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Entangling light in high dimensions

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Entangling light in high dimensions

Bart-Jan Pors

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Entangling light in high dimensions

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Wie zich verwondert, geeft zich rekenschap van een wonder.

M.C. Escher



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CHAPTER 1

Introduction

1.1 Quantum entanglement

Quantum entanglement is a most intriguing feature of certain states of composite quantum systems that contain two (or more) distinct objects [1–3]. This fundamental trait of quantum mechanics causes the information about the properties of the objects to be inextricably linked. The entanglement works in such a way that, when a measurement of the properties of one of the objects is performed, the properties of the other object are immediately altered, even when these objects are separated at arbitrary distances.

The consequences of this phenomenon can be so unimaginable that the intuition we derive from the everyday world around us is often deceived. First discussed in the early 1930s, it stirred a vigorous debate between Schrödinger, Bohr, and Einstein about the fundamental properties of nature, and the acceptability of quantum mechanics as a valid description of natural phenomena. Ever since, entanglement has played a central but controversial role in the development of quantum mechanics, as it clearly exposes the divorce between classical and quantum physics. That is, quantum entanglement refutes the classical idea of a fully deterministic world, by violating what became known as *local realism*: the classical postulates that (i) objects carry properties prior to and independent of measurements (realism) and (ii) measurement results obtained at one location are independent of actions at another (spacelike-separated) location (locality) cannot be maintained jointly.

In spite of several attempts to bridge this emerging gap between classical and quantum physics, it was John S. Bell who first realised that, if entanglement were to exist, it should give rise to correlations between outcomes of measurements that are stronger than any classical theory allows for [4]. His seminal work of 1964 provided the framework to investigate quantum entanglement experimentally. Another 15 years later, some 50 years after the original dispute, the existence of quantum entanglement was first demonstrated in the laboratory [5, 6]. Since then, a multitude of experiments have corroborated these findings, using entangled states of light, electrons, atoms, molecules, and Bose condensates.

In recent years, the view on quantum entanglement has become more pragmatic, when it was realised that quantum entanglement could have application in information science. In principle, the strong quantum correlations exhibited by entangled systems enable computation

and communication tasks that are beyond reach of classical methods [7]. This perspective has proved fruitful and has resulted in such ideas as quantum computation [7], quantum metrology [8], quantum imaging [9], and quantum cryptography [10], to name a few. Although technologically still in its infancy, modern research on entangled systems is largely driven by the prospect of such applications, creating a bustling field of research known as quantum information processing.

1.2 High-dimensional entanglement

In this thesis, we will be concerned with entangled states of light. Historically, quantum states of light, *i.e.*, photons, have played a prominent role in the experimental study of quantum entanglement. For instance, the first Bell tests [5, 6] and first realisations of quantum teleportation and quantum cryptographic key distribution were all performed using entangled two-photon states [11]. The vast majority of this work involves the use of the polarisation degrees of freedom of light [11]. Such polarisation-entangled states are appealing from a conceptual point of view, because the two entangled polarisation eigenmodes form an insightful binary system. From an experimental point of view, such states are readily prepared and managed using conventional polarisation optics.

Recently, however, both theoretical and experimental efforts have shifted towards entanglement in *more than two modes*. This generalised form of entanglement is often referred to as high-dimensional entanglement. The motivation for this work stems from the realisation that, with an increasing number of entangled modes, entanglement becomes correspondingly richer. This is, for instance, reflected in stronger violations of various measures of non-locality, which reveal the complexity of high-dimensional Hilbert spaces [12–15]. From an applications perspective, high-dimensional entanglement also has considerable advantage. For example, it carries promise to provide a larger channel capacity [16, 17] and improved security [18, 19] for quantum communication, and simplifies logic-gates architecture [20].

With increasing dimensionality, however, the analysis of entanglement becomes correspondingly more complex, both theoretically and experimentally. For a start, it is not straightforward to define entanglement criteria in high dimensions. For instance, it is not easy to distinguish unambiguously between classical and quantum correlations [12–15], or to quantify entanglement for high-dimensional mixed states [21]. Second, the number of operations needed to determine properties of the state (such as entanglement witnesses or separability criteria) typically increases steeply with the number of entangled modes. Experimentally, this implies that a large number of resources tends to be needed to investigate high-dimensional entanglement.

Given this situation, it is both interesting and relevant to investigate how much one can learn about high-dimensional entanglement from a limited set of measurements and a small number of resources. Our work aims at exploring this issue.

1.3 Orbital-angular-momentum states

There are several ways to implement high-dimensional two-photon entanglement. One can exploit the temporal [22, 23], frequency [24] or spatial [25–28] properties of the optical field.

It is also possible to use combinations of different degrees of freedom, known as hyperentanglement [29–31].

A promising way to implement high-dimensional entanglement involves the use of the orbital-angular-momentum (OAM) degree of freedom of light [28]. Light beams that carry OAM have a helical wavefront that spirals around the optical axis during propagation [32]. The OAM content of the field is related to the winding number m of the helicity, and amounts to $m\hbar$ per photon. Unlike the *spin* angular momentum of light, which is associated with the polarisation and is limited to a value between $-\hbar$ and $\hbar$, the *orbital* angular momentum content of light is unbounded, in the sense that the number of distinguishable OAM modes that have the same propagation axis is, in principle, infinite. The winding number m , which characterises the OAM eigenmodes, can adopt any integer value between $-\infty$ and ∞ . Moreover, all coaxial OAM modes are mutually orthogonal and they thus span a very high-dimensional Hilbert space.

For the work discussed in this thesis, we follow this approach to high-dimensional entanglement. In the following, we briefly discuss how OAM-entangled states can be generated and how they can be analysed.

The simplest and most common way to produce entangled photon pairs is through the process of spontaneous parametric down-conversion. In this nonlinear process, a “blue” pump photon is spontaneously converted into two “red” photons, while conserving energy, momentum and angular momentum. Typically, the generated photon pair is entangled in many degrees of freedom, like polarisation, frequency and transverse momentum. We focus on the latter aspect, or more specifically on the angular transverse momentum, and configure our system such that it produces a very large number of entangled OAM modes [33–35]. The correlations between the two photons of a pair are tuned such that they have opposite OAM. The number of entangled modes is typically of the order of 60.

As mentioned above, full state determination of high-dimensionally entangled states is not straightforward. Various experimental techniques have been devised to sort the modes of an OAM superposition state by their mode number, some of which have been applied to the quantum regime [36–39]. However, to assess or witness entanglement, this generally does not suffice, as one has to measure observables in various unbiased state bases. In the first experiments on OAM entanglement, which probed a 3-dimensional space, this issue was tackled by cleverly manipulating the OAM analysers [40, 41]. This approach was improved on by using either holographic [42] or interferometric means [43], providing the possibility to perform full state tomography. However, scalability of these methods to systems of larger dimensionality than 3 has not been demonstrated, and may not be realistic. An interesting alternative exploits a combination of the polarisation and OAM mode structure of the field [29]. Recently, OAM and its conjugate variable, the transverse angular position, have been shown to exhibit correlations that are a signature of entanglement [44]. Also in the domain of continuous-variable states, quadrature entanglement has been demonstrated between low-order OAM modes [45, 46]. Obviously, there are many approaches possible, and the challenge is to devise an approach that is easily implementable, reliable and yields as much information as possible.

The approach we follow is to investigate high-dimensional OAM entanglement by making use of only *two* state analysers [47]. Pivotal to the approach is the use of state projecting elements, consisting of an optical phase plate that is lens-coupled to a single-mode fiber. These phase plate are diffractive optical elements, the optical thickness of which has an angular vari-

1. INTRODUCTION

ation. They can be used to imprint an azimuthally dependent phase profile onto a beam, and in this way we can control and modify the OAM content. In view of the fact that the photons of a pair have opposite OAM, we employ complementary phase plates in the two beam lines of our down-conversion setup; that is, the two phase plates are each other's conjugate. Furthermore, the phase plates can be rotated around their central axis. In this way, the OAM projection states can be varied, which allows the analysers to probe a Hilbert space of large dimensionality. The exact size of this Hilbert space is determined the complexity of the spatial pattern of the phase plates.

1.4 Dimensionality and information

Already in the early days, Schrödinger made a notable observation on entanglement, when he stated that “...knowledge of the individual systems can decline to the scantiest, even to zero, while that of the combined system remains continually maximum. The best possible knowledge of a whole does *not* include the best possible knowledge of its parts” [2]. In other words, he established a connection between entanglement and the notion of information. The striking trait of quantum entanglement that full information about the composite system is not identical to full information about the subsystems it consists of, is contradictory to anything we know from classical physics. Or, as Schrödinger concludes: “this is what keeps coming back to haunt us”.

The information content of a general state can be quantified in terms of the entropies of the composite system and the individual subsystems. Probably, the most commonly used measure of this sort is the Von Neumann entropy, defined as $S = -\text{Tr} \rho \log \rho$, with ρ a normalised density operator [7]. It is equally valid, however, to characterise the state in terms of the purity of the combined system and the subsystems, which is defined as $P = \text{Tr}(\rho^2)$.^{*} Note that the purity of ρ yields the average of its eigenvalue probabilities, and in general we will have $0 \leq \text{Tr}(\rho^2) \leq 1$.

Let our two-photon state of interest be described by the normalised density operator ρ_{AB} , and its subsystems by $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$. In this thesis, we will predominantly focus our attention on pure states. In this case, we have

$$P = \text{Tr}_{AB}(\rho_{AB}^2) = 1, \quad \text{and} \quad K = \frac{1}{\text{Tr}_A(\rho_A^2)} = \frac{1}{\text{Tr}_B(\rho_B^2)}. \quad (1.1)$$

The fact that $P = 1$ reflects that ρ_{AB} has a single eigenvalue. The number K , formally a participation ratio and known as the Schmidt number, is given by the inverse of the purity of the subsystems. A very interesting property of this quantity is that it is a measure of the number of entangled modes of the system: as the purity of the subsystems represents the average probability of their eigenvalues, K yields the effective dimensionality of the state [49, 50].

The quantities discussed so far, however, provide information about the properties of the *generated* quantum state. In an actual quantum experiment, one analyses the state with a detection apparatus that has properties of itself. In other words, the detection device determines *how much* one can learn about the system. For instance, a limited detection efficiency, as present

^{*}Note that the Von Neumann entropy and the purity are related to each other and emerge as the first and second order realisations of the generalised Tsallis entropies, respectively [48].

in practically all experiments, sets a limit to the accuracy and reliability of the measurement of system properties.

In particular, the detection apparatus may set a limit to the number of modes of the system that one can distinguish, *i.e.*, an experimental outcome on the number of entangled modes may be mostly determined by the detection system, rather than by the source. It is therefore extremely useful to develop a quantifier for the dimensionality *as measured in an experiment* [51, 52]. This is one of the main issues addressed in this thesis and we will introduce what we refer to as the Shannon dimensionality of measured entanglement.

1.5 Orbital-angular-momentum-entangled states as information carriers

Photons being a natural candidate to transmit information, quantum communication is one of the most promising applications of two-photon entanglement. In 1992, Ekert showed how entanglement could add to the security of quantum communication, as compared to single-photon protocols [10]. Since the proof-of-principles demonstrations [11], impressive technological advance has been made. Nowadays, secure communication distances range up to hundreds of kilometres, both for fiber-based [53–55] and free-space [56–58] transmission links, and even a satellite quantum connection may be within reach [59]. Improving on signal bandwidths and the suppression of losses, however, remains a challenging issue. All this work was done using 2-dimensional entanglement.

Recently, it was realised that quantum-communication protocols could benefit significantly from entanglement in dimensions higher than 2. This extension provides the potential of an increased information capacity [16, 17] and an enhanced security against eavesdropping [18, 19]. Some of these principles have already been realised in the laboratory, implementing the high dimensionality by means of energy-time or OAM entanglement [23, 60–62].

This raises the question whether OAM entanglement could be implemented for quantum communication in a real-world setting. Successful implementation requires that the OAM entanglement can be transported over a sufficiently long distance, via a fiber-based or free-space link. The performance of such a quantum channel, however, is an open issue. Several studies have addressed this aspect for the case of free-space distribution, but there is no unanimity on exactly how robust OAM entanglement is [63–71]. On the one hand, it was suggested that the topological charge of light is a resilient quantity, supporting its use as an information carrier in free-space communication. On the other hand, having the information encoded in the wavefront structure of the field, OAM entanglement may be vulnerable to refractive-index fluctuations that arise due to atmospheric turbulence. In the final two chapters of this thesis, we explore this issue experimentally.

1.6 Outline

This thesis is organised as follows. In Chapter 2, we discuss how we create OAM-entangled photon pairs by means of parametric down-conversion, and we experimentally characterise the emission of our source. In Chapter 3, we introduce our angular state analysers, and quantify the dimensionality of the Hilbert space they probe. Chapter 4 unites the preceding two Chap-

1. INTRODUCTION

ters and demonstrates how to extract high-dimensional OAM entanglement from a down-conversion source by means of such analysers. Furthermore, this Chapter introduces the concept of the Shannon dimensionality of measured entanglement, and experimentally we achieve $D = 3$ and 6. In order to realise OAM entanglement of increasing dimensionality, phase plates with a growing amount of complexity are required. Various fabrication methods of such phase plates are discussed in Chapter 5. In Chapter 6, we extend on the ideas from Chapter 4 and demonstrate how to extract genuinely high-dimensional OAM entanglement. In Chapter 7, we go beyond the use of OAM-entangled states, and take a first step towards incorporation of both the angular and the radial degree of freedom. We discuss some of the consequences of this generalisation and present an experimental demonstration. In the final two Chapters, we study whether two-photon OAM entanglement could be useful for free-space quantum communication. Chapter 8 presents an overview of the theory of optical propagation through the atmosphere, and describes the experimental techniques we have developed to generate realistic atmospheric turbulence in a laboratory environment. In Chapter 9, we present the results of an experimental study of the robustness of OAM entanglement when subject to atmospheric turbulence.

Characterisation of a spontaneous parametric down-conversion source for spatially-entangled photon pairs

2.1 Introduction

Spontaneous parametric down-conversion (SPDC), in the older literature frequently referred to as parametric fluorescence or parametric scattering [72–74], is a second-order nonlinear optical process in which a high-frequency photon spontaneously splits in two lower-frequency photons, such that energy is conserved. It is thereby the inverse process of the more widely known up-conversion processes of second-harmonic generation (SHG) and sum-frequency generation (SFG), where two low-frequency beams are nonlinearly mixed to produce one high-frequency component [75, 76].

The process of SPDC has found common application in various fields of research. In the past, it has been used as a tool to measure the second-order nonlinear optical susceptibility tensor for a variety of materials [77–79]. An advantage of this method over more common methods based on SHG and SFG [80], lies in the fact that in SPDC the conversion efficiency is independent of the pump power, hence obviating the need for detailed knowledge of the pump beam’s characteristics. SPDC is also the initiating process in optical parametric oscillators, as it generates the seed photons from which the coherent output builds up via parametric amplification [74, 81]. Owing to their large frequency tuning range, parametric oscillators are popular sources of tunable, narrowband light in a variety of fields, ranging from coherent anti-Stokes Raman spectroscopy [82] to continuous-variable quantum optics [83]. In recent years, SPDC has made its mark on the fields of quantum optics and quantum information, as a source of quantum states of light. In particular, it has become a standard instrument for the production of entangled photons [84–86], paving the way for such applications as quantum cryptography, quantum teleportation, quantum imaging, etc. [9, 11]. To date, SPDC remains an unparalleled source of entangled photon pairs in terms of brightness, reliability, and universality.

For the experimental work presented in this thesis, we exploit the process of SPDC for the creation of orbital-angular-momentum (OAM) entangled photon pairs. The efficiency of the

2. CHARACTERISATION OF A SPONTANEOUS PARAMETRIC DOWN-CONVERSION SOURCE FOR SPATIALLY-ENTANGLED PHOTON PAIRS

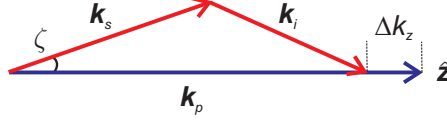


Figure 2.1: **Representation of the pump, signal, and idler wave vectors in SPDC.** The down-converted signal and idler wave vectors $\mathbf{k}_s$ and $\mathbf{k}_i$ point away from the pump wave vector $\mathbf{k}_p$ in such a way that transverse momentum is conserved. The longitudinal mismatch is denoted as Δk_z .

conversion process depends, amongst others, strongly on the wave-vector mismatch $\Delta \mathbf{k}$ between the pump and down-converted photons travelling through the nonlinear medium. This sensitivity on the mismatch sets limits to the spatial distribution in which the down-converted light is emitted. Consequently, when studying spatial (quantum) correlations between the photons in a pair, it is important to have a full understanding of the spatial structure of the SPDC source, and the dependence thereof on phase matching.

In the present Chapter, we present the results of an experimental characterisation of our SPDC source. We study the spatial structure of the down-converted light by imaging the far field of the light emitted from the nonlinear crystal and investigate the dependence of the emission on phase matching. The experimental data are used to estimate the number of entangled modes emitted by our source, characterised by the Schmidt number [50, 87]. This number serves as a benchmark for the work in the ensuing Chapters of this thesis, substantiating some useful assumptions we make regarding the spatial structure of the entanglement.

2.2 Phase matching

The process of SPDC does not exchange energy with the nonlinear crystal, and, consequently, the energy of a pump photon $\hbar\omega_p$ reappears as the sum of the energies of the two generated photons:

$$\omega_p = \omega_s + \omega_i. \quad (2.1)$$

Here, ω_s and ω_i refer to the frequencies of the generated photons, traditionally named signal (*s*) for the high-frequency photon and idler (*i*) for the low-frequency partner.

In the limit that the transverse cross section of the nonlinear crystal is much larger than that of the pump beam, the setup is invariant to translations in the plane of the crystal. Consequently, the transverse component of the wave vector has to be conserved. This implies that the signal and idler photons in a pair have equal but opposite transverse wave vectors. Because of the finite length of the crystal, the longitudinal component of the wave vector does not need to be conserved; the ensuing wave-vector mismatch between the pump, idler, and signal waves is given by (see Fig. 2.1)

$$\Delta \mathbf{k} = \Delta k_z = k_{p,z} - k_{s,z} - k_{i,z}. \quad (2.2)$$

Here, $k_j = 2\pi n_j / \lambda_j$, with n_j the (wavelength-dependent) refractive index of the nonlinear medium and λ_j the wavelength of the light.

The efficiency of any nonlinear optical process depends strongly on this wave-vector mismatch. The emission is brightest if the various fields are coherent over the full length of the

crystal. For that reason, nonlinear optical processes are preferably operated under phase-matched conditions, *viz.*, $\Delta \mathbf{k} \simeq 0$.

Dispersion in the nonlinear material between pump, signal, and idler waves hereby plays an inhibiting role and should be eliminated. This can be achieved with the use of birefringent nonlinear crystals. In such crystals perfect phase matching can be attained by appropriately orienting the crystal axes, and the wave vectors and polarization vectors of the input fields. However, it is often the case that the strongest nonlinearities cannot be addressed in this manner. Consequently, considerable effort has been devoted to develop materials that combine perfect phase matching with a strong nonlinear response, resulting in a multitude of quasi-phase-matched and periodically-poled materials [88, 89].

In up-conversion processes such as SHG and SFG, the concept of phase matching has a clear experimental significance: it simply determines the net power of the up-converted beam. This is due to the fact that the two fixed input waves impose quite a stringent condition on the frequency and directionality of the generated harmonic. When perfect phase matching is not met, the emitted radiation is much weaker and modulates rapidly as a function of the phase mismatch. This behaviour was first reported by Maker *et al.*, who investigated SHG from a crystalline quartz sample [90]. Although quartz does not allow for perfect phase matching, they observed a weak second-harmonic signal that varied in power with the orientation of the crystal, *i.e.*, with $\Delta \mathbf{k}$. Nowadays, this modulation goes by the name of “Maker fringes” [91, 92].

In SPDC, the system has more freedom to obey the phase-matching condition, and a change of the crystal orientation does not necessarily lead to an increase or decrease in the radiated power. Rather, it may result in a directional and/or frequency redistribution of the emitted radiation, since Eqs. (2.1) and (2.2) together allow for a certain freedom in the frequencies and directionalities of the generated photons. This flexibility lies at the heart of the wide tuning range of parametric oscillators. Nevertheless, power variations equivalent to those encountered in SHG and SFG appear when the output power is measured within a narrow spectral bandwidth and within a well-defined range of wave vectors, that is, if one is selective regarding the frequency-mode and spatial-mode structure of the SPDC light.

Let us study the spatial structure of the SPDC emission in closer detail. In a plane-wave description, the down-converted field emanating from a non-linear crystal of length L is of the general form

$$E \propto \int_0^L e^{i\Delta k_z z} dz \propto \frac{\sin(\frac{1}{2}\Delta k_z L)}{\frac{1}{2}\Delta k_z} L. \quad (2.3)$$

This function is strongly peaked around $\Delta k_z = 0$, corresponding to perfect phase matching.

To find the exact spatial distribution of this phase-matched emission, we should take into account the polarization dependence of the problem. Phase matching in SPDC can usually be accomplished in two configurations, known as Type-I and Type-II phase matching. In the latter case, the polarization of the generated signal and idler waves are mutually orthogonal, causing them to experience different refractive indices inside the birefringent nonlinear crystal. Consequently, signal and idler emerge in different directions, in the form of two cones that can be tuned to intersect. Examples of such Type-II rings have appeared on the cover of various books and magazines [11] in the wake of advances in quantum information processing. By selecting light from the intersections of the rings, this configuration serves as a source of

polarization entangled photon pairs [86]. In our pursuit to create OAM-entangled photons, on the contrary, we have chosen to use Type-I phase matching. In this configuration, signal and idler photons have the same polarization and emerge in a single cone coaxial with the pump beam axis. The aperture of the cone can be widened or shrunk by tuning the phase-matching. Of particular interest is the situation that the cone is contracted to the point that the emission is beam-like. This relieves the need of apertures and the accompanying truncation of spatial modes.

Equation (2.3) shows that the strength of the generated field is proportional to the crystal length L , while the width of the phase-matched cone is inversely proportional to L . Away from perfect phase matching, emission is fully inhibited whenever $\Delta k_z L$ equals a multiple of 2π , but revives weakly in between these nodes. These fringes constitute the SPDC analogy of Maker's observation; they have a spatial character and appear as rings around the perfectly phase-matched emission cone, somewhat reminiscent of an Airy diffraction pattern.

In the following Section, we will present experimental data of the spatial structure of the SPDC emission from a Type-I source.

2.3 Experimental results

In our experiment, we use a BBO crystal (β -barium borate, β -BaB₂O₄) that allows for Type-I phase matching by angle tuning of the crystal. Figure 2.2 shows a diagram of our experimental setup. The crystal of length 1 mm is pumped by a weakly focused (waist $w_0 = 250 \mu\text{m}$) Kr⁺ laser beam of 100 mW power at a wavelength of $\lambda = 413 \text{ nm}$. The pump beam is polarised along the extraordinary crystal axis, whereas the signal and idler beams are polarised along the ordinary axis. The phase mismatch can then be written as

$$\Delta k_z = 2\pi \left[\frac{n_e(\lambda_p, \Theta)}{\lambda_p} - \frac{n_o(\lambda_i)}{\lambda_i} - \frac{n_o(\lambda_s)}{\lambda_s} \right], \quad (2.4)$$

where the subscripts (o, e) denote the ordinary and extraordinary crystal axes, respectively. The extraordinary refractive index n_e depends on the tuning angle Θ between the direction of propagation and the optical axis of the crystal. The effective nonlinearity of BBO cut for Type-I collinear phase matching ($\Theta = 28.3^\circ$) equals $d_{\text{eff}} = 1.9 \times 10^{-12} \text{ m/V}$ [93]. Given the 1 mm crystal length, the total yield of the resulting down-converted light is only of the order of 10^4 photon pairs per second per spatial mode per nm bandwidth. This feeble signal must be filtered from the 100 mW pump beam ($\approx 10^{17}$ photons per second), which is achieved by means of a blue laser mirror that reflects the pump beam. Additionally, an interference filter, centred around 826 nm with a 20 nm width, suppresses residual stray (pump) light and ensures that we detect only degenerate photon pairs with $\lambda_i \approx \lambda_s$. We use an intensified CCD camera (Princeton PI-MAX) that is positioned in the Fourier plane of the crystal to detect the SPDC light. This far-field configuration enables us to characterise the angular emission profile of the degenerate down-converted light.

We will work close to the point where $\Delta k_z = 0$ and, furthermore, close to collinearity where signal and idler point in the same direction as the pump beam. Because $\lambda_i \approx \lambda_s \approx 2\lambda_p$, the phase mismatch can be approximated by

$$\Delta k_z \approx |\mathbf{k}_p| - |\mathbf{k}_i| - |\mathbf{k}_s| + \frac{1}{2}(|\mathbf{k}_i| + |\mathbf{k}_s|)\zeta^2, \quad (2.5)$$

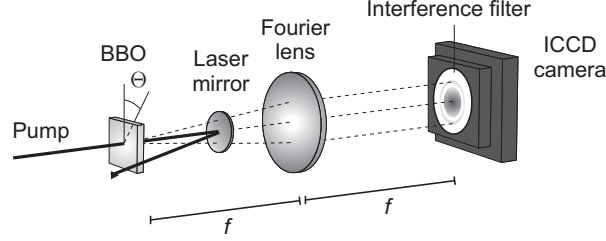


Figure 2.2: **Experimental Setup.** A 100 mW laser beam pumps a 1 mm nonlinear BBO crystal, cut for Type-I phase matching, at 413 nm. The pump beam is removed from the down-converted light by means of a blue laser mirror. A 826 ± 10 nm interference filter selects frequency-degenerate SPDC light ($\lambda_i \simeq \lambda_s$), and the far-field emission profile is recorded by means of an intensified CCD camera.

where ζ is the angle between $\mathbf{k}_p$ and either $\mathbf{k}_i$ or $\mathbf{k}_s$ inside the crystal. Note that in the case of exact degeneracy and collinearity, only the term depending on ζ survives.

We have recorded far-field spatial distributions of the down-converted light for various tuning angles of the crystal. Figures 2.3(a)-(c) give three examples for different phase matching conditions (see below). These intensity profiles are plotted on a logarithmic false-colour scale in order to increase the visual contrast. We observe patterns of concentric rings of diminishing intensity. The bright ring stems from emission that obeys the phase matching condition $\Delta k_z = 0$. As discussed in the previous Section, the emission cone can be opened (see Fig. 2.3(a)) or closed (see Fig. 2.3(b)), depending on the tuning angle of the nonlinear crystal. In Fig. 2.3(c), the phase-matched ring has shrunk to a single spot, corresponding to collinear phase matching. Tilting the crystal even further will take it beyond perfect phase matching, and bright emission will no longer be possible. The fainter rings on the inside and outside of the phase-matched emission ring represent the subsidiary maxima of Eq. (2.3) and manifest a spatial variant of Maker's fringes. As the intensity drops rapidly with increasing wave-vector mismatch, at most two higher-order SPDC rings can be observed, given the background noise level. Although - with a little imagination - a hint of non-phase-matched emission rings may be discerned in observations from Refs. [94, 95], their appearance hasn't been discussed elsewhere. In the footsteps of the work discussed here, a high-quality spatial analysis of SPDC emission was presented in Ref. [96], using a 600 times brighter periodically-poled KTP (Potassium Titanyl Phosphate) source.

Let us compare our experimental results to the field distribution described by Eq. (2.3). Using Eq. (2.5), the intensity profile can be written as

$$I \propto |E|^2 \propto \left[\frac{\sin(\varphi + c\xi^2)}{\varphi + c\xi^2} \right]^2, \quad (2.6)$$

where the variable ξ is the external emission angle in air, given by $\xi^2 = (n\zeta)^2 = \xi_x^2 + \xi_y^2$. The factor $c = (|\mathbf{k}_s| + |\mathbf{k}_i|)L/(2n)^2$ is a constant, equal to $c = 2.29 \times 10^3$ for our experimental parameters (crystal length 1 mm and refractive index $n = 1.66$ for both signal and idler). The parameter $\varphi = (|\mathbf{k}_p| - |\mathbf{k}_s| - |\mathbf{k}_i|)L/2$ determines the opening angle of the emission cone, *i.e.*, it is a measure of the non-collinearity of the process. The radius of the perfectly phase-matched

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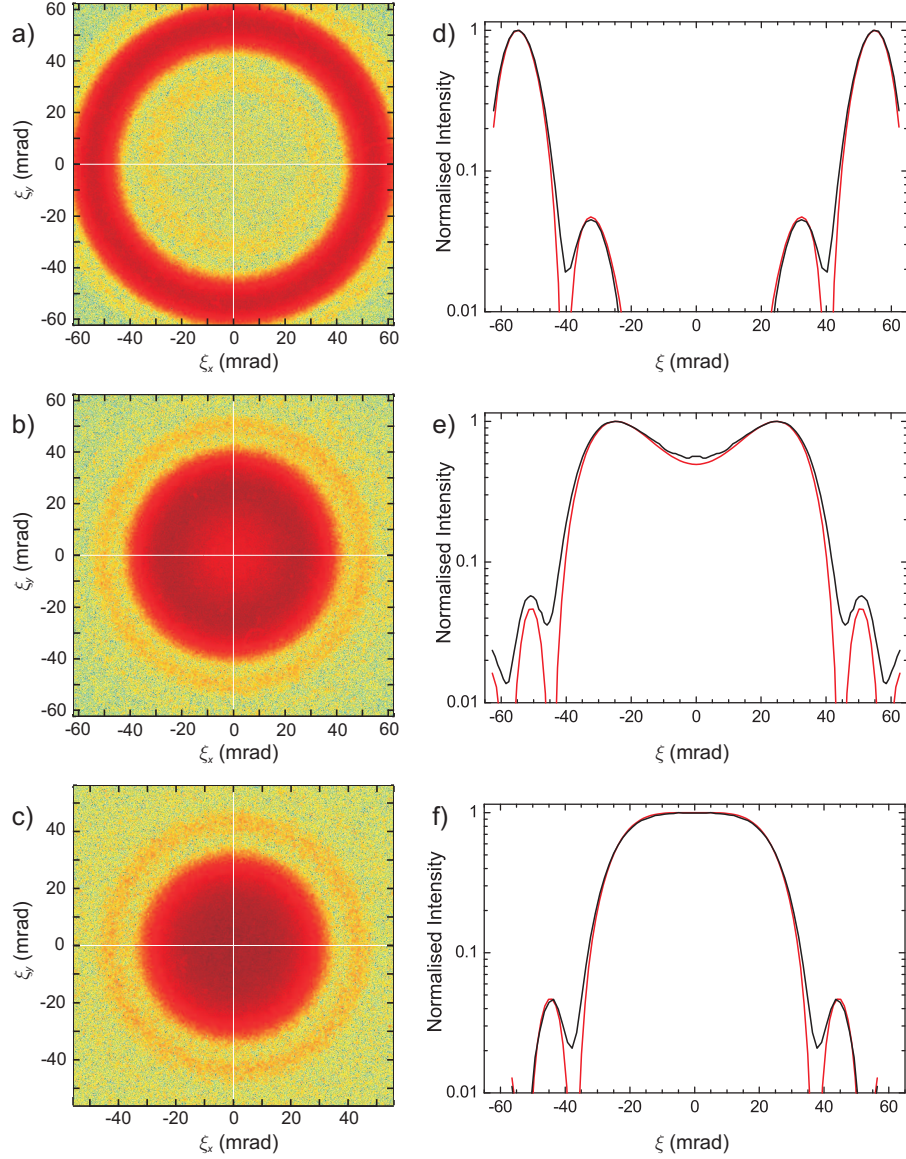


Figure 2.3: **Experimental emission profiles of our Type-I SPDC source.** (a)-(c) Narrow-band far-field spatial profiles around $\lambda = 826$ nm for various values of the phase-matching (see text), plotted on a logarithmic scale. (d)-(f) Their respective cross sections (black curves) were azimuthally averaged and fit to Eq. 2.6 (red curves). Graphs (c) and (f) correspond to collinear alignment.

SPDC ring in the Fourier plane is given by $\xi_R = \sqrt{-\varphi/c}$. The phase-matched cone is open for $\varphi < 0$, with collinearity occurring at $\varphi = 0$. For $\varphi > 0$ perfect phase matching is no longer possible.

Figures 2.3(d)-(f) show the cross sections (solid curves) of Figs. 2.3(a)-(c), respectively. These cross sections, obtained after careful determination of the profile centres, were azimuthally averaged and normalized to maximum intensity. The dashed curves represent the theoretical prescription from Eq. (2.6), where only the “non-collinearity” φ was left as a fitting parameter. From these fits, we found $\varphi = -6.90$ rad and $\varphi = -1.40$ rad for Fig. 2.3(d) and 2.3(e), respectively. For Fig. 2.3(f) we obtained $\varphi = -0.05$ rad, in correspondence with the statement that $\varphi \rightarrow 0$ at collinearity. Similar results were obtained when leaving c as a second fitting parameter.

2.4 Estimate of the number of spatially entangled modes

The intensity measurements in Fig. 2.3 of the parametric fluorescence from the nonlinear crystal are in itself classical. Between the signal and idler photons, however, exist spatial correlations that are essentially of quantum nature. This has been confirmed and exploited in numerous experiments [40, 44, 47, 97], and here, we will adopt a quantum description of the emission without further justification.

