Uniform infinite and Gibbs causal triangulations

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About the cover: Simulation of the outgrowth along the spine of a size-biased critical Galton-Watson tree with off-spring probability $p(n) = 2^{-n-1}$. The embedding is such that, apart from the edges on the spine, the edge length falls off exponentially with the distance to the root.

UNIFORM INFINITE AND GIBBS CAUSAL TRIANGULATIONS

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Abstract

We discuss uniform infinite causal triangulations (UICT) and Gibbs causal triangulations which are probabilistic models for the causal dynamical triangulations (CDT) approach to quantum gravity. Since there is a bijection between causal triangulations and planar rooted trees we first discuss some aspects of random trees. In particular, we describe new methods to obtain the fractal and spectral dimension for a large class of random tree ensembles which in the thermodynamic limit have the property that they posses a unique infinite spine. The results are applied to obtain the spectral dimension of generic and non-generic trees, as well as a model of randomly grown trees. In the following, we discuss in detail the relation between UICT and size-biased critical Galton-Watson processes. This relation is used to prove convergence of the joint rescaled length-area-process to a diffusion process and to derive from this the quantum Hamiltonian of CDT. In what follows, in an alternative construction to the branching process, we propose a growth process which samples sections of UICT by elementary moves in which a single triangle is added with a certain probability. This construction is used to show that the fractal dimension of UICT is almost surely 2, in an alternative derivation to the branching process picture. Furthermore, we also derive convergence results for the rescaled length-area-process of the grown triangulation to a diffusion process leading to an interesting duality relation and a mathematically rigorous derivation of the so-called peeling procedure. In the final part, we discuss Gibbs causal triangulations and using the transfer matrix formalism we show convergence of the partition function to a limiting measure. Further, we analyse the transfer matrix of the Ising model coupled to (Gibbs) causal triangulations and derive several properties of the latter.

Contents

List of Figures

CHAPTER 1

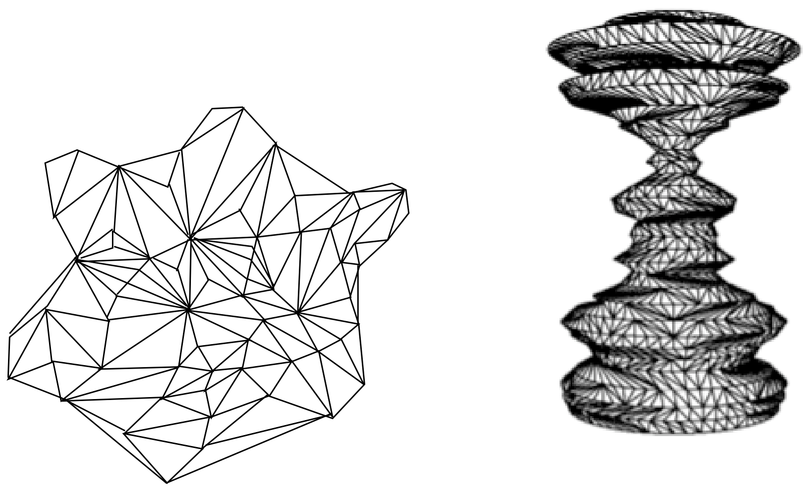
Background and a glimpse of the results

1.1 | Introduction and physical motivation

Models of random graphs are widely studied in the mathematics literature with applications in different areas of physics and sciences in general. In this work we are mainly interested in planar maps which are essentially embedded graphs with the topology of a sphere or a cylinder. The study of planar maps was initiated by Tutte in the 1960s [3, 4, 5, 6] and its first appearance in the physics literature was through the seminal works of 't Hooft [7] and Brezin, Itzykson, Parisi and Zuber [8] in the 1970s which shows the relation to so-called *random matrix models*.

Maybe its most important application in the filed of physics has been as a discretisation of fluctuating surfaces in models of two-dimensional quantum gravity. Firstly, in the 1980s so-called Dynamical Triangulations (DT) were introduced as models of two-dimensional Euclidean quantum gravity (see [9] for an overview), while later, in 1998 Ambjørn and Loll introduced so-called Causal Dynamical Triangulations (CDT) as models of two-dimensional Lorentzian quantum gravity ([10] and see [11, 12] for recent reviews).

