Radicals in arithmetic

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Chapter 7

Enumerating ABC triples

7.1 Introduction

The *radical* $\mathrm{rad}(n)$ of a positive integer n is defined to be the product of the prime numbers dividing n. We say that positive integers a, b, c form an *ABC triple* if they satisfy the following conditions:

- $a + b = c$;
- $a \leq b$;
- $\gcd(a, b, c) = 1$, and
- $\mathrm{rad}(abc) < c$

The two smallest examples of ABC triples are $1 + 8 = 9$ and $5 + 27 = 32$, with radicals $2 \cdot 3 = 6$ and $2 \cdot 3 \cdot 5 = 30$ respectively. There are in fact infinitely many ABC triples. For example, for every positive integer n, the sum $1 + (64^n - 1) = 2^{6n}$ defines an ABC triple since 9 divides $(64^n - 1)$ and we have $\mathrm{rad}((64^n - 1)2^{6n}) \leq \frac{2}{3}(64^n - 1) \leq 2^{6n}$.

The *quality* $q(a, b, c)$ of an ABC triple is defined as

$$q(a, b, c) = \frac{\log(c)}{\log(\mathrm{rad}(abc))}.$$

By the fourth condition for ABC triples, this quality is always greater than 1.

The famous *ABC conjecture* [33] proposed by Masser and Oesterlé in 1985 states that the limsup of the quality of all ABC triples is equal to 1.

Over the years, it has become popular to search for triples with high quality [26, 29]. The current record is held by the triple $2 + 3^{10} \cdot 109 = 23^5$ with quality approximately 1.630, found by Eric Reyssat in 1987.

In this chapter, we report results of the project *ABC@home*, a distributed computing project built on the BOINC platform [1], for which many people worldwide contributed computing resources to enumerate *all* ABC triples with $c < 10^{18}$.

In Section 7.2 we derive an upper bound for the number of such ABC triples, and in Section 7.3 we give the algorithm used by *ABC@home* to perform the enumeration. After that, in Section 7.4 we describe a number of algorithmic implementation details to accelerate the process, and finally Section 7.5 contains an overview of the produced data.

Related efforts have previously been made. In 1993, Elkies and Kanapka used a similar, but unpublished, algorithm to enumerate all ABC triples below 2^{32} with quality above 1.2. Their results are no longer available from their original location, but are mirrored at `http://www.abcathome.com/Elkies1993/`.

In 2007, Jeroen Demeyer computed all ABC triples with $c \leq 2^{67} \approx 1.4 \cdot 10^{20}$ and quality at least 1.4 (see [11]). His results have been incorporated into the tables of known ABC triples with quality at least 1.4 maintained by Nitaj [26] and de Smit [29].

7.2 Bounds

In this section we derive an upper bound for the number of ABC triples.

Theorem 7.1. *For every $\varepsilon > 0$, the number of ABC triples $a + b = c$ with $c < N$ is $O(N^{2/3+\varepsilon})$.*

The main ingredient of the proof is the following theorem.

Theorem 7.2. *Let α be a real number with $0 < \alpha \leq 1$. Then for every $\varepsilon > 0$, the number $X(N, \alpha)$ of positive integers $x < N$ with $\operatorname{rad}(x) < N^\alpha$ is $O(N^{\alpha+\varepsilon})$.*

Proof. This is Theorem 12 from section II.1 in [34]

□

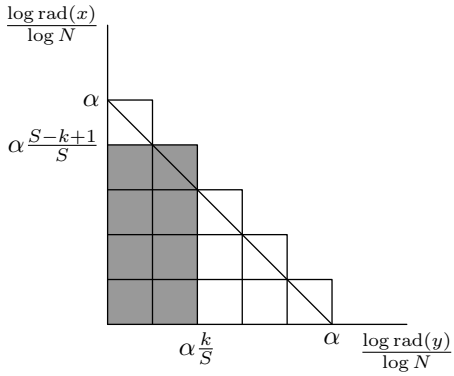
Corollary 7.3. *Let α be a real number with $0 < \alpha \leq 1$. Then for every $\varepsilon > 0$, the number $Y(N, \alpha)$ of pairs of coprime positive integers $x, y < N$ with $\operatorname{rad}(xy) < N^\alpha$ is $O(N^{\alpha+\varepsilon})$.*

Proof. Let $\varepsilon > 0$ be an arbitrary positive real number.

