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Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

Rijk, B. de

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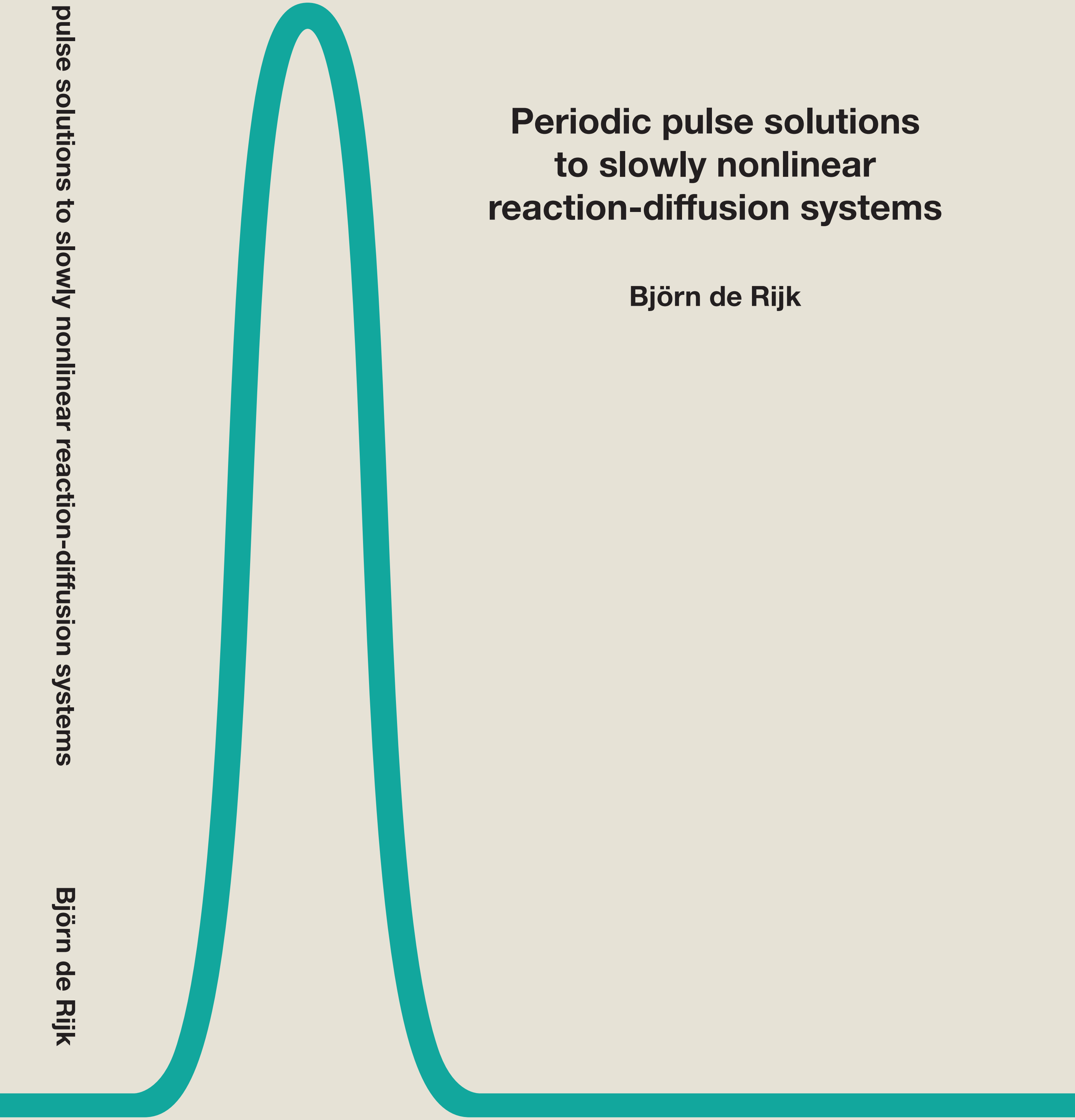

Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

Björn de Rijk

Periodic pulse solutions to slowly nonlinear reaction-diffusion systems

PROEFSCHRIFT

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To my grandparents.

Contents

Abstract

Patterns arise frequently in reaction-diffusion systems with a strong spatial scale separation, which naturally leads to the question of their dynamic stability. The scale separation induces a slow-fast decomposition in both the existence and stability analyses, which reduces complexity. In the existence analysis, patterns can be obtained by concatenating orbit segments of *slow and fast reduced systems*. These patterns exhibit spatially localized fronts and pulses, while they vary slowly in between those localized interfaces – see also Figure 1. In the stability analysis, the slow-fast decomposition manifests itself through a complex-analytic determinant-type function: *the Evans function*, which vanishes on the spectrum of the linearization about the pattern. In many specific models it has been shown using geometric methods that the Evans function factorizes in accordance with the scale separation. This factorization corresponds to a decomposition of the spectrum into slow and fast components, which are explicitly determined by *slow and fast reduced eigenvalue problems* arising in an appropriate singular limit. Thus, the factorization method leads to asymptotic control over the spectrum.

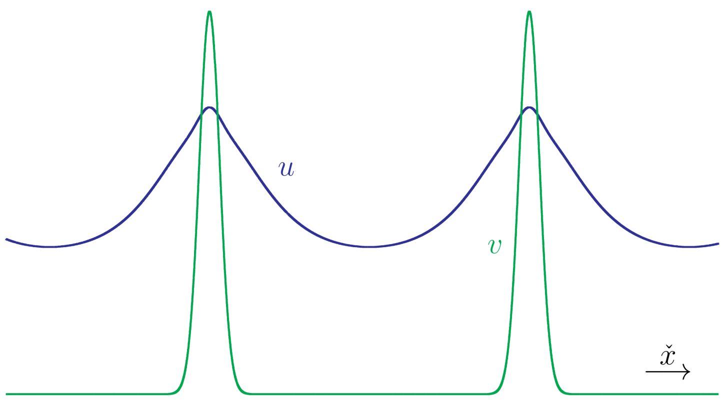


Figure 1: A periodic pulse solution in a reaction-diffusion system with two components. The v-component exhibits localized pulses and the u-component varies slowly.

The geometric factorization method has been developed in the context of *slowly linear* prototype models, in the sense that the associated slow reduced problems are linear. In the context of the periodic pulse solution shown in Figure 1, slow linearity entails that the dynamics of the slow u-component in between localized pulses is driven by linear equations. Recently, the geometric factorization procedure has been generalized to homoclinic pulse solutions in a general class of *slowly nonlinear* reaction-diffusion systems with two components. It has been shown that the dynamics in such systems differs fundamentally from their slowly linear counterparts. In this thesis we study *periodic* pulse solutions to a general class of *multi-component, slowly nonlinear* reaction-diffusion systems. At first sight this seems a straightforward extension of the homoclinic case. However, the geometric factorization method of the Evans function fails for periodic structures in slowly nonlinear systems. In addition, due to translational invariance of the pulse profile in space there is an entire curve of spectrum attached to the origin that shrinks to the origin in the singular limit, whereas for homoclinic pulse solutions there is only a simple eigenvalue residing at the origin. Therefore, in contrast to the homoclinic case, asymptotic control over the spectrum is insufficient to decide upon stability and a local higher-order analysis is required to determine the fine structure of the spectrum about the origin.

In this thesis we develop an alternative, analytic factorization method that does work for periodic structures in the slowly nonlinear regime, but applies more generally beyond this setting, i.e. it formalizes and generalizes the existing (geometric) factorization methods. We derive explicit formulas for the factors of the Evans function, which yield asymptotic control over the spectrum. Moreover, we obtain a leading-order expression for the critical spectral curve attached to origin. Together these spectral approximation results lead to explicit stability and instability criteria in terms of lower-dimensional reduced eigenvalue problems. Furthermore, the analytical grip on the spectrum provides insights into destabilization mechanisms of periodic pulse solutions, especially in the long-wavelength limit. Finally, we mention that, as a prerequisite for the stability analysis, we develop an existence theory for periodic pulse solutions with fine error estimates.

Chapter 1

Introduction

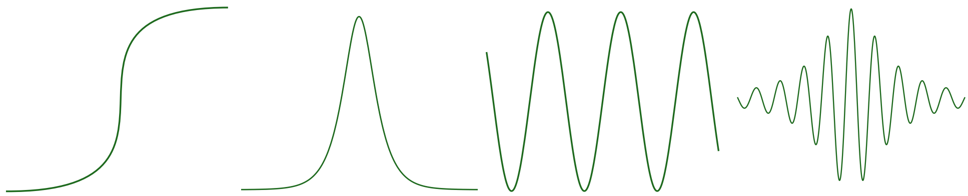
Pattern formation is observed in dynamical processes within various scientific disciplines, including chemistry, biology, neurophysiology, optics and ecology. Reaction-diffusion systems exhibit a large variety of patterns and have attracted much interest as a (simplified) model describing these dynamical processes. For instance, reaction-diffusion systems have been employed to model the propagation of nerve impulses through axons [45], the formation of spots and stripes on animal skin [78], the development of vegetation patterns [63] and the dynamics of flame fronts arising in combustion theory [127].

Turing laid the foundation for reaction-diffusion systems as a prototype model for pattern formation. In [112] he showed that in linear reaction-diffusion systems patterns emerge from a uniform initial condition if two components diffuse at (very) different rates. Later Gierer and Meinhardt extended this to the semi-linear regime [40] – see also [61]. Nowadays, singularly perturbed, semi-linear reaction-diffusion systems on the line of the form,

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H(u,v,\varepsilon), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u,v,\varepsilon), \end{aligned} \qquad u(\check{x},t) \in \mathbb{R}^m, v(\check{x},t) \in \mathbb{R}^n, \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is asymptotically small and $D_{1,2}$ are non-negative diagonal matrices, serve as a paradigmatic class for the study of patterns. Complex patterns often consist of simpler building blocks such as pulses, fronts, periodic wave trains and wave packets that are stationary or propagate with a constant speed – see Figure 1.1. Mathematical understanding of these elementary patterns is essential to gain fundamental insights into the more complex patterns.

Many analytical methods have been developed to construct (elementary) pattern solutions to (1.1) and to study the naturally associated issue of their dynamic stability. All of these methods exploit the presence of the small parameter ε, which induces a reduction of complexity in both the existence and stability analyses.



(a) Front (b) Pulse (c) Wave train (d) Wave packet

Figure 1.1: Elementary patterns

1.1 Existence of patterns

Solutions to (1.1) that are stationary or travel with a constant speed, can be written as $\varphi(\check{x}, t) = Q(\varepsilon^{-1}\check{x} + ct)$, where $Q \colon \mathbb{R} \to \mathbb{R}^{m+n}$ is the wave profile and $c \in \mathbb{R}$ is the wave speed (which is possibly 0). Finding such traveling-wave solutions is equivalent to finding bounded solutions to the ordinary differential equation,

$$\begin{aligned} u_\xi &= \varepsilon p, \\ D_1 p_\xi &= \varepsilon\left(c\varepsilon p + H(u,v,\varepsilon)\right), \\ v_\xi &= q, \\ D_2 q_\xi &= cq + G(u,v,\varepsilon), \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^{2(m+n)}, \tag{1.2}$$

where $\xi = \varepsilon^{-1}\check{x} + ct$. Clearly, a heteroclinic or homoclinic connection in (1.2) gives rise to a (traveling) front or pulse solution to (1.1). Similarly, a periodic orbit in (1.2) yields a periodic wave train solution to (1.1).

There are various approaches to construct solutions to singularly perturbed problems of the form (1.2). We mention the classical technique of rigorous matched asymptotic expansions [31, 68, 71] and the method via nonstandard analysis [20]. In this thesis we adopt a geometric point of view, which originates in the work of Fenichel [34]. Using *geometric singular perturbation theory* [34, 54, 57], solutions to (1.2) can be constructed in the following way. First, *slow and fast reduced systems* are established by taking the limit $\varepsilon \to 0$ in properly scaled versions of (1.2). Then, one obtains a so-called singular orbit, by piecing together orbit segments of these reduced systems – see Figure 1.2. Finally, one proves that an actual solution to (1.2) lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. In this last step exchange lemmas [55, 57, 59, 102] or blow-up techniques [66, 67] can be employed to control the dynamics in a neighborhood of the singular orbit.

Paradigmatic examples for the construction of traveling-wave solutions are the FitzHugh-Nagumo equations [35, 79] for nerve propagation, the Gierer-Meinhardt system [40, 80] in morphogenesis and the Gray-Scott model [43] for autocatalytic reactions. Using geometric singular perturbation theory, traveling pulses [11, 52] and periodic wave trains [110] have been constructed in the FitzHugh-Nagumo equations. Similarly, stationary (multi-)pulse patterns [21, 25, 26, 76] have been obtained in the Gray-Scott and Gierer-Meinhardt models. Yet, the geometric construction of solutions to (1.2) can be performed in a general setting without restricting to one of the aforementioned prototype models. For instance, in [110] one proves the existence of a periodic orbit in (1.2) under the assumption that a singular periodic orbit exists and the slow and fast reduced systems satisfy certain transversality conditions.

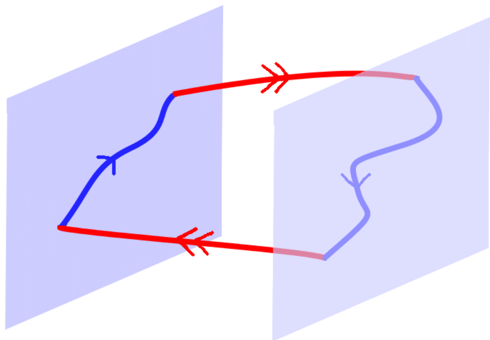


Figure 1.2: A singular periodic orbit consisting of slow orbit segments on two invariant manifolds (blue) and fast heteroclinic transitions (red).

1.2 Stability of patterns

The stability of solutions to (1.1) is in many cases determined by the spectral properties of the linearization of (1.1) about the solution. Yet, it is often difficult – especially in multi-component systems – to obtain the required spectral information to decide upon stability. As in the existence problem (1.2), the presence of the small parameter ε yields a reduction of complexity. Various methods have been developed to capture the underlying structure that facilitates this reduction – see Remark 1.1. The majority of these methods are built on the complex-analytic *Evans function* [1, 5, 33, 38], which vanishes precisely on the critical spectrum. Thus, to decide upon stability, it is sufficient to locate the roots of the Evans function. It was first observed by Alexander, Gardner and Jones [1] in the context of traveling pulses in the FitzHugh-Nagumo equations that the Evans function $\mathcal{E}_\varepsilon$ factorizes into a slow and a fast component,

$$\mathcal{E}_\varepsilon = \mathcal{E}_{s,\varepsilon} \cdot \mathcal{E}_{f,\varepsilon}, \tag{1.3}$$

in accordance with the scale separation in (1.1). The factors $\mathcal{E}_{s,\varepsilon}$ and $\mathcal{E}_{f,\varepsilon}$ correspond to lower-dimensional, slow and fast eigenvalue problems associated with the linearization. Although the geometric arguments behind the decomposition in [1] are very general, they need to be based on an analytical result that indeed controls the relevant slow and fast eigenvalue problems in projective space. In subsequent work, Gardner and Jones [37] validated the geometric argument of [1], and thus the factorization (1.3), in the context of traveling fronts in a predator-prey model. They tracked the slow and fast eigenvalue problems analytically via the so-called *elephant trunk lemma*. Further technical adaptations of the elephant trunk lemma to stability problems associated with localized structures in the Gray-Scott and Fabry-Pérot model have been carried out in [22] and [95], respectively, whereas its extension to periodic wave trains in the FitzHugh-Nagumo equations can be found in [32]. Nowadays, it is widely accepted that the elephant trunk procedure can be mimicked – or better: adapted – for a large class of systems of the form (1.1). However, for every application one should in principle go through one of the extensive proofs developed in the setting of the aforementioned specific systems to check whether technicalities still hold true.

By tracking the fast eigenvalue problem through the elephant trunk procedure, it is possible to derive an explicit analytic fast Evans function $\mathcal{E}_{f,0}$ whose zeros approximate those of $\mathcal{E}_{f,\varepsilon}$. Moreover, an explicit, but meromorphic, slow Evans function $\mathcal{E}_{s,0}$ can be obtained via the so-called *NonLocal Eigenvalue Problem* (NLEP) approach, which was established by Doelman, Gardner and Kaper in [22] in the context of stationary pulse solutions in the Gray-Scott model. Thus, a combination of the elephant trunk procedure and the NLEP approach yields an analytic *reduced Evans function*,

$$\mathcal{E}_0 = \mathcal{E}_{s,0} \cdot \mathcal{E}_{f,0}, \tag{1.4}$$

whose factors $\mathcal{E}_{s,0}$ and $\mathcal{E}_{f,0}$ can be derived explicitly through slow and fast *reduced* eigenvalue problems. These reduced eigenvalue problems are *lower-dimensional* and arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of the full eigenvalue problem. Winding number arguments imply that the roots of the Evans function $\mathcal{E}_\varepsilon$ are approximated by the ones of the reduced Evans function $\mathcal{E}_0$ – see Figure 1.3. The NLEP approach and thus the validation of the decomposition (1.3) and its explicit reduction (1.4), was further developed in the context of localized pulses, fronts or periodic solutions in certain classes of 2- or 3-component reaction-diffusion systems in [21, 23, 30, 114, 116, 120]. It should be remarked that in neither of these papers the elephant trunk procedure is carried out in full analytical detail.

Hence, using the elephant trunk and NLEP procedures, the spectrum of the linearization is approximated by the zeros of the reduced Evans function $\mathcal{E}_0$, which is defined in terms

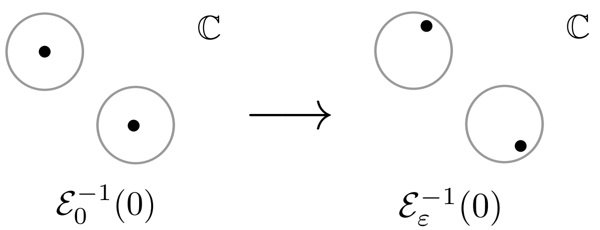
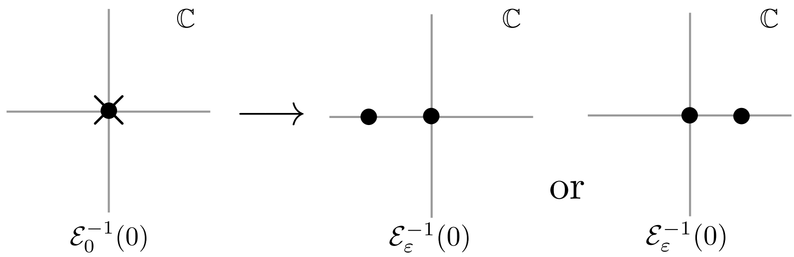


Figure 1.3: Spectral approximation using winding number arguments: the roots of $\mathcal{E}_0$ approximate those of $\mathcal{E}_\varepsilon$.

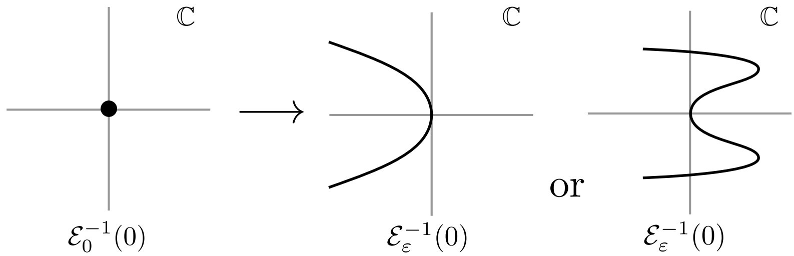
of lower-dimensional, reduced eigenvalue problems. Thus, locating the roots of $\mathcal{E}_0$ yields asymptotic control over the spectrum. In some cases asymptotic spectral control is sufficient to decide upon stability, however the situation is often delicate about the origin – note that the origin must be part of the spectrum due to translational invariance (obtained by shifting the profile in space) – and a local higher-order analysis is required to prove stability. We elaborate on the latter.

The spectrum of the linearization is made up of point spectrum, consisting of isolated eigenvalues of finite multiplicity, and its complement, the essential spectrum – see [98]. Suppose the essential spectrum is confined to the open left half-plane and $\mathcal{E}_0$ has no zeros in the closed right half-plane except a *simple* root at 0. Then, the aforementioned spectral approximation result implies that 0 is an isolated, simple eigenvalue and that the rest of the spectrum lies in the open left half-plane, which yields nonlinear stability of the underlying pattern [44]. This situation occurs for instance in [3, 37, 47].

However, one is often less fortunate. For example, in the stability analyses [10, 53, 126] of traveling pulses in the FitzHugh-Nagumo equations, 0 is a double root of the reduced Evans function $\mathcal{E}_0$, while the essential spectrum is confined to the open left half-plane. Consequently, there are two eigenvalues close to the origin. One of these eigenvalues resides at the origin due to translational invariance, while the position of the other eigenvalue with respect to



(a) Point spectrum associated with fast pulses in the FitzHugh-Nagumo equations



(b) Essential spectrum attached to the origin associated with periodic structures

Figure 1.4: Asymptotic spectral control is insufficient to determine the spectral configuration about the origin.

the imaginary axis is decisive for stability. Yet, asymptotic spectral control is insufficient to determine its location – see Figure 1.4a. The situation becomes even more problematic in the case of *periodic* patterns [32, 114]. Then, the spectrum is composed of curves parameterized over the unit circle S^1 due to Floquet theory [38]. Consequently, all spectrum is essential and one needs to characterize an entire curve of eigenvalues – the so-called *linear dispersion relation* – which is attached to the origin and shrinks to the origin in the limit $\varepsilon \to 0$. Again, asymptotic spectral control is insufficient to ascertain the location of the curve with respect to the imaginary axis – see Figure 1.4b. We elaborate on the various ways that have been developed to overcome these difficulties.

For traveling pulses in the FitzHugh-Nagumo equations, we discuss two approaches to obtain leading-order control over the location of the critical eigenvalue. In [53, 126] one proves that the derivative of the Evans function $\mathcal{E}_\varepsilon$ at 0 is positive, which follows from geometric properties of the pulse profile in the limit $\varepsilon \to 0$. Then, a parity argument implies that the critical eigenvalue is real and negative, which yields nonlinear stability. In [10] one constructs using *Lin's method* [70, 97, 118] a piecewise continuous eigenfunction of the linearization for each prospective eigenvalue λ near the origin. The eigenfunction admits exactly two jumps that occur in the middle of the front and the back of the pulse profile – see Figure 1.5. Finding eigenvalues reduces to identifying values of λ for which these jumps vanish. Melnikov theory provides leading-order expressions for these jumps that can be solved for λ, which yields that the critical eigenvalue lies in the open left half-plane. An advantage of this method over the one in [53, 126] is that one obtains leading-order expressions for the eigenvalues near the origin rather than only their signs. Therefore, it applies more generally to situations where there are more than two eigenvalues near the origin – see Remark 1.3.

In the stability analyses [32, 114] of periodic structures in the FitzHugh-Nagumo and Gierer-Meinhardt models, an entire curve of eigenvalues is attached to the origin, which shrinks to the origin in the limit $\varepsilon \to 0$. As in [10] – but with slightly different methods – one proceeds by

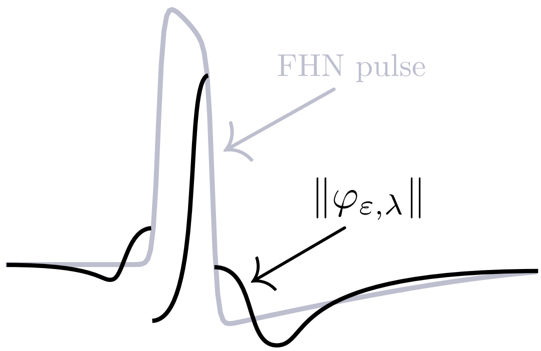


Figure 1.5: A piecewise continuous eigenfunction $\varphi_{\varepsilon,\lambda}$ with jumps occurring in the middle of the front and the back of the fast pulse to the FitzHugh-Nagumo equations.

constructing eigenfunctions corresponding to potential eigenvalues near the origin. A careful matching procedure then gives solvability conditions in terms of the eigenvalue λ, the Floquet multiplier γ and the corresponding eigenfunction. By expanding the eigenfunction in terms of ε, a leading-order expression for the critical spectral curve $\lambda_\varepsilon(\gamma)$ is derived from these solvability conditions. Finally, by complementing the local spectral analysis about the origin with the Evans-function analysis, stability or instability of the periodic pattern is established depending on the system parameters.

Remark 1.1. The concept of the Evans function as a method to determine the spectrum associated with a solution to a system of singularly perturbed reaction-diffusion equations on the line was introduced in [33] and was established as a general and powerful approach in [1, 37, 38, 53]. Core aspects of the NLEP approach have been developed independently in [50, 123]. The SLEP (= Singular Limit Eigenvalue Problem) method [81, 82] is an alternative method that has been linked to the Evans function approach in [49]. In [21], the relation between the Evans function, the NLEP method and the SLEP method is discussed. ■

Remark 1.2. It is a general phenomenon that the presence of essential spectrum about the origin is an issue in the stability analysis of periodic structures. Besides the above-mentioned methods for singularly perturbed reaction-diffusion systems of the form (1.1), let us mention that for periodic waves of conservation laws, Whitham's modulation equations [51, 107] provide an accurate description of the spectral configuration about the origin. Moreover, for periodic wave trains to general reaction-diffusion systems one can compute [28] the derivative $\lambda_*''(0)$ of the critical curve $\lambda_*(\gamma)$ attached to the origin in terms of derivatives of the wave train and the corresponding solution to the adjoint eigenvalue problem. Yet, knowing (the sign of) $\lambda_*''(0)$ is insufficient to control the *entire* spectral curve $\lambda_* : S^1 \to \mathbb{C}$. ■

Remark 1.3. In [50, 64, 122] the stability of multi-pulse solutions to the Gierer-Meinhardt, Gray-Scott and Schnakenberg models is investigated using formal asymptotic expansions. The formal analysis yields the existence of multiple eigenvalues close to the origin and provides leading-order expressions for these eigenvalues. We expect that the above-mentioned approach in [10] using Lin's method could be employed to establish the existence and position of these eigenvalues rigorously. Let us emphasize that, for general semilinear parabolic equations, the stability of multi-pulse solutions bifurcating from a stable primary pulse has already been determined successfully in [97] using Lin's method. ■

1.3 Extension beyond prototype models: slow nonlinearity

The aforementioned spectral methods, including the elephant trunk and NLEP procedures, have been developed in the context of specific (prototype) models, such as the Gray-Scott, Gierer-Meinhardt and FitzHugh-Nagumo equations. These models are of *slowly linear* nature, in the sense that the dynamics of the slow u-components in between localized fast pulses or fronts are driven by linear equations. Thus, the slow reduced system arising in the existence analysis is linear and the slow eigenvalue problem in the stability analysis is autonomous. In the context of the periodic pulse solution shown in Figure 1, slow linearity entails that the dynamics of (1.1) in the rest state $v = 0$ is linear, i.e. the coupling term $H(u, 0, \varepsilon)$ in (1.1) is linear. In recent work [30, 120] an NLEP approach has been carried out for homoclinic pulse solutions to a general class of *slowly nonlinear*, 2-component systems of the form (1.1). Earlier, the stability of fronts was studied in a specific slowly nonlinear model in [23].

The introduction of a slow nonlinearity in (1.1) yields more than just additional technicalities. For instance, it is shown in [119] that, unlike known classical slowly linear examples such as the Gray-Scott and Gierer-Meinhardt models, Hopf bifurcations for homoclinic pulses can be supercritical. Such a bifurcation could even be the first step in a sequence of further bifurcations leading to complex (amplitude) dynamics of a standing solitary pulse – as observed in the simulations in [120].

The slow linearity plays a crucial role in the analysis of the Evans function and its decomposition and reduction. In fact, it is essential for an application of the elephant trunk procedure that the eigenvalue problem is to leading order linear near the boundaries of the spatial domain – as is the case for homoclinic and periodic pulse solutions to the slowly linear Gierer-Meinhardt model in [21, 114]. Although the models in [23, 30, 120] are slowly nonlinear, the elephant trunk procedure is still applicable, because eventually the dynamics becomes linear due to the homoclinic or heteroclinic nature of the patterns. However, the eigenvalue problem associated with *periodic* solutions to slowly nonlinear models is non-autonomous over the *entire* domain, thus obstructing an application of the elephant trunk lemma. This brings us to the main goal of this thesis: *extending the spectral analysis of periodic structures in reaction-diffusion systems of the form* (1.1) *beyond the slowly linear regime.*

1.4 Contents of this thesis

In this thesis we study stationary, spatially periodic pulse solutions to singularly perturbed reaction-diffusion systems of the form (1.1), allowing for general dimensions $n, m \geq 1$ and a large class of nonlinearities H and G – see §1.4.1 for the precise details. The solutions under consideration are spatially symmetric and exhibit exponentially localized pulses in the fast v-components, but admit non-localized behavior in the slow u-components – see Figure 1. In other words, they are in semi-strong interaction [24] (of second order [92]). Our class of equations includes the Gierer-Meinhardt system. Thus, on the one hand, we extend the existence and stability analyses [25, 114] for periodic pulse solutions in the Gierer-Meinhardt equations to the slowly nonlinear and multi-dimensional regime. On the other hand, our work can be considered as the extension from homoclinic to periodic structures within the general, slowly nonlinear class of systems in [30].

However, this thesis is not a straightforward extension of [30, 114], since there is – as outlined in §1.3 – no obvious adaptation of the elephant trunk lemma for spatially periodic patterns in slowly nonlinear systems. Therefore, we present a generalized analytic alternative to both the elephant trunk and NLEP procedures to establish the validity of both the decomposition (1.3) of the Evans function and its singular limit structure (1.4). This analytic method is based on the *Riccati transformation* [12, 13]. This transformation, which satisfies a matrix Riccati equation, diagonalizes the associated eigenvalue problem and thus explicitly separates fast from slow dynamics. The separation yields the factorization of the Evans function (1.3) and provides a framework for the passage to the singular limit (1.4). We emphasize that our factorization procedure applies beyond the current setting of periodic pulse solutions and is therefore interesting in its own right – see Remark 1.4.

Thus, using the analytic factorization method, we obtain a reduced Evans function $\mathcal{E}_0$, whose roots approximate the spectrum and whose factors $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$ are defined in terms of explicit, lower-dimensional reduced eigenvalue problems. Thus, we obtain asymptotic control over the spectrum. However, as mentioned in §1.2, there is a second challenge: asymptotic spectral control is insufficient to decide upon stability, since there is a curve of essential spectrum which shrinks to the origin in the limit $\varepsilon \to 0$. Therefore, we need to complement our Evans-function analysis with a local analysis about the origin in order to obtain leading-order control over this critical spectral curve.

Our analysis of the critical spectral curve is based on [10, 100]. Recall from §1.2 that the stability of (fast) traveling pulses in the FitzHugh-Nagumo equations is decided by the location of a nontrivial eigenvalue near the origin. The method adopted in [10] yields a leading-order expression of this critical eigenvalue in terms of the small parameter ε. Furthermore, in [100] the spectral configuration about the origin is determined for periodic wave trains that accompany homoclinic pulse solutions to general reaction-diffusion systems. An expansion of the critical spectral curve is provided in terms of the period. The situations in [10] and [100] do not directly translate to our situation, since we consider periodics that do not lie in the vicinity of a homoclinic.

Nevertheless, we adopt a similar approach: using Lin's method we obtain a piecewise continuous eigenfunction of the linearization for any potential eigenvalue λ near the origin. In contrast to [100], we do not use the homoclinic limit structure as a framework for the construction of the eigenfunction; instead the singular limit structure serves as a backbone like in [10]. On the other hand, as in [100], Floquet theory yields boundary conditions for the eigenfunction on a single periodicity interval, whereas one requires in [10] that the eigenfunction is exponentially localized on the real line. The construction of the piecewise continuous eigenfunction yields a Lyapunov-Schmidt type reduction procedure: finding the critical spectral curve attached to the origin reduces to equating the jumps to zero. The Fredholm alternative allows us to find expressions for these jumps that can then be solved. Eventually, we obtain a leading-order expression for the critical spectral curve in terms of lower-dimensional, variational equations about the orbit segments that constitute the pulse profile in the limit $\varepsilon \to 0$.

Thus, we gain both asymptotic control over the spectrum through the reduced Evans function and leading-order control over the critical spectral curve. This leads to *explicit* criteria yielding stability and instability of the periodic pulse solution in terms of simpler, lower-dimensional problems. These conditions can be interpreted in more simple cases in which either $n = 1$, $m = 1$, or both $n = m = 1$. In the latter case, we directly recover the expressions obtained in the stability analysis [114] of spatially periodic pulse patterns in the Gierer-Meinhardt equation. The outcome of our spectral analysis shows that the Gierer-Meinhardt setting represents a very special case. The restriction to this specific system obscures the underlying general structure of the reduced Evans function and the critical spectral curve in terms of simpler, lower-dimensional problems. On the other hand, the restriction of (1.1) to a more general, slowly nonlinear, 2-component model as in [30] yields a (relatively) simple instability criterion in terms of the signs of a number of explicit expressions that can be computed with only an asymptotic approximation of the underlying pattern as input. Thereby, we extend a similar result of [30] on homoclinic pulses to periodic structures.

The analytical grip on the spectrum provides insights into destabilization mechanisms of periodic pulse solutions to (1.1). Depending on which one of the aforementioned stability criteria fails, we can identify the type of instability occurring. We establish that generic (primary) instabilities must be of sideband, Hopf or period doubling type, whereas in general reaction-diffusion systems also Turing and fold instabilities are robust for symmetric, spatially periodic patterns [93].

Destabilization mechanisms become rather complex when periodic patterns approach a homoclinic limit. While increasing the wavelength, the character of destabilization alternates between two kinds of Hopf instabilities. This phenomenon is called the *Hopf dance* [27, 115]. It has been analytically established in (slowly linear) Gierer-Meinhardt models in [27] and recovered by numerical methods in the generalized Klausmeier-Gray-Scott model [27, 115]. Both the Hopf dance as well as the *belly dance* [27] – an associated higher order phenomenon – can be analyzed in the general, slowly nonlinear setting of (1.1) by the methods developed here. In addition, we establish an explicit sign criterion to determine whether the homoclinic pulse solution is the last or the first 'periodic' solution to destabilize.

Finally, we comment on the existence of stationary, periodic pulse patterns to (1.1). Our construction of these solutions relies on geometric singular perturbation theory – see §1.1. First, we establish a singular periodic orbit by piecing together orbit segments of slow and fast reduced systems in such a way that they form a closed loop. Then, we prove that an actual periodic orbit lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. The construction respects the symmetry $\check{x} \mapsto -\check{x}$ of system (1.1). Consequently, the periodic pulse solution is spatially symmetric. The existence result is a significant extension of similar results in the literature that only consider 2-component, Gierer-Meinhardt type models [25].

As a final remark, let us mention that we illustrate our existence and stability results by explicit calculations in a slowly nonlinear toy model.

Remark 1.4. The analytic factorization procedure of the Evans function can be outlined in a way that neither depends on the specific structure of the system nor on the specific patterns under consideration. We require that the eigenvalue problem associated with the linearization of (1.1) about the pattern can be written in block-matrix form,

$$\begin{pmatrix} \varphi_x \\ \psi_x \end{pmatrix} = \begin{pmatrix} \sqrt{\varepsilon}A_{11,\varepsilon}(x,\lambda) & \sqrt{\varepsilon}A_{12,\varepsilon}(x,\lambda) \\ A_{21,\varepsilon}(x,\lambda) & A_{22,\varepsilon}(x,\lambda) \end{pmatrix} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, \tag{1.5}$$

in the rescaled spatial variable $x = \varepsilon^{-1}\check{x}$. Consider the reduced eigenvalue problem,

$$\psi_x = A_{22,0}(x,\lambda)\psi, \tag{1.6}$$

in which $A_{22,0}(x,\lambda)$ represents the singular limit of $A_{22,\varepsilon}(x,\lambda)$. Equation (1.6) admits an exponential dichotomy [14, 96] on $\mathbb{R}$ as long as the associated differential operator,

$$\mathcal{L}_\lambda\psi = \psi_x - A_{22,0}(\cdot,\lambda)\psi,$$

is invertible – see also [98]. Since exponential dichotomies are robust against small perturbations, the exponential dichotomy of (1.6) on $\mathbb{R}$ carries over to the perturbed problem,

$$\psi_x = A_{22,\varepsilon}(x,\lambda)\psi, \tag{1.7}$$

for $0 < \varepsilon \ll 1$. This exponential dichotomy on $\mathbb{R}$ of (1.7) allows us to successfully diagonalize the eigenvalue problem (1.5) with the Riccati transformation yielding the factorization (1.3) of the Evans function. In the last step, we approximate the two blocks, in which (1.5) diagonalizes, by their singular limits yielding (1.4). As a consequence, the roots of the Evans function can be approximated by the roots of the reduced Evans function $\mathcal{E}_0$.

We stress that our factorization method applies in particular to the context [30] of *homoclinic* pulse solutions in a large class of 2-component, slowly nonlinear systems. We expect that our method could extend the results in [30] to a multi-component setting. In addition, let us mention that a uv-term in the v-component of (1.1) is not allowed in [30], whereas our method can handle such terms. ■

1.4.1 Setting

In this section we introduce the class of systems under consideration in this thesis. Take $m, n \in \mathbb{Z}_{>0}$ and consider a general reaction-diffusion system in one space dimension with a scale separation in the diffusion lengths (1.1). We assume that the diagonal matrices $D_{1,2}$ in (1.1) are *positive*. Following [30], we write

$$H(u,v,\varepsilon) = H(u,0,\varepsilon) + \tilde{H}_2(u,v,\varepsilon),$$

where $\tilde{H}_2(u,v,\varepsilon) := H(u,v,\varepsilon) - H(u,0,\varepsilon)$, so that $\tilde{H}_2$ vanishes at $v = 0$. To sustain stable localized patterns in semi-strong interaction (of second order [92]) in system (1.1), we allow $\tilde{H}_2(u,v,\varepsilon)$ to scale with ε^{-1} and define

$$H_2(u,v) := \lim_{\varepsilon\to 0} \varepsilon\tilde{H}_2(u,v,\varepsilon).$$

Finally, we write

$$H(u,v,\varepsilon) = H_1(u,v,\varepsilon) + \varepsilon^{-1}H_2(u,v), \tag{1.8}$$

with $H_1(u,v,\varepsilon) := H(u,0,\varepsilon) + [\tilde{H}_2(u,v,\varepsilon) - \varepsilon^{-1}H_2(u,v)]$. By construction $H_2(u,v)$ vanishes at $v = 0$. We assume that $H_1(u,v,\varepsilon)$ and $G(u,v,\varepsilon)$ are smooth functions of ε at $\varepsilon = 0$. Note that we allow for the possibility that $H_2(u,v) \equiv 0$ in the upcoming analysis. We emphasize that, if we have in addition $n = 1$ or $m = 1$, then all patterns are unstable – see Remark 3.22. This confirms the scalings used for classical systems as the Gray-Scott and Gierer-Meinhardt models [21, 22, 50, 123] – see also [30]. For the benefit of our spectral analysis, we need one extra condition on G. That is, G vanishes at $v = 0$. We postpone the discussion of this extra condition to Remark 2.6. In summary, the model class we consider is of the form

$$\begin{aligned} u_t &= D_1 u_{\check{x}\check{x}} - H_1(u,v,\varepsilon) - \varepsilon^{-1}H_2(u,v), \\ v_t &= \varepsilon^2 D_2 v_{\check{x}\check{x}} - G(u,v,\varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, \check{x} \in \mathbb{R}, \tag{1.9}$$

or, in the 'small' spatial scale $x = \varepsilon^{-1}\check{x}$,

$$\begin{aligned} \varepsilon^2 u_t &= D_1 u_{xx} - \varepsilon^2 H_1(u,v,\varepsilon) - \varepsilon H_2(u,v), \\ v_t &= D_2 v_{xx} - G(u,v,\varepsilon), \end{aligned} \qquad u \in \mathbb{R}^m, v \in \mathbb{R}^n, x \in \mathbb{R}, \tag{1.10}$$

in which we will usually work. The aforementioned conditions read:

(S1) ***Conditions on the interaction and diffusion terms***
There exists open, connected sets $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ with $0 \in V$ and $0 \in I$ such that H_1, G and H_2 are C^3 on their domains $U \times V \times I$ and $U \times V$, respectively. Moreover, we have $H_2(u,0) = 0$ and $G(u,0,\varepsilon) = 0$ for all $u \in U$ and $\varepsilon \in I$. Finally, $D_{1,2}$ are positive diagonal matrices.

Remark 1.5. If we have $n = 1$, we can without loss of generality assume $D_2 = 1$ in (1.9) by rescaling the spatial variable $\check{x}$. Similarly, in the case $m = 1$, we can without loss of generality assume $D_1 = 1$ by rescaling the parameter ε. ■

1.4.2 Outline

This thesis is structured as follows. In Chapter 2 we elaborate on the existence of periodic pulse solutions to (1.9) and we obtain fine estimates on the error between the periodic pulse solutions and the associated singular periodic orbit. We apply the existence result to construct periodic pulse solutions in an explicit slowly nonlinear toy model. In Chapter 3 we present the main results of our spectral analysis: the approximation of the spectrum by the roots of the reduced Evans function (1.4) and the expansion of the critical spectral curve. We obtain explicit conditions in terms of simpler, lower-dimensional systems yielding stability. Moreover, we test for instability by calculating the signs of a number of explicit expressions. Finally, we interpret these results in the lower-dimensional regime and apply them to the slowly nonlinear toy model. Chapter 4 contains prerequisites for our spectral analysis. In particular, we provide extensive background on exponential dichotomies and establish the Riccati transform, which provides a natural framework for the factorization of the Evans function – see Remark 1.4. In Chapter 5 we perform the actual spectral analysis and prove our main results. Chapter 6 focusses on destabilization mechanisms of periodic pulse solutions. Finally, in Chapter 7 we elaborate on future research possibilities.

The results presented in Chapter 2 and Sections 3.5, 3.8.1 and 3.8.4 appeared earlier in *Spectra and stability of spatially periodic pulse patterns: Evans function factorization via Riccati transformation* in the SIAM Journal on Mathematical Analysis in 2016 – see [17].

Chapter 2

Existence analysis

2.1 Introduction

In this thesis we are interested in of stationary, spatially periodic pulse solutions to the class of reaction-diffusion systems (1.10), where we assume that the interaction terms satisfy **(S1)**. Such solutions are constant in time and they are periodic and symmetric in space. In addition, the v-components exhibit spatially localized pulses, whereas the u-components are non-localized. We refer to Figure 1 for a plot of the pulse profile in the case $m = n = 1$.

In this chapter we focus on the construction of such solutions. Finding stationary solutions to (1.10) is equivalent to solving the singularly perturbed ordinary differential equation,

$$\begin{aligned} D_1 u_x &= \varepsilon p, \\ p_x &= \varepsilon H_1(u, v, \varepsilon) + H_2(u, v), \\ D_2 v_x &= q, \\ q_x &= G(u, v, \varepsilon), \end{aligned} \qquad (u, p, v, q) \in \mathbb{R}^{2(m+n)}, \tag{2.1}$$

which is R-reversible, where $R\colon \mathbb{R}^{2(m+n)} \to \mathbb{R}^{2(m+n)}$ is the reflection in the space $p = q = 0$. Taking the limit $\varepsilon \to 0$ in properly scaled versions of (2.1) yields slow and fast reduced systems. By piecing together orbit segments of these reduced systems in such a way that they form a closed loop, one obtains a so-called singular periodic orbit. Although this singular periodic orbit is not an actual solution to (2.1), one can prove that (under certain conditions) an actual periodic solution to (2.1) arises from the singular one, provided $\varepsilon > 0$ is sufficiently small.

In this chapter we perform a slow-fast decomposition of (2.1) and construct a singular periodic orbit from the slow and fast reduced systems. Next, we use geometric singular perturbation theory [34, 54, 57] to study the dynamics of system (2.1) in the neighborhood of the singular orbit. Then, we have the ingredients to prove the existence of an actual periodic pulse solution to (2.1) in the vicinity of the singular one. The R-reversibility of system (2.1) plays an essential role in the proof. Therefore, both the periodic pulse solution and its singular limit naturally respect the R-reversibility of system (2.1). Since the stability analysis in Chapter 5 relies crucially on how the periodic pulse solutions are approximated by the singular limit structure, we provide detailed (pointwise) estimates along with the existence result. Finally, we apply the existence result to an explicit slowly nonlinear toy model.

2.2 The singular limit

2.2.1 Slow-fast decomposition

We perform a slow-fast decomposition of the singularly perturbed equation (2.1). Fast and slow reduced systems arise by taking the limit $\varepsilon \to 0$ in properly scaled versions of (2.1). First, if set $\varepsilon = 0$ in (2.1), then the dynamics is given by the *fast reduced system*,

$$\begin{aligned} u_x &= 0, \\ p_x &= H_2(u,v), \\ D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (u,p,v,q) \in \mathbb{R}^{2(m+n)}, \tag{2.2}$$

System (2.2) is governed by the family of $2n$-dimensional systems,

$$\begin{aligned} D_2 v_x &= q, \\ q_x &= G(u,v,0), \end{aligned} \qquad (v,q) \in \mathbb{R}^{2n}, \tag{2.3}$$

parameterised over $u \in U$. Note that (2.3) is R_f-reversible, where $R_f\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the reflection in the space $q = 0$. Moreover, we observe that *the slow manifold*,

$$\mathcal{M} := \{(u,p,0,0) : u \in U, p \in \mathbb{R}^m\},$$

consists entirely of equilibria of (2.2) by assumption **(S1)**. When $\varepsilon > 0$, the manifold $\mathcal{M}$ consists no longer of equilibria, but remains invariant for the dynamics of (2.1). The flow restricted to $\mathcal{M}$ is of order $\mathcal{O}(\varepsilon)$. In the spatial scale $\check{x} = \varepsilon x$, the dynamics of (2.1) on $\mathcal{M}$ is to leading order governed by the *slow reduced system*,

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= H_1(u,0,0), \end{aligned} \qquad (u,p) \in \mathbb{R}^{2m}. \tag{2.4}$$

Note that system (2.4) is R_s-reversible, where $R_s\colon \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ is the reflection in the space $p = 0$.

Although the fast and slow reduced systems (2.2) and (2.4) are simpler, lower-dimensional systems, enough information can be obtained from them to determine the leading-order dynamics of the full system (2.1) close to the slow manifold $\mathcal{M}$ for $0 < \varepsilon \ll 1$ – see §2.3.

2.2.2 Construction of the singular periodic pulse

In this section we construct a singular periodic orbit by concatenating solutions of the fast and slow reduced systems (2.2) and (2.4) in such a way that they form a closed loop. The singular orbit consist of a pulse satisfying the fast reduced system (2.2) and a segment on the invariant slow manifold $\mathcal{M}$, satisfying the slow reduced system (2.4). We emphasize that such a singular orbit is not a solution to (2.1). However, when the singular orbit satisfies certain conditions, we will prove that an actual periodic pulse solution lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small – see §2.4.

The first ingredient for constructing the singular periodic orbit is the existence of a pulse solution in the fast reduced system (2.2). This is ensured by the following assumption.

(E1) ***Existence of a pulse solution to the fast reduced system***
There exists $u_\diamond \in U$ such that system (2.3) has for $u = u_\diamond$ a solution $\psi_h(x, u_\diamond) = (v_h(x, u_\diamond), q_h(x, u_\diamond))$ homoclinic to the hyperbolic saddle 0. The stable manifold $W^s_{u_\diamond}(0)$ intersects the space $\ker(I - R_f)$ transversely in the point $\psi_h(0, u_\diamond)$.

Remark 2.1. In the terminology of [118] homoclinics that lies in the transverse intersection of $W^s_{u_\diamond}(0)$ and $\ker(I - R_f)$ are called *elementary*. In particular, any non-degenerate homoclinic solution is elementary by [118, Lemma 4]. We emphasize that in the case $n = 1$ any homoclinic solution to (2.3) is elementary. ■

Since transverse intersections are robust under perturbations, assumption **(E1)** implies the existence of an open neighborhood $U_h \subset U$ of $u_\diamond$ such that for every $u \in U_h$ there exists a solution $\psi_h(x, u)$ to (2.3), which is homoclinic to the hyperbolic saddle 0, such that $W^s_u(0) \pitchfork \ker(I - R_f) = \{\psi_h(0, u)\}$. The homoclinics $\psi_h(x, u)$ yield solutions,

$$\phi_h(x, u) := \left(u, \int_0^x H_2(u, v_h(z, u))dz, v_h(x, u), q_h(x, u)\right), \quad u \in U_h,$$

to the fast reduced system (2.2), which are homoclinic to $\mathcal{M}$. The homoclinics $\phi_h(x, u)$ take off and touch down on the points $\lim_{x\to\pm\infty} \phi_h(x, u) \in \mathcal{M}$. We define the mapping $\mathcal{J} \colon U_h \to \mathbb{R}^m$ by

$$\mathcal{J}(u) = \int_0^\infty H_2(u, v_h(z, u))dz. \tag{2.5}$$

The m-dimensional graphs $\mathcal{T}_\pm := \{(u, \pm\mathcal{J}(u)) : u \in U_h\}$ on $\mathcal{M}$ are the so-called *take-off and touch-down manifolds*. Since 0 is a hyperbolic saddle in (2.3), there exists constants $C, \mu_h > 0$ such that

$$\|\phi_h(\pm x, u) - (u, \pm\mathcal{J}(u), 0, 0)\| \leq Ce^{-\mu_h x}, \quad x \geq 0, u \in U_h. \tag{2.6}$$

The manifolds $\mathcal{T}_\pm$ allow us to piece the pulse solutions ϕ_h to solutions that lie in $\mathcal{M}$ in order to obtain a singular periodic orbit – see Figure 2.1. Therefore, we shift our attention to the slow reduced system (2.4). Recall that (2.4) is R_s-reversible. In addition, since (2.3) is

R_f-reversible, it holds $R_s[\mathcal{T}_+] = \mathcal{T}_-$. Therefore, to establish a connection between the take-off and touch-down manifolds $\mathcal{T}_\pm$, it is sufficient to find a solution to (2.4) that starts on the touch-down manifold $\mathcal{T}_+$ and crosses $\ker(I - R_s)$ at some point. This is the content of our next assumption.

(E2) ***Existence of connecting orbit in slow reduced system***
There exists a solution $\psi_s(\check{x}) = (u_s(\check{x}), p_s(\check{x}))$ to system (2.4) with initial condition $\psi_s(0) \in \mathcal{T}_+$ and $\psi_s(\ell_0) \in \ker(I - R_s)$ for some $\ell_0 > 0$. Moreover, let $\Phi_s(\check{x}, \check{y})$ be the evolution operator of the associated variational equation,

$$\varphi_{\check{x}} = \mathcal{A}_s(\check{x})\varphi, \quad \varphi \in \mathbb{R}^{2m}, \tag{2.7}$$

with

$$\mathcal{A}_s(\check{x}) := \begin{pmatrix} 0 & D_1^{-1} \\ \partial_u H_1(u_s(\check{x}), 0, 0) & 0 \end{pmatrix}.$$

Denote $u_0 := u_s(0)$, $H_1(u_0, 0, 0) = (h_1, \ldots, h_m)$ and for $i, j \in \{1, \ldots, m\}$ by A_{ij} the $(m \times m)$-submatrix of

$$\Phi_s(\ell_0, 0)\begin{pmatrix} I \\ \partial_u J(u_0) \end{pmatrix},$$

containing rows $\{i, m+1, \ldots, 2m\} \setminus \{m + j\}$. There exists $i_* \in \{1, \ldots, m\}$ such that

$$\sum_{j=1}^{m} (-1)^j h_j \det(A_{i_* j}) \neq 0. \tag{2.8}$$

By concatenating the orbits of ψ_s and ϕ_h, we obtain the *singular periodic pulse*,

$$\phi_{p,0} := \{(\psi_s(\check{x}), 0) : \check{x} \in (0, 2\ell_0)\} \cup \{\phi_h(x, u_0) : x \in \mathbb{R}\} \subset \mathbb{R}^{2(m+n)}, \tag{2.9}$$

consisting of a pulse satisfying the fast reduced system (2.2) and an orbit segment on the slow manifold. We emphasize that $\phi_{p,0}$ is C^1, except at the two *corners* $(u_0, \pm\mathcal{J}(u_0), 0, 0) = (u_s(0), \pm p_s(0), 0, 0)$. Eventually, our goal is to construct a periodic pulse solution to (2.1) in the vicinity of the singular orbit (2.9), provided $0 < \varepsilon \ll 1$. Therefore, we need some robustness of the structure (2.9) under perturbations. Robustness of the pulse ϕ_h is ensured by the transversality condition in **(E1)**. The orbit ψ_s in the slow system (2.4) persists by regular perturbation arguments on the finite interval $[0, 2\ell_0]$. Lastly, to ensure persistence of the connections between $(\psi_s(\check{x}), 0)$ and $\psi_h(x, u_0)$ at the two corners, we impose the technical condition (2.8) in assumption **(E2)**. For $m = 1$ the condition (2.8) is equivalent to the transversality condition,

$$\partial_u \mathcal{J}(u_0) D_1^{-1} \mathcal{J}(u_0) - H_1(u_0, 0, 0) \neq 0, \tag{2.10}$$

of the touch-down curve $\mathcal{T}_+$ and the solution ψ_s at $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ (and of $\mathcal{T}_-$ and ψ_s at $\psi_s(2\ell_0) = R_s\psi_s(0)$) – see Figure 2.1. In the case $m > 1$, the technical condition (2.8) is employed to generate a 'good' set of initial conditions in $\ker(I - R_s)$. This set becomes under the forward flow of the slow reduced system (2.4) an m-dimensional manifold, which contains the solution ψ_s and intersect $\mathcal{T}_-$ transversally. We emphasize that (2.10) is a necessary condition for (2.8) to hold true for any $m \geq 1$ – see identity (2.19) in the proof of Theorem 2.3.

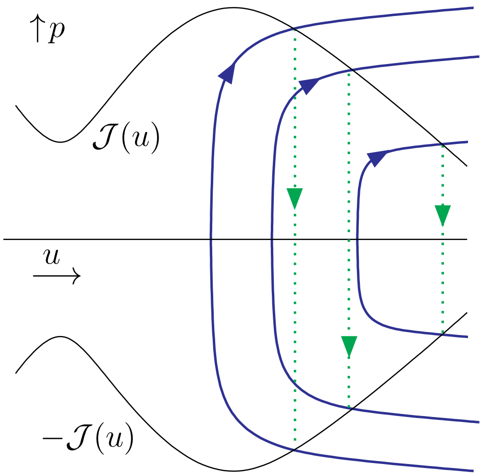


Figure 2.1: Orthogonal projection of three singular periodic orbits on the slow manifold $\mathcal{M}$ in the case $m = n = 1$.

2.3 Dynamics in the vicinity of the slow manifold

Eventually, our goal is prove that close to the singular concatenation (2.9) there exists an actual periodic pulse solution to (2.1), provided $\varepsilon > 0$ is sufficiently small. The singular orbit (2.9) consists of a pulse and an orbit segment on the slow manifold $\mathcal{M}$. Using Grönwall-type arguments it is not difficult to track solutions to (2.1) close to the pulse $\phi_h(x, u_0)$ on an interval $[-X, X]$, where X is ε-independent. However, on the slow manifold $\mathcal{M}$ the dynamics of system (2.1) is of order $O(\varepsilon)$. Thus, to track solutions to (2.1) close to the orbit segment $\{(\psi_s(\check{x}), 0) : \check{x} \in [0, 2\ell_0]\}$ on $\mathcal{M}$, we need approximations on an interval of length $2\ell_0\varepsilon^{-1}$ and Grönwall type estimates fail. Thus, to capture the dynamics in the vicinity of $\mathcal{M}$, we need additional arguments. First, we require $\mathcal{M}$ to be normally hyperbolic.

(S2) ***Normal hyperbolicity***
For each $u \in U$ the symmetric part $\mathrm{Re}(\mathcal{G}(u)) = \frac{1}{2}\left(\mathcal{G}(u) + \mathcal{G}(u)^T\right)$ of $\mathcal{G}(u) := \partial_v G(u, 0, 0)$ is positive definite.

The behavior of singularly perturbed equations of the form (2.1) close to an invariant, normally hyperbolic manifold is described by *Fenichel geometric singular perturbation theory* [34, 54]. Fenichel theory states that the dynamics close to $\mathcal{M}$ is to leading order governed by the fast and slow reduced systems (2.2) and (2.4), respectively. In this section we collect the facts from Fenichel theory needed to prove our existence result.

2.3.1 Fenichel fibering

Let $\mathcal{M}_0$ be a compact $2m$-dimensional submanifold of the slow manifold $\mathcal{M}$. Fenichel theory [34, Theorem 9.1] states that, the manifold $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a manifold $\mathcal{M}_\varepsilon$, which is diffeomorphic to $\mathcal{M}_0$ and locally invariant for the dynamics of (2.1). Since $\mathcal{M}_0$ is itself locally invariant for the dynamics of (2.1), there exists an ε-independent constant $C > 0$ such that $\mathcal{M}_\varepsilon$ has Hausdorff distance $O(e^{-C/\varepsilon})$ from $\mathcal{M}_0$ – see also [19, Theorem 2.1] and [121, Theorem 1].

By assumption **(S2)** any $\psi_0 \in \mathcal{M}_0$ is a saddle-centre equilibrium for system (2.2) having n-dimensional stable and unstable fibers $W_0^{u,s}(\psi_0)$. Fenichel theory [34, Theorem 9.1] states that, for $\varepsilon > 0$ sufficiently small, there exists $\psi_\varepsilon \in \mathcal{M}_\varepsilon$ such that these fibers persist as n-dimensional manifolds $W_\varepsilon^{u,s}(\psi_\varepsilon)$ that have $O(\varepsilon)$-Hausdorff distance to $W_0^{u,s}(\psi_0)$ within an ε-independent neighborhood $\mathcal{D} \subset \mathbb{R}^{2(m+n)}$ of $\mathcal{M}_0$, i.e. the Hausdorff distance between $W_\varepsilon^{u,s}(\psi_\varepsilon) \cap \mathcal{D}$ and $W_0^{u,s}(\psi_0) \cap \mathcal{D}$ is $O(\varepsilon)$. Moreover, we have the following invariance principle called *Fenichel fibering*: if $\psi_\varepsilon(x)$ is a solution to (2.1) lying in $\mathcal{M}_\varepsilon$ for $\varepsilon x \in [0, X]$, where $X > 0$ is ε-independent, then the manifolds,

$$\mathcal{P}_\varepsilon^{u,s} = \bigcup_{\varepsilon x \in [0,X]} W_\varepsilon^{u,s}(\psi_\varepsilon(x)),$$

are locally invariant for the dynamics of (2.1). Moreover, solutions in $\mathcal{P}_\varepsilon^s$ or $\mathcal{P}_\varepsilon^u$ converge to $\mathcal{M}_\varepsilon$ exponentially as $x \to \infty$ or $x \to -\infty$, respectively. Finally, $\mathcal{P}_\varepsilon^{u,s}$ have $O(\varepsilon)$-Hausdorff distance (within $\mathcal{D}$) to the manifolds,

$$\mathcal{P}_0^{u,s} = \bigcup_{\check{x} \in [0,X]} W_0^{u,s}((\psi_0(\check{x}), 0)),$$

where ψ_0 is the solution to the slow reduced system (2.4) governing the leading-order dynamics of ψ_ε. In particular, the stable and unstable manifolds $W_0^s(\mathcal{M}_0)$ and $W_0^u(\mathcal{M}_0)$ defined as the union of the stable and unstable fibers of $\mathcal{M}_0$ in (2.2) persist as locally invariant, stable and unstable manifolds $W_\varepsilon^s(\mathcal{M}_\varepsilon)$ and $W_\varepsilon^u(\mathcal{M}_\varepsilon)$ of $\mathcal{M}_\varepsilon$ in (2.1).

Fenichel fibering gives a detailed description of the behavior of solutions to (2.1) converging to $\mathcal{M}_\varepsilon$. In essence the dynamics is an interplay of the attracting or repelling behavior induced by the fast reduced system (2.2) and the dynamics on $\mathcal{M}_\varepsilon$ described by the slow reduced system (2.4).

2.3.2 Fenichel normal form

Fenichel fibering describes the dynamics of those solutions to (2.1) that converge to $\mathcal{M}_\varepsilon$ as $x \to \pm\infty$. However, to understand the behavior of *any* solution close to $\mathcal{M}_\varepsilon$ it is convenient to put system (2.1) into a canonical form in the neighborhood $\mathcal{D} \subset \mathbb{R}^{2(m+n)}$ of $\mathcal{M}_0$, the so-called *Fenichel normal form* [57, Proposition 1]. For $0 \leq \varepsilon \ll 1$, there exists a C^1-change of coordinates $\Psi_\varepsilon \colon \mathcal{D} \to \mathbb{R}^{2(m+n)}$, depending C^1-smoothly on ε, in which the flow of (2.1) is given by,

$$\begin{aligned} a_x &= A(a,b,c,\varepsilon)a, \\ b_x &= B(a,b,c,\varepsilon)b, \qquad\qquad a,b \in \mathbb{R}^n, c \in \mathbb{R}^{2m}, \\ c_x &= \varepsilon K(c,\varepsilon) + H(a,b,c,\varepsilon)(a \otimes b), \end{aligned} \tag{2.11}$$

where the A, B, K and H are C^1 in their arguments, K maps to $\mathbb{R}^{2m}$, A and B map to the square matrices of order n and H maps to tensors of appropriate rank. Moreover, there exists $\Delta > 0$ and an open and bounded set $U_F \subset \mathbb{R}^{2m}$ such that the image $\Psi_\varepsilon(\mathcal{D})$ contains the compact box,

$$\mathcal{B} := \left\{(a,b,c) : ||a||, ||b|| \leq \Delta, c \in \overline{U_F}\right\}. \tag{2.12}$$

In addition, there exists $C, \mu > 0$, independent of ε, such that

$$\mathrm{Re}(\sigma(A(a,b,c,\varepsilon))) \leq -\mu, \quad \mathrm{Re}(\sigma(B(a,b,c,\varepsilon))) \geq \mu, \tag{2.13}$$

and

$$||H(a,b,c,\varepsilon)(a \otimes b)|| \leq C||a||||b||, \tag{2.14}$$

for all $(a,b,c) \in \mathcal{B}$ and $0 \leq \varepsilon \ll 1$.

In the local *Fenichel coordinates* $\mathcal{M}_\varepsilon$ correspond to the space $a = b = 0$ and the local stable and unstable manifolds $W_\varepsilon^{u,s}(\mathcal{M}_\varepsilon) \cap \mathcal{D}$ of $\mathcal{M}_\varepsilon$ correspond to the spaces $b = 0$ and $a = 0$, respectively. Since system (2.1) is R-reversible, R maps $W_\varepsilon^u(\mathcal{M}_\varepsilon)$ onto $W_\varepsilon^s(\mathcal{M}_\varepsilon)$ and vice versa. Hence, $\ker(I - R) \cap \mathcal{D}$ corresponds to the space $a = b$. Finally, system

$$c_{\check{x}} = K(c, 0), \quad c \in \mathbb{R}^{2m}, \tag{2.15}$$

is equivalent to the slow reduced system (2.4).

In the canonical form (2.11) the dynamics of (2.1) is decomposed in an attracting a-direction, a repelling b-direction and a slowly evolving c-direction.

2.3.3 The Exchange lemma

Through the Fenichel normal form (2.11) one observes that (2.1) exhibits attracting, repelling and slow dynamics. *Exchange lemmas* [55, 57, 59, 102] provide a way to capture this combination of dynamics.

As mentioned before, we need to track solutions close to the orbit segment on the slow manifold of the singular concatenation (2.9) in order to prove our main existence result. For this reason we need the following exchange lemma, which is (naturally) stated in Fenichel coordinates.

Lemma 2.2. *[102, Theorem 2.3] Let $a_* \in \mathbb{R}^n$ with $\|a_*\| < \Delta$ and let $c_0(\check{x})$ be a solution to* (2.15) *such that $c_0(\check{x}) \in U_F$ for $\check{x} \in [0, X]$ with $X > 0$. Let $\mathcal{Z}_\varepsilon$ for $0 \leq \varepsilon \ll 1$ be a submanifold of $\mathbb{R}^{2(m+n)}$ of dimension $n + l$, where $0 \leq l \leq 2m - 1$, satisfying the assertions:*

i. $\mathcal{Z} = \{(a, b, c, \varepsilon) : (a, b, c) \in \mathcal{Z}_\varepsilon\}$ *is itself a manifold;*

ii. $\mathcal{Z}_0$ *meets the space $b = 0$ transversally at the point $(a_*, 0, c_0(0))$.*

Denote by $\mathcal{P}_\varepsilon, 0 \leq \varepsilon \ll 1$ the orthogonal projection of the l-dimensional manifold $\mathcal{Z}_\varepsilon \cap \{(a, 0, c) : a \in \mathbb{R}^n, c \in U_F\}$ on the space $a = b = 0$. We require in addition:

iii. $\mathcal{P}_0$ *is an l-dimensional manifold and the flow of* (2.15) *is not tangent to $\mathcal{P}_0$ at $c_0(0)$.*

Denote by $\mathcal{Z}_\varepsilon^*$ and $\mathcal{P}_\varepsilon^*$ the $(n + l + 1)$- and $(l + 1)$-dimensional manifolds obtained by flowing initial conditions on $\mathcal{Z}_\varepsilon$ and $\mathcal{P}_\varepsilon$ forward in (2.11). *Then, there exists a $(n + l + 1)$-dimensional submanifold $\mathcal{Z}_{1,\varepsilon}$ of $\mathcal{Z}_\varepsilon^*$ and an ε-independent neighborhood $U_1 \subset U_F$ of $c_0(X)$ such that the Hausdorff distance between $\mathcal{Z}_{1,\varepsilon}$ and the $(n + l + 1)$-dimensional manifold,*

$$\{(0, b, c) : b \in \mathbb{R}^n, c \in \mathcal{P}_\varepsilon^* \cap U_1\} \subset W_\varepsilon^u(\mathcal{M}_\varepsilon),$$

is $O(e^{-C/\varepsilon})$, where $C > 0$ is independent of ε. Moreover, trajectories crossing $\mathcal{Z}_{1,\varepsilon}$ remain in the box $\mathcal{B}$ – see (2.12) *– during their excursion from $\mathcal{Z}_\varepsilon$ to $\mathcal{Z}_{1,\varepsilon}$.*

2.4 Main existence result

In this section, we prove that close to the singular concatenation (2.9) there exists an actual periodic pulse solution to (2.1), provided $\varepsilon > 0$ is sufficiently small.

It is a well-known principle that close to a singular periodic orbit, constructed by piecing together orbit segments of the fast and slow reduced systems in such a way that they form a closed loop, one can find an actual periodic orbit, provided $\varepsilon > 0$ is sufficiently small. In [110] this is proved for a large class of slow-fast systems. However, an essential condition for the result in [110] is that the slow components are constant in the fast reduced system. In our case the slow p-components are non-constant along orbits in (2.2). Therefore, the result in [110] is not applicable.

To our knowledge there is no existence result in the literature focusing on periodic solutions in the large class of singularly perturbed systems (2.1) beyond the type of slow-fast systems considered in [110]. However, for the Gierer-Meinhardt equations – see Remark 2.7 – the existence of stationary, spatially periodic pulse solutions is proved in [25]. We emphasize that the framework in [25] differs fundamentally from ours due to a difference in scaling in the p-components and the fact that the u, p, v- and q-components are scalar. In Remark 2.5 we elaborate in more detail on the scaling in the p-components.

We prove the existence of periodic pulse solutions close to (2.9) in the class of systems (2.1) by adapting and extending the techniques in [25, 110] – see Remark 2.4. The proof of our result exploits the fact that every orbit that crosses the space $\ker(I-R)$ twice, must be a closed loop. Therefore, our approach is to start with a 'good' set of initial conditions in $\mathcal{Z} \subset \ker(I-R)$ and track these conditions under the forward flow of (2.1) with the aid of the Exchange Lemma 2.2. We show that the tracked trajectories remain close to the singular orbit (2.9). In particular, we establish that the union of trajectories starting in $\mathcal{Z}$ intersects $\ker(I-R)$ transversally in some point P_ε, which lies close to $\phi_{\mathrm{h}}(0,u_0)$. The desired periodic solution is the one that crosses P_ε.

Theorem 2.3. *Assume **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Then, there exists constants* $C, \mu_0, \varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$ *there exists a solution* $\phi_{\mathrm{p},\varepsilon}(x)$ *to* (2.1) *satisfying the following assertions:*

1. ***Periodicity***
   $\phi_{\mathrm{p},\varepsilon}$ *is* $2L_\varepsilon$*-periodic, where* $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$.
2. ***Reversibility***
   We have $\phi_{\mathrm{p},\varepsilon}(x) = R\phi_{\mathrm{p},\varepsilon}(-x)$ *for* $x \in \mathbb{R}$.
3. ***Singular limit***
   Define for $\theta \geq \mu_0^{-1}$ *the quantity* $\Xi_\theta(\varepsilon) := -\theta\log(\varepsilon)$. *The solution* $\phi_{\mathrm{p},\varepsilon}$ *approximates the pulse as,*
   $$\left\|\phi_{\mathrm{p},\varepsilon}(x) - \phi_{\mathrm{h}}(x,u_0)\right\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon), \Xi_\theta(\varepsilon)], \tag{2.16}$$
   and it approximates the orbit segment on the slow manifold as,
   $$\left\|\phi_{\mathrm{p},\varepsilon}(x) - (\psi_{\mathrm{s}}(\varepsilon x), 0)\right\| \leq C\varepsilon, \quad x \in [\Xi_\theta(\varepsilon), 2L_\varepsilon - \Xi_\theta(\varepsilon)]. \tag{2.17}$$
4. ***Exponential convergence to slow manifold***
   We have the estimate,
   $$d(\phi_{\mathrm{p},\varepsilon}(x), \mathcal{M}) \leq Ce^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}. \tag{2.18}$$

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε.

We start with constructing a good manifold of initial conditions in $\ker(I-R)$. Denote by $e_i, i \in 1,\ldots,m$ the unit basis of $\mathbb{R}^m$. Let $\mathcal{U}$ be the $(m \times (m-1))$-matrix with column vectors $e_1,\ldots,e_{i_*-1}, e_{i_*+1},\ldots,e_m$, where i^* is as in **(E2)**. Consider the $(m+n-1)$-dimensional manifold,

$$\mathcal{Z} := \{(u_{\mathrm{s}}(\ell_0) + u, 0, v, 0) : u \in \mathcal{U}[\mathbb{R}^{m-1}], v \in \mathbb{R}^n\} \subset \ker(I-R).$$

The intersection of $\mathcal{Z}$ and $\mathcal{M}$ equals,

$$\mathcal{P}_0 := \{(u_{\mathrm{s}}(\ell_0) + u, 0) : u \in \mathcal{U}[\mathbb{R}^{m-1}]\} \subset \ker(I - R_s).$$

By assumption **(E2)** $\mathcal{P}_0$ becomes under the forward flow of the slow reduced system (2.4) an m-dimensional manifold $\mathcal{P}_0^*$, which intersects $\mathcal{T}_- = R_s[\mathcal{T}_+]$ transversely at $\psi_s(2\ell_0) = R_s\psi_s(0) = (u_0, -\mathcal{J}(u_0))$. Indeed, we have by condition (2.8)

$$\begin{aligned}
0 &\neq \det\left(\Phi_s(\ell_0,0)\begin{bmatrix} I \\ \partial_u\mathcal{J}(u_0) \end{bmatrix} \middle| \begin{matrix} 0 & \mathcal{U} \\ H_1(u_s(\ell_0),0,0) & 0 \end{matrix} \right) \\
&= \det\left(\begin{matrix} I & D_1^{-1}\mathcal{J}(u_0) \\ \partial_u\mathcal{J}(u_0) & H_1(u_0,0,0) \end{matrix} \middle| \Phi_s(0,\ell_0)\begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \right) \\
&= \det\left(H_1(u_0,0,0) - \partial_u\mathcal{J}(u_0)D_1^{-1}\mathcal{J}(u_0) \middle| \begin{bmatrix} -\partial_u\mathcal{J}(u_0) & I \end{bmatrix}\Phi_s(0,\ell_0)\begin{bmatrix} \mathcal{U} \\ 0 \end{bmatrix} \right)
\end{aligned} \tag{2.19}$$

where we use that $\Phi_s(0,\ell_0)$ induces an isomorphism between the tangent spaces of $\mathcal{P}_0^*$ at $\psi_s(\ell_0)$ and at $\psi_s(0)$ and that the determinant of $\Phi_s(0,\ell_0)$ equals 1.

Eventually, our goal is to show that the $(m+n)$-dimensional manifold $\mathcal{Z}_\varepsilon^*$ obtained by flowing initial conditions on $\mathcal{Z}$ forward in (2.1) intersects the $(m+n)$-dimensional manifold $\ker(I-R)$ transversally within $\mathbb{R}^{2(m+n)}$ close to the point $\phi_h(0,u_0)$. The unique intersection point then yields a periodic solution.

To describe the dynamics on $\mathcal{Z}_\varepsilon^*$ close to $\mathcal{M}$, we apply Fenichel theory – see §2.3. We choose a compact submanifold $\mathcal{M}_0$ of $\mathcal{M}$ that contains the projection of the singular orbit (2.9) on $\mathcal{M}$, i.e. let $\mathcal{M}_0$ be a compact $2m$-dimensional submanifold of $\mathcal{M}$ such that $\mathcal{M}_0$ serves as a neighborhood of the orbit segment $\{\psi_s(\check{x}) : \check{x} \in [0,2\ell_0]\}$ and of the projection $\{(u_0, \int_0^x H_2(u_0, v_h(z,u_0))dz) : x \in \mathbb{R}\}$ of the pulse $\phi_h(x,u_0)$ on $\mathcal{M}$.

By assumption **(S2)** $\mathcal{M}_0$ is normally hyperbolic. So, according to Fenichel theory, $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a manifold $\mathcal{M}_\varepsilon$, which is diffeomorphic to $\mathcal{M}_0$ and locally invariant for the dynamics of (2.1). In addition, $\mathcal{M}_\varepsilon$ has Hausdorff distance $O(e^{-C/\varepsilon})$ to $\mathcal{M}_0$.

To track solutions on $\mathcal{Z}_\varepsilon^*$ we apply the Exchange Lemma 2.2. By switching to Fenichel coordinates in the neighborhood $\mathcal{D}$ of $\mathcal{M}_0$ – see §2.3.2 – it is readily seen that $\mathcal{Z} \subset \ker(I-R)$ intersects the local stable manifold $W_0^s(\mathcal{M}_0) \cap \mathcal{D}$ of $\mathcal{M}_0$ in the fast reduced system (2.2) transversally at $(\psi_s(\ell_0), 0)$. Moreover, the slow reduced flow (2.4) on $\mathcal{M}_0$ is not tangent to $\mathcal{P}_0$ at $\psi_s(\ell_0)$ by (2.19). We conclude that the conditions for the Exchange Lemma 2.2 are satisfied.

Denote by $\mathcal{P}_\varepsilon \subset \mathcal{M}_\varepsilon$ the $(m-1)$-dimensional manifold, where $\mathcal{Z}$ and the local stable manifold $W_\varepsilon^s(\mathcal{M}_\varepsilon) \cap \mathcal{D}$ meet transversally. Moreover, let $\mathcal{P}_\varepsilon^* \subset \mathcal{M}_\varepsilon$ be the m-dimensional manifold obtained by flowing initial conditions on $\mathcal{P}_\varepsilon$ forward in (2.1). Finally, we denote by

$$\mathcal{Y}_\varepsilon := \bigcup_{\varphi \in \mathcal{P}_\varepsilon^*} W_\varepsilon^u(\varphi) \subset W_\varepsilon^u(\mathcal{M}_\varepsilon),$$

the union of unstable fibers in (2.1) with base points in $\mathcal{P}_\varepsilon^* \subset \mathcal{M}_\varepsilon$. Note that $\mathcal{Y}_\varepsilon$ is locally invariant in (2.1) by Fenichel fibering – see §2.3.1. By the Exchange Lemma, there exists an

$(m+n)$-dimensional submanifold $\mathcal{Z}_{1,\varepsilon}$ of $\mathcal{Z}_\varepsilon^*$ and an ε-independent neighborhood $\mathcal{D}_1 \subset \mathcal{D}$ of $(\psi_s(2\ell_0), 0)$ such that the Hausdorff distance between $\mathcal{D}_1 \cap \mathcal{Y}_\varepsilon$ and $\mathcal{Z}_{1,\varepsilon}$ is $O(e^{-C/\varepsilon})$. Moreover, trajectories crossing $\mathcal{Z}_{1,\varepsilon}$ remain in $\mathcal{D}$ during the excursion from $\mathcal{Z}$ to $\mathcal{Z}_{1,\varepsilon}$.

We aim to show that the $(m+n)$-dimensional manifold $\mathcal{Y}_\varepsilon$ intersects $\ker(I-R)$ transversally. Then, by the above closeness estimate the same holds for the $(m+n)$-dimensional manifold $\mathcal{Z}_{1,\varepsilon}^* \subset \mathcal{Z}_\varepsilon^*$ obtained by flowing $\mathcal{Z}_{1,\varepsilon}$ forward in (2.1). Therefore, we determine the singular limit $\mathcal{Y}_0$ of $\mathcal{Y}_\varepsilon$. First, recall that $\mathcal{P}_0^*$ intersects $\mathcal{T}_-$ transversely at $\psi_s(2\ell_0)$. Second, the unstable manifold $W_0^u(\mathcal{M}_0)$ of $\mathcal{M}_0$ in (2.2) intersects $\ker(I-R)$ transversely in an m-dimensional manifold $\mathcal{S}_0 := \{\phi_h(0,u) : u \in U_h\}$ by assumption **(E1)**. The α-limit set of $\mathcal{S}_0$ equals the touch-down manifold $\mathcal{T}_-$ in $\mathcal{M}$. We now put these two items together and conclude that the $(m+n)$-dimensional union,

$$\mathcal{Y}_0 := \bigcup_{\varphi \in \mathcal{P}_0^*} W_0^u(\varphi) \subset W_0^u(\mathcal{M}_0),$$

of unstable fibers in (2.2) with base points in $\mathcal{P}_0^*$ intersects the $(m+n)$-dimensional manifold $\ker(I-R)$ transversally in the point $\phi_h(0,u_0)$.

By Fenichel fibering – see §2.3.1 – the manifolds $\mathcal{Y}_\varepsilon$ and $\mathcal{Y}_0$ have Hausdorff distance $O(\varepsilon)$ in a neighborhood of the intersection point $\phi_h(0,u_0)$. Therefore, provided $\varepsilon > 0$ is sufficiently small, $\mathcal{Y}_\varepsilon$ intersects $\ker(I-R)$ transversally in some point $P_{h,\varepsilon}$, which lies $O(\varepsilon)$-close to $\phi_h(0,u_0)$. Denote by $\phi_{h,\varepsilon}(x)$ the solution to (2.1) with initial condition $\phi_{h,\varepsilon}(0) = P_{h,\varepsilon}$.

Since $\phi_h(x,u_0)$ converges to $(\psi_s(2\ell_0), 0) \in \mathcal{M}$ as $x \to -\infty$, there exists $x_0 > 0$ such that $\phi_h(-x_0,u_0)$ is contained in the neighborhood $\mathcal{D}_1 \subset \mathcal{D}$ of $(\psi_s(2\ell_0), 0)$. Hence, since $\phi_{h,\varepsilon}(0)$ is $O(\varepsilon)$-close to $\phi_h(0,u_0)$ and x_0 is independent of ε, one derives via Grönwall type estimates that $\phi_{h,\varepsilon}(-x_0)$ is contained in $\mathcal{D}_1 \cap \mathcal{Y}_\varepsilon$. Recall that the outcome of the Exchange Lemma is that $\mathcal{Y}_\varepsilon$ has Hausdorff distance $O(e^{-C/\varepsilon})$ from $\mathcal{Z}_{1,\varepsilon}$ in the neighborhood $\mathcal{D}_1$ of $\phi_{h,\varepsilon}(-x_0)$. Thus, using x_0 is ε-independent, we infer, again via Grönwall type estimates, that the Hausdorff distance between $\mathcal{Y}_\varepsilon$ and $\mathcal{Z}_{1,\varepsilon}^*$ is $O(e^{-C/\varepsilon})$ in a neighborhood of $\phi_{h,\varepsilon}(0)$. Therefore, $\mathcal{Z}_{1,\varepsilon}^*$ intersects $\ker(I-R)$ transversally in some point $P_{p,\varepsilon}$, which is $O(\varepsilon)$-close to $\phi_h(0,u_0)$. The solution $\phi_{p,\varepsilon}(x)$ to (2.1) with initial condition $\phi_{p,\varepsilon}(0) = P_{p,\varepsilon}$ is the desired periodic orbit. Indeed, $\phi_{p,\varepsilon}(x)$ crosses $\ker(I-R)$ at $x = 0$ and at some point $x = -L_\varepsilon < 0$, since $\phi_{p,\varepsilon}$ is contained in $\mathcal{Z}_\varepsilon^*$.

