Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Title: Periodic pulse solutions to slowly nonlinear reaction-diffusion systems
Issue Date: 2016-12-22

Chapter 1

Introduction

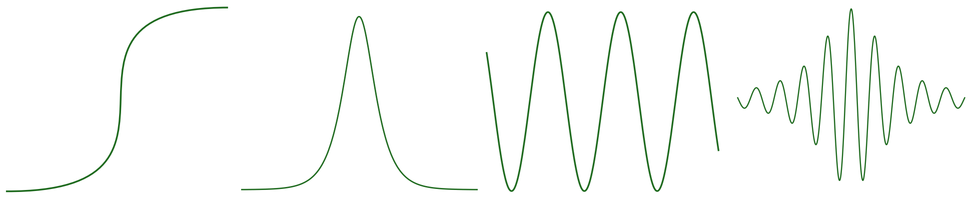
Pattern formation is observed in dynamical processes within various scientific disciplines, including chemistry, biology, neurophysiology, optics and ecology. Reaction-diffusion systems exhibit a large variety of patterns and have attracted much interest as a (simplified) model describing these dynamical processes. For instance, reaction-diffusion systems have been employed to model the propagation of nerve impulses through axons [45], the formation of spots and stripes on animal skin [78], the development of vegetation patterns [63] and the dynamics of flame fronts arising in combustion theory [127].

Turing laid the foundation for reaction-diffusion systems as a prototype model for pattern formation. In [112] he showed that in linear reaction-diffusion systems patterns emerge from a uniform initial condition if two components diffuse at (very) different rates. Later Gierer and Meinhardt extended this to the semi-linear regime [40] – see also [61]. Nowadays, singularly perturbed, semi-linear reaction-diffusion systems on the line of the form,

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H(u,v,\varepsilon), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u,v,\varepsilon), \end{aligned} \qquad u(\check{x},t) \in \mathbb{R}^m, v(\check{x},t) \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is asymptotically small and $D_{1,2}$ are non-negative diagonal matrices, serve as a paradigmatic class for the study of patterns. Complex patterns often consist of simpler building blocks such as pulses, fronts, periodic wave trains and wave packets that are stationary or propagate with a constant speed – see Figure 1.1. Mathematical understanding of these elementary patterns is essential to gain fundamental insights into the more complex patterns.

Many analytical methods have been developed to construct (elementary) pattern solutions to (1.1) and to study the naturally associated issue of their dynamic stability. All of these methods exploit the presence of the small parameter ε, which induces a reduction of complexity in both the existence and stability analyses.



(a) Front (b) Pulse (c) Wave train (d) Wave packet

Figure 1.1: Elementary patterns

1.1 Existence of patterns

Solutions to (1.1) that are stationary or travel with a constant speed, can be written as $\varphi(\check{x}, t) = Q(\varepsilon^{-1}\check{x} + ct)$, where $Q\colon \mathbb{R} \to \mathbb{R}^{m+n}$ is the wave profile and $c \in \mathbb{R}$ is the wave speed (which is possibly 0). Finding such traveling-wave solutions is equivalent to finding bounded solutions to the ordinary differential equation,

$$\begin{aligned} u_\xi &= \varepsilon p, \\ D_1 p_\xi &= \varepsilon\left(c\varepsilon p + H(u, v, \varepsilon)\right), \\ v_\xi &= q, \\ D_2 q_\xi &= cq + G(u, v, \varepsilon), \end{aligned} \qquad (u, p, v, q) \in \mathbb{R}^{2(m+n)}, \tag{1.2}$$

where $\xi = \varepsilon^{-1}\check{x} + ct$. Clearly, a heteroclinic or homoclinic connection in (1.2) gives rise to a (traveling) front or pulse solution to (1.1). Similarly, a periodic orbit in (1.2) yields a periodic wave train solution to (1.1).

There are various approaches to construct solutions to singularly perturbed problems of the form (1.2). We mention the classical technique of rigorous matched asymptotic expansions [31, 68, 71] and the method via nonstandard analysis [20]. In this thesis we adopt a geometric point of view, which originates in the work of Fenichel [34]. Using *geometric singular perturbation theory* [34, 54, 57], solutions to (1.2) can be constructed in the following way. First, *slow and fast reduced systems* are established by taking the limit $\varepsilon \to 0$ in properly scaled versions of (1.2). Then, one obtains a so-called singular orbit, by piecing together orbit segments of these reduced systems – see Figure 1.2. Finally, one proves that an actual solution to (1.2) lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. In this last step exchange lemmas [55, 57, 59, 102] or blow-up techniques [66, 67] can be employed to control the dynamics in a neighborhood of the singular orbit.