Let S be a positive integer, and k an integer with $0 < k \leq S$. Define the set R_k to be

$$R_k = \Big\{ x, y \in \mathbf{Z}_{\geq 1} : \quad \begin{array}{l} x, y < N; \\ x,\ y \text{ coprime}; \\ \operatorname{rad}(x) < N^{\alpha \frac{S-k+1}{S}}; \\ \operatorname{rad}(y) < N^{\alpha \frac{k}{S}} \end{array} \Big\}.$$

In $\log \operatorname{rad}(x)$, $\log \operatorname{rad}(y)$ space, we can depict R_k as the following rectangle.



By using Theorem 7.2 twice, for x and y separately, we can for every $\delta > 0$ bound the order of R_k from above by

$$\#R_k = O(N^{\alpha\frac{S-k+1}{S}+\delta} N^{\alpha\frac{k}{S}+\delta}) = O(N^{\alpha+\frac{1}{S}+2\delta}).$$

Since the union $\bigcup_{0<k\leq S} R_k$ covers the set we are counting in this Corollary, we find $Y(N,\alpha) \leq S \cdot \#R_k = O(N^{\alpha+\frac{1}{S}+2\delta})$.

This holds for every $\delta > 0$ and S, so we can choose them such that $\frac{1}{S} + 2\delta < \varepsilon$ to complete the proof. □

It is often convenient to sort the integers in an ABC triple by radical rather than by size. We use the following notation for that purpose.

Definition 7.4. If $a + b = c$ is a triple of positive integers, let (x, y, z) be a permutation of (a, b, c) such that

$$\mathrm{rad}(x) \leq \mathrm{rad}(y) \leq \mathrm{rad}(z).$$

We then define $x(a,b,c) = x$, $y(a,b,c) = y$ and $z(a,b,c) = z$.

Proof of Theorem 7.1. Let $a + b = c$ be an ABC triple with $c < N$. For brevity, we write $x = x(a,b,c)$, $y = y(a,b,c)$ and $z = z(a,b,c)$.

We have $\mathrm{rad}(xy) < \mathrm{rad}(xz) < \mathrm{rad}(yz)$, so we can derive

$$\begin{aligned} \mathrm{rad}(xy)^3 &< \mathrm{rad}(xy)\mathrm{rad}(xz)\mathrm{rad}(yz) \\ &= \mathrm{rad}(xyz)^2 < c^2 < N^2. \end{aligned}$$

We conclude $\mathrm{rad}(xy) < N^{2/3}$.

Given any two coprime positive integers x, y, there are at most 2 ABC triples $(\{x, y, x+y\},\ \{x, y, |x-y|\})$ that could correspond to this pair x, y, so we get an upper bound

$$\#\{\text{ABC triples } a+b=c<N\} \leq 2\#\{x, y \in \mathbf{Z}_{\geq 1} : x, y < N \text{ and } \mathrm{rad}(xy) < N^{2/3}\}.$$

The theorem now immediately follows from Corollary 7.3. □

Lower bound

The following theorem by Sander Dahmen provides an asymptotic lower bound for the number of ABC triples. It builds on earlier results and methods from van Frankenhuysen [14] and Stewart and Tijdeman [33].

Theorem 7.5 (S. Dahmen, [10])**.** *For every $\varepsilon > 0$ and N large enough, the number of ABC triples $a + b = c$ with $c < N$ is at least* $\exp((\log N)^{1/2-\varepsilon})$.

7.3 Enumeration algorithm

Here we describe the main enumeration algorithm used in the ABC@home project.

The speed of the algorithm we give below is not asymptotically optimal. One could instead enumerate every potential triple and execute a sub-exponential time factoring algorithm such as the Quadratic Sieve, or heuristically the General Number Field Sieve [27], to obtain a run time of $O(N^{2/3+\varepsilon})$ as a consequence of Theorem 7.1.

However, in the search range that is currently feasible, the integers to be considered are small enough that they can be factored much more efficiently using different methods. The algorithm from this section does this using a combination of sieving small factors in blocks of numbers simultaneously, and basic trial division.

Proposition 7.6. *For positive integers a and b, the following algorithm enumerates all squarefree integers x satisfying $a \leq x < b$ in factored form.*

Algorithm 7.7.

1. Create a list of integers $r(n)$ for $a \leq n < b$, initialized to $r(n) = 1$.
2. Create a list of sets $P(n)$ for $a \leq n < b$, initialized to $P(n) = \emptyset$.
3. Loop over all primes p with $p^2 < b$:
   (a) For all $n \equiv 0 \bmod p$ and $a \leq n < b$:
      i. Multiply $r(n)$ with p.
      ii. Add p to $P(n)$.
   (b) For all $n \equiv 0 \bmod p^2$ and $a \leq n < b$:
      i. Set $r(n) = 0$.
4. Loop over all n with $a \leq n < b$ and $r(n) \neq 0$:
   (a) If $r(n) \neq n$, add $n/r(n)$ to $P(n)$.
   (b) Return the squarefree integer $r(n)$ with its prime factors $P(n)$.