All that remains to show is the four assertions in the theorem statement. The second assertion is immediate, since $\phi_{p,\varepsilon}(0) \in \ker(I-R)$. The other assertions require more work.

We start by estimating $\phi_{p,\varepsilon}$ with the pulse solution ϕ_h to the fast reduced system (2.2). Since $\phi_{p,\varepsilon}(0)$ is $O(\varepsilon)$-close to $\phi_h(0,u_0)$, we approximate

$$\left\|\phi_{p,\varepsilon}(x) - \phi_h(x,u_0)\right\| \leq C\varepsilon, \quad x \in [-x_0, 0]. \tag{2.20}$$

Next, we obtain decay estimates of $\phi_{p,\varepsilon}(x)$ to the slow manifold. Without loss of generality we may assume $\phi_{p,\varepsilon}(x)$ is in $\mathcal{D}$ for $x \in [-L_\varepsilon, -x_0]$. Thus, we may express $\phi_{p,\varepsilon}(x)$ in Fenichel coordinates as $\tilde{\phi}_{\mathrm{p},\varepsilon}(x) = (a_{\mathrm{p},\varepsilon}(x), b_{\mathrm{p},\varepsilon}(x), c_{\mathrm{p},\varepsilon}(x)) = \Psi_\varepsilon(\phi_{\mathrm{p},\varepsilon}(x))$ for $x \in [-L_\varepsilon, x_0]$ – see §2.3.2. By [57, Corollary 1] the estimates (2.13) yield a $\mu_0 > 0$, independent of ε, such that

$$\|a_{\mathrm{p},\varepsilon}(x)\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad \|b_{\mathrm{p},\varepsilon}(x)\| \leq Ce^{\mu_0(x+x_0)}, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.21}$$

We prove the fourth assertion. First, $\mathcal{M}_\varepsilon$ corresponds to the space $a = b = 0$ in (2.11). Second, $\mathcal{M}_\varepsilon$ has Hausdorff distance $\mathcal{O}(e^{-C/\varepsilon})$ to $\mathcal{M}_0 \subset \mathcal{M}$. Third, the coordinate transform Ψ_ε is C^1. Combining these items with estimate (2.21) yields the fourth assertion.

We prove the third assertion. We express the pulse solution ϕ_{h} to the fast reduced system (2.2) in Fenichel coordinates as $\tilde{\phi}_{\mathrm{h}}(x) = \Psi_0(\phi_{\mathrm{h}}(x, u_0))$ for $x \leq -x_0$. Observe that $\tilde{\phi}_{\mathrm{h}}(x)$ satisfies (2.11) for $\varepsilon = 0$ and lies in the unstable space $a = 0$. Consequently, we can write $\tilde{\phi}_{\mathrm{h}}(x) = (0, b_{\mathrm{h}}(x), c_0)$, where c_0 is a constant in U_F and $b_{\mathrm{h}}(x)$ satisfies the equation $b_x = B(0, b, c_0, 0)b$, where B is as in (2.11). Clearly, $b_{\mathrm{h}}(x)$ converges exponentially to 0 as $x \to -\infty$. By estimate (2.20) and C^1-smoothness of Ψ_ε in ε, it holds

$$\left\|\tilde{\phi}_{\mathrm{p},\varepsilon}(-x_0) - \tilde{\phi}_{\mathrm{h}}(-x_0)\right\| \leq C\varepsilon. \tag{2.22}$$

Using estimates (2.14), (2.21) and (2.22) we obtain,

$$\begin{aligned} \|c_{\mathrm{p},\varepsilon}(x) - c_0\| &\leq \int_x^{-x_0} \left(\varepsilon\|K(c_{\mathrm{p},\varepsilon}(y), \varepsilon)\| + \left\|H(\tilde{\phi}_{\mathrm{p},\varepsilon}(y), \varepsilon)(a_{\mathrm{p},\varepsilon}(y) \otimes b_{\mathrm{p},\varepsilon}(y))\right\|\right) dy \\ &\quad + \|c_{\mathrm{p},\varepsilon}(-x_0) - c_0\| \\ &\leq C\varepsilon\Xi_\theta(\varepsilon), \end{aligned} \tag{2.23}$$

for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. The difference $g_\varepsilon(x) = b_{\mathrm{p},\varepsilon}(x) - b_{\mathrm{h}}(x)$ satisfies an inhomogeneous equation of the form,

$$g_x = A_\varepsilon(x)g + h_\varepsilon(x),$$

where $A_\varepsilon(x)g_\varepsilon(x) = B(\tilde{\phi}_{\mathrm{h}}(x), 0)g_\varepsilon(x) + (B(0, b_{\mathrm{p},\varepsilon}(x), c_0, 0) - B(\tilde{\phi}_{\mathrm{h}}(x), 0))b_{\mathrm{p},\varepsilon}(x)$ and $h_\varepsilon(x) = (B(\tilde{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) - B(0, b_{\mathrm{p},\varepsilon}(x), c_0, 0))b_{\mathrm{p},\varepsilon}(x)$. Taking x_0 larger if necessary, estimates (2.13), (2.21) and (2.23) yield $\mathrm{Re}(A_\varepsilon(x)) \leq -\mu_0$ and $\|h_\varepsilon(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon)$ for $x \in [-\Xi_\theta(\varepsilon), -x_0]$. Therefore, we conclude using (2.22) that,

$$\|b_{\mathrm{p},\varepsilon}(x) - b_{\mathrm{h}}(x)\| \leq C\varepsilon\Xi_\theta(\varepsilon), \quad x \in [-\Xi_\theta(\varepsilon), -x_0]. \tag{2.24}$$

Estimate (2.16) now follows from C^1-smoothness of Ψ_ε^{-1} in ε together with estimates (2.20), (2.21), (2.23) and (2.24).

We prove (2.17). By (2.14) and (2.21) we have,

$$\left\|H(\tilde{\phi}_{\mathrm{p},\varepsilon}(x))(a_{\mathrm{p},\varepsilon}(x) \otimes b_{\mathrm{p},\varepsilon}(x))\right\| \leq Ce^{-\mu_0 L_\varepsilon}, \quad x \in [-L_\varepsilon, -x_0].$$

Therefore, using Grönwall type estimates, there exists a solution $(0, 0, c_{\mathrm{s},\varepsilon}(x))$ on the invariant manifold $\mathcal{M}_\varepsilon \subset \{a = b = 0\}$ satisfying $\partial_x c = \varepsilon K(c, \varepsilon)$, which is $\mathcal{O}(e^{-\mu_0 L_\varepsilon})$-close to $c_{\mathrm{p},\varepsilon}(x)$

for $x \in [-L_\varepsilon, -x_0]$. The solution $c_{s,\varepsilon}(x)$ is to leading order described by a solution $c_{s,0}(\check{x})$ to $\partial_{\check{x}} c = K(c, 0)$. This results in the estimate,

$$\|c_{p,\varepsilon}(x) - c_{s,0}(\varepsilon x)\| \leq C\varepsilon, \quad x \in [-L_\varepsilon, -x_0]. \tag{2.25}$$

Estimates (2.23) and (2.25) imply $c_{s,0}(0) = c_0$. On the other hand, we have $\Psi_0((\psi_s(2\ell_0), 0)) = \lim_{x\to-\infty} \tilde{\phi}_h(x) = (0, 0, c_0)$. Since system $\partial_{\check{x}} c = K(c, 0)$ corresponds to the slow reduced system (2.4), we have $\Psi_0^{-1}((0, 0, c_{s,0}(\check{x}))) = (\psi_s(\check{x} + 2\ell_0), 0)$ for $\varepsilon^{-1}\check{x} \in [-L_\varepsilon, 0]$. Hence, by C^1-smoothness of Ψ_ε^{-1} in ε, R-reversibility of $\phi_{p,\varepsilon}(x)$, estimates (2.21) and (2.25) and the inequality $\theta \geq \mu_0^{-1}$, we conclude estimate (2.17) holds true.

Finally, we prove the first assertion. On the one hand, we have $p_s(\ell_0) = 0$ and $p_s'(\ell_0) = H_1(u_s(\ell_0), 0, 0) \neq 0$ by (2.19). On the other hand, it holds $\|p_s(\varepsilon L_\varepsilon)\| \leq C\varepsilon$ by (2.17). Thus, an application of the inverse function theorem and the mean value theorem yields $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$. □

Remark 2.4. Although the framework in [25, 110] is different, the idea to track solutions close to the slow manifold with the aid of an appropriate exchange lemma is the same. However, in contrast to [25, 110], we need an exchange lemma that works for systems having non-constant slow components in the fast reduced system – see [102, Section 2.5]. Moreover, as in [25], we exploit that system (2.1) is R-reversible. Yet, transversality arguments differ from [25], since our class of systems admits multidimensional components. ■

Remark 2.5. In principle, the existence problem could also be put in slow-fast form by introducing $p = \varepsilon^{-1/2} D_1 u_x$ instead of $p = \varepsilon^{-1} D_1 u_x$. Then the p-equation reads $p_x = \varepsilon^{3/2} H_1(u, v, \varepsilon) + \varepsilon^{1/2} H_2(u, v)$. This is done in existence analysis of periodic pulse solutions in the Gierer-Meinhardt equations in [25]. The equation for the p-components in the slow reduced system (2.4) would be $p_{\check{x}} = 0$ in that case. This makes the construction of the desired singular periodic orbit, performed in §2.2.2, impossible. Therefore, the scaling regime in (2.1) is the most natural for our set-up. In [25] one avoids setting $\varepsilon = 0$ in the existence analysis and makes a distinction between slow and 'super-slow' behavior. We emphasize that in the spectral stability analysis in Chapter 5 we adopt a similar scaling regime to put the eigenvalue problem in slow-fast form, which is required for an application of the Riccati transform – see also Remark 1.4. ■

Remark 2.6. As mentioned in §1.4.1, our model (1.9) is a reaction-diffusion system (1.1) that allows for semi-strong interaction (1.8), with the extra condition that G vanishes at $v = 0$. For general G, consider a $2n$-dimensional compact submanifold $\mathcal{M}_0$ of $\{(u, p, v, 0) : G(u, v, 0) = 0\} \subset U \times \mathbb{R}^m \times V \times \mathbb{R}^n$. By Fenichel theory [34] $\mathcal{M}_0$ perturbs, for $\varepsilon > 0$ sufficiently small, to a locally invariant manifold $\mathcal{M}_\varepsilon$ in (2.1). This manifold $\mathcal{M}_\varepsilon$ is diffeomorphic to $\mathcal{M}_0$ and lies at Hausdorff distance $O(\varepsilon)$ from $\mathcal{M}_0$. When $\mathcal{M}_0$ can be given as a graph over $(u, p) \in U \times \mathbb{R}^m$, the same holds for $\mathcal{M}_\varepsilon$. Thus, in that case one can change coordinates in (2.1) relative to $\mathcal{M}_\varepsilon$ and we obtain $\mathcal{M}_\varepsilon = \mathcal{M}_0 \subset \{(u, p, 0, 0) : u \in U, p \in \mathbb{R}^m\}$. Therefore, in the existence analysis, the condition that G vanishes at $v = 0$, corresponds to an a priori coordinate change in (2.1).

However, one introduces more than additional technical difficulties in the spectral stability analysis when G does not vanish at $v = 0$. Indeed, without relative coordinates, we do not achieve estimate (2.18), which is essential in our stability analysis. However, applying the coordinate change to equation (1.9) changes its structure fundamentally. In the new coordinates (1.9) is not even of reaction-diffusion type. Hence, we expect that the spectral analysis differs essentially, when G does not vanish at $v = 0$. This is an interesting subject of future research, especially since it includes the possibility of localized patterns with oscillatory tails [11, 30], but is outside the scope of this thesis. ■

Remark 2.7. As mentioned in the introduction in Chapter 1 the class of equations (1.10) includes the generalized Gierer-Meinhardt equations,

$$\begin{aligned} \varepsilon^2 u_t &= u_{xx} - \varepsilon^2 \mu u + \varepsilon u^{\alpha_1} v^{\beta_1}, \\ v_t &= v_{xx} - v + u^{\alpha_2} v^{\beta_2}, \end{aligned} \qquad (u, v) \in \mathbb{R}^2, x \in \mathbb{R}, \tag{2.26}$$

with parameters $\alpha_1 \in \mathbb{R}, \alpha_2 < 0, \beta_{1,2} \in \mathbb{Z}_{>1}$ and $\mu > 0$ satisfying,

$$(\alpha_1 - 1)(\beta_2 - 1) - \alpha_2 \beta_1 > 0.$$

Indeed, it is not difficult to verify that assumptions **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true for (2.26). Thus, Theorem 2.3 reconfirms the existence result of periodic pulse solutions to (2.26) proved in [25]. ■

2.5 Existence in the slowly nonlinear toy problem

In this section, we explicitly construct a singular periodic orbit in the slowly nonlinear toy problem

$$\begin{aligned} \varepsilon^2 u_t &= u_{xx} - \varepsilon^2 \mu \sin(u) - \varepsilon(\nu_2 v^2 + \nu_3 v^3), \\ v_t &= v_{xx} - v + \frac{v^2}{f(u)}, \end{aligned} \qquad (u, v) \in \mathbb{R}^2, x \in \mathbb{R}, \tag{2.27}$$

with $\mu > 0$, $\nu_2, \nu_3 \in \mathbb{R}$ and $f \colon \mathbb{R} \to \mathbb{R}_{>0}$ at least C^3. We then evoke Theorem 2.3 to prove the existence of an actual periodic solution close to the singular one.

For the toy problem (2.27) the fast reduced system reads,

$$\begin{aligned} u_x &= 0, \\ p_x &= \nu_2 v^2 + \nu_3 v^3, \\ v_x &= q, \\ q_x &= v - \frac{v^2}{f(u)}, \end{aligned} \qquad (u, p, v, q) \in \mathbb{R}^4. \tag{2.28}$$

For any $u > 0$, the governing subsystem,

$$\begin{aligned} v_x &= q, \\ q_x &= v - \frac{v^2}{f(u)}, \end{aligned} \qquad (v, q) \in \mathbb{R}^2. \tag{2.29}$$

of (2.28) is Hamiltonian and has a hyperbolic saddle in $(0, 0)$. By a phase-portrait analysis one observes that (2.29) admits for any $u \in \mathbb{R}$ a homoclinic solution to $(0, 0)$. By integrating equation (2.29) an explicit expression for this homoclinic can be found. This results in the pulse solution to (2.28) given by,

$$\phi_{\mathrm{h}}(x, u; v_2, v_3) = \left(u, \int_0^x \left(v_2(v_{\mathrm{h}}(z, u))^2 + v_3(v_{\mathrm{h}}(z, u))^3\right) dz, v_{\mathrm{h}}(z, u), v_{\mathrm{h}}'(z, u)\right),$$

with $v_{\mathrm{h}}(x, u) = \frac{3}{2} f(u)\mathrm{sech}^2(\frac{1}{2}x)$. Consequently, the take-off and touch-down curves on the slow manifold $\mathcal{M}$ are given by,

$$\mathcal{T}_{\pm} = \{(u, \mathcal{J}(u; v_2, v_3)) : u \in (0, \pi)\}, \quad \mathcal{J}(u; v_2, v_3) = \tfrac{3}{5}(f(u))^2 \left(5v_2 + 6v_3 f(u)\right).$$

The slow reduced system,

$$\begin{aligned} u_{\check{x}} &= p, \\ p_{\check{x}} &= \mu \sin(u), \end{aligned} \qquad (u, p) \in \mathbb{R}^2,$$

is also Hamiltonian and can be integrated. This leads to the family of bounded solutions given by the $(4K(k)\mu^{-1/2})$-periodic Jacobi-amplitude functions,

$$\psi_{\mathrm{s}}(\check{x}; k, c, \mu) = (u_{\mathrm{s}}(\check{x}), u_{\mathrm{s}}'(\check{x})), \quad u_{\mathrm{s}}(\check{x}; k, c, \mu) = 2\mathrm{Am}\left(-k\sqrt{\mu}(x - c), k^{-2}\right) + \pi, \tag{2.30}$$

parameterized over $k \in (0, 1)$, where $K(k)$ is the Jacobi complete integral of the first kind. The constant $c \in \mathbb{R}$ with $|c| < K(k)\mu^{-1/2}$ corresponds to the initial translation on the orbit of ψ_{s}. In addition, we take

$$\ell_0 = \ell_0(k, l, c, \mu) := c + \frac{(2l + 1)K(k)}{\sqrt{\mu}} > 0, \tag{2.31}$$

where $l \in \mathbb{Z}_{\geq 0}$ such that it holds $u_s'(\ell_0; k, c, \mu) = 0$.

Constructing a singular periodic orbit now reduced to connecting the solution ψ_{s} with the take-off and touch-down curves $\mathcal{T}_{\pm}$, i.e. finding values for μ, k, c, v_2, v_3 such that,

$$\mathcal{J}(u_{\mathrm{s}}(0; k, c, \mu), v_2, v_3) = u_{\mathrm{s}}'(0; k, c, \mu),$$

Such values can be easily found with a computer software programm like Mathematica. If we have found such values, the singular periodic orbit is given by,

$$\phi_{\mathrm{p},0} = \{(\psi_{\mathrm{s}}(\check{x}; k, c, \mu), 0) : \check{x} \in (0, 2\ell_0)\} \cup \{\phi_{\mathrm{h}}(x, u_{\mathrm{s}}(0; k, c, \mu); v_2, v_3) : x \in \mathbb{R}\},$$

One readily observes that **(S1)**, **(S2)** and **(E1)** are satisfied. For **(E2)** to hold true, we require that the transversality condition

$$\mathcal{J}'(u_{\mathrm{s}}(0;k,c,\mu),v_2,v_3)u_{\mathrm{s}}'(0;k,c,\mu)-\mu\sin(u_{\mathrm{s}}(0;k,c,\mu))\neq 0,$$

is satisfied. Now, it follows from Theorem 2.3 that an actual periodic solution to (2.27) lies in the vicinity of the singular orbit $\phi_{\mathrm{p},0}$.

Chapter 3

Stability results

3.1 Introduction

In this chapter we present the outcomes of our spectral stability analysis performed in Chapter 5. We assume that conditions **(S1)**, **(S2)**, **(E1)** and **(E2)** hold true. Then, Theorem 2.3 provides a reversibly symmetric, $2L_\varepsilon$-periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x)$ to (2.1). This yields a stationary, periodic pulse solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$ to system (1.10). We denote by $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ the corresponding solution to the rescaled system (1.9). The stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ is determined by the spectrum of the linearization $\mathcal{L}_\varepsilon$ of (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$. The (critical) spectrum of the periodic differential operator $\mathcal{L}_\varepsilon$ is a union of curves parameterized over the unit circle S^1 by Floquet theory. Due to translational invariance one of these curves is attached to the origin. The spectral curves can be located by tracing the zeros of the analytic Evans function [38].

When the spectrum of $\mathcal{L}_\varepsilon$ is confined to the left half-plane and bounded away from the imaginary axis, except for a quadratic tangency at the origin, it is known [58, 101, 104] that the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinear diffusively stable as solution to (1.9). Verifying such spectral conditions is in general very hard, especially for multi-component systems. However, as mentioned in the introduction in Chapter 1, the presence of the small parameter ε in (1.9) provides a mechanism to reduce complexity. In the singular limit the Evans function corresponding to the full problem decomposes as a product of a slow and a fast Evans function. The analytic fast and meromorphic slow Evans function are defined in terms of simpler, lower-dimensional eigenvalue problems. The spectrum of $\mathcal{L}_\varepsilon$ can be approximated by the roots of the fast and slow Evans functions. This approximation mechanism provides asymptotic control over the spectrum. However, the critical spectral curve attached to origin shrinks to the origin in the singular limit. Thus, our approximation result is unable to determine the spectral geometry about the origin and asymptotic spectral control is insufficient to establish nonlinear stability. Therefore, we complement our analysis with an expansion of this critical spectral curve.

We start this chapter by linearizing (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$ and characterizing the spectrum of the linearization $\mathcal{L}_\varepsilon$ via Floquet-Bloch decomposition. Then, we provide conditions on the spectrum of $\mathcal{L}_\varepsilon$ yielding nonlinear stability. Subsequently, we introduce the analytic Evans function and reformulate the spectral stability conditions in terms of this function. Next, we state our two main spectral approximation results: the slow-fast decomposition of the Evans function in the singular limit and the expansion of the critical spectral curve. These two results then lead to explicit criteria yielding stability and instability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ in terms of simpler, lower-dimensional eigenvalue problems. Finally, we further simplify these criteria in the case $n = 1$ or $m = 1$ and we illustrate our results by explicit calculations in the slowly nonlinear toy problem (2.27).

3.2 Linearizing about the periodic pulse solution

We linearize system (1.9) about $\check{\phi}_{\mathrm{p},\varepsilon}$ and obtain the periodic differential operator $\mathcal{L}_\varepsilon$ on $C_{\mathrm{ub}}(\mathbb{R},\mathbb{R}^{m+n})$ with domain $C^2_{\mathrm{ub}}(\mathbb{R},\mathbb{R}^{m+n})$ given by

$$\mathcal{L}_\varepsilon\psi = D_\varepsilon\psi_{\check{x}\check{x}} - \mathcal{B}_\varepsilon\psi,$$

with

$$D_\varepsilon := \begin{pmatrix} D_1 & 0 \\ 0 & \varepsilon^2 D_2 \end{pmatrix},$$

and

$$\mathcal{B}_\varepsilon(\check{x}) := \begin{pmatrix} \partial_u H_1(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) + \varepsilon^{-1}\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}) & \partial_v H_1(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) + \varepsilon^{-1}\partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}) \\ \partial_u G(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) & \partial_v G(\check{\phi}_{\mathrm{p},\varepsilon},\varepsilon) \end{pmatrix}, \tag{3.1}$$

where we suppress the $\check{x}$-dependence of $\check{\phi}_{\mathrm{p},\varepsilon}$. Here, $C^k_{\mathrm{ub}}(\mathbb{R},\mathbb{R}^{m+n})$ denotes the Banach space of k times continuously differentiable functions, with derivatives up to order k bounded and uniformly continuous. It is endowed with the supremum norm,

$$\|f\| = \sum_{i=0}^{k} \left\|(\partial_{\check{x}})^i f\right\|_\infty .$$

Note that $\mathcal{L}_\varepsilon$ is closed, densely defined and sectorial by [72, Corollary 3.1.9.ii] and [44, Theorem 1.3.2].

3.2.1 Floquet-Bloch decomposition

By Theorem 2.3, $\mathcal{L}_\varepsilon$ is a $2\ell_\varepsilon$-periodic differential operator, where $\ell_\varepsilon := \varepsilon L_\varepsilon \to \ell_0$ as $\varepsilon \to 0$ with $\ell_0 > 0$ defined in **(E2)**. Therefore, Floquet-Bloch decomposition [38] of $\mathcal{L}_\varepsilon$ yields a family of closed and densely defined operators $\mathcal{L}_{\nu,\varepsilon}$ on $L^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ with domain $H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n})$ given by

$$\mathcal{L}_{\nu,\varepsilon}\psi = D_\varepsilon\left(\partial_{\check{x}} - \frac{i\nu}{2\ell_\varepsilon}\right)^2\psi - \mathcal{B}_\varepsilon\psi, \quad \nu \in [-\pi,\pi],$$

where $L^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ is the space of L^2-integrable functions that are $2\ell_\varepsilon$-periodic and $H^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ is the subspace of $L^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ of functions that have weak derivatives up to order 2. By the Rellich compactness theorem the space $H^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ is compactly embedded in $L^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$. Therefore, $\mathcal{L}_{\nu,\varepsilon}$ has compact resolvent. Consequently, its spectrum is discrete and consists entirely of eigenvalues. The spectrum of $\mathcal{L}_\varepsilon$ is given by the union,

$$\sigma(\mathcal{L}_\varepsilon) = \bigcup_{\nu \in [-\pi,\pi]} \sigma(\mathcal{L}_{\nu,\varepsilon}). \tag{3.2}$$

Indeed, if $\lambda \in \sigma(\mathcal{L}_{\nu,\varepsilon})$ is an eigenvalue and $\varphi \in H^2_{\text{per}}([0, 2\ell_\varepsilon], \mathbb{C}^{m+n})$ denotes the corresponding eigenfunction, then the natural extension of $\varphi(\check{x})e^{-i\nu\check{x}/(2\ell_\varepsilon)}$ to $\mathbb{R}$ yields an eigenfunction of $\mathcal{L}_\varepsilon$. Conversely, given $\lambda \in \sigma(\mathcal{L}_\varepsilon)$, there exists by Floquet theory a $\gamma \in S^1$ and a corresponding eigenfunction $\psi \in C^2_{ub}(\mathbb{R}, \mathbb{C}^{m+n})$ satisfying $\psi(\check{x}) = \gamma\psi(\check{x} + 2\ell_\varepsilon)$ for all $\check{x} \in \mathbb{R}$. The restriction of $\psi(\check{x})e^{i\nu\check{x}/(2\ell_\varepsilon)}$ to $[0, 2\ell_\varepsilon]$ is the eigenfunction of $\mathcal{L}_{\nu,\varepsilon}$, where $e^{i\nu} = \gamma$. The spectral decomposition (3.2) gives rise to the following definition.

Definition 3.1. Let $\nu \in [-\pi, \pi]$ and $\gamma = e^{i\nu} \in S^1$. A point $\lambda \in \sigma(\mathcal{L}_{\nu,\varepsilon})$ is called a *γ-eigenvalue* of $\mathcal{L}_\varepsilon$. The algebraic multiplicity of λ as an eigenvalue of $\mathcal{L}_{\nu,\varepsilon}$ is the *algebraic γ-multiplicity* of λ.

3.3 Nonlinear stability by linear approximation

In this section we collect nonlinear (in)stability results from the literature. More precisely, we present conditions on the spectrum of the linearization $\mathcal{L}_\varepsilon$ of (1.9) about $\check{\phi}_{\text{p},\varepsilon}$ yielding some form of nonlinear stability or instability.

3.3.1 Spectral conditions yielding nonlinear stability

By translational invariance, 0 is always a 1-eigenvalue of $\mathcal{L}_\varepsilon$. Indeed, the restriction of the derivative $\check{\phi}'_{\text{p},\varepsilon}(\check{x})$ to $[0, 2\ell_\varepsilon]$ is contained in the kernel of $\mathcal{L}_{0,\varepsilon}$. If we assume that 0 has algebraic 1-multiplicity 1, then there exists by the implicit function theorem a spectral curve $\lambda_\varepsilon \colon U_\varepsilon \to \mathbb{C}$, where $U_\varepsilon \subset [-\pi, \pi]$ is a neighborhood of 0, such that $\lambda_\varepsilon(0) = 0$ and $\lambda_\varepsilon(\nu)$ is a $e^{i\nu}$-eigenvalue for $\nu \in U_\varepsilon$. By assuming that this critical spectral curve touches the origin in a quadratic tangency and the rest of the spectrum is confined to the left half-plane, bounded away from the imaginary axis, we establish some form of nonlinear stability. This leads to the following definition.

Definition 3.2. The periodic pulse solution $\check{\phi}_{\text{p},\varepsilon}$ to (1.9) is *spectrally stable* if 0 is a simple eigenvalue of $\mathcal{L}_{0,\varepsilon}$ and there exists $\varsigma > 0$, possibly dependent on ε, such that

$$\begin{aligned} \text{Re}(\lambda_\varepsilon(\nu)) &\leq -\varsigma\nu^2, \quad \nu \in U_\varepsilon, \\ \sigma(\mathcal{L}_\varepsilon) \setminus \lambda_\varepsilon[U_\varepsilon] &\subset \{\lambda \in \mathbb{C} : \text{Re}(\lambda) < -\varsigma\}. \end{aligned}$$

Spectral stability of $\check{\phi}_{p,\varepsilon}$ implies nonlinear diffusive stability of $\check{\phi}_{p,\varepsilon}$ with respect to localized perturbations. In addition, an initial displacement of the periodic pulse can be tracked for large times.

Theorem 3.3. *[101, Theorem 1] Suppose $\check{\phi}_{p,\varepsilon}$ is spectrally stable. Take $b \in (0, \frac{1}{2})$. There are $\delta, C > 0$, possibly dependent on ε, such that the following holds. The solution $\check{\phi}(x,t)$ to* (1.9) *with initial condition,*

$$\check{\phi}(\check{x}, 0) = \check{\phi}_{p,\varepsilon}(\check{x} + \theta_0(\check{x})) + v_0(\check{x}),$$

with $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ and $\theta_0 \in H^3(\mathbb{R}, \mathbb{R})$ satisfying $\|\theta_0\rho\|_{H^3}, \|v_0\rho\|_{H^2} \leq \delta$ with $\rho(\check{x}) = (1 + \check{x}^2)^{3/2}$, exists for all times $t \geq 0$ and can be written as

$$\check{\phi}(\check{x}, t) = \check{\phi}_{p,\varepsilon}(\check{x} + \theta(\check{x}, t)) + v(\check{x}, t), \quad t > 0,$$

where $\theta \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}$ and $v \colon \mathbb{R} \times (0, \infty) \to \mathbb{R}^{m+n}$. There exists a constant $\theta_{\lim} \in \mathbb{R}$ such that

$$\sup_{\check{x} \in \mathbb{R}} [|\theta(\check{x}, t) - \theta_{\lim} G(\check{x}, t)| + \|v(\check{x}, t)\|] \leq C(1 + t)^{-1+b}, \quad t > 0,$$

where G is the Gaussian,

$$G(\check{x}, t) = \frac{1}{\sqrt{4\alpha\pi(1 + t)}} e^{-\check{x}^2/(4\alpha(1+t))},$$

with $\alpha := -\lambda_\varepsilon''(0)$. In particular, we have

$$\sup_{\check{x} \in \mathbb{R}} \left\| \check{\phi}(\check{x}, t) - \check{\phi}_{p,\varepsilon} \left(\check{x} + \theta_{\lim} G(\check{x}, t)\right) \right\| \leq C(1 + t)^{-1+b}, \quad t > 0.$$

The above result is to be compared with [104, Theorem 1.1]. Here, the class of allowed perturbations is larger, i.e. one requires $v_0\tilde{\rho} \in H^{1/2+b}(\mathbb{R}, \mathbb{R}^{m+n})$ with $\tilde{\rho}(\check{x}) = 1 + \check{x}^2$. However, in [104] one obtains a weaker decay bound of the form $\sup_{\check{x} \in \mathbb{R}} \|v(\check{x}, t)\| \leq C(1+t)^{-1/2}$. Moreover, in [58] pointwise nonlinear estimates are obtained with respect to perturbations $v_0 \in H^2(\mathbb{R}, \mathbb{R}^{m+n})$ satisfying $\|v_0(\check{x})\| \leq E_0 e^{-\check{x}^2/M}$ or $\|v_0(\check{x})\| \leq E_0(1 + |\check{x}|)^{-r}$ for some $M > 1, r > 2$ and $E_0 > 0$. The decay rates obtained in [58] are comparable to those in Theorem 3.3, yet they are more specific, since they depend pointwise on $\check{x}$. Finally, we emphasize that both in [58] and [104] one does not consider an initial displacement in time in contrast to Theorem 3.3.

As mentioned in the introduction of this chapter, verifying spectral stability is in general very hard. The main outcome of our spectral analysis is explicit conditions in terms of simpler, lower-dimensional eigenvalue problems that yield spectral stability of $\check{\phi}_{p,\varepsilon}$ – see §3.7. This reduction of complexity is achieved by a slow-fast decomposition of the Evans function in the singular limit and an expansion of the critical spectral curve $\lambda_\varepsilon(\nu)$ – see §3.5 and §3.6, respectively.

3.3.2 Spectral conditions yielding nonlinear instability

Spectrum of $\mathcal{L}_\varepsilon$ in the right half-plane yields nonlinear instability of the periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ against localized and non-localized perturbations.

Definition 3.4. The periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) is *spectrally unstable* if there exists $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda) > 0$.

Theorem 3.5. *[75, Section 4] Let $X = H^2(\mathbb{R}, \mathbb{R}^{m+n})$ or $X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$. Suppose the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) is spectrally unstable. Then, there exists $\delta > 0$ and a sequence of solutions $\check{\phi}_n(\check{x}, t), n \in \mathbb{Z}_{>0}$ to (1.9) satisfying $\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon} \in X$ such that*

$$\left\|\check{\phi}_n(\cdot, 0) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X \to 0 \text{ as } n \to \infty,$$

but for all $n \in \mathbb{Z}_{>0}$ there exists $t_n > 0$ such that

$$\begin{aligned} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}\right\|_X &\geq \delta, \text{ in the case } X = H^2(\mathbb{R}, \mathbb{R}^{m+n}), \\ \inf_{\theta \in \mathbb{R}} \left\|\check{\phi}_n(\cdot, t_n) - \check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)\right\|_X &\geq \delta, \text{ in the case } X = C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n}). \end{aligned}$$

We emphasize that in the case of non-localized perturbations, it is important to measure the distance from the perturbation to the family of all translates of the solution rather than to the solution itself. Indeed, any translate $\check{\phi}_{\mathrm{p},\varepsilon}(\cdot + \theta)$ corresponds to a non-localized perturbation. Yet, such a translate is a solution to (1.9) itself. Thus, $\check{\phi}_{\mathrm{p},\varepsilon}$ is never stable against translation of the profile. We stress that the θ-terms in Theorem 3.3 account for translation of the profile.

Using the outcomes of our spectral analysis, we obtain explicit conditions in terms of simpler, lower-dimensional systems yielding spectral instability – see §3.7. In particular, in the case $n = 1$ or $m = 1$, we can test for instability by calculating the signs of a number of explicit integral expressions – see §3.8.

3.4 The Evans function

In this section we introduce the Evans function as a tool to locate the spectrum of the linearization $\mathcal{L}_\varepsilon$. Recall from §3.2.1 that a point $\lambda \in \mathbb{C}$ is in the spectrum of $\mathcal{L}_\varepsilon$ if and only if there exists $\psi \in C^2_{ub}(\mathbb{R}, \mathbb{C}^{m+n})$ such that $\mathcal{L}_\varepsilon \psi = \lambda \psi$. The latter equation can be rewritten as an ODE in the 'small' spatial scale $x = \varepsilon^{-1}\check{x}$ as follows

$$\varphi_x = \mathcal{A}_\varepsilon(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{3.3}$$

with coefficient matrix,

$$\mathcal{A}_\varepsilon(x, \lambda) := \begin{pmatrix} \mathcal{A}_{11,\varepsilon}(x, \lambda) & \mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix},$$

where the blocks are given by

$$
\begin{aligned}
\mathcal{A}_{11,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & \varepsilon D_1^{-1} \\ \varepsilon\left(\partial_u H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\lambda\right)+\partial_u H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\
\mathcal{A}_{12,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \varepsilon\partial_v H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\partial_v H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}, \\
\mathcal{A}_{21,\varepsilon}(x) &:= \begin{pmatrix} 0 & 0 \\ \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) & 0 \end{pmatrix}, \\
\mathcal{A}_{22,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)+\lambda & 0 \end{pmatrix},
\end{aligned}
\tag{3.4}
$$

and $\hat{\phi}_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x))$ is the $2L_\varepsilon$-periodic pulse solution to (1.10). We will refer to (3.3) as the *full eigenvalue problem*. By Floquet Theory bounded solutions to (3.3) must satisfy $\varphi(-L_\varepsilon) = \gamma\varphi(L_\varepsilon)$ for some $\gamma \in S^1$. This fact leads to the definition of the Evans function.

Definition 3.6. Denote by $\mathcal{T}_\varepsilon(x,z,\lambda)$ the evolution operator of system (3.3). The *Evans function* $\mathcal{E}_\varepsilon\colon \mathbb{C}\times\mathbb{C}\to\mathbb{C}$ is given by

$$\mathcal{E}_\varepsilon(\lambda,\gamma) := \det(\mathcal{T}_\varepsilon(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_\varepsilon(0,L_\varepsilon,\lambda)).$$

Proposition 3.7. *The Evans function has the following properties:*

1. *The Evans function is analytic in both λ and γ;*
2. *We have $\lambda \in \sigma(\mathcal{L}_\varepsilon)$ if and only if there exists $\gamma \in S^1$ such that $\mathcal{E}_\varepsilon(\lambda,\gamma) = 0$. In that case, λ is a γ-eigenvalue and its algebraic γ-multiplicity is equal to the multiplicity of λ as a root of $\mathcal{E}_\varepsilon(\cdot,\gamma)$;*
3. *It holds $\overline{\mathcal{E}_\varepsilon(\lambda,\gamma)} = \mathcal{E}_\varepsilon(\overline{\lambda},\overline{\gamma})$ for $\lambda,\gamma \in \mathbb{C}$. Thus, the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is invariant under complex conjugation;*
4. *We have $\mathcal{E}_\varepsilon(\lambda,\gamma) = \mathcal{E}_\varepsilon(\lambda,\overline{\gamma})\gamma^{2(m+n)}$ for $\lambda \in \mathbb{C}$ and $\gamma \in S^1$. Thus, λ is a γ-eigenvalue if and only if it is a $\overline{\gamma}$-eigenvalue.*

Proof. The first two properties are established in [38]. Since (3.3) is a real-valued problem for $\lambda \in \mathbb{R}$, the third property follows by the reflection principle. Finally, since $\phi_{\mathrm{p},\varepsilon}$ is reversibly symmetric by Theorem 2.3, the eigenvalue problem (3.3) is R-reversible at $x = 0$, i.e. it holds $R\mathcal{T}_\varepsilon(x,y,\lambda)R = \mathcal{T}_\varepsilon(-x,-y,\lambda)$ for $x, y \in \mathbb{R}$. This yields the fourth property. □

Proposition 3.7 shows that the spectrum $\sigma(\mathcal{L}_\varepsilon)$ is an at most countable union of curves, each of which is covered twice by the unit circle S^1. The endpoints of the curves are ±1-eigenvalues. Proposition 3.7 and the implicit function theorem yield the following reformulation of the concept 'spectral stability' introduced in §3.3.

Corollary 3.8. *The periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable if and only if*

i. $\mathcal{E}_\varepsilon(\lambda, \gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

ii. $\mathcal{E}_\varepsilon(0, \gamma) \neq 0$ *for all* $\gamma \in S^1 \setminus \{1\}$*;*

iii. $\partial_\lambda \mathcal{E}_\varepsilon(0, 1) \partial_{\gamma\gamma} \mathcal{E}_\varepsilon(0, 1) < 0$.

3.5 The Evans function in the singular limit

In this section we present one of the main outcomes of our spectral stability analysis. We obtain an explicit *reduced Evans function* $\mathcal{E}_0(\lambda, \gamma)$, whose zeros, for γ restricted to S^1, approximate the zeros of the Evans function $\mathcal{E}_\varepsilon(\lambda, \gamma)$, provided that $\varepsilon > 0$ is sufficiently small, yielding asymptotic control over the spectrum. The reduced Evans function is defined in terms of three simpler, lower-dimensional eigenvalue problems. Therefore, the verification of the first spectral stability condition in Corollary 3.8 simplifies to a calculation of the roots of the reduced Evans function, which does not require understanding of the full eigenvalue problem (3.3), but rather of three simpler, lower-dimensional eigenvalue problems. Thus, when proving spectral stability, understanding of the full eigenvalue problem (3.3) is only necessary for λ close to the origin. The results of this local analysis are presented in §3.6.

This section is structured as follows. First, we define the reduced Evans function in terms of three eigenvalue problems, which are obtained by a slow-fast decomposition of the full eigenvalue problem (3.3). Then, we state our main result concerning the approximation of the zeros of $\mathcal{E}_\varepsilon$ by the ones of $\mathcal{E}_0$. Finally, using this approximation result, we simplify the verification of the first spectral stability condition in Corollary 3.8.

3.5.1 The reduced Evans function

The reduced Evans function $\mathcal{E}_0$ is only defined on half-planes C_Λ of the following form.

Notation 3.9. For every $\Lambda < 0$ we denote by C_Λ the open half-plane,

$$C_\Lambda := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > \Lambda\}.$$

The reduced Evans function $\mathcal{E}_0 \colon C_\Lambda \times \mathbb{C} \to \mathbb{C}$ is defined as the product,

$$\mathcal{E}_0(\lambda, \gamma) = (-\gamma)^n \mathcal{E}_{f,0}(\lambda) \mathcal{E}_{s,0}(\lambda, \gamma). \tag{3.5}$$

Here, the analytic map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ is called the *fast Evans function*. It is associated with the *homogeneous fast eigenvalue problem*,

$$\varphi_x = \mathcal{A}_{22,0}(x, u_0, \lambda)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.6}$$

with

$$\mathcal{A}_{22,0}(x,u,\lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, v_\mathrm{h}(x,u), 0) + \lambda & 0 \end{pmatrix}, \quad u \in U_\mathrm{h}.$$

Recall that $U_\mathrm{h}, v_\mathrm{h}(x,u), u_0 = u_\mathrm{s}(0)$ and $u_\mathrm{s}(\check{x})$ are defined in **(E1)** and **(E2)**. System (3.6) arises as an eigenvalue problem, when linearizing $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_\mathrm{h}(x, u_0)$. Indeed, equation (3.6) is equivalent to $\mathcal{L}_f \varphi = \lambda \varphi$, where $\mathcal{L}_f \colon L^2(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ is the closed, densely defined and sectorial operator – see [72, Theorem 3.1.3] and [44, Theorem 1.3.2] – with domain $H^2(\mathbb{R}, \mathbb{R}^n)$ given by

$$\mathcal{L}_f v = D_2 v_{xx} - \partial_v G(u_0, v_\mathrm{h}(\cdot, u_0), 0) v. \tag{3.7}$$

We establish the existence of the fast Evans function.

Proposition 3.10. *There exists $\Lambda < 0$ and an analytic map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$, which has a zero if and only if* (3.6) *admits a non-trivial, exponentially localized solution. In particular, the multiplicity of a root $\lambda \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of λ as an eigenvalue of the sectorial operator $\mathcal{L}_f$, defined in* (3.7).

The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is determined by two eigenvalue problems. The first is the *inhomogeneous fast eigenvalue problem*,

$$\partial_x \mathcal{X} = \mathcal{A}_{22,0}(x,u,\lambda)\mathcal{X} + \mathcal{A}_{21,0}(x,u), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}), \tag{3.8}$$

with

$$\mathcal{A}_{21,0}(x,u) := \begin{pmatrix} 0 & 0 \\ \partial_u G(u, v_\mathrm{h}(x,u), 0) & 0 \end{pmatrix}, \quad u \in U_\mathrm{h}.$$

The matrix system (3.8) describes the dynamics in the limit $\varepsilon \to 0$ of the full eigenvalue problem (3.3). The second is the *slow eigenvalue problem*,

$$\begin{aligned} D_1 u_{\check{x}} &= p, \\ p_{\check{x}} &= (\partial_u H_1(u_\mathrm{s}(\check{x}), 0, 0) + \lambda)\, u, \end{aligned} \qquad (u,p) \in \mathbb{C}^{2m}. \tag{3.9}$$

which arises as an eigenvalue problem when linearizing system $u_t = D_1 u_{\check{x}\check{x}} - H_1(u, 0, 0)$ about the stationary solution $u_\mathrm{s}(\check{x})$ in $L^2_\mathrm{per}([0, 2\ell_0])$. Let $\Lambda < 0$ be as in Proposition 3.10. The *slow Evans function* $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is defined by

$$\mathcal{E}_{s,0}(\lambda, \gamma) = \det\left(\Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda) - \gamma I\right), \tag{3.10}$$

where $\ell_0 > 0$ is as in **(E2)**, $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ is the evolution operator of the slow eigenvalue problem (3.9) and $\Upsilon(u, \lambda)$ is given by

$$\begin{aligned} \Upsilon(u,\lambda) &= \begin{pmatrix} I & 0 \\ \mathcal{G}(u,\lambda) & I \end{pmatrix}, \\ \mathcal{G}(u,\lambda) &= \int_{-\infty}^{\infty} [\partial_u H_2(u, v_\mathrm{h}(x,u)) + \partial_v H_2(u, v_\mathrm{h}(x,u))\mathcal{V}_{in}(x,u,\lambda)]\, dx, \end{aligned} \qquad u \in U_\mathrm{h}, \tag{3.11}$$

where $\mathcal{V}_{in}(x, u, \lambda)$ denotes the upper-left $(n \times m)$-block of the unique matrix solution $\mathcal{X}_{in}(x, u, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8). We collect some properties of the slow Evans functions $\mathcal{E}_{s,0}$.

Proposition 3.11. *The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is well-defined and enjoys the following properties:*

1. *$\mathcal{E}_{s,0}$ is analytic on its domain;*
2. *$\mathcal{E}_{s,0}(\cdot, \gamma)$ is meromorphic on C_Λ for each $\gamma \in \mathbb{C}$ in such a way that the reduced Evans function $\mathcal{E}_0$ is analytic on its domain;*
3. *$\mathcal{E}_{s,0}(\lambda, \cdot)$ is a polynomial of degree $2m$ and it holds $\mathcal{E}_{s,0}(\lambda, \gamma) = \gamma^{2m} \mathcal{E}_{s,0}(\lambda, \overline{\gamma})$ for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ and $\gamma \in S^1$;*
4. *The set of roots,*
$$\bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_{s,0}(\lambda, \gamma) = 0\},$$
is bounded.

The analytic reduced Evans function is defined as the product (3.5) of the meromorphic slow Evans function and the analytic fast Evans function. Thus, when determining the zeros of $\mathcal{E}_0(\cdot, \gamma)$ one should be aware of the possibility of zero-pole cancelation at all points in $\mathcal{E}_{f,0}^{-1}(0)$. The next proposition focuses on this issue.

Proposition 3.12. *Let $\lambda_\diamond$ be a simple zero of $\mathcal{E}_{f,0}$. Then, $\lambda_\diamond$ is also a zero of $\mathcal{E}_0(\cdot, \gamma)$ for any $\gamma \in S^1$ if it holds*

$$\int_{-\infty}^{\infty} \partial_v H_2(u_0, v_\mathrm{h}(z, u_0)) \varphi_{\lambda_\diamond,1}(z) dz = 0, \tag{3.12}$$

where $\varphi_{\lambda_\diamond}(x) = (\varphi_{\lambda_\diamond,1}(x), \varphi_{\lambda_\diamond,2}(x))$ is a non-trivial, exponentially localized solution to (3.6) *at $\lambda = \lambda_\diamond$, or*

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond,2}(z)^* \partial_u G(u_0, v_\mathrm{h}(z, u_0), 0) dz = 0, \tag{3.13}$$

where $\psi_{\lambda_\diamond}(x) = (\psi_{\lambda_\diamond,1}(x), \psi_{\lambda_\diamond,2}(x))$ denotes a non-trivial, exponentially localized solution to the adjoint equation,

$$\varphi_x = -\mathcal{A}_{22,0}(x, u_0, \lambda)^* \varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.14}$$

of (3.6) *at $\lambda = \lambda_\diamond$.*

The orthogonality relations (3.12) and (3.13) imply that there is no zero-pole cancelation. However, the converse is not true as pointed out in §3.8.4. One can show that the integrals in the right hand sides of (3.12) and (3.13) appear as one of multiple factors in the principal part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot,\gamma)$. Although it is possible to write down the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot,\gamma)$ explicitly at a zero $\lambda \in \mathcal{E}_{f,0}^{-1}(0)$, we decide to postpone this to §5.1.2 for the benefit of exposition, since the involved expressions are rather complex (except in the case $m = 1$ – see Proposition 3.28). Eventually, these principal parts provide a tool to determine precisely whether zero-pole cancelation occurs or not. Therefore, Proposition 3.12 is weaker – but better digestible – than the statements in §5.1.2.

Remark 3.13. Note that the fast eigenvalue problem (3.6) at $\lambda = 0$ equals the variational equation,

$$\varphi_x = \mathcal{A}_f(x)\varphi, \quad \varphi \in \mathbb{C}^{2n}, \tag{3.15}$$

about the homoclinic solution $\psi_{\mathrm{h}}(x,u_0)$ to (2.3) at $u = u_0$ with

$$\mathcal{A}_f(x) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 \end{pmatrix}, \quad u \in U_{\mathrm{h}}.$$

Therefore, the derivative of the homoclinic solution $\partial_x\psi_{\mathrm{h}}(x,u_0)$ is a non-trivial, exponentially localized solution to (3.6) at $\lambda = 0$. Thus, it holds $\mathcal{E}_{f,0}(0) = 0$ by Proposition 3.10. Now assume 0 is a simple root of $\mathcal{E}_{f,0}$. Since, we have

$$\int_{-\infty}^{\infty} \partial_v H_2(u_0, v_{\mathrm{h}}(x,u_0))\partial_x v_{\mathrm{h}}(x,u_0)dx = 0,$$

there occurs no zero-pole cancelation at $\lambda = 0$ by Proposition 3.12. We infer $\mathcal{E}_0(0,\gamma) = 0$ for each $\gamma \in S^1$. The latter corresponds to the existence of the critical spectral curve attached to the origin – see §3.5.3. ■

The proof of Propositions 3.10, 3.11 and 3.12 are provided in §5.1.

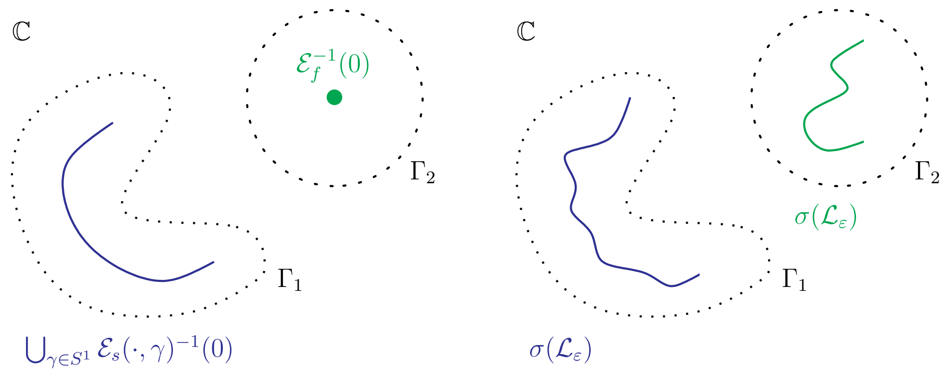
3.5.2 The spectral approximation result

We state our main result concerning the approximation of the zeros of $\mathcal{E}_\varepsilon$ by the ones of $\mathcal{E}_0$.

Theorem 3.14. *Let $\Lambda < 0$ be as in Proposition 3.10. Take a simple closed curve Γ in $C_\Lambda \setminus \mathcal{N}_0$, where*

$$\mathcal{N}_0 := \bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0\}.$$

Then, for $\varepsilon > 0$ sufficiently small, the number of roots (counting multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ and $\mathcal{E}_\varepsilon(\cdot,\gamma)$ interior to Γ coincides for any $\gamma \in S^1$.



(a) Spectrum in the limit $\varepsilon \to 0$ with disjoint contours Γ_1 and Γ_2.

(b) The true spectrum remains in the interior of Γ_1 and Γ_2 for $\varepsilon > 0$ sufficiently small.

Figure 3.1: Approximation of the spectrum $\sigma(\mathcal{L}_\varepsilon)$.

Combining Proposition 3.7 with Theorem 3.14 yields that the number of γ-eigenvalues (counting algebraic γ-multiplicity) of $\mathcal{L}_\varepsilon$ interior to Γ equals the number of roots (counting multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ for any $\gamma \in S^1$. In particular, Theorem 3.14 shows that the spectrum $\sigma(\mathcal{L}_\varepsilon) \cap C_\Lambda$ converges to a subset of $\mathcal{N}_0$ in the limit $\varepsilon \to 0$. Indeed, choose contours close enough to and disjoint from the connected components of $\mathcal{N}_0$, with, say, Hausdorff distance δ. This results in an $\varepsilon_\delta > 0$ such that, if $\varepsilon \in (0, \varepsilon_\delta)$, then $\sigma(\mathcal{L}_\varepsilon) \cap C_\Lambda$ is contained in a δ-neighborhood of $\mathcal{N}_0$.

To see that the singular limit of $\sigma(\mathcal{L}_\varepsilon)$ in fact *equals* $\mathcal{N}_0$, we need the following generalization of Theorem 3.14.

Theorem 3.15. *Let $\Lambda < 0$ be as in Proposition 3.10. Let $S \subset S^1$ be a closed subset. Take a simple closed curve Γ in $C_\Lambda \setminus \mathcal{N}_S$, where*

$$\mathcal{N}_S := \bigcup_{\gamma\in S} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0\}. \tag{3.16}$$

Then, for $\varepsilon > 0$ sufficiently small, the number of roots (including multiplicity) of $\mathcal{E}_0(\cdot,\gamma)$ and $\mathcal{E}_\varepsilon(\cdot,\gamma)$ interior to Γ coincides for any $\gamma \in S^1$.

Theorem 3.15 allows us, by taking $S = \{\gamma\}$ for some $\gamma \in S^1$, to follow individual γ-eigenvalues as they converge to the roots of $\mathcal{E}_0(\cdot,\gamma)$ as $\varepsilon \to 0$ or, equivalently, to establish the convergence of the spectrum $\sigma(\mathcal{L}_{\nu,\varepsilon})$ to the discrete set $\{\lambda \in \mathbb{C} : \mathcal{E}_0(\lambda, e^{i\nu}) = 0\}$.

The proof of Theorem 3.15 is provided in §5.2.

3.5.3 Consequences of the spectral approximation result

The results in §3.5.2 imply that we can approximate the roots of the Evans function, which is defined in terms of the $2(m+n)$-dimensional full eigenvalue problem (3.3), by the roots of the reduced Evans function, which is defined in terms of the $2n$- and $2m$-dimensional, ε-independent, eigenvalue problems (3.6), (3.8) and (3.9). Therefore, this approximation result leads to a reduction in complexity, when verifying the first spectral stability condition in Corollary 3.8.

However, asymptotic spectral control through the reduced Evans function is insufficient to establish spectral stability, since the critical spectral curve attached to the origin shrinks to the origin as $\varepsilon \to 0$. Hence, we cannot determine whether the critical curve lies in the left half-plane and touches the origin in a quadratic tangency – see the second and third condition in Corollary 3.8. Yet, using the approximation results from §3.5.2, we can isolate the critical spectral curve from the rest of the spectrum. All in all, we obtain the following result.

Corollary 3.16. *Suppose the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_{s,0}(0,\gamma) \neq 0$ *for each* $\gamma \in S^1$*;*

iii. $\mathcal{E}_0(\lambda,\gamma) \neq 0$ *for each* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$.

Then, there exists $\sigma_0, \varepsilon_0 > 0$ *such that for each* $\varepsilon \in (0,\varepsilon_0)$ *there exists a* 2π*-periodic, analytic map* $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ *with the following properties:*

1. $\sigma(\mathcal{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} = \lambda_\varepsilon[\mathbb{R}]$*;*
2. $\lambda_\varepsilon(\nu) = \lambda_\varepsilon(-\nu)$ *is a simple zero of* $\mathcal{E}_\varepsilon(\cdot, e^{\pm i\nu})$ *for each* $\nu \in [0,\pi]$*;*
3. $\lambda_\varepsilon(0), \lambda_\varepsilon'(0), \lambda_\varepsilon'(\pi) = 0$*;*
4. $\lambda_\varepsilon(\nu)$ *converges to* 0 *as* $\varepsilon \to 0$ *for each* $\nu \in \mathbb{R}$.

In particular, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable if there exists* $\varsigma > 0$*, possibly dependent on* ε*, such that* $\lambda_\varepsilon(\nu) \leq -\varsigma\nu^2$ *holds for all* $\nu \in [0,\pi]$.

Proof. Since $\mathcal{E}_{s,0}(\cdot,\gamma)$ has no pole at $\lambda = 0$ by Remark 3.13, we deduce that 0 is a simple root of $\mathcal{E}_0(\cdot,\gamma)$ for each $\gamma \in S^1$. In addition, the set $\mathcal{N}_0$, defined in Theorem 3.14, is bounded by Propositions 3.10 and 3.11. So, there exists $\sigma_0 > 0$ such that, if $\mathcal{E}_0(\lambda,\gamma) = 0$ is satisfied for some $\gamma \in S^1$ and $\lambda \in C_\Lambda \setminus \{0\}$, then we have $\mathrm{Re}(\lambda) < -\sigma_0$. Let $\delta \in (0,\sigma_0)$. Theorem 3.14 yields $\varepsilon_\delta > 0$ such that for each $\varepsilon \in (0,\varepsilon_\delta)$ and $\nu \in \mathbb{R}$ there exists precisely one (simple) root $\lambda_\varepsilon(\nu)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ in $B(0,\delta)$. Thus, λ_ε defines a 2π-periodic function from $\mathbb{R}$ to $\mathbb{C}$ satisfying $\lambda_\varepsilon(\nu) \to 0$ as $\varepsilon \to 0$ for each $\nu \in \mathbb{R}$. Since $\mathcal{E}_\varepsilon$ is analytic in both of its arguments by Proposition 3.7 and the root $\lambda_\varepsilon(\nu)$ is simple, it follows by the implicit function theorem that $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{C}$ is analytic.

By Proposition 3.7 it holds

$$0 = \mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{i\nu})e^{-2(m+n)i\nu} = \mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{-i\nu}) = \mathcal{E}_\varepsilon(\overline{\lambda_\varepsilon(\nu)}, e^{i\nu}), \quad \nu \in \mathbb{R}.$$

Thus, by uniqueness of the root of $\mathcal{E}(\cdot, e^{i\nu})$ in $B(0,\delta)$, we conclude that $\lambda_\varepsilon(-\nu) = \lambda_\varepsilon(\nu) = \overline{\lambda_\varepsilon(\nu)}$. Hence, λ_ε must be real-valued and $\lambda_\varepsilon'(0), \lambda_\varepsilon'(\pi) = 0$. Recall from §3.3 that 0 is always a 1-eigenvalue of $\mathcal{L}_\varepsilon$ due to translational invariance, i.e. it holds $\mathcal{E}_\varepsilon(0,1) = 0$. We derive $\lambda_\varepsilon(0) = 0$. The fact that $\mathcal{E}_0(\cdot,\gamma)$ has no roots in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\}$ except 0 yields by Theorem 3.14 that $\mathcal{E}_\varepsilon(\cdot,\gamma)$ has no roots in $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} \setminus B(0,\delta)$ for each $\gamma \in S^1$. This proves $\sigma(\mathcal{L}_\varepsilon) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\sigma_0\} = \lambda_\varepsilon[\mathbb{R}]$. □

3.6 Expansion of the critical spectral curve

We present the second main outcome of our spectral stability analysis; that is, we provide an expansion of the critical spectral curve $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ established in Corollary 3.16.

Theorem 3.17. *Suppose the conditions in Corollary 3.16 are met. Then, provided $\varepsilon > 0$ is sufficiently small, the critical spectral curve $\lambda_\varepsilon \colon \mathbb{R} \to \mathbb{R}$, established in Corollary 3.16, is approximated as*

$$\left|\lambda_\varepsilon(\nu) - \varepsilon^2 \lambda_0(\nu)\right| \le C\varepsilon^3 |\log(\varepsilon)|^5, \tag{3.17}$$

where $C > 0$ is a constant independent of ε and ν and the analytic function $\lambda_0 \colon \mathbb{R} \to \mathbb{R}$ is defined by

$$\lambda_0(\nu) := \frac{\int_{-\infty}^{\infty} \langle \partial_u G(u_0, v_{\mathrm{h}}(x,u_0), 0)^* \psi_{\mathrm{ad},2}(x) x, B(\nu)\rangle\, dx}{\int_{-\infty}^{\infty} \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx}, \tag{3.18}$$

with $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ a non-trivial, exponentially localized solution to the adjoint,

$$\varphi_x = -\mathcal{A}_f(x)^* \varphi, \quad \varphi \in \mathbb{R}^{2n}. \tag{3.19}$$

of the fast variational equation (3.15) *and*

$$\begin{aligned} B(\nu) &:= D_1^{-1} \begin{pmatrix} 0 & I \end{pmatrix} \mathcal{B}(\nu), \\ \mathcal{B}(\nu) &:= \Upsilon_0^{-1} \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0) \end{pmatrix} - \left(I - e^{-i\nu}\Upsilon_0 \Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right)^{-1} \begin{pmatrix} 2D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix}, \\ \Upsilon_0 &:= \begin{pmatrix} I & 0 \\ \partial_u \mathcal{J}(u_0) & I \end{pmatrix}, \end{aligned} \tag{3.20}$$

where $\mathcal{J} \colon U_{\mathrm{h}} \to \mathbb{R}^m$ is defined in (2.5) *and $\Phi_{\mathrm{s}}(\check{x},\check{y})$ is the evolution operator of the slow variational equation* (2.7).

Remark 3.18. The integral $\int_{-\infty}^{\infty} \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx$ in the denominator of $\lambda_0(\nu)$ in Theorem 3.17 arises as a solvability condition for the generalized eigenvalue problem at $\lambda = 0$ associated with the linearization of $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_{\mathrm{h}}(x,u_0)$. Since 0 is a simple zero of the fast Evans function $\mathcal{E}_{f,0}$, this integral is non-zero – see also Proposition 5.21. ■

The critical spectral curve $\lambda_\varepsilon(\nu)$ arises as the solution curve to the equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ about $(\lambda, \nu) = (0, 0)$. The equation $\mathcal{E}_\varepsilon(\lambda, e^{i\nu}) = 0$ is defined in terms of the $2(m+n)$-dimensional full eigenvalue problem (3.3). The leading-order approximation $\lambda_0(\nu)$ of the solution curve $\lambda_\varepsilon(\nu)$, established in Theorem 3.17, is defined in terms of the ε-independent, $2m$-dimensional slow variational equation (2.7) and $2n$-dimensional fast variational equation (3.15). Therefore, Theorem 3.17 yields a reduction of complexity in the local analysis of the full eigenvalue problem (3.3) about $\lambda = 0$ simplifying the verification of the spectral stability conditions in Corollary 3.8. Combining this with Corollary 3.16 leads to a set of spectral stability conditions in terms of simpler, lower-dimensional systems, which we will present in §3.7.

When we have $\mathcal{E}_{s,0}(0, e^{i\nu_\diamond}) = 0$ for some $\nu_\diamond \in \mathbb{R}$, the approximation of $\lambda_\varepsilon(\nu_\diamond)$ in Theorem 3.17 fails. Since it holds $\det\left(I - e^{-i\nu}\Upsilon_0 \Phi_{\mathrm{s}}(2\ell_0, 0)\Upsilon_0\right) = \mathcal{E}_{s,0}(0, e^{-i\nu}) = e^{2im\nu}\mathcal{E}_{s,0}(0, e^{i\nu}) = 0$ by Proposition 3.11, we observe that λ_0 has a pole at $\nu_\diamond$. Yet, for ν away from $\nu_\diamond$, the approximation (3.18) is still valid. This leads to the following generalization of Theorem 3.17.

Theorem 3.19. *Suppose* 0 *is a simple zero of* $\mathcal{E}_{f,0}$*. Let* $\delta > 0$ *and denote*

$$\mathcal{N}_\diamond := \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta := \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta). \tag{3.21}$$

Then, for $\varepsilon > 0$ *sufficiently small, there exists for any* $\nu \in \mathcal{S}_\delta$ *a unique root* $\lambda_\varepsilon(\nu)$ *of* $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ *converging to* 0 *as* $\varepsilon \to 0$*. The root* $\lambda_\varepsilon(\nu)$ *is real-valued and satisfies* (3.17)*, where* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$ *is given by* (3.18) *and* $C > 0$ *is independent of* ε *and* ν*. In addition, the functions* $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ *and* λ_0 *are analytic, even and* 2π*-periodic. Finally, we have* $\lambda_\varepsilon(0) = 0$ *if* $0 \in \mathcal{S}_\delta$*.*

The proof of Theorem 3.19 is provided in §5.3. The proof of Theorem 3.17 follows by combining Corollary 3.16 with Theorem 3.19.

3.7 Explicit criteria for spectral stability and instability

Using the spectral approximation results in §3.5 and §3.6, we obtain explicit conditions in terms of simpler, lower-dimensional problems yielding spectral stability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9). Indeed, using Theorem 3.14 we can approximate the roots of the Evans function by the zeros of the reduced Evans function, which is defined in terms of the $2n$- and $2m$-dimensional eigenvalue problems (3.6), (3.8) and (3.9). This simplifies verifying the first spectral stability condition in Corollary 3.8. Then, using Corollary 3.16, we can isolate the most critical part of the spectrum: the curve $\lambda_\varepsilon(\nu)$ attached to the origin. Theorem 3.17

provides a leading-order approximation of $\lambda_\varepsilon(\nu)$ in terms of the $2m$- and $2n$-dimensional variational equations (2.7) and (3.15). This simplifies verifying the spectral stability conditions in Corollary 3.8 for λ close to the origin.

Thus, we readily obtain the following result by combining Theorems 3.14 and 3.17 and Corollary 3.16.

Corollary 3.20. *Suppose the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_{s,0}(0,\gamma) \neq 0$ *for each* $\gamma \in S^1$*;*

iii. $\mathcal{E}_0(\lambda,\gamma) \neq 0$ *for each* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

iv. $\lambda_0''(0) < 0$*,* $\lambda_0(\pi) < 0$ *and* $\lambda_0'(\nu) \neq 0$ *for each* $\nu \in (0,\pi)$*, where* $\lambda_0 \colon \mathbb{R} \to \mathbb{R}$ *is defined by* (3.17).

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable.*

Observe that, if the conditions in Corollary 3.20 are satisfied, then we obtain by Theorem 3.3 nonlinear diffusive stability of $\check{\phi}_{\mathrm{p},\varepsilon}$ as a solution to (1.9) with $\alpha = -\varepsilon^2\lambda_0''(0) + O(\varepsilon^3|\log(\varepsilon)|^5)$.

Regarding instability, Theorems 3.15 and 3.19 yield the following result.

Corollary 3.21. *If one of the following is true:*

i. There exists $\gamma_\diamond \in S^1$ *and* $\lambda_\diamond \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda_\diamond) > 0$ *satisfying* $\mathcal{E}_0(\lambda_\diamond,\gamma_\diamond) = 0$*;*

ii. It holds $\lambda_0(\nu) > 0$ *for some* $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$*, where* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$ *is given by* (3.17) *and* $\mathcal{N}_\diamond$ *is defined in* (3.21).

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally unstable.*

Thus, if one of the conditions in Corollary 3.21 is satisfied, then the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is nonlinearly unstable against localized and non-localized perturbations by Theorem 3.5.

We emphasize that the conditions in Corollaries 3.20 and 3.21 can be computed with only the singular limit (2.9) of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ as input. More specifically, one needs understanding of the (adjoint) variational equations about the solutions $\psi_{\mathrm{h}}(x,u_0)$ and $\psi_{\mathrm{s}}(\check{x})$ to systems (2.3) at $u = u_0$ and (2.4), respectively, and of the eigenvalue problems arising when linearizing equations $v_t = D_2 v_{xx} - G(u_0,v,0)$ and $u_t = D_1 u_{\check{x}\check{x}} - H_1(u,0,0)$ about the stationary solutions $v_{\mathrm{h}}(x,u_0)$ and $u_{\mathrm{s}}(\check{x})$, respectively.

In the case $n = 1$ or $m = 1$, the conditions for spectral stability and instability in Corollaries 3.20 and 3.21 can be further simplified – see §3.8. In the case $n = 1$, the first condition in Corollary 3.20 is always satisfied and the third condition only has to be checked for the *slow* Evans function. In the case $m = 1$, the second and fourth condition in Corollary 3.20 are satisfied precisely if the signs of three explicit (integral) expressions are equal.

Remark 3.22. As mentioned in §1.4.1, weak coupling $H_2(u, v) \equiv 0$ is allowed in our spectral analysis. In that case, the integral terms $\mathcal{J}(u)$ and $\mathcal{G}(u, \lambda)$ in (2.5) and (3.11) are identically 0, which implies that $\mathcal{E}_{s,0}$ is only determined by the slow eigenvalue problem (3.9). Therefore, $\mathcal{E}_{s,0}$ is analytic on C_Λ and zeros of $\mathcal{E}_{f,0}$ cannot be canceled by poles of $\mathcal{E}_{s,0}$. We conclude that the spectral stability problem fully splits into slow and fast subproblems with no interaction between them. As a consequence, zeros of $\mathcal{E}_{f,0}$ of positive real part yield spectral (and nonlinear) instability by Proposition 3.21. In particular, in the case $n = 1$, the fast Evans function $\mathcal{E}_{f,0}$ always has a zero in the right half-plane, as we will show in Proposition 3.24.

In addition, $\mathcal{E}_{s,0}(\cdot, 1)$ has a root if and only if the slow eigenvalue problem (3.9) admits a $2\ell_0$-periodic solution. Since we have $\mathcal{J}(u_0) = 0$, it holds $\psi_s(0) = (u_0, 0) = \psi_s(2\ell_0)$ by **(E2)**. Thus, the derivative $\psi_s'(\check{x})$ is a $2\ell_0$-periodic solution to (3.9) at $\lambda = 0$. Hence, the reduced Evans function $\mathcal{E}_0(\cdot, 1)$ has a double root at 0. In particular, in the case $m = 1$, Sturm-Liouville theory [7, Theorems 2.4.2 and 2.5.1] implies that there exists a $\lambda_* > 0$ such that (3.9) has a $2\ell_0$-periodic solution at $\lambda = \lambda_*$, because $u_s'(\check{x})$ vanishes at $\check{x} = \ell_0$. Consequently, $\mathcal{E}_{s,0}(\cdot, 1)$ has a zero in the right half-plane.

Consequently, if we have $H_2(u, v) \equiv 0$, then all periodic pulse solutions $\check{\phi}_{p,\varepsilon}$ to (1.9) are spectrally unstable in the case $n = 1$ or $m = 1$. This motivates the scaling in (1.8). ■

Remark 3.23. Suppose the conditions in Theorem 3.17 are met. Observe that the derivative ψ_s' of the solution ψ_s to (2.4) is a solution to the slow variational equation (2.7). By assumption **(E2)** $\psi_s(\check{x})$ intersects the touch-down manifold $\mathcal{T}_+$ at $\check{x} = 0$ in the point $(u_0, \mathcal{J}(u_0))$. Therefore, we have $\psi_s'(0) = (D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$ and by reversible symmetry it holds $\psi_s'(2\ell_0) = R_s\psi_s'(0) = (-D_1^{-1}\mathcal{J}(u_0), H_1(u_0, 0, 0))$. Thus, we deduce

$$\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0 \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} = \begin{pmatrix} -D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} + (\Upsilon_0\Phi_s(2\ell_0, 0) - I) \begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix},$$

where $\mathfrak{a} := \partial_u\mathcal{J}(u_0)D_1^{-1}\mathcal{J}(u_0) - H_1(u_0, 0, 0) \in \mathbb{R}^m$. Rewriting the latter equation gives

$$\begin{aligned}
&\left(1 + e^{-i\nu}\right)\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0\right)^{-1} \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix} \\
&\qquad = e^{-i\nu}\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0, 0)\Upsilon_0\right)^{-1} (\Upsilon_0\Phi_s(2\ell_0, 0) - I) \begin{pmatrix} 0 \\ \mathfrak{a} \end{pmatrix} + \begin{pmatrix} D_1^{-1}\mathcal{J}(u_0) \\ 0 \end{pmatrix},
\end{aligned}$$

for $\nu \in [0, \pi]$. Hence, we obtain the following expression for the quantity $B(\nu)$ in Theorem 3.17

$$B(\nu) = D_1^{-1} \left[\frac{\begin{pmatrix} 0 & I \end{pmatrix}}{1 + e^{-i\nu}} \left(I - e^{-i\nu} \Upsilon_0 \Phi_s(2\ell_0, 0) \Upsilon_0 \right)^{-1} (I - \Upsilon_0 \Phi_s(2\ell_0, 0)) \begin{pmatrix} 0 \\ 2I \end{pmatrix} - I \right] \mathfrak{a},$$

for $\nu \in [0, \pi)$. So, $\mathfrak{a} = 0$ implies $\lambda_0(\nu) = 0$ for any $\nu \in [0, \pi)$ by Theorem 3.17. Therefore, $\mathfrak{a}$ passing through zero, suggests a transition of the critical spectral curve through the imaginary axis. This coincides with a loss of transversality: condition (2.8) in assumption **(E2)** fails if $\mathfrak{a} = 0$ – see §2.2.2. For the case $m = n = 1$, we show in §6.3 that the periodic pulse solution $\check{\phi}_{p,\varepsilon}$ destabilizes through a spatial period doubling bifurcation or sideband instability as $\mathfrak{a}$ passes through zero. ■

3.8 Stability results in lower dimensions

In §3.7 we established explicit conditions yielding spectral stability and instability in terms of the eigenvalue problems (3.6), (3.8) and (3.9) and the variational equations (2.7) and (3.15). In this section we interpret these results in the case $n = 1$ or $m = 1$. Then, the aforementioned systems become 2-dimensional and we can employ techniques tailored for 2-dimensional linear systems to further simplify the spectral (in)stability conditions in Corollaries 3.20 and 3.21.

We proceed as follows. First, we study the slow and fast Evans function and the (leading-order) critical spectral curve $\lambda_0(\nu)$ in the lower-dimensional setting. Subsequently, we interpret the spectral stability conditions in Corollary 3.20 in the case $n = 1$ or $m = 1$. Finally, we present an instability test using parity-type arguments in the regime $n = m = 1$.

Throughout this section we assume without loss of generality $D_1 = 1$ in the case $m = 1$ and $D_2 = 1$ in the case $n = 1$ – see Remark 1.5.

3.8.1 The reduced Evans function

In the case $n = 1$, the homogeneous fast eigenvalue problem (3.6) becomes 2-dimensional. The ordering of the eigenvalues of (3.6), i.e. the roots of the fast Evans function, can be understood with Sturm-Liouville theory. Thus, we obtain the following result.

Proposition 3.24. *Suppose $n = 1$. All zeros of the fast Evans function $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ are real and simple. Moreover, there is precisely one positive zero λ_* of $\mathcal{E}_{f,0}$. Finally, 0 is a root of $\mathcal{E}_{f,0}$.*

Proof. By [60, Theorem 2.3.3] all eigenvalues of the operator $\mathcal{L}_f$, defined in (3.7), are real and simple. In addition, the eigenvalues can be enumerated in strictly decreasing order as $\lambda_N < \ldots < \lambda_0$. The eigenfunction corresponding to $\lambda_i, i = 0, \ldots, N$ has precisely i zeros. Hence, all zeros of $\mathcal{E}_{f,0}$ are real and simple by Proposition 3.10. Furthermore, the derivative

$\partial_x v_h(x, u_0)$ lies in the kernel of $\mathcal{L}_f$. The function $\partial_x v_h(x, u_0)$ has precisely one zero by **(E1)**. So, we derive $\lambda_1 = 0$ and $\lambda_0 > 0$. □

In the case $m = 1$, the slow eigenvalue problem (3.9) becomes 2-dimensional. Since the solution $\psi_s(\check{x})$ to (2.4) crosses the reversible symmetry line $\ker(I - R_s)$ at $\check{x} = \ell_0$ by assumption **(E2)**, it holds $\psi_s(\check{x}) = \psi_s(2\ell_0 - \check{x})$ for each $\check{x} \in [0, 2\ell_0]$. Thus, system (3.9) is R_s-reversible at $\check{x} = \ell_0$, i.e. if $\varphi(\check{x}, \lambda)$ is a solution to (3.9), then so is $\check{x} \mapsto R_s\varphi(2\ell_0 - \check{x}, \lambda)$. Hence, there exists non-trivial solutions $u_+(\check{x}, \lambda)$ and $u_-(\check{x}, \lambda)$ to

$$u_{\check{x}\check{x}} = \left(\frac{\partial H_1}{\partial u}(u_s(\check{x}), 0, 0) + \lambda\right)u, \quad u \in \mathbb{R}, \tag{3.22}$$

which are symmetric and antisymmetric about ℓ_0, respectively. In particular, at $\lambda = 0$ the derivative $u_s'(\check{x})$ is an antisymmetric solution about ℓ_0 to (3.22). A symmetric solution to (3.22) at $\lambda = 0$ can now be found using Rofe-Beketov's formula [7, Chapter 1.9]. This leads to the following result.

Proposition 3.25. *Suppose $m = 1$. Let $u_+(\check{x}, \lambda)$ and $u_-(\check{x}, \lambda)$ be solutions to* (3.22)*, which are symmetric and antisymmetric about ℓ_0, respectively, and have Wronskian 1. The slow Evans function $\mathcal{E}_{s,0} \colon [C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)] \times \mathbb{C} \to \mathbb{C}$ is given by*

$$\mathcal{E}_{s,0}(\lambda, \gamma) = \gamma^2 - \mathfrak{t}(\lambda)\gamma + 1,$$

where $\mathfrak{t} \colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$ is the analytic function given by

$$\begin{aligned} \mathfrak{t}(\lambda) &:= \mathrm{Tr}(\Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda)) \\ &= 2\left[\frac{d}{d\check{x}}[u_+(\check{x}, \lambda)u_-(\check{x}, \lambda)](0) - \mathcal{G}(u_0, \lambda)u_+(0, \lambda)u_-(0, \lambda)\right], \end{aligned} \tag{3.23}$$

with $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$, $\Upsilon(u, \lambda)$ and $\mathcal{G}(u, \lambda)$ defined in §3.5.1. In particular, we find

$$\mathfrak{t}(0) = -2\,(1 + 2\mathfrak{a}\mathfrak{b}),$$

where

$$\begin{aligned} \mathfrak{a} &:= \mathcal{J}'(u_0)\mathcal{J}(u_0) - H_1(u_0, 0, 0), \\ \mathfrak{b} &:= \mathcal{J}(u_0)\int_0^{\ell_0} \frac{(\partial_u H_1(u_s(\check{x}), 0, 0) + 1)[(u_s'(\check{x}))^2 - (H_1(u_s(\check{x}), 0, 0))^2]}{[(u_s'(\check{x}))^2 + (H_1(u_s(\check{x}), 0, 0))^2]^2} d\check{x} \\ &\quad + \frac{H_1(u_0, 0, 0)}{(\mathcal{J}(u_0))^2 + (H_1(u_0, 0, 0))^2}. \end{aligned} \tag{3.24}$$

Finally, $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ is a γ-eigenvalue for some $\gamma \in S^1$ if and only if it holds $\mathfrak{t}(\lambda) \in [-2, 2]$. In that case, we have $2\mathrm{Re}(\gamma) = \mathfrak{t}(\lambda)$.

Proof. The formula (3.23) follows readily by expanding $\mathcal{E}_{s,0}(\lambda,\gamma)$ as a quadratic polynomial in γ and expressing $\mathcal{T}_s(2\ell_0,0,\lambda)$ in terms of symmetric and antisymmetric solutions.

Calculating $\mathrm{t}(0)$ is more elaborate. First, note that the derivative $u_s'(x)$ is a solution to (3.22) at $\lambda = 0$, which is antisymmetric about ℓ_0. By Rofe-Beketov's formula [7, Chapter 1.9] a symmetric solution about ℓ_0 to (3.22) at $\lambda = 0$ is given by

$$\begin{aligned} z(\check{x}) := u_s'(\check{x}) \int_{\check{x}}^{\ell_0} & \frac{(\partial_u H_1(u_s(\check{y}),0,0)+1)[(u_s'(\check{y}))^2 - (H_1(u_s(\check{y}),0,0))^2]}{[(u_s'(\check{y}))^2 + (H_1(u_s(\check{y}),0,0))^2]^2} d\check{y} \\ &+ \frac{H_1(u_s(\check{x}),0,0)}{(u_s'(\check{x}))^2 + (H_1(u_s(\check{x}),0,0))^2}. \end{aligned} \tag{3.25}$$

Note that the Wronskian of z and u_s' has value 1. Second, the matrix function $(\partial_u \psi_h(x,u_0) \mid 0)$ is a solution to the fast inhomogeneous problem (3.8) at $\lambda = 0$ and $u = u_0$. This implies $2\mathcal{J}'(u_0) = \mathcal{G}(u_0,0)$. Putting these two items into (3.23), yields $\mathrm{t}(0) = -2\,(1 + 2\mathfrak{ab})$. □

In §3.8.3 we present an instability test using parity-type arguments. Therefore, we are interested in the asymptotic behavior of the trace map $\mathrm{t}(\lambda)$.