Paradigmatic examples for the construction of traveling-wave solutions are the FitzHugh-Nagumo equations [35, 79] for nerve propagation, the Gierer-Meinhardt system [40, 80] in morphogenesis and the Gray-Scott model [43] for autocatalytic reactions. Using geometric singular perturbation theory, traveling pulses [11, 52] and periodic wave trains [110] have been constructed in the FitzHugh-Nagumo equations. Similarly, stationary (multi-)pulse patterns [21, 25, 26, 76] have been obtained in the Gray-Scott and Gierer-Meinhardt models. Yet, the geometric construction of solutions to (1.2) can be performed in a general setting without restricting to one of the aforementioned prototype models. For instance, in [110] one proves the existence of a periodic orbit in (1.2) under the assumption that a singular periodic orbit exists and the slow and fast reduced systems satisfy certain transversality conditions.

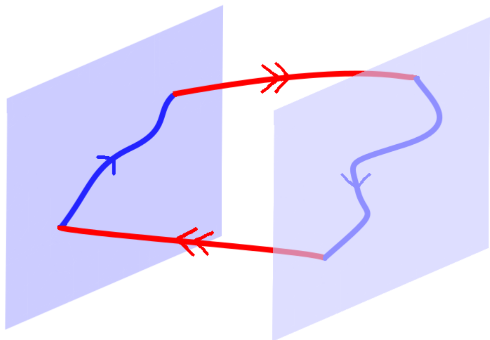


Figure 1.2: A singular periodic orbit consisting of slow orbit segments on two invariant manifolds (blue) and fast heteroclinic transitions (red).

1.2 Stability of patterns

The stability of solutions to (1.1) is in many cases determined by the spectral properties of the linearization of (1.1) about the solution. Yet, it is often difficult – especially in multi-component systems – to obtain the required spectral information to decide upon stability. As in the existence problem (1.2), the presence of the small parameter ε yields a reduction of complexity. Various methods have been developed to capture the underlying structure that facilitates this reduction – see Remark 1.1. The majority of these methods are built on the complex-analytic *Evans function* [1, 5, 33, 38], which vanishes precisely on the critical spectrum. Thus, to decide upon stability, it is sufficient to locate the roots of the Evans function. It was first observed by Alexander, Gardner and Jones [1] in the context of traveling pulses in the FitzHugh-Nagumo equations that the Evans function $\mathcal{E}_\varepsilon$ factorizes into a slow and a fast component,

$$\mathcal{E}_\varepsilon = \mathcal{E}_{s,\varepsilon} \cdot \mathcal{E}_{f,\varepsilon}, \tag{1.3}$$

in accordance with the scale separation in (1.1). The factors $\mathcal{E}_{s,\varepsilon}$ and $\mathcal{E}_{f,\varepsilon}$ correspond to lower-dimensional, slow and fast eigenvalue problems associated with the linearization. Although the geometric arguments behind the decomposition in [1] are very general, they need to be based on an analytical result that indeed controls the relevant slow and fast eigenvalue problems in projective space. In subsequent work, Gardner and Jones [37] validated the geometric argument of [1], and thus the factorization (1.3), in the context of traveling fronts in a predator-prey model. They tracked the slow and fast eigenvalue problems analytically via the so-called *elephant trunk lemma*. Further technical adaptations of the elephant trunk lemma to stability problems associated with localized structures in the Gray-Scott and Fabry-Pérot model have been carried out in [22] and [95], respectively, whereas its extension to periodic wave trains in the FitzHugh-Nagumo equations can be found in [32]. Nowadays, it is widely accepted that the elephant trunk procedure can be mimicked – or better: adapted – for a large class of systems of the form (1.1). However, for every application one should in principle go through one of the extensive proofs developed in the setting of the aforementioned specific systems to check whether technicalities still hold true.

