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Mathematical Beauty and the Evolution of the Standards of Mathematical Proof

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As most mathematicians agree, the beauty of mathematical entities plays an important part in the subjective experience and enjoyment of doing mathematics. Some mathematicians claim also that beauty acts as a guide in making mathematical discoveries and that beauty is an objective factor in establishing the validity and importance of a mathematical result. The combination of subjective and objective aspects makes mathematical beauty an intriguing phenomenon for philosophers as well as mathematicians. This chapter analyzes the concept of mathematical beauty—especially the forms of beauty that mathematicians appreciate in and demand of mathematical proofs. This analysis will help ascertain how mathematical beauty can play both a subjective role in the experience of mathematicians and an objective role in the appraisal of mathematical proofs and other results.

First, let us clarify a few fundamental terms. How should we interpret an observer's claim that a certain entity is beautiful? The most natural interpretation is that the entity has a property named "beauty," which the observer has perceived. I do not regard this interpretation as satisfactory, however. I regard beauty as a value that is projected into or attributed to objects by observers, not a property that intrinsically resides in objects. This philosophical viewpoint is known as projectivism. Whether an observer projects beauty into an object is determined by two factors:

the aesthetic criteria held by the observer, and the object's intrinsic properties.

Aesthetic criteria attach aesthetic value to particular properties of objects: they have the form "Project beauty into an object if, other things being equal, it exhibits property *P*." If an object exhibits properties that are valued by an observer's aesthetic criteria, then the observer will project beauty into that object and describe the object as beautiful. Whereas projections of beauty can in principle be triggered by any property, observers generally attribute aesthetic value to a relatively small family of properties. Typical examples are architectonic or structural traits and ornamental or decorative features. We give the name "aesthetic properties" to the intrinsic properties of an object that evoke an observer's aesthetic response, such as a projection of beauty by the observer into that object.

On this account, to understand the phenomenon of mathematical beauty, we must distinguish the act of projecting the value of beauty into mathematical entities, which mathematicians perform, from the intrinsic properties of the mathematical entities that motivate such a projection. In the terminology that I propose, it is strictly incorrect to say that a mathematical entity has beauty, as though this were a property encountered in the object. Rather, a mathematical entity has certain aesthetic properties, such as simplicity and symmetry. On the strength of perceiving these properties in a mathematical entity, and by virtue of holding to aesthetic criteria that attach value to these properties, an observer is moved to project beauty into the entity. A different observer, holding to different aesthetic criteria, might not do so.

By allowing that observers may hold to different aesthetic criteria, this account explains how mathematicians can disagree about the aesthetic merits of mathematical entities. Two possible patterns of disagreement are particularly interesting. It may be that different mathematicians at the same time hold to different aesthetic criteria. Alternatively, it may be that, while all members of the mathematical community at any one time hold to the same aesthetic criteria, these criteria show historical evolution. In practice, there is wide agreement among mathematicians at any time about the mathematical entities that merit the predicate "beautiful," but the community's aesthetic tastes change with time. We shall examine an instance of this evolution, affecting the standards of mathematical proofs, later in this chapter.

Many alternative philosophical accounts of beauty exist. The one outlined here, I believe, is especially suited to making sense of mathematicians' practice of attributing beauty to mathematical entities as well as the similar practice among scientists of calling theories and experiments beautiful.

Beauty in Mathematical Products and Processes

It is useful to draw a tentative distinction between two classes of mathematical entities to which beauty may be attributed: processes and products. Processes include problem-solving techniques, calculation methods, computer programs, proofs, and all other operations, algorithms, procedures, and approaches used in mathematics. Products, which are outcomes of processes, include numbers, equations, problems, theories, theorems, conjectures, propositions of other sorts, curves, patterns, geometrical figures and constructions, and all other mathematical structures. Entities of both sorts can be regarded as beautiful, but the aesthetic properties of products differ from, and are largely independent of, those of processes.[1]

Let us begin with products. Among the products that mathematicians regard as beautiful are numbers, including individual numbers, such as *e*;[2] classes of numbers, such as the perfect numbers;[3] and arrangements of numbers, such as Pascal's triangles.[4] Mathematicians seem to find numbers beautiful if they show either extreme simplicity or notable richness—for example, if they can be defined in simple terms or generated in a multitude of ways.