Lemma 3.26. *Let $m = 1$. Consider the map $\mathrm{t}\colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$, defined in (3.23). We have $\lim_{\lambda\to\infty} \mathrm{t}(\lambda) = \infty$.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ. Consider system,

$$\begin{aligned} u_{\check{x}} &= \sqrt{\lambda} p, \\ p_{\check{x}} &= \left(\frac{1}{\sqrt{\lambda}} \frac{\partial H_1}{\partial u}(u_s(\check{x}),0,0) + \sqrt{\lambda}\right) u, \end{aligned} \qquad (u,p) \in \mathbb{C}^2, \tag{3.26}$$

with evolution $\mathcal{T}_{s1}(\check{x},\check{y},\lambda)$. Denote by $\mathcal{T}_{s2}(\check{x},\check{y},\lambda)$ the evolution operator of the autonomous system,

$$\begin{aligned} u_{\check{x}} &= \sqrt{\lambda} p, \\ p_{\check{x}} &= \sqrt{\lambda} u, \end{aligned} \qquad (u,p) \in \mathbb{C}^2. \tag{3.27}$$

Proposition 4.1 yields

$$\|\mathcal{T}_{s1}(2\ell_0,0,\lambda) - \mathcal{T}_{s2}(2\ell_0,0,\lambda)\| \le \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0}, \quad \lambda > 0. \tag{3.28}$$

On the other hand, the slow eigenvalue problem (3.9) is equivalent to system (3.26) upon performing a coordinate change. Indeed, it holds

$$C_\lambda \mathcal{T}_{s1}(2\ell_0,0,\lambda) C_\lambda^{-1} = \mathcal{T}_s(2\ell_0,0,\lambda), \quad C_\lambda := \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\lambda} \end{pmatrix}, \quad \lambda > 0. \tag{3.29}$$

We refer to Proposition 5.5 for the fact that, for $\lambda > 0$ sufficiently large, the solution $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8) at $u = u_0$ is exponentially localized with λ-independent decay rates. Hence, $\mathcal{G}(u_0, \lambda)$ remains bounded as $\lambda \to \infty$. Thus, $\mathfrak{t}(\lambda)$ is for $\lambda > 0$ sufficiently large approximated as

$$\left\| \mathfrak{t}(\lambda) - \mathrm{tr}\left(\begin{pmatrix} 1 & 0 \\ \frac{\mathcal{G}(u_0,\lambda)}{\sqrt{\lambda}} & 1 \end{pmatrix} \mathcal{T}_{s2}(2\ell_0, 0, \lambda) \right) \right\| \leq \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0},$$

by (3.28) and (3.29). The latter yields

$$\left\| \mathfrak{t}(\lambda) - e^{2\sqrt{\lambda}\ell_0} \right\| \leq \frac{C}{\sqrt{\lambda}} e^{2\sqrt{\lambda}\ell_0},$$

for $\lambda > 0$ sufficiently large, where we use explicit expressions for the evolution $\mathcal{T}_{s2}(\check{x}, \check{y}, \lambda)$ of system (3.27). We conclude $\mathfrak{t}(\lambda) \to \infty$ as $\lambda \to \infty$. □

Example 3.27. In [114] the spectral stability of spatially periodic pulse patterns is studied, where (1.10) is the generalized Gierer-Meinhardt equation (2.26). Thus, the slow eigenvalue problem (3.9) corresponds to the autonomous system $u_{\check{x}\check{x}} = (\mu + \lambda)u$. The condition $\mathfrak{t}(\lambda) \in [-2, 2]$ in Proposition 3.25 simplifies in that case to

$$2\cosh(2\ell_0\sqrt{\mu + \lambda}) + \frac{\mathcal{G}(u_0, \lambda)\sinh(2\ell_0\sqrt{\mu + \lambda})}{\sqrt{\mu + \lambda}} \in [-2, 2],$$

where $\mathcal{G}(u, \lambda)$ is defined in (3.11) and $\sqrt{\cdot}$ denotes the principal square root. Although derived with a different method, this result agrees with [114, Theorem 1.1.I]. ■

As mentioned in §3.5.1, it is possible to obtain explicit expressions of the principal part of the Laurent series of $\mathcal{E}_{s,0}(\cdot, \gamma)$ at a zero $\lambda \in \mathcal{E}_{f,0}^{-1}(0)$. Because of their complexity the expansions are treated separately in §5.1.2. However, in the case $m = 1$, the expressions simplify significantly. Therefore, it is worthwhile to devote a separate proposition to this case.

Proposition 3.28. *Suppose $m = 1$. Let $\lambda_\diamond$ be a simple zero of $\mathcal{E}_{f,0}$. The singular part of the Laurent expansion at $\lambda = \lambda_\diamond$ of the map $\mathfrak{t}\colon C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0) \to \mathbb{C}$, defined in Proposition 3.25, is given by*

$$\frac{u(2\ell_0, \lambda_\diamond)}{\lambda - \lambda_\diamond} \int_{-\infty}^{\infty} \partial_v H_2(u_0, v_\mathrm{h}(x, u_0)) v_{\lambda_\diamond}(x) dx \int_{-\infty}^{\infty} \tilde{v}_{\lambda_\diamond}(x)^* \frac{\partial G}{\partial u}(u_0, v_\mathrm{h}(x, u_0), 0) dx,$$

where $u(\check{x}, \lambda_\diamond)$ is the solution to (3.22) *at $\lambda = \lambda_\diamond$ having initial values $u(0) = 0, u'(0) = 1$. Moreover, $v_{\lambda_\diamond}$ is an exponentially localized solution to*

$$D_2 v_{xx} = (\partial_v G(u_0, v_\mathrm{h}(x, u_0), 0) + \lambda_\diamond) v, \quad v \in \mathbb{C}^n, \tag{3.30}$$

and $\tilde{v}_{\lambda_\diamond}(x)$ is an exponentially localized solution to the adjoint problem,

$$D_2 v_{xx} = \left(\partial_v G(u_0, v_\mathrm{h}(x, u_0), 0)^* + \overline{\lambda_\diamond}\right) v, \quad v \in \mathbb{C}^n,$$

such that

$$\int_{-\infty}^{\infty} \tilde{v}_{\lambda,\diamond}(x)^* v_{\lambda,\diamond}(x)dx = 1.$$

Proof. The statement is proven in a more general setting in Proposition 5.9. □

3.8.2 The critical spectral curve

In the case $m = 1$, the leading-order approximation (3.18) of the critical spectral curve simplifies. Indeed, the slow variational equation (2.7) becomes 2-dimensional. So, besides the derivative $\psi_s'(\check{x})$, a second, linearly independent solution to (2.7) can be found using Rofe-Beketov's formula. This leads to the following result.

Proposition 3.29. *Let* $m = 1$. *Suppose that* 0 *is a simple zero of* $\mathcal{E}_{f,0}$. *Then, the analytic map* $\lambda_0 \colon \mathbb{R} \setminus \mathcal{N}_\diamond \to \mathbb{R}$, *defined in Theorem 3.19, is given by*

$$\lambda_0(\nu) = \mathfrak{a}\mathfrak{w}\frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}\mathfrak{b}}, \tag{3.31}$$

where $\mathfrak{a}, \mathfrak{b}$ *are defined in* (3.24) *and* $\mathfrak{w}$ *is given by*

$$\mathfrak{w} := -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_h(x,u_0), 0)^* \psi_{ad,2}(x) x dx}{\int_{-\infty}^{\infty} \psi_{ad,2}(x)^* \partial_x v_h(x,u_0) dx}, \tag{3.32}$$

with $\psi_{ad}(x) = (\psi_{ad,1}(x), \psi_{ad,2}(x))$ *a non-trivial, exponentially localized solution to* (3.19). *In the case* $n = 1$, *the expression for* $\mathfrak{w}$ *simplifies to*

$$\mathfrak{w} = -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_h(x,u_0), 0) \partial_x v_h(x,u_0) x dx}{\int_{-\infty}^{\infty} (\partial_x v_h(x,u_0))^2 \, dx}. \tag{3.33}$$

Proof. As in the proof of Proposition 3.25 we observe that, at $\lambda = 0$, the derivative $u_s'(\check{x})$ is a solution to (3.22), which is antisymmetric about ℓ_0, and $z(\check{x})$, given by (3.25), is a solution to (3.22), which is symmetric about ℓ_0. In addition, the Wronskian of $z(\check{x})$ and $u_s'(\check{x})$ equals 1. Expressing the evolution $\Phi_s(2\ell_0, 0)$ of (2.7) in terms of $\psi_s'(0)$ and $(z(0), z'(0))$, simplifies the expression for $B(\nu)$ in (3.20) to

$$\begin{aligned} B(\nu) &= -\left[\mathfrak{a} - \begin{pmatrix} 0 & 1 \end{pmatrix}\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0,0)\Upsilon_0\right)^{-1}\begin{pmatrix} 2\mathcal{J}(u_0) \\ 0 \end{pmatrix}\right], \\ &= -\left[\mathfrak{a} - \frac{4\mathfrak{a}(1+\mathfrak{a}\mathfrak{b})e^{-i\nu}}{\mathcal{E}_{s,0}(0,e^{-i\nu})}\right] \end{aligned}$$

where we use $\mathfrak{b} = z(0)$, $\det\left(I - e^{-i\nu}\Upsilon_0\Phi_s(2\ell_0,0)\Upsilon_0\right) = e^{2i\nu}\mathcal{E}_{s,0}(0,e^{i\nu}) \neq 0$ and $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ by **(E2)**. By Proposition 3.25 it holds $e^{i\nu}\mathcal{E}_{s,0}(0,e^{-i\nu}) = 2(\cos(\nu) + 1 + 2\mathfrak{a}\mathfrak{b})$.

Substituting this into the above expression for $B(\nu)$ leads to the desired formula (3.31) for $\lambda_0(\nu) = -\mathfrak{w}B(\nu)$. Finally, in the case $n = 1$, we observe that $(-\partial_x q_h(x, u_0), \partial_x v_h(x, u_0))$ is a solution to equation (3.19) yielding (3.33). This concludes the proof. □

Remark 3.30. Let $m = 1$. Proposition 3.29 indicates that the geometry of the critical spectral curve attached to the origin is to leading order determined by the expressions $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$. In addition, the value of the slow Evans function $\mathcal{E}_{s,0}(\lambda, \gamma)$ at $\lambda = 0$ is fixed by $\mathfrak{a}$ and $\mathfrak{b}$ using Proposition 3.25. Thus, $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ determine the spectral configuration about the origin and play an important role in destabilization processes – see §6.3. We elaborate on the geometric interpretation of these quantities.

As mentioned in §2.2.2, the quantity $\mathfrak{a}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution ψ_s to (2.4) at $\psi_s(0) = (u_0, \mathcal{J}(u_0))$ and, by symmetry, between the take-off curve $\mathcal{T}_-$ and ψ_s at $\psi_s(2\ell_0) = R_s\psi_s(0)$ – see Figure 3.2. If $\mathfrak{a} = 0$, then $\psi_s(\check{x})$ is tangent to the touch-down curve at $\check{x} = 0$.

The quantity $\mathfrak{b}$ depends on the dynamics in the slow reduced system (2.4) only. Since $\psi_s(\ell_0)$ is contained in $\ker(I - R_s)$ by assumption **(E2)**, the vector $\psi_\diamond = (H_1(u_s(\ell_0), 0, 0)^{-1}, 0)$ is a normal to the tangent space of the curve $\psi_s(\check{x})$ at $\check{x} = \ell_0$ such that $\det(\psi_\diamond \mid \psi_s'(\ell_0)) = 1$. Tracking the tangent space along the flow of (2.4) to $\check{x} = \check{x}_0$, the vector $\psi_\diamond$ becomes $\Phi_s(\check{x}_0, \ell_0)\psi_\diamond$. Since system (2.7) is R_s-reversible at $\check{x} = \ell_0$, the first component $z(\check{x})$ of the solution $\Phi_s(\check{x}, \ell_0)\psi_\diamond$ to (2.7) is symmetric at $\check{x} = \ell_0$. Hence, $z(0)$ equals the quantity $\mathfrak{b}$.

Observe that $z(\check{x})$ has precisely one root between two consecutive zeros of $u_s'(\check{x})$, since the derivative of $u_s'(\check{x})/z(\check{x})$ never vanishes between these two zeros of u_s'. Therefore, given that the orbit of ψ_s in the slow reduced system (2.4) crosses the line $p = 0$ at $u = u_\pm$ with $u_- < u_+$, there is precisely one initial value $u_0 = u_s(0) \in (u_-, u_+)$ for which $\mathfrak{b} = 0$ – see Figure 3.2.

The quantity $\mathfrak{w}$ occurs in [92], where one derives asymptotic interaction laws for quasi-stationary pulse solutions to models of the form (1.9). More precisely, one establishes in [92] an ODE, which describes the (leading-order) evolution of the pulse locations over time, assuming existence and smoothness of the quasi-stationary pulse pattern. The pulse locations of our *stationary*, periodic pulse $\check{\phi}_{p,\varepsilon}(\check{x})$ to (1.9) correspond naturally to an equilibrium of this ODE. The quantity $\mathfrak{w}$ occurs as a factor in the linearization of the ODE about this equilibrium – see [92, Section 6.2.1]. Thus, the sign of $\mathfrak{w}$ corresponds to the character of the equilibrium. Loosely speaking, $\mathfrak{w}$ measures the stability of $\check{\phi}_{p,\varepsilon}(\check{x})$ against perturbations of the pulse locations. This relates to the fact that vanishing of $\mathfrak{w}$ corresponds to a transition of the critical spectral curve through the imaginary axis – see §6.3. ■

3.8.3 Criteria for spectral stability and instability

The results in §3.8.1 and §3.8.2 lead to the following simplification of the spectral stability conditions in Corollary 3.20 in the lower-dimensional setting.

Corollary 3.31. *Suppose $m = 1$ and the following conditions are met:*

i. 0 *is a simple zero of* $\mathcal{E}_{f,0}$*;*

ii. $\mathcal{E}_0(\lambda, \gamma) \neq 0$ *for all* $\gamma \in S^1$ *and* $\lambda \in \mathbb{C} \setminus \{0\}$ *with* $\mathrm{Re}(\lambda) \geq 0$*;*

iii. The quantities $\mathfrak{a}, \mathfrak{b}$ *and* $\mathfrak{w}$*, defined in* (3.24) *and* (3.32)*, have the same (non-zero) sign.*

Then, provided $\varepsilon > 0$ *is sufficiently small, the periodic pulse solution* $\check{\phi}_{\mathrm{p},\varepsilon}$ *to* (1.9) *is spectrally stable.*

Moreover, in the case $n = 1$*, conditions i. and ii. above are satisfied if and only* $\mathfrak{c}_+ = 0$*, with*

$$\begin{aligned} \mathfrak{c}_\pm &:= \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{1}{2\pi i} \oint_{\Gamma_R^\pm} \frac{\partial_\lambda \mathcal{E}_{s,0}(\lambda, e^{i\nu})}{\mathcal{E}_{s,0}(\lambda, e^{i\nu})} d\lambda + 1 \right| d\nu \\ &= \lim_{R\to\infty} \int_0^{2\pi} \left| \frac{1}{2\pi i} \oint_{\Gamma_R^\pm} \frac{\mathfrak{t}'(\lambda)}{\mathfrak{t}(\lambda) + 2\cos(\nu)} d\lambda + 1 \right| d\nu, \end{aligned} \tag{3.34}$$

where $\Gamma_R^\pm$ *is the (counter-clockwise) contour in the complex plane consisting of the circle segment* $\{z \in \mathbb{C} : |z \pm R^{-1}| = R, \mathrm{Re}(z) \geq \mp R^{-1}\}$ *and the line joining the points* $iR \mp R^{-1}$ *and* $-iR \mp R^{-1}$*.*

Proof. Since we have $\mathfrak{a}\mathfrak{b} > 0$, it holds $\mathcal{E}_{s,0}(0, \gamma) \neq 0$ for each $\gamma \in S^1$ by Proposition 3.25. Thus, the first three conditions in Corollary 3.20 are satisfied. Moreover, by Proposition 3.29 we have

$$\lambda_0(\pi) = -\frac{\mathfrak{w}}{\mathfrak{b}}, \qquad \lambda_0''(0) = -\frac{\mathfrak{a}\mathfrak{w}}{2 + 2\mathfrak{a}\mathfrak{b}}, \qquad \lambda_0'(\nu) = -\frac{2\mathfrak{a}(1 + \mathfrak{a}\mathfrak{b})\mathfrak{w}\sin(\nu)}{(1 + 2\mathfrak{a}\mathfrak{b} + \cos(\nu))^2}, \tag{3.35}$$

with $\nu \in \mathbb{R}$. Since $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ are non-zero and have the same sign, the fourth condition in Corollary 3.20 is also satisfied. We conclude that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable.

In the case $n = 1$, 0 is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero λ_* of positive real part by Proposition 3.24. Thus, by Proposition 3.11 the conditions i. and ii. are satisfied if and only if $\mathcal{E}_{s,0}(\lambda, \gamma)$ has precisely one pole of order 1 at $\lambda = \lambda_*$ and no zeros in the closed right half-plane for each $\gamma \in S^1$. Using the argument principle and Proposition 3.25 the latter is the case if and only if $\mathfrak{c}_+ = 0$. □

Thus, in the case $m = n = 1$, we can establish spectral stability by evaluating four expressions $\mathfrak{a}, \mathfrak{b}, \mathfrak{w}$ and $\mathfrak{c}_+$. Even if these four expressions cannot be determined exactly, one can prove spectral stability using rigorously verified computing. To estimate the errors one needs explicit bounds on the solutions $\psi_\mathrm{h}(x, u)$ and $\psi_\mathrm{s}(\check{x})$ to (2.3) at $u = u_0$ and (2.4) that constitute the singular limit (2.9) and on the functions H_1, H_2, G.

On the other hand, the lower-dimensional setting allows us to test for *instability* using parity-type arguments.

Corollary 3.32. *Let $m = n = 1$. If one of the following is true:*

i. *We have $\mathfrak{c}_- \neq 0$, where $\mathfrak{c}_-$ is defined in* (3.34)*;*

ii. *The quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$, defined in* (3.24) *and* (3.32)*, are non-zero and have different signs;*

iii. *We have $\mathcal{J}(u_0) = 0$;*

iv. *It holds $\mathfrak{i} \leq 0$ with*

$$\mathfrak{i} := u(2\ell_0, \lambda_*) \int_{-\infty}^{\infty} \frac{\partial H_2}{\partial v}(u_0, v_{\mathrm{h}}(x, u_0)) v_{\lambda_*}(x) dx \int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_0, v_{\mathrm{h}}(x, u_0), 0) v_{\lambda_*}(x) dx, \tag{3.36}$$

where $\lambda_* > 0$ is as in Proposition 3.24, $u(\check{x}, \lambda)$ is the solution to (3.22) *with initial values $u(0) = 0, u'(0) = 1$ and v_{λ_*} is a normalized, exponentially localized solution (having L^2-norm 1) to* (3.30) *at $\lambda_\diamond = \lambda_*$.*

Then, provided $\varepsilon > 0$ is sufficiently small, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) *is spectrally unstable.*

Proof. First, 0 is a simple zero of $\mathcal{E}_{f,0}$ and $\mathcal{E}_{f,0}$ has only one (simple) zero λ_* of positive real part by Proposition 3.24. Thus, if $\mathfrak{c}_- \neq 0$, then there exists by the argument principle a $\gamma \in S^1$, such that either $\mathcal{E}_{s,0}(\cdot, \gamma)$ has no pole at λ_* or it has a zero $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) > 0$. Thus, it holds either $\mathcal{E}_0(\lambda_*, \gamma) = 0$ or $\mathcal{E}_0(\lambda_0, \gamma) = 0$, which implies by Corollary 3.21 that $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable.

Next, suppose the non-zero quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ have different signs and $1 + \mathfrak{a}\mathfrak{b} > 0$. Then, the calculations (3.35) show that there exists $\nu \in \mathbb{R}$ such that $\lambda_0(\nu) > 0$. By Corollary 3.21 $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable.

Now suppose $1 + \mathfrak{a}\mathfrak{b} \leq 0$ and $\mathfrak{i} > 0$. Then, we have $\mathfrak{t}(0) = -2(1 + 2\mathfrak{a}\mathfrak{b}) \geq 2$ by Proposition 3.25. On the other hand, the quantity $\mathfrak{i}$ corresponds to the singular part of the Laurent expansion of $\mathfrak{t}(\lambda)$ at $\lambda = \lambda_*$ by Proposition 3.28. Thus, if we have $\mathfrak{i} > 0$, there exists by the intermediate value theorem a $\lambda_0 \in (0, \lambda_*)$ such that $\mathfrak{t}(\lambda_0) = 2$. Hence, Proposition 3.25 yields $\mathcal{E}_0(\lambda_0, 1) = 0$. Therefore, $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable by Corollary 3.21.

Suppose $\mathfrak{i} \leq 0$. In the case $\mathfrak{i} = 0$ we have $\mathcal{E}_0(\lambda_*, \gamma) = 0$ for any $\gamma \in S^1$ by Proposition 3.28. On the other hand, it is shown in Lemma 3.26 that $\mathfrak{t}(\lambda)$ tends to infinity as $\lambda \to \infty$. Therefore, the intermediate value theorem implies that, if $\mathfrak{i} < 0$, then there exists $\lambda_0 \in (\lambda_*, \infty)$ such that $\mathfrak{t}(\lambda_0) = 2$. Hence, by Propositions 3.25 and Corollary 3.21 $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable if $\mathfrak{i} \leq 0$.

Finally, in the case $\mathcal{J}(u_0) = 0$, it follows $\mathfrak{a}\mathfrak{b} = -1$ by a direct calculation. Hence, $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally unstable by the analysis in the previous three paragraphs. □

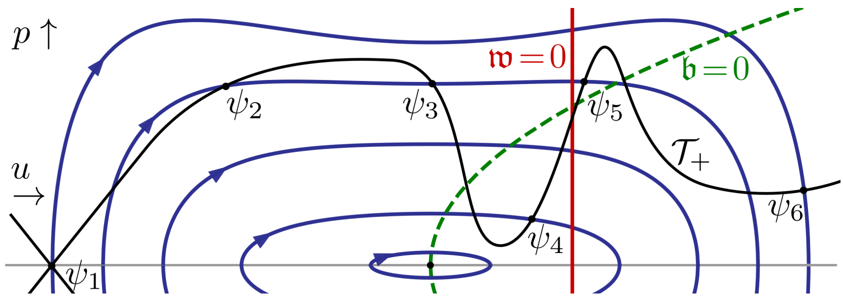


Figure 3.2: Depicted are five orbits of the slow reduced system (2.4) (in purple). The touch-down curve $\mathcal{T}_+$ intersects these orbits transversally at ψ_i, $i = 1, \dots, 6$. The green dashed line corresponds to the initial values such that $\mathfrak{b} = 0$. We have $\mathfrak{b} > 0$ at ψ_1, ψ_2, ψ_3 and ψ_5 and $\mathfrak{b} < 0$ at ψ_4 and ψ_6. The red line corresponds to initial values with $\mathfrak{w} = 0$. We have $\mathfrak{w} > 0$ at ψ_1, ψ_2, ψ_3 and ψ_4 and $\mathfrak{w} < 0$ at ψ_5 and ψ_6. Finally, we have $\mathfrak{a} < 0$ at ψ_1, ψ_3 and ψ_6 and $\mathfrak{a} > 0$ at ψ_2, ψ_4 and ψ_5. The periodic pulse solutions touching-down at ψ_1, ψ_3, ψ_4 and ψ_5 are spectrally unstable by Corollary 3.32. The solutions touching down at ψ_2 and ψ_6 are potentially spectrally stable.

We stress that the value of $\mathfrak{a}, \mathfrak{b}$, $\mathcal{J}(u_0)$ and $\mathfrak{w}$ depends only on the initial value $u_0 = u_s(0)$ of the solution ψ_s to the slow reduced system (2.4) and can directly be read off from the phase plane of (2.4) – see Figure 3.2.

Remark 3.33. If the periodic pulse $\check{\phi}_{p,\varepsilon}$ approaches a homoclinic limit, then the verification of the conditions in Corollaries 3.31 and 3.32 simplifies significantly – see §6.4.6. In particular, we can test for spectral (in)stability by approximating the quantities $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}_-, \mathfrak{i}$ and $\mathfrak{w}$ in the long-wavelength limit. ■

Example 3.34. In [114] the spectral stability of stationary, spatially periodic pulse solutions is studied in the generalized Gierer-Meinhardt equation (2.26). The slow variational equation (2.7) corresponds in this setting to the autonomous equation $u_{\check{x}\check{x}} = \mu u$. The v-component of the homoclinic solution $\psi_h(x, u_0)$ to system (2.3) at $u = u_0$ is given by

$$v_h(x, u_0) = u_0^{-\frac{\alpha_2}{\beta_2 - 1}} w_h(x), \quad w_h(x) := \left(\frac{\beta_2 + 1}{2} \operatorname{sech}^2\left(\frac{(\beta_2 - 1)\, x}{2}\right)\right)^{\frac{1}{\beta_2 - 1}}.$$

Thus, using integration by parts, we calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ in Proposition 3.29,

$$\mathfrak{a} = \mathcal{J}(u_0)\mathcal{J}'(u_0) - \mu u_0, \quad \mathfrak{b} = \frac{\cosh^2\left(\ell_0 \sqrt{\mu}\right)}{4\mu u_0}, \quad \mathfrak{w} = -\frac{\alpha_2 \int_{-\infty}^{\infty} w_h(x)^{\beta_2 + 1} dx}{u_0\,(\beta_2 + 1) \int_{-\infty}^{\infty} \left(w_h'(x)\right)^2 dx},$$

where $\mathcal{J}\colon (0,\infty) \to \mathbb{R}$ is given by

$$\mathcal{J}(u) = \frac{u^{\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2 - 1}}}{2} \int_{-\infty}^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx. \tag{3.37}$$

It holds $\mathfrak{bw} > 0$, since we have $\beta_{1,2} > 1, \alpha_2 < 0$ and $\mu > 0$. In addition, the signs of $\mathfrak{aw}$ and $\mathfrak{ab}$ are equal to the sign of

$$\frac{\mathfrak{a}}{u_0} = \left(\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2 - 1}\right)\left(u_0^{\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2-1} - 1} \int_{-\infty}^{\infty} w_{\mathrm{h}}(x)^{\beta_1} dx\right)^2 - \mu.$$

Thus, the sign of $\frac{\mathfrak{a}}{u_0}$ determines whether condition iii. in Corollary 3.31 is satisfied. The quantity $\frac{\mathfrak{a}}{u_0}$ measures the transversality between the touch-down curve $\mathcal{T}_+$ and the solution ψ_{s} – see Remark 3.30.

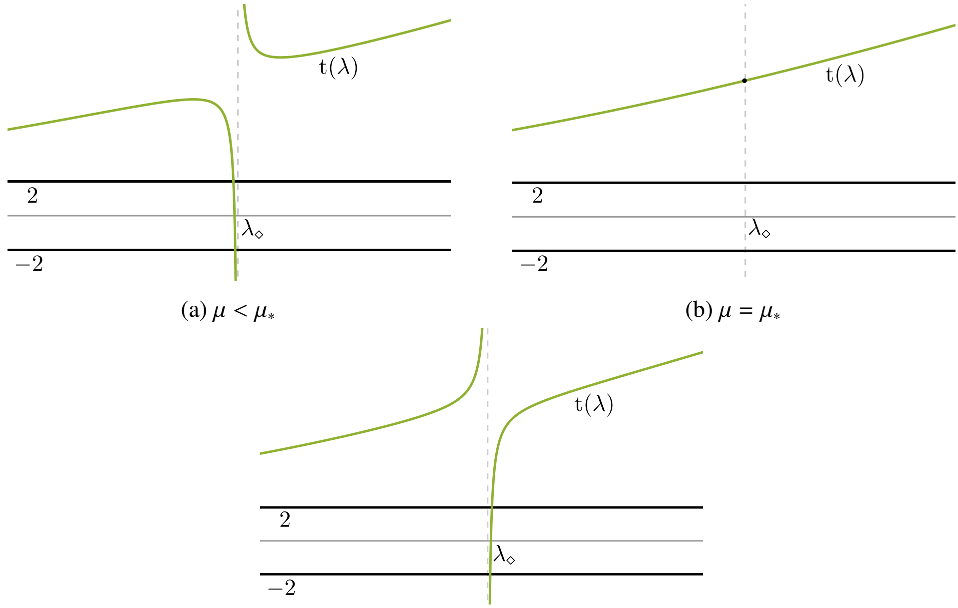
One can verify that the leading-order expression (3.31) of the critical spectral curve coincides with the one in [114] derived with a different method – see §1.2. ■

3.8.4 A closer look at zero-pole cancelation

Proposition 3.28 shows that for $m = 1$ the slow Evans function $\mathcal{E}_{s,0}(\cdot,\gamma)$ has a removable singularity at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$ if and only if one of the identities (3.12), (3.13) holds true or there exists a non-trivial solution to (3.22) at $\lambda = \lambda_\diamond$ with boundary values $u(0) = 0 = u(2\ell_0)$. The set of $\lambda_\diamond \in \mathbb{C}$ for which (3.12) or (3.13) holds true will in general be discrete, since the involved expressions are analytic in $\lambda_\diamond$. Moreover, [128, Theorem 4.3.1-6] shows that this is also the case for the set of $\lambda_\diamond \in C_\Lambda$ for which the boundary value problem (3.22), $u(0) = 0 = u(2\ell_0)$ admits a non-trivial solution. Hence, zero-pole cancelation is a robust phenomenon in the absence of additional structure (such as the translational invariance at $\lambda = 0$ mentioned in Remark 3.13).

Being robust, zero-pole cancelation can still fail in one-parameter families. Suppose equation (1.9) depends on a real parameter μ and $\mathcal{E}_{f,0}$ has a simple zero $\lambda_\diamond$ with $\mathrm{Re}(\lambda_\diamond) > 0$, independent of μ. Denote by $\mathfrak{i}(\mu)$ the singular part of the Laurent expansion at $\lambda = \lambda_\diamond$ of $\mathfrak{t}_\mu(\lambda)$ – see Propositions 3.25 and 3.28. Assume there is a value $\mu_* \in \mathbb{R}$ such that $\mathfrak{i}(\mu_*) = 0$ and $\partial_\mu \mathfrak{i}(\mu_*) \neq 0$. Then, for any $\gamma \in S^1$, $\mathcal{E}_{0,\mu_*}(\lambda_\diamond,\gamma) = 0$ and $\mathcal{E}_{0,\mu}(\lambda_\diamond,\gamma) \neq 0$ for any $\mu \neq \mu_*$ close to μ_*.

The transition of μ through a point μ_* may seem like a blue sky catastrophe, which makes the pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ 'suddenly' spectrally unstable. However, such a transition from cancelation to non-cancelation is caused by unstable spectrum moving through the point $\lambda_\diamond$. This can be seen by noting that the there exists a neighborhood $N \subset \mathbb{R}$ of $\lambda_\diamond$ such that $\mathfrak{t}_\mu(N)$ covers the whole real line as μ approaches μ_* – see Figure 3.3. In particular, $\mathfrak{t}_\mu(N)$ covers the interval $[-2,2]$ as $\mu \to \mu_*$. Thus, by Proposition 3.25 there is a branch of unstable spectrum moving through the point $\lambda_\diamond$. We remark that the orientation of the spectral curve changes in this proces – see Figure 3.3.



(a) $\mu < \mu_*$ (b) $\mu = \mu_*$

(c) $\mu > \mu_*$

Figure 3.3: The trace function $\mathrm{t}_\mu(\lambda)$ about $\lambda_\diamond$.

Example 3.35. We provide an example where zero-pole cancelation fails. Consider the Gierer-Meinhardt equation (2.26), where $\alpha_2 \neq 0$ and $\mu < 0$. We emphasize that in this case the slow reduced system (2.4) is linear of center type. This differs from the 'standard' Gierer-Meinhardt setting considered in [21, 25, 50, 114, 123], where $\mu > 0$ and the slow reduced system is linear of saddle type.

Let $u_0 > 0$. Note that (2.26) satisfies **(S1)**, **(S2)** and **(E1)** with $v_\mathrm{h}(x, u_0) > 0$ for all $x \in \mathbb{R}$. Take $u_1 < 0$ such that $\mu u_1^2 = \mathcal{J}(u_0)^2 + \mu u_0^2$, where $\mathcal{J}\colon (0,\infty) \to \mathbb{R}$ is as in (3.37). Then, assumption **(E2)** is satisfied with $\psi_\mathrm{s}(\check{x})$ the solution to the Hamiltonian system (2.4) with initial condition $\psi_\mathrm{s}(0) = (u_0, \mathcal{J}(u_0))$. Hence, Theorem 2.3 implies that, for $\varepsilon > 0$ sufficiently small, there exists a $2\ell_\varepsilon$-periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(x)$ to (2.26). Moreover, it holds by the Hamiltonian nature of system (2.4)

$$\ell_\varepsilon \to \ell_0 = \ell_0(\mu) := \frac{\pi}{2} + \sin^{-1}\frac{u_0}{\sqrt{\frac{\mathcal{J}(u_0)^2}{\mu} + u_0^2}}, \qquad \text{as } \varepsilon \to 0.$$

In [114, Lemma 3.3] it is shown that $\lambda_\diamond = 1/4(\beta_2+1)^2 - 1 > 0$ is the positive zero of the fast Evans function $\mathcal{E}_{f,0}$. Note that both $\partial_v H_2(u_0, v_\mathrm{h}(x,u_0))$ and $\partial_u G(u_0, v_\mathrm{h}(x,u_0), 0)$ are strictly negative for all $x \in \mathbb{R}$. Moreover, the v-component of any non-trivial solution to (3.6) at $\lambda = \lambda_\diamond$ has no zeros by the proof of Proposition 3.24. Therefore, identities (3.12) and (3.13) are not satisfied. Now assume $\mu > \lambda_\diamond$. The solution to (3.22) at $\lambda = \lambda_\diamond$ with initial values $u(0) = 0, u'(0) = 1$ is given by,

$$u(\check{x}, \lambda_\diamond) = \frac{\sin(\sqrt{\mu - \lambda_\diamond}\,\check{x})}{\sqrt{\mu - \lambda_\diamond}},$$

where $\sqrt{\cdot}$ denotes the principal square root. Clearly, it holds $u(2\ell_0, \lambda_\diamond) = 0$ if and only if

$$\mu = \lambda_\diamond + \left(\frac{k\pi}{2\ell_0(\mu)}\right)^2, \tag{3.38}$$

for some $k \in \mathbb{Z}_{\geq 1}$. Since $\ell_0(\mu) \in (\pi/2, \pi)$ for every $\mu > 0$, equation (3.38) will have a solution $\mu = \mu_k > \lambda_\diamond$ for every $k \in \mathbb{Z}_{\geq 1}$. We conclude with the aid of Proposition 3.28 that, if $\mu = \mu_k$ for some $k \in \mathbb{Z}_{\geq 1}$, then $\mathcal{E}_{s,0}(\lambda, \gamma)$ has a removable singularity at $\lambda = \lambda_\diamond$ and it holds $\mathcal{E}_0(\lambda_\diamond, \gamma) = 0$ for any $\gamma \in S^1$. ∎

3.9 Stability in the slowly nonlinear toy problem

In this section, we derive explicit expressions for the reduced Evans function $\mathcal{E}_0(\lambda, \gamma)$ and the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$, defined in (3.24) and (3.32), in the toy problem (2.27). Then, Corollaries 3.31 and 3.32 can be employed to prove spectral stability or instability of the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}(\check{x})$ constructed in §2.5.

For the toy problem (2.27), the homogeneous fast eigenvalue problem reads,

$$\begin{aligned} v_x &= q, \\ q_x &= \left(1 - 3\operatorname{sech}^2\left(\tfrac{1}{2}x\right) + \lambda\right)v, \end{aligned} \qquad (v, q) \in \mathbb{R}^2, \tag{3.39}$$

where we used the expressions for $v_{\mathrm{h}}(x, u)$ derived in §2.5. Let $\Lambda = -1$. The solutions to (3.39) can be found using Legendre functions – see [120, Section 3.3]. Thus, we establish two non-trivial solutions $\varphi_\pm(x, \lambda)$ to (3.39), whose v-components are given by

$$\begin{aligned} v_\pm(x, \lambda) = e^{\mp\sqrt{\lambda+1}x}\Bigg(&4\lambda\frac{\left(\sqrt{\lambda+1}+3\right)e^{\pm x} + \sqrt{\lambda+1} - 3}{e^{\pm x}+1} \\ &+ 15\left(e^{\pm x}-1\right)\frac{\left(\sqrt{\lambda+1}+1\right)e^{\pm 2x} - \sqrt{\lambda+1}+1}{\left(e^{\pm x}+1\right)^3}\Bigg), \end{aligned} \qquad \lambda \in C_\Lambda,$$

where $\sqrt{\cdot}$ denotes the principal square root. One readily observes $\lim_{x\to\pm\infty}\varphi_\pm(x, \lambda) = 0$. Hence, the fast Evans function $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ is given by the Wronskian of $\varphi_+(x, \lambda)$ and $\varphi_-(x, \lambda)$:

$$\mathcal{E}_{f,0}(\lambda) := 2\lambda\sqrt{\lambda+1}\left(16\lambda^2 - 8\lambda - 15\right),$$

and we find $\mathcal{E}_{f,0}$ has simple roots $-\frac{3}{4}, 0$ and $\frac{5}{4}$. The inhomogeneous fast eigenvalue problem reads

$$\begin{aligned} v_x &= q, \\ q_x &= \left(1 - 3\mathrm{sech}^2\left(\tfrac{1}{2}x\right) + \lambda\right)v + \tfrac{9}{4}\mathrm{sech}^4\left(\tfrac{1}{2}x\right)f'(u), \end{aligned} \qquad (v,q) \in \mathbb{R}^2,$$

where $u > 0$ and $\lambda \in C_\Lambda$. For any $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ its unique solution $\mathcal{X}_{in}(x,u,\lambda)$ can be found using variation of constants. So, the v-component $\mathcal{V}_{in}(x,u,\lambda)$ of $\mathcal{X}_{in}(x,u,\lambda)$ reads,

$$\begin{aligned} \mathcal{V}_{in}(x,u,\lambda) &= f'(u)\frac{v_+(x,\lambda)I(x,\lambda) + v_-(x,\lambda)I(-x,\lambda)}{\mathcal{E}_{f,0}(\lambda)}, \\ I(x,\lambda) &:= -\frac{9}{4}\int_{-\infty}^{x} v_-(y,\lambda)\mathrm{sech}^4\left(\tfrac{1}{2}y\right)dy. \end{aligned}$$

We emphasize that the integral $I(x,\lambda)$ can be evaluated using hypergeometric functions. Yet, the resulting expressions are quite lengthy, so we decide not to provide these. Using the formula for ψ_s in §2.5, we state the slow eigenvalue problem,

$$\begin{aligned} u_{\check{x}} &= p, \\ p_{\check{x}} &= \left(\lambda + \mu\cos\left[2\mathrm{Am}\left(-k\sqrt{\mu}(\check{x}-c), k^{-2}\right) + \pi\right]\right)u \qquad (u,p) \in \mathbb{R}^2, \check{x} \in [0, 2\ell_0], \\ &= \left(\lambda + 2\mu k^2 \mathrm{sn}^2\left[\sqrt{\mu}(\check{x}-c), k^2\right]\right)u, \end{aligned} \tag{3.40}$$

where $\lambda \in C_\Lambda$, $k \in (0,1)$, $c \in \mathbb{R}$ with $|c| < K(k)\mu^{-1/2}$ and $\ell_0 = \ell_0(k,l,c,\mu)$ is defined in (2.31). Here, $K(k)$ is the Jacobi complete integral of the first kind, $\mathrm{Am}(\check{x},k)$ denotes the Jacobi amplitude function and $\mathrm{sn}(\check{x},k)$ is one of the Jacobi elliptic functions. Equation (3.40) is known as Lamé's equation and can be solved explicitly. For $\lambda \in \mathbb{C} \setminus \{0\}$, this is done by first substituting $z = \mathrm{sn}(\sqrt{\mu}(\check{x}-c), k^2)$ and then applying differential Galois theory [65]. This yields two solutions $\psi_\pm(\check{x},\lambda;k,c,\mu)$ to (3.40), which have u-components,

$$u_\pm(\check{x},\lambda;k,c,\mu) := \frac{\sqrt{\frac{\lambda}{k^2\mu} + \mathrm{cn}^2\left(\sqrt{\mu}(\check{x}-c), k^2\right)}}{\exp\left[\pm\sqrt{\frac{\lambda(\lambda-\mu+\mu k^2)}{\mu(\lambda+k^2\mu)}}\Pi\left(\frac{k^2\mu}{\lambda+k^2\mu}, -\mathrm{am}\left(\sqrt{\mu}(\check{x}-c), k^2\right), k^2\right)\right]},$$

where $\Pi(\check{x},\phi,k)$ denotes Legendre's incomplete elliptic integral of the third kind and $\mathrm{cn}(\check{x},k)$ is one of the Jacobi elliptic functions. Thus, we have obtained all ingredients to explicitly calculate the slow Evans function $\mathcal{E}_{s,0}\colon \left[\mathbb{C} \setminus \mathcal{E}_{f,0}^{-1}(0)\right] \times \mathbb{C} \to \mathbb{C}$. Indeed, we obtain

$$\mathcal{E}_{s,0}(\lambda,\gamma) = \det\left(\Upsilon(u_0,\lambda)X(2\ell_0,\lambda)X(0,\lambda)^{-1} - \gamma I\right),$$

with $u_0 = u_0(k,c,\mu) := u_s(0;k,c,\mu)$ – see equation (2.30) – and

$$\begin{aligned} X(\check{x},\lambda) = X(\check{x},\lambda;k,c,\mu) &:= \left(\ \psi_+(\check{x},\lambda;k,c,\mu) \ \middle|\ \psi_-(\check{x},\lambda;k,c,\mu)\ \right), \\ \Upsilon(u,\lambda) = \Upsilon(u,\lambda;\nu_2,\nu_3) &:= \begin{pmatrix} I & 0 \\ \mathcal{G}(u,\lambda;\nu_2,\nu_3) & I \end{pmatrix}, \\ \mathcal{G}(u,\lambda;\nu_2,\nu_3) &:= f(u)\int_{-\infty}^{\infty}\left(3\nu_2\mathrm{sech}^2\left(\tfrac{1}{2}x\right) + \tfrac{27}{4}\nu_3 f(u)\mathrm{sech}^4(\tfrac{1}{2}x)\right)\mathcal{V}_{in}(x,u,\lambda)dx. \end{aligned}$$

So, by multiplying the expressions for the fast and slow Evans functions above, one obtains an explicit reduced Evans function $\mathcal{E}_0(\lambda, \gamma) = -\gamma \mathcal{E}_{s,0}(\lambda, \gamma)\mathcal{E}_{f,0}(\lambda)$.

Our next step is to calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$ in the toy problem (2.27). The slow variational equation about $\psi_s(\check{x}; k, c, \mu)$ – see equation (2.30) – equals the slow eigenvalue problem (3.40) at $\lambda = 0$. Naturally, one of the solutions to (3.40) at $\lambda = 0$ is given by the derivative $\psi_1(\check{x}; k, c, \mu) = \psi_s'(\check{x}; k, c, \mu)$, whose u-component reads

$$u_1(\check{x}; k, c, \mu) = -2k\sqrt{\mu}\,\mathrm{dn}\left(k\sqrt{\mu}(\check{x} - c), k^{-2}\right),$$

where $\mathrm{dn}(\check{x}, k)$ is one of the Jacobi elliptic functions. Note that $u_1(\check{x})$ is antisymmetric about ℓ_0. A second solution $\psi_2(\check{x}; k, l, c, \mu)$, having a symmetric u-component about $\ell_0 = \ell_0(k, l, c, \mu)$, is established via Rofe-Beketov's formula [7, Chapter 1.9]. We gauge ψ_2 such that the Wronskian of ψ_2 and ψ_1 equals 1. The u-component of ψ_2 is given by

$$\begin{aligned} u_2(\check{x}; k, l, c, \mu) = \Big[&\mathrm{cn}\left(\sqrt{\mu}(\check{x} - c), k^2\right)\Big[(k^2 - 1)\sqrt{\mu}(\check{x} - c) + E\left(\mathrm{am}\left(\sqrt{\mu}(\check{x} - c), k^2\right), k^2\right)\Big] \\ &- \mathrm{dn}\left(\sqrt{\mu}(\check{x} - c), k^2\right)\mathrm{sn}\left(\sqrt{\mu}(\check{x} - c), k^2\right) - \alpha(k, l, \mu)u_1(\check{x}; k, c, \mu)\Big]\frac{1}{2k(k^2 - 1)\mu}, \\ \alpha(k, l, \mu) :=\ & \frac{(1 - k^2)(2l + 1)K(k^2) - E\left((l + \frac{1}{2})\pi, k^2\right)}{2k\sqrt{\mu}}, \end{aligned}$$

where $E(\check{x}, k)$ is the Jacobi complete integral of the second kind. Having established the solutions to the slow variational problem, we calculate the quantities $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{w}$:

$$\begin{aligned} \mathfrak{a} &= \frac{72}{25} f(u_0)^3 \left(5\nu_2 + 6\nu_3 f(u_0)\right)\left(5\nu_2 + 9\nu_3 f(u_0)\right) f'(u_0) - \mu \sin(u_0), \\ \mathfrak{b} &= u_2(0; k, l, c, \mu), \qquad \mathfrak{w} = \frac{2f'(u_0)}{f(u_0)}. \end{aligned}$$

Chapter 4

Prerequisites for the spectral stability analysis

In the spectral stability analysis in Chapter 5 we encounter linear ODEs, some of which depend on a small parameter $\varepsilon > 0$ or on a spectral parameter $\lambda \in \mathbb{C}$. In this chapter we collect the necessary techniques to control such systems.

4.1 A Grönwall type estimate for linear systems

In the spectral stability analysis of solutions to singularly perturbed equations one often needs to compare a linear system with its perturbation. Our analysis requires the following approximation result for linear systems, which follows from a direct application of Grönwall's inequality.

Lemma 4.1. *[87, Lemma 1] Let $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ with $a < b$ and $A, B \in C([a,b], \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose there are constants $K, \mu > 0$ such that the evolution operator $T_1(x,y)$ of system,*

$$\varphi_x = A(x)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.1}$$

satisfies

$$\|T_1(x,y)\| \leq Ke^{\mu|x-y|}, \quad x, y \in [a,b]. \tag{4.2}$$

Denote by $T_2(x,y)$ the evolution operator of system,

$$\varphi_x = B(x)\varphi, \quad \varphi \in \mathbb{C}^n. \tag{4.3}$$

It holds

$$\|T_1(x,y) - T_2(x,y)\| \leq K \int_a^b \|A(z) - B(z)\| dz \exp\left(\mu(b-a) + K \int_a^b \|A(z) - B(z)\| dz\right),$$

for $x, y \in [a,b]$.

Remark 4.2. If $M > 0$ is such that $M \geq \sup\{\|A(x)\| : x \in [a,b]\}$, then (4.2) is satisfied for $\mu = M$ and $K = 1$ by Grönwall's inequality. ■

4.2 Asymptotically constant systems

The eigenvalue problems arising in our spectral stability analysis are non-autonomous linear systems of the form,

$$\varphi_x = A(x,\lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.4}$$

depending analytically on a spectral parameter λ. Often we are looking for the eigenvalues $\lambda \in \mathbb{C}$ for which (4.4) admits a non-trivial bounded (or exponentially localized) solution. Therefore, we are interested in the asymptotic behavior of solutions to (4.4).

Linearizing about pulse type solutions leads to eigenvalue problems (4.4) that have an asymptotically constant coefficient matrix. In such systems the asymptotics of solutions is dictated by the behavior of the constant coefficient system at $\pm\infty$ – see also Proposition 4.7. The following result concerns the construction of a unique solution with the highest decay rate to an asymptotically constant system.

Proposition 4.3. *[90, Proposition 1.2] Let $n \in \mathbb{Z}_{>0}, \Omega \subset \mathbb{C}$ open and $A \in C([0,\infty) \times \Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ such that $A(x,\cdot)$ is analytic on Ω for each $x \geq 0$. Suppose that there exists $\mu, K > 0$ and $A_\infty : \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ analytic such that*

$$\|A(x,\lambda) - A_\infty(\lambda)\| \leq Ke^{-\mu x}, \quad x \geq 0, \lambda \in \Omega. \tag{4.5}$$

Furthermore, suppose that the eigenvalue $\mu(\lambda)$ of $A_\infty(\lambda)$ of smallest real part is simple for all $\lambda \in \Omega$. Denote by $v(\lambda)$ an analytic eigenvector of A_∞ corresponding to $\mu(\lambda)$. For any compact subset $\Omega_b \subset \Omega$, there exists $C > 0$, independent of λ, and a unique solution $y(x,\lambda)$ to (4.4) satisfying

$$\left\| e^{-\mu(\lambda)x} y(x,\lambda) - v(\lambda) \right\| \leq Ce^{-\mu x}, \quad x \geq 0, \lambda \in \Omega_b.$$

The solution $y(x,\cdot)$ is analytic on the interior of Ω_b for each $x \geq 0$.

4.3 Exponential dichotomies

Exponential dichotomies enable us to track solutions in linear systems by separating the solution space in solutions that either decay exponentially in forward time or else in backward time. Moreover, their associated projections inherit analytic dependence of the problem on a spectral parameter λ. Therefore, they provide a natural framework [98] to capture the linear dynamics of eigenvalue problems of the form (4.4) arising in our spectral stability analysis.

Definition 4.4. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x,y)$ the evolution operator of (4.1). Equation (4.1) has *an exponential dichotomy on J with constants $K, \mu > 0$ and projections $P(x)\colon \mathbb{C}^n \to \mathbb{C}^n$* if for all $x, y \in J$ it holds

- $P(x)T(x,y) = T(x,y)P(y)$;
- $\|T(x,y)P(y)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x,y)(I - P(y))\| \leq Ke^{-\mu(y-x)}$ for $y \geq x$.

Let $P(x)$ be the family of projections associated with an exponential dichotomy on J. For each $x, y \in J$, we denote by $T^s(x,y) = T(x,y)P(y)$ and $T^u(x,y) = T(x,y)(I - P(y))$ the *stable and unstable evolution* of system (4.1), leaving the projection $P(y)$ implicit.

Below we give a short overview of the properties of exponential dichotomies that we need for our spectral stability analysis. For an extensive introduction on dichotomies the reader is referred to [14, 96]. A generalization of the concept of exponential dichotomies is the notion of *exponential separation*, which is treated in [85]. In particular, one can define *exponential trichotomies* to capture linear systems that exhibit centre behavior in addition to exponential decay in forward and backward time – see §4.4.

4.3.1 Dichotomy projections

Exponential dichotomies on an interval $J \subset \mathbb{R}$ are in general not unique. If $J = [0, \infty)$, then the range of the dichotomy projection corresponds to the space of solutions decaying in forward time and is therefore uniquely determined, whereas its kernel can be any complement.

Lemma 4.5. *[96, Lemma 1.2(ii)] Let $n \in \mathbb{Z}_{>0}$ and $A \in C([0,\infty), \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation* (4.1) *admits an exponential dichotomy on* $[0,\infty)$ *with projections $P(x)$. If $Y \subset \mathbb{C}^n$ satisfies $Y \oplus P(0)[\mathbb{C}^n] = \mathbb{C}^n$, then* (4.1) *admits an exponential dichotomy on* $[0,\infty)$ *with projections $Q(x)$, where $Q(0)$ is the projection on $P(0)[\mathbb{C}^n]$ along Y.*

An autonomous linear system $\varphi_x = A_0\varphi$, where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is hyperbolic, admits an exponential dichotomy on $\mathbb{R}$. The associated dichotomy projection is given by the spectral projection onto the stable eigenspace of A_0. If a non-autonomous linear system (4.1), which admits an exponential dichotomy on $[0,\infty)$, converges to a hyperbolic system as $x \to \infty$, then the dichotomy projections converge to the associated spectral projection.

Lemma 4.6. *[86, Lemma 3.4] Let $n \in \mathbb{Z}_{>0}$ and $A \in C([0,\infty), \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation* (4.1) *admits an exponential dichotomy on* $[0,\infty)$ *with constants $K, \mu > 0$ and projections $P(x)$. In addition, suppose there exists a hyperbolic matrix $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ with spectral gap larger than μ such that*

$$\|A_0\| \leq K, \quad \|A(x) - A_0\| \leq Ke^{-\mu x}, \quad x \geq 0.$$

Then, there exists a constant $C > 0$, depending on n, μ and K only, such that

$$\|P(x) - P_0\| \leq Ce^{-\mu x}, \quad x \geq 0,$$

where P_0 is the spectral projection onto the stable eigenspace of A_0.

4.3.2 Sufficient criteria for exponential dichotomies

As mentioned before, an autonomous linear system $\varphi_x = A_0\varphi$, where $A_0 \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ is hyperbolic, admits an exponential dichotomy on $\mathbb{R}$. This result can be extended to non-autonomous systems (4.1) in at least two ways. First, if the coefficient matrix $A(x)$ converges to a hyperbolic matrix $A_{\pm\infty}$ as $x \to \pm\infty$, then exponential dichotomies for (4.1) on the half-lines $[0, \infty)$ and $(-\infty, 0]$ can be constructed from the exponential dichotomies of the asymptotic systems $\varphi_x = A_{\pm\infty}\varphi$. Second, if $A(x)$ is slowly varying and pointwise hyperbolic, then system (4.1) admits an exponential dichotomy.

In our spectral stability analysis, we use these two results to obtain exponential dichotomies for eigenvalue problems of the form (4.4). We emphasize that both constructions respect analyticity in the spectral parameter λ. We start with the first result that focusses on asymptotically hyperbolic systems.

Proposition 4.7. *[86, Lemma 3.4], [99, Theorem 1] Let $n \in \mathbb{Z}_{>0}$, $\Omega \subset \mathbb{C}$ open and $A \in C([0, \infty) \times \Omega, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ such that $A(x, \cdot)$ is analytic on Ω for each $x \geq 0$. Suppose that there exists constants $\mu, K, \alpha > 0$ and an analytic map $A_\infty : \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that*

i. Identity (4.5) is satisfied for each $x \geq 0$ and $\lambda \in \Omega$;

ii. For any $\lambda \in \Omega$ the matrix $A_\infty(\lambda)$ is hyperbolic with spectral gap larger than α.

System (4.4) admits for any $\lambda \in \Omega$ an exponential dichotomy on $[0, \infty)$ with constants $C(\lambda), \alpha > 0$ and projections $P(x, \lambda)$, whose rank equals the dimension of the stable eigenspace of $A_\infty(\lambda)$. The projections $P(x, \cdot)$ are analytic on Ω for each $x \geq 0$. Moreover, the map $\lambda \mapsto C(\lambda)$ is continuous.

For hyperbolic, constant coefficient systems $\varphi_x = A_0(\lambda)\varphi$ the dichotomy projections equal the spectral projections onto the stable eigenspace of $A_0(\lambda)$. Clearly, this spectral projection inherits its analyticity from A_0. This can be extended to non-autonomous systems of the form (4.4): if $A(x, \lambda)$ varies slowly and is pointwise hyperbolic, then (4.1) admits an exponential dichotomy that has analytic projections close to the spectral projections onto the stable eigenspace of $A(x, \lambda)$.

The latter result is proved in [10, Proposition 6.5] in the setting of the FitzHugh-Nagumo system. In addition, in [14, Proposition 6.1] the result is proved for general systems of the form (4.1). However, the result in [14] lacks the desired closeness estimates on the dichotomy projections and analytic dependence on parameters is not shown. Therefore, we provide a proof of these two facts along the lines of [10, Proposition 6.5].

Proposition 4.8. *Let $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ with $b - a > 2$ and $\Omega \subset \mathbb{C}$ open. Denote $X = [a, b] \times \Omega$ and let $A \in C^1(X, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Assume that $A(x, \cdot)$ is analytic on Ω for each $x \in [a, b]$ and that there exists constants $\alpha > 0$ and $M > 1$ such that:*

i. For each $(x, \lambda) \in X$ the matrix $A(x, \lambda)$ is hyperbolic with spectral gap larger than α;

ii. The matrix function A is bounded by M on X.

There exists $\delta > 0$, depending only on α and M, such that, if we have

$$\sup_{(x,\lambda)\in X} \|\partial_x A(x, \lambda)\| \leq \delta,$$

then (4.4) has an exponential dichotomy on $[a+1, b-1]$ for any $\lambda \in \Omega$ with constants $C, \mu > 0$ and projections $P(x, \lambda)$ such that $P(x, \cdot)$ is analytic on Ω for each $x \in [a+1, b-1]$. In addition, we have $\mu = \frac{1}{2}\alpha$ and C depends only on M, α and n. Finally, for any $(y, \lambda) \in [a+1, b-1] \times \Omega$ we have the estimate,

$$\|P(y, \lambda) - \mathcal{P}(y, \lambda)\| \leq C \sup_{(x,\lambda)\in X} \|\partial_x A(x, \lambda)\|, \tag{4.6}$$

where $\mathcal{P}(x, \lambda)$ is the spectral projection onto the stable eigenspace of $A(x, \lambda)$.

Proof. In the following, we denote by $C > 0$ a constant depending only on M, n and α.

Our approach is to extend system (4.4) to the whole real line, such that it varies only on the finite interval $[a, b]$. We establish an exponential dichotomy for this extended system using [14, Proposition 6.1]. The range or kernel of the dichotomy projections must be analytic for $x \in \mathbb{R} \setminus [a, b]$ by analyticity of the spectral projections. These analyticity properties can be interpolated to the interval $[a, b]$. Finally, to prove the closeness estimate (4.6), we approximate the stable evolution operator of system (4.4) by $\mathcal{P}(x, \lambda)\exp(A(x, \lambda)(x - y))$, using that the derivative of $A(x, \lambda)$ is small.

We introduce a smooth partition of unity $\chi_i \colon \mathbb{R} \to [0, 1]$, $i = 1, 2, 3$ satisfying

$$\sum_{i=1}^{3} \chi_i(x) = 1, \quad |\chi_2'(x)| \leq 2, \quad x \in \mathbb{R},$$
$$\mathrm{supp}(\chi_1) \subset (-\infty, a+1), \quad \mathrm{supp}(\chi_2) \subset (a, b), \quad \mathrm{supp}(\chi_3) \subset (b-1, \infty).$$

The equation,

$$\varphi_x = \mathcal{A}(x, \lambda)\varphi, \quad \varphi \in \mathbb{C}^n, \tag{4.7}$$

with

$$\mathcal{A}(x, \lambda) := \chi_1(x)A(a, \lambda) + \chi_2(x)A(x, \lambda) + \chi_3(x)A(b, \lambda),$$

coincides with (4.4) on $[a+1, b-1]$. We calculate

$$\partial_x \mathcal{A}(x,\lambda) = \begin{cases} \chi_2(x)\partial_x A(x,\lambda), & x \in (a+1, b-1), \\ \chi_2'(x)(A(x,\lambda) - A(a,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [a, a+1], \\ \chi_2'(x)(A(x,\lambda) - A(b,\lambda)) + \chi_2(x)\partial_x A(x,\lambda), & x \in [b-1, b], \\ 0, & \text{otherwise.} \end{cases}$$

First, we have $\|\partial_x \mathcal{A}(x,\lambda)\| \leq 3\delta$ for each $(x,\lambda) \in \mathbb{R}\times\Omega$ by the mean value theorem. Second, by the spectral estimates in [83] the Hausdorff distance between the spectra of $A(a,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for each $(x,\lambda) \in (-\infty, a+1] \times \Omega$. Similarly, the Hausdorff distance between the spectra of $A(b,\lambda)$ and $\mathcal{A}(x,\lambda)$ is smaller than $C\delta^{1/n}$ for every $(x,\lambda) \in [b-1,\infty)\times\Omega$. Hence, for $\delta > 0$ sufficiently small, the matrix $\mathcal{A}(x,\lambda)$ is hyperbolic for each $(x,\lambda) \in \mathbb{R}\times\Omega$ with spectral gap larger than $\frac{1}{2}\alpha$. Third, $\mathcal{A}$ is bounded by M on $\mathbb{R}\times\Omega$. Combining these three items with [14, Proposition 6.1] implies that system (4.7) admits, provided $\delta > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$ with $\mu = \frac{1}{2}\alpha$ and projections $P(x,\lambda)$.

The next step is to prove that the projections $P(x,\cdot)$ are analytic in Ω for each $x \in \mathbb{R}$. Any solution to the constant coefficient system $\psi_x = A(a,\lambda)\psi$ that converges to 0 as $x \to -\infty$ must be in the kernel of the spectral projection $\mathcal{P}(a,\lambda)$ onto the stable eigenspace of $A(a,\lambda)$. Hence, it holds $\ker(\mathcal{P}(a,\lambda)) = \ker(P(a,\lambda))$ by construction of (4.7). Moreover, the spectral projection $\mathcal{P}(a,\cdot)$ is analytic on Ω, since $A(a,\cdot)$ is analytic on Ω. Thus, $\ker(P(a,\lambda))$ and similarly $P(b,\lambda)[\mathbb{C}^n]$ must be analytic subspaces – see [42, Chapter 18] – in $\lambda \in \Omega$. Denote by $T(x,y,\lambda)$ the evolution operator of (4.7), which is by [60, Lemma 2.1.4] analytic in $\lambda \in \Omega$ for each $x, y \in \mathbb{R}$. We conclude that both $\ker(P(a,\lambda))$ and $P(a,\lambda)[\mathbb{C}^n] = T(a,b,\lambda)P(b,\lambda)[\mathbb{C}^n]$ are analytic subspaces in $\lambda \in \Omega$. Therefore, the projection $P(a,\cdot)$ (and thus any projection $P(x,\cdot)$, $x \in \mathbb{R}$) is analytic in Ω.

Finally, we prove that the projections $P(x,\lambda)$ can be approximated by the spectral projections $\mathcal{P}(x,\lambda)$ onto the stable eigenspace of $\mathcal{A}(x,\lambda)$ for any $(x,\lambda) \in \mathbb{R}\times\Omega$. Define $\delta_* := \sup\{\|\partial_x A(x,\lambda)\| : (x,\lambda) \in X\} > 0$. Take $z \in \mathbb{R}$ and $v \in \mathcal{P}(z,\lambda)[\mathbb{C}^n]$. Observe that

$$\hat{\varphi}(x,\lambda) := \mathcal{P}(x,\lambda)e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}v, \quad (x,\lambda) \in \mathbb{R}\times\Omega,$$

satisfies the inhomogeneous equation,

$$\varphi_x = \mathcal{A}(x,\lambda)\varphi + g(x,\lambda),$$

with

$$g(x,\lambda) := e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)}\left[\partial_x\left(\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)\right)(x-z) + \partial_x\mathcal{P}(x,\lambda)\right]v.$$

By uniformity of the bound on the spectral gap of $\mathcal{A}$, there exists a contour $\Gamma \subset \mathbb{C}$, depending only on M, α and n, containing precisely those eigenvalues of $\mathcal{A}(x,\lambda)$ of negative real part for all $(x,\lambda) \in \mathbb{R} \times \Omega$. Thus, we have

$$\mathcal{P}(x,\lambda) = \frac{1}{2\pi i} \oint_\Gamma (w - \mathcal{A}(x,\lambda))^{-1} dw, \quad (x,\lambda) \in \mathbb{R} \times \Omega. \tag{4.8}$$

By [41, Corollary 1.2.4] the norm of the resolvent $(w - \mathcal{A}(x,\lambda))^{-1}$ can be bounded in terms of M, n and the distance $d(w, \sigma(\mathcal{A}(x,\lambda)))$. Hence, choosing the contour Γ appropriately, we observe

$$\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\mathcal{P}(x,\lambda)\| \leq C. \tag{4.9}$$

Since $\mathcal{P}(x,\lambda)$ is the projection onto the stable eigenspace of $\mathcal{A}(x,\lambda)$ and $\mathcal{A}$ is uniformly bounded by M on $\mathbb{R} \times \Omega$ and has a uniform spectral gap larger than $\mu = \frac{1}{2}\alpha$ on $\mathbb{R} \times \Omega$, we have by [41, Theorem 1.2.1] the bound,

$$\sup_{\lambda\in\Omega} \left\| e^{\mathcal{P}(x,\lambda)\mathcal{A}(x,\lambda)(x-z)} \right\| \leq Ce^{-\mu(x-z)}, \quad x \geq z. \tag{4.10}$$

Differentiating identity (4.8) yields

$$\partial_x \mathcal{P}(x,\lambda) = \frac{1}{2\pi i} \oint_\Gamma (w - \mathcal{A}(x,\lambda))^{-1} \partial_x \mathcal{A}(x,\lambda)(w - \mathcal{A}(x,\lambda))^{-1} dw,$$

for each $(x,\lambda) \in \mathbb{R} \times \Omega$. Since the norm of the resolvent $(w - \mathcal{A}(x,\lambda))^{-1}$ can be bounded in terms of M, n and $d(w, \sigma(\mathcal{A}(x,\lambda)))$, we observe that $\sup_{(x,\lambda)\in\mathbb{R}\times\Omega} \|\partial_x \mathcal{P}(x,\lambda)\| \leq C\delta_*$. Thus, combining the latter with (4.9) and (4.10) yields

$$\sup_{\lambda\in\Omega} \|g(x,\lambda)\| \leq C\delta_* \|v\|, \quad x \geq z. \tag{4.11}$$

Take $\xi = z - \log(\delta_*)\mu^{-1} \geq z$. By the variation of constants formula there exists $w \in \mathbb{C}^3$ such that

$$\hat{\varphi}(x,\lambda) = T(x,\xi,\lambda)w + \int_z^x T^s(x,y,\lambda)g(y,\lambda)dy + \int_\infty^x T^u(x,y,\lambda)g(y,\lambda)dy, \tag{4.12}$$

for $x \geq z$ and $\lambda \in \Omega$. Evaluating (4.12) at $x = \xi$, while using (4.9), (4.10) and (4.11), we derive $\|w\| \leq C\delta_* \|v\|$. Thus, applying $I - P(z,\lambda)$ to (4.12) at $x = z$, yields the bound $\|(I - P(z,\lambda))v\| \leq C\delta_* \|v\|$ for every $v \in \mathcal{P}(z,\lambda)[\mathbb{C}^n]$ by (4.10) and (4.11). Similarly, one shows that for every $v \in \ker(\mathcal{P}(z,\lambda))$ we have $\|P(z,\lambda)v\| \leq C\delta_* \|v\|$. Thus, we obtain for any $(z,\lambda) \in \mathbb{R} \times \Omega$

$$\|[P - \mathcal{P}](z,\lambda)\| \leq \|[(I - P)\mathcal{P}](z,\lambda)\| + \|[P(I - \mathcal{P})](z,\lambda)\| \leq C\delta_*.$$

Since (4.7) coincides with (4.4) on $[a+1, b-1]$, we have established the desired exponential dichotomy of (4.1). □

4.3.3 Extending and pasting exponential dichotomies

Once one puts a linear system in the framework of exponential dichotomies, a great technical toolbox becomes available. First, there are several constructions available to extend the interval of the dichotomy. Second, if an equation admits exponential dichotomies on two neighboring intervals, then these dichotomies can be glued together. Third, exponential dichotomies persist under small perturbations of the equation.

In our spectral stability analysis we need to establish exponential dichotomies for eigenvalue problems that have a complicated structure. The aforementioned techniques enable us to built exponential dichotomies for these problems from exponential dichotomies of simpler subproblems. In this section, we treat extending and pasting of exponential dichotomies. In the next section, we consider the persistence of exponential dichotomies against small disturbances.

Every exponential dichotomy can be extended for finite time using Grönwall type estimates.

Lemma 4.9. *[14, p. 13] Let $n \in \mathbb{Z}_{>0}$, $J_2 \subset J_1 \subset \mathbb{R}$ intervals and $A \in C(J_1, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation (4.1) admits an exponential dichotomy on J_2 with constants $K, \mu > 0$ and projections $P_2(x)$. In addition, suppose the length of $J_1 \setminus J_2$ is finite. Take $M > 0$ such that $M \geq \sup\{\|A(x)\| : x \in J_1 \setminus J_2\}$.*

Then, system (4.1) has an exponential dichotomy on J_1 with constants $C, \mu > 0$ and projections $P_1(x)$. The constant C depends on K, μ, M and the length of $J_1 \setminus J_2$ only. Moreover, we have $P_1(x) = P_2(x)$ for all $x \in J_2$.

In the case that the equation is periodic, an exponential dichotomy on a sufficiently large interval can be extended to the whole line.

Lemma 4.10. *[87, Theorem 1] Let $n \in \mathbb{Z}_{>0}$, $T > 0$ and $A \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose that A is T-periodic and that equation (4.1) has an exponential dichotomy on an interval J of length $2T$ with constants $K, \alpha > 0$. Let $M \geq \sup\{\|A(x)\| : x \in \mathbb{R}\}$ and $h := \alpha^{-1}(\sinh^{-1}(4) + \log(K))$.*

If $T > 0$ is so large that $T \geq 2h$, then equation (4.1) has an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu > 0$. We have $\mu = h^{-1}\log 3$ and C depends only on M, K and α.

Exponential dichotomies on two neighboring intervals can be pasted together as long as their spaces of exponential decaying solutions in forward and backward time are complementary.

Lemma 4.11. *Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Let J_1, J_2 be two intervals such that their union equals J and $\max J_1 = b = \min J_2$ for some $b \in \mathbb{R}$. Suppose equation (4.1) has exponential dichotomies on both J_1 and J_2 with constants $K, \mu > 0$ and projections $P_1(x), x \in J_1$ and $P_2(x), x \in J_2$, respectively.*

If $E^u := \ker(P_1(b))$ *and* $E^s := P_2(b)[\mathbb{C}^n]$ *are complementary, then* (4.1) *has an exponential dichotomy on* J *with constants* $K_1, \mu > 0$. *Here,* K_1 *depends only on* K *and* $\|P\|$, *where* P *is the projection on* E^s *along* E^u.

Proof. Let $X(x)$ be the fundamental matrix of (4.1) satisfying $X(b) = I$. Define $P(x) = X(x)PX(x)^{-1}$ for $x \in J$, where P is the projection on E^s along E^u. Observe that $P = P(b)$ has the same range as $P_2(b)$ and the same kernel as $P_1(b)$. Now, the exposition in [14, pp. 16-17] shows that (4.1) has exponential dichotomies on J_1 and on J_2 with constants $K_1, \mu > 0$ and projections $P(x)$ for $x \in J_1$ and $x \in J_2$, respectively. We have $K_1 = K + K^2\|P\| + K^3$. To conclude the proof we need to show that the dichotomy estimates remain true on the union $J = J_1 \cup J_2$. Indeed, take $x \in J_2$ and $y \in J_1$. We estimate

$$\|T(x,y)P(y)\| \leq \|T(x,b)P_2(b)\|\|P\|\|P_1(b)T(b,y)\| \leq K^2\|P\|e^{-\mu(x-y)},$$

where we use $P_2(b)P = P$ and $PP_1(b) = P$. Similarly, one estimates $\|T(y,x)(I - P(x))\| \leq K^2\|P\|e^{-\mu(x-y)}$ for $x \in J_2$ and $y \in J_1$. □

4.3.4 Roughness of exponential dichotomies

Exponential dichotomies are particularly useful to study the spectral properties of perturbed differential equations, since they are robust against small disturbances. This property is often referred to as *roughness*. The following result concerns roughness of exponential dichotomies on arbitrary intervals.

Proposition 4.12. *[14, Proposition 5.1] Let* $n \in \mathbb{Z}_{>0}$. *Take an interval* $J \subset \mathbb{R}$ *and* $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ *such that* (4.1) *has an exponential dichotomy on* J *with constants* $K, \alpha > 0$ *and projections* $P(x)$. *Then, for any* $0 < \varepsilon < \alpha$, *there exists* $\delta > 0$ *depending only on* K, α *and* ε *such that if* $B \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$ *satisfies*

$$\sup_{x\in J} \|A(x) - B(x)\| \leq \delta,$$

then equation (4.3) *has an exponential dichotomy on* J *with constants* $C, \mu > 0$ *and projections* $Q(x)$, *where* $\mu = \alpha - \varepsilon$ *and* C *depends on* K *only. Moreover, for all* $x \in J$ *we have*

$$\|P(x) - Q(x)\| \leq \frac{C\delta}{\alpha}.$$

Proposition 4.12 establishes an exponential dichotomy for any perturbation of an equation admitting an exponential dichotomy. The constructed dichotomy projections are close to the dichotomy projections of the unperturbed equation. The next result shows that the reverse is also true: if two equations that are close to each other admit exponential dichotomies, then the 'gap' between the ranges and the kernels of the dichotomy projections can be estimated.

Lemma 4.13. *Let* $n \in \mathbb{Z}_{>0}$, $a, b \in \mathbb{R}$ *with* $a < b$ *and* $A, B \in C([a,b], \mathrm{Mat}_{n\times n}(\mathbb{C}))$. *Suppose equations* (4.1) *and* (4.3) *have exponential dichotomies on* $[a,b]$ *with constants* $K_{1,2}, \mu_{1,2} > 0$

and projections $P_{1,2}(x)$*. Denote by* $T_{1,2}(x,y)$ *the evolution operators of systems* (4.1) *and* (4.3)*. Let* $\delta \geq 0$ *such that*

$$\|T_1(a,b) - T_2(a,b)\| \leq \delta.$$

Then, for every $v \in E_1^s(a) = P_1(a)[\mathbb{C}^n]$*, there exists* $w \in E_2^s(a) = P_2(a)[\mathbb{C}^n]$ *such that*

$$\|v - w\| \leq (\delta + K_2 e^{-\mu_2(b-a)}) K_1 e^{-\mu_1(b-a)} \|v\|. \tag{4.13}$$

Similarly, for every $v \in E_1^u(b) = \ker(P_1(b))$*, there exists* $w \in E_2^u(b) = \ker(P_2(b))$ *such that* (4.13) *holds true.*

Proof. Let $v \in E_1^s(a)$ and consider $w = T_2(a,b)P_2(b)T_1(b,a)v \in E_2^s(a)$. We estimate

$$\begin{aligned}\|w - v\| &\leq [\|T_2(a,b) - T_1(a,b)\| + \|T_2(a,b)(I - P_2(b))\|] \, \|T_1(b,a)v\| \\ &\leq (\delta + K_2 e^{-\mu_2(b-a)}) K_1 e^{-\mu_1(b-a)} \|v\|.\end{aligned}$$

The other statement is proven in an analogous way. □

In our spectral stability analysis we are interested in non-trivial bounded solutions to eigenvalue problems of the form (4.4). If the eigenvalue problem has an exponential dichotomy on $\mathbb{R}$, then it admits no non-trivial bounded solutions. It is possible to achieve persistence against perturbations of the latter fact under milder conditions than those stated in Proposition 4.12.

Proposition 4.14. *[88, Theorem 1] Let* $n \in \mathbb{Z}_{>0}$ *and* $A, B \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$*. Suppose* A *is bounded on* $\mathbb{R}$ *and system* (4.1) *has an exponential dichotomy on* $\mathbb{R}$ *with constants* $K, \mu > 0$*. Denote by* $T_{1,2}(x,y)$ *the evolution operators of systems* (4.1) *and* (4.3)*, respectively. If there exists* $\tau \geq \mu^{-1}(\sinh^{-1}(4) + \log(K))$ *such that for all* $x, y \in \mathbb{R}$ *with* $|x - y| \leq 2\tau$ *we have*

$$\|T_1(x,y) - T_2(x,y)\| < 1,$$

then (4.3) *admits no non-trivial bounded solutions.*

4.3.5 Inhomogeneous problems

In our spectral stability analysis inhomogeneous problems arise when decomposing complicated eigenvalue problems into a simpler principal part and a remainder. If the associated homogeneous problem admits an exponential dichotomy on $\mathbb{R}$, then the splitting of exponential growth and decay induces a splitting of the integrals in the variation of constants formula leading to a characterisation of the unique bounded solution to the inhomogeneous problem.

This above characterisation allows us to compare bounded solutions to an inhomogeneous problem and its perturbation *on the whole real line*, whereas with Grönwall type arguments, one only obtain sharp estimates on finite intervals. This is the content of the following result.

Proposition 4.15. *Let $n \in \mathbb{Z}_{>0}$, $f, g \in C(\mathbb{R}, \mathbb{C}^n)$ bounded and $A, B \in C(\mathbb{R}, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Suppose equation* (4.1) *has an exponential dichotomy on $\mathbb{R}$ with constants $K, \mu > 0$. Then the inhomogeneous problem,*

$$\omega_x = A(x)\omega + f(x), \quad \omega \in \mathbb{C}^n, \tag{4.14}$$

has a unique bounded solution $\varphi(x)$. Furthermore, if A and B are bounded and $a, b \in \mathbb{R}$ with $a < b$, then, for any bounded solution $\psi(x)$ to the inhomogeneous problem,

$$\omega_x = B(x)\omega + g(x), \quad \omega \in \mathbb{C}^n, \tag{4.15}$$

we estimate for $x \in [a, b]$,

$$\begin{aligned} \|\varphi(x) - \psi(x)\| \leq \frac{K}{\mu}\left(e^{-\mu(x-a)} + e^{-\mu(b-x)}\right)(\|\psi\|\|A - B\| + \|f - g\|) \\ + \frac{2K}{\mu}\left(\|\psi\| \sup_{z\in[a,b]} \|A(z) - B(z)\| + \sup_{z\in[a,b]} \|f(z) - g(z)\|\right). \end{aligned} \tag{4.16}$$

Proof. Denote by $T(x, y)$ the evolution operator of system (4.1). By [14, Proposition 8.2] system (4.14) has a unique bounded solution given by

$$\varphi(x) = \int_{-\infty}^{x} T^s(x, z)f(z)dz + \int_{\infty}^{x} T^u(x, z)f(z)dz, \quad x \in \mathbb{R}.$$

Now, let A and B be bounded and ψ a bounded solution to (4.15). Note that $w: \mathbb{R} \to \mathbb{C}^n$ defined by $w(x) = \varphi(x) - \psi(x)$ is a bounded solution to the inhomogeneous equation,

$$w_x = A(x)w + h(x),$$

where the inhomogeneity $h: \mathbb{R} \to \mathbb{C}^n$ given by $h(x) = (A(x) - B(x))\psi(x) + f(x) - g(x)$ is bounded on $\mathbb{R}$. By applying [14, Proposition 8.2] once again we deduce that $w(x)$ is given by

$$w(x) = \int_{-\infty}^{x} T^s(x, z)h(z)dz + \int_{\infty}^{x} T^u(x, z)h(z)dz, \quad x \in \mathbb{R}. \tag{4.17}$$

Now, let $a, b \in \mathbb{R}$ with $a < b$. Estimate (4.16) for $x \in [a, b]$ is achieved by splitting both integrals in expression (4.17) into two parts. The first integral is split in integrals over $(-\infty, a)$ and over (a, x). Similarly, the second integral is split in integrals over (x, b) and over (b, ∞). This yields four integrals, which can be estimated separately in order to obtain estimate (4.16). □

4.4 Exponential trichotomies

In Chapter 2 we proved the existence of stationary, spatially periodic pulse solutions to (1.10) by separating attracting, repelling and slowly evolving dynamics in the existence problem (2.1).

Naturally, the linearization of system (1.10) about the periodic pulse has a similar structure. Therefore, we encounter eigenvalue problems in our spectral stability analysis that exhibit fast exponential decay in forward and backward time as well as slow 'centre' behavior. Exponential trichotomies capture the dynamics in such linear systems. We employ the following definition.

Definition 4.16. Let $n \in \mathbb{Z}_{>0}$, $J \subset \mathbb{R}$ an interval and $A \in C(J, \mathrm{Mat}_{n\times n}(\mathbb{C}))$. Denote by $T(x,y)$ the evolution operator of (4.1). Equation (4.1) has an *exponential trichotomy* on J with constants $K, \mu > 0$ and projections $P^u(x), P^s(x), P^c(x) \colon \mathbb{C}^n \to \mathbb{C}^n$ if for all $x, y \in J$ it holds

- $P^u(x) + P^s(x) + P^c(x) = I$;
- $P^{u,s,c}(x)T(x,y) = T(x,y)P^{u,s,c}(y)$;
- $\|T(x,y)P^s(y)\|, \|T(y,x)P^u(x)\| \leq Ke^{-\mu(x-y)}$ for $x \geq y$;
- $\|T(x,y)P^c(y)\| \leq K$.

We often use the abbreviations $T^{u,s,c}(x,y) = T(x,y)P^{u,s,c}(y)$ leaving the associated projections of the exponential trichotomy implicit.

If a linear system has a special structure, then exponential trichotomies can be generated explicitly from exponential dichotomies of a subsystem. For instance, consider the upper-triangular block system,

$$\varphi_x = \begin{pmatrix} A(x) & B(x) \\ 0 & C(x) \end{pmatrix} \varphi, \quad \varphi \in \mathbb{C}^{m+n}, \tag{4.18}$$

where A, B, C are bounded matrix functions. If the invariant subsystem $\psi_x = C(x)\psi$ admits an exponential dichotomy on some interval $J \subset \mathbb{R}$ and all solutions to $\omega_x = A(x)\omega$ are bounded on J, then system (4.18) has an exponential trichotomy on J. The latter fact is readily seen using variation of constants formulas. In the spectral stability analysis we use a similar construction to generate exponential trichotomies, see §5.3.2 and §5.3.3.

4.5 The minimal opening between subspaces

The minimal opening [42, Section 13.3] is a quantity measuring the 'gap' between two subspaces.

Definition 4.17. Let $n \in \mathbb{Z}_{>0}$. The *minimal opening* between two non-trivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ is given by

$$\eta(\mathcal{M}, \mathcal{N}) = \inf\{\|x - y\| : x \in \mathcal{M}, y \in \mathcal{N}, \max(\|x\|, \|y\|) = 1\}.$$

The minimal opening has the useful property that the norm of the projection on $\mathcal{M}$ along $\mathcal{N}$ can be bounded in terms of $\eta(\mathcal{M}, \mathcal{N})$. This norm estimate is essential for the application of the "pasting" Lemma 4.11 in our spectral stability analysis.