Proof. If an integer $n < b$ is not squarefree, there is a prime p such that $p^2 \mid n$. Since we then have $p^2 \leq n < b$, we set $r(n)$ to 0 in step 3b. Conversely, if $r(n)$ is 0, it can only have been set to 0 in this step, so n is divisible by the square of a prime. We conclude that this algorithm indeed returns all squarefree integers between a and b.

Finally, note that $r(n)$ divides n, so the quantity $q = n/r(n)$ added to the set $P(n)$ in step 4a is an integer. In fact, q is prime since it has no prime divisors p with $p^2 \leq n/r(n) \leq n < b$. This implies that $P(n)$ indeed contains exactly the prime divisors of $n = r(n) = \mathrm{rad}(n)$. □

Before we give the algorithm used to enumerate ABC triples, we first state and prove a number of elementary propositions that provide limits for steps in the algorithm.

Proposition 7.8. *Let n be a positive integer not divisible by primes p with $p^3 \leq n$. Then n is either the square of a prime or squarefree.*

Proof. Suppose that n is not squarefree and let q be a prime divisor of n with $\mathrm{ord}_q(n) \geq 2$. Because we have $q^3 > n$, we then find that $n/q^2 < n^{1/3}$, so $n/q^2 = 1$ and n is the square of a prime. □

Proposition 7.9. *Let $n > 1$ be an integer not divisible by primes $p < P$. Then n is either a prime power or we have $\mathrm{rad}(n) > P^2$.*

Proof. If n has only one prime divisor, it is a prime power. Otherwise, since n is not 1, the radical of n has at least two prime divisors p and q, with $p \geq P$ and $q > P$, the product of which is greater than P^2. □

Theorem 7.10. *Given integers N, m_x, M_x, m_y, M_y, and squarefree positive integers t, g with $g \mid t$, Algorithm 7.11 lists exactly all ABC triples $a + b = c$ satisfying*

- $c < N$;
- $\gcd(x, t) = g$;
- $m_x \leq \mathrm{rad}(x) < M_x$; *and*
- $m_y \leq \mathrm{rad}(y) < M_y$,

where $x = x(a, b, c)$ and $y = y(a, b, c)$.

Algorithm 7.11.

1. Pre-compute the list of prime numbers below $N^{1/3}$.
2. Generate list L_x of square-free integers r with $\gcd(r, t) = g$ in $[m_x, M_x)$ in factored form.
3. Generate list L_y of square-free integers coprime with g in $[m_y, M_y)$ in factored form.
4. Generate list X of positive integers $< N$ with radical in L_x.
5. Generate list Y of positive integers $< N$ with radical in L_y.
6. Sort X and Y by size.

7. Partition X into subsets $(X_1, \ldots)$ and Y into subsets $(Y_1, \ldots)$.

8. Loop over pairs of sets (X_i, Y_j):

   (a) Generate rectangular table with $r(x, y)$ and $s(x, y)$ for $x \in X_i$, $y \in Y_j$.

   (b) Initialize $r(x, y) = 1$ (partial radical found so far) and $u(x, y) = x + y$ (unfactored part).

   (c) Loop over all primes p up to a given sieve bound:

      i. Sort elements of X_i into their residue classes mod p.

      ii. Loop over $y \in Y_j$:

         A. Find all elements $x \in X_i$ such that $p \mid x + y$, using the pre-sorted list mod p.

         B. Divide all factors p from $u(x, y)$.

         C. Multiply $r(x, y)$ with p.

   (d) Loop over all elements $x + y$ in the table

      i. If x and y are not coprime, skip this triple.

      ii. If $r(x, y) > c/\mathrm{rad}(xy)$, skip this triple.

      iii. Perform trial division by consecutive primes p, updating $r(x, y)$ and $u(x, y)$ as above, until one of the following occurs.

      iv. If $p^3 > u(x, y)$, test if $u(x, y)$ is a square and use Prop. 7.8.

      v. If $p^2 \cdot r(x, y) > c/\mathrm{rad}(xy)$, test if $u(x, y)$ is a prime power and use Prop. 7.9.

      vi. In either case, if $\mathrm{rad}(xyz) > c$ skip this triple. If $\mathrm{rad}(x) < \mathrm{rad}(y) < \mathrm{rad}(z)$ return this triple. Otherwise, skip it.