A second class of mathematical products that are the object of aesthetic assessment are geometrical constructions, such as polygons and tilings;[5] the Platonic solids;[6] and figures and curves exhibiting the golden section.[7] Fractals have become frequent objects of aesthetic appreciation in recent decades.[8] An important aesthetic property of geometrical constructions is symmetry, which can be manifested as regularity, pattern, proportion, or self-similarity.[9]

Lastly, many mathematicians comment upon the aesthetic merits of theorems. According to G. H. Hardy, beautiful theorems are those that exhibit the properties of seriousness, generality, depth, unexpectedness, inevitability, and economy.[10] Hardy regards the theorems that there exist infinitely many primes and that the square root of 2 is irrational as especially beautiful. David Wells has surveyed mathematicians on their opinions of the aesthetic merits of twenty-four well-known theorems.[11] His respondents awarded the highest score to Euler's identity, $e^{i\pi} = -1$. Various further lists of theorems and other mathematical results regarded as beautiful have been compiled.[12]

Mathematical beauty is appreciated not only in pure mathematics, but also in applied fields. Theoretical physicists often claim mathematical beauty for their theories. Their conceptions of mathematical beauty are shaped by the domains of mathematics on which they draw in formulating their theories. These domains vary with time and from one branch of physics to another. In Renaissance natural philosophy, for example, to describe the world mathematically meant to represent it by geometrical figures or, according to other thinkers, by individual numbers. Natural philosophers thus tended to define mathematical beauty in geometrical and numerical terms. Present-day physics also uses a variety of mathematical tools. Elementary particle physics, for example, classifies particles by reference to symmetry groups, and physicists often describe the resulting theories as having mathematical beauty.[13] In most branches of present-day physics, however, to describe the world mathematically means to represent it in mathematical equations. In consequence, physicists nowadays view mathematical beauty as consisting chiefly in the beauty of equations.[14]

Physicists attribute mathematical beauty to equations primarily on the strength of their simplicity properties, including their conciseness, the simplicity of their algebraic form, and the simplicity of any numerical constants that they contain, such as coefficients and powers. Symmetry properties also play a role in physicists' aesthetic appreciation of equations. Equations to which physicists have attributed mathematical beauty include the inverse-square force laws found in classical gravitation theory and elsewhere;[15] Maxwell's equations, which are admired especially for their symmetries; the Schrödinger equation; and the equations of the general theory of relativity.[16]

Some physicists attach epistemological significance to their assessments of the mathematical beauty of theories: they claim to be able to ascertain from an equation's aesthetic properties whether it constitutes a true description of natural phenomena or a fundamental law of nature. A notable example is P. A. M. Dirac, who was inclined to accept physical theories as correct if he found their equations beautiful and reject them otherwise.[17]

This concludes our brief survey of the beauty attributed to mathematical products. Attributions of beauty to mathematical processes have received somewhat less attention. Mathematicians commonly subject three kinds of processes to aesthetic evaluation.

