Uniform infinite and Gibbs causal triangulations

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CHAPTER 6

Discussion and outlook

In this thesis we investigated aspects of uniform infinite causal triangulations (UICT) and Gibbs causal triangulations which are probabilistic formulations of so-called causal dynamical triangulation (CDT) models of quantum gravity.

Since causal triangulations are in bijections with planar rooted trees and thus the UICT measure is equivalent to the uniform measure on planar rooted trees - a size-biased critical Galton-Watson process -, we first focused our studies on random trees. In Chapter 2 we investigated the fractal and spectral dimension of trees which have a unique infinite spine. There are many random tree ensembles where in the continuum limit the measure is almost surely condensed on trees which have this property; examples include generic trees and non-generic critical trees, which are essentially size-biased critical Galton-Watson trees with finite and infinite variance respectively. We obtain an upper and lower bound on the spectral dimension in terms of the fractal dimension and what we call the hull dimension. In the case where the outgrowths along the spine are *i.i.d.* and the outgrowths die out fast enough we obtain simple expressions for the fractal and spectral dimension. We apply these methods to generic and non-generic critical trees as well as a model of randomly grown trees.

In Chapter 3 we discuss in detail the relation between the UICT and size-biased critical Galton-Watson processes. Besides a detailed proof of the existence of the measure we provide results on the convergence of the rescaled boundary length to a diffusion process. Furthermore, it is shown that joint rescaled length-area processe converges to a stochastic integral over the length process. This relation enables us to extract from the Feynman-Kac procedure what is known in the physics literature as the effective quantum Hamiltonian. This provides a mathematically rigorous formulation of certain scaling limits of CDT and provides us with a first connection between previously disconnected works on CDT in the corresponding physics and probability literature.

Chapter 4 is devoted to the investigation of a growth process which samples sections of UICT by two elementary moves in which a single triangle is added with certain probabilities. We show a correspondence of the growth process to a Markov chain $\{M_n\}_{n\geq 0}$ on the positive integers. Using the properties of this Markov chain we estimate the geodesic distance t_n to the initial boundary of a triangle added at after a certain growth time n. This relation is used to prove that the fractal dimension is almost surely $d_h = 2$ in an alternative manner to the corresponding branching process picture. Furthermore, it is shown that the rescaled length process as a function of growth time also converges to a diffusion process. This process can be related to the corresponding length process as a function of geodesic distance via a random time change. This shows a certain duality relation between the growth process and the branching process: in the former, area is fixed (this is simply growth time), while geodesic distance is estimated, whereas in the latter geodesic distance is fixed (this is the generation number) while area is estimated. This duality relation provides us with a rigorous formulation of the so-called peeling procedure for CDT.

In Chapter 5 we discuss a canonical formulation of the causal triangulations ensemble through introduction of a Gibbs measure with Boltzmann factor $e^{-\mu|T|}$, where $|T|$ is the size of the triangulation and μ the cosmological constant. We introduce the transfer matrix formalism and using Krein-Rutman theory, we show the convergence of the free energy to the largest eigenvalue of the transfer matrix. This determines the limiting measure which at the critical value μ_c for the cosmological constant corresponds to UICT. We extend this analysis to the Ising model coupled to causal triangulations. Using Krein-Rutman theory, we determine the regions in the parameter space for which the partition function converges. This is a first step towards determining the critical line and thus towards being able to analyse the phase transition of the model.

In the final sections of chapters 2 - 5 we have already discussed possible extensions of the here developed results. Most interestingly, are, firstly, further application of the results on the spectral dimension of random tree ensembles with a unique infinite spine to obtain a proof for the full parameter range of the attachment and grafting model as well as other models of randomly grown trees such as Ford's α-model [61] and its generalisation, the $\alpha\gamma$–model [76]. Furthermore, there so far only exists an upper bound on the spectral dimension of CDT, $d_s \leq 2$. It would thus also be interesting to extend the previous results to prove that this bound is tight and one has almost surely $d_s = 2$. Secondly, it would be interesting to extend our results on the growth process to generalise previous work by Angel [14] on a growth process for uniform infinite planar triangulations (UIPT) to show convergence of the boundary process to a Lévy process, as well as to extend them to so-called multi-critical models of DT [53, 92] and CDT [93, 94]. These lines of research are currently under investigation.

Another highly interesting direction which we currently follow in collaboration with Stefánsson is the formulation of the scaling limit of CDT in an analogous manner to the Brownian map as described above, in Section 1.2. On the one hand, as we have seen, causal triangulations are in bijection with planar rooted trees and thus the labelling process which enters the formulation of the Brownian map is trivial. This simplifies the formulation of the scaling limit greatly. On the other hand, due to the time-sliced structure of the causal triangulations it is harder to estimate the volume of balls around a randomly chosen point of the finite causal triangulation. Once the convergence to the scaling limit of the causal triangulation is shown, we expect that several properties of the scaling limit, such as its Hausdorff dimension and topology should follow from simple properties of the Brownian excursion.

In summary, we hope to have convinced the reader that probabilistic aspects of CDT and in particular the UICT as well as Gibbs causal triangulations are interesting models to be investigated. These formulations can be seen as interesting probabilistic models on their own right, but are also of essential importance for the physical understanding of CDT, most notably, in context of the study of the spectral dimension of CDT.