By tracking the fast eigenvalue problem through the elephant trunk procedure, it is possible to derive an explicit analytic fast Evans function $\mathcal{E}_{f,0}$ whose zeros approximate those of $\mathcal{E}_{f,\varepsilon}$. Moreover, an explicit, but meromorphic, slow Evans function $\mathcal{E}_{s,0}$ can be obtained via the so-called *NonLocal Eigenvalue Problem* (NLEP) approach, which was established by Doelman, Gardner and Kaper in [22] in the context of stationary pulse solutions in the Gray-Scott model. Thus, a combination of the elephant trunk procedure and the NLEP approach yields an analytic *reduced Evans function*,

$$\mathcal{E}_0 = \mathcal{E}_{s,0} \cdot \mathcal{E}_{f,0}, \tag{1.4}$$

whose factors $\mathcal{E}_{s,0}$ and $\mathcal{E}_{f,0}$ can be derived explicitly through slow and fast *reduced* eigenvalue problems. These reduced eigenvalue problems are *lower-dimensional* and arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of the full eigenvalue problem. Winding number arguments imply that the roots of the Evans function $\mathcal{E}_\varepsilon$ are approximated by the ones of the reduced Evans function $\mathcal{E}_0$ – see Figure 1.3. The NLEP approach and thus the validation of the decomposition (1.3) and its explicit reduction (1.4), was further developed in the context of localized pulses, fronts or periodic solutions in certain classes of 2- or 3-component reaction-diffusion systems in [21, 23, 30, 114, 116, 120]. It should be remarked that in neither of these papers the elephant trunk procedure is carried out in full analytical detail.

Hence, using the elephant trunk and NLEP procedures, the spectrum of the linearization is approximated by the zeros of the reduced Evans function $\mathcal{E}_0$, which is defined in terms

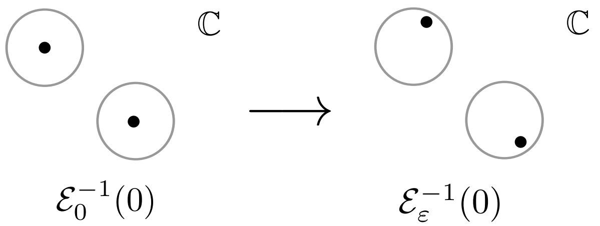
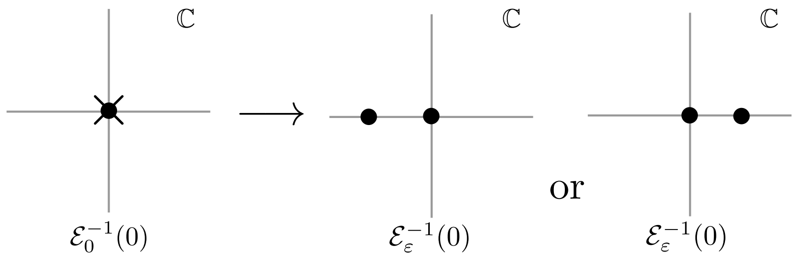


Figure 1.3: Spectral approximation using winding number arguments: the roots of $\mathcal{E}_0$ approximate those of $\mathcal{E}_\varepsilon$.

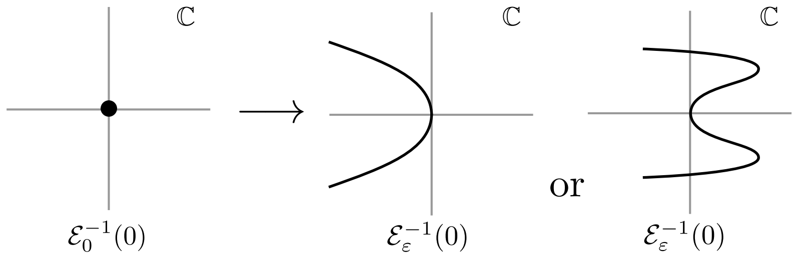
of lower-dimensional, reduced eigenvalue problems. Thus, locating the roots of $\mathcal{E}_0$ yields asymptotic control over the spectrum. In some cases asymptotic spectral control is sufficient to decide upon stability, however the situation is often delicate about the origin – note that the origin must be part of the spectrum due to translational invariance (obtained by shifting the profile in space) – and a local higher-order analysis is required to prove stability. We elaborate on the latter.

The spectrum of the linearization is made up of point spectrum, consisting of isolated eigenvalues of finite multiplicity, and its complement, the essential spectrum – see [98]. Suppose the essential spectrum is confined to the open left half-plane and $\mathcal{E}_0$ has no zeros in the closed right half-plane except a *simple* root at 0. Then, the aforementioned spectral approximation result implies that 0 is an isolated, simple eigenvalue and that the rest of the spectrum lies in the open left half-plane, which yields nonlinear stability of the underlying pattern [44]. This situation occurs for instance in [3, 37, 47].