Proposition 4.18. *Let $n \in \mathbb{Z}_{>0}$. The following assertions hold true.*

1. *If P is a non-trivial projection on $\mathbb{C}^n$, then it holds*

$$\|P\| \leq \frac{1}{\eta(P[\mathbb{C}^n], \ker(P))}.$$

2. *For non-trivial subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathbb{C}^n$ it holds $\eta(\mathcal{M}, \mathcal{N}) \neq 0$ if and only if $\mathcal{M} \cap \mathcal{N} = \{0\}$.*

3. *Let $\mathcal{M}_{1,2}$ and $\mathcal{N}_{1,2}$ be non-trivial subspaces of $\mathbb{C}^n$. Suppose that there exists $0 < \delta < 1$ such that for each $v \in \mathcal{M}_i$ there exists a $w \in \mathcal{N}_i$ such that $\|v - w\| \leq \delta\|v\|$ for $i = 1, 2$. Then, we have the estimate*

$$\eta(\mathcal{N}_1, \mathcal{N}_2) \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + 4\delta.$$

4. *Let $\Omega \subset \mathbb{C}$ be open and connected. Suppose $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$ are continuous families of subspaces on Ω, i.e. there exist continuous families of projections $P_{\mathcal{M}}, P_{\mathcal{N}} \colon \Omega \to \mathrm{Mat}_{n\times n}(\mathbb{C})$ such that $P_{\mathcal{M}}(\lambda)[\mathbb{C}^n] = \mathcal{M}(\lambda)$ and $P_{\mathcal{N}}(\lambda)[\mathbb{C}^n] = \mathcal{N}(\lambda)$ for $\lambda \in \Omega$. Then, the map $\lambda \mapsto \eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda))$ is also continuous on Ω.*

Proof. The first two assertions are derived in [42, p. 396] and [42, Proposition 13.2.1], respectively. For the third assertion take $\epsilon > 0$. There exists $v_1 \in \mathcal{M}_1$ and $v_2 \in \mathcal{M}_2$ with $\max(\|v_1\|, \|v_2\|) = 1$ such that $\|v_1 - v_2\| \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + \epsilon$. Without loss of generality we may assume $\|v_1\| = 1$. By hypothesis there exists $w_1 \in \mathcal{N}_1$ such that $\|v_1 - w_1\| \leq \delta$. Because we have $\delta < 1$, we can normalize w_1 and define $z_1 := \frac{w_1}{\|w_1\|}$. One readily estimates $\|v_1 - z_1\| \leq 2\delta$. Similarly, there exists $w_2 \in \mathcal{N}_2$ such that $\|v_2 - w_2\| \leq \delta$. In the case $\|w_2\| > 1$, take $z_2 := \frac{w_2}{\|w_2\|}$. One easily verifies $\|v_2 - z_2\| \leq 2\delta$. In the case $\|w_2\| \leq 1$, we just take $z_2 := w_2$. Finally, we estimate

$$\eta(\mathcal{N}_1, \mathcal{N}_2) \leq \|z_1 - z_2\| \leq \|v_1 - v_2\| + \|v_1 - z_1\| + \|v_2 - z_2\| \leq \eta(\mathcal{M}_1, \mathcal{M}_2) + 4\delta + \epsilon.$$

Since ϵ is arbitrarily chosen, the second assertion follows. Finally, for the fourth assertion let $P_{\mathcal{M}}(\lambda)$ and $P_{\mathcal{N}}(\lambda)$ be continuous families of projections on Ω with ranges $\mathcal{M}(\lambda)$ and $\mathcal{N}(\lambda)$, respectively. With the aid of identities (13.1.4), (13.2.5) and (13.2.7) in [42] we derive for $\lambda_0 \in \Omega$

$$|\eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda)) - \eta(\mathcal{M}(\lambda_0), \mathcal{N}(\lambda_0))| \leq \sqrt{2}\left(\|P_{\mathcal{M}}(\lambda) - P_{\mathcal{M}}(\lambda_0)\| + \|P_{\mathcal{N}}(\lambda) - P_{\mathcal{N}}(\lambda_0)\|\right).$$

This shows that $\lambda \to \eta(\mathcal{M}(\lambda), \mathcal{N}(\lambda))$ is continuous on Ω. □

4.6 The Riccati transformation

As mentioned in the introduction in Chapter 1, the eigenvalue problem associated with the linearization of system (1.10) about the periodic pulse solution can be put in the following slow-fast block structure,

$$\begin{aligned} \varphi_x &= \epsilon(A_{11}(x,\epsilon)\varphi + A_{12}(x,\epsilon)\psi), \\ \psi_x &= A_{21}(x,\epsilon)\varphi + A_{22}(x,\epsilon)\psi, \end{aligned} \qquad (\varphi,\psi) \in \mathbb{C}^{n_1+n_2}, \tag{4.19}$$

where $0 < \epsilon \ll 1$, $n_1, n_2 \in \mathbb{Z}_{>0}$ and A_{ij} are bounded and continuous matrix functions. The *Riccati transformation* is a tool for diagonalizing linear systems of the form (4.19). This linear non-autonomous transformation, decouples (4.19) into

$$\begin{aligned} \chi_x &= \epsilon\,[A_{11}(x,\epsilon) + A_{12}(x,\epsilon)U_\epsilon(x)]\,\chi, \\ \omega_x &= [A_{22}(x,\epsilon) - \epsilon U_\epsilon(x)A_{12}(x,\epsilon)]\,\omega, \end{aligned} \qquad (\chi,\omega) \in \mathbb{C}^{n_1+n_2}, \tag{4.20}$$

where $U_\epsilon(x)$ is a family of matrix functions satisfying a certain matrix Riccati equation as detailed below. Decoupling the full eigenvalue problem associated with the linearization about the periodic pulse into lower-dimensional, fast and a slow eigenvalue problems leads to a reduction of complexity in the spectral stability analysis. Eventually, the decoupling yields the factorization (1.3) of the Evans function.

Although the construction of the transformation is based on two results of Chang [12, Theorem 1] and [13, Lemma 1], the assumptions on the coefficient matrices in [13] are too restrictive. Therefore, we need a refinement of his statements. For this reason and the fact that the Riccati transformation lies at the core of our analytic factorization method, we present the full construction of the transformation. Moreover, we prove that periodicity of the coefficient matrix implies periodicity of the Riccati transform, which appears to be a new result – see Remark 4.20.

Theorem 4.19. *Let $n_1, n_2 \in \mathbb{Z}_{>0}$, $\epsilon_0 \in \mathbb{R}_{>0}$ and $A_{ij} \in C(\mathbb{R} \times (0,\epsilon_0), \mathrm{Mat}_{n_i \times n_j}(\mathbb{C}))$ such that A_{ij} are bounded by some constant $K > 0$ on $\mathbb{R} \times (0,\epsilon_0)$ for $i, j = 1, 2$. Suppose that*

$$\psi_x = A_{22}(x,\epsilon)\psi, \quad \psi \in \mathbb{C}^{n_2}, \tag{4.21}$$

admits an exponential dichotomy on $\mathbb{R}$ with constants $K, \mu > 0$, independent of ϵ. Then, for $\epsilon > 0$ sufficiently small, there exists continuously differentiable matrix functions $U_\epsilon(x)$ and $S_\epsilon(x)$ satisfying the matrix Riccati equations,

$$\begin{aligned} U &= A_{22}U - \epsilon U A_{11} - \epsilon U A_{12} U + A_{21}, && U \in \mathrm{Mat}_{n_2\times n_1}(\mathbb{C}), \\ S &= \epsilon(A_{11} + A_{12}U)S - S(A_{22} - \epsilon U A_{12}) - A_{12}, && S \in \mathrm{Mat}_{n_1\times n_2}(\mathbb{C}), \end{aligned} \tag{4.22}$$

with the following properties:

1. *U_ϵ and S_ϵ are bounded on $\mathbb{R}$ by some constant, which depends on K and μ only.*
2. *The coordinate transform,*

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = H_\epsilon(x)\begin{pmatrix} \chi \\ \omega \end{pmatrix}, \quad H_\epsilon(x) := \begin{pmatrix} I & -\epsilon S_\epsilon(x) \\ U_\epsilon(x) & I - \epsilon U_\epsilon(x) S_\epsilon(x) \end{pmatrix}, \tag{4.23}$$

diagonalizes system (4.19) *into* (4.20).

3. *The unique bounded solution* Ω_ϵ *to the inhomogeneous matrix problem,*

$$\Omega_x = A_{22}(x,\epsilon)\Omega + A_{21}(x,\epsilon), \quad \Omega \in \mathrm{Mat}_{n_2\times n_1}(\mathbb{R},\mathbb{C}), \tag{4.24}$$

satisfies

$$\|U_\epsilon(x) - \Omega_\epsilon(x)\| \leq C\epsilon|\log(\epsilon)|, \quad x \in \mathbb{R}, \tag{4.25}$$

where $C > 0$ *is a constant depending on* K *and* μ *only.*

4. *Let* $a > 0$*. We have the approximation,*

$$\|U_\epsilon(x)\| \leq C\left[\sup_{y\in[x-a,x+a]}\left(\epsilon\|U_\epsilon(y)\|^2 + \|A_{21}(y,\epsilon)\|\right) + e^{-a\mu/2}\right], \quad x \in \mathbb{R}, \tag{4.26}$$

where $C > 0$ *is a constant depending on* K *and* μ *only.*

5. *If the matrices* $A_{ij}(\cdot,\epsilon)$ *are L-periodic for* $1 \leq i, j \leq 2$*, then the coordinate transform* H_ϵ *is also L-periodic.*

Proof. In the following, we denote by $C > 0$ a constant, which depends on K and μ only.

First, we set up an integral equation for U_ϵ and prove global existence via a contraction argument. Since U_ϵ triangulizes the system, an integral equation for S_ϵ can be derived from the variation of constants formula. The first four properties of U_ϵ and S_ϵ follow readily from the integral equations they satisfy. Finally, periodicity of the transform is proven by exponential separation.

Since A_{11} is bounded by K on $\mathbb{R}\times(0,\epsilon_0)$, the evolution $T_{1,\epsilon}(x,y)$ of system,

$$\varphi_x = \epsilon A_{11}(x,\epsilon)\varphi, \quad \varphi \in \mathbb{C}^{n_1},$$

satisfies

$$\|T_{1,\epsilon}(x,y)\| \leq e^{K\epsilon|x-y|}, \quad x,y \in \mathbb{R}. \tag{4.27}$$

Denote by $T_{2,\epsilon}(x,y)$ the evolution operator of system (4.21). Take $\rho = 8K\mu^{-1}\|A_{21}\|$. The ball $B(0,\rho) \subset C_b(\mathbb{R},\mathrm{Mat}_{n_2\times n_1}(\mathbb{C}))$ is a metric space endowed with the supremum norm. We want to show that the map $\mathcal{A}_\epsilon\colon B(0,\rho) \to B(0,\rho)$ given by

$$\begin{aligned}(\mathcal{A}_\epsilon U)(x) = &\int_{-\infty}^x T^s_{2,\epsilon}(x,y)\left[-\epsilon U(y)A_{12}(y,\epsilon)U(y) + A_{21}(y,\epsilon)\right]T_{1,\epsilon}(y,x)dy \\ &- \int_x^\infty T^u_{2,\epsilon}(x,y)\left[-\epsilon U(y)A_{12}(y,\epsilon)U(y) + A_{21}(y,\epsilon)\right]T_{1,\epsilon}(y,x)dy,\end{aligned}$$

is a well-defined contraction. If $\epsilon > 0$ is sufficiently small, it holds for all $U \in B(0,\rho)$

$$\|\mathcal{A}_\epsilon U\| \leq \frac{2K}{\mu - \epsilon K}\left[\epsilon\rho^2\|A_{12}\| + \|A_{21}\|\right] < \rho,$$

using (4.27) and the exponential dichotomy of (4.21). Therefore, $\mathcal{A}_\epsilon$ is well-defined. Similarly, provided $\epsilon > 0$ is sufficiently small, we estimate for $U_1, U_2 \in B(0,\rho)$

$$||\mathcal{A}_\epsilon U_1 - \mathcal{A}_\epsilon U_2|| \leq \frac{4\epsilon K\rho||A_{12}||}{\mu - \epsilon K}||U_1 - U_2|| < ||U_1 - U_2||.$$

Hence, $\mathcal{A}_\epsilon$ is a contraction mapping. By the Banach fixed point Theorem the integral equation $\mathcal{A}_\epsilon U = U$ has a unique solution $U_\epsilon(x)$ in $B(0,\rho)$. It is readily seen by differentiating this integral equation that U_ϵ satisfies the matrix Riccati equation (4.22). Moreover, U_ϵ is bounded on $\mathbb{R}$ by $\rho \leq 8K^2\mu^{-1}$. Since (4.21) has an exponential dichotomy on $\mathbb{R}$, (4.24) admits a unique bounded solution Ω_ϵ by Proposition 4.15. Since A_{11} is bounded by K, it holds by Proposition 4.1 for $|x-y| \leq \mu^{-1}|\log(\epsilon)|$

$$||T_{1,\epsilon}(x,y) - I|| \leq C\epsilon|\log(\epsilon)|. \tag{4.28}$$

Using $U_\epsilon(x) = (\mathcal{A}_\epsilon U_\epsilon)(x)$ we write

$$\begin{aligned} U_\epsilon(x) - \Omega_\epsilon(x) = &\int_{-\infty}^x T_{2,\epsilon}^s(x,y)A_{21}(y,\epsilon)(T_{1,\epsilon}(x,y) - I)dy \\ &- \int_x^\infty T_{2,\epsilon}^u(x,y)A_{21}(y,\epsilon)(T_{1,\epsilon}(x,y) - I)dy \\ &- \int_{-\infty}^x \epsilon T_{2,\epsilon}^s(x,y)U(y)A_{12}(y,\epsilon)U(y)T_{1,\epsilon}(y,x)dy \\ &+ \int_x^\infty \epsilon T_{2,\epsilon}^u(x,y)U(y)A_{12}(y,\epsilon)U(y)T_{1,\epsilon}(y,x)dy, \end{aligned}$$

We split the interval of integration of the first two integrals in the right hand side of the latter equation. This leads to four integrals over $(-\infty, x-b_\epsilon)$, $(x-b_\epsilon, x)$, $(x, x+b_\epsilon)$ and $(x+b_\epsilon, \infty)$, where $b_\epsilon := \mu^{-1}|\log(\epsilon)|$. Thus, we obtain six integrals, which we estimate separately using (4.27), (4.28) and the bound on U_ϵ. This yields the third property.

The fourth property follows by splitting the interval of integration of the two integrals in the right hand side of the identity $U_\epsilon(x) = (\mathcal{A}_\epsilon U_\epsilon)(x)$. We obtain four integrals over $(-\infty, x-a)$, $(x-a, x)$, $(x, x+a)$ and $(x+a, \infty)$, respectively. We estimate each integral separately using (4.27) and the exponential dichotomy of (4.21). This leads to approximation (4.26).

Since A_{11}, A_{12} and U_ϵ are bounded on $\mathbb{R} \times (0, \epsilon_0)$, the evolution $T_{3,\epsilon}(x,y)$ of system,

$$\chi_x = \epsilon\,[A_{11}(x,\epsilon) + A_{12}(x,\epsilon)U_\epsilon(x)]\,\chi, \quad \chi \in \mathbb{C}^{n_1},$$

is bounded as

$$||T_{3,\epsilon}(x,y)|| \leq e^{\epsilon C|x-y|}, \quad x,y \in \mathbb{R}. \tag{4.29}$$

On the other hand, equation,

$$\omega_x = (A_{22}(x,\epsilon) - \epsilon U_\epsilon(x)A_{12}(x,\epsilon))\omega, \quad \omega \in \mathbb{C}^{n_2}, \tag{4.30}$$

can be seen as a perturbation of (4.21). By Proposition 4.12 it therefore possesses an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu_1 > 0$. Denote by $T_{4,\epsilon}(x,y)$ the evolution operator of system (4.30). We define $S_\epsilon(x)$ via the variation of constants formula,

$$S_\epsilon(x) = -\int_{-\infty}^{x} T_{3,\epsilon}(x,y)A_{12}(y,\epsilon)T^u_{4,\epsilon}(y,x)dy + \int_{x}^{\infty} T_{3,\epsilon}(x,y)A_{12}(y,\epsilon)T^s_{4,\epsilon}(y,x)dy,$$

Using (4.29) and the exponential dichotomy of (4.30) we derive that S_ϵ is bounded on $\mathbb{R}$ by some constant depending on μ and K only. This proves the first property. It is easily verified by differentiation that S_ϵ satisfies the matrix Riccati equation (4.22). Finally, using S_ϵ and U_ϵ satisfy equations (4.22), it is a straightforward calculation to see the change of variables (4.23) transforms system (4.19) into (4.20). This proves the second property.

Only the fifth property remains to be proven. Our plan is to show that system (4.19) is exponentially separated in the sense of [85]. Subsequently, we make use of the fact that exponential separation preserves periodicity. Therefore, denote by $P_\epsilon(x), x \in \mathbb{R}$ the projections corresponding to the exponential dichotomy of system (4.30) on $\mathbb{R}$, established in the latter paragraph. We define the following projections,

$$P_{1,\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & P_\epsilon(0) \end{pmatrix}, \quad P_{2,\epsilon} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{3,\epsilon} = \begin{pmatrix} 0 & 0 \\ 0 & I - P_\epsilon(0) \end{pmatrix}.$$

Denote by $V_{i,\epsilon} \subset \mathbb{C}^{n_1+n_2}$ the range of the projection $P_{i,\epsilon}$ for $i = 1,2,3$. Let m_2 be the rank of $P_\epsilon(0)$. Using (4.29) and the exponential dichotomy of (4.30), we conclude system that (4.20) is, for $\epsilon > 0$ sufficiently small, $(m_2, n_1, n_2 - m_2)$-exponentially separated with respect to the decomposition $V_{1,\epsilon} \oplus V_{2,\epsilon} \oplus V_{3,\epsilon}$. As a result, system (4.19) is also $(m_2, n_1, n_2 - m_2)$-exponentially separated with respect to the decomposition $W_{1,\epsilon} \oplus W_{2,\epsilon} \oplus W_{3,\epsilon}$, where $W_{i,\epsilon}$ is the range of the projection $Q_{i,\epsilon} := H_\epsilon(0)P_{i,\epsilon}H_\epsilon(0)^{-1}$ for $i = 1,2,3$.

Now, suppose $A_{ij}(\cdot,\epsilon)$ are L-periodic for $1 \leq i, j \leq 2$. Let $X_\epsilon(x)$ be the fundamental matrix of system (4.19) with $X_\epsilon(0) = I$. Invoking [9, Corollary 4] gives that $X_\epsilon(\cdot)Q_{2,\epsilon}X_\epsilon(\cdot)^{-1}$ is L-periodic. Denote by $T_\epsilon(x,y)$ the evolution operator of the diagonal system (4.20). We calculate for $x \in \mathbb{R}$

$$\begin{aligned} X_\epsilon(x)Q_{2,\epsilon}X_\epsilon(x)^{-1} &= H_\epsilon(x)T_\epsilon(x,0)P_{2,\epsilon}T_\epsilon(0,x)H_\epsilon(x)^{-1} = H_\epsilon(x)P_{2,\epsilon}H_\epsilon(x)^{-1} \\ &= \begin{pmatrix} I - \epsilon S_\epsilon(x)U_\epsilon(x) & \epsilon S_\epsilon(x) \\ U_\epsilon(x) + \epsilon U_\epsilon(x)S_\epsilon(x)U_\epsilon(x) & \epsilon U_\epsilon(x)S_\epsilon(x) \end{pmatrix}. \end{aligned}$$

Hence, $S_\epsilon, U_\epsilon S_\epsilon, S_\epsilon U_\epsilon$ and $U_\epsilon + \epsilon U_\epsilon S_\epsilon U_\epsilon$ are L-periodic. So, $U_\epsilon S_\epsilon U_\epsilon$ is also L-periodic. Combining this with the L-periodicity of $U_\epsilon + \epsilon U_\epsilon S_\epsilon U_\epsilon$, we conclude that U_ϵ is L-periodic. This implies that H_ϵ is L-periodic, which concludes the proof of the fifth statement. □

Remark 4.20. The periodicity of the transform in Theorem 4.19 is a new discovery to the author's knowledge. It is natural to ask whether there always exists a periodic choice for a coordinate change, which transforms a periodic system into diagonal form. However, it is shown in [84, Chapter 5] that this is not the case. It seems that the periodicity of the coordinate change H_ϵ is due to the special (slow-fast) structure of system (4.19). ■

Remark 4.21. The (m_2, n_1, n_2-m_2)-exponential separation of (4.19) obtained in Theorem 4.19 shows that the solution space of systems of the form (4.19) can be decomposed in fast exponentially decaying solutions in forward and backward time and solutions that vary slowly. This type of decomposition is very similar to the one induced by an exponential trichotomy – see §4.4. Yet, in our definition of exponential trichotomies we do not allow for exponential growth in the centre direction. However, we emphasize that some authors do include this in their definition of exponential trichotomies – see for instance [106]. ■

Remark 4.22. The Riccati transform can be employed to diagonalize general linear equations as pointed out in [4, Remark 4.7]. However, the Riccati solutions can become singular in finite time. We use both the slow-fast structure of (4.19) and the exponential dichotomy of (4.21) to achieve global boundedness of the transformation functions U_ϵ and S_ϵ. ■

Chapter 5

Spectral stability analysis

In this chapter we prove the two main spectral approximation results presented in Chapter 3: we show that the zeros of the Evans function $\mathcal{E}_\varepsilon$ are approximated by the ones of the reduced Evans function $\mathcal{E}_0$ and we derive an expansion of the critical spectral curve attached to the origin. Yet, we start with collecting some properties of the reduced Evans function $\mathcal{E}_0$, which are necessary for the proof of these approximation results.

5.1 The reduced Evans function

In this section we study the reduced Evans function $\mathcal{E}_0$, which is defined in terms of the three eigenvalue problems (3.6), (3.8) and (3.9). Thereby, we provide the proofs of Propositions 3.10, 3.11 and 3.12.

5.1.1 The fast Evans function

The homogeneous fast eigenvalue problem (3.6) arises when linearizing $v_t = D_2 v_{xx} - G(u_0, v, 0)$ about the standing pulse solution $v_\mathrm{h}(x, u_0)$ – see assumption **(E1)**. The homoclinic $\psi_\mathrm{h}(x, u_0) = (v_\mathrm{h}(x, u_0), q_\mathrm{h}(x, u_0))$ to (2.3) at $u = u_0$ converges exponentially to the hyperbolic saddle 0 as $x \to \pm\infty$. Hence, system (3.6) is asymptotically hyperbolic. Consequently, it has exponential dichotomies on both half-lines respecting analyticity in λ. This leads to the construction of the analytic fast Evans function $\mathcal{E}_{f,0}$ which detects the values of λ equation (3.6) has exponentially localized solutions. The above is the content of the following lemma and proposition.

Lemma 5.1. *Let $\mathcal{K} \subset \mathbb{C}^m$ be an open and bounded set containing the orbit segment $\{u_\mathrm{s}(\check{x}) : \check{x} \in [0, 2\ell_0]\}$ such that $\overline{K} \subset U$ – see **(S1)** and **(E2)**. There exists $\Lambda_0 > 0$ such that for $\Lambda \in (-\Lambda_0, 0)$ the spectrum of the matrix,*

$$A(u, \lambda) := \begin{pmatrix} 0 & D_2^{-1} \\ \partial_v G(u, 0, 0) + \lambda & 0 \end{pmatrix}, \tag{5.1}$$

is bounded away from the imaginary axis on $\overline{\mathcal{K}} \times C_\Lambda$ *by some constant* $\mu_r > 0$.

Proof. For $k \in \mathbb{Z}_{>0}$ and a matrix $A \in \mathrm{Mat}_{k\times k}(\mathbb{C})$ denote by $\mathcal{F}(A) = \{v^*Av : v \in \mathbb{C}^k, \|v\| = 1\}$ its field of values. Since $\partial_v G(u, 0, 0)$ has positive definite real part by **(S2)**, the field of values $\mathcal{F}(\partial_v G(u, 0, 0))$ is for every $u \in \overline{K}$ contained in the positive half-plane by [48, Property 1.2.5a]. In fact, by compactness of $\overline{K}$ there exists $\Lambda_0 > 0$ such that we have $\mathcal{F}(\partial_v G(u, 0, 0)) \subset C_{-\Lambda_0}$ for every $u \in \overline{K}$. Let $\Lambda \in (-\Lambda_0, 0)$. For $u \in \overline{K}$ and $\lambda \in C_\Lambda$ we establish using [48, Property 1.2.3] and [48, Corollary 1.7.7]

$$\begin{aligned}\sigma((\partial_v G(u,0,0)+\lambda)D_2^{-1}) &\subset (\mathcal{F}(\partial_v G(u,0,0))+\lambda)\mathcal{F}(D_2^{-1})\\ &\subset \left\{z \in \mathbb{C} : \mathrm{Re}(z) \geq d_{\max}^{-1}(\Lambda_0+\Lambda)\right\},\end{aligned}$$

where $d_{\max}$ is the largest diagonal value of D_2. The eigenvalues of $A(u, \lambda)$ are given by the square roots of the eigenvalues of $(\partial_v G(u, 0, 0) + \lambda)D_2^{-1}$. Therefore, we obtain for $u \in \overline{K}$ and $\lambda \in C_\Lambda$ that any eigenvalue $z \in \sigma(A(u, \lambda))$ satisfies $|\mathrm{Re}(z)| \geq \cos(\pi/4)\sqrt{(\Lambda_0 + \Lambda)/d_{\max}}$, which concludes the proof. □

Proposition 5.2. *Let* $\Lambda \in (-\Lambda_0, 0)$ *with* $\Lambda_0 > 0$ *as in Lemma 5.1. The homogeneous fast eigenvalue problem* (3.6) *admits for every* $\lambda \in C_\Lambda$ *exponential dichotomies on* $[0,\infty)$ *and* $(-\infty, 0]$ *with constants* $C(\lambda), \mu_r > 0$ *and rank n projections* $Q_{f,\pm}(x, \lambda)$*, where* $\mu_r > 0$ *is as in Lemma 5.1. The projections* $Q_{f,\pm}(\pm x, \cdot)$ *are analytic on* C_Λ *for each* $x \geq 0$. *Morover, the map* $\lambda \mapsto C(\lambda)$ *is continuous on* C_Λ.

Let $B_f^{u,s} : C_\Lambda \to \mathrm{Mat}_{2n\times n}(\mathbb{C})$ *be analytic bases such that* $Q_{f,+}(0, \lambda)[\mathbb{C}^{2n}] = B_f^u(\lambda)[\mathbb{C}^n]$ *and* $\ker(Q_{f,-}(0, \lambda)) = B_f^s(\lambda)[\mathbb{C}^n]$ *for* $\lambda \in C_\Lambda$. *The analytic function* $\mathcal{E}_{f,0} : C_\Lambda \to \mathbb{C}$ *given by* $\mathcal{E}_{f,0}(\lambda) = \det(B_f^u(\lambda), B_f^s(\lambda))$ *has the following properties:*

1. $\mathcal{E}_{f,0}(\lambda) = 0$ *if and only if* (3.6) *admits a non-trivial, exponentially localized solution;*
2. $\mathcal{E}_{f,0}(\lambda) \neq 0$ *if and only if* (3.6) *has an exponential dichotomy on* $\mathbb{R}$;
3. *The zero set* $\mathcal{E}_{f,0}^{-1}(0)$ *is discrete and independent of the choice of bases* $B_f^{u,s}$;
4. *The multiplicity of a zero* $\lambda \in C_\Lambda$ *of* $\mathcal{E}_{f,0}$ *coincides with the algebraic multiplicity of* λ *as an eigenvalue of the operator* $\mathcal{L}_f$*, defined in* (3.7).

Proof. By Lemma 5.1 the asymptotic matrix $A(u_0, \cdot)$, defined in (5.1), is hyperbolic on C_Λ with spectral gap larger than μ_r, where u_0 is as in **(E2)**. The stable and unstable eigenspaces of $A(u_0, \lambda)$ have dimension n for any $\lambda \in C_\Lambda$. Moreover, estimate (2.6) implies

$$\left\|\mathcal{A}_{22,0}(x, u_0, \lambda) - A(u_0, \lambda)\right\| \leq Ke^{-\mu_h|x|}, \quad x \in \mathbb{R}, \lambda \in C_\Lambda,$$

where $K > 0$ is a λ-independent constant. Therefore, system (3.6) admits by Proposition 4.7 exponential dichotomies on both half-lines with the desired properties.

By [86, Proposition 2.1] we have $\mathcal{E}_{f,0}(\lambda) \neq 0$ if and only if (3.6) has an exponential dichotomy on $\mathbb{R}$. On the other hand, every exponentially localized solution $\varphi(x,\lambda)$ to (3.6) must satisfy $\varphi(0,\lambda) \in B^u_f(\lambda)[\mathbb{C}^n] \cap B^s_f(\lambda)[\mathbb{C}^n]$. This settles the first two properties. The third and fourth property are the content of [1, Section E]. □

Proposition 5.2 provides the fast Evans function and thereby proves Proposition 3.10.

Definition 5.3. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. The map $\mathcal{E}_{f,0} \colon C_\Lambda \to \mathbb{C}$ given by $\mathcal{E}_{f,0}(\lambda) = \det(B^u_f(\lambda), B^s_f(\lambda))$, obtained in Proposition 5.2, is called the *fast Evans function*.

An important consequence of the exponential dichotomies established in Proposition 5.2 is that the differential operator associated with (3.6) is Fredholm.

Corollary 5.4. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. For each $\lambda \in C_\Lambda$ the bounded operator $\mathcal{L}_\lambda \colon C^1_b(\mathbb{R}, \mathbb{C}^{2n}) \to C_b(\mathbb{R}, \mathbb{C}^{2n})$ given by*

$$\mathcal{L}_\lambda \varphi = \varphi_x - \mathcal{A}_{22,0}(\cdot, u_0, \lambda)\varphi,$$

is Fredholm of index 0. Moreover, $\mathcal{L}_\lambda$ is invertible if and only if $\lambda \in C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$. The multiplicity of a zero $\lambda_\diamond \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ at $\lambda = \lambda_\diamond$.

Proof. This follows readily from Proposition 5.2, [86, Lemma 4.2] and [1, Section E]. We also refer to [6, Section 3.2]. □

5.1.2 The slow Evans function

The slow Evans function $\mathcal{E}_{s,0}$ is explictly given by (3.10). The matrix solution $\mathcal{X}_{in}(x, u_0, \lambda)$ to the inhomogeneous fast eigenvalue problem (3.8) at $u = u_0$ is one of the key ingredients of $\mathcal{E}_{s,0}$. We prove that $\mathcal{X}_{in}(x, u_0, \cdot)$ is meromorphic for each $x \in \mathbb{R}$. Singularities of $\mathcal{X}_{in}(x, u_0, \cdot)$ only occur at the zeros of the fast Evans function $\mathcal{E}_{f,0}$.

Proposition 5.5. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. There exists a unique solution $\mathcal{X}_{in} \colon \mathbb{R} \times U_\mathrm{h} \times [C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)] \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ to the inhomogeneous fast eigenvalue problem (3.8) with the following properties:*

1. *$\mathcal{X}_{in}(x, u_0, \cdot)$ is meromorphic on C_Λ and analytic on $C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$ for all $x \in \mathbb{R}$;*
2. *If $\lambda \mapsto \mathcal{X}_{in}(x, u_0, \lambda)$ has a pole at $\lambda = \lambda_\diamond$, then its order is at most the multiplicity of $\lambda_\diamond$ as a root of $\mathcal{E}_{f,0}$;*
3. *$\mathcal{X}_{in}(\cdot, u_0, \lambda)$ is exponentially localized for each $\lambda \in C_\Lambda \setminus \mathcal{E}^{-1}_{f,0}(0)$. In particular, there exists λ-independent constants $C, \mu_{in} > 0$ such that*

$$\|\mathcal{X}_{in}(x, u_0, \lambda)\| \leq C e^{-\mu_{in}|x|}, \quad x \in \mathbb{R},$$

for all $\lambda \in C_\Lambda$ with $\mathrm{Re}(\sqrt{\lambda}) > C$;

4. *Let $\lambda_\diamond \in C_\Lambda$ be a simple zero of $\mathcal{E}_{f,0}$. Denote by $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ exponentially localized solutions to (3.6) and its adjoint equation* (3.14)*, respectively, at $\lambda = \lambda_\diamond$ satisfying*

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \varphi_{\lambda_\diamond}(z) dz = 1. \tag{5.2}$$

There exists a neighborhood $B_{\lambda_\diamond} \subset C_\Lambda$ of $\lambda_\diamond$ and a mapping $\mathcal{X}_{\lambda_\diamond} : \mathbb{R} \times B_{\lambda_\diamond} \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$, such that

$$\mathcal{X}_{in}(x, u_0, \lambda) = \frac{\varphi_{\lambda_\diamond}(x)}{\lambda - \lambda_\diamond} \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \mathcal{A}_{21,0}(z, u_0) dz + \mathcal{X}_{\lambda_\diamond}(x, \lambda), \quad (x, \lambda) \in \mathbb{R} \times B_{\lambda_\diamond}.$$

Here, $\mathcal{X}_{\lambda_\diamond}(x, \cdot)$ is analytic on $B_{\lambda_\diamond}$ for every $x \in \mathbb{R}$. Moreover, $\mathcal{X}_{\lambda_\diamond}(\cdot, \lambda)$ is exponentially localized for every $\lambda \in B_{\lambda_\diamond}$.

Proof. For $\lambda \in C_\Lambda$, the operator $\mathcal{L}_\lambda$, established in Corollary 5.4, is Fredholm of index 0 and $\mathcal{L}_\lambda$ is invertible if and only if $\mathcal{E}_{f,0}(\lambda) \neq 0$. The multiplicity of a zero $\lambda_\diamond \in C_\Lambda$ of $\mathcal{E}_{f,0}$ coincides with the algebraic multiplicity of the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ at $\lambda = \lambda_\diamond$. Combining the latter with [74, Theorem 1.3.1] settles the first two properties.

We establish the third property. The homogeneous fast eigenvalue problem (3.6) has by Proposition 5.2 an exponential dichotomy on $\mathbb{R}$ for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Thus, since $\mathcal{A}_{21,0}(\cdot, u_0)$ is exponentially localized by **(S1)** and estimate (2.6), the same holds for $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ by Proposition 4.15. The coordinate change $(v, q) \mapsto (v, \sqrt{|\lambda|}w)$ puts system (3.6) into the form,

$$\begin{aligned} D_2 v_x &= \sqrt{|\lambda|} w, \\ w_x &= \left(\frac{\partial_v G(u_0, v_h(x, u_0), 0)}{\sqrt{|\lambda|}} + \frac{\lambda}{\sqrt{|\lambda|}} \right) v, \end{aligned} \qquad (v, w) \in \mathbb{C}^{2n}, \tag{5.3}$$

where we denote by $\sqrt{\cdot}$ the principal square root. By Proposition 4.12 there exists a constant $K > 0$ such that for any

$$\lambda \in \Sigma_K := \left\{ \lambda \in C_\Lambda : \mathrm{Re}\left(\sqrt{\lambda}\right) > K \right\} \subset \left\{ \lambda \in C_\Lambda : |\lambda| > K^2 \right\}, \tag{5.4}$$

system (5.3) admits an exponential dichotomy on $\mathbb{R}$ with constants $K_1, \mu(\lambda) > 0$, where $\mu(\lambda) = \mu_1 \mathrm{Re}(\sqrt{\lambda})$ and $K_1, \mu_1 > 0$ are independent of λ. Therefore, system (3.6) has for each $\lambda \in \Sigma_K$ an exponential dichotomy on $\mathbb{R}$ with constants $K_2(\lambda), \mu(\lambda) > 0$, where $K_2(\lambda) = \sqrt{|\lambda|} K_1$. Note that $\lambda \mapsto \frac{K_2(\lambda)}{\mu(\lambda)}$ is bounded by a λ-independent constant on Σ_K. Combining this fact with Proposition 4.15 yields the third property.

Finally, let $\lambda_\diamond \in C_\Lambda$ be a simple zero of $\mathcal{E}_{f,0}$. By Corollary 5.4 the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = \lambda_\diamond$. Hence, the fourth property follows by an application of Keldysh formula – see [74, Theorem 1.6.5]. □

Remark 5.6. If $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$, then it is always possible to choose exponentially localized solutions $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ to (3.6) and its adjoint equation (3.14) satisfying (5.2). Indeed, the kernels of the operator $\mathcal{L}_{\lambda_\diamond}$ and its adjoint $\mathcal{L}_{\lambda_\diamond}^*$ are one-dimensional by Corollary 5.4. In addition, since equation (3.6) has exponential dichotomies on both half-lines by Proposition 5.2, the same holds for its adjoint (3.14). So, the spaces of exponentially localized solutions to (3.6) and (3.14) are one-dimensional. Now, take non-trivial, exponentially localized solutions $\varphi_{\lambda_\diamond}(x)$ and $\psi_{\lambda_\diamond}(x)$ to (3.6) and (3.14), respectively. Since the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = \lambda_\diamond$ by Corollary 5.4, the generalized eigenvalue problem,

$$\mathcal{L}_{\lambda_\diamond}\varphi = \partial_\lambda \mathcal{L}_{\lambda_\diamond}\varphi_{\lambda_\diamond},$$

has no bounded solutions. Hence, [86, Lemma 4.2] implies

$$0 \neq \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \partial_\lambda \mathcal{L}_{\lambda_\diamond}\varphi_{\lambda_\diamond}(z)dz = \int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \varphi_{\lambda_\diamond}(z)dz. \qquad \blacksquare$$

Remark 5.7. Let $f \in C_b(\mathbb{R}, \mathbb{C}^{2n})$. The Fredholm alternative in [86, Lemma 4.2] states that the inhomogeneous equation,

$$\partial_x \varphi = \mathcal{A}_{22,0}(x, u_0, \lambda)\varphi + f(x), \quad \varphi \in \mathbb{C}^{2n},$$

has a bounded solution if and only if the solvability condition,

$$\int_{-\infty}^{\infty} \psi(x)^* f(x)dx = 0,$$

is satisfied for all bounded solutions ψ to the adjoint equation (3.14). This agrees with the fact that $\mathcal{X}_{in}(x, u_0, \cdot)$ has a removable singularity at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$ if and only if we have

$$\int_{-\infty}^{\infty} \psi_{\lambda_\diamond}(z)^* \mathcal{A}_{21,0}(z, u_0)dz = 0,$$

by the fourth assertion in Proposition 5.5. ■

Remark 5.8. It is possible to obtain expressions for the singular part of the Laurent series of $\mathcal{X}_{in}$ at a zero of $\mathcal{E}_{f,0}$ of higher multiplicity by looking at a canonical system of generalized eigenfunctions. However, for simplicity of exposition we restrict ourselves to the (generic) case of a simple zero of $\mathcal{E}_{f,0}$. The interested reader is referred to [74, Chapter 1] for the general set-up. ■

Using Proposition 5.5, we prove Proposition 3.11.

Proof of Proposition 3.11. Assumption **(S1)**, estimate (2.6) and Proposition 5.5 yield that $\partial_u H_2(u_0, v_\mathrm{h}(\cdot, u_0))$ and $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ are exponentially localized for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Thus, the integral $\mathcal{G}(u_0, \lambda)$ converges for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$ and $\mathcal{E}_{s,0}$ is well-defined.

It is well-known [60, Lemma 2.1.4] that, when the coefficient matrix depends analytically on a parameter, then the evolution is analytic in this parameter too. Combining this with Proposition 5.5 yields the first two properties.

Since the solution $\psi_s(\check{x})$ to the slow reduced system (2.4) crosses $\ker(I - R_s)$ at $\check{x} = \ell_0$ by **(E2)**, the slow eigenvalue problem (3.9) is R_s-reversible at $\check{x} = \ell_0$, i.e. the evolution $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ of (3.9) satisfies $R_s\mathcal{T}_s(2\ell_0, 0, \lambda)R_s = \mathcal{T}_s(0, 2\ell_0, \lambda)$ for each $\lambda \in \mathbb{C}$. Moreover, we have $R_s\Upsilon(u_0, \lambda) = \Upsilon(u_0, \lambda)^{-1}R_s$ and the matrices $\Upsilon(u_0, \lambda)$ and $\mathcal{T}_s(2\ell_0, 0, \lambda)$ have determinant 1 for any $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. This yields the third property.

Proving the fourth property is more elaborate. We denote by $C > 0$ a constant, which is independent of λ and γ. Putting $\check{y} = \sqrt{|\lambda|}\check{x}$ and $p = \sqrt{|\lambda| D_1}r$ rescales the slow eigenvalue problem (3.9) into

$$\begin{aligned} \sqrt{D_1}u_{\check{y}} &= r, \\ \sqrt{D_1}r_{\check{y}} &= \left(\frac{\partial_u H_1\left(u_s\left(|\lambda|^{-1/2}\check{y}\right), 0, 0\right)}{|\lambda|} + \frac{\lambda}{|\lambda|}\right)u, \end{aligned} \qquad (u, r) \in \mathbb{C}^{2m}, \lambda \in \mathbb{C} \setminus \{0\}. \tag{5.5}$$

Denote by $\mathcal{T}_{s1}(\check{y}, \check{z}, \lambda)$ the evolution operator of system (5.5). It holds

$$C_\lambda \Upsilon_1(\lambda)\mathcal{T}_{s1}\left(2\sqrt{|\lambda|}\ell_0, 0, \lambda\right)C_\lambda^{-1} = \Upsilon(u_0, \lambda)\mathcal{T}_s(2\ell_0, 0, \lambda), \quad \lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0), \tag{5.6}$$

with

$$C_\lambda := \begin{pmatrix} I & 0 \\ 0 & \sqrt{|\lambda| D_1} \end{pmatrix}, \quad \Upsilon_1(\lambda) := \begin{pmatrix} I & 0 \\ (|\lambda| D_1)^{-\frac{1}{2}}\mathcal{G}(u_0, \lambda) & I \end{pmatrix}.$$

We regard system (5.5) as a perturbation of

$$\begin{aligned} \sqrt{D_1}u_{\check{y}} &= r, \\ \sqrt{D_1}r_{\check{y}} &= \frac{\lambda}{|\lambda|}u, \end{aligned} \qquad (u, r) \in \mathbb{C}^{2m}, \lambda \in \mathbb{C} \setminus \{0\}. \tag{5.7}$$

Consider the set Σ_K defined in (5.4). Clearly, there exists a $K > 0$ such that (5.7) has for each $\lambda \in \Sigma_K$ an exponential dichotomy on $\mathbb{R}$ with λ-independent constants and rank m projections,

$$P_1(\lambda) = \frac{1}{2}\begin{pmatrix} I & -\frac{|\lambda|}{\lambda}I \\ -\frac{\lambda}{|\lambda|}I & I \end{pmatrix}. \tag{5.8}$$

Taking $K > 0$ larger if necessary, Proposition 4.12 implies that (5.5) admits an exponential dichotomy on $[0, 2\sqrt{|\lambda|}\ell_0]$ with λ-independent constants and corresponding projections $P_2(x, \lambda)$ satisfying

$$\|P_2(x, \lambda) - P_1(\lambda)\| \leq \frac{C}{|\lambda|}, \quad x \in \left[0, 2\sqrt{|\lambda|}\ell_0\right], \lambda \in \Sigma_K. \tag{5.9}$$

One readily observes from (5.8) that there exists bases $B_1^{u,s}(\lambda) \in \mathrm{Mat}_{2m\times m}(\mathbb{C})$ of $P_1(\lambda)[\mathbb{C}^{2m}] = B_1^s(\lambda)[\mathbb{C}^m]$ and $\ker(P_1(\lambda)) = B_1^u(\lambda)[\mathbb{C}^m]$ such that for each $\lambda \in \Sigma_K$ the quantity $\det(B_1^u(\lambda), B_1^s(\lambda))$ is bounded away from 0 by a λ-independent constant. Define $B_2^s(\lambda) = P_2(0,\lambda)B_1^s(\lambda)$ and $B_2^u(\lambda) = (I - P_2(2\sqrt{|\lambda|}\ell_0,\lambda))B_1^u(\lambda)$. By estimate (5.9) it holds

$$\left\|B_2^{u,s}(\lambda) - B_1^{u,s}(\lambda)\right\| \leq \frac{C}{|\lambda|}, \quad \lambda \in \Sigma_K. \tag{5.10}$$

Consider the invertible matrix,

$$\mathcal{H}(\lambda) := \left(\mathcal{T}_{s1}\left(0, 2\sqrt{|\lambda|}\ell_0, \lambda\right) B_2^u(\lambda), B_2^s(\lambda)\right), \quad \lambda \in \Sigma_K.$$

Taking $K > 0$ larger if necessary, Proposition 5.5 yields that $\mathcal{X}_{in}(\cdot, u_0, \lambda)$ is for each $\lambda \in \Sigma_K$ exponentially localized with λ-independent decay rates. Thus, by (5.10) we have

$$\left\|\left(\Upsilon_1(\lambda)\mathcal{T}_{s1}\left(2\sqrt{\lambda}\ell_0, 0, \lambda\right) - \gamma\right)\mathcal{H}(\lambda) - \left(B_1^u(\lambda), -\gamma B_1^s(\lambda)\right)\right\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \lambda \in \Sigma_K.$$

Taking determinants in the previous expression gives

$$\left\|\mathcal{E}_{s,0}(\lambda,\gamma)\det(\mathcal{H}(\lambda)) - (-\gamma)^m \det\left(B_1^u(\lambda), B_1^s(\lambda)\right)\right\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \gamma \in S^1, \lambda \in \Sigma_K,$$

using (5.6). Since $\lambda \mapsto \det(B_1^u(\lambda), B_1^s(\lambda))$ is bounded away from zero on Σ_K by a λ-independent constant and $\det(\mathcal{H}(\lambda))$ is non-zero on Σ_K, the slow Evans function $\mathcal{E}_{s,0}$ has no roots in $\Sigma_K \times S^1$, provided $K > 0$ is sufficiently large. This proves the third property, because $C_\Lambda \setminus \Sigma_K$ is bounded. □

Finally, we establish the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot,\gamma)$ at a simple zero of $\mathcal{E}_{f,0}$ and thereby prove Proposition 3.12.

Corollary 5.9. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Suppose $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$. Let $\varphi_{\lambda_\diamond} = (\varphi_{\lambda_\diamond,1}, \varphi_{\lambda_\diamond,2})$, $\psi_{\lambda_\diamond} = (\psi_{\lambda_\diamond,1}, \psi_{\lambda_\diamond,2})$, $B_{\lambda_\diamond}$ and $\mathcal{X}_{\lambda_\diamond}$ as in Proposition 5.5. Define for $\lambda \in B_{\lambda_\diamond}$*

$$\begin{aligned}
\varphi &:= \int_{-\infty}^{\infty} \partial_v H_2(u_0, v_\mathrm{h}(z))\varphi_{\lambda_\diamond,1}(z)dz \in \mathbb{C}^m,\\
\psi &:= \int_{-\infty}^{\infty} \psi_{\lambda_\diamond,2}(z)^* \partial_u G(u_0, v_\mathrm{h}(z), 0)dz \in \mathrm{Mat}_{1\times m}(\mathbb{C}),\\
\mathcal{G}_a(\lambda) &:= \int_{-\infty}^{\infty} \left[\partial_u H_2(u_0, v_\mathrm{h}(z)) + \partial_v H_2(u_0, v_\mathrm{h}(z))\mathcal{V}_{\lambda_\diamond}(z,\lambda)\right] dz \in \mathrm{Mat}_{m\times m}(\mathbb{C}),
\end{aligned} \tag{5.11}$$

where $\mathcal{V}_{\lambda_\diamond}$ denotes the upper-left $(n \times m)$-block of the $(2n \times 2m)$-matrix $\mathcal{X}_{\lambda_\diamond}$. Moreover, let $(u_i(x,\lambda), p_i(x,\lambda))$, $i = 1, \ldots, 2m$ be a fundamental set of solutions to the slow eigenvalue problem (3.9). Finally, let $C(\lambda,\gamma)$ be the cofactor matrix of

$$\mathcal{U}_a(\lambda,\gamma) := \begin{pmatrix} I & 0 \\ \mathcal{G}_a(\lambda) & I \end{pmatrix} \mathcal{T}_s(2\ell_0, 0, \lambda) - \gamma I \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C}).$$

For all $\gamma \in S^1$, the singular part of the Laurent series of $\mathcal{E}_{s,0}(\cdot,\gamma)$ at $\lambda_\diamond$ is given by

$$\frac{1}{\lambda-\lambda_\diamond}\sum_{i=1}^{2m}(\psi u_i(2\ell_0,\lambda_\diamond))\left(\varphi^{\mathrm{T}}\left[C_{ji}(\lambda_\diamond,\gamma)\right]_{j=m+1}^{2m}\right).$$

Proof. Assume $\lambda_\diamond$ is a simple zero of $\mathcal{E}_{f,0}$. Using the Laurent series of $\mathcal{X}_{in}$ provided in Proposition 5.5, we can split off the singular part of $\mathcal{G}(u_0,\lambda)$ at $\lambda_\diamond$. Indeed, we have

$$\mathcal{G}(u_0,\lambda) = \frac{1}{\lambda-\lambda_\diamond}\varphi\psi + \mathcal{G}_a(\lambda), \qquad \lambda \in B_{\lambda_\diamond}.$$

Using the multi-linearity of the determinant, we expand

$$\begin{aligned}\mathcal{E}_{s,0}(\lambda,\gamma) &= \det\left[\mathcal{U}_a(\lambda,\gamma) + \frac{1}{\lambda-\lambda_\diamond}\begin{pmatrix} 0 & 0 \\ \varphi\psi & 0\end{pmatrix}\mathcal{T}_s(2\ell_0,0,\lambda)\right]\\ &= \det(\mathcal{U}_a(\lambda,\gamma)) + \frac{1}{\lambda-\lambda_\diamond}\sum_{i=1}^{2m}(\psi u_i(2\ell_0,\lambda_\diamond))\left(\varphi^{\mathrm{T}}\left[C_{ji}(\lambda_\diamond,\gamma)\right]_{j=m+1}^{2m}\right),\end{aligned}$$

for $\lambda \in B_{\lambda_\diamond}$ and $\gamma \in S^1$. □

Remark 5.10. In the case $m = 1$, Propositions 3.25 and 3.28 imply that γ appears as a factor in the singular part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot,\gamma)$ at a simple zero $\lambda_\diamond$ of $\mathcal{E}_{f,0}$. Therefore, $\mathcal{E}_{s,0}(\cdot,\gamma)$ has a pole at $\lambda_\diamond$ for *some* $\gamma \in S^1$ if and only if $\mathcal{E}_{s,0}(\cdot,\gamma)$ has a pole at $\lambda_\diamond$ for *all* $\gamma \in S^1$. However, in the general setting of Corollary 5.9, the principal part of the Laurent expansion of $\mathcal{E}_{s,0}(\cdot,\gamma)$ is polynomial in γ. So, it could happen that $\mathcal{E}_{s,0}(\cdot,\gamma)$ has a pole at $\lambda_\diamond$ for all but a discrete set of $\gamma \in S^1$. We expect that such a (non-generic) situation occurs precisely when $\lambda_\diamond$ is a limit point of the zero set $\bigcup_{\gamma\in S^1}\{\lambda \in C_\Lambda : \mathcal{E}_{s,0}(\lambda,\gamma) = 0\}$. ■

5.2 Approximation of the roots of the Evans function

5.2.1 Introduction

In this section we prove Theorem 3.15. Our plan is to factorize the Evans function into a fast and a slow component:

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \mathcal{E}_{f,\varepsilon}(\lambda,\gamma)\mathcal{E}_{s,\varepsilon}(\lambda,\gamma), \tag{5.12}$$

and to approximate the factors by the fast and slow Evans functions $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$. The factorization (5.12) is induced by diagonalizing the full eigenvalue problem (3.3) via the Riccati transform, which is established in §4.6. Rescaling the p-coordinate in (3.3) by a factor $\sqrt{\varepsilon}$ yields the equivalent system,

$$\varphi_x = \begin{pmatrix} \sqrt{\varepsilon}\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) & \sqrt{\varepsilon}\mathcal{A}_{12,\varepsilon}(x) \\ \mathcal{A}_{21,\varepsilon}(x) & \mathcal{A}_{22,\varepsilon}(x,\lambda)\end{pmatrix}\varphi, \quad \varphi = (u,p,v,q) \in \mathbb{C}^{2(m+n)}, \tag{5.13}$$

where $\mathcal{A}_{12,\varepsilon}, \mathcal{A}_{21,\varepsilon}$ and $\mathcal{A}_{22,\varepsilon}$ are as in (3.4) and

$$\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) := \begin{pmatrix} 0 & D_1^{-1} \\ \varepsilon\left(\partial_u H_1(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) + \lambda\right) + \partial_u H_2(\hat{\phi}_{\mathrm{p},\varepsilon}(x)) & 0 \end{pmatrix}.$$

System (5.13) has the required slow-fast form (4.19) for an application of the Riccati transform. Moreover, the evolution matrices of systems (3.3) and (5.13) are similar. Therefore, it holds

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \det(\tilde{\mathcal{T}}_\varepsilon(0,-L_\varepsilon,\lambda) - \gamma\tilde{\mathcal{T}}_\varepsilon(0,L_\varepsilon,\lambda)), \tag{5.14}$$

where $\tilde{\mathcal{T}}_\varepsilon(x,y,\lambda)$ is the evolution operator of system (5.13). Yet, an application of the Riccati transformation to (5.13) is only legitimate when system,

$$\psi_x = \mathcal{A}_{22,\varepsilon}(x,\lambda)\psi, \quad \psi \in \mathbb{C}^{2n}, \tag{5.15}$$

has an exponential dichotomy on $\mathbb{R}$. If λ is not a zero of the fast Evans function, then the homogeneous fast eigenvalue problem (3.6) admits an exponential dichotomy on $\mathbb{R}$ by Proposition 5.2. Using roughness techniques the exponential dichotomy of (3.6) carries over to the perturbed problem (5.15), whenever λ is away from $\mathcal{E}_{f,0}^{-1}(0)$. In that case, system (5.13) diagonalizes via the Riccati transform. Consequently, using the periodicity of system (5.13) and identity (5.14), the Evans function $\mathcal{E}_\varepsilon$ factorizes as (5.12) for λ away from the roots of $\mathcal{E}_{f,0}$.

The two blocks, in which (5.13) diagonalizes, can be approximated in terms of the three eigenvalue problems (3.6), (3.8) and (3.9). This corresponds to approximating the factor $\mathcal{E}_{f,\varepsilon}$ by the fast Evans function $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,\varepsilon}$ by the slow Evans function $\mathcal{E}_{s,0}$. Thus, we obtain the desired approximation of the roots of $\mathcal{E}_\varepsilon$ by the zeros of the reduced Evans function $\mathcal{E}_0(\lambda,\gamma) = (-\gamma)^n\mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda,\gamma)$ using Rouché's Theorem.

This section is structured as follows. We start by showing that the spectrum of the linearization $\mathcal{L}_\varepsilon$ is contained in an ε-independent sector. This provides an important a priori bound on the magnitude of the roots of the Evans function $\mathcal{E}_\varepsilon$. Subsequently, we establish an exponential dichotomy for system (5.15) for λ away from the zeros of $\mathcal{E}_{f,0}$. Then, the Riccati transform yields the desired diagonalization of (5.13) and the factorization of $\mathcal{E}_\varepsilon$. Then, we link the factors $\mathcal{E}_{f,\varepsilon}$ and $\mathcal{E}_{s,\varepsilon}$ to $\mathcal{E}_{f,0}$ and $\mathcal{E}_{s,0}$. Finally, we apply Rouché's Theorem to conclude the proof of Theorem 3.15.

5.2.2 A priori bounds on the spectrum

In §3.2 we established the linearization $\mathcal{L}_\varepsilon$ of (1.9) about the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$. By [44, Theorem 1.3.2], the differential operator $\mathcal{L}_\varepsilon$ is sectorial as a sum of a sectorial and a bounded operator. The bounded part involves multiplication with the matrix function $\mathcal{B}_\varepsilon(x)$, defined in (3.1), which has a norm of order $\mathcal{O}(\varepsilon^{-1})$. Yet, the spectrum of $\mathcal{L}_\varepsilon$ is confined to an ε-independent sector.

Proposition 5.11. *For $\varepsilon > 0$ sufficiently small, there exists constants $\omega \in \mathbb{R}_{>0}$ and $\varpi \in (\pi/2, \pi)$, both independent of ε, such that the sector $\Sigma := \{\lambda \in \mathbb{C} : \lambda \neq \omega, |\arg(\lambda-\omega)| \leq \varpi\} \cup \{\omega\}$ is contained in the resolvent set $\rho(\mathcal{L}_\varepsilon)$.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Our approach is to decompose $\mathcal{L}_\varepsilon$ in more elementary building blocks in order to control the ε^{-1}-terms in $\mathcal{B}_\varepsilon$. First, we show that the operator $\mathcal{L}_{1,\varepsilon}\colon C^2_{ub}(\mathbb{R}, \mathbb{R}^m) \subset C_{ub}(\mathbb{R}, \mathbb{R}^m) \to C_{ub}(\mathbb{R}, \mathbb{R}^m)$ given by

$$\mathcal{L}_{1,\varepsilon} u = D_1 u_{\check{x}\check{x}} + \varepsilon^{-1} \partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot))u,$$

is sectorial with an ε-independent sector. Subsequently, we prove this for $\widehat{\mathcal{L}_\varepsilon}\colon C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \subset C_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \to C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ given by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{L}_{1,\varepsilon} u + \varepsilon^{-1} \partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot))v \\ \varepsilon^2 D_2 v_{\check{x}\check{x}} \end{pmatrix}.$$

Finally, we regard $\mathcal{L}_\varepsilon$ as a perturbation of $\widehat{\mathcal{L}_\varepsilon}$ by a bounded operator with $O(1)$-norm.

Our goal is to show that the spectrum of the periodic differential operator $\mathcal{L}_{1,\varepsilon}$ is contained in an ε-independent sector. By [38, Proposition 2.1] it is sufficient to show that the associated eigenvalue problem,

$$\begin{aligned} \sqrt{D_1} u_{\check{x}} &= \sqrt{\lambda} p, \\ \sqrt{D_1} p_{\check{x}} &= \left(\sqrt{\lambda} + \frac{\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))}{\sqrt{\lambda}\varepsilon} \right) u, \end{aligned} \qquad (u, p) \in \mathbb{C}^{2m}, \tag{5.16}$$

has no non-trivial bounded solutions for λ in some ε-independent sector. Here, $\sqrt{\cdot}$ is the principal square root. Denote by $\mathcal{T}_{1,\varepsilon}(\check{x}, \check{y}, \lambda)$ the evolution operator of system (5.16) and let $\mathcal{T}_1(\check{x}, \check{y}, \lambda)$ be the evolution operator of

$$\begin{aligned} \sqrt{D_1} u_{\check{x}} &= \sqrt{\lambda} p, \\ \sqrt{D_1} p_{\check{x}} &= \sqrt{\lambda} u, \end{aligned} \qquad (u, p) \in \mathbb{C}^{2m}. \tag{5.17}$$

One readily observes that, whenever $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, system (5.17) has an exponential dichotomy on $\mathbb{R}$ with constants $C_0, \mu_\lambda > 0$ with $C_0 = 1$ and $\mu_\lambda := \|D_1\|^{-1/2} \mathrm{Re}(\sqrt{\lambda})$. Since we have $|\varepsilon L_\varepsilon - \ell_0| < C\varepsilon$ by Theorem 2.3, there exists a constant $C_1 > 0$, independent of ε, such that for all $\lambda \in P_1 := \{\mu \in \mathbb{C} : \mathrm{Re}(\sqrt{\mu}) \geq C_1\}$ it holds $h_\lambda := \mu_\lambda^{-1} \sinh^{-1}(4) \leq \ell_\varepsilon := \varepsilon L_\varepsilon$.

Let $\lambda \in P_1$. Using **(S1)** and Theorem 2.3 we estimate

$$\|\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))\| \leq C e^{-\varepsilon^{-1}\mu_0|\check{x}|}, \quad \check{x} \in [-\ell_\varepsilon, \ell_\varepsilon].$$

Let $\check{w}, \check{z} \in \mathbb{R}$ such that $0 \leq \check{z} - \check{w} \leq 2h_\lambda \leq 2\ell_\varepsilon$. Taking into account the $2\ell_\varepsilon$-periodicity of $\check{\phi}_{\mathrm{p},\varepsilon}$, we have

$$\int_{\check{w}}^{\check{z}} \frac{\|\partial_u H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}))\|}{\sqrt{|\lambda|}\varepsilon} d\check{x} \leq \frac{C}{\sqrt{|\lambda|}}.$$

Thus, by Proposition 4.1, we establish

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda) - \mathcal{T}_{1,\varepsilon}(\check{z}, \check{w}, \lambda)\| \leq \frac{C}{\sqrt{|\lambda|}}, \quad \check{w}, \check{z} \in \mathbb{R} \text{ with } |\check{w} - \check{z}| \leq 2h_\lambda,$$

where we use that the evolution operator of (5.17) satisfies

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda)\| \leq C e^{\mathrm{Re}(\sqrt{\lambda})|\check{z}-\check{w}|}, \quad \check{w}, \check{z} \in \mathbb{R}.$$

So, there exists an ε-independent constant $C_2 > 0$ such that, whenever $\lambda \in P_1$ satisfies $|\lambda| > C_2$, then it holds

$$\|\mathcal{T}_1(\check{z}, \check{w}, \lambda) - \mathcal{T}_{1,\varepsilon}(\check{z}, \check{w}, \lambda)\| < 1, \quad \check{w}, \check{z} \in \mathbb{R} \text{ with } |\check{w} - \check{z}| \leq 2h_\lambda. \tag{5.18}$$

Now let Σ_1 be an ε-independent sector disjoint from $B(0, C_2) \cup [\mathbb{C} \setminus P_1]$ – see Figure 5.1. For all $\lambda \in \Sigma_1$, there are no non-trivial, bounded solutions to (5.16) by combining (5.18) with Proposition 4.14. So, by [38, Proposition 2.1] the resolvent set $\rho(\mathcal{L}_{1,\varepsilon})$ contains the ε-independent sector Σ_1.

Consider the elliptic operator $\mathcal{L}_2 \colon C^2_{ub}(\mathbb{R}, \mathbb{R}^n) \subset C_{ub}(\mathbb{R}, \mathbb{R}^n) \to C_{ub}(\mathbb{R}, \mathbb{R}^n)$ given by $\mathcal{L}_2 v = D_2 v_{\check{x}\check{x}}$. Clearly, we have $\rho(\mathcal{L}_2) = \mathbb{C} \setminus \mathbb{R}_{\leq 0} \supset \Sigma_1$. For $\lambda \in \Sigma_1$ the operator on $C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} (\mathcal{L}_{1,\varepsilon} - \lambda)^{-1}(u - \varepsilon^{-1}\partial_v H_2(\check{\phi}_{\mathrm{p},\varepsilon}(\cdot))(\varepsilon^2\mathcal{L}_2 - \lambda)^{-1}(v)) \\ (\varepsilon^2\mathcal{L}_2 - \lambda)^{-1}(v) \end{pmatrix},$$

is an inverse of $\widehat{\mathcal{L}_\varepsilon} - \lambda$. Therefore, the sector Σ_1 is contained in the resolvent set $\rho(\widehat{\mathcal{L}_\varepsilon})$.

Define

$$\mathcal{B}_{b,\varepsilon}(\check{x}) = \begin{pmatrix} \partial_u H_1(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) & \partial_v H_1(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) \\ \partial_u G(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) & \partial_v G(\check{\phi}_{\mathrm{p},\varepsilon}(\check{x}), \varepsilon) \end{pmatrix}.$$

Let $\mathcal{L}_{b,\varepsilon} \colon C_{ub}(\mathbb{R}, \mathbb{R}^{m+n}) \to C_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ be the multiplication operator $[\mathcal{L}_{b,\varepsilon}\varphi](\check{x}) = \mathcal{B}_{b,\varepsilon}(\check{x})\varphi$. By Theorem 2.3 the norm of $\mathcal{L}_{b,\varepsilon}$ is bounded by an ε-independent constant.

Invoking [44, Theorem 1.3.2] and its proof yields the conclusion: the sum $\mathcal{L}_\varepsilon = \widehat{\mathcal{L}_\varepsilon} + \mathcal{L}_{b,\varepsilon}$ with domain $C^2_{ub}(\mathbb{R}, \mathbb{R}^{m+n})$ is sectorial with an ε-independent sector $\Sigma \subset \rho(\mathcal{L}_\varepsilon)$, using that $\|\mathcal{L}_{b,\varepsilon}\|$ is bounded by an ε-independent constant and Σ_1 is independent of ε. □

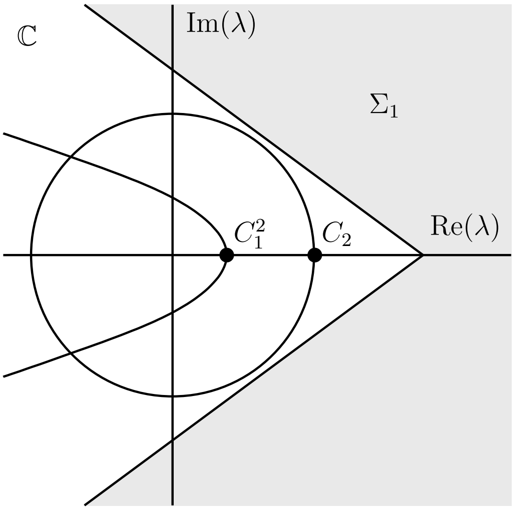


Figure 5.1: Construction of the sector Σ_1 in the proof of Proposition 5.11.

Let $\gamma \in S^1$. By Propositions 3.7 and 5.11 the roots of the Evans function $\mathcal{E}_\varepsilon(\cdot,\gamma)$ in the half-plane C_Λ are confined to an ε- and γ-independent bounded region. In addition, by Propositions 5.2 and 3.11 the same holds for the zeros of the reduced Evans function $\mathcal{E}_0(\cdot,\gamma)$. Thus, we have established the following result.

Corollary 5.12. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. There exists an open and bounded set $\Sigma_{\Lambda,0} \subset C_\Lambda$ such that*

$$\bigcup_{\gamma \in S^1} \{\lambda \in C_\Lambda : \mathcal{E}_0(\lambda,\gamma) = 0 \text{ or } \mathcal{E}_\varepsilon(\lambda,\gamma) = 0\} \subset \Sigma_{\Lambda,0}.$$

Thus, when proving Theorem 3.15, we may without loss of generality restrict ourselves to the set $\Sigma_{\Lambda,0}$ by the a priori bounds in Corollary 5.12.

5.2.3 An exponential dichotomy capturing the fast dynamics

We wish to apply the Riccati transformation to the rescaled full eigenvalue problem (5.13) in order to factorize $\mathcal{E}_\varepsilon$ into a fast and a slow part as in (5.12). However, according to Theorem 4.19 this is only legitimate, when system (5.15) has an exponential dichotomy on $\mathbb{R}$. By Proposition 5.2 the homogeneous fast eigenvalue problem (3.6) admits an exponential dichotomy on $\mathbb{R}$, whenever $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Using roughness techniques the exponential dichotomy of (3.6) carries over to the perturbed problem (5.15). Therefore, we establish the following result.

Notation 5.13. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1 and $\Sigma_{\Lambda,0} \subset C_\Lambda$ as in Corollary 5.12. For $\delta > 0$, we denote

$$\Sigma_{\Lambda,\delta} := \Sigma_{\Lambda,0} \setminus \bigcup_{\lambda \in \mathcal{E}_{f,0}^{-1}(0)} B(\lambda, \delta).$$

Theorem 5.14. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, systems* (3.6) *and* (5.15) *have for all $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with ε- and λ-independent constants $C, \mu_f > 0$.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ and ε.

Our approach is as follows. First, we establish an exponential dichotomy for (5.15) on an interval $[a, 2L_\varepsilon - a]$ for some $a > 0$, using that the coefficient matrix $\mathcal{A}_{22,\varepsilon}(x, \lambda)$ has slowly varying coefficients and is pointwise hyperbolic along the slow manifold. We extend the exponential dichotomy to $[0, 2L_\varepsilon]$. Similarly, we obtain an exponential dichotomy for (5.15) on $[-2L_\varepsilon, 0]$.

Subsequently, we calculate the minimal opening between the kernels and ranges of the dichotomy projections at 0. By approximating system (5.15) by the fast eigenvalue problem (3.6), we show that, whenever λ is contained in $\Sigma_{\Lambda,\delta}$, this minimal opening is substantial. Therefore, Lemma 4.11 provides exponential dichotomies for (5.15) on $[-2L_\varepsilon, 2L_\varepsilon]$ and for (3.6) on $\mathbb{R}$. Finally, we extend the exponential dichotomy of (5.15) to $\mathbb{R}$.

We start by establishing exponential dichotomies for (5.15) on $[0, 2L_\varepsilon]$ and on $[-2L_\varepsilon, 0]$. Theorem 2.3 yields the following estimates,

$$\begin{aligned} \|v_{\mathrm{p},\varepsilon}(x)\| &\leq Ce^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}, \\ \|v'_{\mathrm{p},\varepsilon}(x)\| = \|D_2^{-1} q_{\mathrm{p},\varepsilon}(x)\| &\leq Ce^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}, \qquad x \in [0, 2L_\varepsilon], \\ \|u'_{\mathrm{p},\varepsilon}(x)\| = \varepsilon \left\|D_1^{-1} p_{\mathrm{p},\varepsilon}(x)\right\| &\leq C\varepsilon, \end{aligned}$$

which imply

$$\left\|\partial_x \mathcal{A}_{22,\varepsilon}(x, \lambda)\right\|, \left\|\mathcal{A}_{22,\varepsilon}(x, \lambda) - A(u_{\mathrm{p},\varepsilon}(x), \lambda)\right\| \leq C \max\left\{\varepsilon, e^{-\mu_0 \min\{x, 2L_\varepsilon - x\}}\right\}, \tag{5.19}$$

for $x \in [0, 2L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$, where $A(u, \lambda)$ is defined in (5.1). First, by Theorem 2.3 and Lemma 5.1, there exists an ε-independent constant $\alpha > 0$ such that, for $\varepsilon > 0$ sufficiently small, the matrix $A(u_{\mathrm{p},\varepsilon}(x), \lambda)$ is hyperbolic for each $x \in [0, 2L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$ with spectral gap larger than 2α. Thus, by estimate (5.19) there exists $x_0 > 0$, independent of ε, such that $\mathcal{A}_{22,\varepsilon}$ is hyperbolic on $[x_0, 2L_\varepsilon - x_0] \times \Sigma_{\Lambda,\delta}$ with spectral gap larger than α. Second, $\mathcal{A}_{22,\varepsilon}$ is bounded on $[0, 2L_\varepsilon] \times \Sigma_{\Lambda,\delta}$ by an ε-independent constant using Theorem 2.3. Thus, taking $x_0 > 0$ larger if necessary, Proposition 4.8 and (5.19) yield, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy for system (5.15) on $[x_0, 2L_\varepsilon - x_0]$ with ε- and λ-independent constants. Using Lemma 4.9 we extend this to an exponential dichotomy on $[0, 2L_\varepsilon]$ with constants independent of ε and λ. Similarly, we obtain an exponential dichotomy for (5.15) on $[-2L_\varepsilon, 0]$. We conclude that (5.15) has exponential dichotomies on both $[0, 2L_\varepsilon]$ and $[-2L_\varepsilon, 0]$ for every $\lambda \in \Sigma_{\Lambda,\delta}$ with constants $C, \alpha_f > 0$, independent of ε and λ.

We compare system (5.15) with the homogeneous fast eigenvalue problem (3.6). First, by Theorem 2.3 and estimate (2.6), the corresponding coefficient matrices $\mathcal{A}_{22,\varepsilon}$ and $\mathcal{A}_{22,0}(\cdot, u_0, \cdot)$ are bounded on $\mathbb{R} \times \Sigma_{\Lambda,\delta}$ by a constant $M > 1$, which is independent of ε. Second, Theorem 2.3 yields

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \mathcal{A}_{22,0}(x,u_0,\lambda)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [\log(\varepsilon), -\log(\varepsilon)], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.20}$$

Denote by $\mathcal{T}_r(x,y,\lambda)$ and $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ the evolution operators of (3.6) and (5.15), respectively. Using Lemma 4.1 and (5.20) we estimate

$$\|\mathcal{T}_r(x,y,\lambda) - \mathcal{T}_{f,\varepsilon}(x,y,\lambda)\| < 1, \quad x, y \in [\log(\varepsilon)/4M, -\log(\varepsilon)/4M], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.21}$$

We recall some facts from Proposition 5.2. First, system (3.6) admits for each $\lambda \in C_\Lambda$ exponential dichotomies on both half-lines with constants that depend continuously on λ. Second, the corresponding projections $Q_{f,\pm}(x,\lambda)$ are analytic in λ. Third, the subspaces $E_0^s(\lambda) := Q_{f,+}(0,\lambda)[\mathbb{C}^{2n}]$ and $E_0^u(\lambda) := \ker(Q_{f,-}(0,\lambda))$ are complementary for each $\lambda \in C_\Lambda \setminus \mathcal{E}_{f,0}^{-1}(0)$. Therefore, Proposition 4.18 implies that the continuous map $\eta_r \colon C_\Lambda \to [0,\infty)$ given by the minimal opening $\eta_r(\lambda) = \eta(E_0^s(\lambda), E_0^u(\lambda))$ is bounded away from 0 on the set $\Sigma_{\Lambda,\delta}$. Hence, the projection on $E_0^s(\lambda)$ along $E_0^u(\lambda)$ is well-defined on $\Sigma_{\Lambda,\delta}$ and bounded by a λ-independent constant by Proposition 4.18. Thus, Lemma 4.11 yields for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy of the homogeneous fast eigenvalue problem (3.6) on $\mathbb{R}$ with λ-independent constants.

Denote by $Q_{\pm,\varepsilon}(x,\lambda)$ the projections corresponding to the exponential dichotomies of (5.15) on $[0, 2L_\varepsilon]$ and on $[-2L_\varepsilon, 0]$. Let $\lambda \in \Sigma_{\Lambda,\delta}$. By combining estimate (5.21) with Lemma 4.13, there exists for each $w \in E_\varepsilon^s(\lambda) := Q_{+,\varepsilon}(0,\lambda)[\mathbb{C}^{2n}]$ an element $v \in E_0^s(\lambda)$ such that

$$\|v - w\| \leq C\varepsilon^{\alpha_f/4M}\|w\|. \tag{5.22}$$

Similarly, there exists for each $w \in E_\varepsilon^u(\lambda) := \ker(Q_{-,\varepsilon}(0,\lambda))$ a vector $v \in E_0^u(\lambda)$ such that (5.22) holds true. Therefore, Proposition 4.18 yields the estimate

$$\left|\eta_r(\lambda) - \eta(E_\varepsilon^s(\lambda), E_\varepsilon^u(\lambda))\right| \leq C\varepsilon^{\alpha_f/4M}, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.23}$$

Finally, we establish the desired exponential dichotomy for (5.15) on $\mathbb{R}$. Recall that the map η_r is bounded away from 0 on $\Sigma_{\Lambda,\delta}$. Thus, by estimate (5.23) and Proposition 4.18 one deduces that, for $\varepsilon > 0$ sufficiently small, $E_\varepsilon^s(\lambda)$ and $E_\varepsilon^u(\lambda)$ are complementary on $\Sigma_{\Lambda,\delta}$. So, the projection $Q_\varepsilon(\lambda)$ onto $E_\varepsilon^s(\lambda)$ along $E_\varepsilon^u(\lambda)$ is well-defined for $\lambda \in \Sigma_{\Lambda,\delta}$. In addition, by Proposition 4.18 and (5.23), the norm of Q_ε is bounded on $\Sigma_{\Lambda,\delta}$ by an ε-independent constant. Therefore, Lemma 4.11 implies that (5.15) admits an exponential dichotomy for each $\lambda \in \Sigma_{\Lambda,\delta}$ on $[-2L_\varepsilon, 2L_\varepsilon]$ with λ- and ε-independent constants. Subsequently, for each

$\lambda \in \Sigma_{\Lambda,\delta}$, Lemma 4.10 yields an exponential dichotomy for system (5.15) on $\mathbb{R}$ with λ- and ε-independent constants, where we use that the coefficient matrix $\mathcal{A}_{22,\varepsilon}$ is ε-uniformly bounded on $\mathbb{R} \times \Sigma_{\Lambda,\delta}$. □

In Theorem 5.14 we established exponential dichotomies on $\mathbb{R}$ for the homogeneous fast eigenvalue problem (3.6) and its perturbation (5.15). This enables us to compare solutions to the inhomogeneous fast eigenvalue problem (3.8) and its perturbation,

$$\Psi_x = \mathcal{A}_{22,\varepsilon}(x,\lambda)\Psi + \mathcal{A}_{21,\varepsilon}(x), \quad \Psi \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}). \tag{5.24}$$

Corollary 5.15. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For each $\lambda \in \Sigma_{\Lambda,\delta}$, there exists a unique bounded solution $\Psi_\varepsilon(x,\lambda)$ to (5.24) satisfying*

$$\|\Psi_\varepsilon(x,\lambda) - \mathcal{X}_{in}(x,u_0,\lambda)\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [-L_\varepsilon, L_\varepsilon], \lambda \in \Sigma_{\Lambda,\delta},$$

where $C > 0$ is a λ- and ε-independent constant.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Systems (3.6) and (5.15) have by Theorem 5.14 for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with constants $C, \mu_f > 0$, which are independent of ε and λ. Let $\mu_0 > 0$ be as in Theorem 2.3 and take $\chi := 2/\min\{\mu_f, \mu_0\}$. Theorem 2.3 yields

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \mathcal{A}_{22,0}(x,u_0,\lambda)\right\|, \left\|\mathcal{A}_{21,\varepsilon}(x) - \mathcal{A}_{21,0}(x,u_0)\right\| \leq C\varepsilon|\log(\varepsilon)|,$$

for $x \in [2\chi\log(\varepsilon), -2\chi\log(\varepsilon)]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Now, we apply Proposition 4.15 to the inhomogeneous equations (3.8) and (5.24): there exists a unique bounded solution $\Psi_\varepsilon(x,\lambda)$ to (5.24) satisfying

$$\|\Psi_\varepsilon(x,\lambda) - \mathcal{X}_{in}(x,u_0,\lambda)\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in [\chi\log(\varepsilon), -\chi\log(\varepsilon)], \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.25}$$

where we use that $\mathcal{A}_{22,\varepsilon}, \mathcal{A}_{22,0}(\cdot,u_0,\cdot)$ and $\mathcal{X}_{in}(\cdot,u_0,\cdot)$ are ε-uniformly bounded on $\mathbb{R} \times \Sigma_{\Lambda,\delta}$ and $\mathcal{A}_{21,\varepsilon}$ and $\mathcal{A}_{21,0}(\cdot,u_0)$ are ε-uniformly bounded on $\mathbb{R}$ by Theorem 2.3 and Proposition 5.5. Furthermore, by Theorem 2.3, estimate (2.6) and **(S1)** we have

$$\begin{aligned} \|\mathcal{A}_{21,\varepsilon}(x)\| &\leq Ce^{-\mu_0|x|}, \quad x \in [-L_\varepsilon, L_\varepsilon], \\ \|\mathcal{A}_{21,0}(x,u_0)\| &\leq Ce^{-\mu_h|x|}, \quad x \in \mathbb{R}, \end{aligned} \qquad \lambda \in \Sigma_{\Lambda,\delta}.$$

Combing the latter with Proposition 4.15 implies

$$\begin{aligned} \|\Psi_\varepsilon(x,\lambda)\| &\leq Ce^{-\min\{\mu_f,\mu_0\}|x|/2}, \quad x \in [-L_\varepsilon, L_\varepsilon]. \\ \|\mathcal{X}_{in}(x,u_0,\lambda)\| &\leq Ce^{-\min\{\mu_f,\mu_h\}|x|/2}, \quad x \in \mathbb{R}, \end{aligned} \qquad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.26}$$

which proves that (5.25) actually holds for all $x \in [-L_\varepsilon, L_\varepsilon]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. □

5.2.4 Factorization of the Evans function via the Riccati transform

We employ the Riccati transform to diagonalize the rescaled full eigenvalue problem (5.13). This yields the factorization (5.12) of the Evans function.