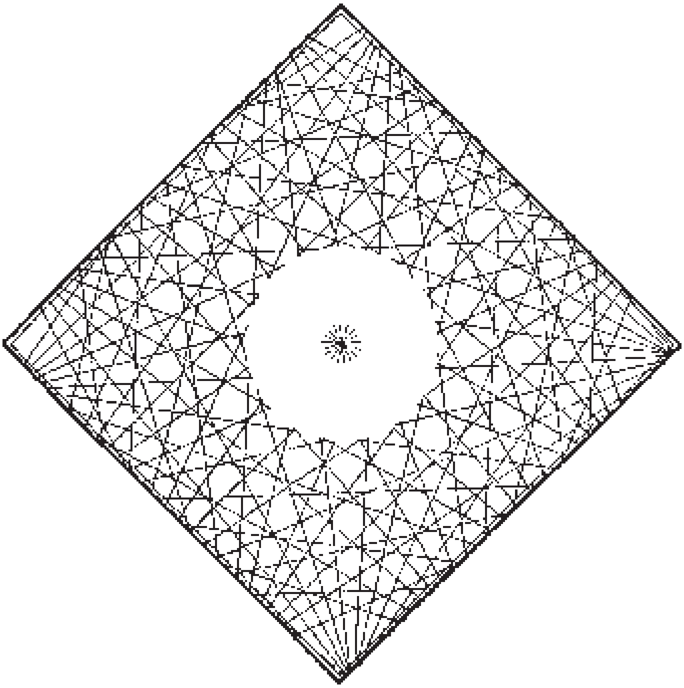
Processes of the first kind consist of calculation methods. Most mathematicians judge calculations that involve analytic methods and yield exact solutions as more beautiful than those involving numerical methods and approximations. Methods that incorporate ingenious shortcuts often strike users as especially beautiful. An example is the method for summing an arithmetic series by pairing off the terms, which according to legend the ten-year-old Carl Friedrich Gauss devised when he was set the task of summing the integers from 1 to 100.[18] Calculation procedures that violate established mathematical rules, by contrast, are usually regarded as ugly, even if they deliver correct results. For example, finite renormalization in quantum electrodynamics—an ad hoc procedure for excising infinities that appear during calculations—evoked aesthetic disapproval when it was first proposed, despite its empirical vindication.[19]

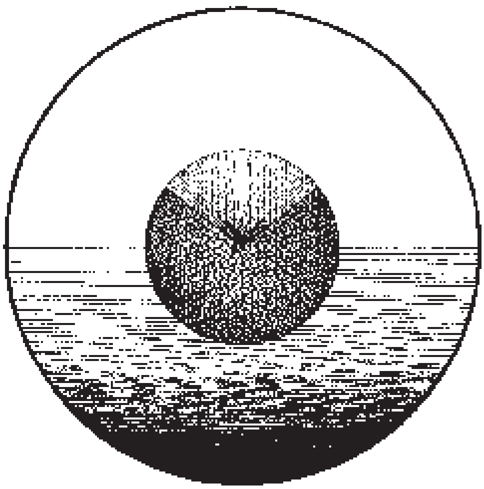
Second, aesthetic considerations play a role in evaluating computer programs and the design of programming languages. Elegance in a computer algorithm is usually associated with the desiderata of transparency and efficiency, whereas a cumbersome program is often described as ugly.[20]

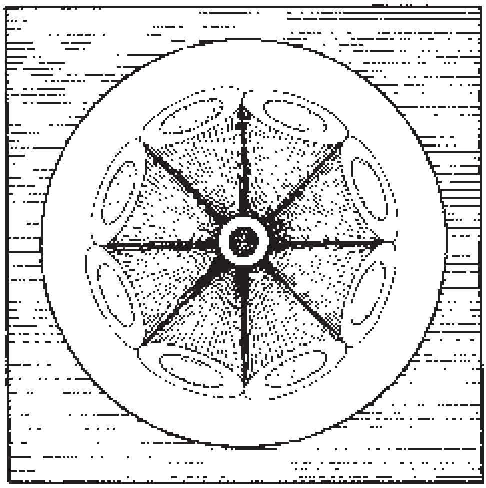
Third, aesthetic considerations play a role in the construction and appraisal of mathematical proofs. Many mathematicians have commented on the aesthetic merits of proofs.[21] Paul Erdős used to claim that God had a book that contained all the most elegant mathematical proofs. When Erdős encountered a proof that he found exceptionally elegant, he declared it to be "straight from The Book."[22] Martin Aigner and Günter M. Ziegler have compiled a collection of proofs that they believe meet this standard.[23] It is on the aesthetic merits of mathematical proofs that we shall concentrate in the remainder of this chapter.