As argued in Section 2.2, signal and idler photons in a pair have equal but opposite transverse momenta. Both signal and idler photons are individually in a superposition of spatial modes, in such a way that their composite state is pure and entangled. In view of our work in the coming Chapters, we will write this entangled state in a cylindrical basis, namely [33, 34],

$$|\Psi\rangle = \sum_{m,p} c_{mp} |m, p\rangle_A | -m, p\rangle_B. \quad (2.7)$$

Here, $|m, p\rangle_A$ ($|m, p\rangle_B$) represents a signal (idler) spatial mode containing one photon, with $p \geq 0$ a radial and $m = -\infty, \dots, \infty$ an azimuthal integer mode index. The entangled state of Eq. (2.7) is represented in its diagonalised Schmidt decomposition, meaning that it is written as a sum over biorthogonal product states (and the modes of photons A and B thus carry the same indices m and p) [98]. The complex expansion coefficients c_{mp} obey the normalization requirement $\sum_{m,p} |c_{mp}|^2 = 1$.

In general, it is not a trivial task to find the mode functions $f_{m,p}(\mathbf{r}) = \langle \mathbf{r} | m, p \rangle$ for down-conversion systems as those described in Section 2.2. Approaches to do so can be found in Refs. [33, 87, 99]. Due to the cylindrical symmetry in Type-I phase matching, however, the azimuthal content is readily obtained; the eigenmodes of rotation around the symmetry axis are of the form $e^{im\theta}$, and thus $f_{m,p}(\mathbf{r}) = f_{m,p}(r)e^{im\theta}$, with θ the azimuthal angle. Modes of this form are also eigenmodes of the OAM operator $\hat{L}_z = -i\partial/\partial\theta$ [100], and $|m, p\rangle$ thus represents a spatial mode containing one photon with OAM $m\hbar$. Note that conservation of OAM ($m_A = -m_B$) was already incorporated in Eq. (2.7).

The coming Chapters will predominantly deal with angular entanglement. Hence, we are particularly interested in the azimuthal m -content of the state. Let us therefore inspect the angular modal spectrum of Eq. (2.7), also known as the *spiral bandwidth* [33, 35]. This quantity

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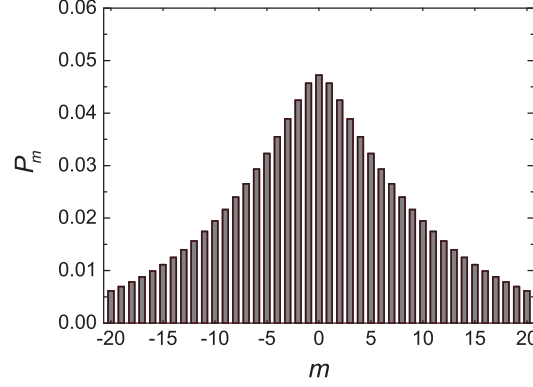


Figure 2.4: **Azimuthal probability distribution of the generated state.** Spiral content $P_m = \sum_p |c_{mp}|^2$ of the Schmidt decomposition (see Eq. (2.7)), calculated for our system. We assumed collinear phase matching and used the experimental parameters as given in Section 2.3.

gives the OAM probability distribution P_m , obtained by summing over the radial part,

$$P_m = \sum_p |c_{mp}|^2. \quad (2.8)$$

Figure 2.4 shows the spiral bandwidth of our source, calculated for collinear phase matching and the experimental values for the pump beam waist w_0 and the crystal length L mentioned in Section 2.3 [35]. We observe a broad spectrum that is symmetric around $m = 0$, which is a consequence of the fact that our Gaussian pump has $m = 0$. The histogram has a considerable width with long tails extending to high m numbers. Obviously, however, the modes in the superposition carry unequal weights.

In principle, the number of entangled modes in Eq. (2.7) can be infinite. However, it is natural to take the relative weight of the modes into account and to define an *effective* number of entangled modes. This is done by the so-called Schmidt number, given by [50, 87]

$$K = \frac{1}{\sum_{m,p} |c_{mp}|^4}, \quad (2.9)$$

For the case that one is only interested in the azimuthal m -content (*i.e.*, the spiral bandwidth) of the state, it can be derived that [34]

$$K_{\text{az}} = \frac{1}{\sum_m P_m^2} \simeq 2\sqrt{K}. \quad (2.10)$$

The Schmidt number is commonly used as a meaningful quantifier of (high-dimensional) entanglement [87].

In principle, if one manages to construct the Schmidt basis $\{|m, p\rangle\}$, the Schmidt decomposition as given in Eq. (2.7) allows to calculate K theoretically, provided one computes enough terms in the expansion. However, in many cases this is not so easy. Moreover, this procedure does not provide a clear way to determine K experimentally. To this end, Refs. [87] and [101]

2.4. ESTIMATE OF THE NUMBER OF SPATIALLY ENTANGLED MODES

presented two independent methods to measure the Schmidt number experimentally. In the following, we use these methods to estimate the Schmidt number for our experimental configuration.

In Ref. [87], Law and Eberly provided an approximation in terms of the ratio of the far-field beam widths σ_{pump} and σ_{SPDC} of the pump and SPDC emission, respectively,

$$K \simeq \frac{1}{4} \left(\frac{\sigma_{pump}}{\sigma_{SPDC}} + \frac{\sigma_{SPDC}}{\sigma_{pump}} \right)^2, \quad (2.11)$$

where $\sigma_{pump} = 2/w_0$ and $\sigma_{SPDC} = \sqrt{4k_p/L}$. Implicit to this result are the assumption of collinear phase matching and the approximation of Eq. (2.6) by a Gaussian of width σ_{SPDC} . Equation (2.11) constitutes in fact a lower limit to K [87]. Filling in the relevant experimental numbers for the pump beam waist, wave number, and crystal length, we arrive at $K = 395$ (and $K_{az} = 40$). Evidently, the widths σ_{pump} and σ_{SPDC} can also be estimated experimentally. In Fig. 2.5 we reproduced the far-field intensity profile of the source for nearly collinear phase-matching. The inset shows the Gaussian profile of the pump beam in the same plane.* The scale of the inset is magnified by a factor of five compared to the main graph. We observe that the pump beam is much more compact than the generated SPDC light. We made a rough estimate of the angular widths by simply taking the full width at half maximum and arrived at $\sigma_{SPDC} = 48$ mrad and $\sigma_{pump} = 1.2$ mrad. Equation (2.11) thus yields $K = 385$ ($K_{az} = 39$), in good agreement with the calculation presented above. As stated earlier, this result based on Eq. (2.11) constitutes a lower boundary to the Schmidt number.

Recently, a more accurate method to measure the Schmidt number was demonstrated in Ref. [101]. Using concepts from classical coherence theory, this method takes into account the detailed spatial structure of the two-photon field. It yields the Schmidt number in terms of the spatial degree of coherence of field,

$$K \simeq \frac{1}{\lambda^2} \frac{[\int I_{NF}(\mathbf{r})d\mathbf{r}]^2}{\int I_{NF}^2(\mathbf{r})d\mathbf{r}} \frac{[\int I_{FF}(\boldsymbol{\xi})d\boldsymbol{\xi}]^2}{\int I_{FF}^2(\boldsymbol{\xi})d\boldsymbol{\xi}} \quad (2.12)$$

where $I_{NF}(\mathbf{r})$ and $I_{FF}(\boldsymbol{\xi})$ are the spatial intensity profiles of the down-converted light measured in the near field and far field of the nonlinear crystal, respectively. In the near field, the SPDC profile simply adopts the Gaussian profile of the pump beam (see inset Fig. 2.5). In the far field, the profile shows the concentric structure as presented in Fig. 2.5. Based on the theoretical descriptions for the near field and far field (see Eq. (2.6)), we calculate the Schmidt number to be $K = 930$. Using the experimental data from Fig. 2.5, we obtain an experimental value of $K = 850$, probably being a slight underestimate due to the noise floor of the measurement.

In summary, the calculated Schmidt number based on Eq. (2.12) is confirmed by our experiment. Moreover, this result is consistent with the lower bound according to Eq. (2.11).

*

For experimental convenience, in reality we did not use the bright pump beam itself for this purpose, but an auxiliary beam that was mode matched to it to a very high degree.

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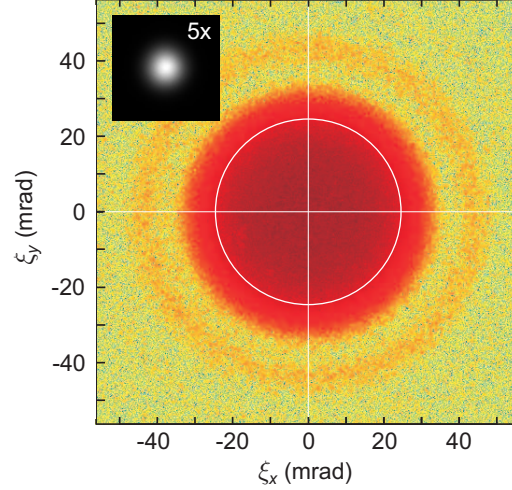


Figure 2.5: **Experimental determination of the Schmidt number.** Narrow-band far-field intensity profile of the SPDC light for collinear phase matching, reprinted from Fig. 2.3(c). The circle indicates the full-width-at-half-maximum intensity level of the central spot. Inset: size of the Gaussian pump beam in the same plane, plotted with five times magnification for comparison. A rough estimate of the Schmidt number is obtained by taking the ratios of the two beam sizes (see Eq. (2.11)), yielding $K = 385$. A more careful analysis based on Eq. (2.12) yields $K = 850$.

2.5 Conclusions

We have characterised the parametric fluorescence from a Type-I SPDC source, which we will employ in the coming Chapters to create OAM-entangled photon pairs. The emission profiles were spatially analysed, and the influence of phase matching on the emission cone aperture was studied. Apart from dominant phase-matched emission, we observed secondary non-phase-matched emission rings, which are a spatial analogy of the well-known Maker fringes exhibited in second-harmonic generation and sum-frequency generation.

Furthermore, we computed the spiral bandwidth of the OAM-entangled modes and estimated the Schmidt number of the generated two-photon field to be 850. The corresponding azimuthal Schmidt number is approximately equal to 60, a number that is sufficiently large for the purpose of our experiments.

CHAPTER 3

Angular phase-plate analysers for measuring the dimensionality of multi-mode fields

Analysers comprised of an angular phase plate and a single-mode fiber have recently been introduced to study the angular profile of optical fields. Here, we quantify the number of degrees of freedom, or modes, that such an analyser can resolve. Its performance is described by means of an angular coherence function and we introduce a novel dimensionality that gives the *effective* number of modes that a given analyser can probe. This quantity can, as we show experimentally, easily be retrieved from a dual analyser setup.

J.B. Pors, A. Aiello, S.S.R. Oemrawsingh, M.P. van Exter, E.R. Eliel, J.P. Woerdman, *Physical Review A* 77, 033845 (2008)

3.1 Introduction

During the last fifteen years, impressive advance has been made on wavefront control of optical fields. A striking example of this progress is found in the technique of adaptive optics imaging [102], where a spatial light modulator [103] or micro-mirror array [104] performs dynamic wavefront corrections on an impinging field. Currently, several devices, known as diffractive optical elements, are available to manipulate or analyse the azimuthal phase profile of a beam. Among these are angular phase plates [105–107] and amplitude holograms [38, 108, 109] or phase holograms [110, 111]. An angular phase plate is a transmissive (or reflective) plate whose optical thickness has a purely angular variation, hence imprinting into a field an azimuthally dependent phase retardation. When the angular variation of the optical thickness is superimposed with a spatial carrier frequency, we deal with a phase hologram.

In recent years, the azimuthal phase dependence of optical fields has drawn much attention, both from a fundamental and applied perspective. It was realised that the azimuthal phase profile of a paraxial electromagnetic field can be identified with the orbital angular momentum carried by that field ($m\hbar$ per photon, with m a discrete index) [32, 37]. Nowadays, orbital-angular-momentum states find their application in optical tweezers [112, 113], in cold-atom physics [114], and in the manipulation of Bose-Einstein condensates [115, 116], where they are utilised to rotate samples.

Orbital-angular-momentum states, of which there are infinitely many, were also addressed in twin-photon experiments [40, 47], motivated by the advantages that quantum entanglement in a high-dimensional mode space might provide for quantum-information science [12]. The experiments employed similar field analysers composed of a diffractive optical element, a focussing lens and a single-mode fiber that is coupled to a photodetector. The important aspect introduced in Ref. [47] was to rotate the diffractive element around the propagation axis of the field. In the current article, we will investigate this class of field analysers, in particular regarding their capability to measure the dimensionality of an incident field by rotating the diffractive element.

As mentioned above, the angular phase operation performed on the field can be realised with either an amplitude or phase hologram [40] or an angular phase plate [47]. Although these devices are in many respects very similar, the use of a phase hologram in a field analyser as described above has a drawback because of the beam deflection that is inherent to its operation; when the hologram is to be rotated, it would imply that the fiber must be translated, which greatly complicates a practical implementation. In contrast, phase plates are purely zero-order devices and hence do not suffer from this disadvantage. We will therefore, without loss of generality, assume that the diffractive phase object be a phase plate.

Thus, the field analyser we are concerned with comprises, successively, a rotatable angular phase plate, a focussing lens, a single-mode fiber, and a photodetector. A crucial property of the analyser is that it performs a single-mode detection for any orientation of the phase plate. This selection goes under the name of ‘spatial filtering’ or ‘projective measurement’ in classical and quantum optics, respectively, and is enforced by the single-mode fiber, which exclusively sustains a Gaussian mode carrying no orbital angular momentum ($m = 0$). Note that the analyser can be applied to both classical and quantum fields, if intensity measurements are performed with a photodiode or single-photon detector, respectively. For the current argument, we will simply speak of a photodetector.

The detection state of the analyser as a whole is given by the fiber's Gaussian mode combined with the angular phase plate's operation. This detection state can be expanded in the orbital-angular-momentum eigenmodes of the field so as to reveal its modal content, with expansion coefficients carrying both amplitude and phase. The amplitudes of these complex coefficients are fixed by the physical profile of the phase object; they are 'engraved' in the plate. The phase components, in contrast, depend on the orientation angle of the device. The analyser's detection state can be readily customised by designing the appropriate phase plate. For instance, pure orbital-angular-momentum states (integer m) can be selected using so-called spiral phase plates of integer order [105]. This kind of plate acts as a pure ladder operator in orbital-angular-momentum space and increases (or decreases) the orbital angular momentum of the field by an integer multiple of $\hbar$ per photon. Field analysers equipped with these plates constitute a special class; their expansion in field eigenmodes contains merely one term, and their operation is therefore invariant under rotation of the plate. It was in fact this kind of transformation that was exploited in Ref. [40] (be it by using a fork-shaped phase hologram, rather than a spiral phase plate). The detection state of a general analyser, however, is typically a superposition of numerous, if not infinitely many orbital-angular-momentum eigenmodes (as, for example, for the *non*-integer spiral phase plate used in Ref. [47]). In that case, the phases of the various modes will evolve each in their individual way when the plate's orientation angle is varied. As a consequence, the detection state alters as the phase plate is rotated and the analyser thus scans a potentially high-dimensional mode space.

In this article, we aim to gain a deeper understanding of this behaviour. In particular we address the question how to quantify the number of spatial modes, or dimensionality, that such a field analyser can resolve. In order to do so, we first represent the single-mode analyser by a mutual coherence function and derive an expansion in orbital-angular-momentum eigenmodes. We then discuss the commonly used *fidelity* dimension D_{fid} , which counts the *total* number of modes that can be observed by this type of field analysers [117]. Subsequently, we introduce a novel measure, D_{eff} , that gives the *effective* number of angular degrees of freedom that can be resolved. It can be interpreted as the number of information channels available, be it in a non-trivial way. The effective dimensionality is, unlike the fidelity dimension, independent of experimental conditions. We show that this number can straightforwardly be retrieved from a dual analyser setup and we present experimental data that confirm this.

3.2 The Heaviside step phase plate

To illustrate our general theory, we will apply our findings at several moments in this paper to a *Heaviside-step-phase-plate* analyser [118, 119]. We therefore first introduce this specific angular phase-plate analyser.

A Heaviside step phase plate is a transmissive (or reflective) plate having an arc sector whose optical thickness is half a wavelength greater than the remainder of the plate (see Fig. 3.1(a)). The part of the field that crosses this arc sector thus flips sign. The length of the arc section producing the π phase shift is given by the parameter Θ . The plate's transmission function can simply be written as:

$$t(\theta, \alpha) = 1 - 2[H(\theta - \alpha) - H(\theta - \alpha - \Theta)]. \quad (3.1)$$

3. ANGULAR PHASE-PLATE ANALYSERS FOR MEASURING THE DIMENSIONALITY OF MULTI-MODE FIELDS

Here $H(x)$ is the Heaviside step function, θ is the azimuthal coordinate and α is the orientation angle of the phase plate. The angles θ and α are both measured from the positive direction of a reference axis and are periodic in 2π . A special case is given by $\Theta = \pi$, in which case the plate consists of two equal halves of phase difference π . The corresponding phase operation connected to such a plate is the well-known Hilbert transformation [120].

Assembling an angular phase plate, a coupling lens, a single-mode fiber, and a photodetector leads to our field analyser. The phase plate and single-mode fiber are placed in their mutual far field, at a focal distance f on either side of the incoupling lens. An illustration of an analyser equipped with a Heaviside plate of arc sector $\Theta = \pi$ is shown in Fig. 3.1(b).

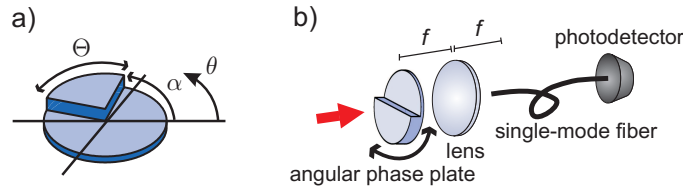


Figure 3.1: **Angular-phase-plate analyser.** (a) Heaviside step phase plate with arc sector Θ producing a phase shift π with respect to the remainder of the plate. The plate orientation is denoted by α , and θ is the azimuthal coordinate. (b) Angular phase-plate analyser with Heaviside step phase plate having $\Theta = \pi$. The impinging field diffracts from the angular phase plate and is coupled to a single-mode fiber by a lens of focal length f . The phase plate can be rotated.

3.3 Detection-state expansion in orbital-angular-momentum eigenmodes

We consider a monochromatic paraxial field of wavelength $\lambda = 2\pi/k$, propagating along the z -axis of an optical system. It can be written in the form:

$$\psi(r, \theta, z, t) = V(r, \theta) \exp[i(kz - \omega t)], \quad (3.2)$$

where $V(r, \theta)$ is the complex amplitude of the field, and (r, θ, z) are cylindrical coordinates defined with respect to the z -axis of the system. We aim to analyse the azimuthal dependence of $V(r, \theta)$ with an field analyser of the kind described above.

The phase plate performs a purely *angular* phase operation on the field that is unitary and is represented by a transmission function $t(\theta, \alpha) = \exp[i\phi(\theta, \alpha)]$, where $\phi(\theta, \alpha)$ describes the azimuthal phase dependence and α is the orientation angle of the plate. We note that *radial* degrees of freedom may in principle be incorporated by allowing for an overall radial dependence that is decoupled from the angular part, yet this is beyond the scope of the current paper. The fiber is placed in the Fourier plane of the phase plate, where the orbital-angular-momentum field components are radially separated. It exclusively supports a single mode, which we approximate by a Gaussian profile $V_0(r)/\sqrt{2\pi}$, where $V_0(r) = (2/w_0) \exp(-r^2/w_0^2)$, and w_0 is the beam waist. The fiber filters this mode, which depends on the radial coordinate only and thus corresponds to the $m = 0$ orbital-angular-momentum component, from an impinging field.

3.3. DETECTION-STATE EXPANSION IN ORBITAL-ANGULAR-MOMENTUM EIGENMODES

We are free to consider the product of the fiber mode and the phase plate's transformation as our detection state. We define the *detection dual field* as

$$U(r, \theta, \alpha) = V_0(r) \frac{1}{\sqrt{2\pi}} t(\theta, \alpha), \quad (3.3)$$

which is the detection state of the composite measurement device. The dual field has a straightforward physical meaning: it is the field emerging from the phase plate when the single-mode fiber is fed in the backward direction (*i.e.*, from the photodetector side) with the fundamental Gaussian. This important property will be exploited later to build an experimental setup for measuring D_{eff} .

The strength of the coupling, quantified by $P(\alpha)$, between the analyser and an impinging field is given by the mode-overlap integral

$$P(\alpha) = \left| \int V^*(r, \theta) U(r, \theta, \alpha) r dr d\theta \right|^2. \quad (3.4)$$

The power measured by the detector can thus be calculated as the overlap integral between the input field $V(r, \theta)$ and the detection dual field $U(r, \theta, \alpha)$. Formulated alternatively, the input field is projected onto the detection state.

Due to the fact that the analyser selects one particular radial mode, that does not depend on the orientation α of the plate, it is justified to restrict our attention to the angular content of the detection state. We therefore define the normalised *angular* detection dual field as $A(\theta, \alpha) = t(\theta, \alpha)/\sqrt{2\pi}$.

An important property of this field is its rotational symmetry,

$$\hat{R}(\alpha) A(\theta, 0) = A(\theta, \alpha) = A(\theta - \alpha, 0), \quad (3.5)$$

where $\hat{R}(\alpha) = \exp(i\alpha \hat{L}_z)$ is the rotation operator representing a counterclockwise rotation about z by an angle α , and $\hat{L}_z = -i\partial/\partial\theta$ is the orbital-angular-momentum operator [100].

Now, let us assume that for a given input field we perform intensity measurements $P(\alpha)$ for several angular settings α of the phase plate. To each plate setting $\alpha = \alpha_i$ corresponds a dual field $A(\theta, \alpha_i)$, and to a whole set of orientations $\{\alpha_1, \alpha_2, \dots\}$ corresponds a set of detection dual fields $\{A(\theta, \alpha_1), A(\theta, \alpha_2), \dots\}$. That is to say that, as α is varied, an ensemble of different realisations of the field $A(\theta, \alpha)$ is constructed. It is customary in optics to describe ensembles by means of their mutual coherence function [121]. Along similar lines, we introduce an angular coherence function [63]: $\gamma(\theta_1, \theta_2) = \langle A(\theta_1, \alpha) A^*(\theta_2, \alpha) \rangle_\alpha$, where the brackets $\langle \dots \rangle_\alpha$ denote averaging with respect to the angle α . Since α is a continuous parameter, we can write this as

$$\gamma(\theta_1, \theta_2) = \frac{1}{2\pi} \int_0^{2\pi} A(\theta_1, \alpha) A^*(\theta_2, \alpha) d\alpha, \quad (3.6)$$

normalised to $\int_0^{2\pi} \gamma(\theta, \theta) d\theta = 1$. This is the first main result of this article: It furnishes an explicit and simple recipe to represent a given analyser by a partially coherent field described by a angular coherence function $\gamma(\theta_1, \theta_2)$.

Next, we apply the methods of image-analysis theory [122] to determine the participating degrees of freedom of such a field. These methods are based on the fact that the mutual

coherence function is a Hilbert-Schmidt kernel, Hermitean and positive semidefinite, which follows from its definition and its rotational symmetry (see Eq. (3.6)) [121]. Then, a modal decomposition is always possible and $\gamma(\theta_1, \theta_2)$ may be expressed as

$$\gamma(\theta_1, \theta_2) = \sum_m \gamma_m u_m(\theta_1) u_m^*(\theta_2). \quad (3.7)$$

The functions $u_m(\theta)$ are the eigenfunctions and the coefficients $\gamma_m \geq 0$ are the eigenvalues of the homogeneous Fredholm integral equation $\int_0^{2\pi} \gamma(\theta, \theta') u_m(\theta') d\theta' = \gamma_m u_m(\theta)$. The modal decomposition is particularly simple thanks to the cylindrical symmetry of the functions $A(\theta, \alpha)$. In fact, the field modes are just the orbital-angular-momentum eigenfunctions of $\hat{L}_z$:

$$u_m(\theta) = \frac{1}{\sqrt{2\pi}} \exp(im\theta), \quad (3.8)$$

with $m = 0, \pm 1, \pm 2, \dots, \pm\infty$. The eigenvalues γ_m are given by the modulus square of the Fourier coefficients of $A(\theta, 0)$,

$$\gamma_m = \frac{1}{2\pi} \left| \int_0^{2\pi} A(\theta, 0) e^{-im\theta} d\theta \right|^2. \quad (3.9)$$

The eigenvalues γ_m give the coupling strength, or sensitivity of the analyser to the field mode $u_m(\theta)$. The set complies the natural normalisation condition

$$\sum_m \gamma_m = 1. \quad (3.10)$$

For the example of an analyser equipped with a Heaviside step phase plate, we find

$$\gamma_m = \begin{cases} (1 - \Theta/\pi)^2, & m = 0, \\ \frac{4}{m^2\pi^2} \sin^2(m\Theta/2), & m \neq 0. \end{cases} \quad (3.11)$$

In Fig. 3.2 we show the spectrum of eigenvalues for $\Theta = \pi$. This distribution contains ample information about the expected performance of the field analyser. For example, if the input field has no angular dependence, it will not couple at all with this analyser, since $\gamma_0 = 0$. Secondly, as the Heaviside plate has an anti-symmetric profile on the domain $0 < \theta < 2\pi$, all even- m terms vanish.

3.4 Dimensionality

In an actual experimental setting, every field mode $u_m(\theta)$ is subject to a certain amount of noise. A mode $u_m(\theta)$ can thus only be detected if the analyser's coupling efficiency to that mode, γ_m , is sufficiently large [123]. Hence, there is a *total* number of detectable modes, that is the number of modes whose detection efficiency exceeds their noise level, which is referred to as the fidelity dimension D_{fid} [117]. One should bear in mind that this measure only has a useful meaning if the field is expanded in the eigenmodes of the system. The fidelity dimension has some merit, as it gives the number of modes needed to describe an analyser's dual field to a noise-limited accuracy [119]. Notwithstanding, it is clear that this dimension is not an absolute measure of the analyser's performance, as the noise level depends on the exact experimental

conditions. For example, the noise could be suppressed by prolonging the measurement time, hence increasing D_{fid} .

Instead, we now introduce an alternative definition that *does* have an absolute meaning. It relies on the fact that, generally, modes do not participate equally. Returning to Fig. 3.2, it is clear that there are two dominant modes ($m = \pm 1$) and two subdominant ones ($m = \pm 3$) (and two subsubdominant ones ($m = \pm 5$), etc). Thus, we expect that the *average* number of detectable modes will be larger than 2, but not much larger. To quantify this number, let us note that if $\gamma(\theta_1, \theta_2)$ had exactly D nonzero eigenvalues *uniformly* distributed, $\gamma_1 = \gamma_2 = \dots = \gamma_D$, the normalisation condition would impose $\gamma_m = 1/D$, ($m = 1, \dots, D$). As a consequence, the Hilbert-Schmidt norm of such a finite-dimensional distribution would just be $\sum_{m=1}^D \gamma_m^2 = 1/D$. Inspired by this result, we define the *average* number of detectable modes, or *effective dimensionality* D_{eff} of the field analyser, as:

$$D_{\text{eff}} = \frac{1}{\sum_{m=-\infty}^{\infty} \gamma_m^2}. \quad (3.12)$$

This dimensionality can be interpreted as the number of channels available for communication purposes [124, 125], although they are not individually accessible. It is worth noting that this definition is actually independent of experimental conditions.

When we apply this formula to the distribution given in Eq. (3.11), we obtain after a straightforward calculation:

$$D_{\text{eff}}(\Theta) = \begin{cases} 1/(1 - 4\Theta/\pi + 6(\Theta/\pi)^2 - (8/3)(\Theta/\pi)^3), & \Theta \in [0, \pi], \\ D_{\text{eff}}(2\pi - \Theta), & \Theta \in [\pi, 2\pi]. \end{cases} \quad (3.13)$$

The effective dimensionality ranges between 1 and 6. In the specific case of the distribution shown in Fig. 3.2, we find $D_{\text{eff}}(\Theta = \pi) = 3$, consistent with our intuitive reasoning.

As a second example, we return to the analyser equipped with a spiral phase plate of integer order n . It selects a pure orbital-angular-momentum state: $\gamma_m = \delta_{nm}$, with δ_{ij} the Kronecker

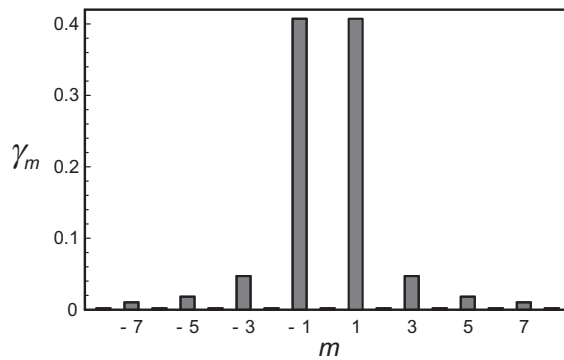


Figure 3.2: **Modal decomposition of the detection dual field for an analyser equipped with a Heaviside step phase plate, with $\Theta = \pi$.** The histogram shows the distribution of the eigenvalues γ_m for the orbital-angular-momentum states $u_m(\theta)$.

delta. The resultant effective dimensionality equals $D_{\text{eff}} = 1$, exposing the inability of this apparatus to probe a multi-dimensional space.

Equation (3.12) is the second main result of this article. It furnishes a simple recipe to calculate the number of modes that an angular phase plate analyser can effectively detect.

3.5 Measuring the effective dimensionality

The effective dimensionality D_{eff} , defined in Eq. (3.12), can actually be measured with the simple experimental setup shown in Fig. 3.3. The setup consists of a mirror-inverted field analyser, oriented at α' , that is imaged by means of a telescope onto a normal field analyser oriented at α . With mirror-inverted field analyser, we mean that the analyser is equipped with an angular phase plate that is a mirrored copy of the normal angular phase plate. More details on the setup can be found in Ref. [119]. With this scheme, we can basically measure the overlap

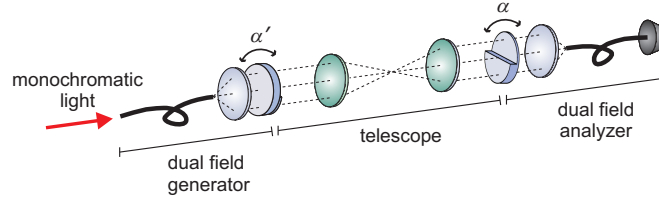


Figure 3.3: **Setup for the determination of the effective dimensionality D_{eff} .** Monochromatic light emerges from a mirror-inverted field analyser oriented at an angle α' , and is coupled into a normal field analyser set at α . The intensity is recorded as α is rotated over 2π . Here, the phase plates have a Heaviside step profile, with $\Theta = \pi$.

between two analyser modes belonging to different phase plate orientations α and α' . From the definition of the detection dual field given previously, it follows that the mirror-inverted analyser generates a dual field $A(\theta, \alpha')$, when fed from the output port of its fiber. This field is imaged onto the second analyser that selects the dual field $A(\theta, \alpha)$ and relays an output signal whose power equals

$$P(\alpha, \alpha') = \left| \int_0^{2\pi} A^*(\theta, \alpha) A(\theta, \alpha') d\theta \right|^2 \equiv |G(\alpha, \alpha')|^2. \quad (3.14)$$

The rotational symmetry (see Eq. (3.5)) yields a direct correspondence between $G(\alpha - \alpha')$ and the analyser's coherence function: $G(\alpha, \alpha') = 2\pi\gamma(-\alpha, -\alpha')$. In fact, it follows that

$$G(\alpha, \alpha') = G(\alpha - \alpha'). \quad (3.15)$$

The coupling strength between the mode generator and mode analyser is, not surprisingly, dependent on the *relative* orientation angle $\alpha - \alpha'$ only. The coherence function $G(\alpha, \alpha')$ is a measure of the *angular sensitivity* of a mode analyser, meaning that it characterises how fast the detection mode changes when the phase plate is rotated.

Figure 3.4 shows intensity measurements obtained with a dual detector setup for our Heaviside step phase plate analyser (with $\Theta = \pi$) [119]. The dots are experimental data and the solid

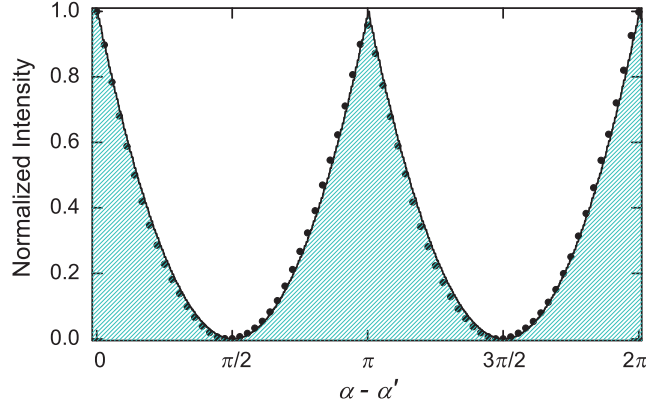


Figure 3.4: **Experimental determination of the effective dimensionality D_{eff} .** Experiment performed with a Heaviside step phase plate analyser with $\Theta = \pi$, by means of the setup depicted in Fig. 3.3. The dots are experimental data (taken from Ref. [119]) and the solid curve the theoretically predicted $|G(\alpha - \alpha')|^2$. The value of D_{eff} is the inverse of the average normalised intensity (equal to 2π divided by the area underneath the normalised curve).

curve gives the theoretical mode overlap $|G(\alpha - \alpha')|^2$. The coherence $G(\alpha - \alpha')$ changes linearly with the difference angle $\alpha - \alpha'$, giving rise to a parabolic intensity curve.