For our purposes we can understand both models as statistical mechanical ensembles of planar triangulations (with topology of a sphere or a cylinder). The difference between the two is given in the configuration space: in DT the ensemble is taken over



(a) A typical planar triangulation.

(b) A typical causal triangulation.

Figure 1.1: Illustration of the typical triangulations entering the DT and CDT ensemble.

all planar triangulations (of a given topology), whereas in CDT the ensemble only includes triangulations with certain "time slices", i.e. strips of topology $S^1 \times [t, t+1]$ as shown in Figure 1.1. The *thermodynamic limit* or *scaling limit* of this model corresponds to the limit when the number of triangles $N \to \infty$.

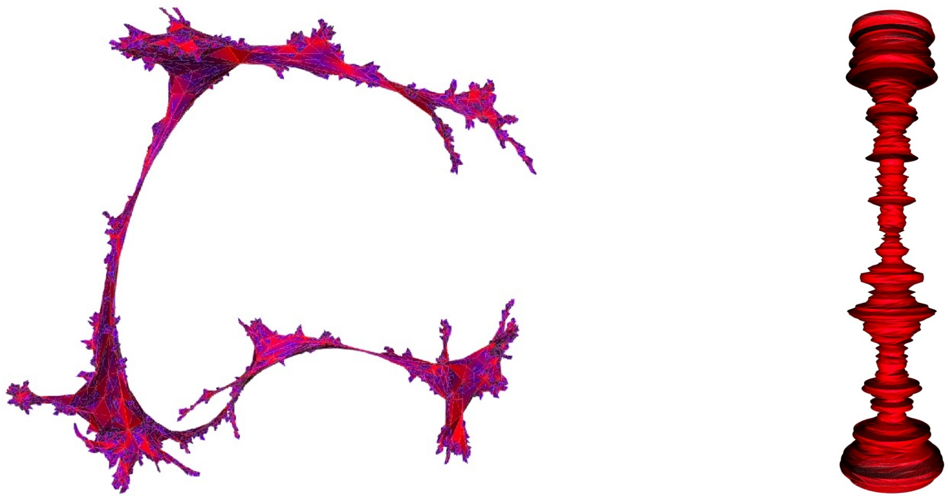
In what follows we are often interested in two "observables", the *fractal dimension* and the *spectral dimension*. For a fixed graph, the *fractal dimension* d_h is defined from the volume of a ball $\mathcal{B}_R$ of radius R through,

$$|\mathcal{B}_R| \approx R^{d_h} \tag{1.1}$$

as $R \to \infty$ where "$\approx$" means that both sides are asymptotically the same in a sense which will be made more precise in later chapters. In the corresponding physics literature the fractal dimension is often referred to as the *Hausdorff dimension*.[1]

The spectral dimension on the other hand is defined through the return probability $p(t)$ of a simple random walk on a graph, i.e. the probability that the walker is back at

[1] The reason for this is that in the scaling limit one expects the fractal dimension of the discrete graph ensemble to correspond to the Hausdorff dimension of the continuum metric space. In the case of the Brownian map both can be calculated and are indeed equal.



(a) A DT geometry. (b) A CDT geometry.

Figure 1.2: Illustration of the random geometry of DT and CDT in the scaling limit. The figures are generated from data of Monte Carlo simulations [1, 2].

its starting point after (diffusion) time t, through

$$p(t) \approx t^{-d_s/2} \tag{1.2}$$

as $t \to \infty$. For fixed graphs under certain regularity conditions there exists a general bound relating the fractal and spectral dimension, due to Coulhon [13, Theorem 7.7]

$$\frac{2d_h}{1+d_h} \leq d_s \leq d_h. \tag{1.3}$$

We will further comment on this relation later. If one has a statistical mechanical ensemble of random graphs, as in the case of DT or CDT, we will most often be interested in knowing the fractal dimension or spectral dimension *almost surely*, i.e. with probability one with respect to the probability measure associated with the random graph ensemble.