Theorem 5.16. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, there exists a function $U_\varepsilon \colon \mathbb{R} \times \Sigma_{\Lambda,\delta} \to \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ such that we have the factorization,*

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \mathcal{E}_{s,\varepsilon}(\lambda,\gamma)\mathcal{E}_{f,\varepsilon}(\lambda,\gamma), \quad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in \mathbb{C},$$

with $\mathcal{E}_{s,\varepsilon}, \mathcal{E}_{f,\varepsilon} \colon \Sigma_{\Lambda,\delta} \times \mathbb{C} \to \mathbb{C}$ given by

$$\begin{aligned}\mathcal{E}_{s,\varepsilon}(\lambda,\gamma) &:= \det(\mathcal{T}_{sd,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{sd,\varepsilon}(0,L_\varepsilon,\lambda)),\\ \mathcal{E}_{f,\varepsilon}(\lambda,\gamma) &:= \det(\mathcal{T}_{fd,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{fd,\varepsilon}(0,L_\varepsilon,\lambda)).\end{aligned}$$

where $\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)$ is the evolution operator of system,

$$\chi_x = \sqrt{\varepsilon}\left(\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda)\right)\chi, \quad \chi \in \mathbb{C}^{2m}, \tag{5.27}$$

and $\mathcal{T}_{fd,\varepsilon}(x,y,\lambda)$ is the evolution operator of system,

$$\omega_x = \left(\mathcal{A}_{22,\varepsilon}(x,\lambda) - \sqrt{\varepsilon}U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x)\right)\omega, \quad \omega \in \mathbb{C}^{2n}. \tag{5.28}$$

In addition, U_ε enjoys the following properties:

1. *U_ε is bounded by an ε-independent constant on its domain $\mathbb{R} \times \Sigma_{\Lambda,\delta}$;*
2. *$U_\varepsilon(\cdot,\lambda)$ is $2L_\varepsilon$-periodic for each $\lambda \in \Sigma_{\Lambda,\delta}$;*
3. *Take*

$$\Xi_\varepsilon := -\frac{12\log(\varepsilon)}{\min\{\mu_{\mathrm{h}}, \mu_0, \mu_f\}},$$

where $\mu_{\mathrm{h}} > 0$ is as in (2.6), $\mu_0 > 0$ is as in Theorem 2.3 and $\mu_f > 0$ is as in Theorem 5.14. It holds,

$$\left.\begin{aligned}\|U_\varepsilon(x,\lambda) - \mathcal{X}_{in}(x,u_0,\lambda)\| &\le C\sqrt{\varepsilon}|\log(\varepsilon)|, & x &\in [0,2L_\varepsilon],\\ \|U_\varepsilon(x,\lambda)\| &\le C\varepsilon^3, & x &\in [\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon],\end{aligned}\right\} \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.29}$$

where $C > 0$ is a λ- and ε-independent constant.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of λ and ε.

System (5.13) is clearly of the slow-fast form (4.19) with coefficient matrices that are ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ by Theorem 2.3. Furthermore, by Theorem 5.14, system (5.15) admits for every $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with λ- and ε-independent constants $C, \mu_f > 0$. Hence, we can apply the Riccati transform to (5.13). Thus, Theorem 4.19 yields matrix functions $H_\varepsilon(x,\lambda) \in \mathrm{Mat}_{2(m+n)\times 2(m+n)}(\mathbb{C})$ and $U_\varepsilon(x,\lambda) \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C})$ such that the change of variables $\varphi(x) = H_\varepsilon(x,\lambda)\psi(x)$ transforms (5.13) into the diagonal system,

$$\psi_x = \begin{pmatrix} \sqrt{\varepsilon}\left(\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda)\right) & 0 \\ 0 & \mathcal{A}_{22,\varepsilon}(x,\lambda) - \sqrt{\varepsilon}U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x) \end{pmatrix}\psi, \quad (5.30)$$

with $\psi \in \mathbb{C}^{2(m+n)}$. The evolution $\mathcal{T}_{d,\varepsilon}(x,y,\lambda)$ of system (5.30) is a block-diagonal matrix with consecutively $\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)$ and $\mathcal{T}_{fd,\varepsilon}(x,y,\lambda)$ on its diagonal. Furthermore, $H_\varepsilon(\cdot,\lambda)$ and $U_\varepsilon(\cdot,\lambda)$ are $2L_\varepsilon$-periodic by Theorem 4.19 for any $\lambda \in \Sigma_{\Lambda,\delta}$. Finally, since $H_\varepsilon(x,\lambda)$ is a product of two triangular matrices with only ones on the diagonal by (4.23), the determinant of $H_\varepsilon(x,\lambda)$ equals 1 for every $(x,\lambda) \in \mathbb{R}\times\Sigma_{\Lambda,\delta}$. Therefore, we obtain the factorization,

$$\mathcal{E}_\varepsilon(\lambda,\gamma) = \det\left(H_\varepsilon(0,\lambda)\left[\mathcal{T}_{d,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{d,\varepsilon}(0,L_\varepsilon,\lambda)\right]H_\varepsilon(L_\varepsilon,\lambda)^{-1}\right) = \mathcal{E}_{s,\varepsilon}(\lambda,\gamma)\mathcal{E}_{f,\varepsilon}(\lambda,\gamma),$$

where we use that the Evans function can be expressed as (5.14).

We establish the above properties of U_ε. The first two properties follow immediately from Theorem 4.19. Furthermore, combining (4.25) with Corollary 5.15, settles the first estimate in (5.29). For the second estimate in (5.29) we use the method of successive approximation. Theorem 2.3 and **(S1)** yield

$$\|\mathcal{A}_{21,\varepsilon}(x)\| \leq Ce^{-\mu_0|x|}, \quad x \in [-L_\varepsilon, L_\varepsilon]. \quad (5.31)$$

Because U_ε is ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$, estimates (4.26) and (5.31) yield $\|U_\varepsilon(x,\lambda)\| \leq C\sqrt{\varepsilon}$ for $x \in [\Xi_\varepsilon/4, 2L_\varepsilon - \Xi_\varepsilon/4]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Thus, employing (4.26) and (5.31) again gives $\|U_\varepsilon(x,\lambda)\| \leq C\varepsilon\sqrt{\varepsilon}$ for $x \in [\Xi_\varepsilon/2, 2L_\varepsilon - \Xi_\varepsilon/2]$ and $\lambda \in \Sigma_{\Lambda,\delta}$. Finally, a third application of (4.26) and (5.31) leads to the second estimate in (5.29). □

Theorem 5.16 provides a diagonalization of the rescaled full eigenvalue problem (5.13) into two lower-dimensional problems (5.27) and (5.28). The diagonalization yields a factorization of the Evans function $\mathcal{E}_\varepsilon$ into two factors $\mathcal{E}_{s,\varepsilon}$ and $\mathcal{E}_{f,\varepsilon}$. By relating (5.27) and (5.28) to the three eigenvalue problems (3.6), (3.8) and (3.9), we link $\mathcal{E}_{s,\varepsilon}$ to the slow Evans function $\mathcal{E}_{s,0}$ and $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$.

First, we consider problem (5.27). Along the pulse, the transformation matrix $U_\varepsilon(x,\lambda)$ is approximated by the solution $\mathcal{X}_{in}(x,u_0,\lambda)$ to the inhomogeneous fast eigenvalue problem (3.8). On the other hand, along the slow manifold, U_ε is small and system (5.27) is a perturbation of the slow eigenvalue problem (3.9). Thus, we observe that both (3.8) and (3.9) govern the leading-order dynamics in system (5.27). This leads to the following approximation result.

Lemma 5.17. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. For $\varepsilon > 0$ sufficiently small, the map $\mathcal{E}_{s,\varepsilon}$, defined in Theorem 5.16, is approximated as*

$$\left|\mathcal{E}_{s,\varepsilon}(\lambda,\gamma) - \mathcal{E}_{s,0}(\lambda,\gamma)\right| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in S^1, \quad (5.32)$$

where $C > 0$ is a constant, which is independent of λ and ε.

Proof. In the following, Ξ_ε is as in Theorem 5.16 and $C > 0$ is a constant, which is independent of ε and λ.

Our approach is as follows. We introduce a splitting of the coefficient matrix of system (5.27) that is consistent with the decay behavior along the slow manifold, i.e. we write

$$\tilde{\mathcal{A}}_{11,\varepsilon}(x,\lambda) + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda) = \mathcal{B}_{1,\varepsilon}(x,\lambda) + \mathcal{B}_{2,\varepsilon}(x,\lambda),$$

with

$$\mathcal{B}_{1,\varepsilon}(x,\lambda) := \begin{pmatrix} 0 & D_1^{-1} \\ \varepsilon\left(\partial_u H_1(u_{\mathrm{p},\varepsilon}(x),0,\varepsilon)+\lambda\right) & 0 \end{pmatrix},$$
$$\mathcal{B}_{2,\varepsilon}(x,\lambda) := \begin{pmatrix} 0 & 0 \\ B_{2,\varepsilon}(x) & 0 \end{pmatrix} + \mathcal{A}_{12,\varepsilon}(x)U_\varepsilon(x,\lambda),$$
$$B_{2,\varepsilon}(x) := \partial_u H_2(u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x)) + \varepsilon\left(\partial_u H_1(u_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x),\varepsilon) - \partial_u H_1(u_{\mathrm{p},\varepsilon}(x),0,\varepsilon)\right).$$

Theorems 2.3 and 5.16 and assumption **(S1)** imply that $\mathcal{B}_{1,\varepsilon}$ and $\mathcal{B}_{2,\varepsilon}$ are ε-uniformly bounded on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$ and it holds

$$\|\mathcal{B}_{2,\varepsilon}(x,\lambda)\| \leq C\varepsilon^3, \quad x\in[\Xi_\varepsilon, 2L_\varepsilon-\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}. \tag{5.33}$$

The splitting gives rise to an intermediate system,

$$\chi_x = \sqrt{\varepsilon}\mathcal{B}_{1,\varepsilon}(x,\lambda)\chi, \quad \chi\in\mathbb{C}^{2m}. \tag{5.34}$$

On the one hand, system (5.34) is a perturbation of (a rescaled version of) the slow eigenvalue problem (3.9). On the other hand, (5.27) and (5.34) are related via the variation of constants formula. This leads to the desired approximation of $\mathcal{E}_{s,\varepsilon}$ by $\mathcal{E}_{s,0}$ on $\Sigma_{\Lambda,\delta}$.

First, we relate systems (5.27) and (5.34) via the variation of constants formula. Denote by $\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)$ and $\mathcal{T}_{is,\varepsilon}(x,y,\lambda)$ the evolution operators of system (5.27) and (5.34), respectively. Lemma 4.1 gives the estimate,

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda) - I\|, \|\mathcal{T}_{is,\varepsilon}(x,y,\lambda) - I\| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|, \quad x,y\in[-\Xi_\varepsilon,\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}. \tag{5.35}$$

On the other hand, upon rescaling the p-coordinate in (5.34), one obtains the Grönwall estimates,

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda)\|, \|\mathcal{T}_{is,\varepsilon}(x,y,\lambda)\| \leq \frac{C}{\sqrt{\varepsilon}} e^{\varepsilon\mu_s|x-y|}, \quad x,y\in\mathbb{R}, \lambda\in\Sigma_{\Lambda,\delta}, \tag{5.36}$$

where $\mu_s > 0$ is a λ- and ε-independent constant. Thus, combining (5.33) and (5.36) with Lemma 4.1 gives

$$\|\mathcal{T}_{sd,\varepsilon}(x,y,\lambda) - \mathcal{T}_{is,\varepsilon}(x,y,\lambda)\| \leq C\varepsilon^2, \quad x,y\in[\Xi_\varepsilon, 2L_\varepsilon-\Xi_\varepsilon], \lambda\in\Sigma_{\Lambda,\delta}, \tag{5.37}$$

where we use that $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ by Theorem 2.3. We apply the variation of constants formula and write

$$\mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda) = \mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda) - \sqrt{\varepsilon} \int_0^{L_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, L_\varepsilon, \lambda)dz, \tag{5.38}$$

for $\lambda \in \Sigma_{\Lambda,\delta}$. Estimates (5.33) and (5.36) yield

$$\left\| \int_{\Xi_\varepsilon}^{L_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, L_\varepsilon, \lambda)dz \right\| \leq C\varepsilon, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.39}$$

Applying (5.36), (5.37) and (5.39) to (5.38) gives

$$\left\| \mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda) - \mathcal{F}_{+,\varepsilon}(\lambda)\mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda) \right\| \leq C\varepsilon\sqrt{\varepsilon}|\log(\varepsilon)|, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.40}$$

where

$$\mathcal{F}_{+,\varepsilon}(\lambda) := I - \sqrt{\varepsilon} \int_0^{\Xi_\varepsilon} \mathcal{T}_{is,\varepsilon}(0, z, \lambda)\mathcal{B}_{2,\varepsilon}(z, \lambda)\mathcal{T}_{sd,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz\mathcal{T}_{is,\varepsilon}(\Xi_\varepsilon, 0, \lambda).$$

Using (5.35), we derive

$$\left\| \mathcal{F}_{+,\varepsilon}(\lambda) - I + \sqrt{\varepsilon} \int_0^{\Xi_\varepsilon} \mathcal{B}_{2,\varepsilon}(z, \lambda)dz \right\| \leq C\varepsilon|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.41}$$

Recall that the $(2n \times 2m)$-matrix $\mathcal{X}_{in}(x, u_0, \lambda)$ is a composition of four block matrices, where $\mathcal{V}_{in}(x, u_0, \lambda)$ is the upper-left $n \times m$-block. Theorems 2.3 and 5.16 and estimates (2.6), (5.26) and (5.41) yield

$$\left\| \mathcal{F}_{+,\varepsilon}(\lambda) - \begin{pmatrix} I & 0 \\ -\sqrt{\varepsilon} \int_0^\infty \left[\partial_u H_2(u_0, v_{\mathrm{h}}(x)) + \partial_v H_2(u_0, v_{\mathrm{h}}(x))\mathcal{V}_{in}(x, u_0, \lambda)\right] dx & I \end{pmatrix} \right\| \leq C\varepsilon|\log(\varepsilon)|^2, \tag{5.42}$$

for any $\lambda \in \Sigma_{\Lambda,\delta}$.

Our next step is to relate systems (3.9) and (5.34). We apply two operations on system (5.34). First, we perform the coordinate change $\chi = C_\varepsilon\tilde{\chi}$, where $C_\varepsilon := \begin{pmatrix} I & 0 \\ 0 & \sqrt{\varepsilon} \end{pmatrix} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C})$. Second, we switch to the spatial scale $\check{x} = \varepsilon x$. Thus, Lemma 4.1 yields the following estimate,

$$\|C_\varepsilon^{-1}\mathcal{T}_{is,\varepsilon}(0, L_\varepsilon, \lambda)C_\varepsilon - \mathcal{T}_s(0, \ell_0, \lambda)\| \leq C\varepsilon, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.43}$$

where $\mathcal{T}_s(\check{x}, \check{y}, \lambda)$ is the evolution operator of the slow eigenvalue problem (3.9).

Finally, we approximate $\mathcal{E}_{s,\varepsilon}$ by the slow Evans function $\mathcal{E}_{s,0}$. Applying (5.40), (5.42) and (5.43) to (5.38) yields

$$\left\| C_\varepsilon^{-1}\mathcal{T}_{sd,\varepsilon}(0, L_\varepsilon, \lambda)C_\varepsilon - \Upsilon_+(\lambda)\mathcal{T}_s(0, \ell_0, \lambda) \right\| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.44}$$

with

$$\Upsilon_+(\lambda) := \begin{pmatrix} I & 0 \\ -\int_0^\infty [\partial_u H_2(u_0, v_h(x)) + \partial_v H_2(u_0, v_h(x))\mathcal{V}_{in}(x, u_0, \lambda)]\,dx & I \end{pmatrix}.$$

Similarly, we derive

$$\left\| C_\varepsilon^{-1} \mathcal{T}_{sd,\varepsilon}(0, -L_\varepsilon, \lambda) C_\varepsilon - \Upsilon_-(\lambda)\mathcal{T}_s(2\ell_0, \ell_0, \lambda) \right\| \leq C\sqrt{\varepsilon}|\log(\varepsilon)|^2, \quad \lambda \in \Sigma_{\Lambda,\delta}, \tag{5.45}$$

with

$$\Upsilon_-(\lambda) := \begin{pmatrix} I & 0 \\ \int_{-\infty}^0 [\partial_u H_2(u_0, v_h(x)) + \partial_v H_2(u_0, v_h(x))\mathcal{V}_{in}(x, u_0, \lambda)]\,dx & I \end{pmatrix}.$$

For any $\lambda \in \Sigma_{\Lambda,\delta}$, we have $\det(\mathcal{T}_s(0, \ell_0, \lambda)) = 1$ and $\Upsilon_+(\lambda)^{-1}\Upsilon_-(\lambda) = \Upsilon(u_0, \lambda)$, where $\Upsilon(u, \lambda)$ is defined in (3.11). Combining the latter with estimates (5.44) and (5.45) yields (5.32). □

It remains to link the factor $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$.

Lemma 5.18. *Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Take $\delta > 0$. There exists $\mu_p > 0$ such that, for $\varepsilon > 0$ sufficiently small, there is a map $h_\varepsilon \colon \Sigma_{\Lambda,\delta} \to \mathbb{C}$ satisfying*

$$\left.\begin{aligned} 0 < |h_\varepsilon(\lambda)| &\leq Ce^{-\mu_p L_\varepsilon}, \\ \left|\mathcal{E}_{f,\varepsilon}(\lambda, \gamma)h_\varepsilon(\lambda) - (-\gamma)^n \mathcal{E}_{f,0}(\lambda)\right| &\leq C\varepsilon^{\mu_p}, \end{aligned}\right. \qquad \lambda \in \Sigma_{\Lambda,\delta}, \gamma \in S^1,$$

where $\mathcal{E}_{f,\varepsilon}$ is as in Theorem 5.16 and $C > 0$ is a constant, which is independent of λ and ε.

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

Our approach is as follows. Using roughness techniques we show that system (5.28) has for each $\lambda \in \Sigma_{\Lambda,\delta}$ an exponential dichotomy on $\mathbb{R}$ with projections $P_{fd,\varepsilon}(x, \lambda)$. Moreover, by Proposition 5.2, the homogeneous fast eigenvalue problem (3.6) admits for every $\lambda \in \Sigma_{\Lambda,\delta}$ exponential dichotomies on $[0, \infty)$ and $(-\infty, 0]$ with λ-independent constants $C, \mu_r > 0$ and projections $Q_{f,\pm}(x, \lambda)$. Recall that the fast Evans function $\mathcal{E}_{f,0}$ is defined in terms of bases $B_f^s(\lambda)$ and $B_f^u(\lambda)$ of $Q_{f,+}(0, \lambda)[\mathbb{C}^{2n}]$ and $\ker(Q_{f,-}(0, \lambda))$, respectively. By comparing system (5.28) to (3.6), we construct bases $B_\varepsilon^{u,s}(\lambda)$ of $P_{fd,\varepsilon}(0, \lambda)[\mathbb{C}^{2n}]$ and $\ker(P_{fd,\varepsilon}(0, \lambda))$, which are close to $B_f^{u,s}(\lambda)$. By tracking the bases $B_\varepsilon^{u,s}(\lambda)$ either forward or backward, we obtain bases of $P_{fd,\varepsilon}(L_\varepsilon, \lambda)[\mathbb{C}^{2n}]$ and $\ker(P_{fd,\varepsilon}(L_\varepsilon, \lambda))$. These bases will form the column vectors of a matrix $\mathcal{H}_\varepsilon(\lambda)$, which connects $\mathcal{E}_{f,\varepsilon}$ to $\mathcal{E}_{f,0}$.

We start by establishing an exponential dichotomy on $\mathbb{R}$ for system (5.28). System (5.15) has by Theorem 5.14 an exponential dichotomy on $\mathbb{R}$ with ε- and λ-independent constants. In addition, by Theorems 2.3 and 5.16, $\mathcal{A}_{12,\varepsilon}$ and U_ε are ε-uniformly bounded on $\mathbb{R}$ and $\mathbb{R} \times \Sigma_{\Lambda,\delta}$, respectively. Therefore, Proposition 4.12 yields that (5.28) has, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy on $\mathbb{R}$ with projections $P_{fd,\varepsilon}(x, \lambda)$ and ε- and λ-independent constants $C, \mu_d > 0$. Since the coefficient matrix of (5.28) is $2L_\varepsilon$-periodic by Theorem 5.16, the projections $P_{fd,\varepsilon}(\cdot, \lambda)$ are also $2L_\varepsilon$-periodic – see [14, Proposition 8.4].

Our next step is to compare system (5.28) to the homogeneous fast eigenvalue problem (3.6). Theorems 2.3 and 5.16 yield

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \sqrt{\varepsilon}U_\varepsilon(x,\lambda)\mathcal{A}_{12,\varepsilon}(x) - \mathcal{A}_{22,0}(x,u_0,\lambda)\right\| \leq C\sqrt{\varepsilon}, \quad x \in [\log(\varepsilon), -\log(\varepsilon)]. \tag{5.46}$$

By **(E1)** there exists an $M > 1$ such that $\mathcal{A}_{22,0}(\cdot, u_0, \cdot)$ is bounded by M on $\mathbb{R}\times\Sigma_{\Lambda,\delta}$. Denote by $\mathcal{T}_r(x,y,\lambda)$ and $\mathcal{T}_{fd,\varepsilon}(x,y,\lambda)$ the evolution operators of (3.6) and (5.28), respectively. Provided that $\varepsilon > 0$ is sufficiently small, Lemma 4.1 and estimate (5.46) imply

$$\|\mathcal{T}_r(x,y,\lambda) - \mathcal{T}_{fd,\varepsilon}(x,y,\lambda)\| < 1, \quad x,y \in \left[\frac{\log(\varepsilon)}{8M}, -\frac{\chi\log(\varepsilon)}{8M}\right], \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.47}$$

Since $\lambda \mapsto B_f^{u,s}(\lambda)$ is analytic on C_Λ by Proposition 5.2, $B_f^{u,s}$ is bounded on $\Sigma_{\Lambda,\delta}$. Now, combine estimate (5.47) and Lemma 4.13: there exists, for $\varepsilon > 0$ sufficiently small, bases $B_\varepsilon^{u,s}\colon \Sigma_{\Lambda,\delta} \to \mathrm{Mat}_{2n\times n}(\mathbb{C})$ of $P_{fd,\varepsilon}(0,\lambda)[\mathbb{C}^{2n}] = B_\varepsilon^s[\mathbb{C}^n]$ and $\ker(P_{fd,\varepsilon}(0,\lambda)) = B_\varepsilon^u[\mathbb{C}^n]$, such that

$$\|B_\varepsilon^{u,s}(\lambda) - B_f^{u,s}(\lambda)\| \leq C\varepsilon^{\mu_r/(8M)}, \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.48}$$

Since $B_f^{u,s}$ is bounded on $\Sigma_{\Lambda,\delta}$, the some holds for $B_\varepsilon^{u,s}$ by (5.48).

Finally, we link $\mathcal{E}_{f,\varepsilon}$ to the fast Evans function $\mathcal{E}_{f,0}$. Define

$$\mathcal{H}_\varepsilon(\lambda) := \left(\mathcal{T}_{fd,\varepsilon}(-L_\varepsilon,0,\lambda)B_\varepsilon^u(\lambda), \mathcal{T}_{fd,\varepsilon}(L_\varepsilon,0,\lambda)B_\varepsilon^s(\lambda)\right), \quad \lambda \in \Sigma_{\Lambda,\delta}.$$

Since $P_{fd,\varepsilon}(\cdot,\lambda)$ is $2L_\varepsilon$-periodic, the first n columns of $\mathcal{H}_\varepsilon(\lambda)$ form a basis of the space $\ker(P_{fd,\varepsilon}(L_\varepsilon,\lambda))$ and the last n columns form a basis of $P_{fd,\varepsilon}(L_\varepsilon,\lambda)[\mathbb{C}^{2n}]$. Thus, $\mathcal{H}_\varepsilon(\lambda)$ is invertible. By Hadamard's inequality we have $|\det(\mathcal{H}_\varepsilon(\lambda))| \leq Ce^{-2n\mu_d L_\varepsilon}$ for each $\lambda \in \Sigma_{\Lambda,\delta}$. Moreover, using that $P_{fd,\varepsilon}(\cdot,\lambda)$ is $2L_\varepsilon$-periodic, we estimate

$$\begin{aligned}\|\mathcal{T}_{fd,\varepsilon}(0,L_\varepsilon,\lambda)\mathcal{T}_{fd,\varepsilon}(-L_\varepsilon,0,\lambda)B_\varepsilon^u(\lambda)\| &\leq Ce^{-2\mu_d L_\varepsilon},\\ \|\mathcal{T}_{fd,\varepsilon}(0,-L_\varepsilon,\lambda)\mathcal{T}_{fd,\varepsilon}(L_\varepsilon,0,\lambda)B_\varepsilon^s(\lambda)\| &\leq Ce^{-2\mu_d L_\varepsilon},\end{aligned} \quad \lambda \in \Sigma_{\Lambda,\delta}. \tag{5.49}$$

We combine estimates (5.48) and (5.49) and derive

$$\left\|\left(\mathcal{T}_{fd,\varepsilon}(0,-L_\varepsilon,\lambda) - \gamma\mathcal{T}_{fd,\varepsilon}(0,L_\varepsilon,\lambda)\right)\mathcal{H}_\varepsilon(\lambda) - \left(B_f^u(\lambda), \gamma B_f^s(\lambda)\right)\right\| \leq C\varepsilon^{\mu_r/(8M)},$$

for $\lambda \in \Sigma_{\Lambda,\delta}$ and $\gamma \in S^1$. Taking determinants and defining $h_\varepsilon(\lambda) := \det(\mathcal{H}_\varepsilon(\lambda))$ concludes the proof. □

Remark 5.19. In the proof of Lemma 5.18, the connection between $\mathcal{E}_{f,\varepsilon}$ and $\mathcal{E}_{f,0}$ is given by the matrix $\mathcal{H}_\varepsilon$. This idea is taken from the proof of [99, Theorem 2]. However, the context in [99] is different: here it is shown that the eigenvalues of a periodic boundary value problem are exponentially close to the eigenvalues of the corresponding unbounded problem. ■

5.2.5 Conclusion

In contrast to the approximation of $\mathcal{E}_{s,\varepsilon}$ by $\mathcal{E}_{s,0}$ in Lemma 5.17, we need to rescale $\mathcal{E}_{f,\varepsilon}$ in Lemma 5.18 by an exponentially small quantity h_ε in order to approximate it by the ε-independent fast Evans function $\mathcal{E}_{f,0}$. This quantity prevents us from directly estimating the Evans function $\mathcal{E}_\varepsilon$ by the reduced Evans function $\mathcal{E}_0(\lambda,\gamma) = (-\gamma)^n\mathcal{E}_{s,0}(\lambda,\gamma)\mathcal{E}_{f,0}(\lambda)$. Nevertheless, it is still possible to compare the zero sets of $\mathcal{E}_\varepsilon$ and $\mathcal{E}_0$ using the classical symmetric version of Rouché's Theorem due to Estermann. This yields the proof of Theorem 3.15.

Proof of Theorem 3.14. Let $\Lambda \in (-\Lambda_0, 0)$ with $\Lambda_0 > 0$ as in Lemma 5.1. Let $S \subset S^1$ be closed. Take a simple closed curve Γ in $C_\Lambda \setminus [\mathcal{N}_S \cup \mathcal{E}_{f,0}^{-1}(0)]$, where $\mathcal{N}_S$ is as in (3.16). Since $\mathcal{E}_\varepsilon(\cdot,\gamma)$ and $\mathcal{E}_0(\cdot,\gamma)$ have no roots in $C_\Lambda \setminus \Sigma_{\Lambda,0}$ for each $\gamma \in S$ by Corollary 5.12, we may assume $\Gamma \subset \Sigma_{\Lambda,0} \setminus [\mathcal{N}_S \cup \mathcal{E}_{f,0}^{-1}(0)]$. Observe that there exists $\delta > 0$ such that $\Gamma \subset \Sigma_{\Lambda,\delta} \setminus \mathcal{N}_S$, since Γ avoids the set of roots of $\mathcal{E}_{f,0}$, which is discrete by Proposition 5.2.

By Propositions 3.7 and 3.11, $\mathcal{E}_\varepsilon(\cdot,\gamma)$ and $\mathcal{E}_0(\cdot,\gamma)$ are analytic on Γ and its interior for each $\gamma \in S$. Furthermore, $\mathcal{E}_0$ is bounded away from 0 on the compact set $\Gamma \times S$, because Γ is disjoint from $\mathcal{N}_S$ and $\mathcal{E}_0$ is analytic on $C_\Lambda \times \mathbb{C}$ by Proposition 3.11. Hence, for $\varepsilon > 0$ sufficiently small, we have

$$\begin{aligned} |\mathcal{E}_\varepsilon(\lambda,\gamma) - \mathcal{E}_0(\lambda,\gamma)| &\leq |\mathcal{E}_{s,\varepsilon}(\lambda,\gamma)\mathcal{E}_{f,\varepsilon}(\lambda,\gamma)h_\varepsilon(\lambda) - (-\gamma)^n\mathcal{E}_{s,0}(\lambda,\gamma)\mathcal{E}_{f,0}(\lambda)| \\ &\quad + (1 - h_\varepsilon(\lambda))|\mathcal{E}_\varepsilon(\lambda,\gamma)| \qquad\qquad \lambda \in \Gamma, \gamma \in S, \\ &< |\mathcal{E}_0(\lambda,\gamma)| + |\mathcal{E}_\varepsilon(\lambda,\gamma)|, \end{aligned}$$

by Theorem 5.16 and Lemmas 5.17 and 5.18. The result follows by an application of the symmetric version of Rouché's Theorem. □

Remark 5.20. The technical Lemmas 5.17 and 5.18 seem to provide a rate at which the spectrum $\sigma(\mathcal{L}_\varepsilon)$ converges to its singular limit. However, the approximations in these lemmas are only valid away from the zeros of the fast Evans function $\mathcal{E}_{f,0}$! So, one can only deduce that spectrum converging to

$$\left\{\lambda \in \mathbb{C} : \mathcal{E}_{s,0}(\lambda,\gamma) = 0 \text{ for some } \gamma \in S^1\right\} \setminus \mathcal{E}_{f,0}^{-1}(0),$$

does this at an algebraic rate of order $O(\sqrt{\varepsilon})$. We expect that this rate is in fact of order $O(\varepsilon)$ and that the square root appears due to the rescaling of the full eigenvalue problem (3.3) in §5.2.1. By making the parameter δ appearing in the proof of Lemma 5.18 dependent on ε, it might be possible to derive an overall rate at which the spectrum $\sigma(\mathcal{L}_\varepsilon)$ converges to its singular limit spectrum. However, this is beyond the scope of this thesis. Yet, we derive in §5.3 that the critical spectral curve, which is attached to the origin, scales with ε^2. This suggest that spectrum converging to the roots of $\mathcal{E}_{f,0}$ does this at an algebraic rate of order $O(\varepsilon^2)$. ■

5.2.6 Discussion

As mentioned in the introduction in Chapter 1, our factorization method via the Riccati transformation of the Evans function offers one unified analytic alternative to both the elephant trunk procedure developed by Alexander, Gardner and Jones [1, 37] and the NLEP approach of [21, 22] – that both have a geometric nature. It is worthwhile to compare and discuss the links between our work and these methods.

Consider a localized pulse solution to a 2-component, singularly perturbed reaction-diffusion system of the form (1.1). When the associated eigenvalue problem has a slow-fast structure (1.5), it is a general phenomenon that it decouples outside the pulse region due to exponential decay of the solution to its asymptotic background state. This yields a decomposition of the solution space into three subspaces $V_{s\pm} \oplus V_{c\pm} \oplus V_{u\pm}$ at both sides ($\pm$) of the pulse region. Here, $V_{s\pm}$ consists of (fast) exponentially decaying solutions, whereas $V_{u\pm}$ consists of exponentially increasing solutions. Lastly, $V_{c\pm}$ consists of solutions that evolve slowly. In the sense of [85], one could say that the eigenvalue problem admits exponential separations with respect to the decompositions $V_{s\pm} \oplus V_{c\pm} \oplus V_{u\pm}$. The difficulty is to 'glue' the subspaces $V_{\cdot+}$ and $V_{\cdot-}$ for $\cdot = u, s, c$ together, yielding an exponential separation of the eigenvalue problem on the whole line. Eventually, this induces a factorization of the Evans function into a fast and slow component.

Gardner and Jones achieved this in [37] by considering the eigenvalue problem in projective space. When the eigenvalue problem is asymptotically of constant coefficient type, one can obtain stable and unstable bundles. These bundles are then split into fast and slow (un)stable subbundles. The elephant trunk lemma is used to track the fast (un)stable bundle through the pulse region. By the control on the fast subbundle, it is possible to approximate the dynamics of the slow (un)stable subbundles. Eventually, this yields a $(1, 2, 1)$-exponential separation (in the sense of [85]) of the eigenvalue problem on $\mathbb{R}$. Note that the 2-dimensional center direction corresponds to the slow (un)stable subbundles. In our stability analysis, the Riccati transformation plays the role of the elephant trunk lemma – see Section 5.2.4. This transformation yields an $(n, 2m, n)$-exponential separation on $\mathbb{R}$ of the eigenvalue problem as long as we are not close to the eigenvalues of the operator $\mathcal{L}_f$, defined in (3.7).

Although the proof of the elephant trunk lemma has been worked out in full detail for some specific 2-component models [22, 32, 37, 95] only, it is widely accepted that the method can be followed for a larger class of systems. However, there are some limitations. For instance, the elephant trunk lemma is only suitable for eigenvalue problems that have an asymptotically constant coefficient matrix. This is neither a restriction for slowly linear systems as the classical Gray-Scott and Gierer-Meinhardt models nor for homoclinic pulses on $\mathbb{R}$. However, the eigenvalue problem associated with spatially periodic patterns in slowly nonlinear systems exhibits non-autonomous behavior in the background state on its domain of periodicity – and thus does not approach a constant coefficient matrix. This prohibits the application of the elephant trunk procedure. Moreover, the elephant trunk lemma is only capable of tracking the 'most unstable' fast solution, which corresponds to the (simple) eigenvalue of largest real part of the asymptotic coefficient matrix. Therefore, it is unclear how to obtain the exponential separation with the elephant trunk method in the multi-component setting $n > 1$.

Furthermore, there is a major difference in the mathematical framework used in [1, 37] and our work. The framework in [1, 37] has a highly geometrical character, whereas our method is of a more analytical nature. Alexander, Gardner and Jones track solutions via vector bundles arising from the projectivized eigenvalue problem. This has the advantage that the generated bundles have a clean and natural characterization as ε tends to zero, whereas the actual solutions of the eigenvalue problem become singular. On the other hand, one could argue that exponential dichotomies provide a natural framework to capture the dynamics of the eigenvalue problem being a non-autonomous linear system, which depends analytically on the spectral parameter λ. The Riccati transformation is naturally formulated in terms of exponential dichotomies and is explicit in terms of the coefficient matrix of the eigenvalue problem. Therefore, the exponential separation of the solution space is much more explicit than in [1, 37], which shortens proofs. Finally, it is interesting to remark that in both the approach initiated by Alexander, Gardner and Jones and our method we need an a-priori ε-independent estimate on the sector containing the spectrum. Our proof of this fact in Proposition 5.11 forms an analytical counterpart to the geometrical proof provided in [1, Proposition 2.2] and [37, Lemma 3.3].

Based on the geometric methods of Alexander, Gardner and Jones [1, 37], the NLEP approach was developed in the context of the stability of homoclinic (multi-)pulse patterns in the Gray-Scott equation [22] and Gierer-Meinhardt-type models [21]. This method established the approximation of the Evans function by the product (1.4) of an analytic fast Evans function and a meromorphic slow Evans function and provided explicit analytic expressions for both factors. The NLEP approach was extended to the spectral analysis of spatially periodic pulse patterns in the generalized Gierer-Meinhardt equations in [114] and to the stability of heteroclinic and homoclinic multi-front patterns in 2- and 3-component bistable systems of FitzHugh-Nagumo-type [23, 116]. Moreover, the method has recently been generalized to the stability of homoclinic pulses in slowly nonlinear systems in [30, 120]. In each of these works, the fast and slow Evans functions are interpreted geometrically in terms of fast and slow transmission functions that encode the passage of specially selected fast and slow basis functions over the fast pulse regions. The expressions for the slow transmission functions include Melnikov-type components. The meromorphic character of the slow Evans function generates the zero-pole cancelation mechanism – also called NLEP paradox – in each of these models. The spectral analysis for periodic pulse solutions developed here shows that these phenomena occur in a broad class of multi-component singularly perturbed reaction-diffusion systems.

Although the present work stands in the tradition of [21, 22, 23, 30, 114, 120], the methods differ fundamentally. Unlike these works, our analysis is based on an intrinsically analytic reduction method. This has the advantage that our spectral analysis allows for non-autonomous behavior of the eigenvalue problem outside the pulse region – a crucial extension in the case of spatially periodic patterns in slowly nonlinear systems. This extended applicability of the present method also plays a role in the spectral analysis of homoclinic patterns as outlined in Remark 1.4. Moreover, in contrast to the present work, the slow and fast eigenvalue problems appearing in [21, 22, 23, 30, 114, 120] are scalar, which significantly simplifies the analysis of these problems. In [21, 22, 114] the slow and fast Evans functions can be explicitly computed in terms of hypergeometric functions, while in [23, 116] the stability of the (multi-)fronts is determined by spectrum near the origin, so that the relevant reductions can be determined in a relatively straightforward manner. An extensive analysis of the multi-component slow and fast eigenvalue problems, as we did in Section 5.1, is thus not necessary in these cases.

5.3 The critical spectral curve

5.3.1 Introduction

In this section we prove Theorem 3.19. Thus, we assume 0 is a simple zero of the fast Evans function $\mathcal{E}_{f,0}$. Moreover, we take $\delta > 0$ and denote

$$\mathcal{N}_\diamond = \left\{\nu \in \mathbb{R} : \mathcal{E}_{s,0}(0, e^{i\nu}) = 0\right\}, \quad \mathcal{S}_\delta = \mathbb{R} \setminus \bigcup_{\nu \in \mathcal{N}_\diamond} (\nu - \delta, \nu + \delta).$$

For each $\nu \in \mathbb{R} \setminus \mathcal{N}_\diamond$ the reduced Evans function $\mathcal{E}_0(\cdot, e^{i\nu})$ has a simple root at 0 by Remark 3.13. Since $\mathcal{E}_0$ is analytic by Proposition 3.11, there exists $\varsigma > 0$ such that there are no other roots of $\mathcal{E}_0(\cdot, e^{i\nu})$ in the closed ball $\overline{B(0, \varsigma)}$ for any $\nu \in \mathcal{S}_\delta$. So, provided $\varepsilon > 0$ is sufficiently small, there exists by Theorem 3.15 a unique (simple) root $\lambda_\varepsilon(\nu)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ in $B(0, \varsigma)$ for each $\nu \in \mathcal{S}_\delta$. By Proposition 3.7 $\lambda_\varepsilon \colon \mathcal{S}_\delta \to B(0, \varsigma)$ is real-valued, 2π-periodic and even. Moreover, since $\mathcal{E}_\varepsilon$ is analytic by Proposition 3.7, $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ is also analytic by the implicit function theorem. By translational invariance, it holds $\lambda_\varepsilon(0) = 0$ if we have $0 \in \mathcal{S}_\delta$. Thus, all that remains to prove Theorem 3.19 is to approximate $\lambda_\varepsilon(\nu)$ for any $\nu \in \mathcal{S}_\delta$ with an error bound that is ν-uniform.

We describe our approach to obtain a leading-order approximation for $\lambda_\varepsilon(\nu)$ for each $\nu \in \mathcal{S}_\delta$. Fix $\nu \in \mathcal{S}_\delta$. On the one hand, since $\mathcal{E}_\varepsilon(\lambda_\varepsilon(\nu), e^{i\nu}) = 0$, the full eigenvalue problem (3.3) admits at $\lambda = \lambda_\varepsilon(\nu)$ a solution $\tilde{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$, which satisfies $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(x + 2L_\varepsilon)$ for each $x \in \mathbb{R}$. On the other hand, the derivative $\phi_{\mathrm{p},\varepsilon}'(x)$ of the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}(x) = (u_{\mathrm{p},\varepsilon}(x), p_{\mathrm{p},\varepsilon}(x), v_{\mathrm{p},\varepsilon}(x), q_{\mathrm{p},\varepsilon}(x))$ to (2.1) is a solution to (3.3) at $\lambda = 0$. Therefore,

$$\psi_{\nu,\varepsilon}(x) := \begin{pmatrix} \tilde{v}_{\nu,\varepsilon}(x) - v_{\mathrm{p},\varepsilon}'(x) \\ \tilde{q}_{\nu,\varepsilon}(x) - q_{\mathrm{p},\varepsilon}'(x) \end{pmatrix},$$

solves the inhomogeneous problem,

$$\psi_x = \mathcal{A}_f(x)\psi + \begin{pmatrix} 0 \\ \mathcal{B}_{\nu,\varepsilon}(x) + \lambda_\varepsilon(\nu)\tilde{v}_{\nu,\varepsilon}(x) \end{pmatrix}, \quad \psi \in \mathbb{C}^{2n}, \tag{5.50}$$

where $\mathcal{A}_f(x)$ is the coefficient matrix of the fast variational equation (3.15) and $B_{\nu,\varepsilon}(x)$ is given by

$$\mathcal{B}_{\nu,\varepsilon}(x) := \begin{pmatrix} \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) \\ \partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x), \varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x, u_0), 0) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \tilde{u}_{\nu,\varepsilon}(x) - u'_{p,\varepsilon}(x) \\ \tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x) \end{pmatrix}.$$

By Proposition 5.2 and Corollary 5.4 the fast variational equation (3.15) has exponential dichotomies on both half-lines and the corresponding differential operator $\mathcal{L}_0$ is Fredholm of index 0. Since 0 is a simple root of $\mathcal{E}_{f,0}$, $\mathcal{L}_0$ has a one-dimensional kernel by Corollary 5.4. So, there exists a non-trivial, exponentially localized solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint problem (3.19), which is unique up to scalar multiples. Applying the solvability condition in [86, Lemma 4.2] to equation (5.50) leads to the key identity,

$$\lambda_\varepsilon(\nu) \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x) dx = -\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \mathcal{B}_{\nu,\varepsilon}(x) dx. \tag{5.51}$$

Hence, to obtain a leading-order expression of $\lambda_\varepsilon(\nu)$, it is sufficient to approximate the two integrals in (5.51). Thus, we need leading-order expressions of the solution $\tilde{\varphi}_{\nu,\varepsilon}(x)$ to (3.3), of the solution $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ to (1.10) and of the difference $\tilde{\varphi}_{\nu,\varepsilon}(x) - \phi'_{\mathrm{p},\varepsilon}(x)$. Clearly, we can approximate $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ by its singular limit – see Theorem 2.3. To obtain leading-order expressions for the other quantities in (5.51), we proceed as follows.

Define

$$D_{\eta,\varepsilon} := \{\lambda \in \mathbb{C} : |\lambda||\log(\varepsilon)| < \eta\}, \tag{5.52}$$

with $\eta > 0$ an ε-independent constant. Moreover, consider the intervals,

$$I_{f,\varepsilon} := [-\Xi_\varepsilon, \Xi_\varepsilon], \quad I_{s,\varepsilon} := [\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \quad \Xi_\varepsilon := -\frac{8\log(\varepsilon)}{\min\{\mu_0, \mu_r, \mu_{\mathrm{h}}\}}, \tag{5.53}$$

with $\mu_{\mathrm{h}} > 0$ as in (2.6), $\mu_0 > 0$ as in Theorem 2.3 and $\mu_r > 0$ as in Lemma 5.1. For any $\nu \in \mathcal{S}_\delta$ and $\lambda \in D_{\eta,\varepsilon}$ we establish a *piecewise continuous* solution $\varphi_{\nu,\varepsilon}(x, \lambda)$ to the full eigenvalue problem (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$ – see Figure 5.2. We explicitly construct $\varphi_{\nu,\varepsilon}$ via Lin's method [10, 70, 118] using the singular limit structure (2.9) the periodic pulse solution $\phi_{\mathrm{p},\varepsilon}$ as our framework.

By Theorem 2.3, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{f,\varepsilon}$ approximated by the pulse solution $\phi_{\mathrm{h}}(x, u_0)$ to the fast reduced system (2.2). Moreover, $\phi_{\mathrm{p},\varepsilon}(x)$ is for $x \in I_{s,\varepsilon}$ approximated by the solution $(\psi_{\mathrm{s}}(\varepsilon x), 0)$ on the slow manifold, where ψ_{s} solves the slow reduced system (2.4). The endpoints of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}$ correspond to the x-values for which $\phi_{\mathrm{p},\varepsilon}(x)$ converges to one of the two non-smooth corners $(u_0, \pm\mathcal{J}(u_0))$ of the singular concatenation (2.9) as $\varepsilon \to 0$.

For $x \in I_{f,\varepsilon}$, we establish a reduced eigenvalue problem by setting ε and λ to 0 in (3.3), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x, u_0)$. The reduced eigenvalue problem admits

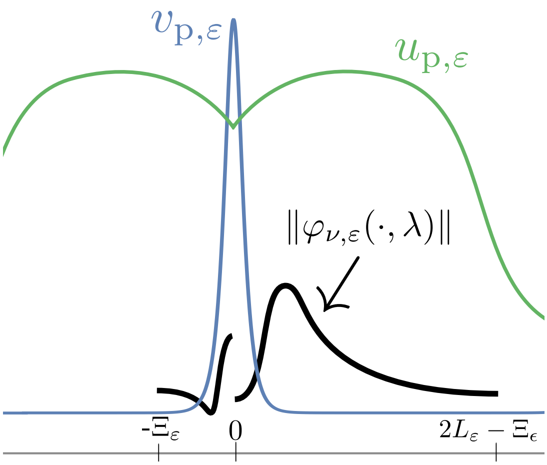


Figure 5.2: A sketch of the piecewise continuous eigenfunction $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ on its domain of definition $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$. Also depicted are the u- and v-component of the periodic pulse solution $\hat{\phi}_{p,\varepsilon}$ (in the case $n = m = 1$).

exponential trichotomies on both half-lines. Hence, one can construct solutions to (3.3) for $\lambda \in D_{\eta,\varepsilon}$ using variation of constants formulas on the intervals,

$$I_{f,\varepsilon}^{-} := [-\Xi_\varepsilon, 0], \quad I_{f,\varepsilon}^{+} := [0, \Xi_\varepsilon]. \tag{5.54}$$

We can control the perturbation terms in these formulas by taking $\eta, \varepsilon > 0$ sufficiently small.

For $x \in I_{s,\varepsilon}$, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in (3.3) is exponentially small by assumption **(S1)** and Theorem 2.3. Thus, we obtain a reduced eigenvalue problem by setting $\mathcal{A}_{21,\varepsilon}(x)$ to 0 in (3.3), while approximating $\phi_{p,\varepsilon}(x)$ by $(\psi_s(\varepsilon x), 0)$. The reduced eigenvalue problem is upper-triangular and the spectrum of the lower-right block has a consistent splitting into n unstable and n stable eigenvalues. This splitting yields the existence of an exponential trichotomy on the interval $I_{s,\varepsilon}$. Thus, one can construct solutions to (3.3) on $I_{s,\varepsilon}$ using the variation of constants formula again.

In summary, we obtain variation of constants formulas for solutions to (3.3) on the three intervals $I_{f,\varepsilon}^{\pm}$ and $I_{s,\varepsilon}$. Matching of these expressions yields for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$ which has a jump at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$. We show that for any $\nu \in \mathcal{S}_\delta$ the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes at a unique λ-value $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$. Thus, since (3.3) is $2L_\varepsilon$-periodic, there exists a continuous solution $\check{\varphi}_{\nu,\varepsilon}$ to (3.3) at $\lambda = \tilde{\lambda}_\varepsilon(\nu)$ satisfying

$$\begin{aligned} \check{\varphi}_{\nu,\varepsilon}(x) &= \varphi_{\nu,\varepsilon}(x,\tilde{\lambda}_\varepsilon(\nu)), && x \in I_{f,\varepsilon} \cup I_{s,\varepsilon}, \\ \check{\varphi}_{\nu,\varepsilon}(x) &= e^{i\nu}\check{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x), && x \in \mathbb{R}, \end{aligned} \quad \nu \in \mathcal{S}_\delta. \tag{5.55}$$

Consequently, $\tilde{\lambda}_\varepsilon(\nu)$ must be a zero of the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$. Since the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ has a unique root $\lambda_\varepsilon(\nu)$ in $B(0,\varsigma)$, we must have $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ for each $\nu \in \mathcal{S}_\delta$. Since the key identity (5.51) is satisfied for any solution $\tilde{\varphi}_{\nu,\varepsilon}$ to (3.3) at $\lambda = \lambda_\varepsilon(\nu)$ satisfying $\tilde{\varphi}_{\nu,\varepsilon}(x) = e^{i\nu}\tilde{\varphi}_{\nu,\varepsilon}(2L_\varepsilon + x)$ for any $x \in \mathbb{R}$, it holds in particular for $\tilde{\varphi}_{\nu,\varepsilon} = \check{\varphi}_{\nu,\varepsilon}$.

The variation of constants formulas provide leading-order control over $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the intervals $I^\pm_{f,\varepsilon}$ and $I_{s,\varepsilon}$. Consequently, we obtain approximations for $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi'_{\mathrm{p},\varepsilon}$ for each $\nu \in \mathcal{S}_\delta$. Substituting these into (5.51) yields the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

This section is structured as follows. First, we establish the aforementioned reduced eigenvalue problems along the pulse (i.e. for $x \in I_{f,\varepsilon}$) and along the slow manifold (i.e. for $x \in I_{s,\varepsilon}$) and we generate exponential trichotomies for these problems. Then, we construct solutions to (3.3) on $I_{s,\varepsilon}$ and $I^\pm_{f,\varepsilon}$ using variation of constants formulas. By matching these solutions at the endpoints of the intervals $I^\pm_{f,\varepsilon}$ and $I_{s,\varepsilon}$ we obtain the desired piecewise continuous solution $\varphi_{\nu,\varepsilon}$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$. We show that there is a unique λ-value for which the jump of $\varphi_{\nu,\varepsilon}(\cdot,\lambda)$ vanishes. Finally, we substitute leading-order approximations of $\check{\varphi}_{\nu,\varepsilon}$ and $\check{\varphi}_{\nu,\varepsilon} - \phi'_{\mathrm{p},\varepsilon}$ into the key identity (5.51) and obtain the desired leading-order expression for $\lambda_\varepsilon(\nu)$.

5.3.2 A reduced eigenvalue problem along the pulse

We establish a reduced eigenvalue problem along the pulse by setting ε and λ to 0 in (3.3), while approximating $\phi_{\mathrm{p},\varepsilon}(x)$ by the pulse $\phi_{\mathrm{h}}(x,u_0)$. Thus, the reduced eigenvalue problem reads

$$\varphi_x = \mathcal{A}_0(x)\varphi, \quad \varphi = (u,p,v,q) \in \mathbb{C}^{2(m+n)}, \tag{5.56}$$

with

$$\mathcal{A}_0(x) := \left(\begin{array}{c|c} \mathcal{A}_1(x) & \mathcal{A}_2(x) \\ \hline \mathcal{A}_3(x) & \mathcal{A}_f(x) \end{array}\right)$$
$$:= \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ \partial_u H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 & \partial_v H_2(u_0, v_{\mathrm{h}}(x,u_0)) & 0 \\ \hline 0 & 0 & 0 & D_2^{-1} \\ \partial_u G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 & \partial_v G(u_0, v_{\mathrm{h}}(x,u_0), 0) & 0 \end{array}\right).$$

Note that (5.56) coincides with the variational equation about the pulse solution $\phi_{\mathrm{h}}(x,u_0)$ to the fast reduced system (2.2).

The u-components of any solution to (5.56) are constant, whereas the p-components are slaved to the other components. Moreover, given the values of the u-components, the dynamics in the v- and q-components is determined by (3.15) via the variation of constants formula. Therefore, the reduced eigenvalue problem (5.56) is governed by the variational equation (3.15) about the homoclinic $\psi_{\mathrm{h}}(x,u_0)$ to (2.3) at $u = u_0$.

Thus, before studying problem (5.56), we study the dynamics of the fast variational equation (3.15). Naturally, the derivative $\partial_x\psi_\mathrm{h}(x,u_0)$ is a non-trivial, exponentially localized solution to (3.15). Moreover, since $\psi_\mathrm{h}(0,u_0)$ is contained in the space $\ker(I-R_f)$ by **(E1)**, system (3.15) is R_f-reversible at $x=0$. We establish exponential dichotomies for (3.15) on both half-lines that respect the reversible symmetry.

Proposition 5.21. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Then, the fast variational equation* (3.15) *admits exponential dichotomies on* $[0,\infty)$ *and* $(-\infty,0]$ *with constants* $C,\mu_r>0$ *and rank* n *projections* $P_{f,\pm}(x)$ *satisfying*

$$\|P_{f,\pm}(\pm x)-\mathcal{P}_f\|\leq Ce^{-\min\{\mu_r,\mu_\mathrm{h}\}x},\quad x\geq 0, \tag{5.57}$$

where $\mu_\mathrm{h}>0$ *is as in* (2.6), $\mu_r>0$ *is as in Lemma 5.1 and* $\mathcal{P}_f$ *denotes the spectral projection onto the stable eigenspace of the asymptotic matrix,*

$$\mathcal{A}_{f,\infty}:=\lim_{x\to\pm\infty}\mathcal{A}_f(x)=\begin{pmatrix}0 & D_2^{-1}\\ \partial_v G(u_0,0,0) & 0\end{pmatrix}. \tag{5.58}$$

The space of exponentially localized solutions to (3.15) *is spanned by* $\kappa_\mathrm{h}(x)=\partial_x\psi_\mathrm{h}(x,u_0)=(\partial_x v_\mathrm{h}(x,u_0),\partial_x q_\mathrm{h}(x,u_0))$. *Similarly, the adjoint* (3.19) *has a non-trivial, exponentially localized solution* $\psi_\mathrm{ad}(x)=(\psi_{\mathrm{ad},1}(x),\psi_{\mathrm{ad},2}(x))$, *which is unique up to scalar multiples and satisfies*

$$\int_{-\infty}^{\infty}\psi_{\mathrm{ad},2}(x)^*\partial_x v_\mathrm{h}(x,u_0)dx\neq 0,\quad \|\psi_\mathrm{ad}(y)\|\leq Ce^{-\mu_r|y|},\quad y\in\mathbb{R}.$$

Moreover, we have the decomposition,

$$\mathbb{C}^{2n}=Y^u\oplus Y^s\oplus Y^c\oplus Y^\perp, \tag{5.59}$$

with $Y^c=\mathrm{Sp}(\kappa_\mathrm{h}(0))$, $Y^\perp=\mathrm{Sp}(\psi_\mathrm{ad}(0))$ *and*

$$\begin{aligned} P_{f,+}(0)[\mathbb{C}^{2n}]&=Y^s\oplus Y^c, & P_{f,-}(0)[\mathbb{C}^{2n}]&=Y^s\oplus Y^\perp,\\ \ker(P_{f,+}(0))&=Y^u\oplus Y^\perp, & \ker(P_{f,-}(0))&=Y^u\oplus Y^c. \end{aligned} \tag{5.60}$$

The spaces $Y^u\oplus Y^s$, $Y^\perp$ *and* Y^c *are pairwise orthogonal and the decomposition* (5.59) *respects the reversible symmetry:*

$$R_f\kappa_\mathrm{h}(0)=-\kappa_\mathrm{h}(0),\quad R_f\psi_\mathrm{ad}(0)=\psi_\mathrm{ad}(0),\quad R_f[Y^s]=Y^u. \tag{5.61}$$

Proof. Since (3.15) coincides with the fast eigenvalue problem (3.6) at $\lambda=0$, Proposition 5.2 provides exponential dichotomies for (3.15) on $[0,\infty)$ and $(-\infty,0]$ with constants $C_r,\mu_r>0$ and rank n projections $P_{f,\pm}(x)$. By (2.6) it holds

$$\left\|\mathcal{A}_f(x)-\mathcal{A}_{f,\infty}\right\|\leq Ke^{-\mu_\mathrm{h}|x|},\quad x\in\mathbb{R},$$

for some $K>0$. Hence, Lemma 4.6 yields estimate (5.57). In addition, by Proposition 5.2, the space of exponentially localized solutions to (3.15) is one-dimensional, because 0 is a simple root of $\mathcal{E}_{f,0}$. Since $\kappa_{\rm h}(x)$ is a non-trivial, exponentially localized solution to (3.15) by **(E1)**, we deduce $Y^c := \mathrm{Sp}(\kappa_{\rm h}(0)) = P_{f,+}(0)[\mathbb{C}^{2n}] \cap \ker(P_{f,-}(0))$.

Define Y^s to be the $(n-1)$-dimensional orthogonal complement of Y^c in $P_{f,+}(0)[\mathbb{C}^{2n}]$. Any solution $\varphi(x)$ to (3.15) with initial condition $\varphi(0) \in Y^s$ decays exponentially to 0 as $x \to \infty$. In addition, since system (3.15) is R_f-reversible at $x = 0$, the solution $R_f\varphi(-x)$ to (3.15) decays exponentially to 0 as $x \to -\infty$. Therefore, $Y^u := R_f[Y^s]$ is contained in $\ker(P_{f,-}(0))$. Since R_f is self-adjoint and $R_f[\kappa_{\rm h}(0)] = -\kappa_{\rm h}(0)$, the n-dimensional space $\ker(P_{f,-}(0))$ arises as the orthogonal sum of Y^c and Y^u.

Because the kernel of the operator $\mathcal{L}_0$ of Fredholm index 0 is one-dimensional by Corollary 5.4, the adjoint $\mathcal{L}_0^*$ has a one-dimensional kernel too. In addition, since equation (3.15) has exponential dichotomies on both half-lines, the same holds for its adjoint (3.19). So, there exists a non-trivial, exponentially localized solution $\psi_{\rm ad}(x)$ to (3.19), which is unique up to scalar multiples. The pointwise inner product of $\psi_{\rm ad}(x)$ with any solution $\varphi(x)$ to (3.15) is constant in x. Thus, the pointwise inner product of $\psi_{\rm ad}(x)$ with solutions $\varphi(x)$ to (3.15) that are decaying to 0 as $x \to \pm\infty$ must equal 0. Hence, the spaces $Y^s \oplus Y^u$, Y^c and $Y^\perp := \mathrm{Sp}(\psi_{\rm ad}(0))$ must be pairwise orthogonal. Since we have the decomposition (5.59), we may without loss of generality assume by Lemma 4.5 that $P_{f,-}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^\perp$ and $\ker(P_{f,+}(0)) = Y^u \oplus Y^\perp$.

Finally, $R_f\psi_{\rm ad}(-x)$ is also an exponentially localized solution to (3.19). This implies $R_f\psi_{\rm ad}(0) = \alpha\psi_{\rm ad}(0)$ for some $\alpha \in \sigma(R_f) = \{\pm 1\}$. On the other hand, since the operator pencil $\lambda \mapsto \mathcal{L}_\lambda$ has algebraic multiplicity 1 at $\lambda = 0$ by Corollary 5.4, the generalized eigenvalue problem,

$$\mathcal{L}_0\varphi = \partial_\lambda \mathcal{L}_0\kappa_{\rm h},$$

has no bounded solutions. Hence, the Fredholm alternative in [86, Lemma 4.2] implies

$$0 \neq \int_{-\infty}^{\infty} \psi_{\rm ad}(x)^* \partial_\lambda \mathcal{L}_0 \kappa_{\rm h}(x) dx = \int_{-\infty}^{\infty} \psi_{{\rm ad},2}(x)^* \partial_x v_{\rm h}(x, u_0) dx.$$

Therefore, $\psi_{{\rm ad},2}(x)$ cannot be even, because $\partial_x v_{\rm h}(x, u_0)$ is an odd function of x. Hence, $\psi_{{\rm ad},2}(x)$ is odd and we establish $R_f\psi_{\rm ad}(0) = \psi_{\rm ad}(0)$. □

The reduced eigenvalue problem (5.56) is governed by the fast variational equation (3.15). More precisely, the evolution operator of (5.56) can be expressed in terms of the evolution operator of (3.15) via variation of constants formulas. Thus, the solution $\kappa_{\rm h}(x) = \partial_x\psi_{\rm h}(x, u_0)$ to (3.15) yields the non-trivial, exponentially localized solution,

$$\varphi_{\rm h}(x) := \begin{pmatrix} \int_\infty^x \mathcal{A}_2(z)\kappa_{\rm h}(z)dz \\ \kappa_{\rm h}(x) \end{pmatrix} = \begin{pmatrix} 0 \\ H_2(u_0, v_{\rm h}(x, u_0)) \\ \partial_x v_{\rm h}(x, u_0) \\ \partial_x q_{\rm h}(x, u_0) \end{pmatrix} = \partial_x \phi_{\rm h}(x, u_0), \tag{5.62}$$

to (5.56). Moreover, since the matrix function $\mathcal{K}_{in}(x) := (\partial_u\psi_{\rm h}(x, u_0) \mid 0)$ solves the inhomogeneous problem,

$$\mathcal{X}_x = \mathcal{A}_f(x)\mathcal{X} + \mathcal{A}_3(x), \quad \mathcal{X} \in \mathrm{Mat}_{2n\times 2m}(\mathbb{C}),$$

we obtain a family of solutions,

$$\Phi_{in}(x) := \begin{pmatrix} I + \int_0^x [\mathcal{A}_2(z)\mathcal{K}_{in}(z) + \mathcal{A}_1(z)]\,dz \\ \mathcal{K}_{in}(x) \end{pmatrix}. \tag{5.63}$$

to (5.56). By **(S1)** and (2.6) there exists a constant $C > 0$ such that

$$\left\| \Phi_{in}(\pm x) - \begin{pmatrix} \Upsilon_{\pm\infty} \\ 0 \end{pmatrix} \right\| \leq Ce^{-\mu_{\mathrm{h}} x}, \quad x \geq 0, \tag{5.64}$$

with

$$\Upsilon_{\pm\infty} := \begin{pmatrix} I & 0 \\ \pm\partial_u \mathcal{J}(u_0) & I \end{pmatrix} \in \mathrm{Mat}_{2m\times 2m}(\mathbb{C}),$$

where $\mathcal{J}\colon U_{\mathrm{h}} \to \mathbb{R}$ is defined in (2.5).

We show that the exponential dichotomies of (3.15), established in Proposition 5.21, yield exponential trichotomies for (5.56) with projections converging to the spectral projections of the asymptotic matrix,

$$\mathcal{A}_\infty := \lim_{x\to\pm\infty} \mathcal{A}_0(x) = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty} \\ 0 & \mathcal{A}_{f,\infty} \end{pmatrix}, \quad \mathcal{A}_{2,\infty} = \begin{pmatrix} 0 & 0 \\ \partial_v H_2(u_0, 0, 0) & 0 \end{pmatrix}, \tag{5.65}$$

where $\mathcal{A}_{f,\infty}$ is defined in (5.58).

Proposition 5.22. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *System* (5.56) *admits exponential trichotomies on* $[0,\infty)$ *and* $(-\infty, 0]$ *with constants* $C, \mu_r > 0$ *and projections* $P_\pm^{u,s,c}(x)$ *satisfying*

$$\left\| P_\pm^{u,s,c}(\pm x) - \mathcal{P}^{u,s,c} \right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x/2}, \quad x \geq 0, \tag{5.66}$$

where $\mu_{\mathrm{h}} > 0$ *is as in (2.6),* $\mu_r > 0$ *is as in Lemma 5.1 and* $\mathcal{P}^u, \mathcal{P}^s$ *and* $\mathcal{P}^c$ *are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix* $\mathcal{A}_\infty$*, respectively. Moreover, it holds*

$$P_-^u(0) = \begin{pmatrix} 0 & \int_{-\infty}^0 \mathcal{A}_2(x)\Phi_{f,-}^u(x,0)dx \\ 0 & I - P_{f,-}(0) \end{pmatrix}, \; P_+^u(0) = \begin{pmatrix} 0 & 0 \\ \int_0^\infty \Phi_{f,+}^u(0,x)\mathcal{A}_3(x)dx & I - P_{f,+}(0) \end{pmatrix},$$
$$P_+^s(0) = \begin{pmatrix} 0 & \int_\infty^0 \mathcal{A}_2(x)\Phi_{f,+}^s(x,0)dx \\ 0 & P_{f,+}(0) \end{pmatrix}, \quad P_-^s(0) = \begin{pmatrix} 0 & 0 \\ \int_0^{-\infty} \Phi_{f,-}^s(0,x)\mathcal{A}_3(x)dx & P_{f,-}(0) \end{pmatrix}, \tag{5.67}$$

where $\Phi_{f,\pm}^{u,s}(x,y)$ *denotes the (un)stable evolution operator of the fast variational equation* (3.15) *under the exponential dichotomies, established in Proposition 5.21, with projections* $P_{f,\pm}(x)$*. Finally, we have the decompositions,*

$$\begin{aligned} \ker(P_+^u(0)) &= P_+^s(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \\ \ker(P_-^s(0)) &= P_-^u(0)[\mathbb{C}^{2(m+n)}] \oplus \Phi_{in}(0)[\mathbb{C}^{2m}], \end{aligned} \tag{5.68}$$

where Φ_{in} is defined in (5.63)*, and*

$$\begin{aligned} P^s_+(0)[\mathbb{C}^{2(m+n)}] &= P^s_+(0)[Z^s] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad Z^s := \{(0,b)\colon b \in Y^s\}, \\ P^u_-(0)[\mathbb{C}^{2(m+n)}] &= P^u_-(0)[Z^u] \oplus \mathrm{Sp}(\varphi_{\mathrm{h}}(0)), \quad Z^u := \{(0,b)\colon b \in Y^u\}. \end{aligned} \tag{5.69}$$

where $Y^{u,s}$ are as in Proposition 5.21 and φ_{h} is defined in (5.62).

Proof. In the following, we denote by $C > 0$ a constant.

The evolution $\Phi_0(x,y)$ of (5.56) can be expressed in terms of the evolution $\Phi_f(x,y)$ of (3.15) as follows

$$\Phi_0(x,y) = \begin{pmatrix} I + \int_y^x \left[\mathcal{A}_2(z)\int_y^z \Phi_f(z,w)\mathcal{A}_3(w)dw + \mathcal{A}_1(z)\right]dz & \int_y^x \mathcal{A}_2(z)\Phi_f(z,y)dz \\ \int_y^x \Phi_f(x,z)\mathcal{A}_3(z)dz & \Phi_f(x,y) \end{pmatrix}. \tag{5.70}$$

By Proposition 5.21 equation (3.15) admits exponential dichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C, \mu_r > 0$ and rank n projections $P_{f,\pm}(x)$ satisfying

$$\left\|P_{f,\pm}(\pm x) - \mathcal{P}_f\right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x}, \quad x \geq 0, \tag{5.71}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$, defined in (5.58). We construct an explicit exponential trichotomy for (5.56) on $(-\infty,0]$ using the matrix functions,

$$\begin{aligned} A(x) &:= \int_{-\infty}^x \mathcal{A}_2(z)\Phi^u_{f,-}(z,x)dx, & B(x) &:= \int_x^0 \Phi^u_{f,-}(x,z)\mathcal{A}_3(z)dz, \\ E(x) &:= \int_0^x \mathcal{A}_2(z)\Phi^s_{f,-}(z,x)dx, & D(x) &:= \int_x^{-\infty} \Phi^s_{f,-}(x,z)\mathcal{A}_3(z)dz. \end{aligned}$$

Clearly, A, B, D and E are bounded on $(-\infty,0]$. We consider their asymptotic behavior. By (2.6) and **(S1)**, it holds

$$\|\mathcal{A}_1(x)\|, \|\mathcal{A}_2(x) - \mathcal{A}_{2,\infty}\|, \|\mathcal{A}_3(x)\|, \|\mathcal{A}_f(x) - \mathcal{A}_{f,\infty}\| \leq Ce^{-\mu_{\mathrm{h}}|x|}. \quad x \in \mathbb{R}, \tag{5.72}$$

By writing $B(x)$ as a sum of two integrals over the intervals $(x, x/2)$ and $(x/2, 0)$ and estimating both integrals independently using (5.72) and the exponential dichotomy of (3.15), we deduce that $B(x)$ converges exponentially to 0 as $x \to -\infty$ with rate $\min\{\mu_r,\mu_{\mathrm{h}}\}/2$. Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 5.1, the matrix $\mathcal{A}_f(x)$ is by (5.72) invertible for $x < 0$ sufficiently small. Thus, for $x \ll 0$ we may write

$$A(x) = \int_{-\infty}^x \mathcal{A}_2(z)\mathcal{A}_f(z)^{-1}\partial_z\Phi^u_{f,-}(z,x)dz.$$

Combining the latter with (5.71) and (5.72), leads, via integration by parts, to the approximations,

$$\|B(x)\|, \left\|A(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I - \mathcal{P}_f)\right\| \leq Ce^{-\min\{\mu_r,\mu_{\mathrm{h}}\}x/2}, \quad x \leq 0. \tag{5.73}$$

Similarly, we derive

$$\|D(x)\|, \left\|E(x) - \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f\right\| \le Ce^{-\min\{\mu_r,\mu_h\}x/2}, \quad x \le 0. \tag{5.74}$$

We define candidate trichotomy projections,

$$P_-^u(x) := \begin{pmatrix} A(x)B(x) & A(x) \\ B(x) & I - P_{f,-}(x) \end{pmatrix}, \quad P_-^s(x) := \begin{pmatrix} E(x)D(x) & E(x) \\ D(x) & P_{f,-}(x) \end{pmatrix}, \quad x \le 0,$$

and we calculate using (5.70)

$$\begin{aligned} P_-^u(x)\Phi_0(x,y) &= \begin{pmatrix} A(x)\Phi_{f,-}^u(x,y)B(y) & A(x)\Phi_{f,-}^u(x,y) \\ \Phi_{f,-}^u(x,y)B(y) & \Phi_{f,-}^u(x,y) \end{pmatrix} = \Phi_0(x,y)P_-^u(y), \\ P_-^s(y)\Phi_0(y,x) &= \begin{pmatrix} E(y)\Phi_{f,-}^s(y,x)D(x) & E(y)\Phi_{f,-}^s(y,x) \\ \Phi_{f,-}^s(y,x)D(x) & \Phi_{f,-}^s(y,x) \end{pmatrix} = \Phi_0(y,x)P_-^s(x), \end{aligned} \quad x \le y \le 0.$$

Since A, B, D and E are bounded on $(-\infty, 0]$, the above calculations imply

$$\left\|P_-^u(x)\Phi_0(x,y)\right\|, \left\|P_-^s(y)\Phi_0(y,x)\right\| \le Ce^{-\mu_r(y-x)}, \quad x \le y \le 0.$$

Define $P_-^c(x) := I - P_-^s(x) - P_-^u(x)$ for $x \le 0$. Observe that

$$P_-^c(x)\Phi_0(x,y) = \begin{pmatrix} E_1(x,y) & E_2(x,y) \\ E_3(x,y) & 0 \end{pmatrix} = \Phi_0(x,y)P_-^c(y), \quad x, y \le 0,$$

where the matrices,

$$\begin{aligned} E_1(x,y) &:= I + \int_y^x \mathcal{A}_1(z)dz + \int_y^{-\infty} \mathcal{A}_2(z)\int_y^z \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz \\ &\quad + \int_x^{-\infty} \mathcal{A}_2(z)\int_z^0 \Phi_{f,-}^u(z,w)\mathcal{A}_3(w)dwdz + \int_y^0 \mathcal{A}_2(z)\int_y^z \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz \\ &\quad + \int_x^0 \mathcal{A}_2(z)\int_z^{-\infty} \Phi_{f,-}^s(z,w)\mathcal{A}_3(w)dwdz, \\ E_2(x,y) &:= \int_y^{-\infty} \mathcal{A}_2(z)\Phi_{f,-}^u(z,y)dz + \int_y^0 \mathcal{A}_2(z)\Phi_{f,-}^s(z,y)dz, \\ E_3(x,y) &:= \int_0^x \Phi_{f,-}^u(x,z)\mathcal{A}_3(z)dz + \int_{-\infty}^x \Phi_{f,-}^s(x,z)\mathcal{A}_3(z)dz. \end{aligned}$$

are bounded on $(-\infty, 0] \times (-\infty, 0]$ by (5.72). Therefore, the projections $P_-^{u,s,c}(x)$ define an exponential trichotomy for equation (5.56) on $(-\infty, 0]$. The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by

$$\mathcal{P}^u = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}(I - \mathcal{P}_f) \\ 0 & I - \mathcal{P}_f \end{pmatrix}, \ \mathcal{P}^s = \begin{pmatrix} 0 & \mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1}\mathcal{P}_f \\ 0 & \mathcal{P}_f \end{pmatrix}, \ \mathcal{P}^c = \begin{pmatrix} I & -\mathcal{A}_{2,\infty}\mathcal{A}_{f,\infty}^{-1} \\ 0 & 0 \end{pmatrix}, \tag{5.75}$$

Thus, the approximations (5.71), (5.73) and (5.74) yield $\|P_-^{u,s,c}(x) - \mathcal{P}^{u,s,c}\| \leq Ce^{\min\{\mu_r,\mu_h\}x/2}$ for $x \leq 0$. Thus, we have obtained the desired exponential trichotomy for (5.56) on $(-\infty, 0]$. The construction of the exponential trichotomy for (5.56) on $[0, \infty)$ is analogous.

Finally, we establish the decompositions (5.68) and (5.69). The upper $(2m \times 2m)$-block of $\Phi_{in}(0)$ is lower-triangular and has determinant 1. Therefore, the columns of $\Phi_{in}(x)$ constitute $2m$ linearly independent solutions to (5.56), which are bounded, but not exponentially localized by (5.64). On the other hand, $P_\pm^{u,s}(0)$ has rank n, since $P_{f,\pm}(0)$ is a rank n projection. This yields the decomposition (5.68). Furthermore, it holds $P_+^s(0)[\mathbb{C}^{2(m+n)}] = P_+^s(0)[\{(0,b) : b \in P_{f,+}(0)[\mathbb{C}^{2n}]\}]$. Since we have $P_{f,+}(0)[\mathbb{C}^{2n}] = Y^s \oplus Y^c$ with $Y^c = \mathrm{Sp}(\kappa_h(0))$ by Proposition 5.21, the decomposition of $P_+^s(0)[\mathbb{C}^{2(m+n)}]$ in (5.69) follows. Analogously, we obtain the decomposition of $P_-^u(0)[\mathbb{C}^{2(m+n)}]$ in (5.69). □

As mentioned in §5.3.1, our goal is to construct a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3), which has a jump at $x = 0$ only. The solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ arises by matching solutions to (3.3), which are defined on the three intervals $I_{f,\varepsilon}^\pm$ and $I_{s,\varepsilon}$, given by (5.53) and (5.54). We match these solutions in such a way that the jump at 0 is confined to the one-dimensional space spannend by $(0, \psi_{\mathrm{ad}}(0))$, where $\psi_{\mathrm{ad}}(x)$ is the solution to the adjoint variational equation (3.15), established in Proposition 5.21. Thus, we need the following lemma.

Lemma 5.23. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Let* Y^s, Y^u, Y^c *and* $Y^\perp$ *be as in Proposition 5.21. Denote by* Q^c *the projection on* Y^c *along* $Y^s \oplus Y^u \oplus Y^\perp$, *by* Q^s *the projection on* Y^s *along* $Y^u \oplus Y^c \oplus Y^\perp$ *and by* Q^u *the projection on* Y^u *along* $Y^s \oplus Y^c \oplus Y^\perp$. *The projections,*

$$\mathcal{Q}^c := \begin{pmatrix} 0 & 0 \\ 0 & Q^c \end{pmatrix}, \; \hat{\mathcal{Q}}^c := \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dx Q^u - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^s + Q^c) \\ 0 & 0 \end{pmatrix},$$
$$\mathcal{Q}^s := \begin{pmatrix} 0 & 0 \\ Q^s \int_0^{-\infty} \Phi_f(0,x)\mathcal{A}_3(x)dx & Q^s \end{pmatrix}, \quad \mathcal{Q}^u := \begin{pmatrix} 0 & 0 \\ Q^u \int_0^\infty \Phi_f(0,x)\mathcal{A}_3(x)dx & Q^u \end{pmatrix}, \tag{5.76}$$

are well-defined and it holds

$$Z^\perp = \ker(\mathcal{Q}^c) \cap \ker(\hat{\mathcal{Q}}^c) \cap \ker(\mathcal{Q}^s) \cap \ker(\mathcal{Q}^u), \quad Z^\perp := \{(0,b) \colon b \in Y^\perp\}. \tag{5.77}$$

Moreover, we have

$$\begin{aligned} &\mathcal{Q}^c\Phi_{in}(0) = 0, \quad \mathcal{Q}^c\varphi_h(0) = \begin{pmatrix} 0 \\ \kappa_h(0) \end{pmatrix}, \quad \hat{\mathcal{Q}}^c \begin{pmatrix} 0 & 0 \\ 0 & I - R_f \end{pmatrix} = 0, \quad \hat{\mathcal{Q}}^c\Phi_{in}(0) = \begin{pmatrix} I \\ 0 \end{pmatrix}, \\ &\hat{\mathcal{Q}}^c = \begin{pmatrix} I & -\int_{-\infty}^0 \mathcal{A}_2(x)\Phi_f(x,0)dx(Q^u + Q^c) - \int_\infty^0 \mathcal{A}_2(x)\Phi_f(x,0)dx Q^s \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{5.78}$$

where φ_h *and* Φ_{in} *are defined in* (5.62) *and* (5.63), *respectively, and* $\kappa_h(x) = \partial_x\psi_h(x, u_0)$.

Proof. The integrals in (5.76) converge by (5.60). Thus, the projections in (5.76) are well-defined. Furthermore, the homoclinic solution $\psi_h(x, u_0)$ to (2.3) at $u = u_0$ satisfies $R_f\psi_h(x, u_0) = \psi_h(-x, u_0)$ for any $x \in \mathbb{R}$ by **(E1)**. Taking derivatives yields

$$R_f\kappa_h(0) = -\kappa_h(0), \quad R_f\kappa_{in}(0) = \kappa_{in}(0), \tag{5.79}$$

where $\kappa_{in}(x) = \partial_u\psi_h(x, u_0)$. Consequently, any column of $\kappa_{in}(0)$ lies in the orthogonal complement of $Y^c = \mathrm{Sp}(\kappa_h(0))$, which is given by $Y^s \oplus Y^u \oplus Y^\perp$ by Proposition 5.21. Hence, we have $Q^c\kappa_{in}(0) = 0$ and the first two identities in (5.78) follow.

The fast variational equation (3.15) is R_f-reversible at $x = 0$ by **(E1)**. Thus, by (5.61) it holds $\Phi_f(-x, 0)Q^u = R_f\Phi_f(x, 0)Q^sR_f$ and $\Phi_f(-x, 0)Q^c = R_f\Phi_f(x, 0)Q^cR_f$ for any $x \geq 0$. Combining the latter with (5.79) leads to the other three identities in (5.78), where we use that $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds for any $x \in \mathbb{R}$ by **(E1)**.

Using (5.60) we immediately establish $Z^\perp \subset \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$. Conversely, assume $(a, b) \in \ker(Q^c) \cap \ker(\hat{Q}^c) \cap \ker(Q^s) \cap \ker(Q^u)$ with $a \in \mathbb{C}^{2m}$ and $b \in \mathbb{C}^{2n}$. Then, it holds

$$Q^cb = 0, \quad a = \int_{-\infty}^{0} \mathcal{A}_2(x)\Phi_f(x, 0)dxQ^ub + \int_{\infty}^{0} \mathcal{A}_2(x)\Phi_f(x, 0)dx(Q^s + Q^c)b,$$
$$Q^sb = -Q^s\int_0^{-\infty} \Phi_f(0, x)\mathcal{A}_3(x)adx, \quad Q^ub = -Q^u\int_0^{\infty} \Phi_f(0, x)\mathcal{A}_3(x)adx.$$

We derive that a is strictly lower-triangular implying $\mathcal{A}_3(x)a = 0$ for any $x \in \mathbb{R}$. Hence, it holds $Q^{u,s,c}b = 0$ yielding $b \in Y^\perp$ and $a = 0$. We conclude $(a, b) \in Z^\perp$. □

5.3.3 A reduced eigenvalue problem along the slow manifold

Along the slow manifold, the v-components of the periodic pulse solution $\phi_{p,\varepsilon}(x)$ are exponentially small and the u-components are approximated by $u_s(\varepsilon x)$ – see Theorem 2.3. Hence, by assumption **(S1)**, the lower-left block $\mathcal{A}_{21,\varepsilon}(x)$ in the full eigenvalue problem (3.3) is exponentially small, whereas the upper-left block $\mathcal{A}_{11,\varepsilon}(x, \lambda)$ is approximated by $\varepsilon\mathcal{A}_s(\varepsilon x)$, where $\mathcal{A}_s$ is the coefficient matrix of the slow variational equation (2.7). Thus, along the slow manifold, we arrive at the reduced eigenvalue problem,

$$\varphi_x = \mathcal{A}_{*,\varepsilon}(x, \lambda)\varphi, \quad \varphi = (u, p, v, q) \in \mathbb{C}^{2(m+n)}, \tag{5.80}$$

with

$$\mathcal{A}_{*,\varepsilon}(x, \lambda) := \begin{pmatrix} \varepsilon\mathcal{A}_s(\varepsilon x) & \mathcal{A}_{12,\varepsilon}(x) \\ 0 & \mathcal{A}_{22,\varepsilon}(x, \lambda) \end{pmatrix}.$$

Due to its upper-triangular block structure, the dynamics in system (5.80) is governed by the blocks on the diagonal via the variation of constants formula. The lower-right block $\mathcal{A}_{22,\varepsilon}(x, \lambda)$ has slowly varying coefficients and is pointwise hyperbolic along the slow manifold. Hence, on the interval $I_{s,\varepsilon}$, defined in (5.53), system (5.15) admits an exponential dichotomy, which yields an exponential trichotomy for the reduced eigenvalue problem (5.80).