Exploiting the correspondence with $\gamma(\alpha, \alpha')$ and its expansion in orbital-angular-momentum eigenmodes, we integrate $G(\alpha - \alpha')$ over the difference angle $\alpha - \alpha'$ and arrive at:

$$D_{\text{eff}} = \frac{2\pi}{\int_0^{2\pi} |G(\alpha - \alpha')|^2 d(\alpha - \alpha')}. \quad (3.16)$$

That is to say, the effective dimensionality D_{eff} of the detector is just equal to 2π times the inverse of the area below the curve of normalised maximum intensity. This shows the experimental relevance of the newly defined effective dimensionality. Applying this strategy to the case shown in Fig. 3.4, we find indeed $D_{\text{eff}} = 3.0$, in agreement with theory. Equation (3.16) is the third major result of this article.

3.6 Discussion

We have demonstrated that the angular phase plate analysers under consideration show multiple aspects regarding dimensionality: they are (i) single-mode projectors, (ii) able to access high-dimensional spaces and, (iii) characterised by an effective dimensionality D_{eff} . Here, we aim to give an intuitive representation of these features.

The key idea in this section is to represent a detection dual field $A(\theta, \alpha)$ by a complex vector in the linear, infinite-dimensional space $\mathcal{U}$ that is spanned by the orbital-angular-momentum modes $u_m(\theta)$. The detection state vector has components along the ‘axes’ $u_m(\theta)$ that carry both amplitude and phase.

The coupling sensitivity of the field analyser to a mode $u_m(\theta)$, given by an amplitude γ_m ,

is set by the physical shape of the phase plate. Thus, as all γ_m are fixed, the modal content of the dual field is fixed. This reflects the single-mode detection of the analyser.

However, the performance of these analysers is not determined by a single value of $A(\theta, \alpha)$ calculated for a given value of the continuous parameter α , but rather by the *whole set* of fields $\{A(\theta, \alpha)\}_\alpha$ obtained by varying α between 0 and 2π . When the phase plate is rotated, the state vector $A(\theta, \alpha)$ redirects, as the phase factors of each field component $u_m(\theta)$ start to change. As a result, the set of fields $\{A(\theta, \alpha)\}_\alpha$ spans a subspace $U_\alpha \subseteq U$ that occupies some ‘volume’ within U . Our effective dimensionality D_{eff} quantifies this volume, by weighing the eigenmodes $u_m(\theta)$ in the expansion of $A(\theta, \alpha)$ by the square of their coefficients γ_m .

In Fig. 3.5 we sketch the behaviour of $A(\theta, \alpha)$ in a cartoon-like manner. For pictorial convenience, we fix this figure to $\dim(U) = 3$ and we draw $A(\theta, \alpha)$ as a real-valued three-dimensional vector. As the parameter α varies, such a vector draws a continuous curve within U , and after a 2π rotation of α it returns at its initial point. The closed curve makes excursions in all three dimensions and so it spans an overall object. When this curve embodies a ball-like volume, as in Fig. 3.5(a), it implies that all γ_m are approximately equal and the effective dimensionality of this volume is about 3. However, the excursions may not be equally strong along all axes. When the spanned structure spans a plate-like volume, squeezed along a certain direction, as shown in Fig. 3.5(b), its effective dimensionality is about 2. Finally, when the curve covered by $A(\theta, \alpha)$ spans a cigar-like space squeezed along two directions, as depicted in Fig. 3.5(c), then $D_{\text{eff}} \simeq 1$.

3.7 Conclusions

The key results of this article are threefold. First, we have shown that an angular phase-plate analyser can be represented by an angular coherence function (Eq. (3.6)) and we have given its expansion in orbital-angular-momentum eigenmodes. Secondly, we have introduced a novel quantity that gives the effective number of modes that an analyser can access (Eq. (3.12)). Unlike the fidelity dimension, which counts the total number of observable modes in the presence of noise, the effective dimensionality does not depend on experimental conditions. It can be seen as the number of communication channels that an analyser sustains. Lastly, it was shown that the effective dimensionality can easily be obtained experimentally. This important feature is expressed by Eq. (3.16).

Moreover, we have suggested an intuitive picture of how to represent the analyser’s detection state as a vector in an infinite-dimensional mode space. The new insight in the properties of angular phase plate analysers paves the way to design analysers that scan high-dimensional mode spaces, applicable to the analysis of both classical and quantum fields.

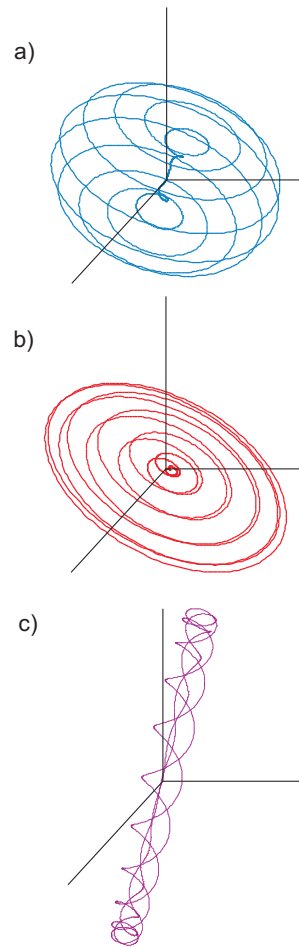


Figure 3.5: **Graphical representation of the effective dimensionality.** Although all three curves, representing different sets of detection dual fields, make excursions along all three axes, only figure (a) spans a 3-dimensional object. The curves in (b) and (c) span objects of dimensionality close to 2 and 1, respectively.

3. ANGULAR PHASE-PLATE ANALYSERS FOR MEASURING THE DIMENSIONALITY OF MULTI-MODE FIELDS

CHAPTER 4

Shannon dimensionality of quantum channels and its application to photon entanglement

We introduce the concept of Shannon dimensionality D as a new way to quantify bipartite entanglement as measured in an experiment. This is applied to orbital-angular-momentum entanglement of two photons, using two state analysers composed of a rotatable angular-sector phase plate that is lens-coupled to a single-mode fiber. We can deduce the value of D directly from the observed two-photon coincidence fringe. In our experiment, D varies between 2 and 6, depending on the experimental conditions. We predict how the Shannon dimensionality evolves when the number of angular sectors imprinted in the phase plate is increased and anticipate that $D \simeq 50$ is experimentally within reach.

J.B. Pors, S.S.R. Oemrawsingh, A. Aiello, M.P. van Exter, E.R. Eliel, G. W. 't Hooft, and J.P. Woerdman, *Physical Review Letters* **101**, 120502 (2008)

(Selected for the Virtual Journal of Quantum Information, October 2008)

4.1 Introduction

Photons can be entangled in various degrees of freedom. The most extensively studied variety involves the polarisation degrees of freedom, of which there are inherently two per photon. In a typical EPR-Bell type experiment, the state analysers are polarisers, and when their relative orientation is scanned, this gives rise to a sinusoidal coincidence fringe [6]. This particular shape is characteristic of the two-dimensional nature of polarisation entanglement.

Recently, much attention has been drawn to bipartite entanglement involving more than two degrees of freedom. With increasing dimensionality, quantum entanglement becomes correspondingly richer. High-dimensional entanglement is predicted to violate locality more strongly and to show more resilience to noise [12, 13]. From an applications perspective, it holds promise for implementing larger alphabets in quantum information, *e.g.*, quantum cryptography [126], and for an increased security against eavesdropping [127]. High-dimensional entanglement can be studied employing the frequency-time [22] or position-momentum degrees of freedom, the latter having been demonstrated for both the transverse linear [25, 26] and orbital-angular-momentum degrees of freedom [40, 47].

It is crucial to have a quantifier of the dimensionality of entanglement as measured in an experiment [51]. In this Letter we introduce such a quantifier, using concepts from classical information theory in the spirit of Shannon [128]. We apply these ideas to orbital-angular-momentum entanglement, inserting appropriate angular state analysers in the beamlines of a parametric down-conversion setup. We have realised a Shannon dimensionality $2 \leq D \leq 6$ and we argue that $D \simeq 50$ is within reach.

4.2 Shannon dimensionality

In classical information theory [128], the number of independent communication channels of a signal is known as the Shannon number. The signal being the state of a physical system, the Shannon number is also referred to as the number of degrees of freedom, or the number of modes, of that system [122, 124]. For example, a signal encoded in the polarisation degrees of freedom of a light beam has a Shannon number equal to 2.

When dealing with a bipartite quantum system in an entangled pure state $|\Psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, the usual measure of the effective dimensionality of the Hilbert space in which the state lives, is given by the Schmidt number K [50]

$$K = \frac{1}{\text{Tr}_A(\rho_A^2)} = \frac{1}{\text{Tr}_B(\rho_B^2)}. \quad (4.1)$$

Here, $\rho_A = \text{Tr}_B(|\Psi\rangle\langle\Psi|)$ and $\rho_B = \text{Tr}_A(|\Psi\rangle\langle\Psi|)$, are the reduced density matrices representing the states of the two sub-systems $A \in \mathcal{H}_A$ and $B \in \mathcal{H}_B$, respectively. Although a system may have infinitely many degrees of freedom, any actual measurement apparatus has effective access only to a finite number of them, say D . Such a dimensionality D is referred to as the Shannon number of the measurement apparatus.

Consider an experiment measuring correlations between the two subsystems A and B . There are two measuring apparatuses, say $\mathcal{P}_A(\alpha)$ and $\mathcal{P}_B(\beta)$, interacting with subsystems A and B , respectively, where α and β label possible settings of the two apparatuses. For a

given setting $\xi \in \{\alpha, \beta\}$, detector $\mathcal{P}_X(\xi)$ is represented by the projection operator $\hat{\Gamma}(\xi) = |X(\xi)\rangle\langle X(\xi)|$, where $X \in \{A, B\}$, and $|X(\xi)\rangle$ is the state in which the system X is left after measurement.

If a Von Neumann-type projective measurement is performed, the set of states $\{|X(\xi)\rangle\}_\xi$ obtained by varying ξ is complete and orthonormal, namely

$$\langle X(\xi)|X(\xi')\rangle = \delta_{\xi\xi'}, \quad \sum_{\xi} \hat{\Gamma}(\xi) = \hat{1}, \quad (4.2)$$

where the measurement operators $\hat{\Gamma}(\xi)$ are Hermitean and idempotent. The number of these operators is equal to the dimension of the Hilbert space of the measured quantum system [129]. However, in many situations non-orthogonal measurements are made and Eqs. (4.2) do not hold [43]. In this case, the number of projection operators $\hat{\Gamma}(\xi)$ does not give the dimension of the Hilbert space of the measured system, and a new criterion must be introduced.

Let us therefore consider finite-dimensional systems, say $\dim(\mathcal{H}_X) = M$, and rewrite Eq. (4.2) for the case of non-orthogonal measurements as

$$\langle X(\xi)|X(\xi')\rangle = g_{\xi\xi'}, \quad \sum_{\xi} \hat{\Gamma}(\xi) = \hat{\gamma}, \quad (4.3)$$

where $G = [g_{\xi\xi'}]$ is a matrix of size $M \times M$, and $\hat{\gamma}$ is an Hermitean operator. The eigenvalues γ_m of $\hat{\gamma}$ give the detector's 'sensitivity' to the corresponding eigenmodes. In general, a detector will not be equally sensitive to all eigenmodes and some γ_m are substantially larger than others. The *effective* dimensionality $D \leq M$ of the Hilbert space $\mathcal{D}$ where the *measured* system lives can be quantified as the Hilbert-Schmidt norm of the eigenvalue distribution

$$D \equiv \frac{1}{\text{Tr}(\hat{\gamma}^2)} = \frac{1}{\sum_m \gamma_m^2}. \quad (4.4)$$

This dimensionality should be interpreted as the effective Shannon number of information channels [122, 128].

The isomorphism of Eq. (4.1) and Eq. (4.4) suggests a relation between the Schmidt number K and the Shannon dimensionality D . The nature of such relation becomes clear if one notes that since the operators $\hat{\Gamma}(\xi)$ are Hermitian and positive semidefinite, the operator $\hat{\gamma}$ may be interpreted as a density matrix acting in $\mathcal{H}_X$ [130]. Thus, if we think of $\hat{\gamma}$ as a reduced density matrix of a bipartite system, then K and D are formally the same. However, it is important to note that while K furnishes the dimensionality of the *generated* entanglement, D gives the effective dimensionality of the space $\mathcal{D}$ that can *potentially* be *probed* and it is a property of the projection apparatus only. The dimensionality of the *measured* entanglement is a joint property of the generated system and analysers, but simply amounts to D as long as $\mathcal{H} \supset \mathcal{D}$.

4.3 Experimental results

Next, we apply our formal theory to an experiment on orbital-angular-momentum entanglement of two photons, in order to illustrate how detector characteristics bound the measured entanglement to an effective Shannon dimensionality D , while probing a generated state with Schmidt number $K \gg D$ (and $\mathcal{H} \supset \mathcal{D}$). Our experimental setup is depicted in Fig. 4.1.

4. SHANNON DIMENSIONALITY OF QUANTUM CHANNELS AND ITS APPLICATION TO PHOTON ENTANGLEMENT

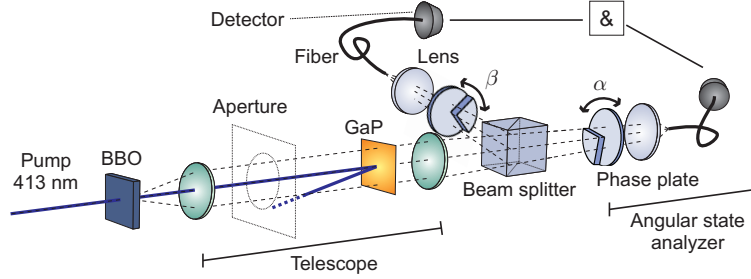


Figure 4.1: **Experimental setup.** Orbital-angular-momentum entangled photons are emitted at 826 nm by a BBO crystal, cut for Type-I collinear phase matching. A thin GaP wafer serves to eliminate the pump beam. The two-photon field can be clipped with an aperture. The twin photons are spatially separated by a beam splitter and imaged on the angular phase plates ($f_2 = 4f_1 = 40$ cm). Just behind the phase plates, the frequency-degenerate photons are selected by interference filters (not shown), centered around 826 nm with a 10 nm width. The phase plates (shown are quarter-sector plates) are oriented at angles α and β , and photon counts are rendered by a coincidence circuit.

Pumping a BBO non-linear crystal with a 150 mW Kr^+ laser beam at $\lambda = 413$ nm, we produce spatially entangled photons by means of spontaneous parametric down conversion. The state we generate is of the form $|\Psi\rangle = \sum_m c_m |m\rangle_A | -m\rangle_B$, where $|m\rangle$ denotes the orbital-angular-momentum eigenmode of order m : $\langle\theta|m\rangle = \exp(im\theta)/\sqrt{2\pi}$, with θ the azimuthal angle [97]. Employing Type-I collinear phase matching, we collect the full emission cone and with the experimental parameters of our setup (beam half-waist at the position of the crystal $w_0 = 250 \mu\text{m}$ and crystal length 1 mm) we obtain an azimuthal Schmidt number $K \simeq 60$ (see Chapter 2). The twin photons are spatially separated by means of a non-polarising beam splitter.

Each arm of the setup contains an angular state analyser, composed of an angular phase plate that is lens-coupled to a single-mode fiber (see Chapter 3) [119]. The angular phase plates carry a purely azimuthal variation of the optical thickness. As in polarisation entanglement [6], the phase plates are rotated around their normals and the photon coincidence rate is recorded as a function of their independent orientations [47].

The combined detection state of the two angular-phase-plate analysers, each acting locally, can be expressed as

$$|A(\alpha)\rangle \otimes |B(\beta)\rangle = \left[\sum_m \lambda_m |m\rangle e^{im\alpha} \right]_A \otimes \left[\sum_m \lambda_m |m\rangle e^{im\beta} \right]_B, \quad (4.5)$$

where α and β denote the orientations of the two phase plates, respectively^{*}. The complex expansion coefficients $\{\lambda_m\}$ are fixed by the physical profile of the angular phase plate and obey the normalisation condition $\sum_m \gamma_m = 1$, where $\gamma_m = |\lambda_m|^2$. In general, the detection state constitutes a non-uniform superposition of orbital-angular-momentum modes. When the angular phase plates are rotated over α or β , respectively, all modes in the superposition rephase with respect to each other, yielding a set of detection states of the type Eq. (4.3). The

^{*}The set of orbital-angular-momentum eigenmodes constitutes the appropriate basis to expand the detection state, for these modes are eigenfunctions of rotation over α , and the set contains the fiber mode $m = 0$.

effective Shannon dimensionality that is so probed is given by Eq. (4.4). It is the average number of modes captured by an analyser when its phase plate is rotated over 360° .

As we have shown in Chapter 3, the Shannon dimensionality is straightforwardly deduced from the shape of the experimental coincidence curve: it is the inverse of the area underneath the peak-normalised coincidence fringe, obtained when rotating one of the phase plates.

In our experiment, we have used angular-sector phase plates; these have a single arc sector, characterised by the angle Θ , whose optical thickness is $\lambda/2$ greater than that of the remainder of the plate [119]. The part of the field that crosses this sector thus flips sign. The phase plates are manufactured from fused-quartz plane-parallel plates, having a wedge angle of $0.25''$. They are processed by a combination of photolithography, wet etching, deposition and lift-off, resulting in a well-defined mesa structure, with a transition region that is typically $20 \mu\text{m}$ wide. The insets of Fig. 4.2 show two such plates; a half-sector plate ($\Theta = \pi$) consisting of two equal halves that are phase shifted by π ; and a quarter-sector plate ($\Theta = \pi/2$) having one quadrant π -phase shifted with respect to the remainder of the plate.

For state analysers that are equipped with such sector phase plates, the Shannon dimensionality is given by (see Eq. (3.13))

$$D(\Theta) = \begin{cases} \left[1 - 4\left(\frac{\Theta}{\pi}\right) + 6\left(\frac{\Theta}{\pi}\right)^2 - \frac{8}{3}\left(\frac{\Theta}{\pi}\right)^3 \right]^{-1}, & \Theta \in [0, \pi], \\ D(2\pi - \Theta), & \Theta \in [\pi, 2\pi]. \end{cases} \quad (4.6)$$

For $\Theta = 0$ we find the trivial result $D = 1$; a planar plate does nothing. For $\Theta = \pi$, *i.e.*, a state analyser equipped with a half-sector plate, we arrive at $D = 3$. For an analyser equipped with a quarter-sector plate we find $D = 6$. This is the maximum value for a single angular-sector phase plate. We note that for our setup indeed $K \gg D$.

In the experiment, we scan one angular-sector phase plate over a 360° rotation, the other remaining fixed, and measure the coincidence rate. In terms of Klyshko's picture of advanced waves [131], valid when $K \gg D$, the resulting shape of the coincidence curves can be explained in terms of the mode overlap of the two state analysers. Figure 4.2(a) shows experimental results obtained with two half-sector plates ($\Theta = \pi$), having a step height of 0.48λ . The data points form a double *parabolic* fringe, consistent with theory (solid curve). The maxima at 0° and 180° are sharply peaked. The zeros of the fringe are very deep; less than 10 counts per 10 s. The maximum coincidence rate is of the order of 6.5×10^3 per 10 s, compared to 10^5 single counts. We verified that the coincidence rate depends on the *relative* orientation between the two phase plates only, the fringe visibility being $>99\%$ for all cases studied. This basis independence is the key aspect of quantum entanglement. From the area underneath the data we deduce the experimental value $D = 3.0$. Note that a parabolic fringe was also reported in Ref. [47], obtained with non-integer spiral phase plates. We conclude that also in that case $D = 3$.

An aperture, positioned inside the telescope, allows us to control the number of detected modes (see Fig. 4.1). Because of the anti-symmetric profile of the half-sector plate, the detection state contains only odd expansion terms (see Eq. (4.5)) in a fashion $\gamma_m = \gamma_{-m}$. When the aperture size is reduced, higher-order orbital-angular-momentum modes are cut off so that, eventually, only the modes $m = 1$ and $m = -1$ survive. We then expect a sinusoidal fringe, analogous to two-dimensional polarisation entanglement [6]. In the experiment, we observe that the coincidence curve is gradually transformed from parabolic to sinusoidal when the

4. SHANNON DIMENSIONALITY OF QUANTUM CHANNELS AND ITS APPLICATION TO PHOTON ENTANGLEMENT

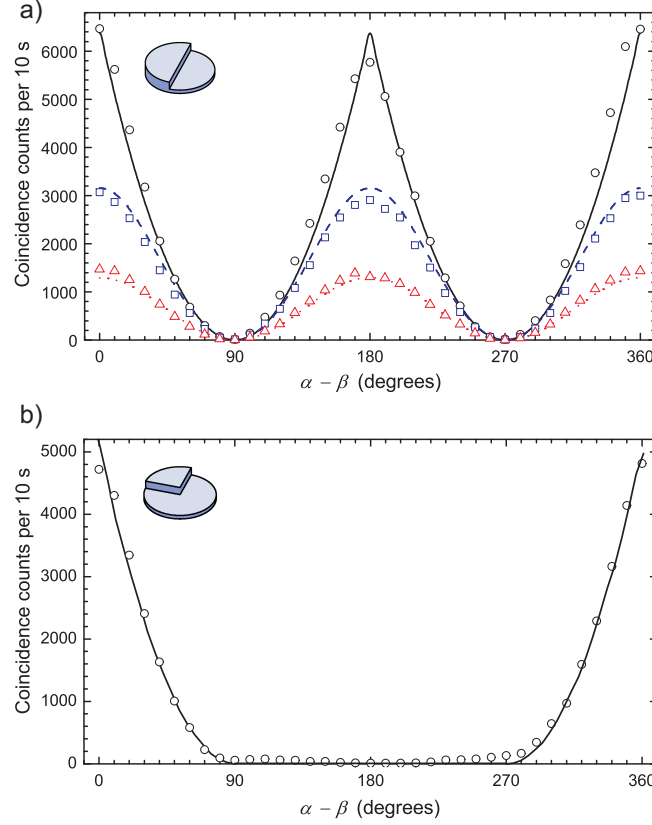


Figure 4.2: **Coincidence count rate vs. the relative orientation of the two state analysers.** Points denote experimental data, the curves are theoretical predictions. **(a)** Half-sector plate. The parabolic fringe (circles) is a signature of a dimensionality larger than two: we find $D = 3.0$. Truncating the number of modes, by closing the aperture, gradually reduces the parabola into a sine of dimensionality 2.0 (triangles). **(b)** Quarter-sector plate. The piece-wise parabolic fringe yields an experimental dimensionality of 5.8 (circles), where theory predicts $D = 6$.

aperture gets smaller (see Appendix). Using an aperture of $600 \mu\text{m}$ diameter, we are in an intermediate regime (squares, $D = 2.1$), while using a $400 \mu\text{m}$ diaphragm yields a curve that resembles a sine very well (triangles, $D = 2.0$). The dashed and dotted curve are theoretical predictions.

To achieve $D = 6$, we use two quarter-sector plates ($\Theta = \pi/2$), carrying an edge discontinuity deviating less than 3% from $\lambda/2$. The circles in Fig. 4.2(b) show our experimental results, revealing a coincidence curve which is parabolic for $|\alpha - \beta| \leq 90^\circ$ and equal to zero otherwise, in agreement with theory (solid curve). We find $D = 5.8$, in very good agreement with the expected value of 6 mentioned above.

The maximum value of the Shannon dimensionality that can be achieved with a phase plate having but a single sector is $D = 6$. Can one reach higher values of D by using plates with

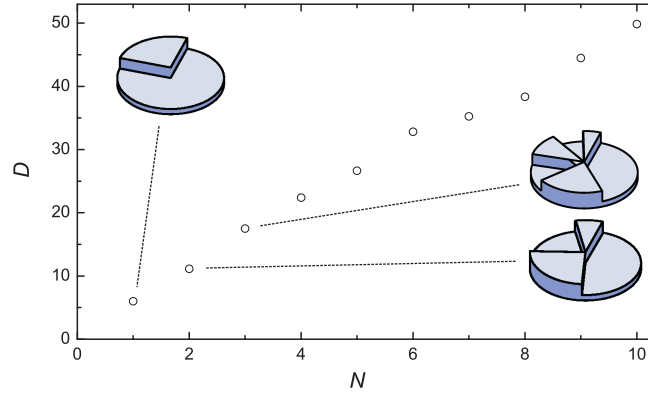


Figure 4.3: **Increasing the Shannon dimensionality with multi-sector phase plates.** Maximum dimensionality that can be accessed with sector phase plates having $2N$ angular sectors alternatingly phase shifted by π . The insets show the optimised plates for $N = 1$, $N = 2$ and $N = 3$.

more sectors? To answer this question, we consider plates with N sectors that are phase shifted by π with respect to interjacent regions. For each choice of sector angles, we calculate the expansion coefficients $\{\lambda_m\}$ and, subsequently, D (see Eq. (4.4) and (4.5)). Next, we maximise D by adjusting the sector angles using a Monte-Carlo random-search algorithm. The result is plotted in Fig. 4.3, showing a graph of the maximum value of D versus the number of mesas N . For 10 such sectors, we find $D = 49.9$. The insets show the optimal phase plates for $N = 1$ (quarter-sector plate), $N = 2$, and $N = 3$.

4.4 Conclusions

In conclusion, we have introduced the effective Shannon dimensionality as a novel quantifier of entanglement as measured in an actual experiment. We have demonstrated its significance to the case of two-photon orbital-angular-momentum entanglement. Using angular-sector phase plates, we have achieved Shannon dimensionalities up to $D = 6$. We anticipate that it is feasible to probe dimensionalities as high as 50, using multi-sector phase plates. These can be manufactured by means of photo- or e-beam lithography as in diffractive-optics technology. Alternatively, the use of adaptive optical devices, such as spatial light modulators or micro-mirror arrays, seems promising, particularly because of their versatility with regard to plate patterns. However, the ultimate limit to the Shannon dimensionality is constrained by the angular Schmidt number of the source; using periodically poled crystals, such as PPKTP, $K \sim 100$ is viable for realistic values of pump-beam waist and crystal length, without loss of count rates [132].

4.5 Appendix

In Fig. 4.2(a), we showed that we can control the Shannon dimensionality probed in our setup by means of a circular, hard-edged aperture. In this Appendix, we present the mathematical foundation for this method.

When facing the question how the mode truncation induced by the aperture will affect the measured dimensionality in our system, it might at first sight seem natural to determine the effect of the clipping on the *generated* entangled state $|\Psi\rangle$. Equivalently, however, the clipping can be applied to the *detection* states $|A(\alpha)\rangle$ and $|B(\beta)\rangle$. This approach will in fact show more fruitful, as these detection states are - unlike the generated state - easily described.

Our strategy will be as follows. The aperture performing the mode truncation is positioned in the far field of the state analyser. We therefore first calculate the Fraunhofer diffraction pattern of the detection state. Since the OAM content of the state is preserved under propagation, we can do this individually for each term $|m\rangle$ in its OAM expansion (carrying probability γ_m). However, dealing with diffraction, we have to take the radial mode properties into account explicitly. Subsequently, we calculate the transmission efficiency W_m for all modes through the aperture of given diameter. Finally, the new expansion coefficients are given by $\tilde{\gamma}_m = W_m \gamma_m / \sum_m W_m \gamma_m$ (where the denominator serves to restore normalisation). These coefficients can be inserted into Eq. (3.14) and (4.4) to find the resulting coincidence-probability curve and dimensionality, respectively.

As mentioned above, the radial and azimuthal mode content must both be taken into account explicitly. We therefore extend the definition of the detection state from Eq. (4.5) with the radial mode structure, as imposed by the single-mode fiber,

$$U(r, \theta, \alpha) = \langle r, \theta | A(\alpha) \rangle = \frac{w_0}{\sqrt{2}} e^{-r^2/w_0^2} \sum_m \lambda_m e^{im(\theta-\alpha)}, \quad (4.7)$$

and similarly for $|B(\beta)\rangle$. Here, (r, θ) are the polar coordinates in the near field of the phase plate. In order to generalise our procedure somewhat, we write each term in the summation over m in Eq. (4.7) as

$$f_m(r, \theta) = f_m(r) e^{im\theta}, \quad (4.8)$$

allowing for an additional parametrisation on the mode number m of the radial part: $f(r) = f_m(r)$. The far field of this expression, exhibited in the plane of the aperture, can be derived via the Fresnel diffraction integral [133]

$$\tilde{f}_m(k, \theta_k) \propto \iint f_m(r, \theta) e^{i(k_x x + k_y y)} dx dy, \quad (4.9)$$

which is simply the Fourier transform of Eq. (4.8). In order to make this expression amenable to the cylindrical symmetry of our system, we rewrite it as

$$\tilde{f}_m(k_r, \theta_k) \propto \iint f_m(r, \theta) e^{ik_r \cos(\theta-\theta_k)} r dr d\theta, \quad (4.10)$$

where (k_r, θ_k) are the polar coordinates in the Fourier plane. Inserting Eq. (4.8), we find

$$\tilde{f}_m(k_r, \theta_k) = \tilde{f}_m(k_r) e^{im\theta_k}, \quad (4.11)$$

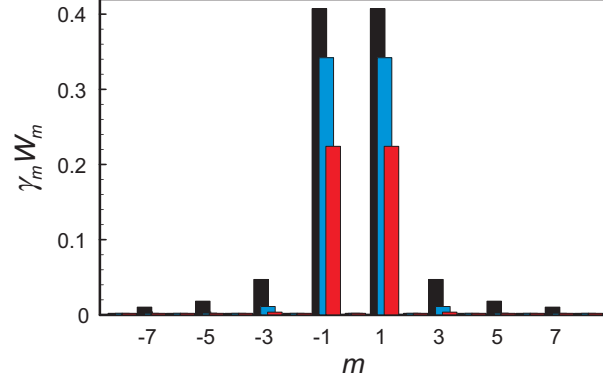


Figure 4.4: **Spectrum of the orbital-angular-momentum eigenvalues for increasing mode truncation.** Distribution of the unnormalised expansion coefficient $\gamma_m W_m$ for no clipping (black), $d = 2.4$ (blue) and $d = 1.6$ (red).

where

$$\tilde{f}_m(k_r) \propto \int_0^\infty f_m(r) \left[\int_0^{2\pi} e^{ik_r \cos(\theta - \theta_k)} e^{im(\theta - \theta_k)} d(\theta - \theta_k) \right] r dr. \quad (4.12)$$

Casting a glance at Eq. (4.8) and Eq. (4.11) learns that they are of a similar form; both near field and far field are separable in a radial and azimuthal part, as may be expected intuitively.

In spite of this apparent simplicity, the radial structure of the field in the plane of the aperture, described by Eq. (4.12), is rather cumbersome to deal with. For the Gaussian profile of Eq. (4.7), it can be worked out by subsequent application of relations (3.915.2) and (6.614.1) from Ref. [134]. After some tedious manipulation we arrive at

$$\tilde{f}_m(k_r) \propto k_r e^{k_r^2 w_o^2 / 8} \left[I_{\frac{1}{2}(m-1)} \left(\frac{k_r^2 w_o^2}{8} \right) - I_{\frac{1}{2}(m+1)} \left(\frac{k_r^2 w_o^2}{8} \right) \right], \quad (4.13)$$

where $I_n(z) = i^{-n} J_n(iz)$ is the n -th modified Bessel function of the first kind [135].

Finally, the transmission function of the hard-edged, circular aperture with radius K_0 is given by

$$T(k_r) = \begin{cases} 1, & 0 \leq k_r \leq K_0, \\ 0, & k_r > K_0. \end{cases} \quad (4.14)$$

The throughput coefficient of mode m through the aperture is thus given by

$$W_m = \frac{\int_0^{K_0} |\tilde{f}_m(k_r)|^2 k_r dk_r}{\int_0^\infty |\tilde{f}_m(k_r)|^2 k_r dk_r}, \quad (4.15)$$

such that $0 < W_m < 1$. The truncated expansion coefficients are now

$$\tilde{\gamma}_m = \frac{W_m \gamma_m}{\sum_m W_m \gamma_m}. \quad (4.16)$$

4. SHANNON DIMENSIONALITY OF QUANTUM CHANNELS AND ITS APPLICATION TO PHOTON ENTANGLEMENT

The coefficients W_m thus function as weighting factors. As higher-order orbital-angular-momentum modes are radially more extended [136], they experience higher losses. Generally, the spectrum of the expansion coefficients $\{\tilde{\gamma}_m\}$ will therefore become narrower when the aperture size is decreased. As a result, the dimensionality shrinks.

We will illustrate this behaviour for the case of the half-sector phase plates from Fig. 4.2(a). We express the size of the aperture in terms of the dimensionless quantity $d = K_0/\sigma_0$, where K_0 is defined as in Eq. (4.14) and $\sigma = 2/w_0$ is the Gaussian beam width in the plane of the aperture. In Fig. 4.4, we have plotted the spectrum of the *unnormalised* OAM expansion coefficients $\gamma_m W_m$ for three situations. As a reference, the spectrum for the case of no clipping is printed in black, analogous to Fig. 3.2. When the aperture is closed to $d = 2.4$ (corresponding to the blue squares in Fig. 4.2(a) using an aperture of radius $300\ \mu\text{m}$ in the experiment), we see that the distribution becomes indeed narrower (blue bars). For $d = 1.6$ (red triangles in Fig. 4.2(a), aperture radius $200\ \mu\text{m}$), the spectrum has practically collapsed to only two contributing terms for $m = -1$ and $m = 1$. This case will give rise to a practically sinusoidal coincidence curve of the form $\cos^2(\alpha - \beta)$, yielding a dimensionality $D \simeq 2$. Note that exact two-qubit entanglement is achieved only in the limit $d \rightarrow 0$.

Fabrication of binary phase plates for the management of orbital-angular-momentum superposition states

5.1 Introduction

The discipline of optical wavefront control is an important branch of optics and photonics where the development of theoretical concepts and technological innovation has evolved hand in hand. The prospects of optical wavefront manipulation have spurred the development of sophisticated technology, which, in turn, has provoked new ideas previously unheard of. Nowadays, wavefront control techniques are indispensable in various branches of science and technology, covering scales both enormous as in astronomical telescopes and tiny as in microscopic imaging.