To analyse the fractal and spectral dimension of DT and CDT one uses their mathematical incarnations, the uniform infinite planar triangulation (UIPT) and the uniform infinite causal triangulation (UICT), as to be described below. As we will see for DT one has $d_h = 4$ while for CDT $d_h = 2$. The spectral dimension has been obtained numerically and is believed to be $d_s = 2$ in both cases. One sees that CDT is tight with the right-hand-side of (1.3) whereas DT is close to the left-hand-side of the bound. We say that a typical DT is fractal while a CDT is rather "surface-like". This is illustrated in Figure 1.2.

1.2 | From uniform infinite planar triangulations to Brownian maps

We now describe the mathematical rigorous model underlying DT. From a statistical physics point of view it is natural to assign a uniform measure to planar triangulations of a fixed size N. It was first shown by Angel and Schramm [14] that the limit $N \to \infty$ of this measure, the so-called uniform infinite planar triangulations (UIPT), exists as a weak limit.

In a later work [15], Angel introduced a growth process which samples UICT. Using this growth process he was able to prove that the area scales as $|\mathcal{B}_R| \approx R^4$ and the boundary length as $|\partial\mathcal{B}_R| \approx R^2$. This proved that the fractal dimension is $d_h = 4$ almost surely (see also [16]). Following this work, Krikun [17] obtained the exact limit theorem for the scaled boundary length $|\partial\mathcal{B}_R|$ (see also [18]).

The result for the spectral dimension is still an unsolved problem. Numerical simulations in the physics literature indicate that $d_s = 2$, but a proof is still missing. A related question is whether random walks on UIPT are almost surely recurrent, which would imply $d_s \leq 2$, or whether they are transient, which would imply $d_s \geq 2$. For planar maps whose vertex degree is bounded from above, it has been shown by Benjamini and Schramm that random walks are recurrent [19] (see also [20] for some recent progress).

Another highly interesting line of research is the study of so-called Brownian maps as pioneered by Le Gall, Miermont and others (see [21] and references therein). At the heart of this work are bijections of certain planar maps with labeled trees. These bijections were introduced by Cori-Vauquelin [22] and Schaeffer [23]. In particular, the Cori-Vauquelin-Schaeffer bijection describes a one-to-one map between (rooted and pointed) quadrangulations of the sphere and well-labelled trees, which are planar rooted trees with integer labels, where the root has label one and labels can only change by an absolute value of one from one vertex to another connected by an edge in the tree. This bijection was extended to more general planar maps, including planar triangulations, by Bouttier, Di Francesco and Guitter [24].

If we take the planar rooted tree and ignore the labels for a moment, we can construct a contour process, the so-called Dyke path, by following the contour of the tree from left to right. In the case of uniform planar trees it can be shown that the rescaled contour process converges to a normalised Brownian excursion. Equipping the Brownian excursion with an equivalence relation which identifies points on the excursion which correspond to equal points in the tree one obtains Aldous' continuum random tree [25]. If we consider the uniform measure on quadrangulations of the sphere of size N, it can be shown that the corresponding well-labeled trees converge to the continuum random tree with Brownian labels as $N \to \infty$. The labelling process determines the equivalence relation and the metric space obtained as the quotient space of the continuum random tree with this equivalence relation is called the *Brownian map* and was first introduced by Marckert and Mokkadem [26] (see [21] for more details). The details of the convergence proofs, in particular, the independence of the limit on a specific sub-sequence, have only been finished recently by Le Gall and Miermont [27, 28].

The above construction can be used to prove a number of interesting properties of the Brownian map. For example it has been shown that the topology of the Brownian map is that of the sphere [29, 30] and that its Hausdorff dimension (in the strict mathematical sense) is 4 [31].

1.3 | Causal triangulations and bijections with planar trees

As for the case of planar triangulations there also exists a bijection between causal triangulations and rooted planar labeled trees. However, in the case of causal triangulations the labelling is trivial and the bijection is thus between rooted causal triangulations and rooted planar trees. This bijection goes back to work by Di Francesco, Guitter and Kristjansen [32] where it was formulated for the dual triangulation. Its formulation in the here presented form was introduced by Malyshev, Yambartsev and Zamyatin [33] and later independently derived by Durhuss, Jonsson and Wheater [34].