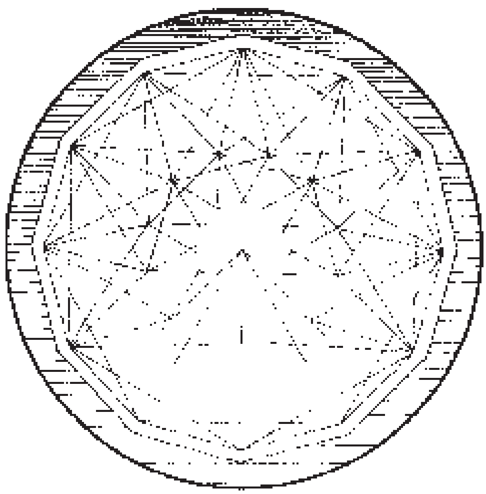
The Evolution of Mathematical Proof

In classical times, the proof of a mathematical theorem was defined as a short, simple series of logical inferences from a set of axioms to the theorem. The series of inferences was required to be sufficiently short and simple that a mathematician could grasp it in a single act of mental apprehension.









There were two reasons for imposing this requirement. First, it was considered essential to the validity of the proof. If a proposed proof was so long or complicated that a mathematician was unable to perceive all its steps in one mental image, then who could say that all its steps held simultaneously? Second, it was felt that only a proof that could be grasped in a single act of mental apprehension provided genuine understanding of the reasons for the truth of the theorem.

Paradigmatic examples of classical proofs are Pythagoras's proof that the square root of 2 is irrational, Euclid's proof that there exist infinitely many primes, and the proofs of geometrical theorems in Euclid's *Elements*. Techniques typically employed in classical proofs include *reductio ad absurdum* and mathematical induction.[24]

Mathematicians' views about beauty in proofs have been influenced by their familiarity with classical proofs. Mathematicians have customarily regarded a proof as beautiful if it conformed to the classical ideals of brevity and simplicity. The most important determinant of a proof's perceived beauty is thus the degree to which it lends itself to being grasped in a single act of mental apprehension.

Some writers acknowledge a second kind of proof, which is as ancient as the classical proof: the picture proof, or proof consisting exclusively of a picture or diagram with little or no accompanying verbal elucidation.[25] A successful picture proof, according to those who recognize this concept, is capable of establishing the truth of a theorem in an indubitable and immediate manner.[26] Other writers deny that a picture is sufficient to establish a mathematical claim because of the nonpropositional and nonrigorous nature of pictures. There is no doubt, however, that picture proofs satisfy the classical requirement that proofs should lend themselves to being grasped in a single act of mental apprehension. Partly on the strength of their power and intuitive nature, many mathematicians find picture proofs aesthetically attractive.

For centuries, all mathematical proofs took the form of the classical proof or the picture proof. In recent decades, however, some mathematicians have claimed to have developed proofs that do not conform to these styles. The new proofs fall into two categories.

The first of these is the long proof. This is a proof that rests on a large number of logical inferences, typically running into the thousands. An example is the proof of Fermat's last theorem given by Andrew Wiles, which fills an issue of the periodical *Annals of Mathematics*.[27] Because of their length, it is difficult to maintain that long proofs lend themselves to being grasped in a single act of mental apprehension. Nonetheless, long proofs share the deductive structure of the classical proof. It may therefore be argued that a long proof consists of a number of sections, each of which satisfies the classical requirement of graspability.

Long proofs have evoked conflicting aesthetic responses. In the eyes of some mathematicians, the length of Wiles's proof of Fermat's last theorem impairs its aesthetic value; other writers grant that it contains elements of beauty.[28]

The second new sort of proof is the computer-assisted proof. In such a proof, the theorem is reduced to a claim about the properties of the elements belonging to a certain large set. As the elements are too numerous for a human to examine, a computer is programmed to verify the claim. The output of the computation is taken as showing that the original theorem holds. The most striking example is the proof of the four-color conjecture given by Kenneth Appel and Wolfgang Haken in 1977. Appel and Haken reduced the conjecture that four colors suffice to color any planar map to a claim about the properties of some 2,000 particular maps. Output from a computer indicated that the claim holds for each of these maps. This output constituted the final step in the proof.[29]

Appel and Haken's proof of the four-color conjecture, and computer-assisted proofs of other theorems that have been proposed subsequently, have provoked much debate about the nature of mathematical proof among philosophers as well as mathematicians.[30] The mathematics community continues to take an ambivalent attitude towards Appel and Haken's proof of the four-color conjecture, as Gian-Carlo Rota describes.[31] On the one hand, virtually all mathematicians profess to be satisfied that a long-standing mathematical question has been settled. On the other hand, many mathematicians seem unwilling to accept the computer-assisted proof as definitive: they continue the search for an argument that uncovers deep reasons for the truth of the four-color conjecture and thus renders the computer-assisted proof superfluous.