However, one is often less fortunate. For example, in the stability analyses [10, 53, 126] of traveling pulses in the FitzHugh-Nagumo equations, 0 is a double root of the reduced Evans function $\mathcal{E}_0$, while the essential spectrum is confined to the open left half-plane. Consequently, there are two eigenvalues close to the origin. One of these eigenvalues resides at the origin due to translational invariance, while the position of the other eigenvalue with respect to



(a) Point spectrum associated with fast pulses in the FitzHugh-Nagumo equations



(b) Essential spectrum attached to the origin associated with periodic structures

Figure 1.4: Asymptotic spectral control is insufficient to determine the spectral configuration about the origin.

the imaginary axis is decisive for stability. Yet, asymptotic spectral control is insufficient to determine its location – see Figure 1.4a. The situation becomes even more problematic in the case of *periodic* patterns [32, 114]. Then, the spectrum is composed of curves parameterized over the unit circle S^1 due to Floquet theory [38]. Consequently, all spectrum is essential and one needs to characterize an entire curve of eigenvalues – the so-called *linear dispersion relation* – which is attached to the origin and shrinks to the origin in the limit $\varepsilon \to 0$. Again, asymptotic spectral control is insufficient to ascertain the location of the curve with respect to the imaginary axis – see Figure 1.4b. We elaborate on the various ways that have been developed to overcome these difficulties.

For traveling pulses in the FitzHugh-Nagumo equations, we discuss two approaches to obtain leading-order control over the location of the critical eigenvalue. In [53, 126] one proves that the derivative of the Evans function $\mathcal{E}_\varepsilon$ at 0 is positive, which follows from geometric properties of the pulse profile in the limit $\varepsilon \to 0$. Then, a parity argument implies that the critical eigenvalue is real and negative, which yields nonlinear stability. In [10] one constructs using *Lin's method* [70, 97, 118] a piecewise continuous eigenfunction of the linearization for each prospective eigenvalue λ near the origin. The eigenfunction admits exactly two jumps that occur in the middle of the front and the back of the pulse profile – see Figure 1.5. Finding eigenvalues reduces to identifying values of λ for which these jumps vanish. Melnikov theory provides leading-order expressions for these jumps that can be solved for λ, which yields that the critical eigenvalue lies in the open left half-plane. An advantage of this method over the one in [53, 126] is that one obtains leading-order expressions for the eigenvalues near the origin rather than only their signs. Therefore, it applies more generally to situations where there are more than two eigenvalues near the origin – see Remark 1.3.

In the stability analyses [32, 114] of periodic structures in the FitzHugh-Nagumo and Gierer-Meinhardt models, an entire curve of eigenvalues is attached to the origin, which shrinks to the origin in the limit $\varepsilon \to 0$. As in [10] – but with slightly different methods – one proceeds by

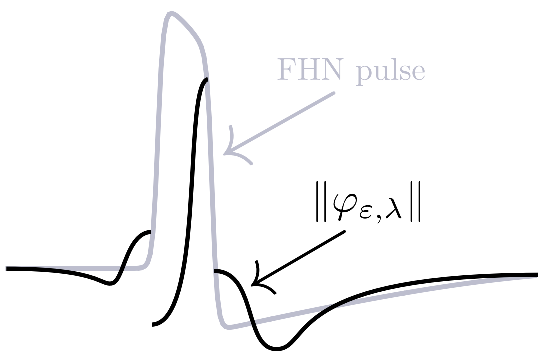


Figure 1.5: A piecewise continuous eigenfunction $\varphi_{\varepsilon,\lambda}$ with jumps occurring in the middle of the front and the back of the fast pulse to the FitzHugh-Nagumo equations.

constructing eigenfunctions corresponding to potential eigenvalues near the origin. A careful matching procedure then gives solvability conditions in terms of the eigenvalue λ, the Floquet multiplier γ and the corresponding eigenfunction. By expanding the eigenfunction in terms of ε, a leading-order expression for the critical spectral curve $\lambda_\varepsilon(\gamma)$ is derived from these solvability conditions. Finally, by complementing the local spectral analysis about the origin with the Evans-function analysis, stability or instability of the periodic pattern is established depending on the system parameters.