One of the first persons to have realised the potential of wavefront manipulation may well have been Frits Zernike, when suggesting that the contrast of a microscope image could be enhanced by modifying the phase distribution of the incoming light with binary phase patterns [137, 138]. This idea that won him the Nobel prize in 1953 has spurred microscopists to develop various contrast- and resolution-enhancement techniques by phase manipulation [120, 139–141]. Similar experimental techniques have found their way to the astronomical sciences, where binary phase manipulation has led to novel forms of coronagraphy [142–144]. In recent years, applications using binary phase modulation have surpassed those of imaging alone, and there has been great demand for binary wavefront control techniques for such diverse purposes as maskless photolithography [145], laser beam focusing [146], optical atom traps [147, 148], and femtosecond pulse shaping [149]. And, to add to the list, the field of quantum optics [27, 150, 151].

In the previous Chapters, we have discussed how to treat angular entanglement of photons and probe orbital-angular-momentum (OAM) superposition states by means of structured binary phase plates. In the pursuit of making high-dimensional entanglement a useful resource for quantum information, we need to tailor the angular structure of these binary phase patterns. In our approach, the detection of ever higher dimensionalities requires plates with a growing amount of detail. In Chapter 6, we will study OAM entanglement using angular phase plates with elaborate azimuthal structure, while two-dimensionally patterned plates

containing both radial and azimuthal structure will come into play in Chapter 7. The growing amount of complexity imposes more and more stringent requirements on the fabrication of these phase plates. In the current Chapter, we present a detailed overview of several such fabrication methods.

Generally speaking, there exist both static and dynamic devices for optical wavefront control. Static devices imprint a fixed phase pattern by reflective or transmissive means. Such hardware optical elements can be produced with high quality and are easy to use. They are the natural choice if one is interested in a specific phase pattern only. On the other hand, there are dynamic devices [103], such as liquid-crystal-based spatial light modulators [152] or electromechanically deformable mirrors [153]. Over the last decade, these latter devices have undergone a tremendous development and they are obviously very appealing for their versatility and broad applicability.

Independent of the choice of implementation, there are two ways to impart the phase delay onto the light: one can use holograms or phase plates. Basically, these techniques are essentially the same, but the spatial carrier frequency present in holograms has some important consequences. In the following, we briefly discuss both techniques.

First, when using holograms, a (computer-generated) diffraction grating casts light with the desired phase modification into a particular diffraction order. A well-known example of such an element is a fork hologram [108, 109, 111], which is commonly used to harness the OAM of light beams. Basically, this is a linear diffraction grating with one or more line dislocations at its centre (the number depending on its vortex charge). Holograms are achromatic devices, in the sense that the phase imprint is independent of the wavelength, although the diffraction angle is. A second attractive property of holograms is their ability to perform relatively clean conversions: as the diffraction order of interest is deflected with respect to the input beam, scattering from imperfections in the hologram is unlikely to end up in this diffraction order. Ideally, a hologram is represented in continuous phase levels, although approximating it by a binary phase profile, or even simply printing it as transparent and opaque regions will do the job - be it at the cost of efficiency.

Second, the phase profile of interest can be imprinted into a beam by means of phase plates. A well-known example of such direct phase imprint is a spiral phase plate, an optical element that has a linearly growing optical thickness along the azimuthal coordinate. Due to this helical phase ramp, it imparts OAM to a beam and, thus, forms the analog of a fork hologram (in fact, can be viewed as a hologram without spatial carrier frequency). A major advantage of phase plates is the fact that no light is lost in discarded diffraction orders, and hence conversion efficiencies approach unity. However, the realisation of a flawless phase plate, either implemented by static or dynamic means, is challenging, for any deficiency will introduce a wavefront distortion in the light beam. Several production methods of hardware phase plates have been proposed, using bleaching [154], moulding [106], and shearing [27, 107] techniques, vapour deposition [155, 156] or UV, laser-beam, and electron-beam lithography [157–159]; even tunable [160, 161] and achromatic designs [162] have been realised.

For our experiments on angular entanglement, hardware-implemented phase plates are the preferred choice to manage OAM superposition states. This choice is motivated by the following two reasons. First, in our experiments the phase patterns are rotated around a central axis, which makes the use of holograms highly undesirable due to the beam deflections inherent to their operation. Second, we strive for a spatial resolution of the phase patterns that is

currently beyond reach of pixelated dynamic devices. These experimental considerations lead to the conclusion that static phase plates are the preferred choice.

For the fabrication of such hardware-implemented phase plates, we have explored various available production methods and developed new technologies ourselves. All our devices are transmissive elements into which the phase information is written as a thickness variation. We focus on binary phase profiles, *i.e.*, profiles carrying regions of 0 and π phase imprint only; this significantly reduces the complexity of the manufacturing process, without restricting our experimental aims.

In a parallel effort, we have explored the capabilities of a software-driven deformable-mirror system, that is composed of an array of 240×200 microscopic mirrors. These mirrors can each and individually be addressed and retracted; consequently, this micro-electromechanical device combines a high reflection efficiency with flexibility in regards to phase patterns.

The present Chapter gives an overview of our efforts. In Sections 5.2 and 5.3, we present the requirements on the phase plates and the methods of characterisation. Section 5.4 presents four different techniques we have explored to manufacture static phase elements. The results of our work on the deformable-mirror system will be presented in Section 5.5.

5.2 Specifications of hardware binary phase plates

Before we describe the actual fabrication methods, we first discuss the most important requirements the phase plates should fulfill. As mentioned above, we restrict ourselves to plates carrying binary phase profiles. In our implementation, the phase profile is realised by taking a plane-parallel plate with a spatially varying thickness. The top surface of the plate is thus surfaced with a landscape of flat plateaus (“mesas”) elevated above flatland valleys. The thickness variations h are such that the resulting phase difference $\Delta\phi = \pi$ at our target wavelength $\lambda = 826$ nm. To fulfill this condition, we must have

$$h = \frac{\lambda}{2(n-1)}, \quad (5.1)$$

with n the refractive index of the material. (For the case of reflective phase elements, as for instance the micro-mirror array to be discussed in Section 5.5, the reflected light makes a double pass through air, leading to a target step height $h = \lambda/4$.)

The following list classifies some critical aspects in the manufacturing process and enumerates the requirements we impose to the quality of the phase representation (see Fig. 5.1).

- **Optical step height t .** The optical thickness t of the elevated steps depends both on the physical step height h and the refractive index n of the material. Accurate control of the step height is therefore a critical parameter in the fabrication process. Furthermore, note that a 1% change of the refractive index (typically of the order of 1.5) results in a 3% error in the phase imprint, due to the factor $(n-1)$ in Eq. (5.1). Accurate knowledge of the refractive index is thus vital. Typically, we strive for phase imprints to within 3% of the target value.
- **Height variance $\langle \Delta t^2 \rangle$.** The phase patterns have the contrasting property that they may contain detailed structures of $1 - 10 \mu\text{m}$ size, yet stretch out over areas as large as 4×4

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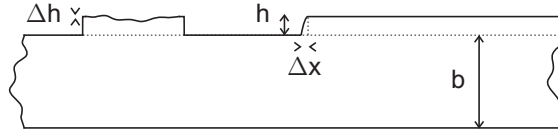


Figure 5.1: **Relevant parameters of a binary phase plate.** The supporting substrate has a thickness b and parallelism $\Delta b/b$. The elevated mesas have a height h (optical height $t = nh$), with height variations Δh . The phase edges have a width Δx .

mm^2 . Ensuring the homogeneity of the phase elevations over such distances is experimentally demanding. We tolerate a maximum phase variation across the plate of typically 3%.

- **Step width Δx .** Ideally, we strive for binary phase patterns that have a zero transition width between the two phase levels. In practice, of course, the step is not infinitely sharp. The critical parameter here is the width of the steps compared to the size of the beam impinging on the plate. The beam has a typical diameter of 2 mm, the step width should be a small fraction of that, especially when the phase pattern is complex.
- **Parallelism $\Delta b/b$.** For our particular application, the parallelism of the supporting substrates is of paramount importance. We impose this constraint in order to avoid beam deflections of the transmitted beam, which cause a major nuisance if one rotates the plates as in our experiments. All phase plate structures are therefore fabricated on high-quality plane-parallel substrates from Bernard Halle Nachfl. GmbH, Berlin, having a wedge angle of less than 2 seconds of arc.

5.3 Characterisation of binary phase plates

After fabricating a phase element, tools are needed to evaluate its performance. Here, we list the characterisation methods we have used.

- Microscope images provide an indication of the overall surface quality of the plates.
- A two-dimensional height profile can be obtained by phase microscopy or white-light interferometry. These methods provide accurate knowledge of the local *physical* step height h as well as the height variance of the protuberances.
- A one-dimensional height profile along a particular trajectory can be obtained by tracing a mechanical stylus across the plate. Such line profiles serve to verify the step height h and provide valuable information about the width Δx of the phase steps. However, as the contact needle may cause slight damage to the surface, scans are better avoided in the region of interest.
- A direct determination of the *optical* step height t can be obtained from the diffraction pattern of a phase plate. In particular, we have found that the far-field intensity pattern of a Gaussian laser beam diffracting from a single phase edge makes a very sensitive

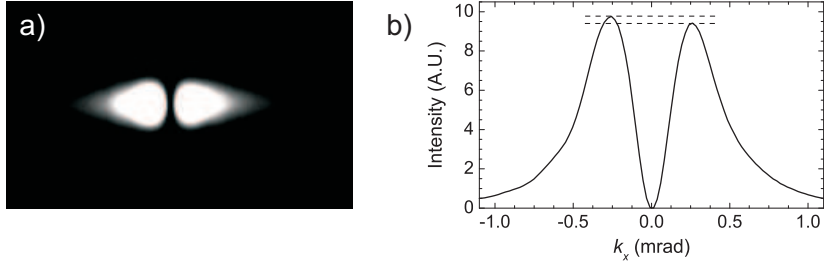


Figure 5.2: **Determination of the optical height of a single phase step.** (a) Experimental far-field profile and (b) horizontal cross section of a Gaussian beam at 826 nm after diffracting from a phase step of height $\Delta\phi = 0.99\pi$.

marker of the optical step height. Figure 5.2(a) shows an experimental diffraction pattern of a Gaussian beam of wavelength 826 nm after traversing such a plate carrying a vertically oriented phase edge. We observe two intensity lobes with a minimum in between. The depth of this minimum depends sensitively on the alignment of the Gaussian beam relative to the phase discontinuity. It becomes zero when the phase edge cuts the Gaussian beam exactly in half. For such balanced illumination, the cross section of the intensity profile along the lobes (see Fig. 5.2(b)) can be written as [163, 164]

$$I_{FF}(k_x) \propto e^{-2\pi^2 w_0^2 k_x^2} [\cos(\Delta\phi/2) - \sin(\Delta\phi/2) \operatorname{erfi}(\pi k_x w_0)]^2, \quad (5.2)$$

where $\operatorname{erfi}(x) = (2/\sqrt{\pi}) \int_0^x e^{t^2} dt$, and w_0 is the beam waist. Note that this far-field distribution has very extended tails reflecting the large wave vector content of the sharp phase step. The ratio of the peak heights of the two lobes depends sensitively on the optical thickness of the step, expressed by the phase difference $\Delta\phi$. When $\Delta\phi = \pi$, the two peaks are equally high. When $\Delta\phi$ deviates from π by 1%, however, the peak ratio already changes by as much as 4%.

5.4 Hardware realisations

In collaboration with the Philips Research Laboratories at Eindhoven, we have developed two production methods for binary phase elements. Additionally, we have worked on two other methods, together with the Laser-Laboratorium Göttingen e.V. and Holoeye Photonics AG in Berlin. Here, we will describe the four manufacturing methods that have been used and discuss their strengths and weaknesses.

5.4.1 Shadow-mask-assisted layer deposition

For the production of the half- and quadrant-sector phase plates used in Chapter 4, we employed a lithographic technique. This was done in collaboration with Philips Research Laboratories. In essence, it comprises three manufacturing steps (see Fig. 5.3(a)):

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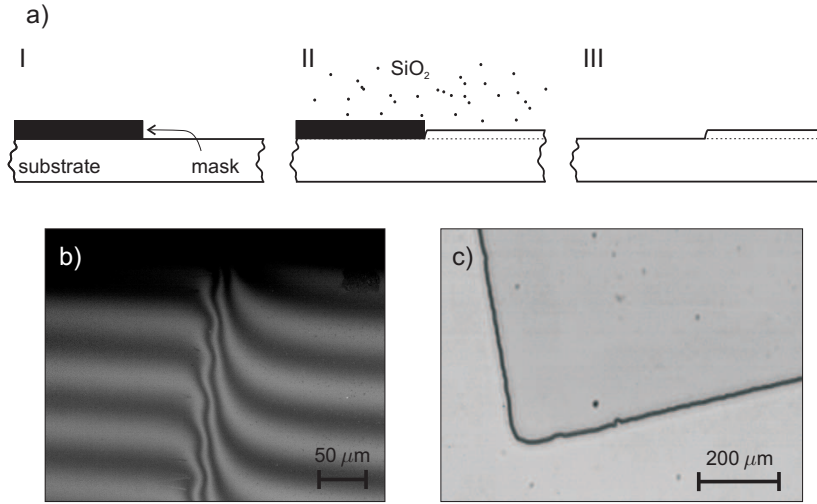


Figure 5.3: **Production of silicon-dioxide-based phase plates.** (a) Schematic of the fabrication process. The glass substrate is covered either with a shadow mask of thickness $25\ \mu\text{m}$ or with a photoresist layer of thickness $1\ \mu\text{m}$ (I) after which SiO_2 is sputtered (II) until the desired step height is reached (III). See text for more details. (b) Phase microscope image of a half-sector phase plate and (c) microscope image of a quadrant-sector phase plate manufactured by this process; these plates have been used for the experiments described in Chapter 4.

1. A metallic shadow mask of the desired plate pattern is placed on top of a fused-quartz substrate. The mask has a thickness of only $25\ \mu\text{m}$ in order to minimise shadow effects at its edges.
2. SiO_2 is sputtered onto the substrate, until it reaches the target step height.
3. The shadow mask is lifted off.

This method tends to yield patterns with a rather large step width Δx as a result of shadowing effects during deposition. Figure 5.3(b) shows a phase microscope image of a step that was deposited using a sizable mask. It indicates that the step has a width of order $50\ \mu\text{m}$. In order to reduce this width, we modified the fabrication procedure and introduced an intermediate photoresist layer in the following way:

1. Spin coat a thin (order $1\ \mu\text{m}$) photoresist layer onto the substrate.
2. Cover the sample with a shadow mask and expose to UV light.
3. Lift the mask and remove the non-exposed regions of photoresist.
4. SiO_2 is sputtered onto the substrate, until it reaches the target step height.
5. Strip off the remaining photoresist.

Using the thin auxiliary sheet of photoresist, which we introduced for the production of quadrant-sector phase plates (see Fig. 5.3(c)), the step width was reduced to approximately $20\text{ }\mu\text{m}$.

The tabulated refractive index of SiO_2 at a wavelength of 826 nm is $n = 1.4529$. In order to produce a π phase difference, the step height of the mesa should therefore be equal to 912 nm . Experimentally, step heights were measured by tracing a mechanical stylus across the edge, yielding values between 920 and 944 nm for different plates. We have used the optical diffraction pattern from a phase step to measure its optical step height. The experimental result is typically to within 3% from specs. Furthermore, by repeating measurements at various locations along the phase edge, we concluded that the step height is homogeneous over a distance of several centimeters. This indicates that the growth rate of the SiO_2 layer is accurately controlled during the deposition process.

The production method discussed above yields excellent results when one is primarily concerned with accuracy and homogeneity of the sputtered mesa. Moreover, the method is simple and can be implemented with relatively small effort. However, in spite of the efforts to mitigate shadow effects during the sputtering process, this fabrication method is limited in terms of resolution. It is therefore not really suited for the production of phase plates with more complicated designs, for instance, plates where several mesas merge in a single point.

5.4.2 Polymer-based photolithography

To fabricate more complex designs, a fully photolithographic method was developed, again in collaboration with Philips Research Laboratories. The basic idea is to use photoresist as a material to create the elevated mesas of our plates. To realise this, we spin coat a layer of precisely tuned optical thickness made of the negative photoresist material SU-8 2002 onto the substrate. Subsequently, we write the phase design by means of soft-contact lithography. Finally, the undeveloped material is etched away to reveal the binary phase pattern.

Although conceptually a clear-cut strategy, it is by no means straightforward to spin coat a layer of controlled, order $1\text{ }\mu\text{m}$ thickness that extends homogeneously over a 1 cm^2 area. Such dimensions are not common practice in photolithography, and several hurdles had to be overcome. The following overview lists the consecutive fabrication steps (see Fig. 5.4(a)):

1. Clear the glass substrates of dust and clean the shadow mask by means of a UV-ozone treatment.
2. Dilute the SU-8 2002 with cyclopentanone (ratio $1 : 0.8$).
3. Spin a coating of SU-8 onto the glass substrate with a Convac Fairchild spinner, during 5 s at 500 rpm and subsequently 45 s at 1850 rpm .
4. Apply a 2 min soft-bake at 95°C on a Präzitherm hotplate (Novoglas Labortechnik, Bern).
5. Bring the mask into soft contact (use a Karl Süss MA8 for mask alignment) and expose sample (88 mJ).
6. Apply a 3 min post-exposure bake-out at 95°C on a Präzitherm hotplate.

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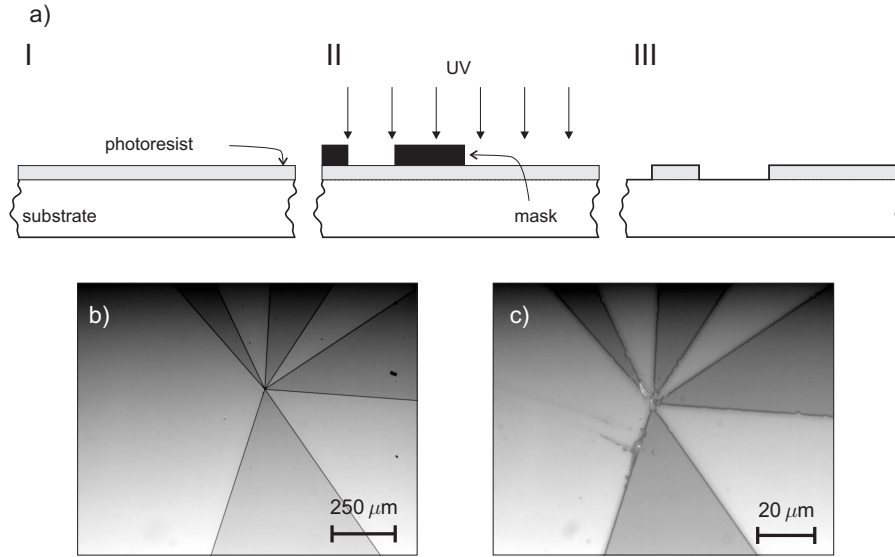


Figure 5.4: **Production of polymer-based phase plates.** (a) Schematic of the fabrication process. (I) A coating of diluted SU-8 is spun onto the glass substrate, tailored to form a homogeneous layer of thickness $h = 707 \pm 10$ nm. (II) A mask of the phase-plate profile is brought into soft contact and the stack is treated photolithographically, after which (III) the sample is baked and the unexposed regions are lifted. (b) Microscope image of the phase plate for a large field of view, and (c) a zoom of the plate centre.

7. Develop sample in SU-8 developer mr-DEV600 (Micro Resist Technology GmbH, Berlin) for 30 s, and flush sample in isopropyl alcohol.
8. Flush sample in demineralised water and dry it.

The refractive index of developed SU-8 at $\lambda = 825$ nm was measured to be $n = 1.584$, in agreement with literature values [165]. In order to produce a phase step of π , the step height should be equal to $h = 707$ nm. Realising a layer of this thickness with SU-8 is not so easy because of the large viscosity of this photoresist. In fact, SU-8 is a standard photoresist for the production of relatively thick ($> 2 \mu\text{m}$) coatings. The cyclopentanone serves as a solvent to lower the viscosity, hence allowing to spin thinner layers. During the bake-out, this solvent evaporates. Another issue common to spin coating is that resist material tends to pile up at the edge of the substrate, which causes the layer to bulge locally. For this reason, a mould was crafted to hold the substrate, such that surplus resist material could overflow. The spinning conditions were optimised for maximum control of the thickness and homogeneity of the layer. The binary amplitude mask was brought into soft contact with the sample, which was then exposed to UV light. The shadow masks were initially printed on thin transparent sheets, however, higher-resolution writing was achieved when printing on quartz crystal. As SU-8 is a negative epoxy-based photoresist, the exposed areas solidify and the non-exposed areas can be etched away by a developer.

Figure 5.4(b) shows an example of a successfully produced plate, composed of four angular

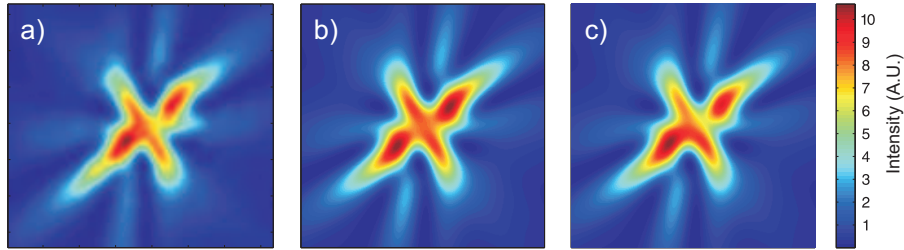


Figure 5.5: **Far-field diffraction pattern of the 4-sector phase plate.** The plate is illuminated with a Gaussian optical beam ($\lambda = 826$ nm) that is centred at the conjunction of the sectors. Profiles shown according to (a) experiment, (b) theory for $\Delta\phi = \pi$, and (c) theory for $\Delta\phi = 1.02\pi$.

sectors that meet each other in the centre. A zoom-in of the central area (see Fig. 5.4(c)), which is the most demanding part to manufacture, shows that small quantities of photoresist have crept into the notches between elevated sectors in an area not larger than $10 \times 10 \mu\text{m}^2$. The steps are less than $2 \mu\text{m}$ wide and shadow effects are invisible on this scale.

The step height was measured to be 720 nm by means of a mechanical stylus, less than 2% away from the target value of 707 nm. Similar results were obtained from measurements on the optical diffraction from a single phase step (made possible by pointing a Gaussian laser beam on an isolated phase edge at the periphery of the plate), yielding phase values to within 2% from π . However, these methods do not provide conclusive information on the optical thickness in the central area of the plate that is of paramount interest. We therefore examined the optical diffraction from the central area, by aligning the beam centre at the conjunction point of the sectors. Figure 5.5(a) shows the experimental result, obtained with a $\lambda = 826$ nm beam of 2 mm diameter. Figures 5.5(b) and (c) present the far-field patterns calculated for a $\Delta\phi = \pi$ and a $\Delta\phi = 1.02\pi$ phase step, respectively. The strong similarity between Figs. (a) and (c) supports the earlier observation that the (optical) step height is 2% too large.

Figure 5.6 shows a tomogram of the plate that was obtained with a white-light interferometer. The colour scale is enhanced to bring out height fluctuations on the photoresist mesas. We observe that the inhomogeneities do not exceed 20 nm, or 3%, over an extended area. Note that, since the low areas are uncoated, we do not expect these areas to show height variations on this scale.

In summary, we have developed a novel method to fabricate binary phase plates with high resolution, using photoresist as a material to create elevated mesas of accurately controlled thickness. The step height is deposited to within 2% from specs over a 1 cm^2 area, with transition edges less than $2 \mu\text{m}$ wide. In principle, this photolithographic technique is low in cost and readily scalable.

5.4.3 UV laser ablation

We have explored the use of a laser ablation technique developed by the Laser Laboratory in Göttingen, Germany, that allows for high-resolution lithographic writing [166]. The method is based on laser ablation of a silicon suboxide coating (SiO_x with $x < 2$) that is strongly absorbing in the UV region due to the oxygen depletion. The manufacturing recipe is outlined as follows:

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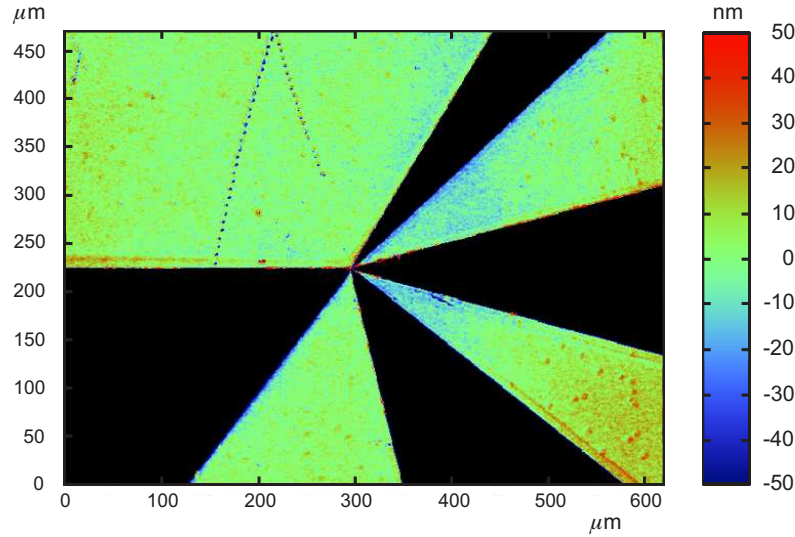


Figure 5.6: **Homogeneity of the SU-8 photoresist coating.** Tomogram of a 4-sector phase plate as measured with a white-light interferometer. The colour scale is enhanced to bring out the homogeneity of the elevated photoresist mesas.

1. First, a silicon suboxide coating (SiO_x with $x < 2$) with a predefined thickness is deposited homogeneously on a fused-silica substrate.
2. Relevant areas of the coating are removed by excimer laser irradiation at 193 nm or 248 nm, leading to a binary depth profile.
3. Finally, the remaining SiO_x -coating is oxidised to UV-transparent SiO_2 by a thermal annealing process.

This method is efficacious in producing high-resolution phase patterns, apt to create structures as small as $1 \mu\text{m}^2$. However, for practical reasons, the lateral dimensions of the whole pattern are limited to $0.5 \times 0.5 \text{ mm}^2$, which is smaller than our typical beam size. Larger patterns can be constructed by patching several such elements together. However, a residual grid of sizeable seams between the individual patches is unavoidable, which makes the resulting structure unsuitable for our purposes.

5.4.4 Laser-beam lithography

For our ultimate attempt we worked together with Holoeys Photonics AG, in Berlin, a company specialising in diffractive optics. Their production technique combines laser-beam lithography with reactive-ion etching and allows for fabrication of structures with sub-micron resolution. Although details of the fabrication procedure are not available, the following list gives an outline (see Fig. 5.7(a)):

1. Coat the glass substrate with a few nm thick sheet of chromium.

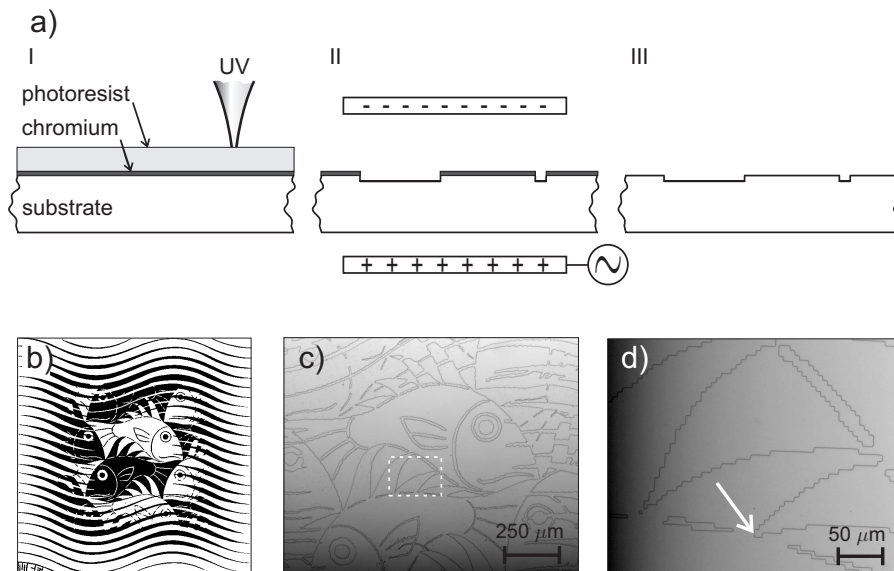


Figure 5.7: **Laser-beam lithographic production of phase plates.** (a) Schematic of the fabrication process. (I) A thin sheet of chromium and a coating of photoresist are deposited on the bare substrate. This stack is inscribed with the plate design of interest by means of a focused UV laser beam. (II) After stripping the photoresist, the relevant areas are excavated to within 4% of the target depth during a reactive-ion etching process. (III) Finally, the remaining chromium is removed, revealing the patterned plate. (b) Picture of the phase plate design, with black and white colours representing excavated and elevated regions, respectively. (c) Microscope image of the fabricated plate, showing the contours of the step edges. The boxed area is enlarged in (d), exposing the pixelation of the fabricated pattern due to the finite resolution of the design from (b). The arrow pointing at the notch between two fins of the fish indicates the point that we have conventionally assigned as the axis of rotation in our experiments described in Chapter 7. Figure (b) after M.C. Escher's "Vissen" © the M.C. Escher Company BV-Baarn-the Netherlands, www.mcescher.com, used with kind permission.

2. Spin a coating of photoresist atop.
3. Write the plate profile lithographically into the photoresist with a focused laser beam.
4. Strip the unexposed photoresist.
5. Use reactive-ion etching to erode the exposed regions to within 4% of the specified depth.
6. Strip the remainder of the chromium mask.

Because of its considerable promise for high-resolution patterning, we have utilised this approach for the production of an intricate binary structure that carries a formidable amount of complexity. The original design comes from the artwork "Vissen", a woodcut by M.C. Escher from 1963 (Fig. 5.7(b)). In our implementation of this pattern, the colours black and white are

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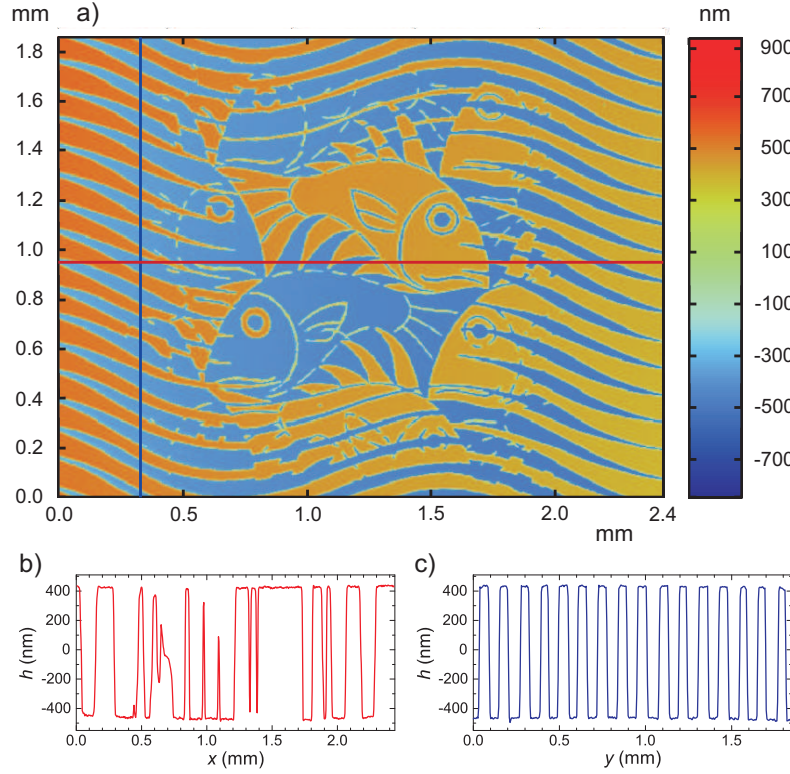


Figure 5.8: **Tomogram of the Escher phase plate.** (a) Height profile of the plate as measured by white-light interferometry. Cross sections along the indicated horizontal and vertical lines are shown in (b) and (c), respectively.

translated into regions of phase $\Delta\phi = 0$ and π . The full relief as realised in our optical device measures $3.4 \times 3.4 \text{ mm}^2$ area (see Fig. 5.7(c)). The magnified microscope image from Fig. 5.7(d) reveals the pixelation of the pattern due to the finite resolution of the design from Fig. 5.7(b); a single pixel corresponds to $4 \times 4 \mu\text{m}^2$. The arrow in Fig. 5.7(d) indicates the position of the axis around which the plate is rotated in our experiments described in Chapter 7.

Figure 5.8 shows a tomogram of the plate as obtained with a white-light interferometer. Cross sections along the indicated horizontal and vertical lines are shown in Fig. 5.8(b) and (c), respectively. The incomplete modulation in the horizontal trace is due to the limited spatial resolution (a few microns) of the white-light interferometer. The physical depth of the excavated areas was measured to be 903 nm. Given the refractive index of the substrate (Suprasil fused silica) of $n = 1.4529$ at $\lambda = 826 \text{ nm}$, this is in excellent agreement with the target value of 912 nm. Furthermore, the depth is very homogeneous across the plate.

This fabrication method yields unprecedented results in terms of height control and resolution. Furthermore, it could also be used to fabricate a mould that functions as a negative master from which multiple copies can be replicated.