Recall that a causal triangulation is composed of slices $S^1 \times [t, t+1]$ as shown in Figure 1.3. The details of the bijection are explained in Chapter 3, in essence it reduces to the following: Starting from a causal triangulation one defines a unique way to cut open the triangulation starting from the root edge and then removes all horizontal as well as every left-most outgoing edge from the triangulation, the result is a planar rooted tree (see Figure 1.3). The way of going from the planar rooted tree to the rooted causal triangulation should now be clear.

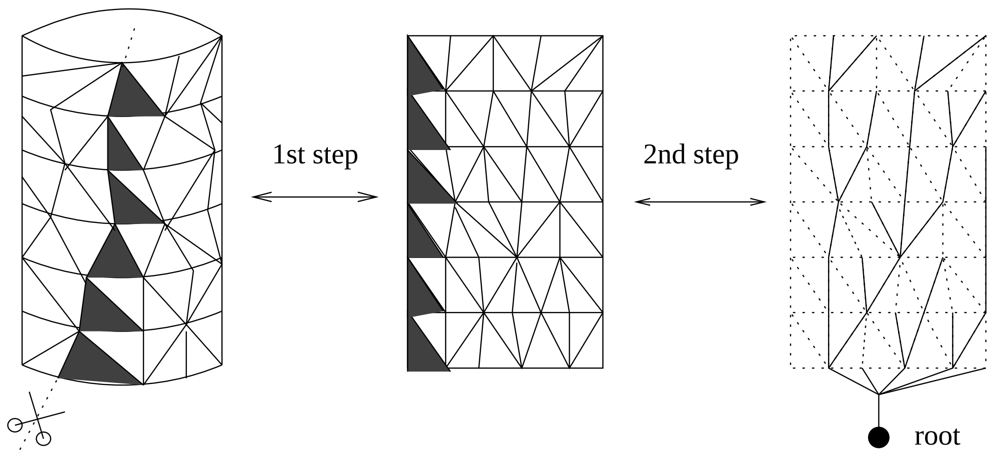


Figure 1.3: Bijection between (rooted) causal triangulations and planar rooted tress.

1.4 | Some aspects of random trees

In the previous section we have seen that causal triangulations are in bijection with planar rooted trees. In this respect it is useful to consider some properties of random planar rooted trees first before moving to the causal triangulations. To do so we associate a probability measure to planar rooted trees to obtain an ensemble of random trees. From a statistical physics point of view the simplest such implementation is via the micro-canonical ensemble. In the micro-canonical ensemble one assigns equal probabilities to all trees with a fixed number N of vertices (or edges) and then takes the thermodynamic limit in which $N \to \infty$. In a rigorous mathematical formulation this corresponds to so-called *uniform infinite planar trees*.

The uniform measure on finite planar rooted trees of size N can be seen to be equivalent to critical Galton Watson trees with offspring distribution $p(n) = 2^{-n-1}$ conditioned to have total progeny N. Recall that a Galton-Watson tree is a ancestor tree where each individual has an independently and identically distributed (i.i.d.) number n of offsprings according to the offspring probability $p(n)$. The process is said to be critical (subcritical or super-critical) if the average number of offsprings $\sum_{n\geq 0} np(n)$ is 1 (less than 1 or bigger than 1). For computational reasons it is often useful to introduce the generating function of the offspring distribution

$$f(s) = \sum_{n=0}^{\infty} p(n) s^n. \tag{1.4}$$

In the generating function language the criticality condition then corresponds to $f'(1) = 1$ and the variance of the offspring distribution is obtain by $f''(1)$.

The generating function for the size of the n-th generation is given by the n-th iterate of the generating function $f(s)$ of the offspring distribution $f_n = f \circ \cdots \circ f$, n-times. A well-known result by Kesten, Ney and Spitzer shows that for $f''(1) < \infty$ and $\eta_0 = 1$ (e.g. see [35])

$$\frac{1}{1 - f_n(s)} = \frac{1}{1-s} + \frac{f''(1)}{2} n + o(n), \quad \text{uniformly for } 0 \leq s < 1. \tag{1.5}$$

This shows that the probability that the population size of n-th generation is bigger than zero is given by $1 - f_n(0) = 2/(f''(1)n)(1 + o(1))$.