Remark 1.1. The concept of the Evans function as a method to determine the spectrum associated with a solution to a system of singularly perturbed reaction-diffusion equations on the line was introduced in [33] and was established as a general and powerful approach in [1, 37, 38, 53]. Core aspects of the NLEP approach have been developed independently in [50, 123]. The SLEP (= Singular Limit Eigenvalue Problem) method [81, 82] is an alternative method that has been linked to the Evans function approach in [49]. In [21], the relation between the Evans function, the NLEP method and the SLEP method is discussed. ■

Remark 1.2. It is a general phenomenon that the presence of essential spectrum about the origin is an issue in the stability analysis of periodic structures. Besides the above-mentioned methods for singularly perturbed reaction-diffusion systems of the form (1.1), let us mention that for periodic waves of conservation laws, Whitham's modulation equations [51, 107] provide an accurate description of the spectral configuration about the origin. Moreover, for periodic wave trains to general reaction-diffusion systems one can compute [28] the derivative $\lambda_*''(0)$ of the critical curve $\lambda_*(\gamma)$ attached to the origin in terms of derivatives of the wave train and the corresponding solution to the adjoint eigenvalue problem. Yet, knowing (the sign of) $\lambda_*''(0)$ is insufficient to control the *entire* spectral curve $\lambda_* : S^1 \to \mathbb{C}$. ■

Remark 1.3. In [50, 64, 122] the stability of multi-pulse solutions to the Gierer-Meinhardt, Gray-Scott and Schnakenberg models is investigated using formal asymptotic expansions. The formal analysis yields the existence of multiple eigenvalues close to the origin and provides leading-order expressions for these eigenvalues. We expect that the above-mentioned approach in [10] using Lin's method could be employed to establish the existence and position of these eigenvalues rigorously. Let us emphasize that, for general semilinear parabolic equations, the stability of multi-pulse solutions bifurcating from a stable primary pulse has already been determined successfully in [97] using Lin's method. ■

1.3 Extension beyond prototype models: slow nonlinearity

The aforementioned spectral methods, including the elephant trunk and NLEP procedures, have been developed in the context of specific (prototype) models, such as the Gray-Scott, Gierer-Meinhardt and FitzHugh-Nagumo equations. These models are of *slowly linear* nature, in the sense that the dynamics of the slow u-components in between localized fast pulses or fronts are driven by linear equations. Thus, the slow reduced system arising in the existence analysis is linear and the slow eigenvalue problem in the stability analysis is autonomous. In the context of the periodic pulse solution shown in Figure 1, slow linearity entails that the dynamics of (1.1) in the rest state $v = 0$ is linear, i.e. the coupling term $H(u, 0, \varepsilon)$ in (1.1) is linear. In recent work [30, 120] an NLEP approach has been carried out for homoclinic pulse solutions to a general class of *slowly nonlinear*, 2-component systems of the form (1.1). Earlier, the stability of fronts was studied in a specific slowly nonlinear model in [23].

The introduction of a slow nonlinearity in (1.1) yields more than just additional technicalities. For instance, it is shown in [119] that, unlike known classical slowly linear examples such as the Gray-Scott and Gierer-Meinhardt models, Hopf bifurcations for homoclinic pulses can be supercritical. Such a bifurcation could even be the first step in a sequence of further bifurcations leading to complex (amplitude) dynamics of a standing solitary pulse – as observed in the simulations in [120].

The slow linearity plays a crucial role in the analysis of the Evans function and its decomposition and reduction. In fact, it is essential for an application of the elephant trunk procedure that the eigenvalue problem is to leading order linear near the boundaries of the spatial domain – as is the case for homoclinic and periodic pulse solutions to the slowly linear Gierer-Meinhardt model in [21, 114]. Although the models in [23, 30, 120] are slowly nonlinear, the elephant trunk procedure is still applicable, because eventually the dynamics becomes linear due to the homoclinic or heteroclinic nature of the patterns. However, the eigenvalue problem associated with *periodic* solutions to slowly nonlinear models is non-autonomous over the *entire* domain, thus obstructing an application of the elephant trunk lemma. This brings us to the main goal of this thesis: *extending the spectral analysis of periodic structures in reaction-diffusion systems of the form* (1.1) *beyond the slowly linear regime.*

1.4 Contents of this thesis

In this thesis we study stationary, spatially periodic pulse solutions to singularly perturbed reaction-diffusion systems of the form (1.1), allowing for general dimensions $n, m \geq 1$ and a large class of nonlinearities H and G – see §1.4.1 for the precise details. The solutions under consideration are spatially symmetric and exhibit exponentially localized pulses in the fast v-components, but admit non-localized behavior in the slow u-components – see Figure 1. In other words, they are in semi-strong interaction [24] (of second order [92]). Our class of equations includes the Gierer-Meinhardt system. Thus, on the one hand, we extend the existence and stability analyses [25, 114] for periodic pulse solutions in the Gierer-Meinhardt equations to the slowly nonlinear and multi-dimensional regime. On the other hand, our work can be considered as the extension from homoclinic to periodic structures within the general, slowly nonlinear class of systems in [30].

