \tag{2.23}$$

for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. The difference $g_\varepsilon(x) = b_{p,\varepsilon}(x) - b_h(x)$ satisfies an inhomogeneous equation of the form,

$$g_x = A_\varepsilon(x)g + h_\varepsilon(x),$$

where $A_\varepsilon(x)g_\varepsilon(x) = B(\tilde{\phi}_h(x), 0)g_\varepsilon(x) + (B(0, b_{p,\varepsilon}(x), c_0, 0) - B(\tilde{\phi}_h(x), 0))b_{p,\varepsilon}(x)$ and $h_\varepsilon(x) = (B(\tilde{\phi}_{p,\varepsilon}(x), \varepsilon) - B(0, b_{p,\varepsilon}(x), c_0, 0))b_{p,\varepsilon}(x)$. Taking x_0 larger if necessary, estimates (2.13), (2.21) and (2.23) yield $\mathrm{Re}(A_\varepsilon(x)) \leq -\mu_0$ and $\|h_\varepsilon(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon)$ for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. Therefore, we conclude using (2.22) that,

$$\|b_{p,\varepsilon}(x) - b_h(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon), -x_0]. \tag{2.24}$$

Estimate (2.16) now follows from C^1-smoothness of Ψ_ε^{-1} in ε together with estimates (2.20), (2.21), (2.23) and (2.24).

We prove (2.17). By (2.14) and (2.21) we have,

$$\left\|H(\tilde{\phi}_{p,\varepsilon}(x))(a_{p,\varepsilon}(x) \otimes b_{p,\varepsilon}(x))\right\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad x \in [-L_\varepsilon, -x_0].$$

Therefore, using Grönwall type estimates, there exists a solution $(0, 0, c_{s,\varepsilon}(x))$ on the invariant manifold $\mathcal{M}_\varepsilon \subset \{a = b = 0\}$ satisfying $\partial_x c = \varepsilon K(c, \varepsilon)$, which is $\mathcal{O}(e^{-\mu_0 L_\varepsilon})$-close to $c_{p,\varepsilon}(x)$

for $x \in [-L_\varepsilon, -x_0]$. The solution $c_{s,\varepsilon}(x)$ is to leading order described by a solution $c_{s,0}(\check{x})$ to $\partial_{\check{x}} c = K(c, 0)$. This results in the estimate,

$$\|c_{p,\varepsilon}(x) - c_{s,0}(\varepsilon x)\| \leq C\varepsilon, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.25}$$

Estimates (2.23) and (2.25) imply $c_{s,0}(0) = c_0$. On the other hand, we have $\Psi_0((\psi_s(2\ell_0), 0)) = \lim_{x\to-\infty} \tilde{\phi}_h(x) = (0, 0, c_0)$. Since system $\partial_{\check{x}} c = K(c, 0)$ corresponds to the slow reduced system (2.4), we have $\Psi_0^{-1}((0, 0, c_{s,0}(\check{x}))) = (\psi_s(\check{x} + 2\ell_0), 0)$ for $\varepsilon^{-1}\check{x} \in [-L_\varepsilon, 0]$. Hence, by C^1-smoothness of Ψ_ε^{-1} in ε, R-reversibility of $\phi_{p,\varepsilon}(x)$, estimates (2.21) and (2.25) and the inequality $\theta \geq \mu_0^{-1}$, we conclude estimate (2.17) holds true.

Finally, we prove the first assertion. On the one hand, we have $p_s(\ell_0) = 0$ and $p_s'(\ell_0) = H_1(u_s(\ell_0), 0, 0) \neq 0$ by (2.19). On the other hand, it holds $\|p_s(\varepsilon L_\varepsilon)\| \leq C\varepsilon$ by (2.17). Thus, an application of the inverse function theorem and the mean value theorem yields $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$. □

Remark 2.4. Although the framework in [25, 110] is different, the idea to track solutions close to the slow manifold with the aid of an appropriate exchange lemma is the same. However, in contrast to [25, 110], we need an exchange lemma that works for systems having non-constant slow components in the fast reduced system – see [102, Section 2.5]. Moreover, as in [25], we exploit that system (2.1) is R-reversible. Yet, transversality arguments differ from [25], since our class of systems admits multidimensional components. ■

