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Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

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To my grandparents.

Contents

Abstract	9
1 Introduction	11
1.1 Existence of patterns	12
1.2 Stability of patterns	13
1.3 Extension beyond prototype models: slow nonlinearity	17
1.4 Contents of this thesis	18
1.4.1 Setting	21
1.4.2 Outline	22
2 Existence analysis	25
2.1 Introduction	25
2.2 The singular limit	26
2.2.1 Slow-fast decomposition	26
2.2.2 Construction of the singular periodic pulse	27
2.3 Dynamics in the vicinity of the slow manifold	29
2.3.1 Fenichel fibering	30
2.3.2 Fenichel normal form	30
2.3.3 The Exchange lemma	31
2.4 Main existence result	32
2.5 Existence in the slowly nonlinear toy problem	38
3 Stability results	41
3.1 Introduction	41
3.2 Linearizing about the periodic pulse solution	42
3.2.1 Floquet-Bloch decomposition	42
3.3 Nonlinear stability by linear approximation	43
3.3.1 Spectral conditions yielding nonlinear stability	43
3.3.2 Spectral conditions yielding nonlinear instability	45
3.4 The Evans function	45
3.5 The Evans function in the singular limit	47
3.5.1 The reduced Evans function	47

3.5.2	The spectral approximation result	50
3.5.3	Consequences of the spectral approximation result	52
3.6	Expansion of the critical spectral curve	53
3.7	Explicit criteria for spectral stability and instability	54
3.8	Stability results in lower dimensions	57
3.8.1	The reduced Evans function	57
3.8.2	The critical spectral curve	61
3.8.3	Criteria for spectral stability and instability	62
3.8.4	A closer look at zero-pole cancelation	66
3.9	Stability in the slowly nonlinear toy problem	68
4	Prerequisites for the spectral stability analysis	71
4.1	A Grönwall type estimate for linear systems	71
4.2	Asymptotically constant systems	72
4.3	Exponential dichotomies	72
4.3.1	Dichotomy projections	73
4.3.2	Sufficient criteria for exponential dichotomies	74
4.3.3	Extending and pasting exponential dichotomies	78
4.3.4	Roughness of exponential dichotomies	79
4.3.5	Inhomogeneous problems	80
4.4	Exponential trichotomies	81
4.5	The minimal opening between subspaces	82
4.6	The Riccati transformation	83
5	Spectral stability analysis	89
5.1	The reduced Evans function	89
5.1.1	The fast Evans function	89
5.1.2	The slow Evans function	91
5.2	Approximation of the roots of the Evans function	96
5.2.1	Introduction	96
5.2.2	A priori bounds on the spectrum	97
5.2.3	An exponential dichotomy capturing the fast dynamics	100
5.2.4	Factorization of the Evans function via the Riccati transform	104
5.2.5	Conclusion	110
5.2.6	Discussion	111
5.3	The critical spectral curve	113
5.3.1	Introduction	113
5.3.2	A reduced eigenvalue problem along the pulse	116
5.3.3	A reduced eigenvalue problem along the slow manifold	123
5.3.4	Construction of a piecewise continuous eigenfunction	127
5.3.5	Conclusion	137
5.3.6	Discussion	140

6 Destabilization mechanisms	143
6.1 Introduction	143
6.2 Classification of codimension-one instabilities	145
6.3 Generic destabilization mechanisms	146
6.3.1 The first destabilization scenario	147
6.3.2 The second destabilization scenario	148
6.3.3 The third destabilization scenario	150
6.4 Destabilization mechanisms in the homoclinic limit	151
6.4.1 Existence of homoclinic pulse solutions	152
6.4.2 Spectral properties of homoclinic pulse solutions	153
6.4.3 Destabilization mechanisms for homoclinic pulse solutions	155
6.4.4 Existence of a family of periodic pulse solutions approaching a homoclinic limit	155
6.4.5 Spectral geometry of long-wavelength periodic pulse solutions	157
6.4.6 Spectral stability of long-wavelength periodic pulse solutions	159
6.4.7 Hopf destabilization in the homoclinic limit	161
6.4.8 Proofs of key results	164
7 Outlook	179
7.1 Breaking the symmetry	179
7.2 Dynamics of periodic pulses upon destabilization	181
7.3 Multiple spatial dimensions	181
Nederlandse samenvatting	193
Dankwoord	197
Curriculum vitae	199