However, this thesis is not a straightforward extension of [30, 114], since there is – as outlined in §1.3 – no obvious adaptation of the elephant trunk lemma for spatially periodic patterns in slowly nonlinear systems. Therefore, we present a generalized analytic alternative to both the elephant trunk and NLEP procedures to establish the validity of both the decomposition (1.3) of the Evans function and its singular limit structure (1.4). This analytic method is based on the *Riccati transformation* [12, 13]. This transformation, which satisfies a matrix Riccati equation, diagonalizes the associated eigenvalue problem and thus explicitly separates fast from slow dynamics. The separation yields the factorization of the Evans function (1.3) and provides a framework for the passage to the singular limit (1.4). We emphasize that our factorization procedure applies beyond the current setting of periodic pulse solutions and is therefore interesting in its own right – see Remark 1.4.

Thus, using the analytic factorization method, we obtain a reduced Evans function $\mathcal{E}_0$, whose roots approximate the spectrum and whose factors $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$ are defined in terms of explicit, lower-dimensional reduced eigenvalue problems. Thus, we obtain asymptotic control over the spectrum. However, as mentioned in §1.2, there is a second challenge: asymptotic spectral control is insufficient to decide upon stability, since there is a curve of essential spectrum which shrinks to the origin in the limit $\varepsilon \to 0$. Therefore, we need to complement our Evans-function analysis with a local analysis about the origin in order to obtain leading-order control over this critical spectral curve.

Our analysis of the critical spectral curve is based on [10, 100]. Recall from §1.2 that the stability of (fast) traveling pulses in the FitzHugh-Nagumo equations is decided by the location of a nontrivial eigenvalue near the origin. The method adopted in [10] yields a leading-order expression of this critical eigenvalue in terms of the small parameter ε. Furthermore, in [100] the spectral configuration about the origin is determined for periodic wave trains that accompany homoclinic pulse solutions to general reaction-diffusion systems. An expansion of the critical spectral curve is provided in terms of the period. The situations in [10] and [100] do not directly translate to our situation, since we consider periodics that do not lie in the vicinity of a homoclinic.

Nevertheless, we adopt a similar approach: using Lin's method we obtain a piecewise continuous eigenfunction of the linearization for any potential eigenvalue λ near the origin. In contrast to [100], we do not use the homoclinic limit structure as a framework for the construction of the eigenfunction; instead the singular limit structure serves as a backbone like in [10]. On the other hand, as in [100], Floquet theory yields boundary conditions for the eigenfunction on a single periodicity interval, whereas one requires in [10] that the eigenfunction is exponentially localized on the real line. The construction of the piecewise continuous eigenfunction yields a Lyapunov-Schmidt type reduction procedure: finding the critical spectral curve attached to the origin reduces to equating the jumps to zero. The Fredholm alternative allows us to find expressions for these jumps that can then be solved. Eventually, we obtain a leading-order expression for the critical spectral curve in terms of lower-dimensional, variational equations about the orbit segments that constitute the pulse profile in the limit $\varepsilon \to 0$.

Thus, we gain both asymptotic control over the spectrum through the reduced Evans function and leading-order control over the critical spectral curve. This leads to *explicit* criteria yielding stability and instability of the periodic pulse solution in terms of simpler, lower-dimensional problems. These conditions can be interpreted in more simple cases in which either $n = 1$, $m = 1$, or both $n = m = 1$. In the latter case, we directly recover the expressions obtained in the stability analysis [114] of spatially periodic pulse patterns in the Gierer-Meinhardt equation. The outcome of our spectral analysis shows that the Gierer-Meinhardt setting represents a very special case. The restriction to this specific system obscures the underlying general structure of the reduced Evans function and the critical spectral curve in terms of simpler, lower-dimensional problems. On the other hand, the restriction of (1.1) to a more general, slowly nonlinear, 2-component model as in [30] yields a (relatively) simple instability criterion in terms of the signs of a number of explicit expressions that can be computed with only an asymptotic approximation of the underlying pattern as input. Thereby, we extend a similar result of [30] on homoclinic pulses to periodic structures.