Remark 2.5. In principle, the existence problem could also be put in slow-fast form by introducing $p = \varepsilon^{-1/2} D_1 u_x$ instead of $p = \varepsilon^{-1} D_1 u_x$. Then the p-equation reads $p_x = \varepsilon^{3/2} H_1(u, v, \varepsilon) + \varepsilon^{1/2} H_2(u, v)$. This is done in existence analysis of periodic pulse solutions in the Gierer-Meinhardt equations in [25]. The equation for the p-components in the slow reduced system (2.4) would be $p_{\check{x}} = 0$ in that case. This makes the construction of the desired singular periodic orbit, performed in §2.2.2, impossible. Therefore, the scaling regime in (2.1) is the most natural for our set-up. In [25] one avoids setting $\varepsilon = 0$ in the existence analysis and makes a distinction between slow and 'super-slow' behavior. We emphasize that in the spectral stability analysis in Chapter 5 we adopt a similar scaling regime to put the eigenvalue problem in slow-fast form, which is required for an application of the Riccati transform – see also Remark 1.4. ■

Remark 2.6. As mentioned in §1.4.1, our model (1.9) is a reaction-diffusion system (1.1) that allows for semi-strong interaction (1.8), with the extra condition that G vanishes at $v = 0$. For general G, consider a $2n$-dimensional compact submanifold $\mathcal{M}_0$ of $\{(u, p, v, 0) : G(u, v, 0) = 0\} \subset U \times \mathbb{R}^m \times V \times \mathbb{R}^n$. By Fenichel theory [34] $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a locally invariant manifold $\mathcal{M}_\varepsilon$ in (2.1). This manifold $\mathcal{M}_\varepsilon$ is diffeomorphic to $\mathcal{M}_0$ and lies at Hausdorff distance $O(\varepsilon)$ from $\mathcal{M}_0$. When $\mathcal{M}_0$ can be given as a graph over $(u, p) \in U \times \mathbb{R}^m$, the same holds for $\mathcal{M}_\varepsilon$. Thus, in that case one can change coordinates in (2.1) relative to $\mathcal{M}_\varepsilon$ and we obtain $\mathcal{M}_\varepsilon = \mathcal{M}_0 \subset \{(u, p, 0, 0) : u \in U, p \in \mathbb{R}^m\}$. Therefore, in the existence analysis, the condition that G vanishes at $v = 0$, corresponds to an a priori coordinate change in (2.1).

However, one introduces more than additional technical difficulties in the spectral stability analysis when G does not vanish at $v = 0$. Indeed, without relative coordinates, we do not achieve estimate (2.18), which is essential in our stability analysis. However, applying the coordinate change to equation (1.9) changes its structure fundamentally. In the new coordinates (1.9) is not even of reaction-diffusion type. Hence, we expect that the spectral analysis differs essentially, when G does not vanish at $v = 0$. This is an interesting subject of future research, especially since it includes the possibility of localized patterns with oscillatory tails [11, 30], but is outside the scope of this thesis. ∎

Remark 2.7. As mentioned in the introduction in Chapter 1 the class of equations (1.10) includes the generalized Gierer-Meinhardt equations,

$$\begin{aligned} \varepsilon^2 u_t &= u_{xx} - \varepsilon^2 \mu u + \varepsilon u^{\alpha_1} v^{\beta_1}, \\ v_t &= v_{xx} - v + u^{\alpha_2} v^{\beta_2}, \end{aligned} \qquad (u,v) \in \mathbb{R}^2, x \in \mathbb{R}, \tag{2.26}$$

with parameters $\alpha_1 \in \mathbb{R}, \alpha_2 < 0, \beta_{1,2} \in \mathbb{Z}_{>1}$ and $\mu > 0$ satisfying,

$$(\alpha_1 - 1)(\beta_2 - 1) - \alpha_2 \beta_1 > 0.$$

Indeed, it is not difficult to verify that assumptions **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true for (2.26). Thus, Theorem 2.3 reconfirms the existence result of periodic pulse solutions to (2.26) proved in [25]. ∎

2.5 Existence in the slowly nonlinear toy problem

In this section, we explicitly construct a singular periodic orbit in the slowly nonlinear toy problem

$$\begin{aligned} \varepsilon^2 u_t &= u_{xx} - \varepsilon^2 \mu \sin(u) - \varepsilon(\nu_2 v^2 + \nu_3 v^3), \\ v_t &= v_{xx} - v + \frac{v^2}{f(u)}, \end{aligned} \qquad (u,v) \in \mathbb{R}^2, x \in \mathbb{R}, \tag{2.27}$$

with $\mu > 0$, $\nu_2, \nu_3 \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}_{>0}$ at least C^3. We then evoke Theorem 2.3 to prove the existence of an actual periodic solution close to the singular one.