In this context, two features of computer-assisted proofs are especially relevant: their fallibility, and their unsusceptibility to being grasped in a single act of mental apprehension.

First, some writers claim that computer-assisted proofs must be regarded as fallible. Our confidence in the truth of the theorem relies on the correctness of a computer program and its implementation in a computer. Many computer programs contain errors, and computers occasionally malfunction. Indeed, some minor defects in the computer program used by Appel and Haken have been discovered by later workers, who have proposed modified programs. In consequence, we cannot rule out that a proposition that has supposedly been verified by computer is in fact false. These considerations emphasize that the logical structure of the computer-assisted proof differs from that of the classical proof, and even from that of the long proof. Some mathematicians have prescribed that, in consequence, computer-assisted demonstrations be sharply and explicitly distinguished from rigorous proofs.[32]

Second, the length of a computer-assisted proof such as that of Appel and Haken means that, almost certainly, no rational agent will ever survey it in its entirety. Grasping such a proof in a single act of mental apprehension—an even more demanding task—seems quite impossible.[33] On the strength of this fact, many mathematicians have argued that computer-assisted proofs do not deliver understanding in the sense in which classical proofs do. Some mathematicians have also claimed that, partly in consequence of the foregoing, computer-assisted proofs cannot be beautiful.[34]

Mathematicians are thus faced with a conundrum: how does one regard and respond to computer-assisted proofs? Perhaps some guidance can be drawn from physics, where a similar intellectual challenge has recently been met.

Visualization and Understanding in Physics and Mathematics

The requirement that a mathematical proof be graspable in a single act of mental apprehension resembles the requirement, traditionally put forward by physicists, that a physical theory should provide a visualization of the phenomena that it treats. Both requirements consist of demands for understanding. In the case of physical theories, the demand for understanding translates into a demand for a visual image of the phenomena; in the case of mathematical proofs, it translates into a demand for an overview of a proof's argument. In both cases, the demand for understanding is expressed primarily in terms of perceptual unity. A visualization of a phenomenon offers a unified view of a physical domain, which cannot be gained from a theory's individual empirical predictions. Similarly, grasping a mathematical proof in a single act of mental apprehension yields a global view of the reasons for the truth of a theorem, which is not obtained by studying the proof's individual steps.

In particular historical periods, the requirements of visualization and graspability have helped define the criteria for acceptability of physical theories and mathematical proofs, respectively. We have already encountered the prescription that proofs must satisfy the standards of brevity and simplicity characteristic of classical proofs. We now review the history of the requirement of visualization in physical theorizing.

The development of classical physics is associated strongly with the provision of visualizations.[35] Newtonian physics was initially found wanting in this respect, as the gravitational force that Isaac Newton attributed to matter was much less amenable to visualization than René Descartes's account of planets carried round the sun by vortices of particles. By the nineteenth century, however, physical theories provided extensive and detailed visualizations of phenomena. Lord Kelvin and other British physicists were especially insistent that understanding of the physical world depended on the availability of mechanical models of physical phenomena.

A particularly stringent test of the capacity to provide visualizations was provided by submicroscopic phenomena. Physics at the end of the nineteenth century visualized subatomic particles as miniature versions of macroscopic bodies, such as billiard balls. These particles were attributed many of the properties of everyday objects: they were pictured as being precisely localized, as having a definite mass, velocity, momentum, and kinetic energy, and as moving in continuous trajectories.

In 1900, in the effort to account for new empirical findings, Max Planck introduced the notion of a fundamental unit or quantum of energy in his theory of the spectrum of black-body radiation. This notion was adopted by Albert Einstein in his 1905 theory of the photoelectric effect and by Niels Bohr in his 1913 model of the atom. Although energy quanta had no natural counterpart in macroscopic physics, these theories retained many of the customary visualizations of submicroscopic phenomena. For instance, Bohr's model of the atom continued to visualize electrons as classical particles.