Proposition 5.24. *Provided $\varsigma, \varepsilon > 0$ are sufficiently small, system* (5.80) *has for every $\lambda \in B(0,\varsigma)$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of ε and λ, and projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$. We have $\mu_s = \frac{1}{2}\mu_r$, where $\mu_r > 0$ is as in Lemma 5.1. The projections $P^{u,s,c}_{*,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in I_{s,\varepsilon}$ and satisfy*

$$\left\|P^{u,s,c}_{*,\varepsilon}(\Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\|, \left\|P^{u,s,c}_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - \mathcal{P}^{u,s,c}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.81}$$

where Ξ_ε is as in (5.53) *and $\mathcal{P}^u, \mathcal{P}^s$ and $\mathcal{P}^c$ are the spectral projections onto the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$, defined in* (5.65).

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε and λ.

We start by establishing an exponential dichotomy for the subsystem (5.15) of the reduced eigenvalue problem (5.80). We define

$$J_{\alpha,\varepsilon} := [\Xi_\varepsilon/\alpha, 2L_\varepsilon - \Xi_\varepsilon/\alpha], \quad \alpha \geq 0.$$

First, by Theorem 2.3 it holds

$$\|u'_{\mathrm{p},\varepsilon}(x)\| = \varepsilon\left\|D_1^{-1}p_{\mathrm{p},\varepsilon}(x)\right\| \leq C\varepsilon, \quad \|v'_{\mathrm{p},\varepsilon}(x)\| = \|D_2^{-1}q_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\partial_x \mathcal{A}_{22,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma).$$

Second, by Theorem 2.3 we have

$$\left\|\hat\phi_{\mathrm{p},\varepsilon}(x) - (u_{\mathrm{p},\varepsilon}(x), 0)\right\| \leq C\varepsilon^2, \quad x \in J_{4,\varepsilon},$$

which implies

$$\left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - A(u_{\mathrm{p},\varepsilon}(x),\lambda)\right\| \leq C\varepsilon, \quad x \in J_{4,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.82}$$

where $A(u,\lambda)$ is defined in (5.1). By Theorem 2.3 and Lemma 5.1, the matrix $A(u_{\mathrm{p},\varepsilon}(x),\lambda)$ is, provided $\varepsilon > 0$ is sufficiently small, hyperbolic for each $x \in J_{4,\varepsilon}$ and $\lambda \in \mathcal{B}(0,\varsigma)$ with spectral gap larger than $\mu_r = 2\mu_s$. So, by (5.82), the same holds for $\mathcal{A}_{22,\varepsilon}(x,\lambda)$, provided $\varepsilon > 0$ is sufficiently small. Third, $\mathcal{A}_{22,\varepsilon}$ is bounded on $J_{4,\varepsilon} \times B(0,\varsigma)$ by an ε-independent constant using Theorem 2.3. Combining these three items with Proposition 4.8 yields, provided $\varepsilon > 0$ is sufficiently small, an exponential dichotomy for system (5.15) on $J_{2,\varepsilon}$ with constants $C, \mu_s > 0$ and projections $\Pi_{f,\varepsilon}(x,\lambda)$. The projections $\Pi_{f,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in J_{2,\varepsilon}$ and satisfy

$$\left\|\Pi_{f,\varepsilon}(x,\lambda) - Q_\varepsilon(x,\lambda)\right\| \leq C\varepsilon, \quad x \in J_{2,\varepsilon}, \lambda \in B(0,\varsigma), \tag{5.83}$$

where $Q_\varepsilon(x,\lambda)$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{22,\varepsilon}(x,\lambda)$. On the other hand, by Theorem 2.3 and estimate (2.6) we have

$$\left\|\hat\phi_{\mathrm{p},\varepsilon}(\Xi_\varepsilon) - (u_0, 0)\right\| \leq C\varepsilon|\log(\varepsilon)|,$$

yielding

$$\left\|\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{A}_{f,\infty}\right\| \leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right), \quad \lambda \in B(0,\varsigma),$$

where $\mathcal{A}_{f,\infty}$ is given by (5.58). Thus, the same bound holds true for the spectral projections associated with $\mathcal{A}_{22,\varepsilon}(\Xi_\varepsilon,\lambda)$ and $\mathcal{A}_{f,\infty}$. Combining the latter with (5.83) yields

$$\left\|\Pi_{f,\varepsilon}(\Xi_\varepsilon,\lambda)-\mathcal{P}_f\right\| \leq C\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right), \quad \lambda \in B(0,\varsigma), \tag{5.84}$$

where $\mathcal{P}_f$ is the spectral projection onto the stable eigenspace of $\mathcal{A}_{f,\infty}$.

The next step is to express the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ of the upper-triangular block system (5.80) in terms of the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ of (5.15) and the evolution $\Phi_s(\check{x},\check{y})$ of the slow variational equation (2.7). Thus, via the variation of constants formula we obtain

$$\mathcal{T}_{*,\varepsilon}(x,y,\lambda)=\begin{pmatrix} \Phi_s(\varepsilon x,\varepsilon y) & \int_y^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}_{f,\varepsilon}(z,y,\lambda)dz \\ 0 & \mathcal{T}_{f,\varepsilon}(x,y,\lambda)\end{pmatrix}. \tag{5.85}$$

We define candidate trichotomy projections,

$$\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & \int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,x,\lambda)dz \\ 0 & \Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix}, \\
P^u_{*,\varepsilon}(x,\lambda) &:= \begin{pmatrix} 0 & \int_{\frac{1}{2}\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz \\ 0 & I-\Pi_{f,\varepsilon}(x,\lambda)\end{pmatrix}, \qquad x \in I_{s,\varepsilon}, \lambda \in B(0,\varsigma), \\
P^c_{*,\varepsilon}(x,\lambda) &:= I-P^s_{*,\varepsilon}(x,\lambda)-P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}$$

where $\mathcal{T}^{u,s}_{f,\varepsilon}(x,y,\lambda)$ denotes the (un)stable evolution under the exponential dichotomy of (5.15) on $J_{2,\varepsilon}$. The projections $P^{u,s,c}_{*,\varepsilon}(x,\cdot)$ are analytic on $B(0,\varsigma)$ for each $x \in I_{s,\varepsilon}$, because the projections $\Pi_{f,\varepsilon}(x,\lambda)$ and the evolution $\mathcal{T}_{f,\varepsilon}(x,y,\lambda)$ are analytic in λ using [60, Lemma 2.1.4]. On the other hand, lemma 4.1 it yields

$$\|\Phi_s(\varepsilon x,\varepsilon z)\| \leq C, \quad x,y \in J_{2,\varepsilon}, \tag{5.86}$$

because it holds $|\varepsilon L_\varepsilon-\ell_0| \leq C\varepsilon$ by Theorem 2.3. Using (5.85) we calculate for $x,y \in I_{s,\varepsilon}$ and $\lambda \in B(0,\varsigma)$

$$\begin{aligned}
P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) &:= \begin{pmatrix} 0 & \int_{2L_\varepsilon-\frac{1}{2}\Xi_\varepsilon}^x \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz \\ 0 & \mathcal{T}^s_{f,\varepsilon}(x,y,\lambda)\end{pmatrix} \\
&= \mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^s_{*,\varepsilon}(y,\lambda), \\
P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda) &:= \begin{pmatrix} 0 & \int_{\frac{1}{2}\Xi_\varepsilon}^y \Phi_s(\varepsilon y,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,x,\lambda)dz \\ 0 & \mathcal{T}^u_{f,\varepsilon}(y,x,\lambda)\end{pmatrix} \qquad x \geq y, \\
&= \mathcal{T}_{*,\varepsilon}(y,x,\lambda)P^u_{*,\varepsilon}(x,\lambda),
\end{aligned}$$

and

$$P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda) := \begin{pmatrix} \Phi_s(\varepsilon x,\varepsilon y) & E_\varepsilon(x,y,\lambda) \\ 0 & 0 \end{pmatrix} = \mathcal{T}_{*,\varepsilon}(x,y,\lambda)P^c_{*,\varepsilon}(y,\lambda), \tag{5.87}$$

$$\begin{aligned} E_\varepsilon(x,y,\lambda) := &- \int_{2L_\varepsilon - \frac{1}{2}\Xi_\varepsilon}^{y} \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z,y,\lambda)dz \\ &- \int_{\frac{1}{2}\Xi_\varepsilon}^{y} \Phi_s(\varepsilon x,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,y,\lambda)dz. \end{aligned}$$

Using estimate (5.86) and the fact that $\mathcal{A}_{12,\varepsilon}$ is ε-uniformly bounded on $J_{2,\varepsilon}$ by Theorem 2.3, the above calculations imply for $x,y \in I_{s,\varepsilon}$ and $\lambda \in B(0,\varsigma)$

$$\left\|P^s_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\|, \left\|P^u_{*,\varepsilon}(y,\lambda)\mathcal{T}_{*,\varepsilon}(y,x,\lambda)\right\| \leq Ce^{-\mu_s(x-y)}, \quad x \geq y,$$

and

$$\left\|P^c_{*,\varepsilon}(x,\lambda)\mathcal{T}_{*,\varepsilon}(x,y,\lambda)\right\| \leq C.$$

Therefore, the projections $P^{u,s,c}_{*,\varepsilon}(x)$ define an exponential trichotomy for equation (5.80) on $I_{s,\varepsilon}$.

Finally, we establish the approximations (5.81). Define $\tilde{J}_\varepsilon := \left[\frac{1}{2}\Xi_\varepsilon, \frac{3}{2}\Xi_\varepsilon\right]$. First, by Lemma 4.1 it holds

$$\|\Phi_s(\varepsilon x,\varepsilon y) - I\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in \tilde{J}_\varepsilon, \tag{5.88}$$

Second, by Theorem 2.3 and estimate (2.6) we have

$$\left\|\hat{\phi}_{p,\varepsilon}(x) - (u_0,0)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x,y \in \tilde{J}_\varepsilon,$$

yielding for $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0,\varsigma)$

$$\left\|\mathcal{A}_{12,\varepsilon}(x) - \mathcal{A}_{2,\infty}\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad \left\|\mathcal{A}_{22,\varepsilon}(x,\lambda) - \mathcal{A}_{f,\infty}\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.89}$$

where $\mathcal{A}_{2,\infty}$ is defined in (5.65). Since $\mathcal{A}_{f,\infty}$ is hyperbolic by Lemma 5.1, the matrix $\mathcal{A}_{22,\varepsilon}(x,\lambda)$ is by (5.89) invertible for each $x \in \tilde{J}_\varepsilon$ and $\lambda \in B(0,\varsigma)$, provided $\varepsilon,\varsigma > 0$ are sufficiently small. Thus, for $\lambda \in B(0,\varsigma)$ we may write

$$\begin{aligned} &\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz \\ &\quad = \int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{A}_{22,\varepsilon}(z,\lambda)^{-1}\partial_z\mathcal{T}_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz\left(I - \Pi_{f,\varepsilon}(\Xi_\varepsilon,\lambda)\right). \end{aligned}$$

Combining the latter with (5.84), (5.88) and (5.89), leads, via integration by parts, to the approximation,

$$\left\|\int_{\frac{1}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon,\varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^u_{f,\varepsilon}(z,\Xi_\varepsilon,\lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}(I - \mathcal{P}_f)\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \tag{5.90}$$

for $\lambda \in B(0, \varsigma)$, where we use $\mu_r = 2\mu_s$. Similarly, we derive

$$\left\| \int_{\frac{3}{2}\Xi_\varepsilon}^{\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz - \mathcal{A}_{2,\infty}\mathcal{A}^{-1}_{f,\infty}\mathcal{P}_f \right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad (5.91)$$

for $\lambda \in B(0, \varsigma)$. On the other hand, (5.86) yields

$$\left\| \int_{2L_\varepsilon - \frac{1}{2}\Xi_\varepsilon}^{\frac{3}{2}\Xi_\varepsilon} \Phi_s(\varepsilon\Xi_\varepsilon, \varepsilon z)\mathcal{A}_{12,\varepsilon}(z)\mathcal{T}^s_{f,\varepsilon}(z, \Xi_\varepsilon, \lambda)dz \right\| \leq C\varepsilon, \quad \lambda \in B(0, \varsigma). \quad (5.92)$$

The spectral projections $\mathcal{P}^{u,s,c}$ on the unstable, stable and neutral eigenspace of the asymptotic matrix $\mathcal{A}_\infty$ are given by (5.75). Thus, the approximations (5.84), (5.90), (5.91) and (5.92) yield $\|P^{u,s,c}_{*,\varepsilon}(\Xi_\varepsilon, \lambda) - \mathcal{P}^{u,s,c}\| \leq C(\varepsilon|\log(\varepsilon)| + |\lambda|)$ for $\lambda \in B(0, \varsigma)$. The other estimate in (5.81) follows analogously. □

5.3.4 Construction of a piecewise continuous eigenfunction

Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and Ξ_ε be as in (3.21), (5.52) and (5.53), respectively. In this section we establish for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x, \lambda)$ to the full eigenvalue problem (3.3) on the interval $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$. The construction of $\varphi_{\nu,\varepsilon}$ is based on Lin's method [10, 70, 100].

Theorem 5.25. *Suppose* 0 *is a simple root of* $\mathcal{E}_{f,0}$. *Take* $\delta > 0$. *Provided* $\eta, \varepsilon > 0$ *are sufficiently small, there exists for every* $\lambda \in D_{\eta,\varepsilon}$ *and* $\nu \in \mathcal{S}_\delta$ *a piecewise continuous solution* $\varphi_{\nu,\varepsilon}(x, \lambda)$ *to the full eigenvalue problem* (3.3) *on* $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$, *which has a jump only at* $x = 0$, *satisfies*

$$\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda),$$

and enjoys the estimates,

$$\left\|\varphi_{\nu,\varepsilon}(x, \lambda) - \varphi_h(x)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad x \in [-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon], \quad (5.93)$$

$$\left\|\varphi_{\nu,\varepsilon}(x, \lambda) - \phi'_{p,\varepsilon}(x) + \varepsilon\Phi_{in}(x)\mathcal{B}(\nu)\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^3 + |\lambda|\right), \quad x \in \left[-\tfrac{\Xi_\varepsilon}{2}, \tfrac{\Xi_\varepsilon}{2}\right], \quad (5.94)$$

where $\mathcal{B}(\nu)$, φ_h *and* Φ_{in} *are defined in* (3.20), (5.62) *and* (5.63), *respectively, and* $C > 0$ *is a constant independent of* ε, λ *and* ν. *In addition, for any* $\nu \in \mathcal{S}_\delta$ *there exists a unique* λ*-value* $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$ *for which the jump of* $\varphi_{\nu,\varepsilon}(\cdot, \lambda)$ *vanishes.*

Proof. In the following, we denote by $C > 0$ a constant, which is independent of ε, λ and ν.

Let $I_{f,\varepsilon} = I^+_{f,\varepsilon} \cup I^-_{f,\varepsilon}$ and $I_{s,\varepsilon}$ be as in (5.53) and (5.54). Our approach is to regard the full eigenvalue problem (3.3) as a perturbation of the reduced eigenvalue problems (5.56) and (5.80) on the intervals $I_{f,\varepsilon}$ and $I_{s,\varepsilon}$, respectively. Propositions 5.22 and 5.24 yield exponential trichotomies for (5.56) and (5.80). For $\lambda \in D_{\eta,\varepsilon}$, this leads to variation of constants formulas for solutions to (3.3) on the three intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}^{\pm}$. We match these solutions at the endpoints $0, \pm\Xi_\varepsilon$ and $2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I_{f,\varepsilon}^{\pm}$ using the estimates (5.66) and (5.81) on the trichotomy projections and identity (5.77). Thus, we obtain for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon, \lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda)$. For each $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ the jump

$$J_{\nu,\varepsilon}(\lambda) := \lim_{x\downarrow 0} \varphi_{\nu,\varepsilon}(x,\lambda) - \lim_{x\uparrow 0} \varphi_{\nu,\varepsilon}(x,\lambda), \tag{5.95}$$

is contained in the one-dimensional space $Z^\perp$, defined in (5.77). Pairing the jump with the solution $\psi_{\mathrm{ad}}(x)$ to the adjoint (3.19), established in Proposition 5.21, leads to an (analytic) equation in λ and ν, which has a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

The variation of constants formulas provide leading-order expressions for $\varphi_{\nu,\varepsilon}(x,\lambda)$ on the three intervals $I_{f,\varepsilon}^{\pm}$ and $I_{s,\varepsilon}$. Finally, since the derivative $\phi_{\mathrm{p},\varepsilon}'(x)$ is a solution to (3.3) at $\lambda = 0$, we can write $\phi_{\mathrm{p},\varepsilon}'(x)$ in terms of similar variation of constants formulas on $I_{f,\varepsilon}^{\pm}$ yielding leading-order approximation for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)$.

Thus, we start by establishing expressions for solutions to the full eigenvalue problem (3.3) along the pulse. We regard (3.3) as the perturbation,

$$\varphi_x = \left(\mathcal{A}_0(x) + \mathcal{B}_{0,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.56). By Theorem 2.3, the perturbation matrix $\mathcal{B}_{0,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_0(x)$ is bounded by

$$\|\mathcal{B}_{0,\varepsilon}(x,\lambda)\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.96}$$

By Proposition 5.22, system (5.56) has exponential trichotomies on $[0,\infty)$ and $(-\infty,0]$ with constants $C, \mu_r > 0$ and projections $P_{\pm}^{u,s,c}(x)$ satisfying (5.66). We denote by $\Phi_{0,\pm}^{s,u,c}(x,y)$ the stable, unstable and neutral evolution operator of system (5.56) under the exponential trichotomies. For convenience, we abbreviate $\Phi_{0,\pm}^{sc}(x,y) = (I - P_{\pm}^{u}(x))\Phi_0(x,y)$ and $\Phi_{0,\pm}^{uc}(x,y) = (I - P_{\pm}^{s}(x))\Phi_0(x,y)$.

We apply the variation of constants formula. Thus, by the decompositions (5.68) and (5.69), any solution $\varphi_{f,\varepsilon}^{+}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I_{f,\varepsilon}^{+}$:

$$\begin{aligned}\varphi_{f,\varepsilon}^{+}(x,\lambda) = \Phi_{0,+}^{u}(x,\Xi_\varepsilon)a_+ + \Phi_{in}(x)b_+ + \int_0^x \Phi_{0,+}^{s}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^{+}(y,\lambda)dy \\ + \varphi_{\mathrm{h}}(x)c_+ + \Phi_{0,+}^{s}(x,0)d_+ + \int_{\Xi_\varepsilon}^x \Phi_{0,+}^{uc}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{f,\varepsilon}^{+}(y,\lambda)dy,\end{aligned} \tag{5.97}$$

for some $a_+ \in P_+^u(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_+ \in \mathbb{C}^{2m}$, $c_+ \in \mathbb{C}$ and $d_+ \in Z^s$, where Z^s is defined in (5.69). Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.96) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi^+_{f,\varepsilon}(x,\lambda)$ to (5.97) on $I^+_{f,\varepsilon}$ using the contraction mapping principle. Note that $\varphi^+_{f,\varepsilon}(x,\lambda)$ is linear in (a_+, b_+, c_+, d_+) and satisfies the bound,

$$\sup_{x\in I^+_{f,\varepsilon}} \|\varphi^+_{f,\varepsilon}(x,\lambda)\| \leq C\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.98}$$

by estimate (5.96), taking $\eta, \varepsilon > 0$ smaller if necessary. Similarly, by (5.68) and (5.69), any solution $\varphi^-_{f,\varepsilon}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I^-_{f,\varepsilon}$:

$$\begin{aligned}\varphi^-_{f,\varepsilon}(x,\lambda) = {} & \Phi^s_{0,-}(x,-\Xi_\varepsilon)a_- + \Phi_{in}(x)b_- + \int_0^x \Phi^u_{0,-}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^-_{f,\varepsilon}(y,\lambda)dy \\ & + \varphi_{\mathrm{h}}(x)c_- + \Phi^u_{0,-}(x,0)d_- + \int_{-\Xi_\varepsilon}^x \Phi^{sc}_{0,-}(x,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi^-_{f,\varepsilon}(y,\lambda)dy,\end{aligned} \tag{5.99}$$

for some $a_- \in P^s_-(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_- \in \mathbb{C}^{2m}$, $c_- \in \mathbb{C}$ and $d_- \in Z^u$, where Z^u is defined in (5.69). There exists for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi^-_{f,\varepsilon}(x,\lambda)$ to (5.99) on $I^-_{f,\varepsilon}$, which is linear in (a_-, b_-, c_-, d_-) and satisfies the bound,

$$\sup_{x\in I^-_{f,\varepsilon}} \|\varphi^-_{f,\varepsilon}(x,\lambda)\| \leq C\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.100}$$

taking $\eta, \varepsilon > 0$ smaller if necessary.

Our next step is to obtain expressions for solutions to the full eigenvalue problem (3.3) along the slow manifold. We regard (3.3) as the perturbation,

$$\varphi_x = \left(\mathcal{A}_{*,\varepsilon}(x,\lambda) + \mathcal{B}_{*,\varepsilon}(x,\lambda)\right)\varphi, \quad \varphi \in \mathbb{C}^{2(m+n)},$$

of the reduced eigenvalue problem (5.80). By Theorem 2.3 it holds

$$\|u_{\mathrm{p},\varepsilon}(x) - u_{\mathrm{s}}(\varepsilon x)\| \leq C\varepsilon, \quad \|v_{\mathrm{p},\varepsilon}(x)\| \leq C\varepsilon^2, \quad x \in I_{s,\varepsilon}.$$

Therefore, by **(S1)** the perturbation matrix $\mathcal{B}_{*,\varepsilon}(x,\lambda) := \mathcal{A}_\varepsilon(x,\lambda) - \mathcal{A}_{*,\varepsilon}(x,\lambda)$ is bounded by

$$\left\|\mathcal{B}_{*,\varepsilon}(x,\lambda)\right\| \leq C\varepsilon\left(\varepsilon + |\lambda|\right), \quad x \in I_{s,\varepsilon}, \lambda \in \mathbb{C}. \tag{5.101}$$

By Proposition 5.24 system (5.80) admits for every $\lambda \in D_{\eta,\varepsilon}$ an exponential trichotomy on $I_{s,\varepsilon}$ with constants $C, \mu_s > 0$, independent of ε and λ, and projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ satisfying (5.81). We denote by $\mathcal{T}^{s,u,c}_{*,\varepsilon}(x,y,\lambda)$ the stable, unstable and neutral evolution operator of system (5.80) under the exponential trichotomy.

We apply the variation of constants formula. Thus, any solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (3.3) must satisfy the following integral equation on $I_{s,\varepsilon}$:

$$\begin{aligned}\varphi_{s,\varepsilon}(x,\lambda) = {} & \mathcal{T}^s_{*,\varepsilon}(x,\Xi_\varepsilon,\lambda)f + \mathcal{T}^c_{*,\varepsilon}(x,\Xi_\varepsilon,\lambda)h + \int_{\Xi_\varepsilon}^x \mathcal{T}^{sc}_{*,\varepsilon}(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy \\ & + \mathcal{T}^u_{*,\varepsilon}(x,2L_\varepsilon - \Xi_\varepsilon,\lambda)g + \int_{2L_\varepsilon-\Xi_\varepsilon}^x \mathcal{T}^u_{*,\varepsilon}(x,y,\lambda)\mathcal{B}_{*,\varepsilon}(y,\lambda)\varphi_{s,\varepsilon}(y,\lambda)dy,\end{aligned} \tag{5.102}$$

for some $f \in P^s_{*,\varepsilon}(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $g \in P^u_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$ and $h \in P^c_{*,\varepsilon}(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$. Provided $\eta, \varepsilon > 0$ are sufficiently small, there exists by (5.101) for any $\lambda \in D_{\eta,\varepsilon}$ a unique solution $\varphi_{s,\varepsilon}(x,\lambda)$ to (5.102) on $I_{s,\varepsilon}$ using the contraction mapping principle. The solution $\varphi_{s,\varepsilon}(x,\lambda)$ is linear in (f,g,h) and enjoys the bound,

$$\sup_{x \in I_{s,\varepsilon}} \|\varphi_{s,\varepsilon}(x,\lambda)\| \leq C\left(\|f\| + \|g\| + \|h\|\right), \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.103}$$

using estimate (5.101) and the fact that $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ by Theorem 2.3.

Now, we match the solutions $\varphi^\pm_{f,\varepsilon}(x,\lambda)$ and $\varphi_{s,\varepsilon}(x,\lambda)$, given by (5.97), (5.99) and (5.102), at the endpoints $x = \pm\Xi_\varepsilon$ and $x = 2L_\varepsilon - \Xi_\varepsilon$ of the intervals $I_{s,\varepsilon}$ and $I^\pm_{f,\varepsilon}$. Applying the projection $P^s_{*,\varepsilon}(\Xi_\varepsilon,\lambda)$ to the difference $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition,

$$\begin{aligned} f &= \mathcal{H}^1_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+), \\ \|\mathcal{H}^1_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.104}$$

where we use (5.66), (5.68), (5.81), (5.96) and (5.98) to obtain the bound on the linear map $\mathcal{H}^1_{\varepsilon,\lambda}$. Similarly, applying $P^u_{*,\varepsilon}(\Xi_\varepsilon,\lambda)$ to $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition,

$$\begin{aligned} a_+ &= \mathcal{H}^2_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+, f, g, h), \\ \|\mathcal{H}^2_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+, f, g, h)\| &\leq C\left[\varepsilon\left(\varepsilon + |\lambda|\right)\left(\|f\| + \|g\| + \|h\|\right)\right. \\ &\quad \left. +\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right)\right], \end{aligned} \tag{5.105}$$

where we use (5.66), (5.68), (5.81), (5.96), (5.98), (5.101), (5.103) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bound on the linear map $\mathcal{H}^2_{\varepsilon,\lambda}$. Finally, applying $P^c_{*,\varepsilon}(\Xi_\varepsilon,\lambda)$ to $\varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda) - \varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda)$ yields the matching condition,

$$\begin{aligned} h &= \begin{pmatrix} \Upsilon_\infty b_+ \\ 0 \end{pmatrix} + \mathcal{H}^3_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+), \\ \|\mathcal{H}^3_{\varepsilon,\lambda}(a_+, b_+, c_+, d_+)\| &\leq \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_+\| + \|b_+\| + |c_+| + \|d_+\|\right), \end{aligned} \quad \lambda \in D_{\eta,\varepsilon}, \tag{5.106}$$

where we use (5.64), (5.66), (5.81), (5.96) and (5.98) to obtain the bound on the linear map $\mathcal{H}^3_{\varepsilon,\lambda}$. Note that $\mathcal{H}^{1,2,3}_{\varepsilon,\lambda}$ are analytic in λ, because the perturbations matrices $\mathcal{B}_{0,\varepsilon}(x,\lambda)$ and $\mathcal{B}_{*,\varepsilon}(x,\lambda)$, the projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ and the evolution $\mathcal{T}_{*,\varepsilon}(x,y,\lambda)$ are analytic in λ by Proposition 5.24 and [60, Lemma 2.1.4].

Take $\nu \in \mathcal{S}_\delta$. We obtain the following matching conditions for any $\lambda \in D_{\eta,\varepsilon}$ by applying the projections $P^{u,s,c}_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$ to the difference $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) - e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon,\lambda)$:

$$\begin{aligned} g &= \mathcal{H}^4_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-), \\ \|\mathcal{H}^4_{\varepsilon,\lambda}(a_-, b_-, c_-, d_-)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\left(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|\right), \end{aligned} \tag{5.107}$$

$$\begin{aligned} a_- &= \mathcal{H}^5_{\varepsilon,\lambda}(a_-,b_-,c_-,d_-,f,g,h), \\ \|\mathcal{H}^5_{\varepsilon,\lambda}(a_-,b_-,c_-,d_-,f,g,h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|)\right. \\ &\qquad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right], \end{aligned} \tag{5.108}$$

$$\begin{aligned} \mathcal{T}^c_{*,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \Xi_\varepsilon, \lambda)h &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty}b_- \\ 0 \end{pmatrix} + \mathcal{H}^6_{\varepsilon,\lambda}(a_-,b_-,c_-,d_-,f,g,h), \\ \|\mathcal{H}^6_{\varepsilon,\lambda}(a_-,b_-,c_-,d_-,f,g,h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a_-\| + \|b_-\| + |c_-| + \|d_-\|)\right. \\ &\qquad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right], \end{aligned} \tag{5.109}$$

where we use (5.64), (5.66), (5.68), (5.81), (5.96), (5.100), (5.101), (5.103) and $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ to obtain the bounds on the linear maps $\mathcal{H}^{4,5,6}_{\varepsilon,\lambda}$, which are analytic in λ. We introduce the short-hand notation $a = (a_+, a_-)$, $b = (b_+, b_-)$, $c = (c_+, c_-)$ and $d = (d_+, d_-)$. Substituting (5.106) into (5.109) yields a linear map $\mathcal{H}^7_{\varepsilon,\lambda}$, which is analytic in λ, satisfying

$$\begin{aligned} \begin{pmatrix} \Phi_s(2\ell_0, 0)\Upsilon_\infty b_+ \\ 0 \end{pmatrix} &= e^{i\nu}\begin{pmatrix} \Upsilon_{-\infty}b_- \\ 0 \end{pmatrix} + \mathcal{H}^7_{\varepsilon,\lambda}(a,b,c,d,f,g,h) \\ \|\mathcal{H}^7_{\varepsilon,\lambda}(a,b,c,d,f,g,h)\| &\leq C\left[(\varepsilon|\log(\varepsilon)| + |\lambda|)(\|a\| + \|b\| + \|c\| + \|d\|)\right. \qquad \lambda \in D_{\eta,\varepsilon}, \\ &\qquad \left. + (\varepsilon + |\lambda|)(\|f\| + \|g\| + \|h\|)\right], \end{aligned} \tag{5.110}$$

where we use (5.87), $|\varepsilon L_\varepsilon - \ell_0| \leq C\varepsilon$ and the bound,

$$\|\Phi_s(\varepsilon x, \varepsilon y) - I\| \leq C\varepsilon|\log(\varepsilon)|, \quad |x - y| \leq 2\Xi_\varepsilon, \tag{5.111}$$

which follows from Proposition 4.1. The matching conditions (5.104), (5.105), (5.106), (5.107), (5.108) and (5.110) constitute a system of 6 linear equations in 11 variables. One readily observes that, provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for $a_\pm, f, g, h$ and b_- yielding linear maps $\mathcal{H}^{8,9}_{\varepsilon,\lambda}$, which are analytic in λ and satisfy

$$\begin{aligned} (f, g, a) &= \mathcal{H}^8_{\varepsilon,\lambda}(b_+, c, d), \\ (h, b_-) &= \left(\Upsilon_\infty b_+, e^{-i\nu}\Upsilon_\infty \Phi_s(2\ell_0, 0)\Upsilon_\infty b_+\right) + \mathcal{H}^9_{\varepsilon,\lambda}(b_+, c, d), \qquad \lambda \in D_{\eta,\varepsilon}, \\ \|\mathcal{H}^{8,9}_{\varepsilon,\lambda}(b_-, c, d)\| &\leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)(\|b_-\| + \|c\| + \|d\|). \end{aligned} \tag{5.112}$$

Thus, since the projections $P^{u,s,c}_{*,\varepsilon}(x,\lambda)$ are complementary, we observe that (f, g, h, a, b_-) satisfies (5.112) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon, \lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon, \lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon, \lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon, \lambda)$ hold true.

Our next step is to match the solutions $\varphi^\pm_{f,\varepsilon}(x,\lambda)$, given by (5.97) and (5.99), at $x = 0$ such that the jump $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ is confined to the one-dimensional space $Z^\perp$, which is defined in (5.77). First, we apply the projections $Q^{u,s}$, given by (5.76). By (5.60) and (5.67) it holds

$$Q^s P^s_-(0) = Q^s, \quad Q^s P^u_+(0) = 0, \quad (I - Q^s P^s_+(0))[Z^s] = 0. \tag{5.113}$$

Applying the projection Q^s to the difference $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition,

$$\begin{aligned} d_+ &= \mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d), \\ \|\mathcal{H}^{10}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left(\|a\|+\|b\|+\|c\|+\|d\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \quad (5.114)$$

where we use (5.68), (5.69), (5.96), (5.98), (5.100) and (5.113) to obtain the bound on the linear map $\mathcal{H}^{10}_{\varepsilon,\lambda}$, which is analytic in λ. Similarly, applying Q^u to $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$, we establish a linear map $\mathcal{H}^{11}_{\varepsilon,\lambda}$, which is analytic in λ, satisfying

$$\begin{aligned} d_- &= \mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d), \\ \|\mathcal{H}^{11}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left(\|a\|+\|b\|+\|c\|+\|d\|\right), \end{aligned} \qquad \lambda \in D_{\eta,\varepsilon}, \quad (5.115)$$

Next, we apply the projections Q^c and $\hat{Q}^c$, given by (5.76). By (5.60) and (5.67) it holds

$$Q^cP^u_-(0) = Q^c = Q^cP^s_+(0), \quad \hat{Q}^cP^{sc}_-(0) = \hat{Q}^c = \hat{Q}^cP^{uc}_+(0). \qquad (5.116)$$

Applying Q^c to the difference $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields the matching condition,

$$c_+ = c_-, \qquad (5.117)$$

where we use (5.78) and (5.116). Finally, applying $\hat{Q}^c$ to $\varphi^+_{f,\varepsilon}(0,\lambda)-\varphi^-_{f,\varepsilon}(0,\lambda)$ yields for $\lambda \in D_{\eta,\varepsilon}$ the matching condition,

$$\begin{aligned} \begin{pmatrix} b_+ - b_- \\ 0 \end{pmatrix} &= \int_{-\Xi_\varepsilon}^0 \hat{Q}^c\Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\mathrm{h}}(y)c_-dy + \mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d) \\ &\quad - \int_{\Xi_\varepsilon}^0 \hat{Q}^c\Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\mathrm{h}}(y)c_+dy, \qquad \lambda \in D_{\eta,\varepsilon}, \quad (5.118) \\ \|\mathcal{H}^{12}_{\varepsilon,\lambda}(a,b,c,d)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\left[\left(\|a\|+\|b\|+\|d\|\right)\right. \\ &\quad \left.+|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)|+|\lambda|\right)\|c\|\right], \end{aligned}$$

where we use (5.78), (5.96), (5.98), (5.100) and (5.116) to obtain the bound on the linear map $\mathcal{H}^{12}_{\varepsilon,\lambda}$, which is analytic in λ.

We wish to approximate the integral expressions in (5.118). Therefore, we split the perturbation $\mathcal{B}_{0,\varepsilon}(y,\lambda)$ in an ε-dependent and λ-dependent part, i.e. it holds

$$\left\|\mathcal{B}_{0,\varepsilon}(y,\lambda)-\mathcal{B}_{0,\varepsilon}(y,0)-\lambda\mathcal{B}_*\right\| \leq C\varepsilon|\lambda|, \quad y \in I_{f,\varepsilon}, \lambda \in \mathbb{C}, \qquad (5.119)$$

with

$$\mathcal{B}_* := \begin{pmatrix} 0 & 0 \\ 0 & B_* \end{pmatrix} \in \mathrm{Mat}_{2(n+m)\times 2(n+m)}(\mathbb{C}), \quad B_* := \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix} \in \mathrm{Mat}_{2n\times 2n}(\mathbb{C}).$$

First, we approximate the λ-dependent part of the integrals in (5.118). Recall that system (3.15) is R_f-reversible at $x = 0$ by **(E1)**. Thus, the evolution $\Phi_f(x, y)$ of (3.15) satisfies $R_f \Phi_f(x, y) R_f = \Phi_f(-x, -y)$ for any $x, y \in \mathbb{R}$. Hence, using (5.70) we calculate

$$\Phi_0(0, x)\mathcal{B}_*\varphi_{\mathrm{h}}(x) = \begin{pmatrix} \int_x^0 \mathcal{A}_2(y)\Phi_f(y, x)B_*\kappa_{\mathrm{h}}(x)dy \\ \Phi_f(0, x)B_*\kappa_{\mathrm{h}}(x) \end{pmatrix} = \begin{pmatrix} -\int_{-x}^0 \mathcal{A}_2(y)\Phi_f(y, -x)B_*\kappa_{\mathrm{h}}(-x)dy \\ R_f\Phi_f(0, -x)B_*\kappa_{\mathrm{h}}(-x) \end{pmatrix}$$
$$= \begin{pmatrix} -I & 0 \\ 0 & R_f \end{pmatrix} \Phi_0(0, -x)\mathcal{B}_*\varphi_{\mathrm{h}}(-x),$$

where we use that $R_f B_* = -B_* R_f$, $\mathcal{A}_2(x)R_f = \mathcal{A}_2(x)$, $R_f\kappa_{\mathrm{h}}(x) = -\kappa_{\mathrm{h}}(-x)$ and $\mathcal{A}_2(x) = \mathcal{A}_2(-x)$ holds true for any $x \in \mathbb{R}$ by **(E1)**. Combining the latter identity with (5.78) yields

$$\hat{Q}^c \left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0, y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy \right] = 0. \tag{5.120}$$

Next, we approximate the ε-dependent part of the integrals in (5.118). This can be done by using that the derivative $\phi'_{\mathrm{p},\varepsilon}(x)$ is a solution to (3.3) at $\lambda = 0$. Thus, $\phi'_{\mathrm{p},\varepsilon}(x)$ satisfies the integral equation (5.97) on $I^+_{f,\varepsilon}$ at $\lambda = 0$, i.e. we have for $x \in I^+_{f,\varepsilon}$

$$\begin{aligned} \phi'_{\mathrm{p},\varepsilon}(x) = \Phi^u_{0,+}(x, \Xi_\varepsilon)a_{\mathrm{p},+} + \Phi_{in}(x)b_{\mathrm{p},+} + \int_0^x \Phi^s_{0,+}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_{\mathrm{h}}(x)c_{\mathrm{p},+} + \Phi^s_{0,+}(x, 0)d_{\mathrm{p},+} + \int_{\Xi_\varepsilon}^x \Phi^{uc}_{0,+}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy, \end{aligned} \tag{5.121}$$

for some constants $a_{\mathrm{p},+} \in P^u_+(\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},+} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},+} \in \mathbb{C}$ and $d_{\mathrm{p},+} \in Z^s$, where we suppress their ε-dependence for notational convenience. Similarly, it holds for $x \in I^-_{f,\varepsilon}$

$$\begin{aligned} \phi'_{\mathrm{p},\varepsilon}(x) = \Phi^s_{0,-}(x, -\Xi_\varepsilon)a_{\mathrm{p},-} + \Phi_{in}(x)b_{\mathrm{p},-} + \int_0^x \Phi^u_{0,-}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ + \varphi_{\mathrm{h}}(x)c_{\mathrm{p},-} + \Phi^u_{0,-}(x, 0)d_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^x \Phi^{sc}_{0,-}(x, y)\mathcal{B}_{0,\varepsilon}(y, 0)\phi'_{\mathrm{p},\varepsilon}(y)dy, \end{aligned} \tag{5.122}$$

for some $a_{\mathrm{p},-} \in P^s_-(-\Xi_\varepsilon)[\mathbb{C}^{2(m+n)}]$, $b_{\mathrm{p},-} \in \mathbb{C}^{2m}$, $c_{\mathrm{p},-} \in \mathbb{C}$ and $d_{\mathrm{p},-} \in Z^u$. By applying suitable projections, we obtain leading-order approximations for the constants $a_{\mathrm{p},\pm}, b_{\mathrm{p},\pm}, c_{\mathrm{p},\pm}$ and $d_{\mathrm{p},\pm}$. This leads to the desired approximations for the integrals in (5.118).

First, Theorem 2.3 and **(S1)** yield

$$\left\| \phi'_{\mathrm{p},\varepsilon}(\pm\Xi_\varepsilon) - \varepsilon \begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0, 0, 0) \\ 0 \\ 0 \end{pmatrix} \right\| \leq C\varepsilon^2 |\log(\varepsilon)|, \tag{5.123}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ solves the differential equation (2.1). By applying the projections $P^u_+(\Xi_\varepsilon)$ and $P^c_+(\Xi_\varepsilon)$ to (5.121) at $x = \Xi_\varepsilon$, we derive via (5.68) and (5.69)

$$a_{\mathrm{p},+} = P^u_+(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon), \quad P^c_+(\Xi_\varepsilon)\Phi_{in}(\Xi_\varepsilon)b_{\mathrm{p},+} = P^c_+(\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(\Xi_\varepsilon).$$

Similarly, we apply $P_-^s(-\Xi_\varepsilon)$ and $P_-^c(-\Xi_\varepsilon)$ to (5.122) at $x = -\Xi_\varepsilon$ yielding

$$a_{\mathrm{p},-} = P_-^s(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon), \quad P_-^c(-\Xi_\varepsilon)\Phi_{in}(-\Xi_\varepsilon)b_{\mathrm{p},-} = P_-^c(-\Xi_\varepsilon)\phi'_{\mathrm{p},\varepsilon}(-\Xi_\varepsilon).$$

Combining the latter two identities with (5.64), (5.66), (5.75) and (5.123) gives

$$\|a_{\mathrm{p},\pm}\| \leq C\varepsilon, \quad \left\| b_{\mathrm{p},\pm} - \varepsilon\Upsilon_{\mp\infty}\begin{pmatrix} \pm D_1^{-1}\mathcal{J}(u_0) \\ H_1(u_0,0,0)\end{pmatrix}\right\| \leq C\varepsilon^2|\log(\varepsilon)|. \tag{5.124}$$

Recall that we have $\varphi_{\mathrm{h}}(x) = \partial_x\phi_{\mathrm{h}}(x,u_0)$. Thus, by Theorem 2.3 it holds

$$\left\|\phi'_{\mathrm{p},\varepsilon}(x) - \varphi_{\mathrm{h}}(x)\right\| \leq C\varepsilon|\log(\varepsilon)|, \quad x \in I_{f,\varepsilon}, \tag{5.125}$$

where we use that $\phi_{\mathrm{p},\varepsilon}$ and ϕ_{h} solve (2.1) and (2.2), respectively. Next, we apply Q^c to (5.121) and (5.122) at $x = 0$, yielding

$$c_{\mathrm{p},+} = \frac{\left\langle \left(\begin{smallmatrix} 0 \\ \kappa_{\mathrm{h}}(0)\end{smallmatrix}\right), \phi'_{\mathrm{p},\varepsilon}(0)\right\rangle}{\|\kappa_{\mathrm{h}}(0)\|^2} = c_{\mathrm{p},-}, \quad \left|c_{\mathrm{p},\pm} - 1\right| \leq C\varepsilon, \tag{5.126}$$

by (5.78), (5.116) and (5.125). Finally, applying $\hat{Q}^c$ to (5.121) and (5.122) at $x = 0$, gives the identity,

$$\begin{aligned}\begin{pmatrix} b_{\mathrm{p},+} - b_{\mathrm{p},-} \\ 0\end{pmatrix} = \hat{Q}^c\Bigg[&\Phi_{0,-}^s(0,-\Xi_\varepsilon)a_{\mathrm{p},-} + \int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy \\ &- \Phi_{0,+}^u(0,\Xi_\varepsilon)a_{\mathrm{p},+} - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\phi'_{\mathrm{p},\varepsilon}(y)dy\Bigg],\end{aligned}$$

by (5.78) and (5.116). Using (5.96), (5.124) and (5.125), we approximate both sides of the latter identity, yielding

$$\begin{aligned}&\left\|\hat{Q}^c\left[\int_{-\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_{\mathrm{h}}(y)dy - \int_{\Xi_\varepsilon}^0 \Phi_0(0,y)\mathcal{B}_{0,\varepsilon}(y,0)\varphi_{\mathrm{h}}(y)dy\right] - \varepsilon\begin{pmatrix} 2D_1^{-1}\mathcal{J}(u_0) \\ 0 \\ 0 \\ 0\end{pmatrix}\right\| \\ &\quad \leq C\varepsilon^2|\log(\varepsilon)|^2,\end{aligned} \tag{5.127}$$

which gives together with (5.119) and (5.120) the desired leading-order expressions of the integrals in (5.118).

Thus, the matching conditions (5.112), (5.114), (5.115), (5.117) and (5.118) constitute a system of 10 linear equations in 11 variables. Provided $\eta, \varepsilon > 0$ are sufficiently small, this system can be solved for a, b, c_-, d, f, g, h yielding analytic linear maps $\mathcal{H}_{\varepsilon,\lambda}^{13}, \mathcal{H}_{\varepsilon,\lambda}^{14}$ and $\mathcal{H}_{\varepsilon,\lambda}^{15}$

for $\lambda \in D_{\eta,\varepsilon}$ satisfying

$$
\begin{aligned}
(a,d,f,g,h) &= \mathcal{H}^{13}_{\varepsilon,\lambda}(c_+),\\
c_- &= c_+,\\
b_+ &= 2\varepsilon\left(I - e^{-i\nu}\Upsilon_\infty\Phi_s(2\ell_0,0)\Upsilon_\infty\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0)\\ 0\end{pmatrix}c_+ + \mathcal{H}^{14}_{\varepsilon,\lambda}(c_+),\\
b_- &= 2\varepsilon\left(e^{i\nu}\Upsilon_{-\infty}\Phi_s(0,2\ell_0)\Upsilon_{-\infty} - I\right)^{-1}\begin{pmatrix} D_1^{-1}\mathcal{J}(u_0)\\ 0\end{pmatrix}c_+ + \mathcal{H}^{15}_{\varepsilon,\lambda}(c_+),\\
\|\mathcal{H}^{13}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)|c_+|,\\
\|\mathcal{H}^{14,15}_{\varepsilon,\lambda}(c_+)\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2|c_+|,
\end{aligned}
\tag{5.128}
$$

where we use (5.119), (5.120), (5.127) and the fact that $\det(I - e^{-i\nu}\Upsilon_\infty\Phi_s(2\ell_0,0)\Upsilon_\infty) = e^{2im\nu}\mathcal{E}_{s,0}(0,e^{i\nu})$ and $\det(e^{i\nu}\Upsilon_{-\infty}\Phi_s(0,2\ell_0)\Upsilon_{-\infty} - I) = \mathcal{E}_{s,0}(0,e^{i\nu})$ are bounded away from 0 by a ν-independent constant.

Recall that (f,g,h,a,b_-) satisfy (5.112) if and only if both $\varphi_{s,\varepsilon}(\Xi_\varepsilon,\lambda) = \varphi^+_{f,\varepsilon}(\Xi_\varepsilon,\lambda)$ and $\varphi_{s,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda) = e^{i\nu}\varphi^-_{f,\varepsilon}(-\Xi_\varepsilon,\lambda)$ hold true. Moreover, by identity (5.77), (a,b,c,d) satisfy (5.114), (5.115), (5.117) and (5.118) if and only if the jump $\varphi^+_{f,\varepsilon}(0,\lambda) - \varphi^-_{f,\varepsilon}(0,\lambda)$ lies in $Z^\perp$. Thus, take $c_+ := c_{p,+}$ and define quantities $a_\pm, b_\pm, c_-, d_\pm, f, g$ and h through (5.128), where we suppress their ε-, λ- and ν-dependence for notational convenience. Then, (5.97), (5.99) and (5.102) define for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) on $I_{f,\varepsilon} \cup I_{s,\varepsilon}$, which has a jump only at $x = 0$ in the space $Z^\perp$ and satisfies $\varphi_{\nu,\varepsilon}(-\Xi_\varepsilon,\lambda) = e^{i\nu}\varphi_{\nu,\varepsilon}(2L_\varepsilon - \Xi_\varepsilon,\lambda)$.

Now, estimate (5.93) follows readily by approximating the coefficients (a,b,c,d,f,g,h) in the variation of constants formulations (5.97), (5.99) and (5.102) of the solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ using (5.96), (5.98), (5.100), (5.101), (5.103), (5.126) and (5.128).

Next, we show that for any $\nu \in \mathcal{S}_\delta$ the jump $J_{\nu,\varepsilon}(\lambda)$, defined in (5.95), of $\varphi_{\nu,\varepsilon}(x,\lambda)$ at $x = 0$ vanishes for a unique λ-value in $D_{\eta,\varepsilon}$. Fix $\nu \in \mathcal{S}_\delta$. The jump $J_{\nu,\varepsilon}(\lambda)$ can be expressed as the difference of the two variation of constants formulas (5.97) and (5.99) at $x = 0$ with coefficients $a_\pm, b_\pm, c_\pm$ and $d_\pm$ defined through (5.128) and $c_+ = c_{p,+}$. We observe that $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, because the perturbation $\mathcal{B}_{0,\varepsilon}(x,\lambda)$ and the linear maps $\mathcal{H}^{13}_{\varepsilon,\lambda}, \mathcal{H}^{14}_{\varepsilon,\lambda}$ and $\mathcal{H}^{15}_{\varepsilon,\lambda}$ are analytic in λ. For any $\lambda \in D_{\eta,\varepsilon}$ the jump is approximated by

$$
\begin{aligned}
&\left\| J_{\nu,\varepsilon}(\lambda) - d_+ + d_- - \lambda\int_\infty^0 \Phi^{uc}_{0,+}(0,y)\mathcal{B}_*\varphi_h(y)dy - \lambda\int_{-\infty}^0 \Phi^{sc}_{0,-}(0,y)\mathcal{B}_*\varphi_h(y)dy\right\|\\
&\quad \leq C|\log(\varepsilon)|^2\left(\varepsilon + |\lambda|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right),
\end{aligned}
\tag{5.129}
$$

using (5.93), (5.96), (5.119) and (5.128). By Proposition 5.21 we have $\psi_{ad}(0) \in \ker(P_{f,+}(0)^*) \cap P_{f,-}(0)^*[\mathbb{C}^{2n}]$. Therefore, it holds

$$
Z^\perp \subset \ker(P^s_+(0)^*) \cap \ker(P^u_-(0)^*),
\tag{5.130}
$$

by (5.67). The jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ of $\varphi_{\nu,\varepsilon}(x,\lambda)$ at $x = 0$ vanishes if and only if

$$\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle = 0. \tag{5.131}$$

With the aid of (5.130) we calculate

$$\begin{aligned} &\left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, \int_\infty^0 \Phi_{0,+}^{uc}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy - \int_{-\infty}^0 \Phi_{0,-}^{sc}(0,y)\mathcal{B}_*\varphi_{\mathrm{h}}(y)dy \right\rangle \\ &\quad = -\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx. \end{aligned}$$

Combining the latter with (5.129) yields

$$\begin{aligned} &\left\| \left\langle \begin{pmatrix} 0 \\ \psi_{\mathrm{ad}}(0) \end{pmatrix}, J_{\nu,\varepsilon}(\lambda) \right\rangle + \lambda \int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx \right\| \qquad \lambda \in D_{\eta,\varepsilon}, \\ &\quad \leq C|\log(\varepsilon)|^2 \left(\varepsilon + |\lambda| \left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)\right), \end{aligned}$$

since $d_+ \in Z^s$ and $d_- \in Z^u$ are in the orthogonal complement of $Z^\perp$ by Proposition 5.21. Hence, because the λ- and ε-independent integral $\int_{-\infty}^\infty \langle \psi_{\mathrm{ad},2}(x), \partial_x v_{\mathrm{h}}(x,u_0)\rangle\, dx$ is non-zero by Proposition 5.21 and the jump $J_{\nu,\varepsilon}$ is analytic on $D_{\eta,\varepsilon}$, Rouché's Theorem implies that equation (5.131) has, provided $\eta, \varepsilon > 0$ are sufficiently small, a unique solution $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$.

Our last step is to prove estimate (5.94). Fix $\nu \in \mathcal{S}_\delta$. First, we establish the a priori bound,

$$\left\|\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right\| \leq C\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \quad x \in I_{f,\varepsilon}, \lambda \in D_{\eta,\varepsilon}, \tag{5.132}$$

using (5.93) and (5.125). By subtracting (5.121) from (5.97) and (5.122) from (5.99), we obtain variation of constants formulas for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)$ on $I_{f,\varepsilon}^+$ and $I_{f,\varepsilon}^-$, respectively. Our approach is to obtain leading-order expressions for the coefficients $a_\pm - a_{\mathrm{p},\pm}$, $b_\pm - b_{\mathrm{p},\pm}$, $c_\pm - c_{\mathrm{p},\pm}$ and $d_\pm - d_{\mathrm{p},\pm}$ in these variation of constants formulas. By (5.124), (5.126) and (5.128) it holds

$$\begin{aligned} c_\pm - c_{\mathrm{p},\pm} &= 0, \\ \left\|a_\pm - a_{\mathrm{p},\pm}\right\| &\leq C|\log(\varepsilon)|\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right), \qquad \lambda \in D_{\eta,\varepsilon}, \\ \left\|b_\pm - b_{\mathrm{p},\pm} + \mathcal{B}(\nu)\right\| &\leq C|\log(\varepsilon)|^2\left(\varepsilon|\log(\varepsilon)| + |\lambda|\right)^2, \end{aligned} \tag{5.133}$$

where $\mathcal{B}(\nu)$ is defined in (3.20). Estimating $d_\pm - d_{\mathrm{p},\pm}$ is more elaborate. Note that the jump $J_{\nu,\varepsilon}(\lambda) \in Z^\perp$ lies in the kernels of Q^u and Q^s by (5.77). Thus, to estimate $d_+ - d_{\mathrm{p},+}$, we apply the projection Q^s to

$$J_{\nu,\varepsilon}(\lambda) = \lim_{x\downarrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right) - \lim_{x\uparrow 0}\left(\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{\mathrm{p},\varepsilon}'(x)\right), \quad \lambda \in D_{\eta,\varepsilon},$$

yielding

$$\begin{aligned} d_+ - d_{p,+} = &\int_{-\Xi_\varepsilon}^{0} \Phi_{0,-}^s(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi_{p,\varepsilon}'(y)\right]dy \\ &- Q^s \int_{\Xi_\varepsilon}^{0} \Phi_{0,+}^c(0,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi_{p,\varepsilon}'(y)\right]dy \\ &+ \Phi_{0,-}^s(0,-\Xi_\varepsilon)(a_- - a_{p,-}), \end{aligned}$$

by (5.68), (5.113) and (5.133). Therefore, (5.93), (5.96), (5.119), (5.132) and (5.133) imply

$$\left\|d_+ - d_{p,+}\right\| \leq C|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^2 + |\lambda|\right), \quad \lambda \in D_{\eta,\varepsilon}. \tag{5.134}$$

Subtracting (5.121) from (5.97) gives for each $\lambda \in D_{\eta,\varepsilon}$ a variation of constants formula for $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{p,\varepsilon}'(x)$ on $I_{f,\varepsilon}^+$:

$$\begin{aligned} \varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{p,\varepsilon}'(x) = &\Phi_{0,+}^u(x,\Xi_\varepsilon)(a_+ - a_{p,+}) + \Phi_{in}(x)(b_+ - b_{p,+}) + \Phi_{0,+}^s(x,0)(d_+ - d_{p,+}) \\ &+ \int_0^x \Phi_{0,+}^s(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi_{p,\varepsilon}'(y)\right]dy \\ &+ \int_{\Xi_\varepsilon}^x \Phi_{0,+}^{uc}(x,y)\left[\mathcal{B}_{0,\varepsilon}(y,\lambda)\varphi_{\nu,\varepsilon}(y,\lambda) - \mathcal{B}_{0,\varepsilon}(y,0)\phi_{p,\varepsilon}'(y)\right]dy, \end{aligned}$$

where we use $c_+ = c_{p,+}$. Applying (5.93), (5.96), (5.119), (5.132), (5.133) and (5.134) to the latter identity yields the approximation (5.94) on $[0, \Xi_\varepsilon/2]$. The proof of (5.94) on $[-\Xi_\varepsilon/2, 0]$ is analogous. □

Remark 5.26. The proof of Theorem 5.25 provides a Lyapunov-Schmidt type reduction procedure. Finding a bounded solution to the full eigenvalue problem (3.3) amounts to inverting the operator $\mathcal{L}_\varepsilon - \lambda$ defined in §3.2. By constructing the piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to (3.3) via Lin's method, we invert a certain part of $\mathcal{L}_\varepsilon - \lambda$ and we obtain a one-dimensional reduced equation (5.131) describing the remaining unsolved part.

Thus, solving (5.131) for λ yields the desired simple eigenvalue $\lambda_\varepsilon(\nu)$ of $\mathcal{L}_\varepsilon$ about the origin. A leading-order expression of $\lambda_\varepsilon(\nu)$ can be obtained by calculating the leading order of the ε- and λ-dependent parts of (5.131). Alternatively, we use the key identity (5.51) to derive a leading-order expression for $\lambda_\varepsilon(\nu)$ – see §5.3.5. ■

5.3.5 Conclusion

In this section we provide the proof of Theorem 3.19. Let $\mathcal{S}_\delta$, $D_{\eta,\varepsilon}$ and Ξ_ε be as in (3.21), (5.52) and (5.53), respectively. In Theorem 5.25 we constructed for any $\lambda \in D_{\eta,\varepsilon}$ and $\nu \in \mathcal{S}_\delta$ a piecewise continuous solution $\varphi_{\nu,\varepsilon}(x,\lambda)$ to the full eigenvalue problem (3.3) on the interval $[-\Xi_\varepsilon, 2L_\varepsilon - \Xi_\varepsilon]$ which has a jump only at $x = 0$. In addition, we obtained leading-order expressions for $\varphi_{\nu,\varepsilon}(x,\lambda)$ and $\varphi_{\nu,\varepsilon}(x,\lambda) - \phi_{p,\varepsilon}'(x)$.

Moreover, we proved in Theorem 5.25 that for any $\nu \in \mathcal{S}_\delta$ there is a unique λ-value $\tilde{\lambda}_\varepsilon(\nu) \in D_{\eta,\varepsilon}$ for which the jump of $\varphi_{\nu,\varepsilon}(x,\lambda)$ vanishes. As mentioned in §5.3.1 this λ-value coincides with the unique root $\lambda_\varepsilon(\nu)$ of the Evans function $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$ about the origin. We extend the *continuous* solution $\varphi_{\nu,\varepsilon}(x,\tilde{\lambda}_\varepsilon(\nu))$ to the whole real line via (5.55). In §5.3.1 we derived an identity (5.51) for $\lambda_\varepsilon(\nu)$ in terms of this extended solution $\check{\varphi}_{\nu,\varepsilon}$ to (3.3). Plugging the leading-order expressions for $\check{\varphi}_{\nu,\varepsilon}(x)$ and $\check{\varphi}_{\nu,\varepsilon}(x) - \phi'_{\mathrm{p},\varepsilon}(x)$ into (5.51) yields the desired approximation (3.17) of $\lambda_\varepsilon(\nu)$.

Proof of Theorem 3.19. In the following, we denote by $C > 0$ a constant, which is independent of ε and ν.

In §5.3.1 we established a $\varsigma > 0$ such that, provided $\varepsilon > 0$ is sufficiently small, there exists for any $\nu \in \mathcal{S}_\delta$ a unique (real) root $\lambda_\varepsilon(\nu) \in B(0,\varsigma)$ of $\mathcal{E}_\varepsilon(\cdot, e^{i\nu})$. We showed that the function $\lambda_\varepsilon \colon \mathcal{S}_\delta \to \mathbb{R}$ is analytic, even and 2π-periodic and satisfies $\lambda_\varepsilon(0) = 0$ whenever $0 \in \mathcal{S}_\delta$.

Fix $\nu \in \mathcal{S}_\delta$. Consider the solution $\varphi_{\nu,\varepsilon}(x,\tilde{\lambda}_\varepsilon(\nu))$ to the full eigenvalue problem (3.3), established in Theorem 5.25, and define $\check{\varphi}_{\nu,\varepsilon}$ by (5.55). Clearly, $\check{\varphi}_{\nu,\varepsilon}$ is a solution to (3.3) on the whole real line. In §5.3.1 we showed that it holds $\lambda_\varepsilon(\nu) = \tilde{\lambda}_\varepsilon(\nu)$ and that the key identity (5.51) is satisfied for $\check{\varphi}_{\nu,\varepsilon}(x) = (\tilde{u}_{\nu,\varepsilon}(x), \tilde{p}_{\nu,\varepsilon}(x), \tilde{v}_{\nu,\varepsilon}(x), \tilde{q}_{\nu,\varepsilon}(x))$. To obtain a leading-order expression for $\lambda_\varepsilon(\nu)$ we approximate the integrals in (5.51) using Theorem 5.25.

First, Theorem 2.3 and estimate (5.93) imply that $\check{\varphi}_{\nu,\varepsilon}$ and $\phi_{\mathrm{p},\varepsilon}$ are bounded on $\mathbb{R}$ by a constant independent of ε and ν. On the other hand, the solution $\psi_{\mathrm{ad}}(x) = (\psi_{\mathrm{ad},1}(x), \psi_{\mathrm{ad},2}(x))$ to the adjoint equation (3.19) satisfies

$$\|\psi_{\mathrm{ad}}(x)\| \leq C e^{-\mu_r |x|}, \quad x \in \mathbb{R},$$

by Proposition 5.21. Thus, using estimate (5.93) we approximate

$$\left\| \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \tilde{v}_{\nu,\varepsilon}(x) dx - \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_x v_{\mathrm{h}}(x,u_0) dx \right\| \leq C|\log(\varepsilon)| \left(\varepsilon|\log(\varepsilon)| + |\lambda_\nu(\varepsilon)|\right), \tag{5.135}$$

In addition, by estimate (5.94) and Theorem 2.3 we have

$$\begin{aligned} &\left\| \int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \left(\partial_v G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) - \partial_v G(u_0, v_{\mathrm{h}}(x,u_0),0)\right) \left(\tilde{v}_{\nu,\varepsilon}(x) - v'_{p,\varepsilon}(x)\right) dx \right\| \\ &\qquad \leq C\varepsilon |\log(\varepsilon)|^2 \left(\varepsilon^2 |\log(\varepsilon)|^3 + |\lambda_\varepsilon(\nu)|\right) \end{aligned} \tag{5.136}$$

where we use that $\psi_{\mathrm{ad},2}(x)$ is odd by Proposition 5.21, $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even by Theorem 2.3, $v_{\mathrm{h}}(x,u_0)$ is even by **(E1)** and the v-components of $\Phi_{in}(x)\mathcal{B}(\nu)$ are even by **(E1)**. Integration by parts gives

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_{\mathrm{ad},2}(x)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon) \left(\tilde{u}_{\nu,\varepsilon}(x) - u'_{\mathrm{p},\varepsilon}(x)\right) dx \\ &\qquad = -\varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{x} \psi_{\mathrm{ad},2}(y)^* \partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(y),\varepsilon) D_1^{-1} \left(\tilde{p}_{\nu,\varepsilon}(x) - p'_{\mathrm{p},\varepsilon}(x)\right) dy dx, \end{aligned}$$

since $\psi_{\mathrm{ad},2}(x)$ is odd and $\hat{\phi}_{\mathrm{p},\varepsilon}(x)$ is even. Applying estimate (5.94) and Theorem 2.3 to the latter yields

$$\begin{aligned}\Bigg\|\int_{-\infty}^{\infty} &\psi_{\mathrm{ad},2}(x)^*\partial_u G(\hat{\phi}_{\mathrm{p},\varepsilon}(x),\varepsilon)\left(\tilde{u}_{\nu,\varepsilon}(x)-u_{\mathrm{p},\varepsilon}'(x)\right)dx\\ &-\varepsilon^2\int_{-\infty}^{\infty}\int_{-\infty}^{x}\psi_{\mathrm{ad},2}(y)^*\partial_u G(u_0,v_{\mathrm{h}}(y,u_0),0)dydxB(\nu)\Bigg\| \\ &\leq C\varepsilon|\log(\varepsilon)|\left(\varepsilon^2|\log(\varepsilon)|^3+|\lambda_\nu(\varepsilon)|\right),\end{aligned} \tag{5.137}$$

with $B(\nu)$ defined in (3.20), where we use $\psi_{\mathrm{ad},2}(x)$ is odd, $v_{\mathrm{h}}(x,u_0)$ is even and the p-component of $(I-\Phi_{in}(x))\mathcal{B}(\nu)$ is odd by **(E1)**. Finally, since the integral $\int_{-\infty}^{\infty}\psi_{\mathrm{ad},2}(x)^*\partial_x v_{\mathrm{h}}(x,u_0)dx$ is non-zero by Proposition 5.21, the key identity (5.51) in combination with the estimates (5.135), (5.136) and (5.137) gives

$$\begin{aligned}&\left\|\lambda_\varepsilon(\nu)+\varepsilon^2\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{x}\psi_{\mathrm{ad},2}(y)^*\partial_u G(u_0,v_{\mathrm{h}}(y,u_0),0)dydxB(\nu)}{\int_{-\infty}^{\infty}\psi_{\mathrm{ad},2}(x)^*\partial_x v_{\mathrm{h}}(x,u_0)dx}\right\|\\ &\quad\leq C\varepsilon^3|\log(\varepsilon)|^5.\end{aligned}$$

The latter yields the leading-order expression (3.17) of $\lambda_\varepsilon(\nu)$ by switching the order of integration in the numerator using that $\psi_{\mathrm{ad},2}$ is odd and $v_{\mathrm{h}}(x,u_0)$ is even. □

Remark 5.27. In the proof of Theorem 3.19 we have obtained for any $\nu\in\mathcal{S}_\delta$ an eigenfunction,

$$\psi_{\nu,\varepsilon}(\check{x}):=\begin{pmatrix}\tilde{u}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x})\\ \tilde{v}_{\nu,\varepsilon}(\varepsilon^{-1}\check{x})\end{pmatrix}e^{i\nu\check{x}/2\ell_\varepsilon}\in H^2_{\mathrm{per}}([0,2\ell_\varepsilon],\mathbb{C}^{m+n}),$$

corresponding to the eigenvalue $\lambda_\varepsilon(\nu)$ of the operator $\mathcal{L}_{\nu,\varepsilon}$ defined in §3.2.1. The approximations in Theorem 5.25 and its proof provide leading-order control over this eigenfunction. We observe that $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by $(0,\partial_x v_{\mathrm{h}}(\varepsilon^{-1}\check{x},u_0))$ along the pulse. The derivative $\partial_x v_{\mathrm{h}}(x,u_0)$ corresponds to the translational eigenfunction at $\lambda=0$ of the linearization of $v_t=D_2v_{xx}-G(u_0,v,0)$ about the standing pulse solution $v_{\mathrm{h}}(x,u_0)$. Thus, along the pulse, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is independent of ν. On the other hand, along the slow manifold, i.e. for $\varepsilon\check{x}\in I_{s,\varepsilon}$, $\psi_{\nu,\varepsilon}(\check{x})$ is approximated by the u-components of

$$2\varepsilon e^{i\nu\check{x}/2\ell_\varepsilon}\Phi_{\mathrm{s}}(\check{x},0)\Upsilon_0\left(I-e^{-i\nu}\Upsilon_0\Phi_{\mathrm{s}}(2\ell_0,0)\Upsilon_0\right)^{-1}\begin{pmatrix}D_1^{-1}\mathcal{J}(u_0)\\ 0\end{pmatrix},$$

by (5.87), (5.101), (5.102), (5.103), (5.111), (5.112) and (5.128), where $\mathcal{J}$ is given by (2.5), $\Phi_{\mathrm{s}}(\check{x},\check{y})$ is the evolution (2.7) and Υ_0 is defined in (3.20). Thus, along the slow manifold, the leading-order dynamics of the eigenfunction $\psi_{\nu,\varepsilon}$ is dictated by the slow variational equation (2.7) and the value of ν. ■

5.3.6 Discussion

Our approach to expanding the critical spectral curve relies on Lin's method. As mentioned in the introduction in Chapter 1 a similar approach is employed in [10, 100] to determine the spectral geometry about the origin. In this section we compare the analyses in [10, 100] with ours.

In [100] one considers $2L$-periodic wave trains to general reaction-diffusion systems that converge to a homoclinic pulse solution in the long-wavelength limit $L \to \infty$. An expansion of the critical spectral curve is obtained in terms of the period L. It is assumed that the translational eigenvalue at the origin corresponding to the limiting homoclinic pulse is simple. Therefore, the variational equation about the homoclinic pulse has exponential dichotomies on both half-lines such that the spaces of solutions decaying as $x \to \infty$ and $x \to -\infty$ have a one-dimensional intersection. Thus, one obtains a decomposition (5.59) of the solution space as exhibited by our fast variational equation (3.15).

The variational equation about the limiting homoclinic serves as the backbone for the construction of solutions to the eigenvalue problem associated with the periodic wave train. Using Lin's method a piecewise continuous eigenfunction $\varphi_\nu(x)$ is constructed on $[-L, L]$ for any $\nu \in \mathbb{R}$ that has a jump at 0 and satisfies $\varphi(L) = e^{i\nu}\varphi(-L)$. The exponential dichotomies of the variational equation about the homoclinic control the dynamics of the eigenvalue problem on the growing interval $[-L, L]$. The jump at 0 depends on the spectral parameter λ, the period L and the Floquet exponent ν, because the eigenvalue problem is a (λ, L^{-1})-perturbation of the homoclinic variational equation. Using Melnikov theory the jump can be equated to 0 yielding an expansion of the critical spectral curve in terms of e^{-L}.

In our work there are *two* systems that serve as the backbone for the construction of solutions to the full eigenvalue problem (3.3): the reduced eigenvalue problems (5.56) and (5.80) which describe the leading-order dynamics along the fast pulse and along the slow manifold. In contrast to [100], the reduced eigenvalue problems admit exponential *tri*chotomies in accordance with the slow-fast structure of the eigenvalue problem (3.3). Moreover, the full eigenvalue problem (3.3) is a (λ, ε)-perturbation of the reduced eigenvalue problems. As a result, the jump of the obtained piecewise continuous eigenfunction in our work depends on ε, λ and ν. The center dynamics captured by the exponential trichotomies prevents the critical curve from being exponentially small in terms of the period as in [100]; instead the curve scales with ε^2.

In [10] the location of a critical eigenvalue near the origin is determined in the context of fast traveling pulses (with oscillatory tails) in the FitzHugh-Nagumo equations. Again, Lin's method is employed to obtain a leading-order expression for this critical eigenvalue in terms of the small parameter ε. Similar to our work, the slow-fast structure yields a framework for the construction of a piecewise continuous eigenfunction to the associated eigenvalue problem. This framework consists of *four* (reduced) eigenvalue problems arising along the fast front and back and along the orbit segments on the slow manifolds which together constitute the pulse profile in the limit $\varepsilon \to 0$. However, in contrast to our work, it is sufficient to distinguish between center-stable dynamics and unstable dynamics in the eigenvalue problem. Thus, the introduction of an exponential weight yields exponential *di*chotomies for the reduced eigenvalue problems.

Lin's method then yields a piecewise continuous eigenfunction that has *two* ε- and λ-dependent jumps in the middle of the front and the back. Thus, Lyapunov-Schmidt reduction leads to a quadratic equation in λ rather than a linear one as in [100] and our work. One root of the quadratic corresponds to the translational eigenvalue sitting at the origin. The second root corresponds to the critical, non-trivial eigenvalue that scales with ε in the monotone case, while the scaling in the oscillatory case is $\varepsilon^{2/3}$.

In the aforementioned spectral analyses, the fine structure of the spectrum about the origin is decisive for stability, but not detectable in the relevant asymptotic limit. In these cases Lin's method proves to be a powerful tool to determine how the spectrum locally perturbs from the asymptotic limit. Therefore, we expect that Lin's method can be applied to a wide range of spectral perturbation problems – see also Remark 1.3.

Chapter 6

Destabilization mechanisms

6.1 Introduction

In this chapter we focus on instabilities of periodic pulse solutions to (1.9) as system parameters are varied. To describe the spectral geometry as the periodic pulse destabilizes, we need as much analytical grip as possible. Therefore, we restrict ourselves to the case $m = n = 1$ – see §3.8. We assume that equation (1.9) depends on a real parameter μ. An generic instability occurs at $\mu = \mu_*$ if one of the spectral stability criteria in Corollary 3.8 fails at $\mu = \mu_*$, while the others are still valid. Depending on which one of these criteria fails, we can identify the type of instability occurring when μ passes through μ_*.

Verification of the three spectral stability criteria in Corollary 3.8 requires explicit knowledge of the Evans function $\mathcal{E}_\varepsilon(\lambda, \gamma)$. In Chapter 3 we approximated the roots of the Evans function $\mathcal{E}_\varepsilon(\lambda, \gamma)$ by the zeros of the reduced Evans function $\mathcal{E}_0(\lambda, \gamma) = -\gamma\mathcal{E}_{f,0}(\lambda)\mathcal{E}_{s,0}(\lambda, \gamma)$ which is defined in terms of three simpler, lower-dimensional eigenvalue problems. This leads to asymptotic control over the spectrum and simplifies the verification of the first spectral stability criterium in Corollary 3.8. Moreover, we obtained higher-order control over the spectrum about the origin: we derived a leading-order expression $\lambda_0(\nu)$ for the critical spectral curve attached to the origin, which shrinks to the origin as $\varepsilon \to 0$. The latter simplifies the verification of the spectral stability criteria in Corollary 3.8 further, which eventually leads to spectral stability criteria in terms of simpler, lower-dimensional problems – see Corollaries 3.20 and 3.31.

The zeros of the fast Evans function $\mathcal{E}_{f,0}$ will in general depend on the parameter μ. However, by Proposition 3.24 the relative position of these zeros with respect to the origin is fixed, i.e. no root of the fast Evans function can pass through the origin as we vary μ. Thus, by the aforementioned spectral approximation results, generic instabilities occur if either the curve $\lambda_0(\nu)$ or a curve $\lambda_*(\nu)$ satisfying $\mathcal{E}_{s,0}(\lambda_*(\nu), e^{i\nu}) = 0$ transits through the imaginary axis as we vary μ. By Proposition 3.25 and 3.29 this is precisely the case if one of the following two scenarios occurs:

1. One of the quantities $\mathfrak{a}, \mathfrak{b}$ or $\mathfrak{w}$, defined in (3.24) and (3.32), changes sign as we vary μ;

2. For some $\gamma \in S^1$, there is a complex conjugate pair of roots of $\mathcal{E}_{s,0}(\cdot, \gamma)$ moving through the imaginary axis $i\mathbb{R} \setminus \{0\}$ as we vary μ.

By employing Proposition 3.29, we study the spectral configuration about the origin in detail in the first scenario. We establish that the instabilities are of sideband or period doubling type if $\mathfrak{a}$ or $\mathfrak{w}$ changes sign and of Hopf type if $\mathfrak{b}$ changes sign. Moreover, the second destabilization scenario above corresponds to a Hopf instability. We conclude that the only possible primary codimension-one instabilities occurring are of sideband, Hopf or period doubling type.

This second destabilization scenario has been studied in great detail in [27] for the Gierer-Meinhardt equations (2.26) when periodic pulse solutions approach a homoclinic limit. While decreasing the wave number k, the character of destabilization alternates between two kinds of Hopf instabilities. One in which the destabilization is caused by a conjugated pair of 1-eigenvalues crossing the imaginary axis, allowing for perturbations that are exactly in phase with the periodic solution. The other Hopf instability corresponds to a conjugated pair of -1-eigenvalues crossing the imaginary axis, allowing for antiphase perturbations. In (k, μ)-space the curves $\mathcal{H}_{\pm 1}$ corresponding to ± 1-Hopf instabilities intersect infinitely often as they oscillate about each other while both converging to the Hopf destabilization point of the homoclinic limit solution on the line $k = 0$. This phenomenon is called the *Hopf dance*. In the singular limit $\varepsilon \to 0$ the two curves $\mathcal{H}_{\pm 1}$ cover the boundary of the region of stable pulse solutions. The boundary is non-smooth at the (transversal) intersection points of $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$. This corresponds to an associated higher order phenomenon: *the belly dance*. The analysis of these phenomena in the Gierer-Meinhardt system relies crucially on the specific characteristics of the equations; in particular, on the fact that the slow dynamics away from the pulses are driven by linear equations.

We employ our spectral methods to show that both the Hopf and belly dance are persistent mechanisms that occur in the general class (1.9) of *slowly nonlinear* systems – see §1.3. Second, we wish is to identify whether the limiting homoclinic pulse is the last 'periodic' pulse to become unstable as we vary μ. This was conjectured by W.M. Ni in the context of the Gierer-Meinhardt equations [80]. We establish an explicit sign criterion to determine whether the homoclinic pulse solution is the last or the first to destabilize.

This chapter is structured as follows. First, we provide a complete overview of the possible codimension-one instabilities for periodic pulse solutions to (1.9). Then, we study the spectral geometry in the two generic destabilization scenarios above and identify the type of instability occurring. Subsequently, we switch to the regime where the periodic pulse approaches a homoclinic limit. Before we study destabilization mechanisms in the homoclinic limit, we collect results from the literature concerning the existence and spectral properties of homoclinic pulse solutions to (1.9). Next, we provide the leading and next order geometry of the spectral curves crossing the imaginary axis, when periodic pulse solutions undergo a Hopf destabilization in the homoclinic limit. This key result then yields the existence of the Hopf and belly dance destabilization mechanisms and leads to a criterion which determines whether the homoclinic pulse is the last or the first periodic pulse solution to destabilize.

6.2 Classification of codimension-one instabilities

Let $\check{\phi}_{\mathrm{p},\varepsilon}$ be a periodic pulse solution to (1.9), established in Theorem 2.3. We assume that equation (1.9) depends on a real parameter μ. The periodic pulse $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable if the three conditions in Corollary 3.8 are satisfied. A codimension-one instability of $\check{\phi}_{\mathrm{p},\varepsilon}$ occurs if one of these conditions fails as we vary μ, while the others are still valid. Denote by $\mathcal{E}_{\varepsilon,\mu}(\lambda,\gamma)$ the associated Evans function (depending on μ). Suppose one of the conditions in Corollary 3.8 is violated by a pair $(\lambda_*,\nu_*) \in i\mathbb{R} \times [-\pi,\pi]$ at $\mu = \mu_*$. Consequently, it holds $\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*}) = 0$. If we have $\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*}) \neq 0$, the implicit function theorem yields a local expansion of the marginally stable spectral curve $\lambda_c(\nu)$ through λ_*:

$$\lambda_c(\nu) = \lambda_* + \frac{a_2}{2!}(\nu-\nu_*)^2 + \frac{a_4}{4!}(\nu-\nu_*)^4 + \mathcal{O}\left((\nu-\nu_*)^6\right),$$

with $a_2, a_4 \in \mathbb{C}$. Note that Proposition 3.7 implies that the odd coefficients in the expansion of $\lambda_c(\nu)$ must be zero. The leading coefficient a_2 can be computed through implicit differentiation:

$$a_2 = \frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})e^{2i\nu_*}}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})}.$$

In the case $a_2 = 0$, we have

$$a_4 = \frac{-\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})e^{4i\nu_*}}{\partial_\lambda \mathcal{E}_{\varepsilon,\mu_*}(\lambda_*, e^{i\nu_*})}.$$

This gives rise to the following classification of codimension-one instabilities – see [93, Section 3.3].

- γ_*-*Hopf.* The second and third condition in Corollary 3.8 are satisfied and the first condition is violated by a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1}) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})\gamma_*^{\pm2}}{\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}{\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(\pm\lambda_*, \gamma_*^{\pm1})}\right] \neq 0.$$

- *Spatial period doubling.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at $\gamma = -1$ so that

$$\mathcal{E}_{\varepsilon,\mu_*}(0,-1) = 0, \quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,-1)\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,-1) < 0, \quad \partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(0,-1) \neq 0.$$

- γ_*-*Turing.* The first and third condition in Corollary 3.8 are satisfied and the second condition is violated at a unique pair $\gamma_*^{\pm} \in S^1 \setminus \{\pm1\}$ satisfying

$$\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*^{\pm1}) = 0, \quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*^{\pm1})\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*^{\pm1})\gamma_*^{\pm2} < 0, \quad \partial_\mu\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma_*^{\pm1}) \neq 0.$$

- *Sideband.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1)=0,\quad \partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,1)\partial_{\gamma\gamma\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1)>0,\quad \partial_{\gamma\gamma\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1)\neq 0.$$

- *Fold/Pitchfork.* The first and second condition in Corollary 3.8 are satisfied and the third condition is violated so that

$$\partial_\lambda\mathcal{E}_{\varepsilon,\mu_*}(0,1)=0,\quad \partial_{\lambda\lambda}\mathcal{E}_{\varepsilon,\mu_*}(0,1),\partial_{\gamma\gamma}\mathcal{E}_{\varepsilon,\mu_*}(0,1),\partial_{\lambda\mu}\mathcal{E}_{\varepsilon,\mu_*}(0,1)\neq 0.$$

Using the spectral stability results from Chapter 3 one easily verifies that the only possible primary codimension-one instabilities are of sideband, Hopf or period doubling type.