For the toy problem (2.27) the fast reduced system reads,

$$\begin{aligned} u_x &= 0, \\ p_x &= \nu_2 v^2 + \nu_3 v^3, \\ v_x &= q, \\ q_x &= v - \frac{v^2}{f(u)}, \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^4. \tag{2.28}$$

For any $u > 0$, the governing subsystem,

$$\begin{aligned} v_x &= q, \\ q_x &= v - \frac{v^2}{f(u)}, \end{aligned} \qquad (v,q) \in \mathbb{R}^2. \tag{2.29}$$

of (2.28) is Hamiltonian and has a hyperbolic saddle in $(0,0)$. By a phase-portrait analysis one observes that (2.29) admits for any $u \in \mathbb{R}$ a homoclinic solution to $(0,0)$. By integrating equation (2.29) an explicit expression for this homoclinic can be found. This results in the pulse solution to (2.28) given by,

$$\phi_{\mathrm{h}}(x,u;\nu_2,\nu_3) = \left(u, \int_0^x \left(\nu_2 (v_{\mathrm{h}}(z,u))^2 + \nu_3 (v_{\mathrm{h}}(z,u))^3\right) dz, v_{\mathrm{h}}(z,u), v_{\mathrm{h}}'(z,u)\right),$$

with $v_{\mathrm{h}}(x,u) = \frac{3}{2} f(u) \mathrm{sech}^2(\frac{1}{2}x)$. Consequently, the take-off and touch-down curves on the slow manifold $\mathcal{M}$ are given by,

$$\mathcal{T}_{\pm} = \{(u, \mathcal{J}(u;\nu_2,\nu_3)) : u \in (0,\pi)\}, \quad \mathcal{J}(u;\nu_2,\nu_3) = \tfrac{3}{5}(f(u))^2 \left(5\nu_2 + 6\nu_3 f(u)\right).$$

The slow reduced system,

$$\begin{aligned} u_{\check{x}} &= p, \\ p_{\check{x}} &= \mu \sin(u), \end{aligned} \qquad (u,p) \in \mathbb{R}^2,$$

is also Hamiltonian and can be integrated. This leads to the family of bounded solutions given by the $(4K(k)\mu^{-1/2})$-periodic Jacobi-amplitude functions,

$$\psi_{\mathrm{s}}(\check{x};k,c,\mu) = (u_{\mathrm{s}}(\check{x}), u_{\mathrm{s}}'(\check{x})), \quad u_{\mathrm{s}}(\check{x};k,c,\mu) = 2\mathrm{Am}\left(-k\sqrt{\mu}(x-c), k^{-2}\right) + \pi, \tag{2.30}$$

parameterized over $k \in (0,1)$, where $K(k)$ is the Jacobi complete integral of the first kind. The constant $c \in \mathbb{R}$ with $|c| < K(k)\mu^{-1/2}$ corresponds to the initial translation on the orbit of ψ_{s}. In addition, we take

$$\ell_0 = \ell_0(k,l,c,\mu) := c + \frac{(2l+1)K(k)}{\sqrt{\mu}} > 0, \tag{2.31}$$

where $l \in \mathbb{Z}_{\geq 0}$ such that it holds $u_s'(\ell_0;k,c,\mu) = 0$.

Constructing a singular periodic orbit now reduced to connecting the solution ψ_{s} with the take-off and touch-down curves $\mathcal{T}_{\pm}$, i.e. finding values for μ, k, c, ν_2, ν_3 such that,

$$\mathcal{J}(u_{\mathrm{s}}(0;k,c,\mu),\nu_2,\nu_3) = u_{\mathrm{s}}'(0;k,c,\mu),$$

Such values can be easily found with a computer software programm like Mathematica. If we have found such values, the singular periodic orbit is given by,

$$\phi_{\mathrm{p},0} = \{(\psi_{\mathrm{s}}(\check{x};k,c,\mu),0) : \check{x} \in (0,2\ell_0)\} \cup \{\phi_{\mathrm{h}}(x,u_{\mathrm{s}}(0;k,c,\mu);\nu_2,\nu_3) : x \in \mathbb{R}\},$$

One readily observes that **(S1)**, **(S2)** and **(E1)** are satisfied. For **(E2)** to hold true, we require that the transversality condition

$$\mathcal{J}'(u_{\rm s}(0;k,c,\mu), v_2, v_3)u_{\rm s}'(0;k,c,\mu) - \mu\sin(u_{\rm s}(0;k,c,\mu)) \neq 0,$$

is satisfied. Now, it follows from Theorem 2.3 that an actual periodic solution to (2.27) lies in the vicinity of the singular orbit $\phi_{\rm p,0}$.