5.5 Dynamic devices

In a parallel effort to the various static phase elements discussed so far, we have also investigated the possibilities of dynamic phase-adaptive devices. Known as spatial light modulators, these devices can coarsely be subdivided in liquid-crystal arrays [103, 152] and deformable mirrors [153]. Both technologies have reached an impressive state-of-the-art and have their own advantages and disadvantages. When it comes to operational speed, both techniques perform quite similarly. In terms of resolution and lifetime expectations, liquid-crystal spatial light modulators probably make the better choice. Deformable-mirror systems, on the other hand, do not suffer from unwanted polarisation effects, a common drawback of early generation liquid-crystal arrays. Moreover, as they simply function by specular reflection, they are phase-only instruments by definition and can attain a reflectivity close to 100%. We have explored in particular these deformable-mirror devices.

There is a variety of such systems available, based on the use of independent mirror segments, flexible membranes, piezoelectric and electrostrictive materials, or ferrofluidic solutions [153]. Recently, micro-electromechanical systems have been introduced as a very interesting implementation. Such systems are composed of large arrays of micron-sized mirrors that act as independent piston and/or tip-tilt actuators. The $10^4 - 10^5$ mirrors combine a high reflectivity with a spatial resolution hitherto restricted to liquid-crystal arrays. Fabricated with established micromachining technology, these micro-mirror arrays (MMA) have the potential of breaking the high price threshold of conventional spatial light modulators.

We have explored the performance of a MMA phase-former kit developed by the Fraunhofer Institute for Photonic Microsystems in Dresden [167, 168]. In spite of being a device in development, it has already been successfully implemented in femtosecond laser pulse shaping, holographic beam shaping, and vision correction for the human eye [168, 169]. Here, we investigate its suitability for our experiments on spatial quantum entanglement.

The micro-mirror array consists of a square array of 240×200 mirrors of size $40 \times 40 \mu\text{m}^2$ each (see Fig. 5.9(a)). The mirrors are suspended from their support posts by flexible hinges and can each be deflected over a range of up to 400 nm (see Fig. 5.9(b) and (c)). This is suitable for a 2π phase modulation in the visible spectral region. The mirror array is fabricated from a monolithic aluminium-metal-alloy substrate integrated on top of a CMOS circuitry. The retraction of the mirrors is realised electrostatically by applying a voltage between the mirror surface and an address electrode underneath. The chip can be driven at a frame rate of 500 Hz, it has 8 bit resolution and can be addressed by means of a FireWire connection. The main disadvantage of the device is its long recovery time after deflection, necessary to prevent permanent deformation of the microscopic hinges due to stress. For this reason, the device cannot be operated in continuous mode but has to be cycled with a 1 : 17 on/off rate. The maximum on-time is 1 s. Furthermore, the array behaves as a diffraction grating due to its periodic structure. The intensity reflecting into the zeroth diffraction order of interest is approximately 75%, limited mostly by the fact that the fill factor is 81%.

We have tested the performance of the device by analysing the diffraction pattern of a laser beam after reflection at the mirror array. The Gaussian beam, emerging from a helium-neon laser ($\lambda = 633 \text{ nm}$), covered 4 mm^2 of the mirror array. We loaded predetermined phase profiles like spirals and binary steps onto the device, with the phase patterns carefully aligned to the beam centre. For example, Figs. 5.10(b) and (c) show the calculated and experimental

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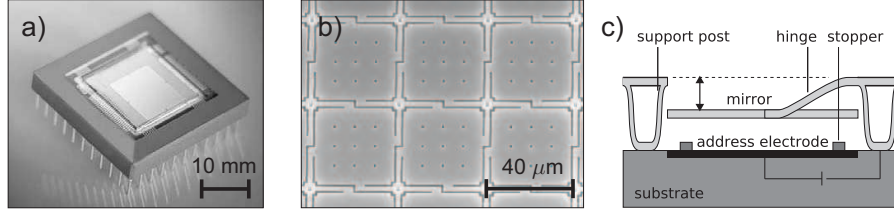


Figure 5.9: **Micro-mirror array.** (a) Image of the device, comprising 200×240 individually addressable piston mirrors. (b) Magnified image of the $40 \times 40 \mu\text{m}^2$ mirror surfaces, suspended from their support posts by flexible hinges. (c) Side-view schematic of a single mirror that can be deflected by addressing a voltage between mirror and electrode. Graphs (a) and (b) reproduced from Ref. [168], (c) based on Ref. [164].

far-field intensity profiles, respectively, when loading a 4-sector binary profile onto the device. The profile (having different sector angles than those in Fig. 5.4) was composed of alternating sectors with zero and $\lambda/4$ deflection level. We observe that the experimental result resembles the calculated pattern, but not very well.

After careful evaluation, we discovered that the lack of agreement can be attributed to the fact that the incoming light does not only reflect off the mirror surfaces, but also off the support posts and hinges. Consequently, there will be an interference effect between light reflecting from the deflected mirrors and light reflecting from the support posts they hinge from. Thus, the amount of light diffracting into the zeroth order depends on the deflection level of the mirrors. Figure 5.10(a) supports this claim, showing experimental data of the reflectivity into the zeroth diffraction order as a function of the uniform deflection of the full mirror array. We observe that the reflection efficiency varies significantly, dropping to a minimum of 50% when the deflection is of order $\lambda/4$; at this point, the reflections from the mirrors and support posts are exactly out of phase. In fact, a simple model of the system using relevant dimensions and phase shifts for the mirrors, support posts and hinges, gives an excellent description of the measurements (solid curve) [164].

These results provide the following insight: when we write a binary pattern where one of the areas has zero deflection and the remaining area deflection $\lambda/4$, the reflectivities of these two areas will be unequal. This thwarts a faithful representation of the 4-sector profile. This problem can be overcome by tuning the deflection levels of the low and high sectors to the point that their reflectivities are balanced, yet their phase difference remains locked at $\Delta\phi = \pi$. The horizontal dashed line in Fig. 5.10(a) indicates this mode of operation. Obviously, this remedy goes at the cost of the overall reflection efficiency. Figure 5.10(d) shows that the configuration indeed leads to much improved correspondence between experiment and calculations.

The inability to function in continuous mode and the apparent tendency to couple phase operations with amplitude effects lead to the conclusion that available MMA technology is not yet sufficiently developed to be used in quantum-optical experiments. However, a recently published progress report [170] describes a mirror design in which the support posts and hinges are hidden underneath the mirrors; this composition eliminates diffraction from suspension elements and thus resolves the deflection-dependent reflectivity. Moreover, it dramatically increases the fill factor and thus the reflection efficiency. Furthermore, improve-

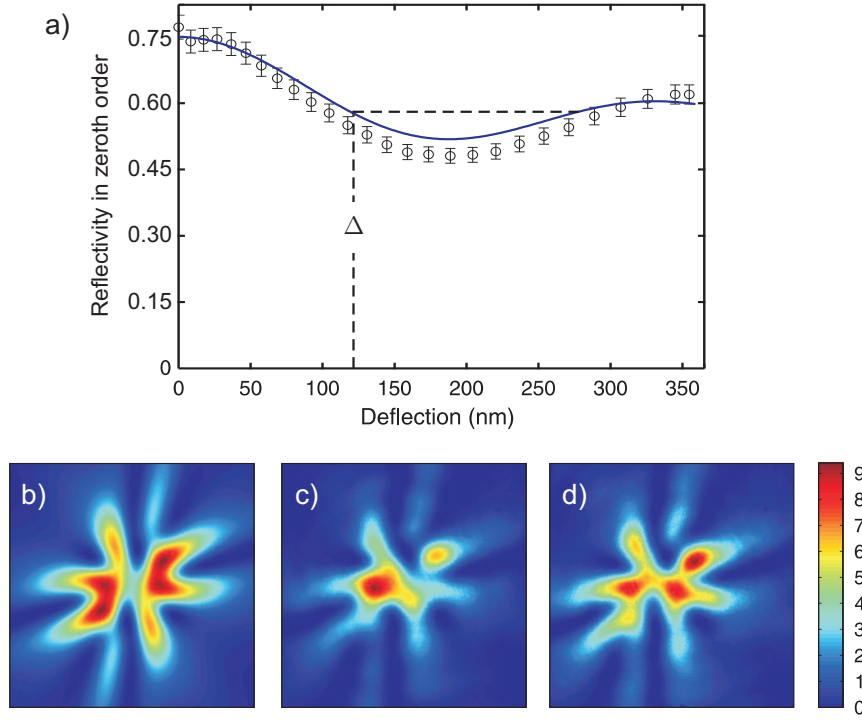


Figure 5.10: **Performance of the micro-mirror array.** (a) Reflection efficiency of a 633 nm beam into the zeroth diffraction order as a function of the deflection of the full mirror array, according to experiment (dots) and theoretical model (curve). (b) Theoretical far-field pattern of a Gaussian beam reflecting from the micro-mirror array displaying a 4-sector binary phase profile. (c) Experimental far-field pattern obtained by representing the 4-sector profile by alternating zones of zero and $\lambda/4$ deflection. (d) Same for alternating zones deflected by Δ and $\Delta + \lambda/4$ such that the reflectivities are balanced. All graphs based on Ref. [164].

ments have been made with the use of alternative actuator materials, like amorphous TiAl or mono-crystalline silicon (m-Si). This is expected to yield superior mechanical drift stability [171, 172].

5.6 Conclusions

We have reviewed a number of experimental realisations of binary phase plates with complex phase patterns. Four out of five methods were realised using hardware transmissive devices, based on deposition of SiO₂, photoresist-based lithography, UV laser ablation techniques, and laser beam lithography, respectively. The fifth method was based on a software-driven flexible micro-mirror array. Excellent performance was achieved with several of these methods, the laser-beam lithographic technique outshining all.

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CHAPTER 6

High-dimensional entanglement with orbital-angular-momentum states of light

Using a measurement-based approach, we extract and engineer high-dimensional orbital-angular-momentum entanglement of photon pairs that emerge from a parametric-down-conversion source. By means of two angular-state analysers, in essence composed of a rotatable multi-sector phase plate and a single-mode fiber, we perform selective projective measurements that maximise the Shannon dimensionality D of measured entanglement. The multi-sector phase plates have a binary phase profile along the azimuthal coordinate, and the arc sector sizes are optimised so as to maximise D . We find that the maximum dimensionality increases linearly with the number of sectors N . The potential of our method is illustrated with an experiment for $N = 4$, yielding $D = 16.5$.

B.-J. Pors, F. Miatto, G.W. 't Hooft, E.R. Eliel, J.P. Woerdman, under review for Journal of Optics, special issue on Orbital Angular Momentum

6.1 Introduction

High-dimensional bipartite entanglement has been predicted to yield a serious speed-up of many quantum-computational protocols as compared to two-qubit entanglement [17, 19, 20, 60]. As a potential candidate for such high-dimensional bipartite entanglement, orbital-angular-momentum (OAM) states of light have received a lot of attention [28]. So far, quantum cryptography [62] and quantum coin tossing [61] have been experimentally realised using OAM-entangled qutrits.

For any application of these methodologies, the generation and management of OAM-entangled states is obviously of paramount importance [173]. The most common way to generate OAM-entangled photons is through the process of spontaneous parametric down-conversion. In a typical setting, the generated spectrum of the OAM correlations is broad with long tails towards high OAM values, as determined by phase matching in the nonlinear crystal. The exact shape of the spectrum can be tailored by controlling the focusing [33] or wavefront [174–176] of the laser beam pumping the nonlinear crystal, or controlling the phase-matching characteristics of the crystal by, for instance, temperature tuning [35]. Alternatively, the problem can be addressed by designing periodically-poled nonlinear crystals with a tailored modulation of the nonlinearity. However, the fabrication of such artificial crystals is challenging [177, 178].

An alternative, measurement-based approach to tailoring entangled superposition states is based on entanglement extraction [179]. This method makes use of selective projective measurements that filter those components of a pure entangled state that are of interest, and discard others. The entanglement can be enhanced (entanglement distillation) or reduced (entanglement dilution). This idea has been used to create “triggered” OAM-entangled qutrits [180, 181].

In Chapter 4, we have used essentially these techniques to extract high-dimensional OAM entanglement from a parametric-down-conversion source of entangled photon pairs. We addressed the question how to quantify the entanglement, and we introduced the Shannon dimensionality D of measured entanglement. This quantity, which may be viewed as a “filtered” Schmidt number, gives the effective number of entangled modes that are detected by the measurement apparatus [50, 87]. We demonstrated the principles of our method, and experimentally achieved $D = 3$ and $D = 6$. In the present Chapter, we expand on this idea and demonstrate its potential to extract genuinely high-dimensional OAM entanglement.

Our method is based on the use of two identical state projectors that selectively filter specific OAM components of the entangled state emitted by the down-conversion source. Such an analyser is in essence composed of a single-mode fiber and a rotatable angular phase plate, which is a diffractive optical element carrying a varying optical thickness along the azimuthal coordinate. Specifically, we use plates that carry a binary azimuthal phase profile, having N elevated arc sectors that produce a π phase shift with respect to the recessed areas between them (see Fig. 6.1). Depending on the exact distribution of the sectors, the projection state is generally an extended superposition of OAM eigenmodes, which varies with the orientation of the plate; by rotating the phase plate around its central pivot, the analyser probes a Hilbert space of large dimensionality (see Chapter 3). In this Chapter, we explore and demonstrate the potential of such multi-sector phase plates to extract high-dimensional OAM entanglement. We optimise the multi-sector profiles for given N so as to maximise the Shannon dimensionality of measured entanglement, and illustrate our results with an experiment.

This Chapter is organised as follows. In Section 6.2, we give an outline of our experimental method. We introduce the multi-sector phase plates in Section 6.3. In Section 6.4, we present the results of a numerical optimisation of the sector profiles, so as to maximise the Shannon dimensionality for a given number of sectors. Our experimental results are presented in Section 6.5.

6.2 Setting

In this Section, we briefly review the basic principles behind our experiments. Using a down-conversion source (see Section 6.5), we create photon pairs that are entangled in their OAM degree of freedom in a fashion^{*} [33]

$$|\Psi\rangle = \sum_{m=-\infty}^{\infty} c_m |m\rangle_A \otimes |-m\rangle_B, \quad (6.1)$$

where $|m\rangle_A$ ($|m\rangle_B$) indicates the Schmidt mode containing one photon in arm A (B) with quantised orbital angular momentum $m\hbar$, where $\langle\theta|m\rangle = \exp(im\theta)/\sqrt{2\pi}$ and θ is the azimuthal angle. The complex expansion coefficients $\{c_m\}$ obey the normalisation condition $\sum_m |c_m|^2 = 1$. The entanglement is analysed by means of two angular-phase-plate analysers that are equipped with multi-sector phase plates. We measure the coincidence rate as a function of the orientations α and β of the two phase plates. The coincidence probability is given by

$$P(\alpha, \beta) = |\langle A(\alpha) | \otimes \langle B(\beta) | \Psi \rangle|^2, \quad (6.2)$$

where the projection state associated with the analyser in arm A is described by

$$|A(\alpha)\rangle = \sum_m \lambda_m |m\rangle e^{im\alpha}, \quad (6.3)$$

and similarly for $|B(\beta)\rangle$. Here, the complex coefficients $\lambda_m = \langle m | A(\alpha = 0) \rangle$ are fully determined by the physical profile of the phase plate (and not by its orientation); see Eq. (6.8). The absolute squares of these coefficients, $\gamma_m = |\lambda_m|^2$, yield the coupling strengths of an OAM mode $|m\rangle$ to the analyser, where normalisation prescribes that $\sum_m \gamma_m = 1$ (*i.e.*, neglecting photon losses).

Equation (6.1) enunciates the fact that the two entangled photons possess complementary OAM. We therefore use two complementary phase plates on the two phase-plate analysers, obeying the symmetry $\lambda_m^A = (\lambda_{-m}^B)^*$. Here, $\{\lambda_m^A\}$ and $\{\lambda_m^B\}$ indicate the expansion coefficients for the projection states in arm A and B , respectively. Given this symmetry, the coincidence probability described by Eq. (6.2) can be shown to take the form

$$P(\alpha - \beta) = \sum_{m, m'} \gamma_m \gamma_{m'} c_m c_{m'}^* e^{i(m-m')(\alpha-\beta)}. \quad (6.4)$$

Not surprisingly, this expression shows that the coincidence probability depends both on the source properties (through the product $c_m c_{m'}^*$) and on the detectors' properties (through the

^{*}In fact, the photon pairs are entangled in both transverse spatial degrees of freedom. However, we will restrict ourselves to a specific single radial mode and consider the azimuthal content of the quantum correlations only.

product $\gamma_m \gamma_{m'}$). We will use this expression to compare with our experimental results presented in Section 6.5.

We will work in the regime that the number of generated entangled modes is much larger than the number of modes our analysers have access to. We may then safely assume that the products $c_m c_{m'}^*$ vary negligibly over the range where the products $\gamma_m \gamma_{m'}$ are nonzero. As shown in Chapter 4, the coincidence probability $P(\alpha - \beta)$ then directly yields the Shannon dimensionality D of the measured entanglement through

$$D = \frac{2\pi P_{\max}}{\int_0^{2\pi} P(\alpha - \beta) d(\alpha - \beta)}, \quad (6.5)$$

where $P_{\max} = P(\alpha = \beta)$ is the maximum coincidence probability. In this regime, the Shannon dimensionality can also be expressed as

$$D = \frac{1}{\sum_m \gamma_m^2}. \quad (6.6)$$

That is, D is fully determined by the analyser properties. The dimensionality D gives the effective number of entangled modes that are detected by the measurement apparatus. Note that D is maximum for a flat-topped distribution of $\{\gamma_m\}$ (i.e., $\gamma_m = 1/M$ for some set of OAM modes and $\gamma_m = 0$ for all others). This case corresponds to entanglement extraction of maximally-entangled qudits, yielding $D = M$. In general, however, for a non-uniform distribution of $\{\gamma_m\}$ we find $D < M$.

The aim of the current Chapter is to present methods to maximise the Shannon dimensionality using multi-sector phase plates. In this fashion, we maximise the amount of extracted entanglement, using phase elements only. We do so by optimising the optical profile of the multi-sector phase plates. In the next Section, we will consider these multi-sector phase plates in closer detail.

6.3 Binary sector phase plates

Let us consider an angular phase plate with an optical phase profile $f(\theta - \alpha)$ along the azimuthal coordinate θ , where α indicates the orientation of the plate. When such a plate is part of an angular state analyser, the projection mode is described by

$$A(\theta, \alpha) = \langle \theta | A(\alpha) \rangle = \frac{1}{\sqrt{2\pi}} e^{if(\theta - \alpha)}. \quad (6.7)$$

This representation allows us to write the OAM expansion coefficients $\{\lambda_m\}$ from Eq. (6.3) as

$$\lambda_m = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} A(\theta, 0) e^{-im\theta} d\theta, \quad (6.8)$$

i.e., $\{\lambda_m\}$ are simply the Fourier expansion coefficients of the plate profile. The OAM coupling strengths $\gamma_m = |\lambda_m|^2$ are thus obtained from the mode overlap between an OAM mode and the projection mode $A(\theta, 0)$.

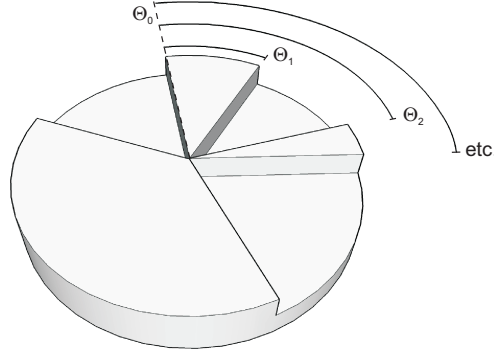


Figure 6.1: **Binary multi-sector phase plate with $N = 3$ elevated angular sectors.** The phase imprint around the azimuthal coordinate jumps alternately between 0 and π at the wavelength of interest. The $2N$ sector angles are denoted as Θ_n , with $\Theta_0 = 0$ by definition.

Next, let us specify the properties of our multi-sector phase plates. Figure 6.1 shows an impression of a $N = 3$ sector phase plate. Such a plate has N elevated arc sectors that produce a π phase shift at the wavelength of interest with respect to the recessed sections between them. The azimuthal mode function corresponding to an angular state projector equipped with such a multi-sector phase plate can be written as

$$A(\theta, 0) = \frac{1}{\sqrt{2\pi}} \begin{cases} (-1)^n, & \Theta_{n-1} \leq \theta < \Theta_n, \quad 1 \leq n < 2N-1, \\ 1, & \Theta_{2N-1} \leq \theta < 2\pi, \end{cases} \quad (6.9)$$

where the angles $\{\Theta_n\}$ represent the $2N$ phase edges between the arc sectors, obeying $\Theta_n < \Theta_{n+1} < 2\pi$ and $\Theta_0 = 0$ by definition (see Fig. 6.1). By application of Eq. (6.8), the OAM expansion coefficients λ_m are expressed in a series of integrals with alternating signs (because of the π phase shift),

$$2\pi\lambda_m = \int_{\Theta_0=0}^{\Theta_1} e^{-im\theta} d\theta - \int_{\Theta_1}^{\Theta_2} e^{-im\theta} d\theta + \dots - \int_{\Theta_{2N-1}}^{2\pi} e^{-im\theta} d\theta. \quad (6.10)$$

Evaluation of the integrals yields the OAM coupling strengths γ_m^* ,

$$\gamma_m = \begin{cases} \left[1 + \frac{1}{\pi} \sum_{n=0}^{2N-1} (-1)^n \Theta_n \right]^2, & \text{for } m = 0, \\ \frac{1}{\pi^2 m^2} \sum_{n,k=0}^{2N-1} (-1)^{n+k} \cos[m(\Theta_n - \Theta_k)], & \text{otherwise.} \end{cases} \quad (6.12)$$

* Calculation of these coupling coefficients may be computationally demanding, as the number of terms in the summation scales with N^2 . However, we can rewrite the cosine of a difference angle as $\cos[m(\Theta_n - \Theta_k)] = \cos(m\Theta_n) \cos(m\Theta_k) + \sin(m\Theta_n) \sin(m\Theta_k)$ and arrive at

$$\gamma_m = \left[\sum_{n=0}^{2N-1} \frac{(-1)^n \cos(m\Theta_n)}{\pi m} \right]^2 + \left[\sum_{n=0}^{2N-1} \frac{(-1)^n \sin(m\Theta_n)}{\pi m} \right]^2. \quad (6.11)$$

The computational complexity of this alternative but equivalent representation scales with N only.

We observe that γ_0 is simply related to the difference between the sums of the elevated and recessed arc sections of the plate. The $1/m^2$ dependence of γ_m for $m \neq 0$ stems from the hard-edged transitions between the 0 and π phase shifted sections.

6.4 Optimisation of binary sector phase plates

It is our aim to maximise the dimensionality

$$D = \frac{1}{\sum_m \gamma_m^2}$$

that can be attained using binary sector phase plates. As discussed before, D is maximised if all γ_m have the value $1/M$ or 0, with M the number of contributing modes. However, such a uniform eigenvalue distribution cannot be realised by shaping the phase profile of the plate only; also an amplitude variation would be required. The problem at hand is thus to maximise D with a multi-sector phase plate, by optimising the distribution of the $2N$ sector angles $\{\Theta_0, \dots, \Theta_{2N-1}\}$. The resulting binary profile gives rise to a distribution of $\{\gamma_m\}$ that is as flat-topped as possible for a pure phase plate. Qualitatively, this situation is realised by randomising the $2N$ sector angles. Quantitatively, however, the optimisation of the sector angles is by no means trivial, as we will see below.

6.4.1 Single-sector phase plates

To commence, we first explore the case $N = 1$, *i.e.*, a phase plate with a single elevated arc sector. In Chapter 3, we already derived an explicit expression for D in terms of the opening angle Θ of the arc sector,

$$D(\Theta) = \begin{cases} 1/[1 - 4\Theta/\pi + 6(\Theta/\pi)^2 - (8/3)(\Theta/\pi)^3], & \Theta \in [0, \pi], \\ D(2\pi - \Theta), & \Theta \in [\pi, 2\pi]. \end{cases} \quad (6.13)$$

Figure 6.2(a) shows a plot of this result. We observe that the Shannon dimensionality D ranges between 1 and 6, the maximum being reached with the quadrant phase plate we used in Chapter 4.

Let us consider this case of a quadrant sector ($\Theta = \pi/2$) in more detail. Of the effective dimensionality $D = 6$ that this plate supports, four dimensions can be ascribed to mutually orthogonal projection states, being the states corresponding to the four plate orientations $\alpha = 0, \pi/2, \pi$ and $3\pi/2$. The states corresponding to any other setting of α , however, cannot be represented as a superposition of these four states alone. In fact, by varying the orientation of the plate, we scan past an infinite set of additional projection states that are each distinct, but mutually non-orthogonal. We therefore decompose the projection states $\{|A(\alpha)\rangle\}$ in the orthogonal and complete OAM basis $\{|m\rangle\}$, which constitutes an appropriate basis due to the cylindrical symmetry of the problem (see Eq. (6.8)). Figure 6.2(b) shows a histogram of the eigenvalues $\{\gamma_m\}$. We find that we need an infinite number of OAM modes to fully describe $|A(\alpha)\rangle$. We note, however, that the eigenvalues have very different weights. In the definition of D , the eigenvalues are precisely weighed by their strengths, yielding an *effective* dimensionality of $D = 6$. A similar argument can be made for the half-sector phase plate ($\Theta = \pi$), which

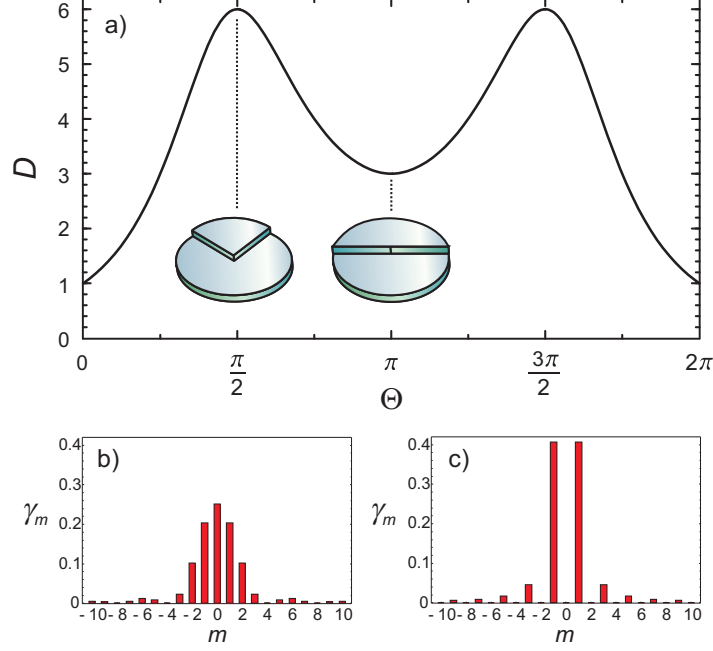


Figure 6.2: **Dimensionality of a single-sector phase plate.** (a) Dimensionality as a function of the angular sector size Θ . The insets show a quadrant-sector phase plate for $\Theta = \pi/2$ and a half-sector phase plate for $\Theta = \pi$. (b) Spectrum of the OAM coupling strengths γ_m for a quadrant-sector phase plate ($\Theta = \pi/2$). (c) Spectrum for a half-sector phase plate ($\Theta = \pi$).

possesses two orthogonal orientations for $\alpha = 0$ and $\alpha = \pi/2$, yet gives rise to $D = 3$ (see Fig. 6.2(c) for the eigenvalue spectrum).

Six is the maximum dimensionality that can be attained with a binary phase plate containing a single elevated arc sector. In order to achieve higher dimensionalities, we thus have to consider plate profiles with increased complexity. The natural thing to do is to add binary sectors to the plate and optimise the set of sector angles. In the following, we will present an effective routine to determine the optimised sector profiles.

6.4.2 Multi-sector phase plates

In order to maximise D , we seek the minimum of the sum $\sum_m \gamma_m^2$ (see Eq. (6.6)). As all elements γ_m^2 of the sum are non-negative quantities, we can therefore minimise the elements themselves. However, the minimisation of these elements is non-trivial, since the variables they depend upon are shared amongst all. We start out from the expression for γ_m as given in Eq. (6.12). The $m = 0$ component γ_0 causes no serious difficulties. Taking the square for $m \neq 0$ yields

$$\gamma_m^2 = \frac{1}{\pi^4} \sum_{n,k,n',k'=0}^{2N-1} \frac{(-1)^{n+k+n'+k'} \cos[m(\Theta_n - \Theta_k)] \cos[m(\Theta_{n'} - \Theta_{k'})]}{m^4}. \quad (6.14)$$

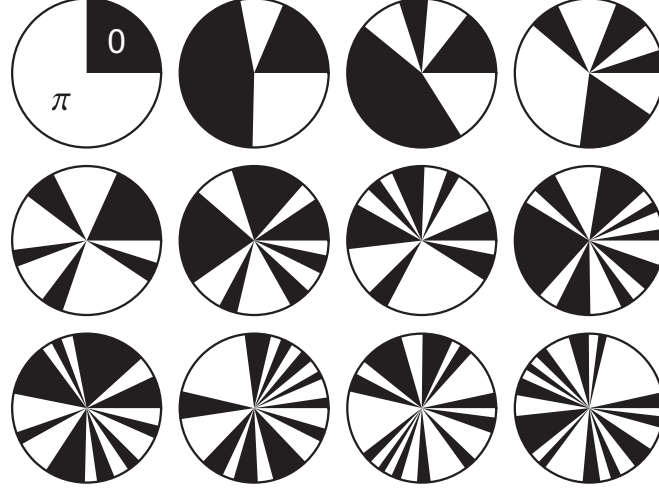


Figure 6.3: **Optimal multi-sector phase plate patterns.** The optimal plate pattern for $N = 1 - 12$. Black and white zones indicate regions of 0 and π phase imprint, respectively.

The complexity of the minimisation arises from the fact that Eq. (6.14) contains all combinatorial permutations of the periodic difference angles $\Theta_n - \Theta_k$ and $\Theta_{n'} - \Theta_{k'}$. In spite of considerable effort, we have not found an analytic solution to the maximisation of D , and it is not obvious that such a solution exists.

Therefore, we have adopted a numerical approach. In Chapter 4, we presented the maximum dimensionality as a function of the number of sectors N , as found with a Monte Carlo search routine. These results are shown in Fig. 6.5 as triangles. We observe that with 10 sectors we would attain $D = 50$. However, apart from showing that the dimensionality increases with N , the exact dependence remained unclear.

The Monte Carlo routine based on Eq. (6.14) is computationally rather demanding. An alternative approach involves writing the inverse dimensionality as a polynomial series of order four (see Appendix),

$$\frac{1}{D} = \frac{1}{24\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \chi^2 (2\pi - \chi)^2 + \gamma_0^2, \quad (6.15)$$

where $\chi = \Theta_n - \Theta_k + \Theta_{n'} - \Theta_{k'} \bmod 2\pi$. This simplification yields a huge speedup when applying a search routine. For example, it reduced the search time for the optimal sector profile for $N = 9$ from almost three days to less than three minutes.

Figure 6.3 shows the optimal sector profiles we found for $N = 1 - 12$, based on Eq. (6.15). Black and white sectors represent zones of π phase difference. We note that these solutions are not unique; the optimal solution is in fact $2 \times 2 \times 2N$ -fold degenerate. Namely, one could (i) interchange the 0 and π phase shifted sectors, (ii) reverse the handedness from clockwise to counterclockwise, and (iii) one could choose any of the $2N$ sector angles as the starting point Θ_0 . Figure 6.4 presents the OAM distributions $\{\gamma_m\}$ for the optimal sector profiles of Fig. 6.3

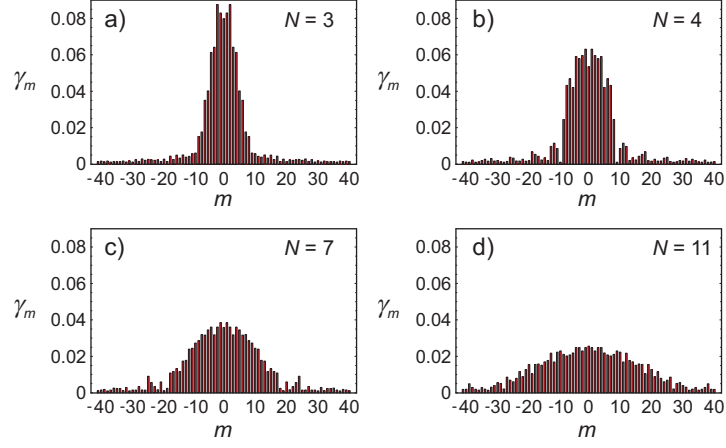


Figure 6.4: **Orbital-angular-momentum distribution of optimal multi-sector plates.** (a)-(d) Histograms of the eigenvalues $\{\gamma_m\}$ of the optimal plate patterns from Fig. 6.3 for $N = 3, 4, 7$, and 11 .

for $N = 3, 4, 7$, and 11 . We observe that with increasing N , the spectrum gradually becomes broader and flatter. In Fig. 6.5, we present the dependence of the maximum dimensionality on the number of sectors N (circles), corresponding to the optimal plate profiles from Fig. 6.3. Note that the results obtained with our accelerated routine are superior to those found previously for $N > 5$ (triangles). Strikingly, we observe that the growth rate is essentially linear, at least up to $N = 14$, the largest value of N for which we applied our routine (solid line). The slope of the linear dependence is 5.9 ± 0.02 . Note that the linear behaviour is not exact: the data are scattered closely around the solid line.

6.5 Experimental results

In this Section, the potential of multi-sector phase plates for extracting genuinely high-dimensional OAM entanglement is illustrated by an experiment. The experimental setup we use is largely the same as that described in Chapter 4 (see Fig. 4.1). A periodically-poled KTP crystal of length 2 mm serves as a Type-I parametric down-conversion source of OAM-entangled photon pairs at 826 nm, prepared in a state described by Eq. (6.1). The effective number of azimuthal entangled modes emitted by the source, characterised by the azimuthal Schmidt number K , is of the order of $K = 30$ [50, 87].