The existence of the limit $N \to \infty$ as a weak limit of the critical Galton Watson trees with finite variance $f''(1) < \infty$ conditioned to have total progeny N was first proven by Kennedy [36] and later by Aldous and Pitman [37]. It was shown there that the resulting measure, the size-biased critical Galton-Watson tree, is condensed almost surely on trees which have a unique infinite and non-backtracking path to infinity, the so-called *spine*. The (finite) trees attached to the vertices along the spine are (unconditioned) critical Galton-Watson trees. The number of finite trees attached to each vertex on the spine are i.i.d. with generating function $f'(s)$. Alternatively, the same measure can be obtained by taking a critical Galton-Watson tree and condition it on non-extinction.

In the physics literature size-biased Galton-Watson trees fall into the class of so-called *simply generated trees* [38] and the specific model of a size-biased critical Galton-Watson tree with finite variance the ensemble goes under the name of *generic trees* [39]. In the quantum gravity literature those models are also referred to as branched polymers and were used as an effective model to describe the so-called branched polymer phase of quantum gravity [40, 41, 42].

The simple structure of trees make them an easy toy model to study aspects of random geometry. For instance, the fractal dimension and spectral dimension can be computed as $d_h = 2$ and $d_s = 4/3$ almost surely. The former result for d_h is well known and a proof can for instance be found in [34]; the latter result for d_s is due to Barlow, Kumagai and Fujii [43, 44]. Alternative to the techniques developed by Barlow, Coulhon, Croydon, Kumagai and others [45, 43, 46, 44, 47], there is also an approach using generating functions of the return probabilities

$$Q(x) = \sum_t p(t)(1-x)^{t/2} \tag{1.6}$$

due to Duurhus, Jonsson and Wheater [48, 39, 34]. The spectral dimension is then defined through

$$Q(x) \approx x^{-1+d_s/2}, \quad x \to 0, \tag{1.7}$$

provided $Q(x)$ diverges as $x \to 0$. This definition is slightly weaker than the common definition through (1.2).

In Chapter 2, we provide a generalisation of the results of Duurhus, Jonsson and Wheater [39] for a general class of trees with a unique infinite spine. In doing so we introduce what we call the hull dimension. It is defined in a similar manner as the fractal dimension, but using the volume of the *hull* $\bar{\mathcal{B}}_R$, the outgrowths from vertices on the spine within distance R from the root,

$$|\bar{\mathcal{B}}_R| \approx R^{\bar{d}_h}, \quad R \to \infty. \tag{1.8}$$

In Theorem 2.2.1 we prove that for trees with a unique infinite spine one has

$$\frac{2d_h}{1+d_h} \leq d_s \leq \frac{2\bar{d}_h}{1+\bar{d}_h} \tag{1.9}$$

Furthermore, in Theorem 2.2.2 we show that in the case when the outgrowths $\mathcal{A}^i$ from vertices s_i along the spine are i.i.d. and the probability that the size of each outgrowth is bigger than N falls off as $N^{-\alpha}$, then

$$\bar{d}_h \leq \frac{1}{\alpha}, \qquad d_s \leq \frac{2}{1+\alpha}. \tag{1.10}$$

If in addition the probability that the height of an outgrowth from a vertex on the spine exceeds n falls off as n^{-1} then one has equality in (1.10). One can understand this result in the way that if the finite outgrowths die out fast enough then $\mathcal{B}_R$ and $\bar{\mathcal{B}}_R$ are close in size and thus $d_h = \bar{d}_h$.