   (e) Repeat the above for a table of elements $|x - y|$ for $x \in X_i$, $y \in Y_j$.

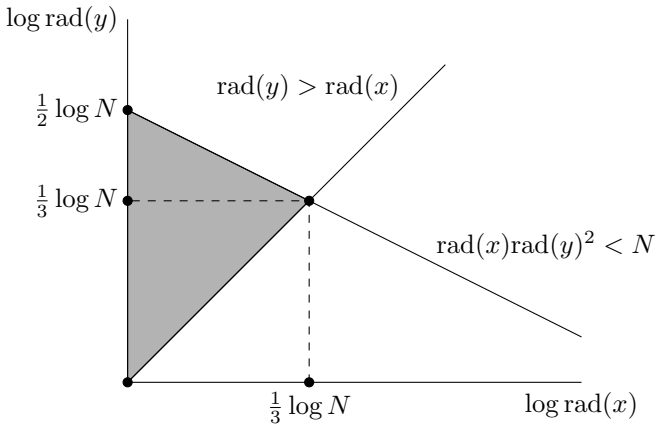
Proof. From the block structure of the algorithm it is clear that all potential triples with radicals of x and y in the right range are considered. Propositions 7.8 and 7.9 then ensure that the right ones are returned.

The remaining point of attention is verifying that the list of pre-computed primes in step 1 is long enough. For listing the squarefree integers using Algorithm 7.7 a list of primes up to the square root of the upper bound of the interval is sufficient. Since the radicals of x and y are at most $N^{1/3}$ and $N^{1/2}$, respectively, we primes up to $N^{1/4}$ suffice for this step.

The trial division loop only needs to loop over primes up to $N^{1/3}$ since the loop terminates at the latest when $u(x, y)$ is not divisible by any primes p with $p^3 \leq u(x, y) < N$ according to Proposition 7.8. □

We use this algorithm to enumerate *all* ABC triples with $c < N$.

The range of possible values for pairs $(\mathrm{rad}(x), \mathrm{rad}(y))$ is determined by the inequalities $\mathrm{rad}(x) < \mathrm{rad}(y)$ and $\mathrm{rad}(x)\mathrm{rad}(y)^2 < N$:



We have covered this triangle by rectangles, and have distributed the resulting work units over participating clients using the BOINC framework.

7.4 Implementation details

In this section we describe a number of implementation details that have a significant impact on performance.

Small prime pre-selection

Algorithm 7.11 processes a block with a prescribed value for $\gcd(x, t)$. This allows us to ensure $\gcd(x, y)$ will not contain primes dividing t, which makes the number of discarded potential triples due to a common factor in x and y significantly smaller. In practice, we take $t = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. Adding more primes produced no significant extra speed-up.

Pre-sorting

In step 6 of Algorithm 7.11, we sort X and Y by size prior to dividing them into smaller sets. After doing this, some sub-blocks $X_i \times Y_j$ will only contain integers so small that they can only lead to triples with c smaller than the radicals considered in this work unit. We can discard these sub-blocks early.

Division strength reduction

In the sieving stage of the enumeration algorithm, we compute the remainders of a large number of integers modulo a relatively small set of primes. We have implemented an approach described in [15].

Theorem 7.12 (Granlund, Montgomery, [15]). *Suppose m, p, k are non-negative integers such that $p \neq 0$ and*

$$2^{M+k} \leq mp \leq 2^{M+k} + 2^k.$$

Then $\lfloor n/p \rfloor = \lfloor mn/2^{M+k} \rfloor$ for every integer n with $0 \leq n < 2^M$.

We have pre-computed such integers m and k for every prime $p < 10^6$ with $M = 64$. This allows us to replace an integer division by p by a multiplication with m followed by an integer division by a power of two.

A remainder operation of n divided by p can then be performed by computing $n - p\lfloor n/p \rfloor$.

Divisibility testing

In the trial division stage, we test divisibility of a large number of integers by a relatively small number of primes.

Proposition 7.13. *Let $p < 2^M$ be an odd prime number. Let q satisfy $0 < q < 2^M$ and $pq \equiv 1 \bmod 2^M$. Then for every integer n with $0 \leq n < 2^M$ we have*

$$p \mid n \iff nq \bmod 2^M < \left\lfloor \frac{2^M}{p} \right\rfloor.$$

Here $nq \bmod 2^M$ is the unique non-negative integer smaller than 2^M that is congruent to $nq \bmod 2^M$.