The analytical grip on the spectrum provides insights into destabilization mechanisms of periodic pulse solutions to (1.1). Depending on which one of the aforementioned stability criteria fails, we can identify the type of instability occurring. We establish that generic (primary) instabilities must be of sideband, Hopf or period doubling type, whereas in general reaction-diffusion systems also Turing and fold instabilities are robust for symmetric, spatially periodic patterns [93].

Destabilization mechanisms become rather complex when periodic patterns approach a homoclinic limit. While increasing the wavelength, the character of destabilization alternates between two kinds of Hopf instabilities. This phenomenon is called the *Hopf dance* [27, 115]. It has been analytically established in (slowly linear) Gierer-Meinhardt models in [27] and recovered by numerical methods in the generalized Klausmeier-Gray-Scott model [27, 115]. Both the Hopf dance as well as the *belly dance* [27] – an associated higher order phenomenon – can be analyzed in the general, slowly nonlinear setting of (1.1) by the methods developed here. In addition, we establish an explicit sign criterion to determine whether the homoclinic pulse solution is the last or the first 'periodic' solution to destabilize.

Finally, we comment on the existence of stationary, periodic pulse patterns to (1.1). Our construction of these solutions relies on geometric singular perturbation theory – see §1.1. First, we establish a singular periodic orbit by piecing together orbit segments of slow and fast reduced systems in such a way that they form a closed loop. Then, we prove that an actual periodic orbit lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. The construction respects the symmetry $\check{x} \mapsto -\check{x}$ of system (1.1). Consequently, the periodic pulse solution is spatially symmetric. The existence result is a significant extension of similar results in the literature that only consider 2-component, Gierer-Meinhardt type models [25].

As a final remark, let us mention that we illustrate our existence and stability results by explicit calculations in a slowly nonlinear toy model.

Remark 1.4. The analytic factorization procedure of the Evans function can be outlined in a way that neither depends on the specific structure of the system nor on the specific patterns under consideration. We require that the eigenvalue problem associated with the linearization of (1.1) about the pattern can be written in block-matrix form,

$$\begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon} A_{11,\varepsilon}(x,\lambda) & \sqrt{\varepsilon} A_{12,\varepsilon}(x,\lambda) \\ A_{21,\varepsilon}(x,\lambda) & A_{22,\varepsilon}(x,\lambda) \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{1.5}$$

in the rescaled spatial variable $x = \varepsilon^{-1}\check{x}$. Consider the reduced eigenvalue problem,

$$\psi_x = A_{22,0}(x,\lambda)\psi, \tag{1.6}$$

in which $A_{22,0}(x,\lambda)$ represents the singular limit of $A_{22,\varepsilon}(x,\lambda)$. Equation (1.6) admits an exponential dichotomy [14, 96] on $\mathbb{R}$ as long as the associated differential operator,

$$\mathcal{L}_\lambda \psi = \psi_x - A_{22,0}(\cdot,\lambda)\psi,$$

is invertible – see also [98]. Since exponential dichotomies are robust against small perturbations, the exponential dichotomy of (1.6) on $\mathbb{R}$ carries over to the perturbed problem,

$$\psi_x = A_{22,\varepsilon}(x,\lambda)\psi, \tag{1.7}$$

for $0 < \varepsilon \ll 1$. This exponential dichotomy on $\mathbb{R}$ of (1.7) allows us to successfully diagonalize the eigenvalue problem (1.5) with the Riccati transformation yielding the factorization (1.3) of the Evans function. In the last step, we approximate the two blocks, in which (1.5) diagonalizes, by their singular limits yielding (1.4). As a consequence, the roots of the Evans function can be approximated by the roots of the reduced Evans function $\mathcal{E}_0$.

We stress that our factorization method applies in particular to the context [30] of *homoclinic* pulse solutions in a large class of 2-component, slowly nonlinear systems. We expect that our method could extend the results in [30] to a multi-component setting. In addition, let us mention that a uv-term in the v-component of (1.1) is not allowed in [30], whereas our method can handle such terms. ∎