These early quantum theories suffered from various shortcomings: they lacked generality and achieved only limited empirical success. In 1925 Werner Heisenberg gave a more systematic and empirically adequate theory of submicroscopic phenomena based on the notion of the quantum. This theory, named matrix mechanics, restricted itself to relating the magnitudes of observable parameters to one another. Subatomic particles were treated in this theory as abstract entities, whose properties ensured that certain measurements had particular outcomes, but of which no visualization was provided. Matrix mechanics thereby marked a break in the tradition that theories in submicroscopic physics should offer visualizations of phenomena in macroscopic terms.

Although the empirical merits of matrix mechanics soon became clear, the theory was initially not well received. Many physicists found matrix mechanics aesthetically repulsive, partly because of their unfamiliarity with matrices. Some also felt that, because of its abstract form, the theory failed to provide an understanding of submicroscopic phenomena.

The link between understanding and visualization was drawn particularly strongly by Erwin Schrödinger.[36] Guided by this principle, Schrödinger developed in 1927 an alternative quantum theory of subatomic particles, named wave mechanics, that was based on the Schrödinger equation. Although this theory was found to be empirically equivalent to matrix mechanics, it seemed to offer a visualization of submicroscopic phenomena in classical terms. Schrödinger interpreted each solution of the Schrödinger equation as describing a matter wave with a particular frequency and visualized a subatomic particle as a wave packet formed by the superposition of multiple matter waves. On the strength of its visualizing power, as well as of its more familiar mathematical form, wave mechanics was quickly hailed as much more aesthetically attractive than matrix mechanics.

Further work showed that the promise of consistent visualization in classical terms offered by wave mechanics was illusory. Empirical evidence established that subatomic particles have no such properties as a determinate position, velocity, momentum, or energy. While physicists continued to value wave mechanics as a variant of matrix mechanics, with useful mathematical and heuristic properties, most rejected the visualizations proposed by Schrödinger. Instead, they embraced a statistical interpretation of the Schrödinger equation that lent itself to no visualization. The Copenhagen interpretation of quantum theory, which became the dominant view of the physics community in the 1930s, maintained this statistical reading.

The fact that quantum theory demonstrated great empirical success but failed to offer visualizations of physical phenomena posed a dilemma for the physics community. Some physicists continued to reject quantum theory as an acceptable physical theory because of its abstract form, despite its empirical performance. Many of these physicists couched their objections to the theory in aesthetic terms. Others felt that the lack of visualization, however unappealing, was a price worth paying for an empirically adequate theory of submicroscopic phenomena.

By the 1960s the overwhelming empirical success of quantum theory had reshaped the physicists' criteria of theory choice. The demand that physical theories should provide visualizations of phenomena was first relaxed and then abandoned. Physicists even began to express regret that this requirement had ever carried weight in physical theorizing, regarding it as having hindered the development of physics. The demand for understanding, which in classical physics had been tied to the provision of visualizations, was now interpreted more loosely. While some physicists argued that understanding was an inappropriate goal for science and that physical theories should be judged only on their empirical performance, others maintained that quantum theory provided understanding of submicroscopic phenomena to some degree, albeit not in visual terms.[37] Simultaneously, the aesthetic objections to the theory softened, and some physicists even began to grant that quantum theory could be beautiful.

In what respects does the rise of quantum theory in physics resemble the introduction of the computer-assisted proof in mathematics? In each domain, an intellectual construct of a particular form—the visualizing theory in physics and the classical proof in mathematics—had demonstrated, over centuries, great empirical success in solving relevant problems. The community had come to regard these intellectual constructs as yielding understanding and as exhibiting beauty. In the twentieth century, however, the community found itself facing problems that constructs of these forms seemed incapable of solving. In physics, the problems arose from new empirical data concerning submicroscopic phenomena, for which no visualization could be provided; in mathematics, they were constituted by a class of conjectures, including the four-color conjecture, for which no classical proof could be found. In time these problems were solved by constructs of a new form: abstract quantum theories in physics and computer-assisted proofs in mathematics. Whereas the empirical success of these new constructs was soon acknowledged, they were initially regarded as violating established criteria of acceptability, which had been shaped by contributions in the classical style. Many workers doubted that the new constructs provided understanding and, moreover, found them aesthetically unattractive. Gradually, however, the established criteria of acceptability were relaxed and revised in response to the empirical success of the new constructs. At present, this process has advanced further in the case of abstract theories in physics than in the case of the computer-assisted proof in mathematics. Quantum theories now raise few objections among physicists, who have partly redefined both their concept of understanding and their aesthetic criteria for theory choice in the light of their success. If the parallel continues to hold, we can forecast that the computer-assisted proof will remold the concept of understanding and aesthetic criteria in mathematics and will gain an acceptance similar to that of quantum theories in physics.