Proof. Multiplication by p gives a permutation of $\mathbf{Z}/2^M\mathbf{Z}$ and maps the integers between 0 and $\lfloor 2^M/p \rfloor$ to the multiples of p between 0 and 2^M. Multiplication by q gives the inverse permutation, so the proposition follows. □

By precomputing q and $\lfloor 2^M/p \rfloor$ for all primes $p < 10^6$ and $M = 64$, we can implement divisibility tests by p as a 64-bit multiplication and a comparison.

Delayed bound checking

The tests 8.d.iv and 8.d.v in the inner loop that check if we can stop processing the current potential triple are relatively expensive. Instead of performing these tests after every prime p, we process 16 primes before every test. This empirically proved a good trade-off between not testing too many primes, and not testing the bounds too often.

To accommodate this, we have to make the list of pre-computed primes 15 elements longer than just the primes up to $N^{1/3}$.

Prime power testing

In step 8.d.v we have to test if a number is a prime power. Due to the delayed bound checking described before, we have already tested divisibility by (at least) the first 16 primes, so we need only consider powers of primes at least 57. Since 57^{11} is greater than 10^{18}, we additionally only need check up to 10th powers.

We perform tests for squares and cubes by first checking if the number is a square or cube, respectively, modulo 63, and if so, doing a binary search through the possible range of roots. These two tests check for any squares and cubes, not just prime powers.

After repeatedly taking 2nd and 3rd roots, we check if the remaining integer is a 5th or 7th power of a prime by table lookup in the precomputed set of 902 such prime powers.

7.5 Data

As part of the ABC@home project a large number of volunteers have executed the algorithm described in the previous sections. The entire search space for the enumeration algorithm has been split up into a large number of so-called *workunits*. Each of these workunits has been sent to multiple clients to ensure their outputs match.

Since many workunits are expected to contain no triples, this output check is not yet sufficiently strong to ensure the clients have properly and completely searched their section of the search space. To that end, the clients report not only the triples they found in their section of the search space, but also a number of internal statistics and counters. This is possible since the algorithms are fully deterministic and platform independent.

The search has resulted in a total of 14 482 065 ABC-triples below 10^{18}. In this section we show a number of tables and graphs highlighting parts of these data. The full dataset is available for download from the website at:

`http://www.abcathome.com/data/`

The first figure (Figure 7.14) shows the number of triples below each power of ten up to 10^{18}. Figure 7.15 shows the number of triples of a given size graphically, and additionally shows how many of these triples have quality larger than $1.1, \ldots, 1.5$.

The next tables show a selection of data on how often specific values of a, b and c occur in the set of found triples. Figure 7.16 does this for a set of small values of a. Figures 7.17, 7.18, 7.19 show the most common values for a, b and c for $c < 10^{14}$, and $c < 10^{16}$ and $c < 10^{18}$, respectively.

The next two tables switch to triples avoiding certain primes. Specifically, Figure 7.20 gives the number of triples below 10^{18} where $\mathrm{rad}(abc)$ is coprime to a selection of small integers. Next, Figure 7.21 gives (indirectly) for each integer p the smallest triple which is not divisible by any odd primes up to and including p.

Figure 7.22 concludes the set of tables with a list of all seven pairs of triples found with identical quality.

Cover illustration

The image on the cover, which is reproduced on the opposite page, illustrates the distribution of $\mathrm{rad}(abc)$ over $\mathrm{rad}(a)$, $\mathrm{rad}(b)$ and $\mathrm{rad}(c)$. More precisely, let S be the set of triples x, y, z of (not necessarily positive) coprime integers satisfying $x + y = z > 0$ and $\mathrm{rad}(|xyz|) < \max(|x|, |y|, |z|)$.

Note that if we would restrict to positive integers here, we would get regular ABC triples. As it is defined, the set S has 6-fold symmetry: given $(x, y, z) \in S$, every permutation of $\{|x|, |y|, |z|\}$ leads to exactly one triple in S by choosing proper signs.