Abstract

Patterns arise frequently in reaction-diffusion systems with a strong spatial scale separation, which naturally leads to the question of their dynamic stability. The scale separation induces a slow-fast decomposition in both the existence and stability analyses, which reduces complexity. In the existence analysis, patterns can be obtained by concatenating orbit segments of *slow and fast reduced systems*. These patterns exhibit spatially localized fronts and pulses, while they vary slowly in between those localized interfaces – see also Figure 1. In the stability analysis, the slow-fast decomposition manifests itself through a complex-analytic determinant-type function: *the Evans function*, which vanishes on the spectrum of the linearization about the pattern. In many specific models it has been shown using geometric methods that the Evans function factorizes in accordance with the scale separation. This factorization corresponds to a decomposition of the spectrum into slow and fast components, which are explicitly determined by *slow and fast reduced eigenvalue problems* arising in an appropriate singular limit. Thus, the factorization method leads to asymptotic control over the spectrum.

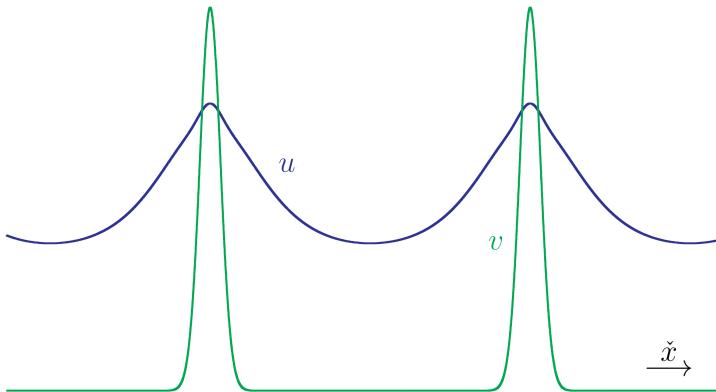


Figure 1: A periodic pulse solution in a reaction-diffusion system with two components. The v -component exhibits localized pulses and the u -component varies slowly.

The geometric factorization method has been developed in the context of *slowly linear* prototype models, in the sense that the associated slow reduced problems are linear. In the context of the periodic pulse solution shown in Figure 1, slow linearity entails that the dynamics of the slow u -component in between localized pulses is driven by linear equations. Recently, the geometric factorization procedure has been generalized to homoclinic pulse solutions in a general class of *slowly nonlinear* reaction-diffusion systems with two components. It has been shown that the dynamics in such systems differs fundamentally from their slowly linear counterparts. In this thesis we study *periodic* pulse solutions to a general class of *multi-component, slowly nonlinear* reaction-diffusion systems. At first sight this seems a straightforward extension of the homoclinic case. However, the geometric factorization method of the Evans function fails for periodic structures in slowly nonlinear systems. In addition, due to translational invariance of the pulse profile in space there is an entire curve of spectrum attached to the origin that shrinks to the origin in the singular limit, whereas for homoclinic pulse solutions there is only a simple eigenvalue residing at the origin. Therefore, in contrast to the homoclinic case, asymptotic control over the spectrum is insufficient to decide upon stability and a local higher-order analysis is required to determine the fine structure of the spectrum about the origin.

In this thesis we develop an alternative, analytic factorization method that does work for periodic structures in the slowly nonlinear regime, but applies more generally beyond this setting, i.e. it formalizes and generalizes the existing (geometric) factorization methods. We derive explicit formulas for the factors of the Evans function, which yield asymptotic control over the spectrum. Moreover, we obtain a leading-order expression for the critical spectral curve attached to origin. Together these spectral approximation results lead to explicit stability and instability criteria in terms of lower-dimensional reduced eigenvalue problems. Furthermore, the analytical grip on the spectrum provides insights into destabilization mechanisms of periodic pulse solutions, especially in the long-wavelength limit. Finally, we mention that, as a prerequisite for the stability analysis, we develop an existence theory for periodic pulse solutions with fine error estimates.