As discussed in Section 6.2, conservation of OAM in the down-conversion process requires that the two state analysers are equipped with complementary phase plates. That is, the phase plate in arm A must be the phase conjugate of the one in arm B . Evidently, the phase conjugate of a multi-sector phase plate is simply obtained by interchanging the 0 and π phase-shifted sectors. However, this implies that the original and the conjugate phase profile are identical up to an overall phase factor of π . Since our coincidence-detection scheme is insensitive to overall phase factors, this means that we can simply use identical multi-sector phase plates in both arms.

6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

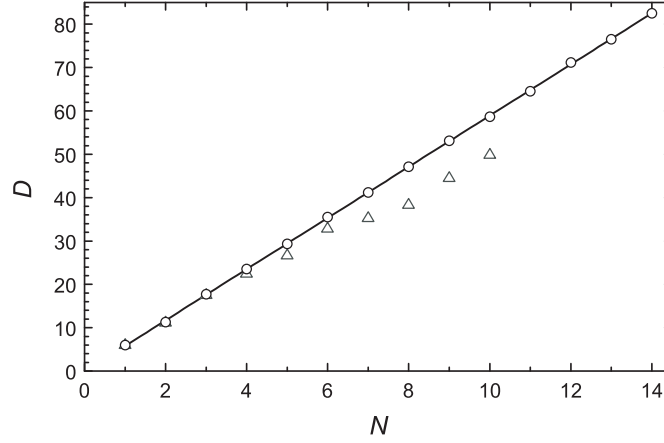


Figure 6.5: **Dimensionality of the optimal multi-sector phase plates.** Maximum dimensionality D as a function of the number of sectors N . (Triangles) Results as presented in Chapter 4 using a Monte Carlo search routine with Eq. (6.14) as a starting point. (Circles) Results based on the use of a search routine with Eq. (6.15) as a starting point. These data closely follow a linear dependence with slope 5.9 (solid line). The corresponding plate profiles are shown in Fig. 6.3.

In Fig. 6.6, we first present the theoretical predictions for the coincidence-probability curves as expected for the optimal plate profiles from Fig. 6.3, for $N = 1$ (dotted), $N = 3$ (dashed), and $N = 11$ (solid). We observe that the coincidence probability is zero or close to zero over a large domain of orientations and is strongly peaked around $\alpha = \beta$. The width of this parabolic peak narrows with increasing number of sectors N . Note that the Shannon dimensionality D is inversely proportional to the area underneath the curve, as described by Eq. (6.5).

We have performed the experiment with multi-sector phase plates of $N = 4$, the fabrication of which is discussed in Chapter 5. The four elevated sectors have an optical thickness $\delta = 1.02\pi$. The sector sizes do not exactly match those of the optimal profile from Fig. 6.3, because the production process was initiated before the calculations presented in Section 6.4 were completed. As a result, the calculated dimensionality of these plates amounts to $D = 23.0$, as compared to 23.5 for the optimal profile. The inset in Fig. 6.7(a) shows the design of the plates, which is remarkably different from the optimal plate shown in Fig. 6.3.

Figure 6.7(a) shows the experimental coincidence curve. We observe a curve that is strongly peaked around $\alpha = \beta$. Note that the single count rates at both detectors (2.6×10^6 per 8 s) are independent of the phase plate orientations. Given these high single count rates and the finite 2.7 ns coincidence detection window, accidental coincidence counts arising from photons that do not belong to a pair are unavoidable; we corrected for these accidentals. At the peak of the coincidence curve, we record about 1.7×10^5 coincidence counts per 8 s. Across the domain $25^\circ < \alpha - \beta < 335^\circ$, the count rate does not exceed 2×10^3 for each setting of $\alpha - \beta$. We thus achieve a visibility as high as 99%. We verified that the coincidence curve depends on the relative orientation between the two phase plates $\alpha - \beta$ only, in agreement with Eq. (6.4). The solid curve represents the theoretical prediction, for which a trivial vertical scaling factor

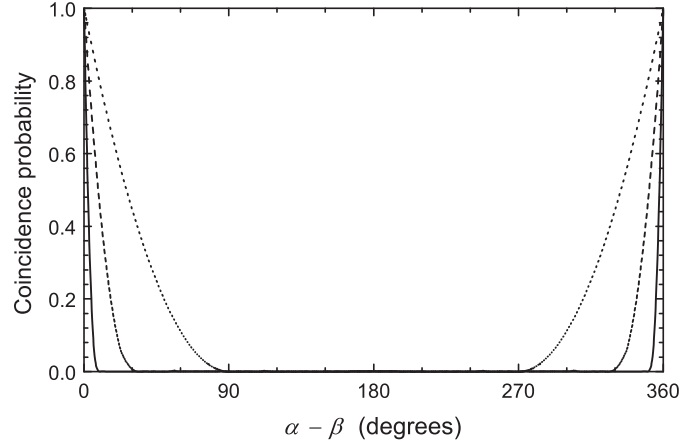


Figure 6.6: **Theoretical peak-normalised coincidence probability for multi-sector phase plates.** Peak-normalised coincidence curves for $N = 1$ (dotted), $N = 3$ (dashed), and $N = 11$ (solid), as a function of the relative orientation $\alpha - \beta$ between the two phase plates.

was used as the only fit parameter. We observe that experiment and theory are in excellent agreement with each other.

Based on Eq. (6.5), the experimental value of the dimensionality we obtain from the data equals $D = 16.5$. This number should be compared to the theoretical value of 23.0 mentioned above. The difference between these two numbers can be ascribed to two reasons. First, the experiment is very sensitive to misalignment of the phase plates. The discrepancy between theory and experiment at the bottom of the curve is a direct result hereof (see Fig. 6.7(b) for a rescaled version of this region). Second, in our derivation of Eq. (6.6) we assumed the source to be ideal, in the sense that it generates an infinite number of entangled modes (and the products $c_m c_{m'}^*$ from Eq. (6.4) could hence be considered constant). With an increasing number of sectors on the phase plates, however, we approach the limit of validity of this approximation. This causes the peak around $\alpha = \beta$ to be somewhat blunt rather than sharply spiked. Given the fact that D scales linearly with the maximum coincidence count rate (see Eq. (6.5)), the dimensionality that is experimentally realised is fairly sensitive to this effect. Increasing the Schmidt number of the source, for instance by using a shorter down-conversion crystal, would resolve this issue.

Finally, we note that the coincidence-probability curve and the accompanying Shannon dimensionality are remarkably robust against deviations of the phase step height from its target value $\delta = \pi$. For the 2% deviation of our $N = 4$ plates ($\delta = 1.02\pi$), the effect is negligible. Even for a 10% deviation, the calculated dimensionality drops only by 3%. This insensitivity makes our method very robust against possible flaws of the plates that may arise in the fabrication process.

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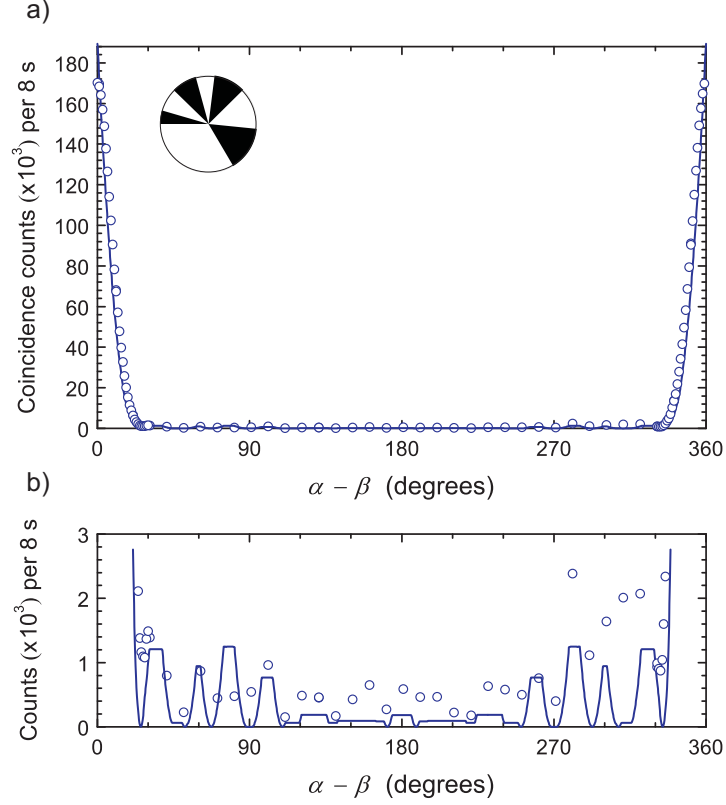


Figure 6.7: **Experimental coincidence curve.** (a) Coincidence rate per 8 s as a function of the relative phase plate orientation $\alpha - \beta$ between the two state analysers, performed with two identical $N = 4$ sector phase plates (see inset). The circles represent experimental data, the solid curve represents our theoretical model. The theoretically predicted dimensionality equals $D = 23.0$. Experimentally, we obtain $D = 16.5$. (b) Magnification of the bottom of figure (a), showing the noise level of the measurements.

6.6 Conclusions

In conclusion, we have introduced multi-sector phase plates as a powerful tool to tailor high-dimensional OAM entanglement of photon pairs. We have presented a numerical study of the dimensionality that can be extracted using angular state analysers with multi-sector phase plates. Qualitatively, the maximum dimensionality is obtained with phase profiles that carry the least amount of azimuthal symmetry, hence yielding a broad and flat OAM eigenvalue spectrum. Quantitatively, we have found that the maximum dimensionality grows linearly with the number of sectors N . We experimentally demonstrated our approach using phase plates with $N = 4$ sectors.

These results demonstrate the potential of our phase-plate technology for extracting high-dimensional OAM entanglement. They represent an important step towards harnessing the full power of high-dimensional OAM entanglement for quantum information processing.

6.7 Appendix

In this Appendix, we present a derivation of the dimensionality D as given in Eq. (6.15). We start out from the definition of D (see Eq. (6.6)), using the expression for γ_m^2 from Eq. (6.14). First, notice that the spectrum of the OAM coupling probabilities γ_m is symmetric around $m = 0$, because the sector phase plates have no net phase ramp along the azimuthal coordinate θ . We can thus write $1/D = \sum_{m=-\infty}^{\infty} \gamma_m^2 = 2 \sum_{m=1}^{\infty} \gamma_m^2 + \gamma_0^2$. Next, we can work out the infinite series over m in Eq. (6.14) by exploiting the standard expansion

$$\sum_{m=1}^{\infty} \frac{\cos(mx)}{m^4} = \frac{\pi^4}{90} - \frac{\pi^2 x^2}{12} + \frac{\pi |x|^3}{12} - \frac{x^4}{48}, \quad (6.16)$$

which is a fourth order polynomial in x valid for $-2\pi \leq x \leq 2\pi$ [134]. Using the trigonometric identity $2 \cos a \cos b = \cos(a+b) + \cos(a-b)$ to rewrite Eq. (6.14), it follows that one can write

$$\frac{1}{D} = \frac{1}{\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \left(\frac{\pi^4}{45} - \frac{\pi^2 \chi^2}{6} + \frac{\pi \chi^3}{6} - \frac{\chi^4}{24} \right) + \gamma_0^2, \quad (6.17)$$

where we defined

$$\chi = \Theta_n - \Theta_k + \Theta_{n'} - \Theta_{k'} \mod 2\pi. \quad (6.18)$$

Here, the modulo 2π ensures that χ is bounded between 0 and 2π , as is required for the application of Eq. (6.16). This function can further be simplified by appreciating that (i) the term $\pi^4/45$ cancels out after the remaining summations, and (ii) $4\pi^2 \chi^2 - 4\pi \chi^3 + \chi^4 = \chi^2 (2\pi - \chi)^2$. It then follows that

$$\frac{1}{D} = \frac{1}{24\pi^4} \sum_{n,k,n',k'=0}^{2N-1} (-1)^{n+k+n'+k'} \chi^2 (2\pi - \chi)^2 + \gamma_0^2, \quad (6.19)$$

with χ as in Eq. (6.18). Finally, Eq. (6.19) possesses many symmetries, which can be exploited to remove redundant terms from the series. Consider the set of indices $\{n, k, n', k'\}$ with $0 \leq n, k, n', k' \leq 2N-1$, then, for instance,

1. the elements $\{a, a, a, a\}$ yield 0,
2. the elements $\{a, a, b, b\}$ yield 0,
3. swapping (i) the first and the third index, $\{a, b, c, d\} \rightarrow \{c, b, a, d\}$; (ii) the second and the fourth index, $\{a, b, c, d\} \rightarrow \{a, d, c, b\}$; or (iii) both, $\{a, b, c, d\} \rightarrow \{c, d, a, b\}$, all yield the same result.

By accounting for such symmetries and multiple identical occurrences, the computational effort is strongly reduced.

6. HIGH-DIMENSIONAL ENTANGLEMENT WITH ORBITAL-ANGULAR-MOMENTUM STATES OF LIGHT

Beyond angular entanglement

7.1 Introduction

In the community working on spatial entanglement as a resource for high-dimensional quantum information processing, particular emphasis has been put on the orbital-angular-momentum content of the field [40, 44, 47]. However, there is no intrinsic argument to restrict oneself to the azimuthal content of the field and neglect the radial content [42, 111]; the choice for this restriction seems to be merely motivated by the fact that there is no obvious approach to deal with the radial degree of freedom. It may well be, however, that exploitation of the radial content of the field provides for interesting opportunities and enhanced performance.

Also in this thesis we have focused our attention on angular entanglement. However, this is an unnecessary restriction, as our ideas on the Shannon dimensionality of measured entanglement surpass this particular application. Here, we make some first steps towards the combined use of both angular and radial degrees of freedom in bipartite spatial entanglement, and explore some of the resulting richness and associated challenges.

7.2 Setting

The approach we choose to follow - obviously one of many - is to leave the essence of our experimental scheme the same, *i.e.*, to use two spatial-mode analysers with conjugate phase plates that can be rotated relative to each other, but to enrich the structure of the phase plates. That is to say, we incorporate the radial degree of freedom in our experiments by using phase plates carrying both angular and radial phase information. The normalised detection mode function of the angular state analyser in arm *A* for given orientation α can then be written as

$$A(\mathbf{r}, \alpha) = \frac{w_0}{\sqrt{2}} e^{-r^2/w_0^2} e^{if(\mathbf{r}, \alpha)}, \quad (7.1)$$

and similarly for $B(\mathbf{r}, \beta)$. Here, $\mathbf{r} = (r, \theta)$ are the polar coordinates in the transverse plane, $f(\mathbf{r}, \alpha) = f(r, \theta - \alpha)$ is the phase profile of the phase plate and w_0 is the Gaussian beam waist setting the radial size of the mode.

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The effective dimensionality probed by one such analyser can be obtained from its auto-correlation function $G(\alpha, \alpha') = G(\alpha - \alpha') = \int A(\mathbf{r}, \alpha) A^*(\mathbf{r}, \alpha') d\mathbf{r}$, through (see Chapter 3),

$$D_{\text{eff}} = \frac{2\pi}{\int_0^{2\pi} |G(\alpha - \alpha')|^2 d(\alpha - \alpha')} . \quad (7.2)$$

In other words, the effective dimensionality is found by averaging the overlap probability $|G(\alpha - \alpha')|^2$ over the relative analyser settings $\alpha - \alpha'$, *regardless of the exact phase profile of the plate*. We conclude that the framework we developed to define the effective dimensionality of a state analyser remains intact. Note, however, that the radial coordinate becomes an essential variable that needs to be taken into account explicitly. That being said, it is immediately apparent that the radial size of the detection mode, set by the Gaussian beam waist w_o , has become a critical parameter.

Our experiment is essentially the same as the one discussed in the previous Chapter, with our generalised phase plates installed in the state analysers. The two-photon coincidence probability that we measure can be written as (see Chapter 4)

$$P(\alpha - \beta) \propto \left| \int d\mathbf{r}_1 d\mathbf{r}_2 \Psi(\mathbf{r}_1, \mathbf{r}_2) A^*(\mathbf{r}_1, \alpha) B^*(\mathbf{r}_2, \beta) \right|^2, \quad (7.3)$$

where $\Psi(\mathbf{r}_1, \mathbf{r}_2) = \langle \mathbf{r}_1, \mathbf{r}_2 | \Psi \rangle$ represents the pure spatially-entangled state as given in Eq. (2.7) from the Chapter 2. We define the Shannon dimensionality of measured entanglement as (see Chapter 4)

$$D = \frac{2\pi P_{\text{max}}}{\int_0^{2\pi} P(\alpha - \beta) d(\alpha - \beta)}, \quad (7.4)$$

where $P_{\text{max}} = P(\alpha = \beta)$ represents the maximum coincidence probability.

7.3 Escher phase plates

It may be worthwhile to study which 2-D plate profiles maximise this quantity, in a fashion similar to our work on multi-sector phase plates from Chapter 7. For the current purpose, we manufactured a binary phase structure with formidable complexity, using the work “Vissen” by the artist M.C. Escher for the phase plate design (see Fig. 7.1(a)). In our implementation of this pattern, the colours black and white are translated into regions of phase $\Delta\phi = 0$ and π at the wavelength of interest ($\lambda = 826$ nm). The phase plates were realised as a height relief; they were manufactured in glass and measured 3.5×3.5 mm² in total (see Chapter 5). The detection mode size w_o used in our experiments measured $800 \mu\text{m}$, and is indicated by the circle in Fig. 7.1(a). The cross hairs indicate the rotation centre of the plate. The magnified microscope image from Fig. 7.1(b) reveals the pixelation of the pattern due to the finite resolution of the design (800×800 pixels, pixel size $4.4 \times 4.4 \mu\text{m}^2$). Figure 7.1(c) shows an image of the Escher phase plate, which is back-illuminated by a Gaussian beam with $w_o = 800 \mu\text{m}$. The cross hairs indicate the centre of mass of the Gaussian beam, to which the rotation centre of the phase plate should be aligned.

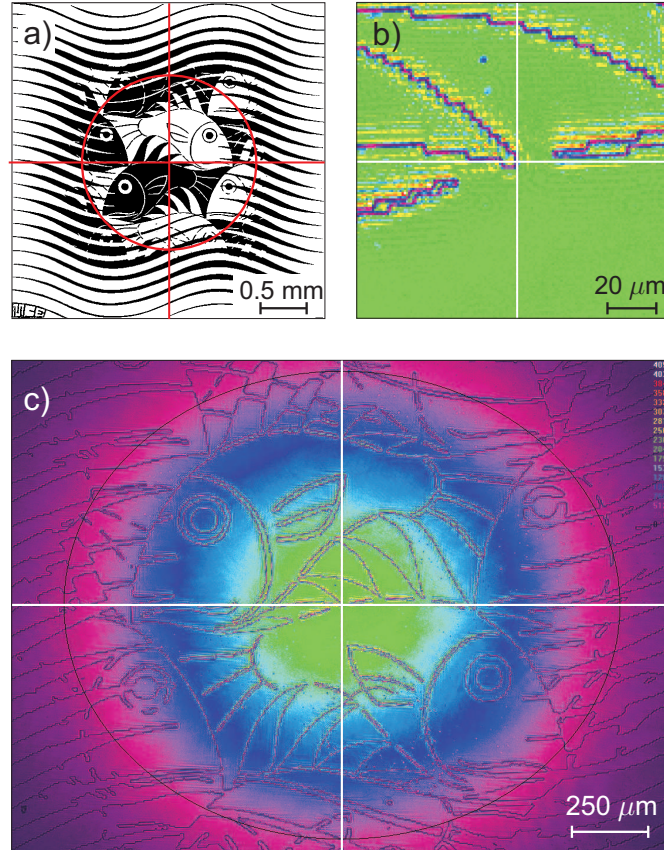


Figure 7.1: **Escher phase plate.** (a) Picture of the phase plate design, with black and white colours representing regions that impart a 0 and π phase shift at the wavelength of interest, respectively. The cross hairs indicate the centre of rotation of the plate. The circle indicates the beam waist w_0 of the detection mode as used in the experiment. (b) Microscope image of the central area. (c) Image of the Escher phase plate, which is back-illuminated by a Gaussian beam with a beam waist w_0 of the same size as indicated in (a). The cross hairs indicate the centre of mass of the Gaussian beam, to which the rotation centre of the phase plate should be aligned with μm precision. Figure (a) after M.C. Escher's "Vissen" © the M.C. Escher Company BV-Baarn-the Netherlands, www.mcescher.com, used with kind permission.

7.4 Relevant parameters and coincidence probability

As mentioned above, two new experimental parameters come into play when allowing the phase plates to have additional radial phase content, being the choice of the rotation centre and the detection beam waist w_o . Furthermore, at variance with the experiments discussed in Chapters 4 and 7, it turns out that we have to take the spatial properties of the down-conversion source into account explicitly when dealing with our Escher phase plates. In the following, we discuss how the choice of (i) rotation centre, (ii) detection beam waist and (iii) bandwidth of the source affect the coincidence probability and the corresponding Shannon dimensionality.

(i) Unlike purely angular phase plates (like, for instance, the multi-sector phase plates discussed in the Chapter 7), the Escher plate does not have a natural centre around which to rotate. We thus have to *select* a centre. In Fig. 7.1(a) and (b), the location of the plate centre we opted for in our experiments is indicated by the cross hairs. A different choice of the plate centre, however, has an immediate impact on the coincidence probability. In Fig. 7.2(a), we plot the calculated coincidence-probability curves $P(\alpha - \beta)$ (see Eq. (7.3)) for two different choices of alignment of the two plate plates. The blue solid curve corresponds to correct alignment according to our definition from Fig. 7.1. The red dashed curve is obtained when displacing the rotation centre of the plates by $20 \mu\text{m}$ to the top. We observe that the exact details of the coincidence curve depend on the transverse positioning of the phase plates. The dimensionality corresponding to these two coincidence curves varies from $D = 8.3$ for our experimental parameters (blue solid) to $D = 9.6$ (green dotted).

(ii) Exploiting the radial degree of freedom, the beam size w_o of the detection modes has become a critical parameter. In Fig. 7.2(b), we plot the calculated coincidence-probability curves for various beam sizes w_o . The blue solid curve corresponds to $w_o = w_{\text{exp}} = 800 \mu\text{m}$, the value we used in our experiments. The red dashed and green dotted curves correspond to $w_o = 0.8w_{\text{exp}}$ and $w_o = 1.2w_{\text{exp}}$, respectively. We observe that the shape, modulation depth and peak width of the curve depend sensitively on w_o . As a result, also the Shannon dimensionality varies significantly, ranging from $D = 8.3$ for our experimental parameters (blue solid) to $D = 6.3$ (red dashed) and 10.1 (green dotted).

(iii) The spatially-entangled photon pairs are experimentally created by means of parametric down conversion in a 2 mm nonlinear periodically-poled KTP crystal (see Fig. 4.1 from Chapter 4). In Chapter 2, we experimentally investigated the emission of such a down-conversion source. Our analysis led us to conclude that the detailed spatial structure of the emission could safely be ignored, since our source generates an abundant number of entangled modes compared to the modal bandwidth of our *angular*-phase-plate analysers (concisely expressed in the statement that the Schmidt number of the source $K \gg D_{\text{eff}}$). In the present discussion regarding the *Escher* phase plates, however, this statement is no longer justified, as we will now argue. In Fig. 7.3(a), we show a measurement of the angular emission profile of our periodically-poled KTP source, having a Schmidt number of order $K_{\text{exp}} \simeq 450$. The angular-acceptance profile of the detection mode is shown in Fig. 7.3(b). Although the total angular spread of the emitted down-conversion light is larger than the angular spread of the detection mode, the intensity of the emission cannot be considered uniform for those angles probed by the analyser. This means that we have to take the spatial structure of the emission, contained in $\Psi(\mathbf{r}_1, \mathbf{r}_2)$, explicitly into account.

Figure 7.2(c) shows how the calculated coincidence probability depends on the number of

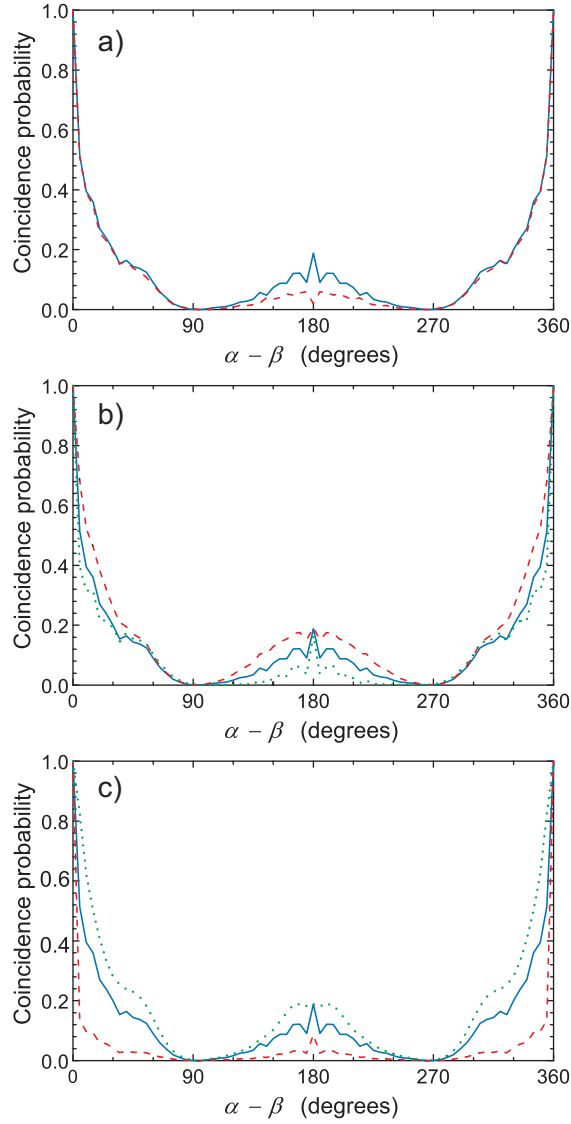


Figure 7.2: **Calculated normalised coincidence-probability curves for various experimental parameters.** (a) Dependence on the transverse position of phase plates' rotation axis for alignment according to the convention as shown in Fig. 7.1(b) (blue solid); and a $20\ \mu\text{m}$ vertical displacement to the top (red dashed). (b) Dependence on the radial beam size w_o of the two detection modes for the experimental value $w_o = w_{\text{exp}} = 800\ \mu\text{m}$ (blue solid); $w_o = 0.8w_{\text{exp}}$ (red dashed); and $w_o = 1.2w_{\text{exp}}$ (green dotted). (c) Dependence on the modal content of the two-photon field for our experimental conditions with Schmidt number $K = K_{\text{exp}} \sim 450$ (blue solid); an ideal source with $K \rightarrow \infty$ (red dashed); and $K \rightarrow K_{\text{exp}}/2$ (green dotted). The blue solid curves in (a)-(c) are identical and based on the parameters used in the experiment.

7. BEYOND ANGULAR ENTANGLEMENT

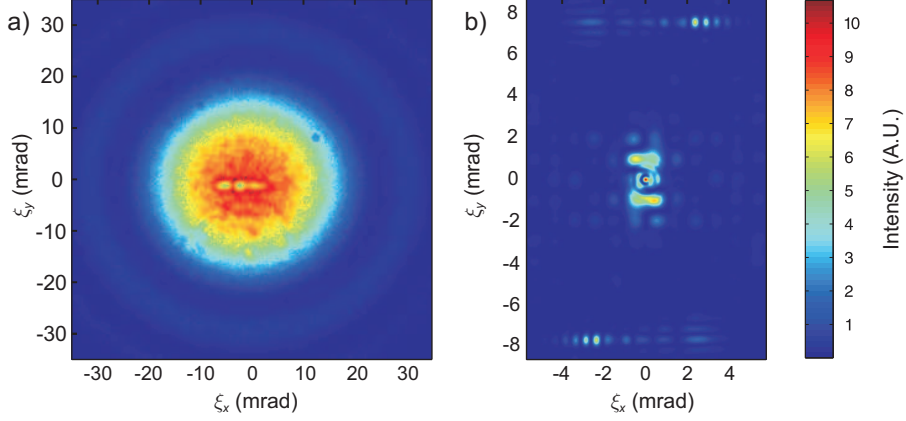


Figure 7.3: **Angular intensity profiles of down-conversion source and detection modes.** (a) Measured angular emission profile of the PPKTP down-conversion source (crystal length 2 mm, pump beam waist $250\ \mu\text{m}$). (b) Measured angular acceptance profile of an Escher phase plate analyser.

generated entangled modes. The blue solid curve is calculated for our experimental conditions corresponding to Fig. 7.3(a), yielding $D = 8.3$. The red dashed curve represents the case of an ideal source, *i.e.*, a source that emits an infinite number of entangled modes ($K \rightarrow \infty$). This curve is remarkably different and the dimensionality increases dramatically to $D = 25.3$. The green dotted curve, on the contrary, demonstrates that the dimensionality drops to $D = 5.7$ when the number of generated modes is reduced by a factor of 2 compared to our experimental settings ($K \rightarrow K_{\text{exp}}/2$).

7.5 Experimental results

In the following, we present our experimental results. Figure 7.4 shows the experimental coincidence curve as a function of the relative orientation $\alpha - \beta$ between the two Escher phase plates. Around $\alpha = \beta$, we record a maximum coincidence rate of $P_{\text{max}} = 4.8 \times 10^3\ \text{s}^{-1}$, compared to an (orientation-independent) single-count rate of $S_c = 1.6 \times 10^5\ \text{s}^{-1}$ at either detector. The minima of the curve around $\alpha - \beta = 90^\circ$ and 270° are deep and measure $\sim 10\ \text{counts s}^{-1}$ only. We corrected for an accidental coincidence rate of $S_c^2 \tau = 69\ \text{s}^{-1}$, arising due to the finite coincidence detection window $\tau = 2.7\ \text{ns}$. The solid curve represents the theoretical calculation based on Eq. (7.3), using the experimental values for the beam waist w_0 and the phase plate centre discussed in Fig. 7.2. The spatial structure of the entangled state $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ was explicitly taken into account. We observe that the experimental data and theoretical model are in reasonable agreement. Deviations between the two are largest around $\alpha - \beta = 180^\circ$. This region, with reference to Fig. 7.2, is most sensitive to experimental deficiencies. The Shannon dimensionality we obtain from the experimental data set amounts to $D_{\text{exp}} = 5.5$, whereas the theoretical model yields $D_{\text{th}} = 8.3$.

It is natural to wonder why the Shannon dimensionality D has this low value. Would it not be reasonable to expect that D would be much increased by using phase plates that have

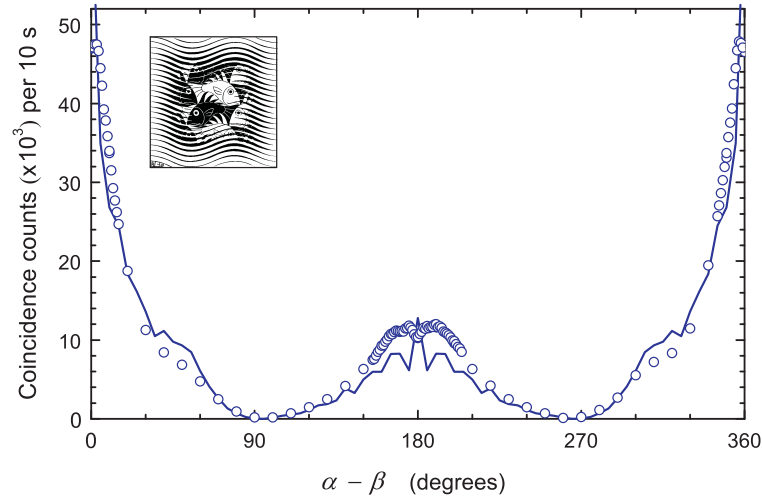


Figure 7.4: **Experimental coincidence curve.** Measured coincidence count rate as a function of the relative orientation of the two Escher phase plates. Circles denote experimental data; the solid curve is a theoretical calculation.

complex spatial structure in both the radial and azimuthal directions (rather than just the azimuthal direction)? The answer to this question is unambiguously positive. However, the phase plates that we use in the current experiment are, when small-scale details are discarded, much alike the half-sector phase plates as used in Chapter 4. Such plates yield $D = 3$. We thus have to conclude that our choice of phase-plate design, although artistically attractive, is far from optimal from the point of view of exploiting the second transverse dimension of the phase plate.

7.6 Conclusions

Exploiting phase plates that carry 2-D structure, we have explored the possibility to harness spatial entanglement of two photons beyond the angular regime. As such, our experimental results constitute a first step towards the combined use of both radial and azimuthal degrees of freedom in bipartite transverse-mode entanglement.

7. BEYOND ANGULAR ENTANGLEMENT

Atmospheric turbulence and optical propagation

8.1 Introduction

In this Chapter, we present an outline of the theory of light propagation through a turbulent atmosphere. Historically, this topic has received much attention from the astronomy community, as the quality of night-sky observations is degraded by atmospheric perturbations. As early as the 1600s, Christiaan Huygens noted when observing the sun that “that very small inequality on his Surface, which is discovered by the Telescopes [...], which makes Men fancy they see boiling Seas and belching Mountains of Fire, is nothing but the trembling Motion of the Vapours our Atmosphere is full of near the Earth; which is likewise the Cause of the Stars twinkling” [182]. Since the days of Huygens, the field of astronomy has witnessed tremendous advance in observation techniques. Nowadays, large ground-based telescopes are often equipped with adaptive optics in order to unwrap image distortions and retrieve the unperturbed light from distant sources [102, 183, 184].