If (x, y, z) is a triple in S, define $r = \mathrm{rad}(|xyz|)$. Then because x, y, z are coprime, we have $\mathrm{rad}(|x|)\mathrm{rad}(|y|)\mathrm{rad}(z) = r$ and therefore

$$\frac{\log \mathrm{rad}(|x|)}{\log(r)} + \frac{\log \mathrm{rad}(|y|)}{\log(r)} + \frac{\log \mathrm{rad}(z)}{\log(r)} = 1.$$

If we take an equilateral triangle T with sides $\frac{2}{3}\sqrt{3}$, and P any point inside T, then the sum of the distances of P to the three sides of T equals 1, and these distances uniquely define P. We can therefore interpret the three fractions $\frac{\log \mathrm{rad}(|x|)}{\log(r)}$, $\frac{\log \mathrm{rad}(|y|)}{\log(r)}$, $\frac{\log \mathrm{rad}(z)}{\log(r)}$ as coordinates in T.

If the radical of a triple is almost entirely concentrated in a single integer, the triple lies near a corner of T. If on the other hand the radical is distributed evenly among the three integers, the triple will lie near the center of T. The image now displays the distribution over T of triples in S with $z < 10^{18}$.

Acknowledgements

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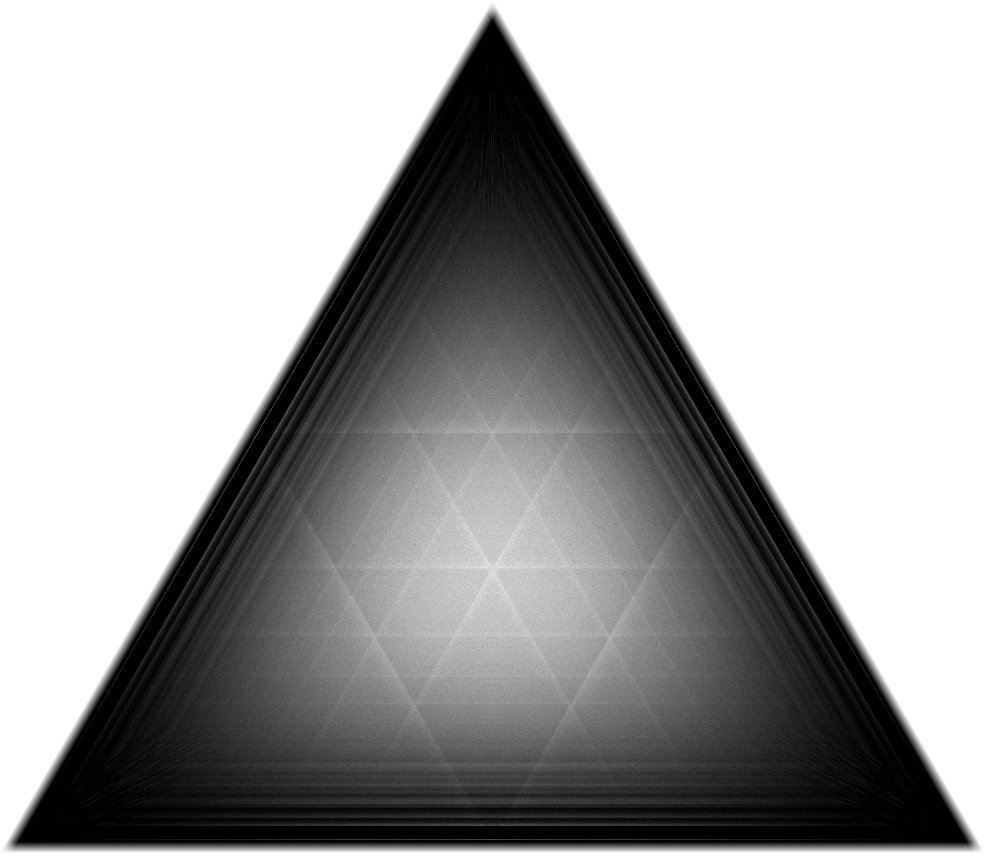

n	#
1	1
2	6
3	31
4	120
5	418
6	1 268
7	3 499
8	8 987
9	22 316
10	51 677
11	116 978
12	252 856
13	528 275
14	1 075 319
15	2 131 671
16	4 119 410
17	7 801 334
18	14 482 065

Figure 7.14: Number of triples below 10^n.