The Aesthetic Induction in Mathematics

In order to account for the evolution of scientists' criteria of theory choice, including the changes provoked by the rise of quantum theory, I have proposed an explanatory model that goes under the name "aesthetic induction."[38] The aesthetic induction is the procedure by which scientists attribute weightings to aesthetic properties of theories. Scientists at a given time attach aesthetic value to an aesthetic property roughly in proportion to the degree of empirical success scored up to that time by the set of all past theories that exhibit the property. Thus, if a property is exhibited by a set of empirically very successful theories, scientists attach great aesthetic value to it and see theories that exhibit that property as beautiful. If a property has no association with empirical success—either because theories exhibiting that property have been demonstrated inadequate or because such theories have as yet no empirical track record—scientists attach no aesthetic value to it and thus feel no aesthetic attraction for theories that exhibit it. This procedure results in shifts in the ascription of aesthetic value to aesthetic properties, which determine the scientific community's aesthetic preferences among theories.

There is considerable historical evidence that the aesthetic preferences of physicists in theory choice evolve in accord with the aesthetic induction. The reception of quantum theory constitutes a good illustration. From the time of Newton to the end of the nineteenth century, physical theories that offered visualizations of phenomena built up a strong empirical track record. By the end of that period, physicists attributed great aesthetic value to visualization, as the aesthetic induction predicts. Quantum theory provided no visualizations of submicroscopic phenomena. When it was first put forward, in consequence, quantum theory was resisted on aesthetic grounds. As quantum theory demonstrated continued empirical success, its aesthetic properties reshaped the community's aesthetic criteria. Physicists first began to regard the abstractness of quantum theory in a less negative light and subsequently began to attach a degree of positive aesthetic value to it.

In classical physics, visualization was regarded as a prerequisite for understanding. It may therefore be that the aesthetic induction operates also on standards of scientific understanding: in other words, the criteria that determine whether a theory is deemed to provide an understanding of phenomena may evolve in response to the empirical success of theories, in accord with the aesthetic induction.[39] If this is true, a deep link exists between the concept of scientific understanding and conceptions of the beauty of scientific theories.

On the basis of the reception of computer-assisted proofs, I conjecture that the evolution of aesthetic criteria applied to mathematical proofs is also governed by the aesthetic induction. This suggests that mathematicians' aesthetic preferences evolve in response to the perceived practical utility of mathematical constructs—a conclusion that contradicts both the view that mathematicians' aesthetic preferences are innate and the view that they are disinterested with respect to practical utility.[40] A proof's property of being graspable by a single act of mental apprehension is linked, as we have seen, to the provision of understanding. It may thus be that conceptions of understanding evolve in accord with the aesthetic induction in mathematics as well as in physics.

Further evidence that conceptions of mathematical beauty evolve under the influence of the aesthetic induction is provided by the gradual acceptance of new classes of numbers in mathematics, such as negative, irrational, and imaginary numbers. Each of these classes of numbers had to undergo a gradual process of acceptance: whereas initially each new class of numbers was regarded with aesthetic revulsion, in due course—as it demonstrated its empirical applicability in mathematical theorizing—it came to be attributed growing aesthetic merit.[41]

In conclusion, we have uncovered both a pattern in the evolution of conceptions of mathematical beauty and deep similarities between the aesthetic criteria of mathematicians and those of natural scientists. It will be interesting to see whether the hypothesis that conceptions of mathematical beauty evolve in accord with the aesthetic induction is supported by further evidence from the history of mathematics.

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