As of late, the emerging field of (quantum) optical communications has triggered renewed interest in atmospheric turbulence, because it hinders the fidelity of information transfer across free-space channels. It is in this context that we will present a study on the suitability of orbital-angular-momentum entanglement for free-space quantum communication in Chapter 9. In the present Chapter, we set the stage for this work. We present a simple atmospheric model and treat the basics of the Kolmogorov theory of turbulence. Next, we introduce an optical description of the loss of coherence this inflicts on light beams. We will establish a connection to Chapter 9 where applicable. Finally, we compare different techniques to produce controlled and realistic atmospheric turbulence. In particular, we discuss the characteristics of our home-built turbulence generator.

8.2 Kolmogorov turbulence

Whirled up through meteorological processes, the earth's atmosphere is in constant motion. Large scale wind flows connect meteorological zones of low and high pressure. Under turbulent conditions, such flows turn unstable, as local fluctuations of the mean velocity will cause

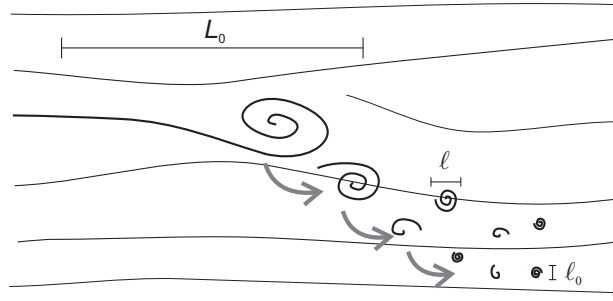


Figure 8.1: **Illustration of the emergence of turbulence in the atmosphere.** Large-scale wind flows of size L_0 turn unstable. The resulting eddies of dimension ℓ sequentially disintegrate, a process that continues down to the damping scale ℓ_0 .

the current to break down into a cascade of eddies. Here, we give a short overview of the Kolmogorov model of turbulence [185, 186].

Let us start out with a laminar wind field that we describe macroscopically in terms of its velocity v , the kinematic viscosity η and its typical length scale L_0 . Turbulent motion is a stochastic process in the sense that density and velocity fluctuations emerge randomly. Now suppose a fluctuation of the velocity v_ℓ occurs in a domain of size ℓ within this field. The kinetic energy (per unit mass) of this fluctuation is equal to v_ℓ^2 , and the typical time scale attached to its occurrence is of order $\tau = \ell/v_\ell$. Thus the energy per unit time transported by this fluctuation can be estimated at

$$\mathcal{E} \sim v_\ell^2/\tau = v_\ell^3/\ell. \quad (8.1)$$

On the other hand, the occurrence of velocity gradients, given by v_ℓ/ℓ , cause different domains to shear. This results in viscous damping. We find that the amount of energy per unit time ε that is dissipated from this fluctuation is given by

$$\varepsilon \sim \eta v_\ell^2/\ell^2. \quad (8.2)$$

Obviously, the fluctuation can only survive if $\mathcal{E} > \varepsilon$. In a more formal formulation, this is expressed in the condition that the parameter $\text{Re} = v_\ell \ell/\eta$, also known as the Reynolds number, is sufficiently large [187].

If the Reynolds number is sufficiently large ($\text{Re} \gtrsim 4000$), the fluctuation will in turn be unstable and break down into structures of smaller scale. Extrapolating this reasoning, we infer that the energy is transferred along a whole cascade of eddies of progressively decreasing size, down to a length scale ℓ_0 at which energy is dissipated. Hence, the atmosphere shows turbulent behaviour on length scales between ℓ_0 and L_0 , known as the inner and outer scales of turbulence, respectively [186]. Typical values for the outer scale amount to 20 m, whereas the inner scale is of order 1 mm. An illustration of the emergence of turbulence in the atmosphere is given in Fig. 8.1.

Because of the stochastic nature of atmospheric turbulence, any description is necessarily statistical. Consider the fluctuations of some physical property $f(\boldsymbol{\rho})$ at a point $\boldsymbol{\rho} = (x, y, z)$. Examples of such properties are the aforementioned velocity, or the density. Its statistical char-

acteristics are conveniently described in terms of the *structure function* [185]

$$D(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \langle [f(\boldsymbol{\rho}_1) - f(\boldsymbol{\rho}_2)]^2 \rangle, \quad (8.3)$$

where $\langle \dots \rangle$ denotes averaging over time. This quantity plays a pivotal role in the theory of turbulence, and is often more expedient than the related correlation function

$$C(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \langle [f(\boldsymbol{\rho}_1) - \langle f(\boldsymbol{\rho}_1) \rangle] [f(\boldsymbol{\rho}_2) - \langle f(\boldsymbol{\rho}_2) \rangle] \rangle. \quad (8.4)$$

Under the fair assumption that the turbulence is homogeneous (*i.e.*, the same at all locations) and isotropic (*i.e.*, the same in all directions), one can show that $D(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = D(|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|)$. Roughly speaking, the structure function characterises the strength of fluctuations of $f(\boldsymbol{\rho})$ with a length scale comparable to $|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|$ [188].

In order to find an explicit expression for the structure function $D_v(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$ for the velocity $v(\boldsymbol{\rho})$, we make the following two observations. First, the dominant contribution to the velocity difference between points $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ comes from eddies of size $\ell \simeq |\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2|$, due to the stochastic nature of the perturbations. Second, we note that the dissipation of structures of length scales $\ell > \ell_0$ is marginal. Therefore, the energy they receive from larger perturbations upstream the cascade must be almost the same as the energy they transfer downwards, and Eq. (8.1) must thus hold at all length scales but for the smallest. Based on these observations, we then find that

$$D_v(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) \propto (|\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2| \mathcal{E})^{2/3}, \quad (8.5)$$

This result is known as “Kolmogorov’s two-thirds law” [185] and can also be obtained through a dimensional analysis.

8.3 Propagation of light through a turbulent atmosphere

8.3.1 Phase structure function

We will now consider the propagation of light through such a turbulent medium [188–190]. We shall restrict ourselves to the regime of weak turbulence, meaning that we take into account weak phase perturbations only, which occur due to variations of the atmospheric density along the beam trajectory. Consequences of phase perturbations leading to such effects as scintillation are a second-order effect and can be ignored as long as diffraction arising from the phase aberrations is negligible. This model is a good approximation for most applications and is supported by a variety of experimental characterisations of the atmosphere [191–193]. It is further corroborated by the fact that adaptive optics works.

In this regime, the cumulative effect of the turbulence along the entire propagation path can be treated as a phase perturbation $\phi(\mathbf{r})$ at the output plane, where $\mathbf{r}_i = (r_i, \theta_i)$ are transverse polar coordinates (see Fig. 8.2). This constitutes a significant conceptual simplification.

We are interested in the stochastic time-evolution of the phase correlation between two points $\mathbf{r}_1$ and $\mathbf{r}_2$ in the output plane. This can be formalised in terms of the time-averaged coherence function

$$C(\mathbf{r}_1, \mathbf{r}_2) = \langle e^{i\phi(\mathbf{r}_1) - i\phi(\mathbf{r}_2)} \rangle, \quad (8.6)$$

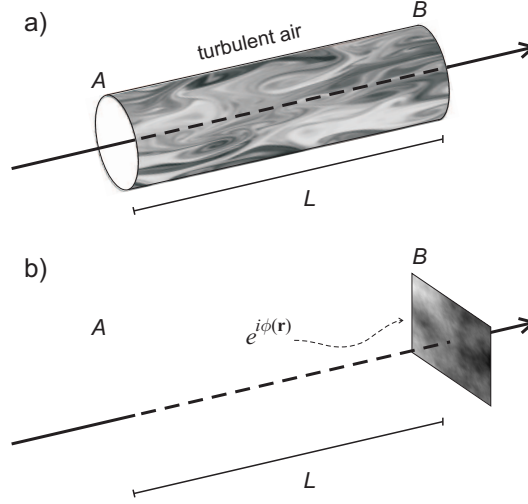


Figure 8.2: **Beam propagation through turbulent air.** In the weak turbulence regime, phase perturbations $\phi(\mathbf{r})$ are taken into account, but amplitude perturbations, which are of second order, are neglected. (a) The cumulative effect of turbulence along the entire propagation length L can be reduced to (b) a single phase operation at the output plane B .

where $\langle \dots \rangle$ denotes averaging over time. By virtue of the central limit theorem, which states that for Gaussian random processes one can approximate $\langle e^{ix} \rangle = e^{-\frac{1}{2}\langle |x|^2 \rangle}$, Eq. (8.6) can be rewritten as

$$\langle e^{i\phi(\mathbf{r}_1) - i\phi(\mathbf{r}_2)} \rangle = e^{-\frac{1}{2}\langle [\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2)]^2 \rangle} = e^{-\frac{1}{2}D_\phi(|\mathbf{r}_1 - \mathbf{r}_2|)}, \quad (8.7)$$

where $D_\phi(|\mathbf{r}_1 - \mathbf{r}_2|)$ is the structure function for the phase (see Eq. (8.3)). The final objective is thus to find an explicit expression for the phase structure function.

The phase difference accumulated over a propagation length L between two parallel trajectories at a distance $|\mathbf{r}_1 - \mathbf{r}_2|$ from each other is simply given by

$$\phi(\mathbf{r}_1) - \phi(\mathbf{r}_2) = k \int_0^L [n(\mathbf{r}_1, z) - n(\mathbf{r}_2, z)] dz, \quad (8.8)$$

where $k = 2\pi/\lambda$. This reveals that the structure function for the phase is intimately related to that of the refractive index $n(\mathbf{r})$. Indeed, it can be shown that the structure functions of both the density and the refractive index follow a similar two-thirds power law as derived for the velocity in Eq. (8.5) [188]. Exploiting this knowledge, one may derive the structure function for the phase, which is conventionally written in the following form [188, 190, 194]

$$D_\phi(|\mathbf{r}_1 - \mathbf{r}_2|) = 6.88 \left(\frac{|\mathbf{r}_1 - \mathbf{r}_2|}{r_0} \right)^{5/3}. \quad (8.9)$$

We observe that the phase structure function goes with the five-thirds power of the distance between two points, a corollary of the Kolmogorov assumptions. The distance r_0 , named the

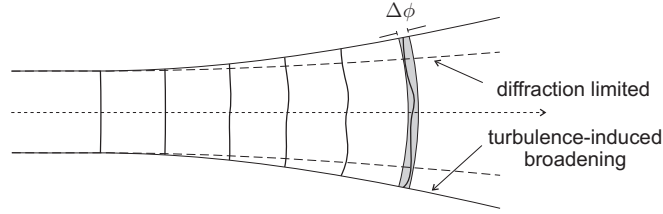


Figure 8.3: **Propagation of a diffraction-limited beam through the turbulent atmosphere.** Under influence of turbulent scattering, the smooth wavefront of the light beam becomes deformed. The directionality of the beam is reduced, causing a wave-vector spread and a broadening of the beam when averaged over time. The time-averaged phase front is ill-defined, having a phase variance $\Delta\phi$ when averaged across the beam.

Fried parameter after its introducer [189, 194], manifests the optical consequences of the induced phase distortions: it plays the role of a correlation length, and the constant 6.88 is chosen such that r_0 is that distance over which the wavefront is distorted by approximately 1 radian of root-mean-square phase difference. That is, in the absence of turbulence $r_0 \rightarrow \infty$, but when turbulence becomes stronger, the spatial coherence is reduced and hence r_0 shortens.

It is evident that the Fried parameter plays an important role in the field of astronomical observations. While telescopes have become ever bigger to increase their light gathering ability, the image resolution they obtain has not kept pace. The resolution is limited by atmospheric wavefront degradation and it is the Fried parameter that determines the largest aperture size that ensures a diffraction-limited imaging. For astronomical observations at visible wavelengths, typical values of the Fried parameter range between 3 cm at sea level to 30 cm at high altitude [183, 195].

Finally, we note that the model presented here constitutes the canonical Kolmogorov model of turbulence, which behaves similarly at all length scales (i.e., inner scale $\ell_0 \rightarrow 0$ and outer scale $L_0 \rightarrow \infty$). Several refinements of the model have been made to incorporate asymptotic behaviour for finite inner and outer scales [196, 197], but in many cases Eq. (8.9) suffices. In Section 8.4.2 we will discuss the validity of the model for the experimental configurations used in Chapter 9.

8.3.2 Beam propagation

We now turn to the problem of beam propagation through a turbulent atmosphere. Consider a Gaussian beam with a beam waist w_0 . The refractive index variations along its trajectory give rise to anomalous refraction, i.e., locally producing wavefront deflections and lensing effects [198]. These random distortions vary over time and, as a consequence, the time-average wave-vector spread of the beam widens. This results in a net beam broadening (see Fig. 8.3).

From the beam broadening we can determine the relation between the Fried parameter and the beam waist,

$$\frac{w_0}{r_0} = \frac{\sqrt{(w_{le}/w_{dl})^2 - 1}}{3.0}, \quad (8.10)$$

with w_{dl} and w_{le} the $1/e$ far-field radius of the diffraction-limited beam and long-exposure

8. ATMOSPHERIC TURBULENCE AND OPTICAL PROPAGATION

broadened beam, respectively [199]. See Fig. 9.2 for an experimental example of turbulence-induced broadening.

For a horizontal line of sight parallel to the earth's surface, the atmosphere is relatively uniform. For beam propagation in the horizontal direction the Fried parameter can be expressed as [200]

$$r_0 = 3.02 \left(k^2 L C_n^2 \right)^{-3/5}. \quad (8.11)$$

It thus depends on the wavelength of the light via $k = 2\pi/\lambda$, the propagation length L and the turbulence conditions. The latter are contained in the refractive-index structure constant C_n^2 , which is a measure of the strength of the refractive-index inhomogeneities. Typical values of C_n^2 range between 10^{-17} and $10^{-12} \text{ m}^{-2/3}$ for very mild and quite severe conditions, respectively. C_n^2 is largest within a few meters above the ground and drops rapidly with increasing altitude. As a numerical example, let us assume moderate inhomogeneities, $C_n^2 = 10^{-14} - 10^{-15} \text{ m}^{-2/3}$, and a visible wavelength of 550 nm. We find a Fried parameter of 1 – 4 cm for a propagation distance of 10 km. This number is larger than the size of our pupils, which explains why, under normal conditions, we do not notice turbulence effects with our bare eyes. Furthermore, Eq. (8.11) shows that the structure constant and the propagation length can be exchanged to yield the same value of the Fried parameter. We will use this property in the next Chapter to scale up our experimental conditions to real-life proportions, and estimate the outdoors propagation length feasible for quantum communication.

For a vertical line of sight from ground level through the entire column of the atmosphere, the structure constant obviously depends on the altitude: $C_n^2 = C_n^2(z)$. For this case, Eq. (8.11) can be generalised to

$$r_0 = \left[0.423 k^2 z \int_0^1 d\xi \xi^{5/3} C_n^2(\xi z) \right]^{-3/5} \quad (8.12)$$

Typically, reported values of the Fried parameter for vertical propagation through the atmosphere are of the order of 3–30 cm, depending on weather conditions and the elevation of the observation station [183, 195].

At any given instant in time, the wavefront at the output plane of a turbulent trajectory may well be perturbed, but it remains well-defined. The time-averaged effect of turbulence, in contrast, cannot be described by a pure phase imprint only, as can be seen from the coherence function for the phase (see Eqs. (8.7) and (8.9)). This suggests that the spatial coherence of the beam is (partially) reduced.

One can quantify the loss of coherence in terms of the variance $\Delta\phi$ of the phase fluctuations in the beam. When naively calculating the phase variance along a light path, however, one will find that it tends to blow up. Nonetheless, this has little physical consequences; what matters is the variance of the phase *difference* between two parallel paths (see Eq. (8.8)). This is the case because the common phase component across the beam does not degrade the coherence and can thus be omitted. The variance of the phase difference is finite and well-defined [201].

The phase variance can be obtained from the beam broadening discussed above, or more precisely, from the Strehl ratio,

$$S = \frac{I_T}{I_0}, \quad (8.13)$$

which is experimentally accessible in a straightforward fashion; I_T and I_0 are simply the peak

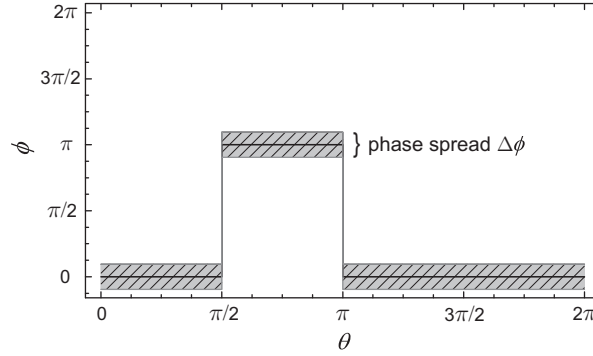


Figure 8.4: **Phase blurring of an analyser mode under influence of atmospheric turbulence.** The black line gives the initial phase profile of an analyser equipped with a quadrant phase plate (inset). Due to turbulence, the time-averaged phase profile is no longer well-defined, but has a certain phase spread instead.

intensities of the turbulence-broadened and diffraction-limited beams monitored in the far-field (see Fig. 9.2). The phase variance $\Delta\phi$ is given by [202]

$$S = e^{-(\Delta\phi)^2}. \quad (8.14)$$

Typical values of the Strehl ratios for the experiments described in Chapter 9 lie in the range $0.3 \lesssim S < 1$, the lower value being reached at the maximum turbulence strength at our disposal. This corresponds to a phase variance of $0 < \Delta\phi \lesssim 2\pi/7$.

It is instructive to consider the influence of this “phase smearing” on the orbital-angular-momentum detection states used in the next Chapter. In Fig. 8.4, we plot the azimuthal phase profile of the detection state for an analyser equipped with a quadrant phase plate. The black line represents the unperturbed phase profile, being a binary function between $\phi = 0$ and $\phi = \pi$. Due to the turbulence, the time-averaged wavefront gets spread out; the phase is no longer well-defined, but rather has a certain phase spread corresponding to the variance $\Delta\phi$. This fuzziness is tantamount to a loss of modal resolution of the analyser.

8.4 Laboratory implementations of turbulence

When studying the effect of atmospheric turbulence on beam propagation, one would be tempted to simply launch a light beam through the atmosphere. There are several reasons, however, to work with artificial turbulence, the primary one being the ability to control and tune the turbulence characteristics [203]. For instance, one may want to simulate various turbulence strengths, propagation lengths or altitudes. There exist two methods to produce tailored turbulence: by means of 2-D random phase screens and by means of 3-D turbulence cells.

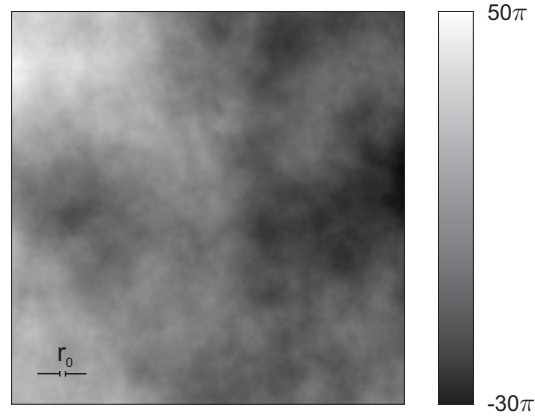


Figure 8.5: **Turbulence-induced phase distortion.** Example of a random phase screen profile, the phase distribution of which follows Kolmogorov statistics. Black and white shades represent various levels of phase imprint between -30π and 50π . The bar shown in the bottom left gives an estimate of the Fried parameter. This pattern was designed with software from the Max-Planck-Institut für Astronomie, Heidelberg [213].

8.4.1 Random phase screens

The first method makes use of a phase mask that carries a random phase profile, as would be imparted by the atmosphere. Advanced computational techniques exist to design realistic Kolmogorov phase patterns (see Fig. 8.5) [204]. These patterns can then be fabricated as a static diffractive element [205–207], or be printed on phase modulating devices such as spatial light modulators [208] or deformable mirrors [209]. To mimic the dynamics of the atmosphere, the phase pattern is continuously refreshed (dynamic devices), or rotated through the beam at the typical wind speed (static screens). The validity of the latter *modus operandi* is established by the *hypothesis of frozen turbulence* proposed by G. I. Taylor [210]. This hypothesis relates the turbulent fluctuations in time to those in space, and assumes their interchangeability. Poor Kolmogorov statistics due to a repetitive reoccurrence of the rotating pattern can be mitigated by using two static phase screens that rotate at slightly different frequencies.

With current technology, it is possible to micro-machine these random phase screens at small dimensions, which has enabled them to become a customary tool to test miniaturised adaptive optics eventually meant for large ground-based telescopes [211, 212].

8.4.2 Turbulence cells

In the second method, real, physical turbulence is created in a mixing chamber, by intermixing hot and cold air to bring about random refractive-index variations [203]. Alternatively, the air can be replaced by a liquid. This is particularly helpful for the creation of strong turbulence, because the refractive-index contrast in fluids can be much larger [214]. A major advantage of turbulence cells is that the generated turbulence naturally follows a Kolmogorov distribution. Full control of the desired inner and outer scales, however, is usually quite challenging.

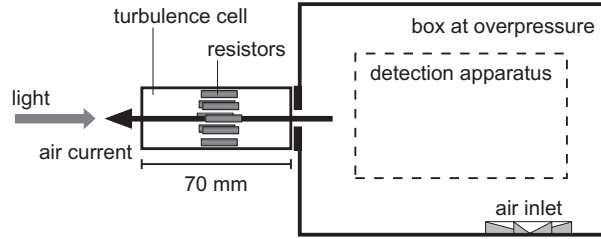


Figure 8.6: **Turbulence cell used in our experiments.** The turbulence cell is composed of a glass tube (length 70 mm, diameter 26 mm) inside which several resistors are mounted, producing up to 60 W of heat. A jet of cold air is directed straight into the turbulence cell. Hence, hot and cold air are intermixed inside the tube, bringing about dynamic turbulence. The cold air emanates from a closed box that is maintained at overpressure and contains the detection apparatus. Light transits the turbulence cell and enters the box through the same aperture that serves as the outlet of cold air.

The results presented in the Chapter 9 were performed with a home-built turbulence cell. The cell is made of a 7 cm long, 26 mm inner diameter glass tube, containing 8 resistors that produce up to 60 W of heat (see Fig. 8.6). A gentle flow of room temperature air is driven straight through the tube. We can tune the turbulence strength by varying the heating power and air flow through the tube.

The airflow is generated as follows: our detection apparatus is enclosed in a box that is kept at overpressure. A small aperture in one side of the box allows the optical beam to enter and the air to leave it. The use of this box as a source of cold air has two advantages. First, no heat can leak into the box and cause the optical equipment to lose alignment due to thermal drift. Second, the optical beam and cold air jet propagate along the same axis, avoiding directional inhomogeneities and asymmetric convection currents. Residual beam deflections in the vertical direction, which easily emerge due to upward movement of hot air, can be nullified by setting a lower limit to the air flow and optimising the beam pointing through the cell.

For reasons of experimental convenience, the beam diameter in the experiments in Chapter 9 is set to 2 mm. As this size is comparable to the inner scale of turbulence (as mentioned above typically of order 1 mm), we cannot expect to realise a turbulence strength w_0/r_0 much bigger than 1. In the experiments, we restricted ourselves to controllable and reproducible turbulence strengths between $0 < w_0/r_0 < 0.65$.

Both the inner and outer scale of turbulence are finite in our turbulence cell. When modeling our experiments, we should therefore question the validity of the most basic phase structure function described by Eq. (8.9), which was derived for $\ell_0 \rightarrow 0$ and $L_0 \rightarrow \infty$. From the literature, it is known that the effect of a finite inner scale ℓ_0 on the phase structure function is usually modest [197]. A finite outer scale L_0 , on the contrary, may have considerable influence. We therefore estimate the outer scale of our system. The size of the largest eddies that can form is constrained by the size of the turbulence cell and a conservative estimate would therefore be that $L_0 = 26$ mm, being the tube diameter. (In fact, the air is blown out of the tube and remains turbulent at some distance from it, suggesting that the outer scale is probably larger). The derivation of the theoretical curves in Fig. 9.4 from Chapter 9 is based on Eq. (8.9). We see that for large w_0/r_0 the canonical Kolmogorov theory indeed slightly overes-

timates the turbulence in our system. Using advanced Kolmogorov modeling for finite outer scale [196, 197], however, we find that the value $L_0 = 26$ mm underestimates the turbulence by a factor of 2. The agreement between experimental and theoretical curves is sufficiently good to warrant our use of the canonical Kolmogorov theory.

8.5 Conclusions

We have outlined the Kolmogorov model of turbulence and introduced the phase structure function, which describes the loss of coherence an optical beam experiences when propagating through a turbulence medium. The relevant parameter in the model is the Fried parameter, which plays the role of a coherence length. Furthermore, we have discussed the home-built turbulence generator we will use in the next Chapter and examined its properties. It produces Kolmogorov-like turbulence up to considerable strengths and thus forms an attractive system to study orbital-angular-momentum transport through the atmosphere.

Transport of orbital-angular-momentum entanglement through a turbulent atmosphere

We demonstrate experimentally how orbital-angular-momentum entanglement of two photons evolves under the influence of atmospheric turbulence. Experimental results are in excellent agreement with our theoretical model, which combines the formalism of two-photon coincidence detection with a Kolmogorov description of atmospheric turbulence. We express the robustness to turbulence in terms of the dimensionality of the measured correlations. This dimensionality is surprisingly robust: scaling up our system to real-life dimensions, a horizontal propagation distance of 2 km seems viable.

Based on: B.-J. Pors, C.H. Monken, E.R. Eliel, and J.P. Woerdman, arXiv:0909.3750v1 [quant-ph] (2009); under review for Optics Express

9.1 Introduction

Quantum communication by means of entangled photon pairs allows for an intrinsically secure transmission of data, by distributing the pairs via a free-space or fiber channel to distant parties [10]. Most popular is polarisation entanglement, which has dimensionality 2. Higher dimensionalities can be achieved using orbital-angular-momentum (OAM) entanglement [40, 215] or energy-time entanglement [22, 60]; this route provides for a larger channel capacity and an increased security against eavesdroppers [17, 19]. However, the performance of a real-world high-dimensional quantum channel is an open issue. Here, we address this issue for the case of OAM entanglement distribution via a free-space channel.

For quantum communication to be of practical relevance, it is imperative that the entanglement between the photons carrying the information survives over a reasonably long propagation distance. Entanglement distribution over fiber-based transmission lines has proven to be feasible over distances over a hundred kilometers [53–55]. However, the use of free-space links is needed when considering such purposes as airplane and satellite quantum links or hand-held communication devices [56–58].

The increased quantum-channel capacity that is available when encoding the information in the OAM of the entangled photons was argued to be severely limited in a practical free-space link, due to atmospheric turbulence that causes wavefront distortions. Several studies have addressed this aspect [63–71], but there is no unanimity on exactly *how* sensitive OAM entanglement is to atmospheric perturbations. So far, no experimental verdict has been obtained to clarify this issue.

In this Chapter, we present the first such experiment. We start with bipartite OAM entanglement of effective dimensionality 6, and demonstrate how the corresponding correlations evolve when one of the photons traverses a turbulent atmosphere, emulated by controlled mixing of cold and hot air. Our experimental results are in excellent agreement with our theoretical model, which combines a Kolmogorov description of atmospheric turbulence with our formalism of bi-photon correlation detection.

9.2 Experimental setting

Our experimental setup is depicted in Fig. 9.1. A PPKTP down-conversion crystal of length 2 mm is pumped by the single transverse mode output of a Kr^+ laser operating at 413 nm. It emits correlated photon pairs with complementary OAM at 826 nm, in a state of the form^{*} [33]

$$|\Psi\rangle = \sum_{m,p} c_{mp} |m, p\rangle |-m, p\rangle. \quad (9.1)$$

Here, $|m, p\rangle$ indicates the Schmidt mode containing one photon with orbital angular momentum $m\hbar$, with p the radial mode index, and we can write $\langle \mathbf{r} | m, p \rangle = \langle r | m, p \rangle e^{im\theta} / \sqrt{2\pi}$. For our source, the total number of entangled azimuthal modes, the so-called angular Schmidt number K , is of order 30 [34]. The correlated photons are spatially separated by a 50/50 beam splitter.

^{*} Within the mode space our analysers have access to, we can safely approximate c_{mp} to be independent of m .

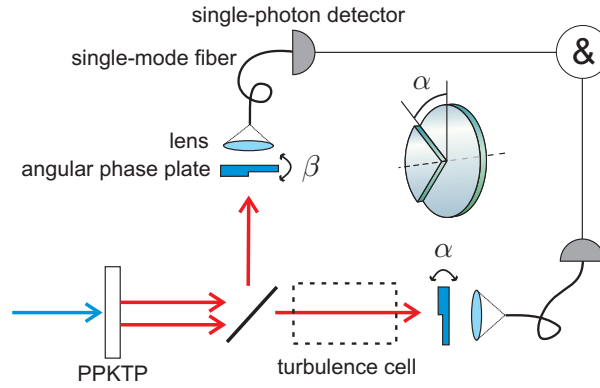


Figure 9.1: **Experimental setup.** A type-I PPKTP crystal emits two frequency-degenerate photons ($\lambda = 826$ nm) that are entangled in their OAM degree of freedom. A beam splitter serves to separate the twin photons spatially. The entanglement is analysed by two angular-phase-plate projectors, variably oriented at α and β , respectively, which are linked to a coincidence circuit. Each phase plate has one elevated quadrant sector with optical thickness $\lambda/2$ (inset). In one of the beam lines we place a turbulence cell.

The entanglement is analysed by means of two state projectors, which are composed of an angular phase plate that is lens-coupled to a single-mode fiber, and a single-photon counter. The two phase plates are identical and carry a purely azimuthal variation of their optical thickness: they have one elevated quadrant sector with an optical thickness that is $\lambda/2$ larger than that of the remainder of the plate (see inset Fig. 9.1). The two phase plates can be rotated around their normals over an angle α and β , respectively. The detection state $|A(\alpha)\rangle$ (or $|B(\beta)\rangle$) of one such analyser is a high-dimensional superposition of OAM modes, the relative phases of which depend on the orientation of the plate. It can be written as $\langle \mathbf{r} | A(\alpha) \rangle = (2/w_0) \exp(-r^2/w_0^2) \sum_m \lambda_m e^{im(\theta+\alpha)}$, where the Gaussian factor describes the fiber mode profile with field radius w_0 , and the summation over the orbital-angular-momentum states describes the phase imprint imparted by the phase plate. When rotating the quadrant phase plate over 2π rad, the analyser scans a mode space of dimensionality $D = 6$ (see Chapter 3).

9.3 Turbulence cell

In one of the beam lines, we place a turbulence cell where cold and hot air are mixed to bring about random variations of the refractive index that vary over time (Fig. 9.2(a)). We can tune the strength of the turbulence by varying the heating power and air flow through the cell. Similar cells have been used as a realistic emulation of atmospheric turbulence [203]. Figure 9.2(b) gives an impression of the cell's functioning: We inject one of the analysers backwards with diode laser light and monitor the beam, which traverses the turbulence cell, in the far field. We do this for two cases; the analyser is equipped with no phase plate (top row), or with the quadrant phase plate (bottom row). We observe that the input beams (left column) become deformed by the refractive index fluctuations, as can be seen when taking a 10 ms snapshot (middle column). Time averaging these fluctuations over 10 s reveals a beam broadening that

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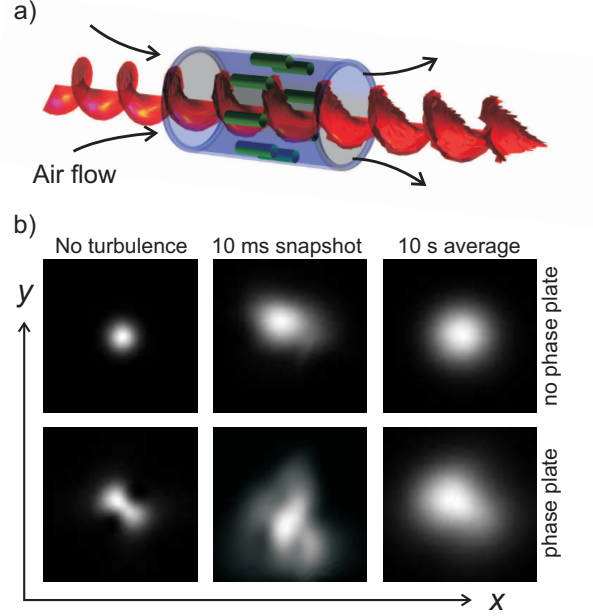


Figure 9.2: **Beam corruption after passage through the turbulence cell.** (a) Impression of how an OAM eigenmode, having a helical wavefront, gets distorted when transiting the turbulence cell. The cell consists of a 7 cm long, 26 mm diameter glass tube, containing several resistors that produce up to 60 W of heat. A gentle flow of room temperature air is driven through the tube. (b) Far-field intensity patterns of the analyser, which is fed backwards with diode laser light at 826 nm. The analyser is equipped with no phase plate (top row), or quadrant phase plate (with its sector aligned along the Cartesian axes) (bottom row). The diffraction limited patterns (left column) get perturbed when turbulence is switched on (middle column): for mild turbulence, the dominant effect is a randomly evolving beam deflection; for the more severe turbulence conditions used here ($w_0/r_0 = 0.65$), the beam profile can get significantly distorted. Taking a 10 s time average reveals an isotropic beam broadening (right column). The apparent asymmetry along the diagonal in the bottom left and right windows is due to the 3% discrepancy of the quadrant phase step from the ideal value of π .

is spatially isotropic (right column).