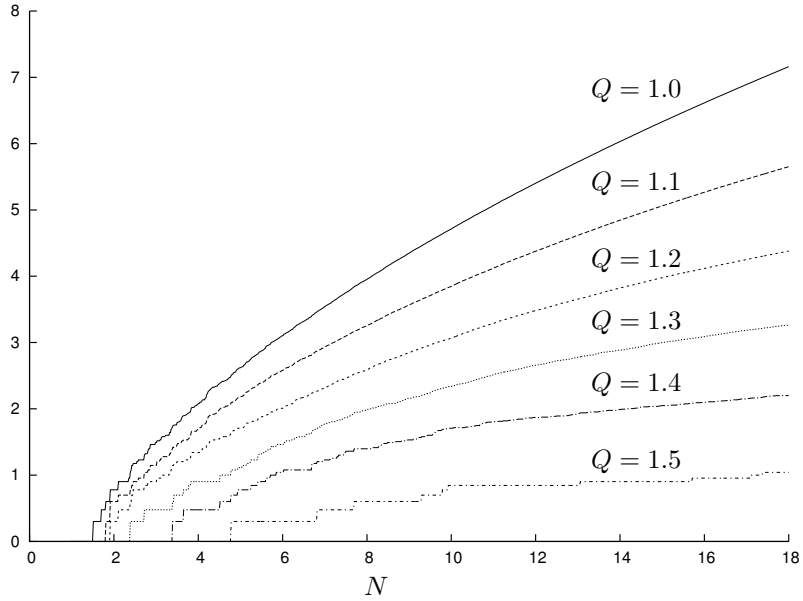

Figure 7.15: ${}^{10}\log \#\{\text{triples with } c < 10^N, \text{quality} > Q\}$

a	#	a	#
1	45 603	15	222
2	1 965	16	2 026
3	2 936	17	2 347
4	1 967	18	119
5	3 288	19	2 141
6	117	20	132
7	3 233	25	3 696
8	1 849	27	2 875
9	3 044	30	8
10	143	32	2 065
11	2 929	64	2 006
12	98	125	3 435
13	2 655	128	1 894
14	127	256	2 175

Figure 7.16: Number of triples below 10^{18} with given a.

a	${}^{10}\log a$	#	b	${}^{10}\log b$	#	c	${}^{10}\log c$	#
1	0.0	9255	7^{16}	13.5	817	5^{20}	14.0	1236
5^4	2.8	866	5^{19}	13.3	753	7^{16}	13.5	1095
5^8	5.6	864	3^{29}	13.8	714	13^{12}	13.4	821
5^6	4.2	862	11^{13}	13.5	693	3^{28}	13.4	801
5^{12}	8.4	851	5^{20}	14.0	662	5^{18}	12.6	791
7^4	3.4	846	13^{12}	13.4	619	3^{29}	13.8	704
7^6	5.1	825	3^{28}	13.4	606	11^{12}	12.5	650
5^2	1.4	819	5^{18}	12.6	570	11^{13}	13.5	637
7^2	1.7	812	7^{15}	12.7	570	7^{15}	12.7	623
5^9	6.3	800	17^{11}	13.5	535	5^{19}	13.3	611
5^{10}	7.0	795	3^{27}	12.9	510	2^{46}	13.8	586
7^3	2.5	794	2^{45}	13.5	463	3^{27}	12.9	550
7^8	6.8	789	23^{10}	13.6	463	2^{44}	13.2	543
5^3	2.1	757	3^{26}	12.4	445	23^{10}	13.6	532
7^5	4.2	750	2^{46}	13.8	438	17^{11}	13.5	497

Figure 7.17: Most frequent values of a, b, c among triples with $c < 10^{14}$

a	$^{10}\log a$	#	b	$^{10}\log b$	#	c	$^{10}\log c$	#
1	0.0	21025	5^{22}	15.4	1716	7^{18}	15.2	2046
5^{8}	5.6	1930	3^{33}	15.7	1598	5^{22}	15.4	1837
5^{6}	4.2	1916	11^{15}	15.6	1591	11^{15}	15.6	1811
5^{12}	8.4	1885	7^{18}	15.2	1587	13^{14}	15.6	1656
5^{4}	2.8	1874	13^{14}	15.6	1380	3^{32}	15.3	1628
7^{6}	5.1	1869	3^{32}	15.3	1352	3^{33}	15.7	1606
7^{4}	3.4	1860	5^{21}	14.7	1262	19^{12}	15.3	1476
7^{8}	6.8	1849	7^{17}	14.4	1103	5^{21}	14.7	1409
7^{12}	10.1	1810	11^{14}	14.6	1083	2^{53}	16.0	1326
5^{2}	1.4	1802	3^{31}	14.8	1069	17^{13}	16.0	1304
7^{2}	1.7	1733	2^{52}	15.7	1056	17^{12}	14.8	1252
5^{9}	6.3	1731	5^{20}	14.0	1039	5^{20}	14.0	1236
5^{10}	7.0	1710	19^{12}	15.3	1038	3^{30}	14.3	1182
7^{3}	2.5	1698	3^{30}	14.3	974	11^{14}	14.6	1149
7^{10}	8.5	1684	13^{13}	14.5	912	2^{52}	15.7	1148