1.4.1 Setting

In this section we introduce the class of systems under consideration in this thesis. Take $m, n \in \mathbb{Z}_{>0}$ and consider a general reaction-diffusion system in one space dimension with a scale separation in the diffusion lengths (1.1). We assume that the diagonal matrices $D_{1,2}$ in (1.1) are *positive*. Following [30], we write

$$H(u,v,\varepsilon) = H(u,0,\varepsilon) + \tilde{H}_2(u,v,\varepsilon),$$

where $\tilde{H}_2(u,v,\varepsilon) := H(u,v,\varepsilon) - H(u,0,\varepsilon)$, so that $\tilde{H}_2$ vanishes at $v = 0$. To sustain stable localized patterns in semi-strong interaction (of second order [92]) in system (1.1), we allow $\tilde{H}_2(u,v,\varepsilon)$ to scale with ε^{-1} and define

$$H_2(u,v) := \lim_{\varepsilon\to 0} \varepsilon \tilde{H}_2(u,v,\varepsilon).$$

Finally, we write

$$H(u, v, \varepsilon) = H_1(u, v, \varepsilon) + \varepsilon^{-1} H_2(u, v), \tag{1.8}$$

with $H_1(u, v, \varepsilon) := H(u, 0, \varepsilon) + [\tilde{H}_2(u, v, \varepsilon) - \varepsilon^{-1} H_2(u, v)]$. By construction $H_2(u, v)$ vanishes at $v = 0$. We assume that $H_1(u, v, \varepsilon)$ and $G(u, v, \varepsilon)$ are smooth functions of ε at $\varepsilon = 0$. Note that we allow for the possibility that $H_2(u, v) \equiv 0$ in the upcoming analysis. We emphasize that, if we have in addition $n = 1$ or $m = 1$, then all patterns are unstable – see Remark 3.22. This confirms the scalings used for classical systems as the Gray-Scott and Gierer-Meinhardt models [21, 22, 50, 123] – see also [30]. For the benefit of our spectral analysis, we need one extra condition on G. That is, G vanishes at $v = 0$. We postpone the discussion of this extra condition to Remark 2.6. In summary, the model class we consider is of the form

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H_1(u, v, \varepsilon) - \varepsilon^{-1} H_2(u, v), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u, v, \varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, \check{x} \in \mathbb{R}, \tag{1.9}$$

or, in the 'small' spatial scale $x = \varepsilon^{-1} \check{x}$,

$$\begin{aligned} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u, v, \varepsilon) - \varepsilon H_2(u, v), \\ v_t &= D_2 v_{xx} - G(u, v, \varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, x \in \mathbb{R}, \tag{1.10}$$

in which we will usually work. The aforementioned conditions read:

(S1) ***Conditions on the interaction and diffusion terms***
There exists open, connected sets $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ with $0 \in V$ and $0 \in I$ such that H_1, G and H_2 are C^3 on their domains $U \times V \times I$ and $U \times V$, respectively. Moreover, we have $H_2(u, 0) = 0$ and $G(u, 0, \varepsilon) = 0$ for all $u \in U$ and $\varepsilon \in I$. Finally, $D_{1,2}$ are positive diagonal matrices.

Remark 1.5. If we have $n = 1$, we can without loss of generality assume $D_2 = 1$ in (1.9) by rescaling the spatial variable $\check{x}$. Similarly, in the case $m = 1$, we can without loss of generality assume $D_1 = 1$ by rescaling the parameter ε. ■

1.4.2 Outline

This thesis is structured as follows. In Chapter 2 we elaborate on the existence of periodic pulse solutions to (1.9) and we obtain fine estimates on the error between the periodic pulse solutions and the associated singular periodic orbit. We apply the existence result to construct periodic pulse solutions in an explicit slowly nonlinear toy model. In Chapter 3 we present the main results of our spectral analysis: the approximation of the spectrum by the roots of the reduced Evans function (1.4) and the expansion of the critical spectral curve. We obtain explicit conditions in terms of simpler, lower-dimensional systems yielding stability. Moreover, we test for instability by calculating the signs of a number of explicit expressions. Finally, we interpret these results in the lower-dimensional regime and apply them to the slowly nonlinear toy model. Chapter 4 contains prerequisites for our spectral analysis. In particular, we provide extensive background on exponential dichotomies and establish the Riccati transform, which provides a natural framework for the factorization of the Evans function – see Remark 1.4. In Chapter 5 we perform the actual spectral analysis and prove our main results. Chapter 6 focusses on destabilization mechanisms of periodic pulse solutions. Finally, in Chapter 7 we elaborate on future research possibilities.

The results presented in Chapter 2 and Sections 3.5, 3.8.1 and 3.8.4 appeared earlier in *Spectra and stability of spatially periodic pulse patterns: Evans function factorization via Riccati transformation* in the SIAM Journal on Mathematical Analysis in 2016 – see [17].