Proposition 6.1. *Suppose $m=n=1$. The periodic pulse solution $\check{\phi}_{p,\varepsilon}(\check{x})$ to* (1.9) *cannot be destabilized through a Turing or fold instability.*

Proof. In the case of a γ_*-Turing instability, $\mathcal{E}_{\varepsilon,\mu_*}(0,\cdot)$ has double roots $\gamma_*^{\pm 1}$ and 1 with $\gamma_*\in S^1\setminus\{1\}$. However, this is impossible, since $\mathcal{E}_{\varepsilon,\mu_*}(0,\gamma)$ is a quartic polynomial in γ by Proposition 3.11. In the case of a fold instability, 0 is a double root of the reduced Evans function $\mathcal{E}_{0,\mu_*}(\cdot,1)$ by Theorem 3.15. Since 0 is a simple root of the fast Evans function $\mathcal{E}_{f,0,\mu_*}$ by Proposition 3.24, the slow Evans function $\mathcal{E}_{s,0,\mu_*}(\cdot,1)$ also has a root 0. Thus, Proposition 3.25 yields $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*)=-1$. So, by Corollary 3.32 there exists a λ in the spectrum $\sigma(\mathcal{L}_\varepsilon)$ with $\mathrm{Re}(\lambda)>0$. Hence, the first condition in Corollary 3.8 is not satisfied, which contradicts the occurrence of a fold instability. □

To identify which one of the three remaining instabilities occurs when the periodic pulse $\check{\phi}_{p,\varepsilon}$ destabilizes does not require control over the full Evans function $\mathcal{E}_\varepsilon$. In the next section we show that generically it is sufficient to track the quantities $\mathfrak{a},\mathfrak{b}$ and $\mathfrak{w}$ and roots of the slow Evans function $\mathcal{E}_{s,0}$ as we vary μ.

6.3 Generic destabilization mechanisms

Let $\check{\phi}_{p,\varepsilon}$ be a periodic pulse solution to (1.9), established in Theorem 2.3. We assume that equation (1.9) depends on a real parameter μ. In the introduction in §6.1 we observed that generically instabilities occur precisely if either one of the quantities $\mathfrak{a}(\mu)$, $\mathfrak{b}(\mu)$ or $\mathfrak{w}(\mu)$, defined in (3.24) and (3.32), changes sign or, for some $\gamma_*\in S^1$, there is a complex conjugate pair of roots of the slow Evans function $\mathcal{E}_{s,0,\mu}(\cdot,\gamma_*)$ moving through the imaginary axis $i\mathbb{R}\setminus\{0\}$ as μ passes through some value μ_*. Thus, we distinguish between the following generic destabilization scenarios:

(D1) $\mathfrak{w}(\mu_*)=0$, $\partial_\mu\mathfrak{w}(\mu_*)\neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{b}(\mu_*)>0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda,\gamma)\neq 0$ for all $\gamma\in S^1$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$;

(D2) $\mathfrak{b}(\mu_*)=0$, $\partial_\mu\mathfrak{b}(\mu_*)\neq 0$, $\mathfrak{a}(\mu_*)\mathfrak{w}(\mu_*)>0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda,\gamma)\neq 0$ for all $\gamma\in S^1$ and $\lambda\in\mathbb{C}$ with $\mathrm{Re}(\lambda)\geq 0$ and $\lambda\neq 0$;

(D3) $\mathfrak{a}(\mu_*) = 0$, $\partial_\mu \mathfrak{a}(\mu_*) \neq 0$, $\mathfrak{b}(\mu_*)\mathfrak{w}(\mu_*) > 0$ and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $\gamma \in S^1$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $\lambda \neq 0$;

(D4) There is a unique quadruple $(\pm\lambda_*, \gamma_*^{\pm 1})$ with $\lambda_* \in i\mathbb{R} \setminus \{0\}$ and $\gamma_* \in S^1$ satisfying

$$\mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1}) = 0, \quad \mathrm{Re}\left[\frac{\partial_{\gamma\gamma}\mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})\gamma_*^{\pm 2}}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}\right] < 0, \quad \mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}{\partial_\lambda \mathcal{E}_{s,0,\mu_*}(\pm\lambda_*, \gamma_*^{\pm 1})}\right].$$

In addition, $\mathfrak{a}(\mu_*)$, $\mathfrak{b}(\mu_*)$ and $\mathfrak{w}(\mu_*)$ have the same non-zero sign and $\mathcal{E}_{s,0,\mu_*}(\lambda, \gamma) \neq 0$ for all $(\lambda, \gamma) \in S^1 \times \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$ and $(\lambda, \gamma) \neq (\pm\lambda_*, \gamma_*^{\pm 1})$.

In this section we identify the type of instability occurring in these four scenarios. Clearly, the following result is an immediate consequence of Theorems 3.15 and 3.17 and Proposition 3.29.

Corollary 6.2. *Assume $m = n = 1$ and **(D4)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (1.9) *destabilizes through a γ_ε-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon - \gamma_*| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

The remainder of this section is devoted to the identification of the type of instability occurring in the three other scenarios, which requires detailed control over the spectral geometry about the origin.

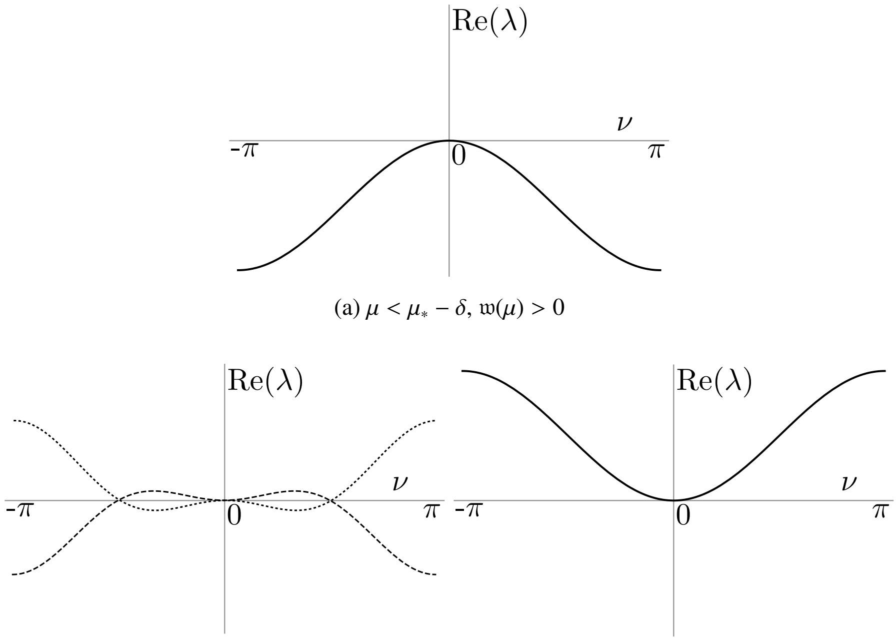
6.3.1 The first destabilization scenario

Let **(D1)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) > 0$. Then, there exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that it holds $\mathcal{E}_{\mathrm{s},0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$. Thus, by Corollary 3.16, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum for any $\mu \in M$. In addition, $\lambda_{\varepsilon,\mu}$ is real-valued and analytic and, by Proposition 3.29, we have the leading-order approximation,

$$\lambda_{\varepsilon,\mu}(\nu) = \varepsilon^2 \mathfrak{a}(\mu)\mathfrak{w}(\mu)\frac{\cos(\nu) - 1}{1 + \cos(\nu) + 2\mathfrak{a}(\mu)\mathfrak{b}(\mu)} + O\left(\varepsilon^3 \left|\log(\varepsilon)\right|^5\right), \tag{6.1}$$

for any $\mu \in M$ and $\nu \in \mathbb{R}$. So, given $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, for $\mu \in M$ with $|\mu - \mu_*| > \delta$ the approximation (6.1) gives the spectral configuration depicted in Figures 6.1a and 6.1c. Hence, $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and unstable for $\mu > \mu_* + \delta$. For $|\mu - \mu_*| \leq \delta$ our leading-order approximation (6.1) is insufficient to determine the precise position of the critical spectral curve with respect to the imaginary axis. However, since $\lambda_{\varepsilon,\mu}$ is real-valued for any $\mu \in M$ and Turing instabilities do not occur by Proposition 6.1, we have obtained the following result.

Proposition 6.3. *Assume $m = n = 1$ and **(D1)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to* (1.9) *destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*



(a) $\mu < \mu_* - \delta$, $\mathfrak{w}(\mu) > 0$

(b) $|\mu - \mu_*| \leq \delta$, $\mathfrak{w}(\mu) \approx 0$

(c) $\mu > \mu_* + \delta$, $\mathfrak{w}(\mu) < 0$

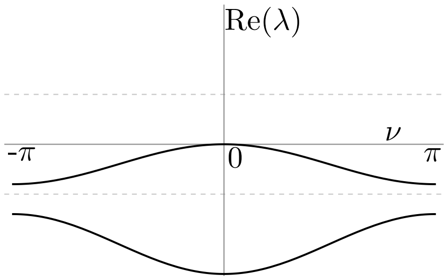
Figure 6.1: The spectral geometry about the origin is depicted in the first generic destabilization scenario **(D1)** with $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{w}(\mu_*) > 0$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

6.3.2 The second destabilization scenario

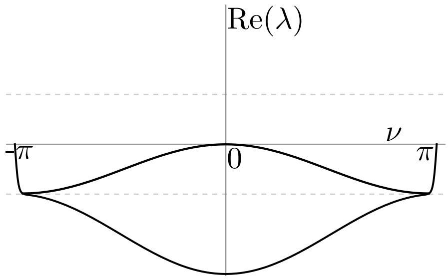
Let **(D2)** hold true and assume without loss of generality $\mathfrak{a}(\mu_*)\partial_\mu \mathfrak{b}(\mu_*) > 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that $\mathfrak{a}(\mu)\mathfrak{w}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda, \gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0, \delta)$ with $\mathrm{Re}(\lambda) \geq 0$. In addition, it holds $\mathcal{E}_{s,0,\mu}(0, \gamma) \neq 0$ for any $\gamma \in S^1$ and $\mu \in M$ with $\mu < \mu_* - \delta$ by Proposition 3.25. So, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum by Corollary 3.16 for any $\mu \in M$ with $\mu < \mu_* - \delta$. In that situation $\lambda_{\varepsilon,\mu}(\nu)$ is by Proposition 3.29 approximated by (6.1) – see Figure 6.2a. Denote

$$\nu_\diamond(\mu) := \arccos\left(\max\{-1 - 2\mathfrak{a}(\mu)\mathfrak{b}(\mu), -1\}\right), \quad \mu \in M.$$

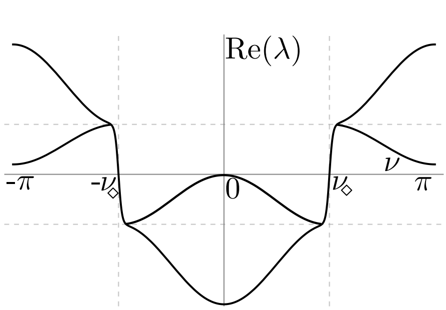
For any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists by Theorem 3.19 and Proposition 3.29 a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0, \delta)$ that is approximated by (6.1) – see Figure 6.2d. Combining this with Proposition 3.25 implies that for any $\mu \in M$ and $\nu \in [-\pi, \pi]$ there are precisely two $e^{i\nu}$-eigenvalues of positive real part if $|\nu| > \nu_\diamond(\mu) + \delta$ and no $e^{i\nu}$-eigenvalues of positive real part if $|\nu| < \nu_\diamond(\mu) - \delta$ – see Figure 6.2c.



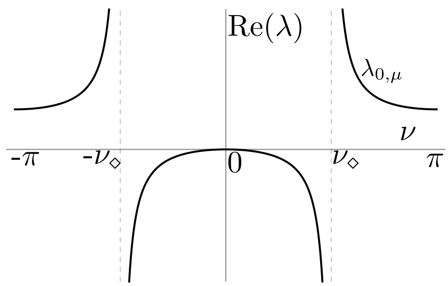
(a) $\mu < \mu_* - \delta$, $\mathfrak{b}(\mu) > 0$



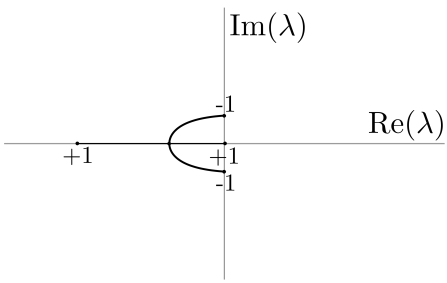
(b) $|\mu - \mu_*| < \delta$, $\mathfrak{b}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



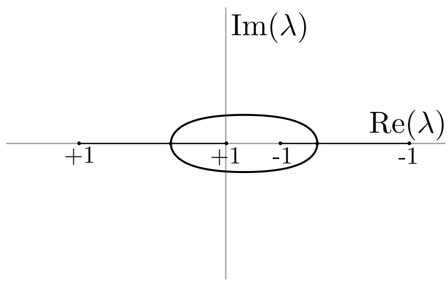
(c) $\mu > \mu_* + \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$ with singularities $\pm\nu_\diamond(\mu)$



(e) Spectrum for $|\mu - \mu_*| < \delta$, $\mathfrak{b}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(f) Spectrum for $\mu > \mu_* - \delta$, $\mathfrak{b}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$

Figure 6.2: The spectral geometry about the origin is depicted in the second destabilization scenario **(D2)** with $\mathfrak{a}(\mu_*)\partial_\mu\mathfrak{b}(\mu_*) > 0$. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = \mathcal{O}(\varepsilon^2)$.

Therefore, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ is spectrally stable for $\mu \in M$ with $\mu < \mu_* - \delta$ and there is unstable spectrum for $\mu > \mu_* + \delta$. In particular, we observe that $e^{i\nu}$-eigenvalues with $|\nu \pm \pi| < \delta$ are in the right half-plane strictly before $e^{i\nu}$-eigenvalue with $|\nu| < \delta$ as μ increases. Thus, a sideband instability cannot occur.

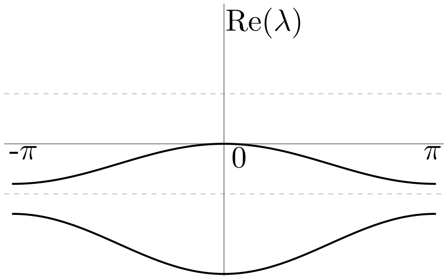
Now suppose a spatial period doubling bifurcation occurs at $\mu = \mu_\varepsilon$. By the previous observations there are precisely two -1-eigenvalues in the right half-plane for $\mu \in M$ with $\mu > \mu_* + \delta \geq \mu_\varepsilon$. By definition of a period doubling bifurcation, the most unstable one of these -1-eigenvalues must have crossed the imaginary axis at the origin. Since the spectrum is symmetric in the real axis – see Proposition 3.7 – the same holds for the other -1-eigenvalue. If the -1-eigenvalues cross simultaneously, then $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ has a root 1 of multiplicity two and a root -1 of multiplicity four, which is impossible, since $\mathcal{E}_{\varepsilon,\mu_\varepsilon}(0,\cdot)$ is a quartic polynomial by Proposition 3.11. If one -1-eigenvalue crosses first, then, by the implicit function theorem and symmetry of the spectrum in the real axis, this -1-eigenvalue is attached to a spectral branch that lies on the real axis. So, if the second -1-eigenvalue crosses at $\mu = \tilde{\mu}_\varepsilon > \mu_\varepsilon$, then $\mathcal{E}_{\varepsilon,\tilde{\mu}_\varepsilon}(0,\cdot)$ has double roots 1 and -1 and simple roots $\gamma^{\pm 1}$ for some $\gamma \in S^1 \setminus \{\pm 1\}$, which is again impossible. We conclude that a period doubling bifurcation cannot occur. So, by Proposition 6.1 a Hopf instability occurs – see Figure 6.2b. Thus, we obtain the following result.

Proposition 6.4. *Assume $m = n = 1$ and **(D2)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) destabilizes through a γ_ε-Hopf instability at $\mu = \mu_\varepsilon$ with $\gamma_\varepsilon \in S^1$ satisfying $|\gamma_\varepsilon + 1| < \delta$ and $|\mu_\varepsilon - \mu_*| < \delta$.*

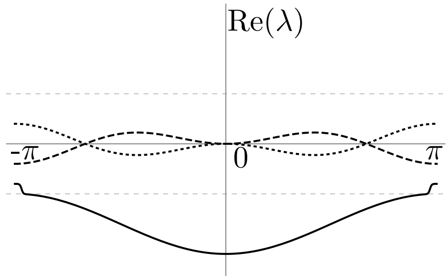
6.3.3 The third destabilization scenario

Let **(D3)** hold true and assume without loss of generality $\mathfrak{w}(\mu_*)\partial_\mu \mathfrak{a}(\mu_*) > 0$. Take $\delta > 0$. There exists a neighborhood $M \subset \mathbb{R}$ of μ_* such that $\mathfrak{w}(\mu)\mathfrak{b}(\mu) > 0$, $1 + \mathfrak{a}(\mu)\mathfrak{b}(\mu) > 0$ and $\mathcal{E}_{s,0,\mu}(\lambda,\gamma) \neq 0$ for any $\gamma \in S^1$, $\mu \in M$ and $\lambda \in \mathbb{C} \setminus B(0,\delta)$ with $\mathrm{Re}(\lambda) \geq 0$. As in the second destabilization scenario **(D2)**, for any $\mu \in M$ with $\mu < \mu_* - \delta$, the critical spectral curve $\lambda_{\varepsilon,\mu}(\nu)$ attached to the origin is an isolated part of the spectrum and it is approximated by (6.1) – see Figure 6.3a. Also similar to scenario **(D2)**, we establish that for any $\mu \in M$ with $\mu > \mu_* - \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there exists a unique root $\lambda_{\varepsilon,\mu}(\nu)$ of $\mathcal{E}_{\varepsilon,\mu}(\cdot, e^{i\nu})$ in $B(0,\delta)$ that is approximated by (6.1) – see Figure 6.3d. Combining this with Proposition 3.25 implies that for any $\mu \in M$ with $\mu > \mu_* + \delta$ and $\nu \in [-\pi, \pi]$ with $|\nu \pm \nu_\diamond(\mu)| > \delta$ there is precisely one $e^{i\nu}$-eigenvalue of positive real part. This excludes the possibility of a Hopf destabilization. So, by Proposition 6.1 either a sideband instability or period doubling bifurcation occurs – see Figure 6.3. Thus, we obtain the following result.

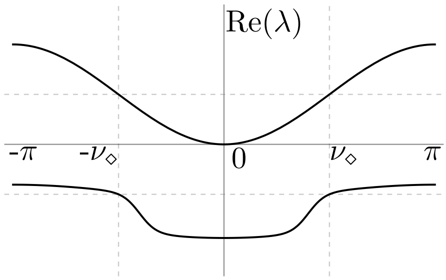
Proposition 6.5. *Assume $m = n = 1$ and **(D3)** holds true. For any $\delta > 0$ there exists an $\varepsilon_0 > 0$ such that, provided $\varepsilon \in (0, \varepsilon_0)$, the periodic pulse solution $\check{\phi}_{\mathrm{p},\varepsilon}$ to (1.9) destabilizes through a sideband instability or spatial period doubling bifurcation at $\mu = \mu_\varepsilon$ satisfying $|\mu_\varepsilon - \mu_*| < \delta$.*



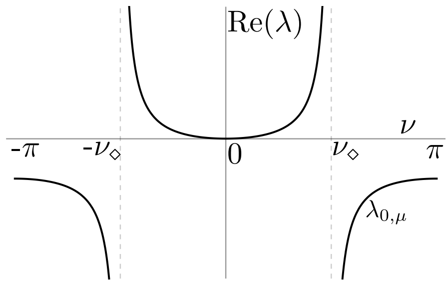
(a) $\mu < \mu_* - \delta$, $\mathfrak{a}(\mu) > 0$



(b) $|\mu - \mu_*| < \delta$, $\mathfrak{a}(\mu) \approx 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(c) $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$



(d) Graph of $\lambda_{0,\mu}(\nu)$ for $\mu > \mu_* + \delta$, $\mathfrak{a}(\mu) < 0$ and $\mathfrak{a}(\mu)\mathfrak{b}(\mu) > -1$ with singularities $\pm\nu_\diamond(\mu)$

Figure 6.3: The spectral geometry about the origin is depicted in the third destabilization scenario **(D3)**. The area between the horizontal dashed lines correspond to the regime $\mathrm{Re}(\lambda) = \mathcal{O}(\varepsilon^2)$. In the second panel, the dotted curve corresponds to the case of a spatial period doubling bifurcation and the dashed curve to a sideband instability.

6.4 Destabilization mechanisms in the homoclinic limit

In this section we are interested in the destabilization mechanisms of periodic pulse solutions to (1.9) approaching a homoclinic limit. We assume that (1.9) depends on a real parameter μ. It is well-known [39, 99] that the spectral curves corresponding to the periodic pulse shrink to the eigenvalues associated with the limiting homoclinic as the wavelength tends to infinity. This process is of particular interest, when it occurs in the vicinity of a destabilization of the homoclinic pattern.

Generic instabilities of symmetric homoclinic pulse solutions are either of Hopf, saddle-node or pitchfork type [30]. A saddle-node or pitchfork bifurcation occurs if a (simple) real eigenvalue passes through the origin as we vary μ. At a Hopf destabilization a pair of complex conjugate eigenvalues transits through the imaginary axis as we vary μ.

Suppose the homoclinic pulse destabilizes at $\mu = \mu_*$. Since the spectral curves corresponding to a long-wavelength periodic pulse lie close to the eigenvalues associated with the homoclinic, the periodic pulse is also unstable for certain μ-values close to μ_*. However, whether the periodic pulse solution also *destabilizes* at some μ-value close to μ_* depends on the position of the critical spectral curve attached to 0 – see §3.6. We establish that the relative position of the critical curve with respect to the imaginary axis does not change in the homoclinic limit.

If the critical spectral curve is confined to the left half-plane and the homoclinic pulse undergoes a Hopf instability at $\mu = \mu_*$, then the long-wavelength periodic pulse solution also destabilizes at some μ-value close to μ_*. The character of destabilization alternates between two kinds of Hopf instabilities as the wavelength tends to infinity. As explained in the introduction §6.1 the latter is called the 'Hopf dance' and the associated higher order phenomenon the 'belly dance'.

In general it is quite challenging to determine the spectral structure, when a periodic pulse solution approaches a homoclinic limit. However, the spectral reduction mechanisms in Chapter 3 for periodic pulses and in [30] for homoclinic pulses allow us to describe this process in great detail in the singular limit $\varepsilon \to 0$. In this limit it is therefore possible to prove the occurrence of the Hopf and belly dance destabilization mechanisms.

This section is structured as follows. We start by collecting results from the literature concerning the existence and spectral properties of homoclinic pulse solutions to (1.9). Second, we construct a family of periodic pulse solutions to (2.1) that converges to a homoclinic pulse. Third, we study the geometry of the spectral curves associated with the periodic pulses in the long-wavelength limit. Then, using these spectral results, we prove the occurrence of the Hopf and belly dance destabilization mechanisms. In addition, we establish an explicit sign criterion to determine whether the limiting homoclinic pulse solution is the last (or the first) 'periodic' pattern to destabilize in the case of a Hopf destabilization.

6.4.1 Existence of homoclinic pulse solutions

In Chapter 2 we constructed a singular periodic orbit by concatenating a pulse solution to the fast reduced systems (2.2) and an orbit segment on the slow manifold $\mathcal{M}$, satisfying the slow reduced system (2.4), in such a way that they form a closed loop. Then, we proved that an actual periodic pulse solution to (2.1) lies in the vicinity of the singular one, provided $\varepsilon > 0$ is sufficiently small. Similarly, one can construct a singular homoclinic orbit by gluing a pulse solution to the fast reduced system (2.2) to a solution to the slow reduced system (2.4) that converges to a fixed point on $\mathcal{M}$. In the case $m = n = 1$ one proves in [30] the existence of an actual homoclinic solution close to the singular one:

Theorem 6.6. *[30, Theorem 2.1] Let $m = n = 1$ and assume **(S1)**, **(S2)** and **(E1)** hold true. Suppose there exists a solution $\psi_\infty(\check{x}) = (u_\infty(\check{x}), p_\infty(\check{x}))$ to (2.4), which intersects the touch down curve $\mathcal{T}_+$ transversally at $\check{x} = 0$ and satisfies $\lim_{\check{x}\to\infty} p_\infty(\check{x}) = 0$.*

Then, for any $\delta > 0$ there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$ there exists a homoclinic solution $\phi_{\infty,\varepsilon}(x)$ to (2.1) *satisfying the following assertions:*

1. ***Reversibility***
   We have $\phi_{\infty,\varepsilon}(x) = R\phi_{\infty,\varepsilon}(-x)$ for $x \in \mathbb{R}$, where $R\colon \mathbb{R}^4 \to \mathbb{R}^4$ is the reflection in the space $p = q = 0$.

2. ***Singular limit***
   The Hausdorff distance between the orbit of $\phi_{\infty,\varepsilon}$ in $\mathbb{R}^4$ and the singular concatenation

$$\{(u_\infty(\check{x}), \pm p_\infty(\check{x}), 0, 0) : \check{x} \geq 0\} \cup \{\phi_{\mathrm{h}}(x, u_\infty(0)) : x \in \mathbb{R}\}, \tag{6.2}$$

   is smaller than δ.

6.4.2 Spectral properties of homoclinic pulse solutions

Suppose $\phi_{\infty,\varepsilon}(x)$ is a homoclinic pulse solution established in Theorem 6.6 with singular limit (6.2). Let $\check{\phi}_{\infty,\varepsilon}(\check{x})$ be the corresponding solution to (1.9). To study destabilization mechanisms as periodic pulse solutions to (1.9) approach a homoclinic limit, we need analytical grip on the spectrum of the linearization about $\check{\phi}_{\infty,\varepsilon}$. We linearize system (1.9) about $\check{\phi}_{\infty,\varepsilon}$ and obtain a differential operator $\mathcal{L}_{\infty,\varepsilon}$ on the space $C_{ub}(\mathbb{R}, \mathbb{R}^2)$. By [72, Theorem 3.1.9.ii] and [44, Theorem 1.3.2] $\mathcal{L}_{\infty,\varepsilon}$ is a closed, densely defined and sectorial operator with domain $C^2_{ub}(\mathbb{R}, \mathbb{R}^2)$. The eigenvalue problem $\mathcal{L}_{\infty,\varepsilon}\varphi = \lambda\varphi$ can be written as a first order system,

$$\varphi_x = \mathcal{A}_{\infty,\varepsilon}(x, \lambda)\varphi, \quad \varphi \in \mathbb{R}^4. \tag{6.3}$$

As in Chapter 3, we define an analytic Evans function in terms of (6.3) that locates the (critical) spectrum of $\mathcal{L}_{\infty,\varepsilon}$. Since $\check{\phi}_{\infty,\varepsilon}(x)$ is homoclinic, the limits $\lim_{x\to\pm\infty} \mathcal{A}_{\infty,\varepsilon}(x, \lambda) = \mathcal{A}_{*,\varepsilon}(\lambda)$ exist. Write $u_* = \lim_{\check{x}\to\infty} u_\infty(\check{x})$. Because $(u_*, 0)$ is a hyperbolic saddle in system (2.4), there exists $\Lambda < 0$ such that

$$-\min\{\partial_v G(u_*, 0, 0), \partial_u H_1(u_*, 0, 0)\} < \Lambda < 0.$$

One readily observes that the matrix $\mathcal{A}_{*,\varepsilon}(\lambda)$ is hyperbolic on the half-plane C_Λ. Hence, by Proposition 4.7, system (6.3) admits for $\lambda \in C_\Lambda$ exponential dichotomies on both half-lines $[0, \infty)$ and $(-\infty, 0]$ such that the associated projections are analytic in λ. Note that the dichotomy constants depend on ε and λ. Denote by $\varphi^s_{1,\varepsilon}(x, \lambda)$ and $\varphi^s_{2,\varepsilon}(x, \lambda)$ two solutions that span the space of exponentially decaying solutions to (6.3) as $x \to \infty$. Similarly, let $\varphi^u_{1,\varepsilon}(x, \lambda)$ and $\varphi^u_{2,\varepsilon}(x, \lambda)$ span the space of exponentially decaying solutions as $x \to -\infty$. By [98] the spectrum in C_Λ is located by the analytic Evans function $\mathcal{E}_{\infty,\varepsilon}\colon C_\Lambda \to \mathbb{C}$ given by

$$\mathcal{E}_{\infty,\varepsilon}(\lambda) = \det\left(\varphi^s_{1,\varepsilon}(0, \lambda) \mid \varphi^s_{2,\varepsilon}(0, \lambda) \mid \varphi^u_{1,\varepsilon}(0, \lambda) \mid \varphi^u_{2,\varepsilon}(0, \lambda)\right).$$

More precisely, a point $\lambda \in C_\Lambda$ is in the spectrum $\sigma(\mathcal{L}_{\infty,\varepsilon})$ if and only if we have $\mathcal{E}_{\infty,\varepsilon}(\lambda) = 0$. We emphasize that the spectrum of $\mathcal{L}_{\infty,\varepsilon}$ in C_Λ consists of point spectrum only – see [98].

Similarly to the case of periodic pulse solutions – see Chapter 3 – we can define an explicit reduced Evans function $\mathcal{E}_{\infty,0}\colon C_\Lambda \to \mathbb{C}$, whose zeros approximate those of $\mathcal{E}_{\infty,\varepsilon}$, provided that $\varepsilon > 0$ is sufficiently small. Again, the reduced Evans function reflects the slow-fast structure of the eigenvalue problem (6.3). Thus, the analytic map $\mathcal{E}_{\infty,0}$ is given by the product,

$$\mathcal{E}_{\infty,0}(\lambda) = \mathcal{E}_{\infty,f}(\lambda)\mathcal{E}_{\infty,s}(\lambda). \tag{6.4}$$

Here, the analytic fast Evans function $\mathcal{E}_{\infty,f}\colon C_\Lambda \to \mathbb{C}$ locates the eigenvalues $\lambda \in \mathbb{C}$ of the homogeneous fast eigenvalue problem,

$$\varphi_x = \mathcal{A}_{22,0}(x, u_\infty(0), \lambda)\varphi, \quad \varphi \in \mathbb{C}^2. \tag{6.5}$$

The slow Evans function $\mathcal{E}_{\infty,s}\colon C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0) \to \mathbb{C}$ is defined in terms of the inhomogeneous fast eigenvalue problem (3.8) and the slow eigenvalue problem,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(\check{x}, \lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \qquad \mathcal{A}_\infty(\check{x}, \lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\infty(\check{x}), 0, 0) + \lambda & 0 \end{pmatrix}. \tag{6.6}$$

Note that the coefficient matrix of (6.6) converges as $\check{x} \to \infty$ to the asymptotic matrix,

$$\mathcal{A}_*(\lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_*, 0, 0) + \lambda & 0 \end{pmatrix}, \tag{6.7}$$

which is hyperbolic on C_Λ with eigenvalues $\pm\sqrt{\partial_u H_1(u_*, 0, 0) + \lambda}$. An application of Proposition 4.3 yields a unique analytic solution $\varphi_\infty(\check{x}, \lambda) = (\hat{u}_\infty(\check{x}, \lambda), \hat{p}_\infty(\check{x}, \lambda))$ to (6.6) that satisfies

$$\lim_{\check{x}\to\infty} \hat{u}_\infty(\check{x}, \lambda) e^{\check{x}\sqrt{\partial_u H_1(u_*,0,0)+\lambda}} = 1, \quad \lambda \in C_\Lambda. \tag{6.8}$$

Thus, the slow Evans function is explicitly given by

$$\mathcal{E}_{\infty,s}(\lambda) = \det\left(\varphi_\infty(0, \lambda) \mid \Upsilon(u_\infty(0), \lambda) R_s \varphi_\infty(0, \lambda)\right),$$

where the term $\Upsilon(u, \lambda)$ is defined in (3.11). We emphasize that the slow Evans function $\mathcal{E}_{\infty,s}$ is meromorphic on C_Λ such that the product $\mathcal{E}_{\infty,0}$ given in (6.4) is analytic on C_Λ. Having defined the reduced Evans function $\mathcal{E}_{\infty,0}$, we state the approximation result.

Theorem 6.7. *[30, Section 4] Let Γ be a simple closed curve, contained in $C_\Lambda \setminus \mathcal{E}_{\infty,0}^{-1}(0)$. For $\varepsilon > 0$ sufficiently small, the number of zeros of $\mathcal{E}_{\infty,\varepsilon}$ interior to Γ equals the number of zeros of $\mathcal{E}_{\infty,0}$ interior to Γ including multiplicity.*

By [30, Lemma 5.9] the slow Evans function at 0 can be expressed as

$$\mathcal{E}_{\infty,s}(0) = -2\mathfrak{d}_\infty \mathfrak{a}_\infty, \tag{6.9}$$

with

$$\mathfrak{d}_\infty := -\mathcal{J}(u_\infty(0)), \quad \mathfrak{a}_\infty := \mathcal{J}'(u_\infty(0))\mathcal{J}(u_\infty(0)) - H_1(u_\infty(0), 0, 0), \tag{6.10}$$

where $\mathcal{J}\colon U_\mathrm{h} \to \mathbb{R}$ is defined in (2.5). This leads to the following result, whose proof is along the lines of the proof of Proposition 3.32.

Proposition 6.8. *[30, Section 5] There exists a positive root of* $\mathcal{E}_{\infty,s}$ *if it holds* $\mathfrak{a}_\infty \mathfrak{d}_\infty < 0$ *or* $(u_\infty(0) - u_*)\mathfrak{d}_\infty < 0$*, where* $\mathfrak{d}_\infty$ *and* $\mathfrak{a}_\infty$ *are defined in* (6.10).

As in Proposition 3.24 one establishes in [30, Section 4] that the roots of the fast Evans function $\mathcal{E}_{\infty,f}$ are real and simple. In particular, 0 is a root of $\mathcal{E}_{\infty,f}$ and there is precisely one positive zero $\lambda_{\infty,1} > 0$. Let $(v_{\infty,1}(x), q_{\infty,1}(x))$ be an eigenfunction of (6.5) corresponding to $\lambda_{\infty,1}$. By [30, Lemma 5.1] the slow Evans function $\mathcal{E}_{\infty,s}$ has a pole at $\lambda_{\infty,1}$ if and only if the generic condition $\mathfrak{i}_\infty \neq 0$ is satisfied, where

$$\mathfrak{i}_\infty := \hat{u}_\infty(0, \lambda_{\infty,1}) \int_{-\infty}^{\infty} v_{\infty,1}(x) \frac{\partial H_2}{\partial v}(u_\star, v_{\mathrm{h}}(x, u_\star))dx \int_{-\infty}^{\infty} v_{\infty,1}(x) \frac{\partial G}{\partial u}(u_\star, v_{\mathrm{h}}(x, u_\star), 0)dx, \tag{6.11}$$

where $u_\star := u_\infty(0)$ and $\hat{u}_\infty(\check{x}, \lambda)$ denotes the u-coordinate of the solution $\varphi_\infty(\check{x}, \lambda)$ to (6.6). Thus, due to zero-pole cancelation, the reduced Evans function $\mathcal{E}_{\infty,0}$ has a zero at $\lambda_{\infty,1}$ if and only if $\mathfrak{i}_\infty = 0$.

6.4.3 Destabilization mechanisms for homoclinic pulse solutions

We study codimension-one instabilities of the homoclinic pulse solution $\check{\phi}_{\infty,\varepsilon}$ to (1.9), which is established in Theorem 6.6. Since the critical spectrum of $\mathcal{L}_{\infty,\varepsilon}$ is given by $\mathcal{E}_{\infty,\varepsilon}^{-1}(0)$, an instability occurs precisely if a root of the Evans function $\mathcal{E}_{\infty,\varepsilon}$ moves through the imaginary axis as we vary a real parameter μ. By Theorem 6.7 the roots of $\mathcal{E}_{\infty,\varepsilon}$ are approximated by the roots of the reduced Evans function $\mathcal{E}_{\infty,0}(\lambda) = \mathcal{E}_{\infty,f}(\lambda)\mathcal{E}_{\infty,s}(\lambda)$. One establishes in [30, Section 4] that the roots of the fast Evans function $\mathcal{E}_{\infty,f}$ are real and simple and that their relative location with respect to the origin is fixed as we vary μ. In addition, 0 is always a root of $\mathcal{E}_{\infty,f}$. Thus, generic instabilities occur precisely if roots of the slow Evans function $\mathcal{E}_{\infty,s}$ transit through the imaginary axis as we vary μ. Thus, by identity (6.9) we distinguish between the following generic destabilization scenarios:

1. One of the quantities $\mathfrak{a}_\infty$ or $\mathfrak{d}_\infty$, defined in (6.10), changes sign as we vary μ;

2. There is a complex conjugate pair of roots of $\mathcal{E}_{\infty,s}$ moving through the imaginary axis $i\mathbb{R} \setminus \{0\}$ as we vary μ.

In [30] one establishes that the homoclinic pulse undergoes a Hopf destabilization in the second scenario. Moreover, a saddle-node or pitchfork bifurcation occurs if $\mathfrak{a}_\infty$ changes sign.

6.4.4 Existence of a family of periodic pulse solutions approaching a homoclinic limit

In this section we establish with the aid of Theorems 2.3 and 6.6 a family of periodic pulse solutions to (2.1) approaching a homoclinic pulse solution in the long-wavelength limit. Key to the construction of such a family is the existence of a saddle in the slow reduced system (2.4).

(E3) ***Existence of saddle in the slow reduced system***
There exists $u_* \in U$ such that $\psi_* := (u_*, 0)$ is a hyperbolic saddle in (2.4). In addition, the touch-down curve $\mathcal{T}_+ = \{(u, \mathcal{J}(u)) : u \in U_\mathrm{h}\}$ intersects the stable manifold $W^s(\psi_*)$ transversally in some point ψ_0.

Theorem 6.9. *Let $m = n = 1$ and assume **(S1), (S2), (E1)** and **(E3)** hold true. Let $\psi_\infty(\check{x})$ be the solution to* (2.4) *in $W^s(\psi_*)$ with initial condition $\psi_\infty(0) = \psi_0$. There exists $\ell_0, \varepsilon_0 > 0$ such that the following assertions hold true:*

1. ***Saddle dynamics in slow reduced system***
For $\ell \in (\ell_0, \infty)$ there exists a solution $\psi_\ell(\check{x}) = (u_\ell(\check{x}), p_\ell(\check{x}))$ to (2.4) *that intersects $\mathcal{T}_+$ transversally at $\check{x} = 0$ and crosses the line $p = 0$ at $\check{x} = \ell$. In addition, $\psi_\ell(\check{x})$ converges as $\ell \to \infty$ to $\psi_\infty(\check{x})$ for each $\check{x} \in [0, \ell]$.*

2. ***Existence of family of periodic pulse solutions***
For $(\ell, \varepsilon) \in (\ell_0, \infty) \times (0, \varepsilon_0)$ there exists a reversibly symmetric, $2L_{\ell,\varepsilon}$-periodic pulse solution $\phi_{\ell,\varepsilon}$ to (2.1)*, whose orbit converges in the Hausdorff distance to the singular concatenation,*
$$\{(u_\ell(\check{x}), p_\ell(\check{x}), 0, 0) : \check{x} \in (0, 2\ell)\} \cup \{\phi_\mathrm{h}(x, u_\ell(0)) : x \in \mathbb{R}\}, \tag{6.12}$$
as $\varepsilon \to 0$ and whose period satisfies $\varepsilon L_{\ell,\varepsilon} \to \ell$ as $\varepsilon \to 0$.

3. ***Long wavelength limit***
For every $\varepsilon \in (0, \varepsilon_0)$ the family of solutions $\phi_{\ell,\varepsilon}$ converges pointwise on $[0, L_{\ell,\varepsilon}]$ to a reversibly symmetric, homoclinic pulse solution $\phi_{\infty,\varepsilon}$ to (2.1) *as $\ell \to \infty$. Moreover, $\phi_{\infty,\varepsilon}$ converges in Hausdorff distance to the singular concatenation,*
$$\{(u_\infty(\check{x}), \pm p_\infty(\check{x}), 0, 0) : \check{x} \in (0, \infty)\} \cup \{\phi_\mathrm{h}(x, u_\infty(0)) : x \in \mathbb{R}\}, \tag{6.13}$$
as $\varepsilon \to 0$.

Proof. The first assertion is immediate by Hamiltonian nature of the planar system (2.4). For any fixed $\ell > \ell_0$ the existence of a periodic pulse solution $\phi_{\ell,\varepsilon}(x)$ for $0 < \varepsilon \ll 1$ follows from Theorem 2.3. Following the proof of Theorem 2.3, one observes that the ε-bound is in fact ℓ-uniform. This establishes the second assertion. The existence of the homoclinic pulse solution $\phi_{\infty,\varepsilon}(x)$ for $0 < \varepsilon \ll 1$ follows from Theorem 6.6. Now fix $\varepsilon \in (0, \varepsilon_0)$. From the proof of Theorem 2.3 we deduce that the pointwise limits $\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x)$ exist for each $x \in \mathbb{R}$ and must lie on the stable manifold $W^s(\phi_{*,\varepsilon})$ in (2.1), where $\phi_{*,\varepsilon} \in \mathcal{M}$ is a saddle converging to $(\psi_*, 0)$ as $\varepsilon \to 0$. Moreover, the limiting orbit $\{\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x) : x \in \mathbb{R}\}$ is reversibly symmetric. On the other hand, the proof of Theorem 6.6 – see [30, Theorem 2.1] – shows that the 2-dimensional manifold $W^s(\phi_{*,\varepsilon})$ intersects the reversible symmetry plane $p = q = 0$ transversely in $\phi_{\infty,\varepsilon}(0)$. This intersection point is locally unique in a small ε- and ℓ-independent neighborhood of $\phi_{\infty,\varepsilon}(0)$. Thus, we conclude that for $x \in [0, L_{\ell,\varepsilon}]$ the pointwise limits $\lim_{\ell\to\infty} \phi_{\ell,\varepsilon}(x)$ are given by the homoclinic $\phi_{\infty,\varepsilon}(x)$. □

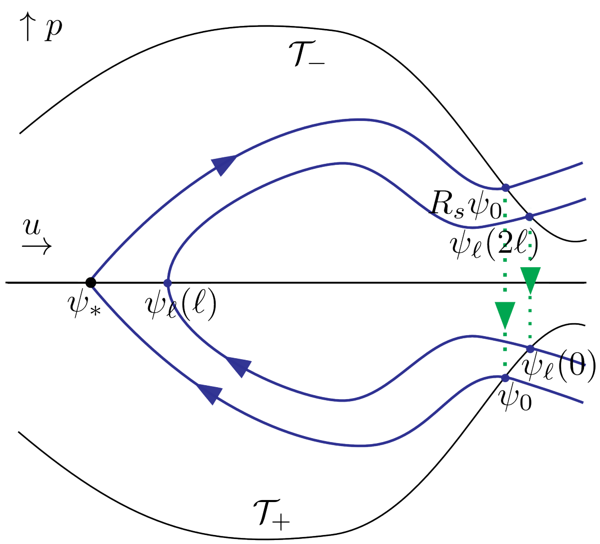


Figure 6.4: Depicted are the orthogonal projections of the singular periodic orbit (6.12) and the singular homoclinic orbit (6.13) onto the slow manifold $\mathcal{M}$ and the take-off and touch-down curves $\mathcal{T}_\pm$.

Remark 6.10. Theorem 6.9 proves that for fixed $\varepsilon \in (0, \varepsilon_0)$ the orbit of the periodic pulse $\phi_{\ell,\varepsilon}$ converges to the orbit of the homoclinic $\phi_{\infty,\varepsilon}$ as $\ell \to \infty$. If we subsequently take the limit $\varepsilon \to 0$, we obtain the singular concatenation (6.13). On the other hand, the orbit of $\phi_{\ell,\varepsilon}$ converges to (6.12) in the limit $\varepsilon \to 0$. Taking subsequently the long-wavelength limit $\ell \to \infty$ yields again (6.13). Thus, we may conclude that the limits $\lim_{\varepsilon\to 0}\lim_{\ell\to\infty}\phi_{\ell,\varepsilon}$ and $\lim_{\ell\to\infty}\lim_{\varepsilon\to 0}\phi_{\ell,\varepsilon}$ with respect to Hausdorff metric on $\mathbb{R}^4$ are equal. ■

6.4.5 Spectral geometry of long-wavelength periodic pulse solutions

Let $n = m = 1$ and assume **(S1)**, **(S2)**, **(E1)** and **(E3)** hold true. For fixed $\varepsilon \in (0, \varepsilon_0)$, Theorem 6.9 provides a family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$ to (1.9) converging pointwise to a homoclinic pulse solution $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. For any $\ell \in (\ell_0, \infty)$ we denote by $\mathcal{E}_{\ell,\varepsilon}$ the Evans function associated with the spectrum of the linearization of (1.9) about $\check{\phi}_{\ell,\varepsilon}$ and by

$$\mathcal{E}_{\ell,0}(\lambda, \gamma) = -\gamma\mathcal{E}_{\ell,f}(\lambda)\mathcal{E}_{\ell,s}(\lambda, \gamma),$$

the corresponding reduced Evans function – see §3.4 and §3.5.1.

We are interested in Hopf destabilization of long-wavelength periodic pulses $\check{\phi}_{\ell,\varepsilon}$, $\ell \gg 0$. Such a destabilization is caused by two complex conjugate curves of spectrum moving through the imaginary axis away from the origin – see §6.2. Since these spectral curves converge [39, 99] to the eigenvalues associated with the homoclinic limit as $\ell \to \infty$, Hopf destabilizations of

$\check{\phi}_{\ell,\varepsilon}$ occur in the vicinity of a Hopf instability of $\check{\phi}_{\infty,\varepsilon}$ as long as the critical spectral curve is confined to the left half-plane. Hopf instabilities of the homoclinic pulse occur when a conjugate pair of roots $\lambda_{\infty,\pm}$ of $\mathcal{E}_{\infty,s}$ moves through the imaginary axis.

Thus, to understand the character of the Hopf destabilization of long-wavelength periodic pulses, we need to control three spectral curves. First, we are interested in the position of the critical spectral curve attached to the origin for $\ell \gg 0$. Second, we need to understand the geometry of the spectral curves that shrink to $\lambda_{\infty,\pm}$ as $\ell \to \infty$. The first curve is by Theorem 3.17 and Proposition 3.29 to leading order approximated by the quantity $\lambda_{0,\ell}(\nu)$, defined in (3.31). The other two curves will be embedded in the set $\{\lambda \in \mathbb{C} : \mathcal{E}_{\ell,s}(\lambda, \gamma) = 0, \gamma \in S^1\}$ as $\varepsilon \to 0$ by Theorem 3.14 and Proposition 3.24.

Regarding the first spectral curve, we have the following result.

Theorem 6.11. *Suppose that the quantities $\mathfrak{a}_\infty$ and $\mathfrak{d}_\infty$, defined in* (6.10)*, are non-zero. Let $\omega_* := \sqrt{\partial_u H_1(u_*, 0, 0)}$ and take $\varsigma_* \in (0, \omega_*)$. Then, for $0 \ll \ell < \infty$, the analytic curve $\lambda_{0,\ell}(\nu)$, given by* (3.31)*, can be expanded in terms of $e^{-2\omega_*\ell}$ as*

$$\left| \lambda_{0,\ell}(\nu) - \frac{2\mathfrak{w}_\infty \omega_* e^{-2\omega_*\ell} (\cos(\nu) - 1)}{\mathfrak{d}_\infty} \right| \leq C e^{-(2\omega_* + \varsigma_*)\ell}, \tag{6.14}$$

where $C > 0$ is independent of ℓ and ν and

$$\mathfrak{w}_\infty := -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_\infty(0), v_{\mathrm{h}}(x, u_\infty(0)), 0) \partial_x v_{\mathrm{h}}(x, u_\infty(0)) x dx}{\int_{-\infty}^{\infty} (\partial_x v_{\mathrm{h}}(x, u_\infty(0)))^2 \, dx}. \tag{6.15}$$

Remark 6.12. In [100] one studies the critical spectral curve associated with long-wavelength periodic solutions to reaction-diffusion systems without assuming the presence of a small parameter ε. Thus, the above result could also have been obtained by taking the singular limit $\varepsilon \to 0$ of the expansion in [100, Theorem 5.5]. However, we stress that one should check whether the error estimates in [100] are in fact ε-uniform. ■

The second key result reveals the leading and next order geometry of the other two spectral curves converging to the eigenvalues $\lambda_{\infty,\pm}$ as $\ell \to \infty$.

Theorem 6.13. *Let $\lambda_\infty \in C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0)$ be a simple zero of $\mathcal{E}_{\infty,s}$ satisfying*

$$-4\mathrm{Re}(\lambda_\infty)\omega_*^2 < \mathrm{Im}(\lambda_\infty)^2, \tag{6.16}$$

where $\omega_* := \sqrt{\partial_u H_1(u_*, 0, 0)}$. Define $\omega_\infty := \sqrt{\partial_u H_1(u_*, 0, 0) + \lambda_\infty}$. Take ς_* and ς_∞ such that

$$0 < \varsigma_* < \omega_* < \varsigma_\infty < \mathrm{Re}(\omega_\infty).$$

For $0 \ll \ell < \infty$ there exists an analytic curve $\lambda_\ell \colon [-1, 1] \to \mathbb{C}$ satisfying the following assertions:

1. *For each* $\gamma \in S^1$ *the point* $\lambda_\ell(\mathrm{Re}(\gamma))$ *is the unique zero of* $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ *converging to* λ_∞ *as* $\ell \to \infty$;

2. *The curve* λ_ℓ *can be expanded in terms of* $e^{-2\omega_*\ell}$ *as*

$$\lambda_\ell(\gamma_r) = \lambda_\infty + L_1 e^{-2\omega_*\ell} + \mathcal{R}_{2,\ell}(\gamma_r),$$
$$L_1 := \frac{2\left(\omega_* \lim_{\check{x}\to\infty} (u_\infty(\check{x}) - u_*)\, e^{\omega_*\check{x}}\right)^2}{\mathfrak{a}_\infty \mathcal{E}'_{\infty,s}(\lambda_\infty)} \Bigg([\hat{u}_\infty(0,\lambda_\infty)]^2\, \partial_u \mathcal{G}(u_\infty(0),\lambda_\infty) \\ + 2\int_0^\infty \partial_{uu} H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})\,[\hat{u}_\infty(\check{x},\lambda_\infty)]^2\, d\check{x}\Bigg), \tag{6.17}$$

where $\mathfrak{a}_\infty$ *is defined in* (6.10) *and the remainder* $\mathcal{R}_{2,\ell}(\gamma_r)$ *is bounded as* $|\mathcal{R}_{2,\ell}(\gamma_r)| \leq C \max\left\{e^{-3\varsigma_*\ell}, e^{-2\varsigma_\infty\ell}\right\}$ *with* $C > 0$ *independent of* ℓ *and* γ_r. *Moreover,* $\hat{u}_\infty(\check{x},\lambda)$ *denotes the u-coordinate of the unique solution* $\varphi_\infty(\check{x},\lambda)$ *to* (6.6) *satisfying* (6.8) *and* $\tilde{u}_\infty(\check{x})$ *is the solution to the initial value problem,*

$$\tilde{u}_{\check{x}\check{x}} = \partial_u H_1(u_\infty(\check{x}),0,0)\tilde{u}, \qquad \tilde{u}(0) = 1, \quad \tilde{u}'(0) = \mathcal{J}'(u_\infty(0));$$

3. *The derivatives of* λ_ℓ *at* $\gamma_r \in [-1,1]$ *are approximated by*

$$\begin{aligned} \left|\lambda'_\ell(\gamma_r) - \frac{4\omega_\infty e^{-2\omega_\infty\ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right| &\leq Ce^{-(2\varsigma_\infty+\varsigma_*)\ell}, \\ \left|\lambda''_\ell(\gamma_r) - \left(\frac{4\omega_\infty e^{-2\omega_\infty\ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right)^2 \left(\frac{-2\ell}{\omega_\infty} + \frac{1}{\omega_\infty^2} - \frac{\mathcal{E}''_{\infty,s}(\lambda_\infty)}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}\right)\right| &\leq Ce^{-(4\varsigma_\infty+\varsigma_*)\ell}, \end{aligned} \tag{6.18}$$

with $C > 0$ *independent of* ℓ *and* γ_r.

The quantities $\pm\omega_*$ in Theorems 6.11 and 6.13 correspond to the eigenvalues of the linearization about the fixed point $(u_*, 0)$ in the slow reduced system (2.4). Moreover, $\pm\omega_\infty$ are the spatial eigenvalues of the asymptotic system obtained by taking the limit $\check{x} \to \pm\infty$ in the slow eigenvalue problem (6.6) at $\lambda = \lambda_\infty$. Furthermore, the condition (6.16) is equivalent to $\omega_* < \mathrm{Re}(\omega_\infty)$. In particular, any $\lambda_\infty \in i\mathbb{R} \setminus \{0\}$ satisfies (6.16).

The proofs of Theorems 6.11 and 6.13 are provided in §6.4.8.

6.4.6 Spectral stability of long-wavelength periodic pulse solutions

Consider the family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$, established in Theorem 6.9, converging pointwise to the homoclinic limit $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. The fact that the spectral curves corresponding to $\check{\phi}_{\ell,\varepsilon}$ shrink to the eigenvalues associated with the homoclinic $\check{\phi}_{\infty,\varepsilon}$ as $\ell \to \infty$, does not imply that spectral stability properties of the homoclinic are inherited by the periodic pulses – see [100]. This depends on the location of critical spectral curve attached to the origin.

By Theorem 6.11 the relative location of the critical curve with respect to the imaginary axis does not alter as $\ell \to \infty$ under the generic assumption that the quantities $\mathfrak{a}_\infty, \mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, defined in (6.10) and (6.15), are non-zero. Depending on the sign of these quantities, long-wavelength periodic pulses inherit the (spectral) stability properties of the limiting homoclinic.

Corollary 6.14. *Suppose the slow Evans function $\mathcal{E}_{\infty,s}$ has no roots $\lambda \in \mathbb{C}$ with* $\mathrm{Re}(\lambda) \geq 0$ *and the quantities $\mathfrak{i}_\infty, \mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, defined in* (6.10)*,* (6.11) *and* (6.15)*, are non-zero. Then, there exists $\ell_0 > 0$ such that for each $\ell \in (\ell_0, \infty)$ the following holds true.*

1. *If $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$ have the same sign, then the periodic pulse solution $\check{\phi}_{\ell,\varepsilon}$ to* (1.9) *is spectrally stable, provided $\varepsilon > 0$ is sufficiently small.*
2. *If $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$ have different signs, then $\check{\phi}_{\ell,\varepsilon}$ is spectrally unstable, provided $\varepsilon > 0$ is sufficiently small.*

Proof. Observe that the quantity $\mathfrak{i}_\ell$, defined in (3.36), converges to $\mathfrak{i}_\infty$ as $\ell \to \infty$ by Theorem 6.9. Thus, by Propositions 3.24 and 3.28, $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ has precisely one pole in the right half-plane for any $\gamma \in S^1$ and $\ell > 0$ sufficiently large. In addition, all roots of $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ in the right half-plane converge to roots of $\mathcal{E}_{\infty,s}$ as $\ell \to \infty$ by Theorem 6.13. Therefore, using Proposition 3.24, we conclude that $\mathcal{E}_{\ell,0}(\cdot,\gamma)$ has no roots $\lambda \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(\lambda) \geq 0$ for any $\gamma \in S^1$ and $\ell > 0$ sufficiently large. In addition, 0 is a simple root of $\mathcal{E}_{\ell,f}$ and $\mathcal{E}_{\ell,s}(0,\gamma) \neq 0$ for each $\gamma \in S^1$ and $\ell > 0$ sufficiently large.

Hence, spectral stability is determined by the position of the critical spectral curve $\lambda_{\varepsilon,\ell}(\nu)$ attached to the origin by Proposition 3.16, which is approximated by the curve $\lambda_{0,\ell}(\nu)$, defined in (3.31), by Proposition 3.29. By Theorem 6.11 the sign of $\lambda_{0,\ell}(\nu)$ and its derivatives is determined by the signs of $\mathfrak{d}_\infty$ and $\mathfrak{w}_\infty$, provided $\ell > 0$ is sufficiently large. This proves the result. □

We stress that the conditions in Corollary 6.14 comprise some form of nonlinear stability for the homoclinic $\check{\phi}_{\infty,\varepsilon}$ to (1.9). Indeed, these conditions imply that $\mathcal{E}_{\infty,0}$ has no zeros $\lambda \in \mathbb{C} \setminus \{0\}$ with $\mathrm{Re}(\lambda) \geq 0$ and 0 is a simple root of $\mathcal{E}_{\infty,0}$ – see §6.4.2. Hence, the same holds for $\mathcal{E}_{\infty,\varepsilon}$, provided $\varepsilon > 0$ is sufficiently small, by Theorem 6.7. So, there exists $\beta > 0$ such that all $\lambda \in \sigma(\mathcal{L}_{\infty,\varepsilon}) \setminus \{0\}$ satisfy $\mathrm{Re}(\lambda) < -\beta$ and $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}_{\infty,\varepsilon}$. The latter implies by [44, Section 5.1] nonlinear stability with asymptotic phase. On the other hand, spectral (in)stability implies nonlinear (in)stability for the *periodic* pulse solution $\check{\phi}_{\ell,\varepsilon}$ by the analysis in §3.3. Thus, Corollary 6.14 can be employed to test whether or not nonlinear stability of the homoclinic $\check{\phi}_{\infty,\varepsilon}$ implies nonlinear stability of the nearby periodics $\check{\phi}_{\ell,\varepsilon}$, $\ell \gg 0$.

6.4.7 Hopf destabilization in the homoclinic limit

Consider the family of periodic pulse solutions $\check{\phi}_{\ell,\varepsilon}(\check{x})$, established in Theorem 6.9, converging pointwise to the homoclinic limit $\check{\phi}_{\infty,\varepsilon}(\check{x})$ as $\ell \to \infty$. In this section we study the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$, when the homoclinic $\check{\phi}_{\infty,\varepsilon}$ undergoes a Hopf destabilization. In §6.4.5 we reasoned that the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$ is determined by the geometry of three spectral curves: the critical spectral curve attached to the origin and the two spectral curves converging to the critical eigenvalues associated with the homoclinic. We employ Theorems 6.11 and 6.13 to control these spectral curves.

Thus, let $\lambda_\infty \in C_\Lambda$ be a simple zero of $\mathcal{E}_{\infty,s}$ in the vicinity of the imaginary axis $i\mathbb{R} \setminus \{0\}$ such that $\lambda_\infty \notin \mathcal{E}_{\infty,f}^{-1}(0)$ and the condition (6.16) is satisfied. We infer from Theorem 6.13 that there is a unique curve $\lambda_\ell \colon [-1,1] \to \mathbb{C}$ of zeros of $\mathcal{E}_{\ell,s}$ shrinking to λ_∞ as $\ell \to \infty$ exponentially with rate $-2\omega_* \ell$. By (6.18) the curve λ_ℓ is to leading order a straight line that rotates with frequency $\mathrm{Im}(\omega_\infty)/\pi$ and whose length decays exponentially with rate $-2\mathrm{Re}(\omega_\infty)\ell$ as $\ell \to \infty$. Therefore, the point on the curve with largest real part will generically be one of the endpoints $\lambda_\ell(\pm 1)$. The following result shows that this is actually always the case – see Figure 6.5.

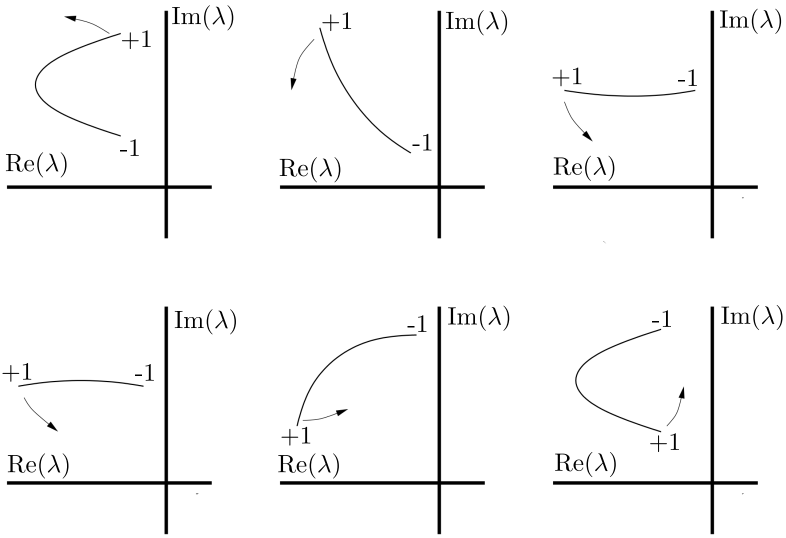


Figure 6.5: Depicted is a series of snapshots of the spectral curve λ_ℓ as ℓ increases. The pictures are corrected for exponential shrinking of the curve. Note that the spectral curve is to leading order a straight line that rotates and its 'belly' always points to the left. The point on the curve with largest real part is always one of the endpoints $\lambda_\ell(\pm 1)$.

Corollary 6.15. *Let $\lambda_\infty \in C_\Lambda \setminus \mathcal{E}_{\infty,f}^{-1}(0)$ be a simple zero of $\mathcal{E}_{\infty,s}$ satisfying* (6.16). *For $0 \ll \ell < \infty$ the point of largest real part on $\lambda_\ell([-1,1])$, where $\lambda_\ell \colon [-1,1] \to \mathbb{C}$ is established in Theorem 6.13, is always one of the endpoints $\lambda_\ell(\pm 1)$. In particular, consider the quantity*

$$\chi_\ell := \frac{4\omega_\infty e^{-2\omega_\infty \ell}}{\mathcal{E}'_{\infty,s}(\lambda_\infty)}, \tag{6.19}$$

with $\omega_\infty := \sqrt{\partial_u H_1(u_*,0,0) + \lambda_\infty}$. If $\mathrm{Re}(\chi_\ell) \neq 0$, then $\lambda_\ell(\mathrm{sgn}(\mathrm{Re}(\chi_\ell)))$ is the point of largest real part on $\lambda_\ell([-1,1])$.

Proof. By (6.18) the curve $\lambda_\ell(\gamma_r)$ is to leading order a straight line. Its orientation is determined by the argument of the quantity χ_ℓ. Thus, in the case $\chi_\ell \notin i\mathbb{R}$, it is clear that $\lambda_\ell(\mathrm{sgn}(\mathrm{Re}(\chi_\ell)))$ must be the endpoint of largest real part. Now suppose $\chi_\ell \in i\mathbb{R}$. Since λ_∞ is a simple zero of $\mathcal{E}_{\infty,s}$, χ_ℓ is non-zero. Thus, we have $\chi_\ell^2 < 0$. By (6.18) the quadratic deformation of the curve λ_ℓ is to leading order determined by the quantity $-2\chi_\ell^2 \ell \omega_\infty^{-1}$, which has strictly positive real part. Hence, we derive $\mathrm{Re}(\lambda_\ell(\pm 1)) \geq \mathrm{Re}(\lambda_{\ell_v}(\gamma_r))$ for all $\gamma_r \in [-1,1]$. This concludes the proof. □

Now suppose equation (1.9) depends on a real parameter μ. We make the following assumption:

(HO) There is $\mu_* \in \mathbb{R}$ and a unique pair $\pm\lambda_\infty$ with $\lambda_\infty \in i\mathbb{R} \setminus \{0\}$ satisfying $\mathcal{E}_{\infty,s,\mu_*}(\pm\lambda_\infty) = 0$ and

$$\mathrm{Re}\left[\frac{\partial_\mu \mathcal{E}_{\infty,s,\mu_*}(\lambda_\infty)}{\partial_\lambda \mathcal{E}_{\infty,s,\mu_*}(\lambda_\infty)}\right] < 0.$$

In addition, we have $\mathfrak{i}_\infty(\mu_*) \neq 0$, $\mathfrak{d}_\infty(\mu_*)\mathfrak{w}_\infty(\mu_*) > 0$ and $\mathcal{E}_{\infty,s,\mu_*}(\lambda) \neq 0$ for all $\lambda \in \mathbb{C} \setminus \{\pm\lambda_\infty\}$ with $\mathrm{Re}(\lambda) \geq 0$.

The condition **(HO)** implies that the homoclinic $\check{\phi}_{\infty,\varepsilon}$ undergoes a Hopf destabilization at a μ-value close to μ_* – see §6.4.2 and §6.4.3. The assumption $\mathfrak{d}_\infty(\mu_*)\mathfrak{w}_\infty(\mu_*) > 0$ in **(HO)** yields that the critical spectral curve associated with $\check{\phi}_{\ell,\varepsilon}$ is confined to the left half-plane by Corollary 6.14 for $\ell > 0$ sufficiently large. Hence, the long-wavelength periodic pulse $\check{\phi}_{\ell,\varepsilon}$ also undergoes a Hopf destabilization at a μ-value close to μ_*, since two spectral curves corresponding to $\check{\phi}_{\ell,\varepsilon}$ converge to the critical eigenvalues of the homoclinic $\check{\phi}_{\infty,\varepsilon}$ by Theorems 3.15, 6.7 and 6.13 as $\ell \to \infty$. The (leading-order) geometry of these spectral curves given in Theorem 6.13 and Corollary 6.15 determines the type of Hopf instability and whether the homoclinic pulse solution is the last (or first) periodic pulse to destabilize – see Figure 6.6. Thus, Theorems 3.15, 6.7, 6.11 and 6.13 and Corollary 6.15 yield the following result.

Corollary 6.16. *Assume (HO) and fix $\delta > 0$. Then, there exists $\ell_0 > 0$ such that for each $\ell \in (\ell_0, \infty)$ the following holds true for $\varepsilon > 0$ sufficiently small:*

1. *The homoclinic pulse solution $\check{\phi}_{\infty,\varepsilon}$ to* (1.9) *undergoes a Hopf destabilization at $\mu = \mu_{\infty,\varepsilon}$ with $|\mu_{\infty,\varepsilon} - \mu_*| < \delta$;*

2. *The periodic pulse solution* $\check{\phi}_{\ell,\varepsilon}$ *to* (1.9) *undergoes a* γ_ℓ*-Hopf destabilization at* $\mu = \mu_{\ell,\varepsilon}$ *with* $|\mu_{\ell,\varepsilon} - \mu_*| < \delta$. *It holds either* $|\gamma_\ell - 1| < \delta$ *or* $|\gamma_\ell + 1| < \delta$;

3. *If the real part of* $\chi_\ell = \chi_\ell(\mu_*)$, *defined in* (6.19), *is non-zero, then we have* $|\gamma_\ell - \mathrm{sgn}(\mathrm{Re}(\chi_\ell))| < \delta$;

4. *If the quantity* $L_1 = L_1(\mu_*)$, *defined in* (6.17), *is non-zero, then it holds* $\mathrm{sgn}(\mu_{\infty,\varepsilon} - \mu_{\ell,\varepsilon}) = \mathrm{sgn}(L_1)$, *i.e. the homoclinic pulse solution is the last to destabilize if* $L_1 > 0$.

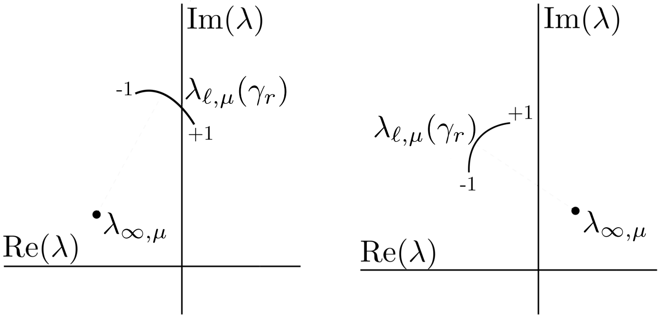


Figure 6.6: The homoclinic pulse solution undergoes a Hopf destabilization at $\mu = \mu_{\infty,\varepsilon}$. We denote by $\lambda_{\ell,\mu}$ the unique spectral curve converging to one of the critical eigenvalues $\lambda_{\infty,\mu}$ as $\ell \to \infty$. The left panel shows the spectral configuration in the case $L_1(\mu_*) > 0$ for $\mu < \mu_{\infty,\varepsilon}$: any (long-wavelength) periodic pulse solution destabilizes at some μ-value $\mu_{\ell,\varepsilon}$ smaller than $\mu_{\infty,\varepsilon}$. The right panel shows the spectral configuration in the case $L_1(\mu_*) < 0$ for $\mu > \mu_{\infty,\varepsilon}$: the homoclinic pulse is unstable, while there are still long-wavelength periodic pulse solutions that are spectrally stable.

Corollary 6.16 implies that, as the wave number $k = \ell^{-1}$ decreases, the character of destabilization of $\check{\phi}_{\ell,\varepsilon}$ alternates between ± 1-Hopf instabilities in the limit $\varepsilon \to 0$. This has the following implications for the region of stable pulse solutions in (k, μ)-space, which is also known as the *Busse balloon* [8, 27, 115]. By Corollary 6.16 the boundary $\{(\ell^{-1}, \mu_{\ell,\varepsilon}) : \ell \in (\ell_0, \infty)\}$ of the Busse balloon is in the limit $\varepsilon \to 0$ covered by two curves $\mathcal{H}_{\pm 1}$ corresponding to ± 1-Hopf instabilities of $\check{\phi}_{\ell,\varepsilon}$. The curves $\mathcal{H}_{\pm,1}$ intersect infinitely often as they oscillate about each other while both converging to the point $\lim_{\varepsilon \to 0}(0, \mu_{\infty,\varepsilon}) = (0, \mu_*)$ on the line $k = 0$. Moreover, Corollary 6.15 implies that in the limit $\varepsilon \to 0$ the boundary of the Busse balloon is non-smooth at the intersection points of $\mathcal{H}_{+1}$ and $\mathcal{H}_{-1}$. Thus, we have established the occurrence of the Hopf and belly dance destabilization mechanisms – see §6.1 – for the general class (1.10) of slowly nonlinear systems.

It was conjectured by W.M. Ni in the context of the Gierer-Meinhardt equations [80] that the homoclinic pulse solution is the last ‘periodic’ pulse to become unstable as we vary μ – see also [27, Remark 5.4]. Preliminary numerical simulations in the slowly nonlinear toy model (2.27) indicate that there exists parameter regimes, where the quantity L_1, defined in (6.17), has negative sign upon destabilization. This suggests that Ni's conjecture does not hold beyond the slowly linear Gierer-Meinhardt equations. We stress that a structural difference can be readily observed between both cases: the derivative $\partial_{uu}H_1(u_\infty(\check{x}),0,0)$ in (6.17) vanishes in the slowly linear case.

6.4.8 Proofs of key results

In this section we prove Theorems 6.11 and 6.13. Our approach is as follows. Let λ_∞ be a simple root of $\mathcal{E}_{\infty,s}$ satisfying (6.16). We want to understand the geometry of the critical curve $\lambda_{0,\ell}(\nu)$, defined in (3.31), and of the unique solution curve $\lambda_\ell(\gamma)$, satisfying $\mathcal{E}_{\ell,s}(\lambda_\ell(\gamma),\gamma) = 0$ for each $\gamma \in S^1$, which converges to λ_∞ as $\ell \to \infty$. By Propositions 3.25 and 3.29 we have

$$\lambda_{0,\ell}(\nu) = \mathfrak{a}_\ell \mathfrak{w}_\ell \frac{\cos(\nu) - 1}{2e^{-i\nu}\mathcal{E}_{\ell,s}(0,e^{i\nu})}, \tag{6.20}$$

where

$$\begin{aligned}
\mathfrak{a}_\ell &:= \mathcal{J}'(u_\ell(0))\mathcal{J}(u_\ell(0)) - H_1(u_\ell(0),0,0),\\
\mathfrak{w}_\ell &:= -\frac{\int_{-\infty}^{\infty} \frac{\partial G}{\partial u}(u_\ell(0), v_{\mathrm{h}}(x,u_\ell(0)),0)\partial_x v_{\mathrm{h}}(x,u_\ell(0)) x dx}{\int_{-\infty}^{\infty} \left[\partial_x v_{\mathrm{h}}(x,u_\ell(0))\right]^2 dx}.
\end{aligned}$$

One readily observes $\mathfrak{a}_\ell \to \mathfrak{a}_\infty$ and $\mathfrak{w}_\ell \to \mathfrak{w}_\infty$ as $\ell \to \infty$ by Theorem 6.9. Thus, to prove Theorems 6.11 and 6.13, we need to relate the periodic slow Evans function $\mathcal{E}_{\ell,s}$ to the homoclinic slow Evans function $\mathcal{E}_{\infty,s}$. The homoclinic slow Evans function $\mathcal{E}_{\infty,s}$ is defined in terms of the unique solution $\varphi_\infty(\check{x},\lambda)$ to the *homoclinic slow eigenvalue problem* (6.6) that satisfies (6.8). Our approach is to find an analytic solution $\varphi_\ell(\check{x},\lambda)$ to the *periodic slow eigenvalue problem*,

$$\varphi_{\check{x}} = \mathcal{A}_\ell(\check{x},\lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \qquad \mathcal{A}_\ell(\check{x},\lambda) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\ell(\check{x}),0,0) + \lambda & 0 \end{pmatrix}, \tag{6.21}$$

which is (pointwise) close to $\varphi_\infty(\check{x},\lambda)$ and decays exponentially on $[0,2\ell]$. Recall from §3.8.1 that system (6.21) is R_s-reversible at $\check{x} = \ell$, i.e. the evolution $\mathcal{T}_\ell(\check{x},\check{y},\lambda)$ of (6.21) satisfies $R_s\mathcal{T}_\ell(\check{x},\check{y},\lambda)R_s = \mathcal{T}_\ell(2\ell-\check{x},2\ell-\check{y},\lambda)$ for $\check{x},\check{y} \in [0,2\ell]$. In particular, $\varphi_\ell^r(\check{x},\lambda) := R_s\varphi_\ell(2\ell-\check{x},\lambda)$ is also a solution to (6.21). Now, to relate the periodic slow Evans function $\mathcal{E}_{\ell,s}$ to $\mathcal{E}_{\infty,s}$, we multiply $\mathcal{E}_{\ell,s}(\lambda,\gamma)$ with the ($\check{x}$-independent) Wronskian $\mathcal{W}_\ell(\lambda) := \det(\varphi_\ell(\check{x},\lambda) \mid \varphi_\ell^r(\check{x},\lambda))$. Using the 2-linearity of the determinant and $\det(\Upsilon(u,\lambda)), \det(\mathcal{T}_\ell(\check{x},\check{y},\lambda)) = 1$ for all $\check{x},\check{y} \in [0,2\ell]$, $\lambda \in C_\Lambda$ and $u \in U_{\mathrm{h}}$, we derive the key identity,

$$\gamma^{-1}\mathcal{E}_{\ell,s}(\lambda,\gamma)\mathcal{W}_\ell(\lambda) := 2\mathrm{Re}(\gamma)\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda), \tag{6.22}$$

where $\mathcal{K}_\ell \colon C_\Lambda \to \mathbb{C}$ is defined by

$$\begin{aligned}
\mathcal{K}_\ell(\lambda) = {}& \det\left(\varphi_\ell(0,\lambda) \mid \Upsilon(u_\ell(0),\lambda)R_s\varphi_\ell(0,\lambda)\right)\\
&+ \det\left(\Upsilon(u_\ell(0),\lambda)\varphi_\ell(2\ell,\lambda) \mid R_s\varphi_\ell(2\ell,\lambda)\right).
\end{aligned} \tag{6.23}$$

Since $\varphi_\ell(\check{x}, \lambda)$ decays exponentially as $\check{x} \to \infty$, one observes that the right hand side of (6.22) converges to the homoclinic slow Evans function $\mathcal{E}_{\infty,s}(\lambda)$ as $\ell \to \infty$. This leads to the desired approximation (6.14) of $\lambda_{0,\ell}(\nu)$ in Theorem 6.11.

To prove Theorem 6.13, we apply the implicit function theorem on (6.22). This yields the existence of a curve $\lambda_\ell \colon [-1, 1] \to \mathbb{C}$ such that for each $\gamma \in S^1$ the point $\lambda_\ell(\mathrm{Re}(\gamma))$ is the unique zero of $\mathcal{E}_{\ell,s}(\cdot, \gamma)$ converging to λ_∞ as $\ell \to \infty$. To calculate the leading-order difference $\lambda_\ell(\mathrm{Re}(\gamma)) - \lambda_\infty$ in order to prove (6.17), we need the leading order of the differences $\varphi_\ell(\check{x}, \lambda) - \varphi_\infty(\check{x}, \lambda)$ and $\psi_\ell(\check{x}) - \psi_\infty(\check{x})$ of the solutions to the slow eigenvalue problems and the slow reduced system, respectively. Finally, identity (6.18) is proved by implicit differentiation of identity (6.22).

Thus, the set-up of this section is as follows. First, we will establish a leading-order expression for the difference $\psi_\ell(\check{x}) - \psi_\infty(\check{x})$ of the solutions to the slow reduced system (2.4). This allows us to approximate $u_\ell(0)$ by $u_\infty(0)$ in (6.23). Second, we construct the desired solution $\varphi_\infty(\check{x}, \lambda)$ to (6.21) that is close to the solution $\varphi_\infty(\check{x}, \lambda)$ to (6.6) and decays exponentially on $[0, 2\ell]$. At the same time, we establish a leading-order expression for the difference $\varphi_\ell(\check{x}, \lambda) - \varphi_\infty(\check{x}, \lambda)$. Finally, we provide the proofs of Theorems 6.11 and 6.13 using the approach described above.

Approximations in the slow reduced subsystem

We start by collecting some basic facts for the situation described in §6.4.4. Recall the definition of ς_* and ω_* provided in Theorems 6.11 and 6.13. Since $\psi_* = (u_*, 0)$ is a hyperbolic saddle in (2.4) by **(E3)**, we have

$$\|\psi_\infty(\check{x}) - \psi_*\| \leq C e^{-\varsigma_* \check{x}}, \quad \check{x} \geq 0, \tag{6.24}$$

where $C > 0$ is a constant. The eigenvectors of the linearization of (2.4) about ψ_* are given by $w_\pm := (1, \pm\omega_*)$. We obtain by the stable manifold theorem:

$$\left\| e^{\omega_* \check{x}}(\psi_\infty(\check{x}) - \psi_*) - \alpha_* w_- \right\|, \left\| e^{\omega_* \check{x}} \psi_\infty'(\check{x}) + \alpha_* \omega_* w_- \right\| \leq C e^{-\varsigma_* \check{x}}, \quad \check{x} \geq 0, \tag{6.25}$$

where $\alpha_* \in \mathbb{R} \setminus \{0\}$ is given by

$$\alpha_* := \lim_{\check{x} \to \infty} e^{\omega_* \check{x}}(u_\infty(\check{x}) - u_*).$$

It is well-known that in a neighborhood of the point ψ_* one can give growth and decay rates of solutions to the (un)stable manifolds, see for example [56, Proposition 3.1]. Using these bounds one can estimate the distance between ψ_ℓ and ψ_∞ in terms of the 'time of flight' ℓ. Indeed, it holds for $0 \ll \ell < \infty$

$$\|\psi_\ell(\check{x}) - \psi_\infty(\check{x})\| \leq C e^{-\varsigma_*(2\ell - \check{x})}, \quad \check{x} \in [0, 2\ell], \tag{6.26}$$

with $C > 0$ a constant independent of ℓ.

We need a leading-order expression for the difference $\psi_\ell(\check{x}) - \psi_\infty(\check{x})$. Identity (6.26) gives an a priori estimate for this quantity, which is used in the proof of the next proposition.

Proposition 6.17. *For $0 \ll \ell < \infty$ we have the following expansion,*

$$\psi_\ell(\check{x}) = \psi_\infty(\check{x}) - \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix} 1 \\ \mathcal{J}'(u_\infty(0)) \end{pmatrix} + \mathcal{R}_{1,\ell}(\check{x}), \quad \check{x} \in [0,\ell], \tag{6.27}$$

where $\mathfrak{a}_\infty$ is defined in (6.10) *and the remainder $\mathcal{R}_\ell \colon [0,\ell] \to \mathbb{C}^2$ is bounded by $\|\mathcal{R}_\ell(\check{x})\| \leq Ce^{-\varsigma_*(3\ell-\check{x})}$ with $C > 0$ independent of ℓ, and $\Phi_\infty(\check{x},\check{y})$ denotes the evolution operator of the variational equation of* (2.4) *about ψ_∞,*

$$\theta_{\check{x}} = \mathcal{A}_\infty(\check{x})\theta, \quad \theta \in \mathbb{R}^2, \qquad \mathcal{A}_\infty(\check{x}) := \begin{pmatrix} 0 & 1 \\ \partial_u H_1(u_\infty(\check{x}),0,0) & 0 \end{pmatrix}. \tag{6.28}$$

Proof. In the following, we denote by $C > 0$ a constant independent of ℓ.