We describe our cell by the Kolmogorov theory of turbulence [188]. This standard model treats the optical effects of the atmosphere at any moment as a random phase operation $e^{i\phi(\mathbf{r})}$, the time evolution of which follows a Gaussian distribution. It is conveniently described in terms of its coherence function, given by

$$\langle e^{i\phi(\mathbf{r}_1) - i\phi(\mathbf{r}_2)} \rangle_t = e^{-\frac{1}{2} 6.88 \left[\frac{r_1 - r_2}{r_0} \right]^{5/3}}, \quad (9.2)$$

where $\langle \dots \rangle_t$ denotes averaging over time [189]. The relevant parameter in this model is the Fried parameter r_0 , being the transverse distance over which the beam profile gets distorted by approximately 1 rad of root-mean-square phase aberration [189]. In the absence of turbulence $r_0 \rightarrow \infty$, but when turbulence becomes stronger, the spatial coherence is reduced and hence

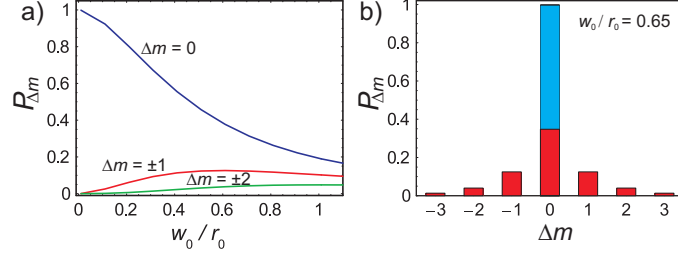


Figure 9.3: **Mode scattering due to turbulence.** (a) Time-averaged survival probability of an analyser's OAM eigenmode $m = m_0$ as a function of turbulence strength (blue). The red and green curves denote turbulence-induced coupling probabilities to neighbouring modes for $\Delta m = \pm 1$ and $\Delta m = \pm 2$, respectively. (b) Time-averaged spreading of the m_0 OAM eigenmode (blue bar) over its neighbours for $w_0/r_0 = 0.65$ (red bars).

r_0 shortens. From the Gaussian beam broadening in Fig. 9.2(b) (top row) we can determine the relation between the Fried parameter and the $1/e$ beam size w_0 ,

$$\frac{w_0}{r_0} = \frac{\sqrt{(w_{le}/w_{dl})^2 - 1}}{3.0}, \quad (9.3)$$

with w_{dl} and w_{le} the $1/e$ far-field radius of the diffraction limited beam and long-exposure broadened beam, respectively [199].

We calculated the effect of Kolmogorov turbulence on a single beam with a mode profile described by $|A(\alpha)\rangle$. The blue curve in Fig. 9.3(a) shows the survival probability of an OAM eigenmode $m = m_0$ upon passing through a turbulent atmosphere as described by Eq. (9.2). The survival probability degrades gradually for increasing turbulence strength. We note that this decay depends on the ratio w_0/r_0 only and not on the specific OAM eigenvalue m_0 , provided that the propagation distance L is small compared to the diffraction length $z_R = \pi w_0^2/\lambda$. Furthermore, the turbulence produces a coupling between the orthogonal OAM modes, leading to a non-vanishing mode overlap between the m_0 eigenmode and its neighbours $\Delta m = \pm 1$ (red) and $\Delta m = \pm 2$ (green). A different perspective on this mode mixing is presented in histogram Fig. 9.3(b), which shows how an OAM eigenmode (blue bar) spreads out over its neighbouring azimuthal modes for $w_0/r_0 = 0.65$ (red bars). We note that normalisation is not preserved, because some intensity is scattered to radial modes that are not sustained by the single-mode fiber. This illustrates the importance of taking into account the radial content of the generated two-photon state and the analysers' detection states when dealing with OAM modes in the presence of turbulence.

Although turbulence acts as a decohering process when time averages are observed, in real time it simply imprints a phase perturbation on the beam. We note that it is possible to fully undo these perturbations if one could monitor and unwrap the wavefront deformations in real time by means of a phase corrector. This can be done using modern adaptive-optics techniques [216].

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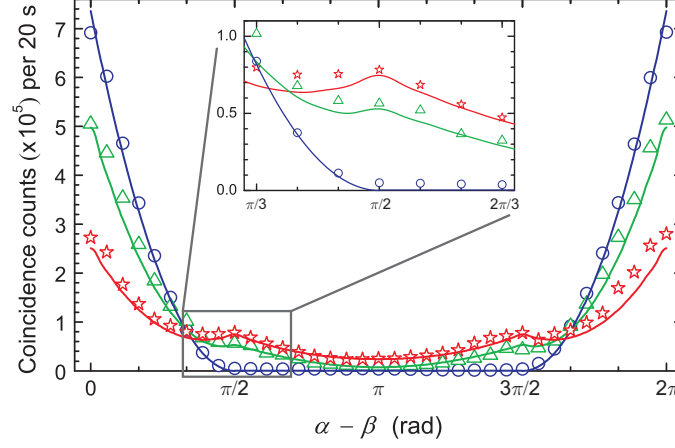


Figure 9.4: **Survival of OAM coincidence curves under influence of turbulence.** Experimental coincidence rates (data points) and theoretical predictions (curves) obtained with two quadrant-sector phase plates for: no turbulence (blue), $w_0/r_0 = 0.30$ (green) and $w_0/r_0 = 0.65$ (red). The inset shows a blow-up of the wiggles around $\alpha - \beta = \pi/2$.

9.4 Results

In the experiment, the phase plates are rotated around their normals, and the time-averaged photon coincidence probability

$$P(\alpha - \beta) = \langle |\langle A(\alpha) | \langle B(\beta) | \hat{S}_A | \Psi \rangle|^2 \rangle_t \quad (9.4)$$

is recorded as a function of their independent orientations. Here, the time-averaged behaviour of the turbulence scattering operator $\hat{S}_A$ working on channel A can be described in terms of its coherence function Eq. (9.2),

$$\langle \hat{S}_A | A(\alpha) \rangle \langle A(\alpha) | \hat{S}_A^\dagger \rangle_t = \int d\mathbf{r}_1 d\mathbf{r}_2 |\mathbf{r}_1\rangle \langle \mathbf{r}_1 | A(\alpha) \rangle \langle A(\alpha) | \mathbf{r}_2 \rangle \langle \mathbf{r}_2 | \langle e^{i\phi(\mathbf{r}_1) - i\phi(\mathbf{r}_2)} \rangle_t. \quad (9.5)$$

Figure 9.4 shows our main experimental results. In the absence of turbulence, we observe a piecewise-parabolic coincidence curve (blue circles), *i.e.*, the coincidence rate follows a parabolic dependence for $|\alpha - \beta| \leq \pi/2$ and is zero elsewhere (see Chapter 4). The coincidence rate depends on the relative orientation of the phase plates only. We have investigated how the coincidence rates evolve for 6 different turbulence strengths, two of them shown in Fig. 9.4: $w_0/r_0 = 0.30$ (green triangles) and $w_0/r_0 = 0.65$ (red stars). The latter strength was also used for Fig. 9.2(b) and 9.3(b). Note that the 20 s integration time used in the experiment assures isotropic sampling of the wavefront fluctuations (see Fig. 9.2(b)). We observe a partial “smoothing” of the coincidence curve, which is excellently described by our theoretical predictions based on Eqs. (9.2) and (9.4), without any fit parameter. The turbulence-induced wiggles at $|\alpha - \beta| = \pi/2$ are reproduced remarkably well (see inset).

We stress that, while the coincidence count rates exhibit this rich behaviour, the single

count rates on both detectors are independent of the orientations of the phase plates and independent of the turbulence strength.

9.5 Discussion

Figure 9.4 shows that the *coherence* of the two-photon state is partly conserved even in the presence of rather strong turbulence. However, the turbulence inevitably damages the *purity* of the quantum state to some degree. Naturally, this damage, or equivalently the mixedness of the state, increases with increasing turbulence strength.

Quantifying entanglement for mixed states is a notoriously hard problem, especially if the entanglement is high dimensional [21, 217, 218]. Nevertheless, for mixed states of two qubits, some mathematical techniques exist to quantify the entanglement [219]. Two recent theoretical studies used this approach to investigate the robustness of entanglement between *two* spatial (OAM) modes against transmission through a turbulent noise channel [71, 220]. In our experiment, however, we are explicitly in the regime of *high-dimensional* OAM entanglement. Consequently, it is near impossible to extract a proper entanglement measure from the experimental data at hand. This notwithstanding, the prospect of a high dimensionality is the very motivation to study OAM entanglement in the first place.

In the following, we therefore take a first step towards analysing mixed high-dimensional entanglement, and apply the techniques developed in Chapter 4 for pure entangled states to the partially mixed states we are dealing with here. We attempt to quantify the robustness of the correlations in terms of the Shannon dimensionality D , as introduced in Chapter 4. It is an operationally defined measure and gives the effective number of modes the combined analysers have access to when scanning over their possible settings, *viz.* the phase-plate orientations. For two identical analysers in the *absence of turbulence*, we can express D in terms of the *pure* detection state operator ρ_A (or ρ_B), where $\rho_A = |A(\alpha)\rangle\langle A(\alpha)|$, as

$$D = \frac{1}{\text{Tr}[(\langle \rho_A \rangle_\alpha)^2]}. \quad (9.6)$$

Here, $\langle \rho_A \rangle_\alpha$ is the density operator obtained by averaging ρ_A over all phase-plate orientations α .

In the *presence of turbulent scattering*, however, the detection state becomes randomly time dependent: $\rho_A = \hat{S}_A |A(\alpha)\rangle\langle A(\alpha)| \hat{S}_A^\dagger = \rho_A(t)$. The relevant detection state operator is therefore not ρ_A , but rather $\langle \rho_A \rangle_t$, *i.e.*, the density operator averaged over time. In general, $\langle \rho_A \rangle_t$ is no longer a single-mode projector, but just a positive operator. In other words, when averaging over the random fluctuations, the detection state becomes multimode. We can attach a -non-utilisable- dimensionality $D^{-1} = \text{Tr}[(\langle \rho_A \rangle_t)^2]$ to this detection operator, which gives the effective number of modes captured by the analyser for fixed orientation α . The mixed nature of the detection operator blurs the analyser's modal resolution when scanning its orientation setting α .

In analogy to Eq. (9.6), the total number of modes captured by the analyser when scanning its orientation setting α is given by $\text{Tr}[(\langle \rho_A \rangle_{t,\alpha})^2]$, where $\langle \rho_A \rangle_{t,\alpha}$ denotes the average of the detection state operator ρ_A over time t and orientation α . However, we need to compensate for the contribution that arises from the degradation of resolution due to turbulence. Therefore,

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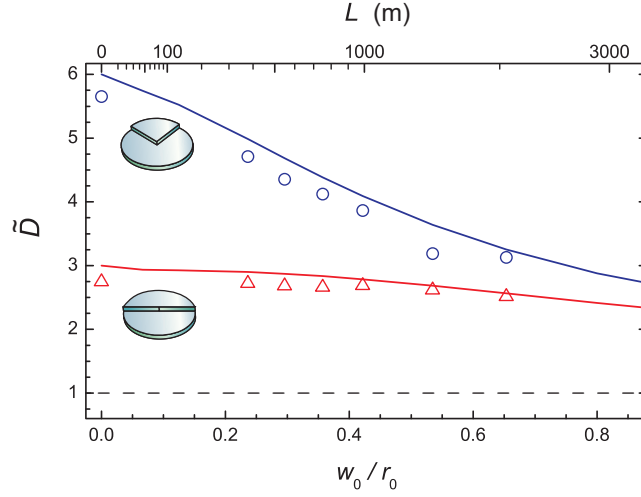


Figure 9.5: **Decay of Shannon dimensionality.** Experimental (data points) and theoretical (curves) dimensionality as a function of turbulence strength, for two quadrant-sector phase plates (circles) and two half-sector phase plates (triangles). The turbulence strength is expressed in the ratio w_0/r_0 (lower abscissa) and in the corresponding real-life propagation distance L (upper abscissa).

in the presence of turbulence, the Shannon dimensionality for *mixed* detection states is written as ^{*}

$$\tilde{D} = \frac{\text{Tr}[(\langle \rho_A \rangle_t)^2]}{\text{Tr}[(\langle \rho_A \rangle_{t,\alpha})^2]}. \quad (9.7)$$

This expression can be understood as being the ratio between the dimensionality of the operator $\langle \rho_A \rangle_{t,\alpha}$ that is *averaged over* α and the dimensionality of $\langle \rho_A \rangle_t$ for *fixed* α . Note that in the limit of no turbulence, this result reduces to Eq. (9.6).

It is worth noting that the numerator in Eq. (9.7) is independent of the specific phase plate in use. It can be shown that its evolution under the action of turbulence follows the survival probability discussed in Fig. 9.3(a). The denominator in Eq. (9.7), on the other hand, does depend on the specific phase plate in use.

Continuing with our naive approach, we can extract $\tilde{D}$ straightforwardly from the experimental coincidence curves in Fig. 9.4. Working in the regime $K \gg D$ and using identical phase-plate analysers in both arms, it can be shown that the numerator in Eq. (9.7) is associated to the maximum coincidence probability, and the denominator in Eq. (9.7) is associated to the average coincidence probability. It then follows that $\tilde{D} = 2\pi N_{\max}/A$, where $N_{\max}$ is the maximum coincidence rate and A is the area underneath the curve.

Figure 9.5 shows how $\tilde{D}$ evolves for increasing turbulence strength according to theory and experiment. In the absence of turbulence, we find an experimental value $D = 5.7$ vs. a theoretical prediction $D = 6$ (see Chapter 4). As the turbulence strength increases, the modal

^{*}We note that in our experiment we have turbulence in one arm only. For this case, the Shannon dimensionality can be generalised as $\tilde{D} = \text{Tr}(\langle \rho_A \rangle_t \rho_B) / \text{Tr}(\langle \rho_A \rangle_{t,\alpha} \langle \rho_B \rangle_\beta)$. It can be shown that this reduces to Eq. (9.7) when one has similar but weaker turbulence in both arms.

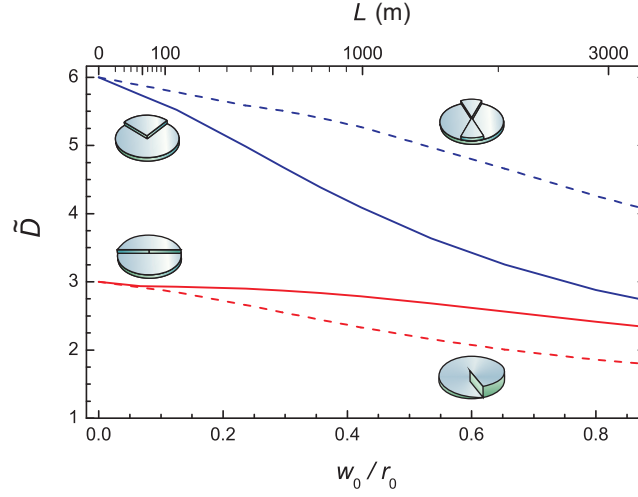


Figure 9.6: **Robustness of the Shannon dimensionality for various OAM superposition states.** Decay of the dimensionality for two quadrant phase plates (blue solid) and two double-octant plates (blue dashed), both of initial dimensionality $D = 6$. Similarly, decay of the dimensionality for half-sector phase plates (red solid) and half-integer spiral phase plates (red dashed), both of initial dimensionality $D = 3$.

resolution of the analysers degrades, constraining the dimensionality to smaller values, ultimately to $\tilde{D} = 1$.^{*} The number of modes is reduced by $\sim 50\%$ to $\tilde{D} = 3.1$ when $w_0/r_0 = 0.65$. Considering the severity of the wavefront distortions (see Fig. 9.2(b)), we conclude that our dimensionality $\tilde{D}$ is surprisingly robust. For comparison, we also plotted our experimental results obtained with two half-sector phase plates, having one semicircle phase shifted by π (see inset Fig. 9.5). In this case we observe that the dimensionality, initially at a value $D = 3$, decays considerably more slowly. This indicates that the resilience to atmospheric turbulence is quite sensitive to the nature of the OAM superposition state, an aspect also noted in Ref. [68].

To further substantiate this observation, we consider two distinct OAM superposition states that have the same zero-turbulence dimensionality, and compute how their dimensionality decays for increasing turbulence strength (see Fig. 9.6). For instance, we compare the decay of the dimensionality for the case $D = 6$, using analysers equipped with (i) *quadrant* phase plates and (ii) phase plates having two opposing *octants* of optical thickness $\lambda/2$ (see inset Fig. 9.6 for the plate profile). These two types of phase plates bear a large similarity; the OAM superposition state corresponding to the double-octant phase plate has an identical OAM eigenvalue distribution as the superposition of the quadrant phase plate, albeit with twice the mode spacing. This double mode spacing reflects the two-fold symmetry of the double-octant plate as compared to the quadrant phase plate.

Fig. 9.6 shows that the dimensionality $\tilde{D}$ decays considerably more slowly for the double-octant plates (dashed blue curve) as compared to the quadrant phase plates (solid blue curve). A qualitative understanding of this difference in robustness can be obtained from the data

^{*}In this limit for extreme turbulence, the azimuthal fingerprint of the analyser mode is fully wiped out. The detection state thus becomes circularly isotropic, leading to $D = 1$.

represented in Fig. 9.3(b). First of all, the dominant effect of turbulence is a loss of modal strength ($\Delta m = 0$ scattering). The non-conserving OAM scattering probability decays rapidly as a function of $|\Delta m|$. This spreading of a mode applies to any initial value of m . Therefore, neighbouring modes in the OAM superposition associated with a certain phase-plate analyser affect each other strongly when separated by $|\Delta m| = 1$ (as for the quadrant phase plates) and much weaker when separated by $|\Delta m| = 2$ (as for the double-octant phase plates). This explains the robustness of the latter phase plates as compared to the standard quadrant phase plates.

As a second example, we compare OAM superposition states for the case $D = 3$, associated to (i) our half-sector phase plates (solid red curve) and (ii) half-integer spiral phase plates having a helical phase ramp of optical height $\lambda/2$ (dashed red curve). Since the OAM eigenvalue spectrum of the half-sector phase plate has twice the modal spacing of the half-integer spiral phase plate, the argumentation given above applies also here. Naturally, the argument can be extended, suggesting that OAM superposition states can be designed that have an optimal robustness against atmospheric perturbations.

So far, we have expressed the turbulence strength in terms of the ratio w_0/r_0 . However, this quantity allows us to estimate the propagation distance L that can be reached outside the laboratory, since the Kolmogorov theory (see Eq. (9.2)) used to describe our data is also a fair description of a real-life atmosphere [188]. For horizontal propagation, the Fried parameter can be expressed as $r_0 = 3.02(k^2 L C_n^2)^{-3/5}$, with $k = 2\pi/\lambda$ the wavenumber of the light, L the propagation length and C_n^2 the structure constant quantifying the phase perturbations [200]. To put this correspondence in perspective, we consider a wavelength $\lambda = 1550$ nm in the transmission window of the atmosphere and assume moderate ground-level perturbations ($C_n^2 = 10^{-14} \text{ m}^{-2/3}$) [221] and a beam size $w_0 = 6$ cm. At $w_0/r_0 = 0.65$, where $\tilde{D}$ has decayed to 50% of its initial level, we find a propagation length of 2 km (satisfying the requirement $L < z_R$). This distance would suffice for use in a metropolitan environment. For vertical propagation from ground level through the entire column of the atmosphere, the Fried parameter is typically of the order of 5-15 cm, depending on the elevation and weather conditions [183, 195]. For the turbulence strength and beam size mentioned above, the Fried parameter equals 9 cm, suggesting that also a satellite communication link may be viable.

9.6 Conclusions

We have presented the first experimental data on the transmission of OAM-entangled photons through a turbulent atmosphere. We have found that the shape of the coincidence curve is quite robust under the action of the turbulence, and that the robustness can be enhanced by judiciously designing the OAM superposition that acts as an information carrier. Present-day adaptive-optical techniques are sufficiently developed that OAM entanglement for free-space distribution is viable.

9.7 Appendix

In this Appendix, we show the full measurement set of the coincidence count rates used for Figs. 9.4 and 9.5. Figures 9.7 and 9.8 show data obtained with half-sector phase plates and quadrant-sector phase plates, respectively. The agreement between experimental data (open

circles) and theoretical predictions (solid curves) is seen to be excellent. Note that the theoretical curves require just a single fit parameter; a trivial vertical scaling factor, which is determined for the initial case of no turbulence (see Figs. 9.7(a) and 9.8(a)) and kept fixed for increasing turbulence strengths (see Figs. 9.7(b)-(g) and 9.8(b)-(g)).

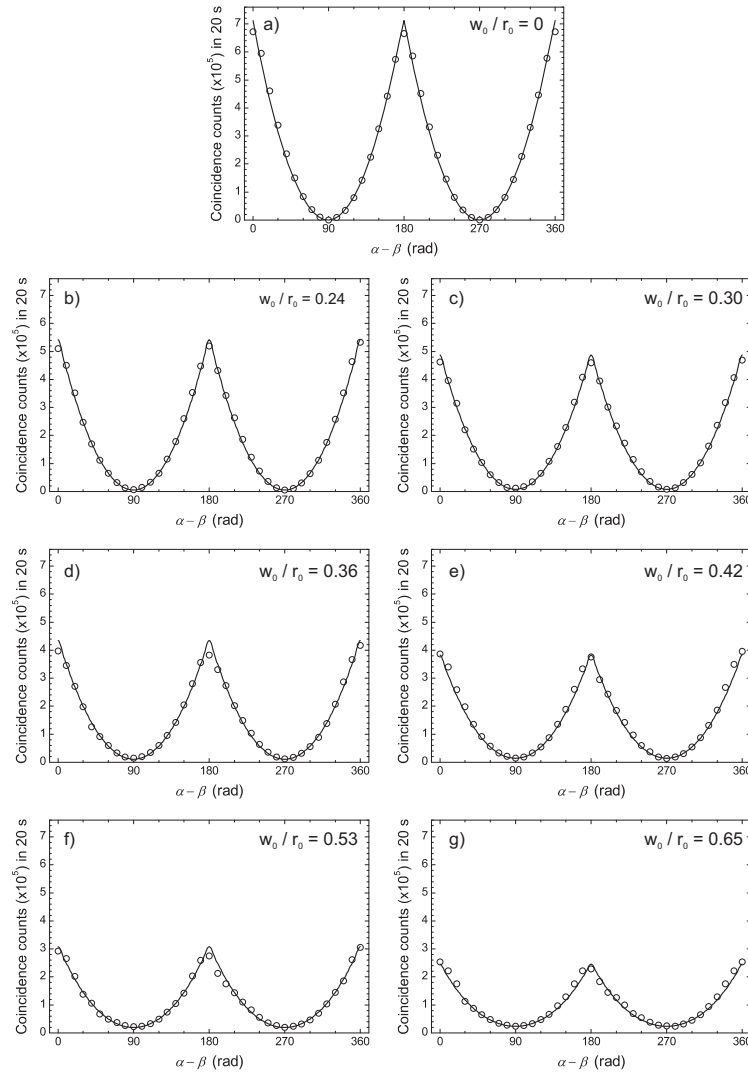


Figure 9.7: **Coincidence curves for two half-sector phase plates.** Circles denote experimental data points measured during a 20 s collection time. Curves denote theoretical predictions, using no fit parameter other than a trivial scaling factor that is determined in the absence of turbulence and kept fixed for all other graphs. (a) $w_0/r_0 = 0$ (no turbulence), (b) $w_0/r_0 = 0.24$, (c) $w_0/r_0 = 0.30$, (d) $w_0/r_0 = 0.36$, (e) $w_0/r_0 = 0.42$, (f) $w_0/r_0 = 0.53$, (g) $w_0/r_0 = 0.65$.

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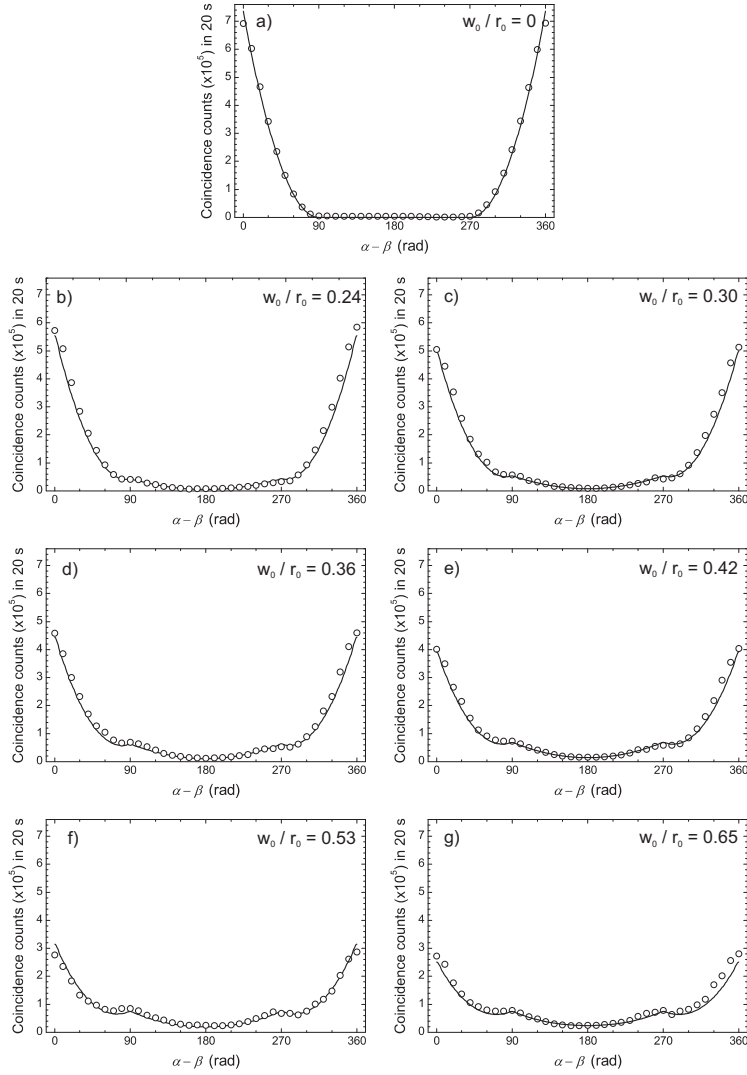


Figure 9.8: **Coincidence curves for two quadrant-sector phase plates.** Circles denote experimental data points measured during a 20 s collection time. Curves denote theoretical predictions, using no fit parameter other than a trivial scaling factor that is determined in the absence of turbulence and kept fixed for all other graphs. (a) $w_0/r_0 = 0$ (no turbulence), (b) $w_0/r_0 = 0.24$, (c) $w_0/r_0 = 0.30$, (d) $w_0/r_0 = 0.36$, (e) $w_0/r_0 = 0.42$, (f) $w_0/r_0 = 0.53$, (g) $w_0/r_0 = 0.65$.

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Samenvatting

In dit proefschrift onderzoeken we de rijke eigenschappen van hoogdimensionale verstrengeling van fotonparen. Experimenteel hebben we deze hoge dimensionaliteit tot stand gebracht door gebruik te maken van de baanimpulsmomenttoestanden van het licht. De nadruk in dit proefschrift ligt op de vraag hoe de dimensionaliteit van de gemeten verstrengeling gekwantificeerd kan worden. Daarnaast bespreken we de produktiemethode van de optische faseplaten die we hanteren om impulsmomenttoestanden van licht te manipuleren. Tot slot presenteren we een haalbaarheidsstudie van de toepasbaarheid van baanimpulsmomentverstrengeling in quantum communicatie door de atmosfeer. Een kort overzicht van de hoofdstukken:

- Hoofdstuk 1 leidt het proefschrift in en de schetst de motivatie achter het onderzoek naar impulsmomentverstrengeling van licht in hoge dimensies.
- Hoofdstuk 2 bespreekt de experimentele technieken die aangewend zijn voor de productie van impulsmomentverstrengelde fotonparen. Daarbij maken we gebruik van parametrische down-conversie in een niet-lineair kristal. We karakteriseren de emissie van het kristal aan de hand van de klassieke intensiteitsdistributie en maken een schatting van het aantal ruimtelijk verstrengelde modes, ofwel van het Schmidt-getal.
- In Hoofdstuk 3 introduceren we de angulaire-mode-analysatoren die we gebruiken om multimodale impulsmomenttoestanden te analyseren. Deze analysatoren bestaan uit een angulaire faseplaat, een focuseringslens, een mono-mode fiber en een detector. De corresponderende detectietoestand is zuiver (mono-mode), maar hangt af van de vorm van de faseplaat. Door de faseplaat te draaien om haar centrale as kunnen we de detectietoestand geleidelijk variëren. Zodoende spannen we een grote set van (niet-orthogonale) detectietoestanden op en scannen we een hoogdimensionale modale ruimte. We onderzoeken de grootte van die ruimte en drukken die uit in een effectieve dimensionaliteit.
- Hoofdstuk 4 verenigt de voorgaande twee hoofdstukken en bespreekt hoe impulsmomentverstrengeling tussen fotonparen kan worden geëxtraheerd uit een down-conversie bron met behulp van twee angulaire-mode-analysatoren. Daarbij introduceren we het concept van de Shannon dimensionaliteit D van gemeten verstrengeling. Dit is een operationele maat van verstrengeling, die lijkt op het Schmidt-getal, maar zich toespitst op het gedetecteerde i.p.v. het gegenereerde deel van de verstrengelde toestand. In het experiment realiseren we $D = 3$ en $D = 6$ met behulp van twee types binaire faseplaten.

SAMENVATTING

- Met het doel werkelijk hoogdimensionale angulaire verstrengeling te extraheren, zijn faseplaten van toenemende complexiteit vereist. In Hoofdstuk 5 bespreken we diverse fabricagemethoden van binaire faseplaten, ontwikkeld in samenwerking met verschillende externe producenten. Tevens hebben we de toepasbaarheid van een dynamische fasemodulator onderzocht.
- In Hoofdstuk 6 bouwen we voort op de principes uit Hoofdstuk 4, en onderzoeken hoe binaire faseplaten kunnen dienen om werkelijk hoogdimensionale impulsmomentverstrengeling te extraheren. Daarbij maken we gebruik van zogeheten multisector-faseplaten. Deze binaire faseplaten bevatten N angulaire sectoren die een halve golf-lengte dikker zijn dan de tussenliggende regionen. We bespreken een numerieke methode om het faseplaat patroon voor gegeven aantal sectoren te optimaliseren, met als inzet het behalen van de hoogste Shannon dimensionaliteit. We geven een experimentele demonstratie van deze concepten en realiseren impulsmomentverstrengeling met een hoge dimensionaliteit van $D = 16, 5$.
- Een groot gedeelte van het onderzoek aan hoogdimensionale ruimtelijke verstrengeling beperkt zich tot het angulaire gedeelte van het tweefotonveld, zo ook de eerste hoofdstukken van dit proefschrift. In Hoofdstuk 7 zetten we een eerste stap op weg naar het gebruik van zowel de angulaire als radiële ruimtelijke vrijheidsgraad. Daarvoor maken we gebruik van binaire faseplaten met een 2-D profiel. We bespreken de relevante parameters die door deze uitbreiding een rol gaan spelen en geven een experimentele demonstratie.
- Er bestaan speculaties dat baanimpulsmomentverstrengeling veelbelovend is voor quantumcommunicatie. Het uitgebreide alfabet dat deze vrijheidsgraad ter beschikking stelt zou een vergrote kanaalcapaciteit en een sterk verminderde gevoeligheid voor af-luisteren mogelijk maken. Echter, de voorspellingen over de vraag hoe goed verstrengelde impulsmomenttoestanden te transporteren zijn lopen sterk uiteen. Hoofdstuk 8 bereidt een experimentele studie van deze vraag voor met betrekking tot transmissie door de lucht. Het hoofdstuk biedt enerzijds een overzicht van de theorie van atmosferische turbulentie, uitgedrukt in de Kolmogorov theorie. Anderzijds geeft het een beschrijving van de experimentele technieken die wij ontwikkeld hebben om in een laboratoriumomgeving realistische atmosferische turbulentie te genereren.
- Hoofdstuk 9 presenteert de uitkomsten van een experimentele studie naar de robuustheid van baanimpulsmomentverstrengeling onder invloed van een turbulente atmosfeer. We drukken de robuustheid uit in de Shannon dimensionaliteit en vinden dat deze afneemt voor toenemende turbulentiesterktes. Door de experimentele turbulentieparameters te vertalen naar de buitenwereld, vinden we dat de bruikbare horizontale propagatieafstand enkele kilometers bedraagt, terwijl in de verticale richting de gehele atmosferische kolom doorsneden kan worden. Tot slot doen we een voorspelling over hoe de robuustheid vergroot kan worden door slim ontwerp van de impulsmoment-superpositietoestanden.

Curriculum Vitæ

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List of publications

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Physical Review Letters **101**, 120502 (2008).
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LIST OF PUBLICATIONS

- High-dimensional entanglement with orbital-angular-momentum states of light,
B.-J. Pors, F. Miatto, G.W. 't Hooft, E.R. Eliel, and J.P. Woerdman,
Under review for Journal of Optics, Special Edition on Orbital Angular Momentum.
(Chapter 6 of this thesis)
- Transport of orbital-angular-momentum entanglement through a turbulent atmosphere,
B.-J. Pors, C.H. Monken, E.R. Eliel, and J.P. Woerdman,
Based on arXiv:0909.3750v1 [quant-ph]; under review for Optics Express.
(Chapter 9 of this thesis)
- Fabrication of binary phase plates for the management of orbital-angular-momentum superposition states,
B.-J. Pors, W.M. Martam, C.A. van den Heuvel, H. de Koning, G.W. 't Hooft, E.R. Eliel, and J.P. Woerdman,
In preparation for publication.
(Chapter 5 of this thesis)
- Two-variable spatial entanglement,
B.-J. Pors, E.R. Eliel, and J.P. Woerdman,
In preparation for publication.
(Chapter 7 of this thesis)

Nawoord

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