Figure 7.18: Most frequent values of a, b, c among triples with $c < 10^{16}$

a	$^{10}\log a$	#	b	$^{10}\log b$	#	c	$^{10}\log c$	#
1	0.0	45603	7^{21}	17.7	3731	5^{24}	16.8	4104
5^{6}	4.2	3999	5^{25}	17.5	3448	7^{21}	17.7	4075
5^{12}	8.4	3995	11^{17}	17.7	3340	13^{16}	17.8	3830
7^{12}	10.1	3973	13^{16}	17.8	3006	7^{20}	16.9	3566
5^{8}	5.6	3969	5^{24}	16.8	2960	3^{36}	17.2	3399
7^{6}	5.1	3946	3^{37}	17.7	2950	5^{25}	17.5	3287
7^{8}	6.8	3919	7^{20}	16.9	2927	11^{17}	17.7	3154
5^{4}	2.8	3914	3^{36}	17.2	2741	19^{14}	17.9	2987
7^{4}	3.4	3873	11^{16}	16.7	2381	3^{37}	17.7	2870
5^{2}	1.4	3696	19^{14}	17.9	2293	11^{16}	16.7	2838
7^{10}	8.5	3661	17^{14}	17.2	2245	17^{14}	17.2	2654
7^{2}	1.7	3636	3^{35}	16.7	2194	31^{12}	17.9	2518
5^{10}	7.0	3586	2^{59}	17.8	2187	13^{15}	16.7	2391
5^{15}	10.5	3560	13^{15}	16.7	2177	2^{59}	17.8	2390
5^{16}	11.2	3538	5^{23}	16.1	2163	29^{12}	17.5	2369

Figure 7.19: Most frequent values of a, b, c among triples with $c < 10^{18}$

n	#
2	0
3	756 946
5	2 523 717
7	4 194 390
11	6 804 914
13	7 769 311
15	126 233
17	9 207 072
19	9 744 974
21	208 359
23	10 586 016
35	702 418
55	1 152 234
$3 \cdot 5 \cdot 7$	33 105
$3 \cdot 5 \cdot 11$	56 056
$3 \cdot 5 \cdot 7 \cdot 11$	14 314
$3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	6 913

Figure 7.20: Number of triples below 10^{18} with rad(abc) coprime to n.

p	triple	quality
3	$4 + 121 = 125$	1.0271
5	$169 + 343 = 512$	1.1987
7	$128 + 4913 = 5041$	1.0945
17	$751 + 130321 = 131072$	1.1486
19	$2048 + 705233 = 707281$	1.0237
29	$263 + 3442688 = 3442951$	1.0037
41	$271 + 38272753 = 38273024$	1.0642
73	$137 + 46268279 = 46268416$	1.0165
137	$8192 + 26171619209 = 26171627401$	1.0044
601	$3539721569 + 562949953421312 = 562953493142881$	1.0895
4871	none with $c < 10^{18}$	

Figure 7.21: Smallest triples with rad(abc) not divisible by any odd primes $\leq p$.

$$
\begin{aligned}
128 + 3645 &= \\
648 + 3125 &= 3773 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 = 2310
\end{aligned}
$$

$$
\begin{aligned}
27 + 12005 &= \\
125 + 11907 &= 12032 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 47 = 9870
\end{aligned}
$$

$$
\begin{aligned}
637 + 52488 &= \\
2704 + 50421 &= 53125 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 = 46410
\end{aligned}
$$

$$
\begin{aligned}
729 + 212960 &= \\
81920 + 131769 &= 213689 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 89 = 205590
\end{aligned}
$$

$$
\begin{aligned}
8281 + 218700 &= \\
32500 + 194481 &= 226981 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 61 = 166530
\end{aligned}
$$

$$
\begin{aligned}
254800 + 23882769 &= \\
2843100 + 21294469 &= 24137569 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 181 = 8400210
\end{aligned}
$$

$$
\begin{aligned}
4645188 + 113348636875 &= \\
20095029775 + 93258252288 &= 113353282063 \\
\mathrm{rad}(abc) &= 2 \cdot 3 \cdot 5 \cdot 7 \cdot 17 \cdot 47 \cdot 53 \cdot 59 \cdot 61 = 32005439130
\end{aligned}
$$

Figure 7.22: Pairs of ABC triples below 10^{18} with the same quality.