Define $\theta_\ell(\check{x}) = \psi_\ell(\check{x}) - \psi_\infty(\check{x})$ for $\check{x} \in [0,\ell]$. Our approach is to obtain a leading-order expression for $\theta_\ell(\check{x})$ using Lin's method [70, 118]. Note that θ_ℓ solves the boundary value problem,

$$\begin{aligned} \theta_{\check{x}} &= \mathcal{A}_\infty(\check{x})\theta + g_0(\theta,\check{x}), \\ \theta(0) + \psi_\infty(0) &\in T_+, \qquad (6.29) \\ \theta(\ell) + \psi_\infty(\ell) &\in \ker(I - R_s), \qquad (6.30) \end{aligned}$$

where $g_0 \colon \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$g_0(\theta,\check{x}) := f(\psi_\infty(\check{x}) + \theta) - f(\psi_\infty(\check{x})) - \mathcal{A}_\infty(\check{x})\theta.$$

Our plan is to study the inhomogeneous equation,

$$\theta_{\check{x}} = \mathcal{A}_\infty(\check{x})\theta + g(\check{x}), \quad \theta \in \mathbb{R}^2. \tag{6.31}$$

with $g \in C([0,\ell],\mathbb{R}^2)$ first. Using the exponential dichotomy of the variational equation, we construct a solution operator to (6.31). Subsequently, we substitute $g_0(\theta,\check{x})$ for $g(\check{x})$ and formulate an integral formulation for $\theta_\ell(\check{x})$ that is of fixed point type. This enables us to obtain a leading-order expression for $\theta_\ell(\check{x})$.

We establish an exponential dichotomy for the variational equation (6.28). First, the matrix function $\mathcal{A}_\infty(\check{x})$ converges as $\check{x} \to \infty$ to the asymptotic matrix $\mathcal{A}_*$. More precisely, by (6.24) it holds for $\check{x} \geq 0$

$$\|\mathcal{A}_\infty(\check{x}) - \mathcal{A}_*\| \leq Ce^{-\varsigma_*\check{x}}.$$

Second, the derivative $\psi_\infty'(\check{x})$ is a solution to (6.28), which is bounded as $\check{x} \to \infty$. Combining these items with Proposition 4.7 yields an exponential dichotomy of (6.28) on $[0,\infty)$ with constants $C, \varsigma_* > 0$ and projections $P_\infty(\check{x})$. By Lemma 4.5 we may without loss of generality assume that $P_\infty(0)$ is the projection on $\mathrm{Sp}(\psi_\infty'(0))$ along $\mathrm{Sp}(1,\mathcal{J}'(u_\infty(0)))$, since the stable manifold $W^s(\psi_*)$ intersects the touch-down curve $\mathcal{T}_+$ transversally in $\psi_\infty(0)$ by **(E3)**. In addition, Lemma 4.6 yields the estimate,

$$\|P_\infty(\check{x}) - P_*\| \leq Ce^{-\varsigma_* \check{x}}, \quad \check{x} \geq 0, \tag{6.32}$$

where P_* denotes the spectral projection of $\mathcal{A}_*$ on $\mathrm{Sp}(w_-)$ along $\mathrm{Sp}(w_+)$.

We proceed by constructing a solution operator to the boundary value problem (6.29)-(6.30). Denote by $\Phi_\infty^{u,s}(\check{x},\check{y})$ the (un)stable evolution operator of (6.28) under the exponential dichotomy. The bounded, linear solution operator $W_\ell\colon \ker(P_*) \times P_\infty(0)[\mathbb{R}^2] \times C([0,\ell],\mathbb{R}^2) \to C([0,\ell],\mathbb{R}^2)$ given by

$$W_\ell(a,b,g)[\check{x}] = \Phi_\infty^u(\check{x},\ell)a + \Phi_\infty^s(\check{x},0)b + \int_0^{\check{x}} \Phi_\infty^s(\check{x},z)g(z)dz - \int_{\check{x}}^{\ell} \Phi_\infty^u(\check{x},z)g(z)dz,$$

solves (6.31). Since G is C^3 on its domain by **(S1)**, the homoclinic solution $\kappa_\mathrm{h}(x,u) = (v_\mathrm{h}(x,u), q_\mathrm{h}(x,u))$ to (2.3) is C^3 on its domain $\mathbb{R} \times U_\mathrm{h}$. Therefore, $\mathcal{J}$ is C^3 on U_h. We expand $\mathcal{J}(u)$ in the neighborhood U_h of $u_\infty(0)$ with Taylor's Theorem as

$$\mathcal{J}(u) = \mathcal{J}(u_\infty(0)) + \mathcal{J}'(u_\infty(0))(u - u_\infty(0)) + h(u - u_\infty(0)), \quad u \in U_\mathrm{h},$$

where $h(u - u_\infty(0)) \leq C|u - u_\infty(0)|^2$. Since $\psi_\infty(0)$ equals $(u_\infty(0), \mathcal{J}(u_\infty(0))) \in T_+$, $\theta(\check{x}) = W_\ell(a,b,g)[\check{x}]$ satisfies condition (6.29) if and only if there exists $\rho \in U_\mathrm{h} - u_\infty(0)$ such that

$$\Phi_\infty^u(0,\ell)a + b - \int_0^\ell \Phi_\infty^u(0,z)g(z)dz = \rho\begin{pmatrix} 1 \\ \mathcal{J}'(u_\infty(0)) \end{pmatrix} + \begin{pmatrix} 0 \\ h(\rho) \end{pmatrix}. \tag{6.33}$$

For a vector $w := (w_1, w_2) \in \mathbb{R}^2$ we denote by $w^\perp$ the vector $(-w_2, w_1)$, which is perpendicular to w. Taking the inner product on both sides of (6.33) with $\psi_\infty'(0)^\perp$ yields

$$\left\langle \Phi_\infty^u(0,\ell)a - \int_0^\ell \Phi_\infty^u(0,z)g(z)dz, \psi_\infty'(0)^\perp \right\rangle = \rho\mathfrak{a}_\infty + h(\rho)u_\infty'(0). \tag{6.34}$$

Since $\mathcal{T}_+$ intersects the stable manifold $W^s(\psi_*)$ transversally by **(E3)**, the quantity $\mathfrak{a}_\infty$ is non-zero. Therefore, the right hand side of (6.34) defines an invertible function in ρ on a neighborhood of 0. Hence, there exists an ℓ-independent neighborhood A_0 of $0 \in \ker(P_0) \times C([0,\ell],\mathbb{R}^2)$ and a Lipschitz continuous map $\rho\colon A_0 \to \mathbb{R}$ such that $\rho(a,g)$ satisfies (6.34) and is bounded by

$$|\rho(a,g)| \leq C(e^{-\varsigma_*\ell}\|a\| + \|g\|). \tag{6.35}$$

Now substitute $\rho(a,g)$ in (6.33) and apply $P_\infty(0)$ on both sides. This gives rise to Lipschitz continuous map $b\colon A_0 \to P_\infty(0)[\mathbb{R}^2]$ satisfying

$$b(a,g) = \frac{-h(\rho(a,g))}{\mathfrak{a}_\infty}\psi_\infty'(0), \quad \|b(a,g)\| \leq C(e^{-\varsigma_*\ell}\|a\| + \|g\|)^2, \tag{6.36}$$

using that $P_\infty(0)$ projects on $\mathrm{Sp}(\psi_\infty'(0))$ along $\mathrm{Sp}(1, \mathcal{J}'(u_\infty(0)))$. By construction $\theta[\check{x}] = W_\ell(a, b(a,g), g)[\check{x}]$ satisfies (6.33) and thus (6.29). Similarly, $\theta[\check{x}] = W_\ell(a, b(a,g), g)[\check{x}]$ satisfies condition (6.30) if there exists $\beta \in \mathbb{R}$ such that

$$(I - P_\infty(\ell))a + \Phi_\infty^s(\ell, 0)b(a,g) + \int_0^\ell \Phi_\infty^s(\ell, z)g(z)dz + \psi_\infty(\ell) - \psi_* = \beta\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{6.37}$$

By estimate (6.32) it holds

$$\|(I - P_\infty(\ell))w_+ - w_+\| \leq Ce^{-\varsigma_*\ell}, \tag{6.38}$$

Estimate (6.38) shows that the inner product $\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), [(I - P_\infty(\ell))w_+]^\perp \rangle$ is to leading order given by the non-zero quantity $-\omega_*$. Thus, taking the inner product on both sides of (6.37) with $[(I - P_\infty(\ell))w_+]^\perp$ yields a Lipschitz continuous map $\beta \colon A_0 \to \mathbb{R}$ given by

$$\beta(a,g) = \frac{\left\langle \Phi_\infty^s(\ell,0)b(a,g) + \int_0^\ell \Phi_\infty^s(\ell,z)g(z)dz + \psi_\infty(\ell) - \psi_*, [(I - P_\infty(\ell))w_+]^\perp \right\rangle}{\left\langle \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), [(I - P_\infty(\ell))w_+]^\perp \right\rangle},$$

satisfying for $(a,g), (a_1, g) \in A_0$

$$|\beta(a,g)| \leq C(e^{-\varsigma_*\ell} + \|g\| + e^{-2\varsigma_*\ell}\|a\|), \quad |\beta(a,g) - \beta(a_1,g)| \leq Ce^{-\varsigma_*\ell}\|a - a_1\|, \tag{6.39}$$

by estimate (6.24). Now substitute $\beta(a,g)$ in (6.37) and apply $I - P_\infty(\ell)$ on both sides. This yields

$$a = (P_\infty(\ell) - P_*)a - (I - P_\infty(\ell))\left[\psi_\infty(\ell) - \psi_* - \beta(a,g)\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\right] \tag{6.40}$$

One readily verifies that the right hand side of (6.40) defines a contraction mapping in a for $\ell > 0$ sufficiently large, using estimates (6.32) and (6.39). Therefore, there exists by the Banach fixed point theorem an ℓ-independent neighborhood A_b of $0 \in C([0,\ell], \mathbb{R}^2)$ and a Lipschitz continuous map $a \colon A_b \to \ker(P_*)$ such that $a(g)$ satisfies equation (6.40) for each $g \in A_b$. The map a enjoys the bound

$$\|a(g)\| \leq C(e^{-\varsigma_*\ell} + \|g\|) \tag{6.41}$$

We conclude that the Lipschitz continuous map $W_{1,\ell} \colon A_b \to C([0,\ell], \mathbb{R}^2)$ given by $W_{1,\ell}(g) = W_\ell(a(g), b(a(g), g), g)$ satisfies (6.29)-(6.31). Therefore, θ_ℓ is the unique solution to the fixed point problem

$$\theta = W_{1,\ell}(g_0(\theta, \cdot)). \tag{6.42}$$

By shrinking A_b if necessary, it is not difficult to verify that the right hand side of (6.42) defines indeed a contraction mapping in $\theta \in C([0,\ell], \mathbb{R}^2)$.

Finally, the above fixed point arguments provide a mechanism to expand θ_ℓ in terms of $\ell \gg 0$. The first observation is that a priori the norm of $\theta_\ell(\check{x})$ is bounded by $Ce^{-\varsigma_*(2\ell - \check{x})}$ by estimate (6.26). Thus, the map $\hat{g}\colon [0,\ell] \to \mathbb{R}^2$ defined by $\hat{g}(\check{x}) = g_0(\theta_\ell(\check{x}),\check{x})$ is bounded by $Ce^{-2\varsigma_*(2\ell-\check{x})}$. We invoke the bounds (6.35), (6.36), (6.39) and (6.41) on the maps ρ, b, β and a to obtain the estimates

$$\|a(\hat{g})\| \leq Ce^{-\varsigma_*\ell}, \quad |\rho(a(\hat{g}),\hat{g})| \leq Ce^{-2\varsigma_*\ell},$$
$$\|b(a(\hat{g}),\hat{g})\| \leq Ce^{-4\varsigma_*\ell}, \quad |\beta(a(\hat{g}),\hat{g})| \leq Ce^{-\varsigma_*\ell}.$$

Combining the latter estimates with (6.25), (6.32) and (6.38) results in the expansions

$$\beta(a(\hat{g}),\hat{g}) = \frac{\alpha_* \langle w_-, w_+^\perp\rangle e^{-\omega_*\ell}}{\left\langle \begin{pmatrix}1\\0\end{pmatrix}, w_+^\perp\right\rangle} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right) = 2\alpha_* e^{-\omega_*\ell} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right),$$
$$a(\hat{g}) = (I - P_*)\left[\beta(a(\hat{g}),\hat{g})\begin{pmatrix}1\\0\end{pmatrix}\right] + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right) = \alpha_* w_+ e^{-\omega_*\ell} + \mathcal{O}\left(e^{-2\varsigma_*\ell}\right).$$

Substituting these expansions in $\theta_\ell = W_\ell(a(\hat{g}), b(a(\hat{g}),\hat{g}),\hat{g})$ yields

$$\psi_\ell(\check{x}) = \psi_\infty(\check{x}) + \alpha_*\Phi_\infty^u(\check{x},\ell)w_+e^{-\omega_*\ell} + \mathcal{O}\left(e^{-\varsigma_*(3\ell-\check{x})}\right), \quad \check{x} \in [0,\ell]. \tag{6.43}$$

Note that $P_\infty(\check{x})$ is the projection on $\mathrm{Sp}(\psi_\infty'(\check{x}))$ along $\mathrm{Sp}\left(\Phi_\infty(\check{x},0)\begin{pmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{pmatrix}\right)$. Thus, we estimate with the aid of (6.25)

$$\begin{aligned}\Phi_\infty^u(\check{x},\ell)w_+ &= \frac{\langle w_+, \psi_\infty'(\ell)^\perp\rangle}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{pmatrix}\\ &= \frac{-2\omega_*^2\alpha_*e^{-\omega_*\ell}}{\mathfrak{a}_\infty}\Phi_\infty(\check{x},0)\begin{pmatrix}1\\ \mathcal{J}'(u_\infty(0))\end{pmatrix} + \mathcal{O}\left(e^{-\varsigma_*(2\ell-\check{x})}\right),\end{aligned} \tag{6.44}$$

for $\check{x} \in [0,\ell]$. Combining (6.43) and (6.44) yields (6.27). □

Remark 6.18. The proof of Proposition 6.17 is based on [118, Theorem 6]. The fundamental difference with [118] is that it is not the existence of θ_ℓ that is of our interest, but the leading-order behavior. Moreover, we have nonlinear boundary conditions in contrast to [118]. ■

Approximation in slow eigenvalue problems

We proceed by constructing an analytic solution $\varphi_\ell(\check{x},\lambda)$ to (6.21) that is close to the solution $\varphi_\infty(\check{x},\lambda)$ to (6.6) and decays exponentially on $[0,2\ell]$. At the same time, we establish a leading-order expression for the difference $\varphi_\ell(\check{x},\lambda) - \varphi_\infty(\check{x},\lambda)$. We start by collecting some facts about the solution $\varphi_\infty(\check{x},\lambda)$ to (6.6). Recall that the coefficient matrix of (6.6) converges as $\check{x} \to \infty$ to the asymptotic matrix $\mathcal{A}_*(\lambda)$, defined in (6.7), which is hyperbolic on C_Λ. The eigenvalues of $\mathcal{A}_*(\lambda)$ are given by $\pm\omega(\lambda)$ and corresponding eigenvectors are $v_\pm(\lambda) := (1, \pm\omega(\lambda))$, where

$$\omega(\lambda) := \sqrt{\partial_u H_1(u_*,0,0) + \lambda},$$

denotes the principal square root. Note that both $\omega(\lambda)$ and $v_\pm(\lambda)$ are analytic on C_Λ. Choose an open and bounded subset $C_{b,\Lambda} \subset C_\Lambda$. An application of Proposition 4.3 yields the following estimate,

$$\|e^{\omega(\lambda)\check{x}}\varphi_\infty(\check{x},\lambda) - v_-(\lambda)\| \leq Ce^{-\varsigma_*\check{x}}, \quad \check{x} \geq 0, \lambda \in C_{b,\Lambda}, \tag{6.45}$$

where $C > 0$ is a constant independent of λ.

We are now ready to prove the existence of the desired solution $\varphi_\ell(\check{x}, \lambda)$ to (6.21). To state the result, we take $\delta > 0$ such that we have

$$\mu(\lambda) := \mathrm{Re}(\omega(\lambda)) - \delta > 0,$$

for all λ in the bounded set $C_{b,\Lambda}$.

Proposition 6.19. *For* $0 \ll \ell < \infty$, *there exists a solution* $\varphi_\ell \colon [0, 2\ell] \times C_{b,\Lambda} \to \mathbb{C}^2$ *to the periodic slow eigenvalue problem* (6.21)*, satisfying the bounds*

$$\begin{aligned} \|\varphi_\ell(\check{x}, \lambda)\| &\leq Ce^{-\mu(\lambda)\check{x}}, \\ \|\varphi_\ell(0, \lambda) - \varphi_\infty(0, \lambda)\| &\leq Ce^{-2\min\{\varsigma_*, \mu(\lambda)\}\ell}, \\ \|\varphi_\ell(\ell, \lambda) - \varphi_\infty(\ell, \lambda)\| &\leq Ce^{-(\varsigma_* + \mu(\lambda))\ell}, \end{aligned} \qquad \begin{aligned} \check{x} &\in [0, 2\ell], \\ \lambda &\in C_{b,\Lambda}, \end{aligned} \tag{6.46}$$

where $C > 0$ *is a constant independent of* ℓ *and* λ. *Moreover,* $\varphi_\ell(\check{x}, \cdot)$ *is analytic on* $C_{b,\Lambda}$ *for each* $\check{x} \in [0, 2\ell]$. *Finally, we have the expansion for* $\lambda \in C_{b,\Lambda}$

$$\begin{aligned} &\varphi_\ell(0, \lambda) - \varphi_\infty(0, \lambda) = \\ &\qquad \int_0^\ell Q_\infty(\lambda) \mathcal{T}_\infty(0, \check{y}, \lambda) \left[\mathcal{A}_\ell(\check{x}, \lambda) - \mathcal{A}_\infty(\check{x}, \lambda)\right] \varphi_\infty(\check{y}, \lambda) d\check{y} + \mathcal{R}_{1,\ell}(\lambda), \end{aligned} \tag{6.47}$$

where $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda)$ *denotes the evolution operator of system* (6.6)*,* $Q_\infty(\lambda)$ *is an analytic projection along* $\mathrm{Sp}(\varphi_\infty(0, \lambda))$ *and the remainder* $\mathcal{R}_{1,\ell} \colon C_{b,\Lambda} \to \mathbb{C}^2$ *is bounded as* $\|\mathcal{R}_{1,\ell}(\lambda)\| \leq C \max\left\{e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right\}$.

Proof. In the following, we denote by $C > 0$ a constant independent of ℓ and λ.

Our approach is to regard the periodic slow eigenvalue problem (6.21) as the perturbation,

$$\varphi_{\check{x}} = \left(\mathcal{A}_\infty(\check{x}, \lambda) + \mathcal{H}_\ell(\check{x})\right)\varphi, \quad \varphi \in \mathbb{C}^2,$$

of system (6.6) on $[0, \ell]$ and as the perturbation,

$$\varphi_{\check{x}} = \left(\mathcal{A}_\infty(-\check{x}, \lambda) + \mathcal{H}_\ell(\check{x})\right)\varphi, \quad \varphi \in \mathbb{C}^2,$$

of system,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(-\check{x}, \lambda)\varphi, \quad \varphi \in \mathbb{C}^2, \tag{6.48}$$

on $[-\ell, 0)$, where $\mathcal{H}_\ell \colon [-\ell, \ell] \to \mathrm{Mat}_2(\mathbb{C})$ is given by,

$$\mathcal{H}_\ell(\check{x}) := \begin{cases} \mathcal{A}_\ell(\check{x}, \lambda) - \mathcal{A}_\infty(\check{x}, \lambda), & \check{x} \in [0, \ell] \\ \mathcal{A}_\ell(2\ell + \check{x}, \lambda) - \mathcal{A}_\infty(-\check{x}, \lambda), & \check{x} \in [-\ell, 0) \end{cases},$$

By estimate (6.26) the norm of $\mathcal{H}_\ell$ satisfies

$$\|\mathcal{H}_\ell\| \leq Ce^{-\varsigma_* \ell}. \tag{6.49}$$

Let X_b be the space of bounded functions $[-\ell, \ell] \to \mathbb{C}^2$ that are continuous, except for a possible discontinuity at 0. Our plan is to obtain exponential dichotomies for equations (6.6) and (6.48) first. The exponential dichotomies yield a solution operator to the inhomogeneous problem,

$$\varphi_{\check{x}} = \mathcal{A}_\infty(|\check{x}|, \lambda)\varphi + G(\check{x}), \quad \varphi \in \mathbb{C}^2, \tag{6.50}$$

with $G \in X_b$ using the variation of constants formula. Then, using Lin's method [70, 100], we construct a solution operator to (6.50) that satisfies a matching condition at the endpoints $\check{x} = \ell$ and $\check{x} = -\ell$. Finally, we substitute $\mathcal{H}_\ell(\check{x})\varphi$ for $G(\check{x})$ in (6.50) and obtain a solution operator to (6.21). We apply the latter solution operator to the initial condition $\varphi_\infty(0, \lambda)$ to establish the existence of the desired solution $\varphi_\ell(\check{x}, \lambda)$.

We establish exponential dichotomies for the homoclinic slow eigenvalue problems (6.6) and (6.48). By Proposition 4.7 and estimate (6.26), system (6.6) has for $\lambda \in C_{b,\Lambda}$ an exponential dichotomy on $[0, \infty)$ with constants $C, \mu(\lambda) > 0$. The corresponding projections $\mathcal{P}_\infty(\check{x}, \lambda)$ can be chosen analytic on $C_{b,\Lambda}$. Moreover, since $\mathcal{A}_*(\lambda)$ is hyperbolic with spectral gap larger than $\mu(\lambda) \geq \varsigma_*$ and $\mathcal{A}_*$ is bounded on $C_{b,\Lambda}$, Lemma 4.6 and (6.26) yield

$$\|\mathcal{P}_\infty(\check{x}, \lambda) - \mathcal{P}_*(\lambda)\| \leq Ce^{-\varsigma_* \check{x}}, \quad \check{x} \geq 0, \lambda \in C_{b,\Lambda}, \tag{6.51}$$

where $\mathcal{P}_*(\lambda)$ denotes the analytic spectral projection of $\mathcal{A}_*(\lambda)$ on $\mathrm{Sp}(v_-(\lambda))$ along $\mathrm{Sp}(v_+(\lambda))$. Moreover, since we have $R_s v_-(\lambda) = v_+(\lambda)$, the identity,

$$R_s \mathcal{P}_*(\lambda) R_s = I - \mathcal{P}_*(\lambda), \tag{6.52}$$

holds for each $\lambda \in C_\Lambda$. Denote by $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda)$ the evolution operator of system (6.6). By [60, Lemma 2.1.4] $\mathcal{T}_\infty(\check{x}, \check{y}, \cdot)$ is analytic on C_Λ, since $\mathcal{A}_\infty(\check{x}, \cdot)$ is analytic on C_Λ.

Using the reversible symmetry R_s, system (6.48) can be fully described in terms of system (6.6). Indeed, for the evolution $\mathcal{T}_{\infty,r}(\check{x}, \check{y}, \lambda)$ of system (6.48) it holds $\mathcal{T}_\infty(\check{x}, \check{y}, \lambda) = R_s \mathcal{T}_{\infty,r}(-\check{x}, -\check{y}, \lambda) R_s$. Consequently, system (6.48) has for any $\lambda \in C_{b,\Lambda}$ an exponential dichotomy on $(-\infty, 0]$ with constants $C, \mu(\lambda) > 0$. The corresponding projections $\mathcal{P}_{\infty,r}(\check{x}, \lambda)$ satisfy $\mathcal{P}_{\infty,r}(\check{x}, \lambda) = I - R_s \mathcal{P}_\infty(-\check{x}, \lambda) R_s$ for $\check{x} \leq 0$. Moreover, by (6.52) it holds

$$\|\mathcal{P}_{\infty,r}(\check{x}, \lambda) - \mathcal{P}_*(\lambda)\| \leq Ce^{\varsigma_* \check{x}}, \quad \check{x} \leq 0, \lambda \in C_{b,\Lambda}, \tag{6.53}$$

We proceed by constructing a solution operator to the periodic slow eigenvalue problem (6.21).

Consider $W_\ell(\lambda)\colon \mathbb{C}^2 \times \mathbb{C}^2 \times X_b \to X_b$ to (6.50) given by

$$\begin{aligned} W_\ell(\lambda)(a,b,G)[\check{x}] &= \mathcal{T}^u_\infty(\check{x},\ell,\lambda)a + \mathcal{T}^s_\infty(\check{x},0,\lambda)b + \int_0^{\check{x}} \mathcal{T}^s_\infty(\check{x},\check{y},\lambda)G(\check{y})d\check{y} \\ &\quad - \int_{\check{x}}^{\ell} \mathcal{T}^u_\infty(\check{x},\check{y},\lambda)G(\check{y})d\check{y}, && \check{x} \in [0,\ell], \\ W_\ell(\lambda)(a,b,G)[\check{x}] &= -\mathcal{T}^s_{\infty,r}(\check{x},-\ell,\lambda)a - \int_{\check{x}}^{0} \mathcal{T}^u_{\infty,r}(\check{x},\check{y},\lambda)G(\check{y})d\check{y} \\ &\quad + \int_{-\ell}^{\check{x}} \mathcal{T}^s_{\infty,r}(\check{x},\check{y},\lambda)G(\check{y})d\check{y}, && \check{x} \in [-\ell,0), \end{aligned}$$

where $\mathcal{T}^{u,s}_\infty(\check{x},\check{y},\lambda)$ and $\mathcal{T}^{u,s}_{\infty,r}(\check{x},\check{y},\lambda)$ denote the (un)stable evolution operator of systems (6.6) and (6.48) under the exponential dichotomies established above. Note that W_ℓ is an analytic operator on $C_{b,\Lambda}$, since the evolutions $\mathcal{T}_\infty(\check{x},\check{y},\cdot)$ and the projections $\mathcal{P}_\infty(\check{x},\cdot)$ are analytic. By (6.51) and (6.53) it holds

$$\|\mathcal{P}_\infty(\ell,\lambda) - \mathcal{P}_{\infty,r}(-\ell,\lambda)\| \leq Ce^{-\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}. \tag{6.54}$$

We conclude that the analytic linear operator $A_{1,\ell}(\lambda) := I - \mathcal{P}_\infty(\ell,\lambda) + \mathcal{P}_{\infty,r}(-\ell,\lambda)$ is invertible for $\ell > 0$ sufficiently large. Now define the analytic linear operator $A_{2,\ell}(\lambda)\colon \mathbb{C}^2 \times X_b \to \mathbb{C}^2$ by

$$A_{2,\ell}(\lambda)(b,G) = A_{1,\ell}(\lambda)^{-1}\left(W_\ell(\lambda)(0,b,G)[-\ell] - W_\ell(\lambda)(0,b,G)[\ell]\right).$$

One readily verifies that the analytic linear operator $W_{2,\ell}(\lambda)\colon \mathbb{C}^2 \times X_b \to X_b$ defined by $W_{2,\ell}(\lambda)(b,G) = W_\ell(\lambda)(A_{2,\ell}(\lambda)(b,G),b,G)$ is linear and satisfies

$$W_{2,\ell}(\lambda)(b,G)[-\ell] = W_{2,\ell}(\lambda)(b,G)[\ell], \quad b \in \mathbb{C}^2, G \in X_b, \lambda \in C_{b,\Lambda}. \tag{6.55}$$

Moreover, we have the estimates

$$\begin{aligned} \|A_{2,\ell}(\lambda)(b,G)\| &\leq C(e^{-\mu(\lambda)\ell}\|b\| + \|G\|), \\ \|W_{2,\ell}(\lambda)(b,G)[\check{x}]\| &\leq \begin{cases} C(e^{-\mu(\lambda)\check{x}}\|b\| + \|G\|), & \check{x} \in [0,\ell], \\ C(e^{-\mu(\lambda)(2\ell+\check{x})}\|b\| + \|G\|), & x \in [-\ell,0), \end{cases} \end{aligned} \tag{6.56}$$

for $b \in \mathbb{C}^2, G \in X_b, \lambda \in C_{b,\Lambda}$. Denote by $W_{3,\ell}(\lambda)\colon X_b \to X_b$ the analytic linear map $W_{3,\ell}(\lambda)(w) = W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot w)$, where $\cdot$ denotes pointwise multiplication, i.e. $(\mathcal{H}_\ell \cdot w)[\check{x}] = \mathcal{H}_\ell(\check{x})w(\check{x})$. By (6.49) we have the estimate,

$$\|W_{3,\ell}(\lambda)\| \leq Ce^{-\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}.$$

Hence for $\ell > 0$ sufficiently large, the map $I - W_{3,\ell}(\lambda)$ is invertible. Finally, consider the analytic linear map $W_{4,\ell}(\lambda)\colon \mathbb{C}^2 \to X_b$ given by $W_{4,\ell}(\lambda)(b) = (I - W_{3,\ell}(\lambda))^{-1}(W_{2,\ell}(\lambda)(b,0))$. One readily checks that

$$W_{4,\ell}(\lambda)(b) = W_{2,\ell}(\lambda)(b, \mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b)), \quad b \in \mathbb{C}^2, \lambda \in C_{b,\Lambda}, \tag{6.57}$$

is satisfied. Define the map $\zeta\colon [0,2\ell) \to [-\ell,\ell]$ by

$$\zeta(\check{x}) = \begin{cases} \check{x}, & \check{x} \in [0,\ell] \\ \check{x} - 2\ell, & \check{x} \in (\ell, 2\ell) \end{cases}.$$

By identities (6.55) and (6.57) we have $W_{4,\ell}(\lambda)(b)[\ell] = W_{4,\ell}(\lambda)(b)[-\ell]$. We conclude for every $\lambda \in C_{b,\Lambda}, b \in \mathbb{C}^2$ and $\ell > 0$ sufficiently large, that $W_{4,\ell}(\lambda)(b)[\zeta(\check{x})]$ is a solution to (6.21) on $[0,2\ell)$ that can be extended to $[0,2\ell]$.

Next, we apply the solution operator $W_{4,\ell}$ to initial condition $b_\lambda := \varphi_\infty(0,\lambda) \in \mathbb{C}^2$ and consider the solution

$$\varphi_\ell(\check{x},\lambda) := W_{4,\ell}(\lambda)(b_\lambda)[\zeta(\check{x})],$$

to (6.21). Note that $\varphi_\ell(\check{x},\cdot)$ is analytic on $C_{b,\Lambda}$, since both $W_{4,\ell}$ and $\varphi_\infty(0,\lambda)$ are analytic on $C_{b,\Lambda}$. Using (6.49), (6.56) and identity (6.57) we estimate

$$\begin{aligned} \|\varphi_\ell(\check{x},\lambda)\| &\leq \|W_{2,\ell}(\lambda)(b_\lambda,0)[\zeta(\check{x})]\| + \|W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\zeta(\check{x})]\| \\ &\leq C\left[e^{-\mu(\lambda)\check{x}} + e^{-\varsigma_*\ell}\int_0^{2\ell}\left(e^{-\mu(\lambda)|\check{x}-\check{y}|} + e^{-\mu(\lambda)(|\ell-\check{x}|+|\ell-\check{y}|)}\right)\|\varphi_\ell(\check{y},\lambda)\|d\check{y}\right], \end{aligned} \tag{6.58}$$

for $\check{x} \in [0,2\ell], \lambda \in C_{b,\Lambda}$. Applying [15, Lemma III.2.1] on the integral inequality (6.58) yields

$$\|\varphi_\ell(\check{x},\lambda)\| \leq Ce^{-\mu(\lambda)\check{x}}, \quad \check{x} \in [0,2\ell], \lambda \in C_{b,\Lambda}, \tag{6.59}$$

provided $\ell > 0$ is sufficiently large. Moreover, we approximate with the aid of (6.54)

$$\begin{aligned} &\|A_{2,\ell}(\lambda)(b_\lambda,0) - T_\infty^s(\ell,0,\lambda)b_\lambda\| \\ &\quad = \|(\mathcal{P}_\infty(\ell,\lambda) - \mathcal{P}_{\infty,r}(-\ell,\lambda))A_{1,\ell}(\lambda)^{-1}T_\infty^s(\ell,0,\lambda)b_\lambda\| \leq Ce^{-(\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \tag{6.60}$$

for $\lambda \in C_{b,\Lambda}$. On the other hand, using (6.49) and (6.59) we estimate

$$\begin{aligned} \|W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\ell]\| &\leq Ce^{-\varsigma_*\ell}\int_0^{2\ell} e^{-\mu(\lambda)|\ell-\check{y}|}\|\varphi_\ell(\check{y},\lambda)\|d\check{y} \\ &\leq Ce^{-(\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \tag{6.61}$$

for $\lambda \in C_{b,\Lambda}$. Using identity (6.57) and estimates (6.60) and (6.61) we expand $\varphi_\ell(\check{x},\lambda)$ at $\check{x} = \ell$ as follows

$$\begin{aligned} \varphi_\ell(\ell,\lambda) &= W_{2,\ell}(\lambda)(b_\lambda,0)[\ell] + W_{2,\ell}(\lambda)(0,\mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda))[\ell] \\ &= T_\infty^s(\ell,0,\lambda)b_\lambda + \mathcal{O}\left(e^{-(\mu(\lambda)+\varsigma_*)\ell}\right) \\ &= \varphi_\infty(\ell,\lambda) + \mathcal{O}\left(e^{-(\mu(\lambda)+\varsigma_*)\ell}\right), \end{aligned}$$

for $\lambda \in C_{b,\Lambda}$. Similarly, using identity (6.57) and estimates (6.26), (6.49) and (6.60) we expand $\varphi_\ell(\check{x}, \lambda)$ at $\check{x} = 0$ as follows for $\lambda \in C_{b,\Lambda}$

$$\begin{aligned}
\varphi_\ell(0,\lambda) &= W_{2,\ell}(\lambda)(b_\lambda, \mathcal{H}_\ell \cdot W_{2,\ell}(\lambda)(b_\lambda, 0))[0] \\
&\qquad\qquad + W_{2,\ell}(\lambda)(0, \mathcal{H}_\ell \cdot W_{2,\ell}(\lambda)(0, \mathcal{H}_\ell \cdot W_{4,\ell}(\lambda)(b_\lambda)))[0] \\
&= \mathcal{P}_\infty(0,\lambda) b_\lambda - \int_0^\ell \mathcal{T}^u_\infty(0,\check{y},\lambda)\mathcal{H}_\ell(\check{y})\mathcal{T}^s_\infty(\check{y},0,\lambda) b_\lambda d\check{y} + O\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right) \\
&= \varphi_\infty(0,\lambda) - \int_0^\ell \mathcal{T}^u_\infty(0,\check{y},\lambda)\mathcal{H}_\ell(\check{y})\varphi_\infty(\check{y},\lambda) d\check{y} + O\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right) \\
&= \varphi_\infty(0,\lambda) + O\left(e^{-2\varsigma_*\ell}\right),
\end{aligned}$$

where we used that $\mu(\lambda) > \varsigma_*$. □

Since system (6.21) is R_s-reversible at $\check{x} = \ell$, $\varphi^r_\ell(\check{x}, \lambda) = R_s \varphi_\ell(2\ell - \check{x}, \lambda)$ is a also solution to (6.21). The next proposition shows that $\varphi_\ell(\check{x}, \lambda)$ and $\varphi^r_\ell(\check{x}, \lambda)$ are linearly independent and approximates their Wronskian $\mathcal{W}_\ell(\lambda)$.

Corollary 6.20. *For $0 \ll \ell < \infty$ the ($\check{x}$-independent) Wronskian $\mathcal{W}_\ell(\lambda) = \det(\varphi_\ell(\check{x}, \lambda) \mid \varphi^r_\ell(\check{x}, \lambda))$ is approximated by*

$$\|\mathcal{W}_\ell(\lambda) - E_\ell(\lambda)\| \leq C e^{-(2\mu(\lambda)+\varsigma_*)\ell}, \quad \lambda \in C_{b,\Lambda}, \tag{6.62}$$

where $C > 0$ is a constant independent of ℓ and λ and $E_\ell : C_{b,\Lambda} \to \mathbb{C}$ is the non-zero analytic map given by $E_\ell(\lambda) = 2\omega(\lambda)e^{-2\omega(\lambda)\ell}$.

Proof. Combining estimates (6.45) and (6.46) yields

$$\left|\det(\varphi_\ell(\ell,\lambda) \mid R_s\varphi_\ell(\ell,\lambda)) - e^{-2\omega(\lambda)\ell} \det(v_-(\lambda) \mid R_s v_-(\lambda))\right| \leq C e^{-(2\mu(\lambda)+\varsigma_*)\ell},$$

which concludes the proof. □

Conclusion

With the preparatory work done in the previous sections, we are able to prove Theorems 6.11 and 6.13 using the aforementioned approach.

Proof of Theorem 6.11. In the following, we denote by $C > 0$ a constant independent of ℓ. First, using (6.26) and (6.46) we approximate

$$\left|\mathcal{K}_\ell(0) - \mathcal{E}_{\infty,s}(0)\right| \leq C e^{-2\varsigma_*\ell},$$

where $\mathcal{K}_\ell(\lambda)$ is defined in (6.23). Combining the latter with (6.22) and (6.62) yields

$$\left|e^{-i\nu}\mathcal{E}_{\ell,s}(0, e^{i\nu})\mathcal{W}_\ell(0) - \mathcal{E}_{\infty,s}(0)\right| \leq C e^{-2\varsigma_*\ell}, \quad \nu \in \mathbb{R}. \tag{6.63}$$

On the other hand, by (6.26) it holds

$$|\mathfrak{a}_\ell - \mathfrak{a}_\infty|, |\mathfrak{w}_\ell - \mathfrak{w}_\infty| \le Ce^{-2\varsigma_* \ell}. \tag{6.64}$$

Finally, applying (6.9), (6.62), (6.63) and (6.64) on identity (6.20) establishes the desired approximation (6.14). □

Proof of Theorem 6.13. In the following, we denote by $C > 0$ a constant independent of ℓ and λ. Let $\lambda_\infty \in C_\Lambda$ be a simple zero of $\mathcal{E}_{\infty,s}$ satisfying (6.16). Then, we take $C_{b,\Lambda} \subset C_\Lambda$ an open and bounded neighborhood of λ_∞ of $\mathcal{E}_{\infty,s}$ such that it holds $\mathrm{Re}(\omega(\lambda)) > \omega_*$ for all $\lambda \in C_{b,\Lambda}$. We chose $\delta > 0$ such that

$$2\delta < \varsigma_*, \quad \mu(\lambda) := \mathrm{Re}(\omega(\lambda)) - \delta > \omega_*,$$

for all λ in $C_{b,\Lambda}$.

We are looking for zeros of $\mathcal{E}_{\ell,s}(\cdot,\gamma)$ close to λ_∞ for $0 \ll \ell < \infty$ and $\gamma \in S^1$. In other words, we are looking for solutions $\lambda \in C_{b,\Lambda}$ in a neighborhood of λ_∞ to the equation

$$0 = \mathcal{E}_{\ell,s}(\lambda,\gamma). \tag{6.65}$$

By multiplying (6.65) with the non-zero (see Corollary 6.20) quantity $\gamma^{-1}\mathcal{W}_\ell(\lambda)$ on both sides, we obtain the equivalent equation,

$$0 = 2\mathrm{Re}(\gamma)\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda), \quad \lambda \in C_{b,\Lambda}, \quad \gamma \in S^1, \tag{6.66}$$

see also (6.22). Using (6.26) and (6.46) we approximate

$$\left|\mathcal{K}_\ell(\lambda) - \mathcal{E}_{\infty,s}(\lambda)\right| \le Ce^{-2\varsigma_*\ell}, \quad \lambda \in C_{b,\Lambda}. \tag{6.67}$$

Note that both $\mathcal{W}_\ell$ and $\mathcal{K}_\ell$ are analytic on $C_{b,\Lambda}$, since $\varphi_\ell(\check{x},\cdot)$ and $\Upsilon(u,\cdot)$ are. By shrinking $C_{b,\Lambda}$ if necessary, the approximations (6.62) and (6.67) provide bounds for the derivatives of the analytic maps $\mathcal{W}_\ell$ and $\mathcal{K}_\ell$ via the estimates,

$$\begin{aligned} \left|\frac{\partial^i}{\partial\lambda^i}\left(\mathcal{K}_\ell(\lambda) - \mathcal{E}_{\infty,s}(\lambda)\right)\right| &\le Ce^{-2\varsigma_*\ell}, \\ \left|\frac{\partial^i}{\partial\lambda^i}\left(\mathcal{W}_\ell(\lambda) - E_\ell(\lambda)\right)\right| &\le Ce^{-(2\mu(\lambda)+\varsigma_*)\ell}, \end{aligned} \qquad i = 0,1,2, \quad \lambda \in C_{b,\Lambda}. \tag{6.68}$$

Consider the analytic function $\eta_\ell \colon C_{b,\Lambda} \times \mathbb{C} \to \mathbb{C}$ given by $\eta_\ell(\lambda,\gamma_r) = 2\gamma_r\mathcal{W}_\ell(\lambda) - \mathcal{K}_\ell(\lambda)$. Let $\mathcal{D} \subset \mathbb{C}$ be open and bounded such that it contains the closed unit circle. Provided $\ell > 0$ is sufficiently large, we have by (6.62) and (6.67)

$$|\eta_\ell(\lambda,\gamma_r) + \mathcal{E}_{\infty,s}(\lambda)| < |\mathcal{E}_{\infty,s}(\lambda)|,$$

for each $\gamma_r \in \mathcal{D}$ and λ on the boundary of some sufficiently small disk $\mathcal{B} \subset C_{b,\Lambda}$ around λ_∞. Thus, by Rouché's Theorem there exists for each $\gamma_r \in \mathcal{D}$ a unique zero $\lambda_\ell(\gamma_r) \in \mathcal{B}$ of $\eta_\ell(\cdot,\gamma_r)$, which satisfies

$$|\lambda_\ell(\gamma_r) - \lambda_\infty| \le Ce^{-2\varsigma_*\ell}. \tag{6.69}$$

By estimate (6.68) it holds

$$\left|\partial_\lambda \eta_\ell(\lambda,\gamma_r) - \mathcal{E}'_{\infty,s}(\lambda)\right| \leq C e^{-2\varsigma_* \ell}, \quad \lambda \in \mathcal{B}, \gamma_r \in \mathcal{D}.$$

Hence, using the (analytic) Implicit Function Theorem and the fact that $\mathcal{E}'_{\infty,s}(\lambda_\infty) \neq 0$, we conclude that the map $\lambda_\ell \colon \mathcal{D} \to \mathbb{C}$ is analytic. Implicit differentiation of identity (6.66) yields the derivatives

$$\begin{aligned} \lambda'_\ell(\gamma_r) &= \frac{2\mathcal{W}_\ell(\lambda_\ell(\gamma_r))}{\mathcal{K}'_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}'_\ell(\lambda(\gamma_r))}, \\ \lambda''_\ell(\gamma_r) &= \lambda'_\ell(\gamma_r)\frac{4\mathcal{W}'_\ell(\lambda_\ell(\gamma_r)) - \left[\mathcal{K}''_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}''_\ell(\lambda(\gamma_r))\right]\lambda'_\ell(\gamma_r)}{\mathcal{K}'_\ell(\lambda(\gamma_r)) - 2\gamma_r \mathcal{W}'_\ell(\lambda(\gamma_r))}, \end{aligned} \qquad \gamma_r \in \mathcal{D}.$$

Approximating these derivatives with (6.69) and (6.68) leads to (6.18). Next, we expand $\mathcal{K}_\ell$ in an ℓ-independent neighborhood V_∞ of λ_∞ with Taylor's Theorem as

$$\mathcal{K}_\ell(\lambda) = \mathcal{K}_\ell(\lambda_\infty) + (\lambda - \lambda_\infty)\mathcal{K}'_\ell(\lambda_\infty) + \hat{\mathcal{K}}_\ell(\lambda - \lambda_\infty), \quad \lambda \in V_\infty, \tag{6.70}$$

with $\|\hat{\mathcal{K}}_\ell(\lambda - \lambda_\infty)\| \leq C|\lambda - \lambda_\infty|^2$. By (6.69) and the ℓ-independence of V_∞ we can substitute $\lambda_\ell(\gamma_r)$ for λ in (6.70) for $\ell > 0$ sufficiently large. Thus, using estimates (6.62), (6.69) and (6.68) we arrive at

$$\begin{aligned} 0 &= 2\gamma_r \mathcal{W}_\ell(\lambda_\ell(\gamma_r)) - \mathcal{K}_\ell(\lambda_\ell(\gamma_r)) \\ &= -\mathcal{K}_\ell(\lambda_\infty) - (\lambda_\ell(\gamma_r) - \lambda_\infty)\mathcal{E}'_{\infty,s}(\lambda_\infty) + \mathcal{O}\left(e^{-4\varsigma_*\ell}, e^{-2\omega(\lambda_\infty)\ell}\right). \end{aligned} \tag{6.71}$$

Hence, we obtain the desired leading-order expression for $\lambda_\ell(\gamma_r) - \lambda_\infty$ by calculating the leading order of $\mathcal{K}_\ell(\lambda_\infty)$. First, since G is C^3 on its domain by **(S1)**, the solutions $\kappa_{\mathrm{h}}(x,u)$ and $\mathcal{X}_{in}(x,u,\lambda)$ to (2.3) and to (3.8) are C^2 on their domains $\mathbb{R} \times U_{\mathrm{h}}$ and $\mathbb{R} \times U_{\mathrm{h}} \times C_{b,\Lambda}$. Therefore, Υ is C^2 on $U_{\mathrm{h}} \times C_{b,\Lambda}$. Thus, by shrinking the ℓ– and λ-independent neighborhood U_∞ of $u_\infty(0)$ if necessary, we expand

$$\Upsilon(u,\lambda) = \Upsilon(u_\infty(0),\lambda) + \partial_u \Upsilon(u_\infty(0),\lambda)(u - u_\infty(0)) + \tilde{\Upsilon}(u,\lambda), \quad u \in U_\infty, \tag{6.72}$$

where $\|\tilde{\Upsilon}(u,\lambda)\| \leq C|u - u_\infty(0)|^2$. With the aid of identities (6.27), (6.46) and (6.72) we expand

$$\begin{aligned} \mathcal{K}_\ell(\lambda) &= \det\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) \\ &\quad + \det\left(\varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda)\right)\right) \\ &\quad + (u_\ell(0) - u_\infty(0))\det\left(\varphi_\infty(0,\lambda) \mid \partial_u\Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) \\ &\quad + \mathcal{E}_{\infty,s}(\lambda) + \mathcal{O}\left(e^{-4\varsigma_*\ell}\right) \\ &= 2\det\left(\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) + \mathcal{E}_{\infty,s}(\lambda) \\ &\quad - \frac{2\omega_*^2\alpha^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\det\left(\varphi_\infty(0,\lambda) \mid \partial_u\Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right) + \mathcal{O}\left(e^{-3\varsigma_*\ell}\right), \end{aligned} \tag{6.73}$$

where we used $[\Upsilon(u_\infty(0),\lambda)]^{-1} = \Upsilon(u_\infty(0),\lambda)R_s$, $\det(\Upsilon(u_\infty(0),\lambda)) = 1$, $\det(R_s) = -1$ and the 2-linearity of the determinant. Our aim is to approximate $\varphi_\ell(0,\lambda_\infty) - \varphi_\infty(0,\lambda_\infty)$ in (6.73). First, recall that H_1 is C^3 on its domain. Fix $\check{x} \in [0, \ell]$ Using Taylor's Theorem and estimate (6.26) we approximate

$$|\partial_u H_1(u_\ell(\check{x}),0,0) - \partial_u H_1(u_\infty(\check{x}),0,0) - \partial_{uu} H_1(u_\infty(\check{x}),0,0)(u_\ell(\check{x}) - u_\infty(\check{x}))| \leq C e^{-2\varsigma_*(2\ell-\check{x})}. \tag{6.74}$$

By estimate (6.27) and (6.74) we obtain

$$\begin{aligned} &\mathcal{A}_\ell(\check{x},\lambda) - \mathcal{A}_\infty(\check{x},\lambda) \\ &\quad = -\frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}\partial_{uu}H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})}{\mathfrak{a}_\infty}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \mathcal{O}\left(e^{-\varsigma_*(3\ell-\check{x})}\right), \end{aligned} \tag{6.75}$$

for $\check{x} \in [0, \ell]$. Subsequently, we combine (6.47) and (6.75) to obtain a leading-order approximation of $\varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda)$ for $\lambda \in C_{b,\Lambda}$

$$\begin{aligned} \varphi_\ell(0,\lambda) - \varphi_\infty(0,\lambda) &= -\int_0^\ell \mathcal{Q}_\infty(\lambda)\mathcal{T}_\infty(0,\check{y},\lambda)\left(\mathcal{A}_\ell(\check{x},\lambda) - \mathcal{A}_\infty(\check{x},\lambda)\right)\varphi_\infty(\check{y},\lambda)d\check{y} \\ &\quad + \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right) \\ &= \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty}\int_0^\infty \mathcal{Q}_\infty(\lambda)\mathcal{T}_\infty(0,\check{y},\lambda)\mathcal{Z}(\check{y},\lambda)d\check{y} + \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda)\ell}\right), \end{aligned} \tag{6.76}$$

where we denote

$$\mathcal{Z}(\check{x},\lambda) := \begin{pmatrix} 0 \\ \partial_{uu}H_1(u_\infty(\check{x}),0,0)\tilde{u}_\infty(\check{x})\hat{u}_\infty(\check{x},\lambda) \end{pmatrix}, \quad \check{x} \geq 0.$$

Since the determinant $\mathcal{E}_{\infty,s}(\lambda_\infty) = \det\left(\varphi_\infty(0,\lambda) \mid \Upsilon(u_\infty(0),\lambda)R_s\varphi_\infty(0,\lambda)\right)$ equals 0, the vectors $\Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)$ and $\varphi_\infty(0,\lambda_\infty)$ are scalar multiples of each other. As the u-coordinate of both vectors are equal, we have in fact $\varphi_\infty(0,\lambda_\infty) = \Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)$. Moreover, $\mathcal{Q}_\infty(\lambda)$ is a projection along $\mathrm{Sp}(\varphi_\ell(0,\lambda))$. Therefore, the determinant $\det(\mathcal{Q}_\infty(\lambda)w \mid \varphi_\ell(0,\lambda))$ equals $\det(w \mid \varphi_\ell(0,\lambda))$ for any vector $w \in \mathbb{C}^2$ and $\lambda \in C_{b,\Lambda}$. Using the latter two observations and $\det(\mathcal{T}_\infty(0,\check{y},\lambda)) = 1$, we simplify the determinant

$$\begin{aligned} &\det\left(\mathcal{Q}_\infty(\lambda_\infty)\mathcal{T}_\infty(0,\check{y},\lambda_\infty)\mathcal{Z}(\check{y},\lambda_\infty) \mid \Upsilon(u_\ell(0),\lambda_\infty)R_s\varphi_\ell(0,\lambda_\infty)\right) \\ &\quad = \det\left(\mathcal{T}_\infty(0,\check{y},\lambda_\infty)\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(0,\lambda_\infty)\right) = \det\left(\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(\check{y},\lambda_\infty)\right). \end{aligned} \tag{6.77}$$

Finally, using (6.73), (6.76) and (6.77), we rewrite (6.71) as

$$\begin{aligned} \lambda_\ell(\gamma_r) - \lambda_\infty &= -\frac{\mathcal{K}_\ell(\lambda_\infty)}{\mathcal{E}'_{\infty,s}(\lambda_\infty)} + \mathcal{O}\left(e^{-4\varsigma_*\ell}\right) \\ &= \frac{2\omega_*^2\alpha_*^2 e^{-2\omega_*\ell}}{\mathfrak{a}_\infty\mathcal{E}'_{\infty,s}(\lambda_\infty)}\Bigg(\det\left(\varphi_\infty(0,\lambda_\infty) \mid \partial_u\Upsilon(u_\infty(0),\lambda_\infty)R_s\varphi_\infty(0,\lambda_\infty)\right) \\ &\qquad -2\int_0^\infty \det\left(\mathcal{Z}(\check{y},\lambda_\infty) \mid \varphi_\ell(\check{y},\lambda_\infty)\right)d\check{y}\Bigg) + \mathcal{O}\left(e^{-3\varsigma_*\ell}, e^{-2\mu(\lambda_\infty)\ell}\right) \end{aligned}$$

$$= \frac{2\omega_*^2 \alpha^2 e^{-2\omega_* \ell}}{\mathfrak{a}_\infty \mathcal{E}'_{\infty,s}(\lambda_\infty)} \left(2 \int_0^\infty \partial_{uu} H_1(u_\infty(\check{x}), 0, 0) \tilde{u}_\infty(\check{x}) \left[\hat{u}_\infty(\check{x}, \lambda_\infty)\right]^2 d\check{x} \right.$$
$$\left. + \left[\hat{u}_\infty(0, \lambda_\infty)\right]^2 \partial_u \mathcal{G}(u_\infty(0), \lambda_\infty) \right) + \mathcal{O}\left(e^{-3\varsigma_* \ell}, e^{-2\mu(\lambda_\infty)\ell}\right),$$

which concludes the proof of identity (6.17). □

Chapter 7

Outlook

In this chapter we outline possible future research topics.

7.1 Breaking the symmetry

This thesis focusses on stationary, spatially symmetric, periodic pulse solutions in the singularly perturbed reaction-diffusion system (1.9). Such solutions arise naturally, because the existence problem (2.1) is R-reversible. Therefore, the associated eigenvalue problem (3.3) is also R-reversible. These symmetries can be broken by adding *advection* terms to system (1.9) or by studying *traveling*-wave solutions to (1.9) instead of stationary ones. We emphasize that in some applications advection terms occur naturally, leading to reaction-advection-diffusion models [2, 63]. It is therefore an interesting and relevant question how symmetry breaking affects our analysis.

Our existence analysis in Chapter 2 relies heavily on the R-reversibility of system (2.1): we exploit that any orbit that crosses $\ker(I - R)$ twice must be periodic. In the absence of such a symmetry, additional transversality arguments are required to construct a periodic orbit using geometric singular perturbation theory. If the singular periodic orbit consists of fast heteroclinic connections and orbit segments on slow invariant manifolds, then the required transverse intersections (of the stable and unstable foliations of the slow manifolds) are often already present in the fast reduced systems arising in the limit $\varepsilon \to 0$ – see for instance [110, Section 7]. On the other hand, if the singular periodic orbit is a concatenation of a *homoclinic* connection with an orbit segment on the slow manifold – as is the case in our work – then these transversal intersections exists only for $\varepsilon > 0$ and tools like Melnikov theory for slowly varying systems [94] can be employed to find them – see for instance [26, 108] for constructions of periodic traveling waves in the two-component (Klausmeier-)Gray-Scott model. Moreover, controlling the periodic orbits close to the transverse intersections might be subtile [108]. It remains an open problem whether the techniques in [26, 108] can be extended to the general class of multi-component systems (1.9) with additional advection terms.

In general, the spectrum associated with periodic wave trains to reaction-advection-diffusion systems consists of continuous images of the unit circle S^1 [38]. The presence of symmetry yields degenerate spectrum: the image of S^1 covers each curve of spectrum twice so that any γ-eigenvalue is also a $\overline{\gamma}$-eigenvalue – see Proposition 3.7. Thus, breaking the symmetry changes the structure of the spectrum fundamentally. Yet, we expect that the present spectral techniques extend to the non-symmetric case without any problems. Let us elaborate on this claim. Recall that our spectral analysis is based on two reduction results: the approximation of the roots of the Evans function by the ones of the reduced Evans function and the expansion of the critical spectral curve attached to the origin. First, our Evans-function analysis is based on the results in [17], where we do not assume that the periodic pulse solutions are symmetric. We observe that R-reversibility of the eigenvalue problem is not essential for the decomposition of the Evans function and its reduction. This makes an extension to models with advection terms straightforward – see also [108].

Second, our expansion method of the critical spectral curve is based on the analyses in [10, 100] using Lin's method – see §5.3.6. Both in [10] and in [100] the eigenvalue problem does not admit a (reversible) symmetry, since one considers *traveling* waves. Therefore, we expect that the present expansion method remains valid in the non-symmetric case. Yet, we foresee that the outcome of the analysis will be different: we conjecture that the critical curve is not confined to the real axis and scales with ε instead of ε^2. Indeed, the essential spectrum is no longer degenerate and the $O(\varepsilon)$-terms in the expansion of the critical curve will no longer vanish due to parity arguments. Our hypothesis is further strengthened by the fact that the critical spectral curve associated with periodic traveling waves in the FitzHugh-Nagumo equations scales with ε and is non-real – see [32].

The non-degeneracy of the spectrum in the non-symmetric case affects the destabilization mechanisms discussed in Chapter 6. Numerical investigations in the Klausmeier-Gray-Scott system indicate that the Hopf and belly dance destabilization mechanisms break down in the presence of $O(1)$ advection: the boundary of the Busse ballon consists no longer of curves of ± 1-Hopf instabilities in the limit $\varepsilon \to 0$ and the codimension-two points disappear. Instead, the boundary is smooth in the limit $\varepsilon \to 0$ and consists of oscillating curves of γ-Hopf instabilities, where γ can be *any* Floquet multiplier in S^1. It remains an open problem to confirm this analytically.

In the non-symmetric case there are three types of robust instabilities: Hopf, sideband or spatial-temporal period doubling – see [93]. As far as the author knows, there is no numerical evidence that periodic pulse solutions can destabilize through a fold or Turing instability in the absence of symmetry (provided $\varepsilon > 0$ is sufficiently small). This suggests that, as in the symmetric case, fold and Turing instabilities cannot occur. However, analytical grip on the spectrum in the non-symmetric case is needed to confirm this hypothesis.

7.2 Dynamics of periodic pulses upon destabilization

The explicit insights in the spectral geometry in Chapters 3 and 6 is a key to understanding the weakly nonlinear dynamics of periodic pulse solutions to (1.9) as they become spectrally unstable. A first step in this direction has been taken in [119], in which a normal form approach associated with a Hopf destabilization of *homoclinic* pulses in 2-component, slowly nonlinear models of the form (1.9) is developed. Unlike known classical slowly linear examples such as the Gray-Scott and Gierer-Meinhardt models, the Hopf bifurcation for homoclinic pulses can be supercritical. It can even be the first step in a sequence of further bifurcations that leads to complex (amplitude) dynamics of a standing solitary pulse – as observed in the simulations of [120]. We expect that the weakly nonlinear dynamics of periodic pulse solutions to (1.9) beyond their destabilization is also very rich – and thus an interesting direction of future research – as indicated by the Hopf and belly dance destabilization mechanisms described in Chapter 6 and the fact that the pulses that together form the periodic pattern are in semi-strong interaction [24, 92]. We stress that it is still unknown whether the Hopf and belly dance destabilization mechanisms generalize to systems (1.9) with multiple components (the regime $n > 1$ or $m > 1$).

7.3 Multiple spatial dimensions

In this thesis we focus on solutions to singularly perturbed reaction-diffusion systems *on the line*. However, in some applications [46, 73, 89, 109], the associated reaction-diffusion models are naturally formulated on the plane or an unbounded cylinder. This give rise to the following class,

$$\begin{aligned} u_t &= D_1 \Delta u - H(u, v, \varepsilon), \\ v_t &= \varepsilon^2 D_2 \Delta v - G(u, v, \varepsilon), \end{aligned} \qquad u(\check{x}, t) \in \mathbb{R}^m, v(\check{x}, t) \in \mathbb{R}^n, \quad \check{x} \in \mathbb{R} \times \Omega, \tag{7.1}$$

of reaction-diffusion systems, where $0 < \varepsilon \ll 1$ and $\Omega \subset \mathbb{R}^k$ can be a bounded or unbounded domain. Spatially multi-dimensional systems of the form (7.1) are far less well understood than their one-dimensional counterparts (1.1). Obviously, solutions to (1.1) give rise to *striped* solutions to (7.1) by trivially extending them into a transverse spatial direction. Similarly, by switching to polar coordinates one can construct *spots*, which are constant along concentric circles. Using singular perturbation techniques, one can construct more elaborate solutions to (7.1) like spiral waves and target patterns – see [113] and references therein.

In the stability analysis of stripes and spots one proceeds by applying the Fourier transform in the transverse or radial direction – see for instance [36, 77, 105, 111]. Consequently, the associated eigenvalue problem depends on one additional parameter, but is still a singularly perturbed *ordinary* differential equation. Therefore, we expect that many of the spectral reduction techniques presented in this thesis can be employed to determine the stability of spot and stripe solutions to (7.1). Pioneering work in that direction can be found in [29, 109, 117] in the context spots and stripes in FitzHugh-Nagumo and Gray-Scott type models.

If a solution to (7.1) has a more elaborate structure, then such a reduction via the Fourier transform is impossible and the associated eigenvalue problem is a *partial* differential equation. In some specific cases, the eigenvalue problem can be reduced to a *scalar*, non-local PDE. Via this non-local problem one can prove spectral stability of the underlying pattern – see for instance [124, 125], where spectral stability is established for (asymmetric) spotty patterns in the Gray-Scott and Gierer-Meinhardt models on the plane.

However, it is still an open problem whether the spectral reduction results presented in this thesis have infinite-dimensional counterparts. We emphasize that many of the employed ODE-techniques carry over to the PDEs: exponential dichotomies [44], Lin's method [96] and the Evans function [18, 69] can be utilized by rewriting the eigenvalue problem as an evolution equation in the spatial variables corresponding to the unbounded directions. This so-called *spatial dynamics approach* was introduced in [62] – see also [16, 91, 103] and references therein. All in all, this could provide the desired framework that facilitates a reduction induced by the small parameter ε in (7.1), possibly through a factorization of the Evans function. Eventually, this might lead to a systematic approach for studying the (spectral) stability of patterns in singularly perturbed systems of the form (7.1) that allow for multiple components, multiple spatial dimensions and slow nonlinearities.

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Nederlandse samenvatting

Periodieke pulsoplossingen in langzaam-niet-lineaire reactie-diffusiesystemen

In velerlei dynamische processen vindt patroonvorming plaats – denk aan de geleiding van zenuwimpulsen door een axon, het ontstaan van vlekken en strepen op dierenvacht of de ontwikkeling van vegetatiepatronen aan de rand van een woestijn. *Reactie-diffusiesystemen* zijn wellicht de simpelste partiële differentiaalvergelijkingen die zulke patroonvormende processen beschrijven. Alan Turing was de eerste die in 1952 liet zien dat patronen uit een uniforme toestand kunnen ontstaan in lineaire reactie-diffusiesystemen met twee componenten. Één van de componenten, de activator, bevordert de groei van beide componenten, terwijl de ander, de inhibitor, zorgt voor afremming. Turing toonde aan dat wanneer de inhibitor veel gemakkelijker diffundeert dan de activator er een terugkoppelingsmechanisme ontstaat wat patroonvorming stimuleert. In 1972 generaliseerde Gierer en Meinhardt dit resultaat naar niet-lineaire reactie-diffusiesystemen. Tegenwoordig worden niet-lineaire reactie-diffusiesystemen met meerdere componenten gebruikt als een paradigmatische klasse voor de bestudering van patroonvorming.

Het bovengenoemde verschil in diffusiesnelheden kan worden gemodelleerd door een (asymptotisch) kleine parameter ε in het systeem te introduceren. Over het algemeen is het uitermate lastig patronen in reactie-diffusievergelijkingen analytisch te beschrijven – helemaal wanneer de vergelijking meerdere componenten heeft. Echter, de kleine parameter induceert een tweedeling tussen langzaam en snel gedrag in het systeem. Door deze tweedeling uit te buiten, kunnen we de wiskundige analyse vergemakkelijken. Immers, door de limiet $\varepsilon \to 0$ in verschillende schalingen van het systeem te nemen ontstaan zogenaamde *gereduceerde, langzame en snelle systemen*. Deze gereduceerde systemen zijn lager-dimensionaal en daarom beter geschikt voor wiskundige analyse. Vaak is het voldoende deze gereduceerde systemen te begrijpen om het gedrag van patronen in het volledige reactie-diffusiesysteem te beschrijven.

Via reductiemethodes is er de afgelopen decennia veel vooruitgang geboekt in de analyse van patroonoplossingen in reactie-diffusiesystemen met een kleine parameter – en dan met name de stationaire en uniform-lopende patronen. Zo kan een dergelijk patroon geconstrueerd worden uit simpelere bouwstenen op de volgende manier. Eerst plakken we simpelweg oplossingen van gereduceerde, langzame en snelle systemen aan elkaar zodat er een gesloten baan ontstaat – zie Figuur 1.2. Hoewel deze baan geen oplossing is van het oorspronkelijke systeem, kan men via *geometrische singuliere-storingsrekening* aantonen dat er een echte oplossing bestaat in een omgeving van deze baan, onder de voorwaarde dat de parameter ε voldoende klein is. Gezien de oplossing ontstaat uit een samenvoeging van snelle en langzame onderdelen, bestaat zij uit sterk gelokaliseerde, spitse stukken en vlakke segmenten tussen de gelokaliseerde gedeeltes – zie bijvoorbeeld Figuur 1.

Vaak is men niet alleen geïnteresseerd in de gevonden patroonoplossing, maar tevens in het gedrag van dichtbij zijnde oplossingen van het reactie-diffusiesysteem. Immers, als oplossingen in de omgeving van het patroon divergeren met het verstrijken van de tijd, dan is het patroon instabiel onder kleine verstoringen. Zo'n patroon zal dus niet in de praktijk waargenomen worden, ook al is zijn existentie wiskundig bewezen. Om de *stabiliteit* van een patroonplossing te bepalen zijn we geïnteresseerd in de leidende orde dynamica in een omgeving van de oplossing, welke wordt verkregen door het reactie-diffusiesysteem rondom de oplossing te lineariseren – het lineaire gedrag zal namelijk voor kleine verstoringen dominant zijn. De gelineariseerde vergelijking is een evolutievergelijking van de vorm $u_t = \mathcal{L}u$, waarbij $\mathcal{L}$ een onbegrensde differentiaaloperator is, gedefinieerd op een geschikte functieruimte. Het *spectrum* van de verworven differentiaaloperator $\mathcal{L}$ is vaak bepalend voor stabiliteit, maar het is meestal zeer lastig de benodigde spectrale informatie te verkrijgen. Immers, het spectrum van $\mathcal{L}$ bestaat uit alle $\lambda \in \mathbb{C}$ waarvoor het *eigenwaardeprobleem* $\mathcal{L}u = \lambda u$ inverteerbaar is. Dit eigenwaardeprobleem is weliswaar een lineaire, *gewone* differentiaalvergelijking, maar – zeker in het geval van meerdere componenten – is het nog steeds erg lastig zo'n niet-autonoom systeem op te lossen.

Desalniettemin kunnen we ook hier gebruik maken van de tweedeling tussen langzaam en snel gedrag in het systeem. Voor patroonoplossingen van verschillende specifieke reactie-diffusiesystemen is er via geometrische methodes aangetoond dat het eigenwaardeprobleem versimpelt in gereduceerde, langzame en snelle eigenwaardeproblemen. Deze versimpeling manifesteert zich via een complex-analytische, determinantfunctie: de *Evansfunctie*, wiens nulpunten overeenkomen met het spectrum van de differentiaaloperator $\mathcal{L}$. De kleine parameter induceert een expliciete *factorisatie* van de Evansfunctie in langzame en snelle componenten, die corresponderen met gereduceerde, langzame en snelle eigenwaardeproblemen. Deze factorisatie geeft in de limiet $\varepsilon \to 0$ aanleiding tot een decompositie van het spectrum in snelle en langzame segmenten, die bepaald kunnen worden aan de hand van de gereduceerde eigenwaardeproblemen, welke lager-dimensionaal en daarom beter behapbaar zijn. Dit leidt tot *asymptotische controle* over het spectrum.

Echter, asymptotische controle over het spectrum is dikwijls onvoldoende om stabiliteit van de patroonoplossing aan te tonen: door *translatie-invariantie* – elke ruimtelijke translatie van het patroon is namelijk weer een oplossing van het reactie-diffusiesysteem – bevindt er zich spectrum nabij de oorsprong in het complexe vlak. In de limiet $\varepsilon \to 0$ convergeert dit spectrum naar de oorsprong toe en dus is asymptotische controle onvoldoende om de fijnstructuur van de spectrum nabij de oorsprong te bepalen. Echter, de precieze locatie van spectrum ten opzichte van de imaginaire as is doorslaggevend voor stabiliteit. Daarom dient in veel gevallen de Evansfunctieanalyse gecomplementeerd te worden met een lokale analyse van het spectrum nabij de oorsprong.

De geometrische factorisatiemethode van de Evansfunctie is ontwikkeld in de context van patroonoplossingen in *langzaam-lineaire* prototypesystemen – zoals de FitzHugh-Nagumo-, Gray-Scott- en Gierer-Meinhardtvergelijkingen. In zulke systemen is de gereduceerde langzame dynamica lineair van aard. Recentelijk is de geometrische factorisatiemethode gegeneraliseerd voor de stabiliteitsanalyse van stationaire, homocliene pulsoplossingen in een klasse van langzaam-*niet*-lineaire reactie-diffusiesystemen met twee componenten. Het is aangetoond dat de dynamica in zulke systemen fundamenteel anders – en rijker – is dan in langzaam-lineaire systemen.

In dit proefschrift bestuderen we stationaire, ruimtelijk *periodieke* pulsoplossingen in langzaam-niet-lineaire reactie-diffusievergelijkingen met een *willekeurig* aantal componenten. Hoewel er op het eerste oog weinig verschil lijkt te zijn met de situatie van homocliene pulsoplossingen, is de geometrische factorisatiemethode in ons geval niet toepasbaar. Bovendien raakt er een spectrale kromme aan de oorsprong, die in de limiet $\varepsilon \to 0$ ineenkrimpt tot een punt. Hierdoor is een additionele analyse van het spectrum nabij de oorsprong noodzakelijk, terwijl in het homocliene geval er zich slechts een simpele, geïsoleerde eigenwaarde op de oorsprong bevindt.

Voor de stabiliteitsanalyse van periodieke pulsoplossingen zijn dus nieuwe reductiemethodes nodig. In dit proefschrift presenteren we een alternatieve, *analytische* factorisatiemethode voor de Evansfunctie. Deze analytische methode werkt, in tegenstelling tot zijn geometrische tegenhanger, voor periodieke patronen in langzaam-niet-lineaire systemen. Bovendien kan de methode onafhankelijk van specifieke systemen of patroonoplossingen worden geformuleerd en werkt zij voor een willekeurig aantal componenten. Daarmee bewijst onze analytische methode ook haar nut buiten de context van periodieke pulsoplossingen. Ten tweede ontrafelen we in dit proefschrift de fijnstructuur van het spectrum nabij de oorsprong door lokaal het inverteren van het oneindigdimensionale eigenwaardenprobleem te reduceren tot het oplossen van een eendimensionale vergelijking. Door de termen in de laatstgenoemde vergelijking uit te rekenen in de limiet $\varepsilon \to 0$ verkrijgen we een expansie van de kritieke spectrale kromme die aan de oorsprong raakt.

De twee reductiemechanismen uit de vorige alinea geven voldoende controle over het spectrum om expliciete *stabiliteitscriteria* op te stellen in termen van gereduceerde, lager-dimensionale eigenwaardeproblemen. Zo kan men stabiliteit van stationaire, periodieke pulsoplossingen in een tweecomponentensysteem bewijzen door een aantal voorwaarden te checken in scalaire Sturm-Liouvilleproblemen. Bovendien vinden we instabiliteitscriteria in termen van het teken van een aantal expliciete uitdrukkingen die uitgerekend kunnen worden met slechts de asymptotische benadering van het pulsprofiel als input. Tot slot leidt de controle over het spectrum tot inzichten in *destabilisatiemechanismen* van periodieke pulsoplossingen. Het is mogelijk de verschillende soorten instabiliteiten te karakteriseren aan de hand van de voorgenoemde stabiliteitscriteria. In het bijzonder kunnen we de destabilisatiemechanismen van een periodieke pulsoplossing die convergeert naar een homocliene puls zeer gedetailleerd beschrijven.

De opzet van dit proefschrift is als volgt. In Hoofdstuk 1 geven we een literatuuroverzicht van de huidige reductiemethodes voor de existentie- en stabiliteitsanalyse van patroonoplossingen in reactie-diffusiesystemen met een kleine parameter. We leggen uit waarom deze reductiemethodes niet geschikt zijn voor de stabiliteitsanalyse van periodieke pulsoplossingen in langzaam-niet-lineaire systemen. Tevens introduceren we de algemene klasse van langzaam-niet-lineaire reactie-diffusiesystemen met een willekeurig aantal componenten die we beschouwen in dit proefschrift. In Hoofdstuk 2 construeren we periodieke pulsoplossingen in deze klasse van reactie-diffusiesystemen met behulp van geometrische singuliere-storingsrekening. In Hoofdstuk 3 presenteren we het hoofdresultaat van dit proefschrift: expliciete stabiliteits- en instabiliteitscriteria in termen van lager-dimensionale eigenwaardeproblemen. De volgende twee hoofdstukken zijn gewijd aan het bewijs van ons hoofdresultaat. In Hoofdstuk 4 behandelen we de voorkennis noodzakelijk voor de stabiliteitsanalyse. Hoofdstuk 5 bevat dan de werkelijke stabiliteitsanalyse: we factoriseren de Evansfunctie via onze alternatieve, analytische reductiemethode en we expanderen de kritieke spectrale kromme. In Hoofdstuk 6 concentreren we ons op destabilisatiemechanismen van periodieke pulsoplossingen. Tot slot bevat Hoofdstuk 7 een overzicht van toekomstige onderzoeksmogelijkheden.

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Curriculum vitae

Björn de Rijk was born on the 9th of October, 1988 in Rijswijk, the Netherlands. In 2007 he obtained his gymnasium diploma at the Interconfessioneel Makeblijde College in Rijswijk. In the same year he enrolled at Leiden University and in 2010 he received a bachelor's degree in mathematics (cum laude). During his bachelor studies he worked as an exam trainer teaching classes of high school students in preparation for their final exams. Moreover, as a student ambassador he provided information to and organized activities for prospective mathematics students. In 2009 he was elected as a member of student party BeP to the faculty council of the faculty of science. In 2012 he obtained his master's degree in mathematics (cum laude) taking courses with an emphasis on functional analysis. His master's thesis is titled *The Order Bicommutant* and was written under the supervision of dr. M. de Jeu. It was awarded the GQT student prize for best master's thesis in a field related to geometry and quantum theory. During his master's studies Björn became student member of the examination appeals board.

In 2012 he commenced his Ph.D. research at the Mathematical Institute of Leiden University under the guidance of prof. dr. A. Doelman and prof. dr. J.D.M. Rademacher, which culminated in this thesis. His research project was partly funded by the Dutch science foundation (NWO) through the NDNS+-cluster. During his Ph.D. studies Björn attended, organized and spoke at various seminars and colloquia in Leiden, Delft, Bremen, Stuttgart, Berlin, Providence, Boston, Troy and Lunteren. He was one of the organizers of the Ph.D. colloquium at the Leiden Mathematical Institute, which aims to connect Ph.D. students from different fields. He gave contributed talks and poster presentations at international conferences in Cambridge, Snowbird, Lyon and Philadelphia. At the SIAM NW14 Conference in Cambridge he won the poster prize. In 2014 he earned a travel grant for a four month research visit to Brown University in Providence. This resulted in a successful research project under the supervision of prof. dr. B. Sandstede leading to a journal publication [10] that is not included in this thesis.

Björn also assisted in the teaching of many bachelor and master courses, including the national Mastermath functional analysis course. He was an instructor of the numeracy course at Leiden University College in The Hague in 2013. Moreover, he lectured ordinary differential equations several times in the absence of the senior lecturer. After obtaining his doctor's degree, Björn will proceed with post-doctoral research at Stuttgart University in the group of prof. dr. G. Schneider.

Propositions

1. Periodic pulse solutions exists in a large class of multi-component, slowly nonlinear reaction-diffusion systems. They arise as perturbations from a singular periodic orbit consisting of a pulse satisfying a fast reduced system and an orbit segment on an invariant manifold satisfying a slow reduced system. *(This thesis, chapter 2.)*

Propositions 2.-8. concern periodic pulse solutions in multi-component, slowly nonlinear reaction-diffusion systems as described in Proposition 1.

2. In order to establish nonlinear diffusive stability of periodic pulse solutions one needs both asymptotic control over the spectrum of the linearization about the pulse solution as well as leading-order control over the critical spectral curve attached to the origin, which shrinks to the origin in the asymptotic limit. *(This thesis, chapter 3.)*
3. Explicit asymptotic control over the spectrum can be obtained by factorizing the Evans function via the Riccati transform. *(This thesis, chapter 5.)*
4. Explicit leading-order control over the critical spectral curve can be obtained by a Lyapunov-Schmidt reduction procedure using Lin's method. *(This thesis, chapter 5.)*
5. Verifying nonlinear diffusive stability of a periodic pulse solution with m slow and n fast components requires explicit knowledge of the dynamics in m- and n-dimensional eigenvalue problems only. *(This thesis, chapter 3.)*
6. A two-component periodic pulse solution cannot be destabilized through a Turing or fold instability, whereas these instabilities are robust for symmetric, spatially periodic patterns in reaction-diffusion systems. *(This thesis, chapter 6.)*
7. The Hopf and belly dance destabilization mechanisms occur in every two-component, slowly nonlinear reaction-diffusion systems. *(This thesis, chapter 6.)*
8. Whether the homoclinic limit is the last 'periodic' pulse solution to destabilize depends on the slow nonlinearity. *(This thesis, chapter 6.)*

9. Techniques for partial differential equations are often developed in the context of specific prototype equations, which facilitates the readability of papers. The drawback is that wider applicability of these techniques is often claimed, but not checked in detail.
10. In mathematics, just like in sports, natural ability alone is not enough to excel.
11. The work of a mathematician can rarely be applied to a real life problem directly. Yet, many solutions to real life problems have their roots in mathematics.
12. There is no clear distinction between fundamental and applied research. It is all a matter of perspective.

Stellingen

1. Periodieke pulsoplossingen bestaan in een grote klasse van langzaam-niet-lineaire reactie-diffusiesystemen met meerdere componenten. Deze oplossingen komen tot stand als verstoringen van een singuliere periodieke baan bestaande uit een puls die voldoet aan een gereduceerd snel systeem en een baansegment op een invariante variëteit die voldoet aan een gereduceerd langzaam systeem. *(Dit proefschrift, hoofdstuk 2.)*

Stellingen 2.-8. hebben betrekking op periodieke pulsoplossingen in langzaam-niet-lineaire reactie-diffusie-systemen met meerdere componenten zoals beschreven in Stelling 1.

2. Om niet-lineaire diffusieve stabiliteit van periodieke pulsoplossingen te bewijzen is zowel asymptotische controle over het spectrum van de linearisatie rondom de pulsoplossing vereist als leidende orde controle over de kritieke spectrale kromme die raakt aan de oorsprong en in de asymptotische limiet ineenkrimpt tot een punt. *(Dit proefschrift, hoofdstuk 3.)*

3. Expliciete asymptotische controle over het spectrum kan worden verkregen door de Evansfunctie te factoriseren via de Riccatitransformatie. *(Dit proefschrift, hoofdstuk 5.)*

4. Expliciete leidende orde controle over de kritieke spectrale kromme kan worden verkregen via een Lyapunov-Schmidtreductieprocedure door gebruik te maken van Lins methode. *(Dit proefschrift, hoofdstuk 5.)*

5. Expliciete kennis van de dynamica in zekere m- and n-dimensionale eigenwaardeproblemen is voldoende om te verifiëren of een periodieke pulsoplossing met m langzame en n snelle componenten niet-lineair diffusief stabiel is. *(Dit proefschrift, hoofdstuk 3.)*

6. Een periodieke pulsoplossing met twee componenten kan niet destabiliseren via een Turing- of foldinstabiliteit, terwijl deze instabiliteiten robuust zijn voor symmetrische, ruimtelijk periodieke oplossingen in reactie-diffusiesystemen. *(Dit proefschrift, hoofdstuk 6.)*

7. De Hopf en belly dance destabilisatiemechanismen vinden in elk langzaam-niet-lineair reactie-diffusiesysteem met twee componenten plaats. *(Dit proefschrift, hoofdstuk 6.)*

8. Het hangt van de langzaam-niet-lineaire termen af of de homocliene limiet de laatste 'periodieke' pulsoplossing is die haar stabiliteit verliest. *(Dit proefschrift, hoofdstuk 6.)*

9. Technieken voor partiële differentiaalvergelijkingen worden vaak ontwikkeld in de context van specifieke prototypesystemen, wat de leesbaarheid van artikelen ten goede komt. De schaduwzijde hiervan is dat er vaak geclaimd wordt dat deze technieken breder toepasbaar zijn, maar dit niet wordt geverifieerd.

10. In de wiskunde is, net zoals bij sport, aanleg alleen onvoldoende om de top te bereiken.

11. Het werk van een wiskundige kan bijna nooit direct in de praktijk toegepast worden. Toch vinden veel oplossingen voor problemen uit de praktijk hun oorsprong in de wiskunde.

12. Er is geen duidelijk onderscheid te maken tussen fundamenteel en toegepast onderzoek. Het is een kwestie van perspectief.