In a first application, in Section 2.3, we employ these results to determine the fractal and spectral dimension of generic and critical non-generic trees. The case of generic trees was already discussed above. Critical non-generic trees correspond to critical size-biased Galton-Watson trees with infinite variance, where the offspring probability

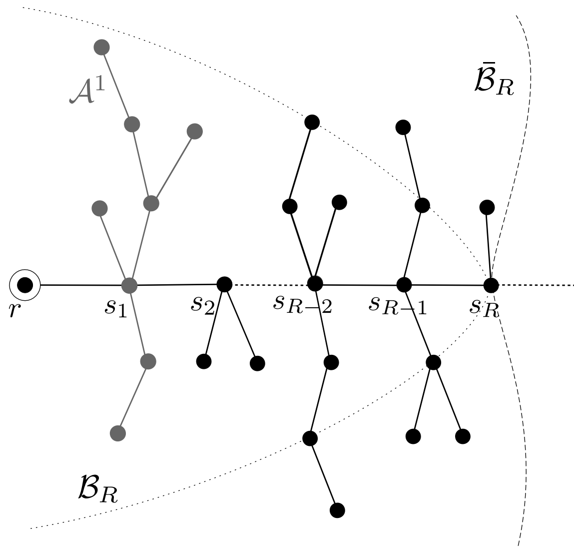


Figure 1.4: An illustration of a tree with a unique infinite spine with vertices $r, s_1, s_2, \ldots$. Shown are also the ball $\mathcal{B}_R$ and the hull $\bar{\mathcal{B}}_R$.

falls off with a power-law behaviour, $p(n) \sim n^{-\beta}$, and $\beta \in (2, 3]$. In the probability literature those random tree ensembles also go under the name of α-stable trees. Their fractal and spectral dimensions are almost surely given by

$$d_h = \frac{\beta - 1}{\beta - 2} \quad \text{and} \quad d_s = \frac{2(\beta - 1)}{2\beta - 3} \tag{1.11}$$

These values were conjectured in the physics literature [49, 50] and first proven by Croydon and Kumagai [46]. In Theorem 2.3.2, we provide an alternative and much simplified proof of these results using Theorem 2.2.2 (though slightly weaker than the proof by Croydon and Kumagai in the sense as described above). Another attractive feature is that our proof has a very intuitive and geometric interpretation.

In a second application, in Section 2.4, we analyse the fractal and spectral dimension of a model of randomly grown trees, the so-called attachment and grafting model [51]. In Theorem 2.4.1, we give an almost surely upper bound on the fractal and spectral dimension for the full parameter range of the model (which is believed to be tight) and determine their values almost surely for a smaller range of parameters.

1.5 | Uniform infinite causal triangulations and their properties

As for the planar triangulations and planar rooted trees one can assign a probability measure to the causal triangulations by introducing the uniform measure on causal triangulations with size (number of triangles) N. The limit $N \to \infty$ then corresponds to what is called *Uniform infinite causal triangulations* (UICT). Since causal triangulations are in bijection with rooted planar trees and since (apart from the boundaries) there are twice as many triangles in the triangulation as vertices in the tree, the UICT measure is equivalent to the uniform measure on infinite planar trees, i.e. the size-biased critical Galton-Watson tree with offspring distribution $p(n) = 2^{-n-1}$. This relation was first noticed by Yambartsev and Krikun [52] and independently proven by Duurhus, Jonsson and Wheater [34] on basis of their earlier work on generic trees [39]. In Chapter 3, we give an alternative proof taking as a starting point the original results by Kennedy [36], Aldous and Pitman [37].

In the remainder of Chapter 3 we are interested in the convergence of the joint boundary length and volume (area) process. Consider a ball $\mathcal{B}_R$ of radius R around the root. From the result of the previous section we know the scaling of the boundary length $|\partial\mathcal{B}_R| \approx R$ and area $|\mathcal{B}_R| \approx R^2$. We then show in Theorem 3.4.1 that the rescaled boundary length process converges to a diffusion process L_τ with Itô's equation

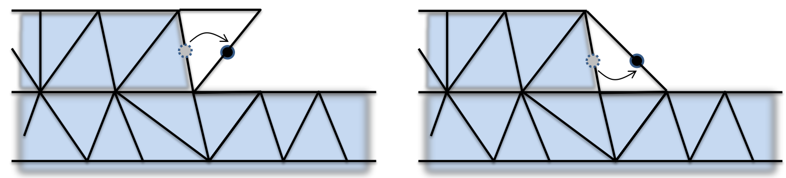
$$dL_\tau = 2d\tau + \sqrt{2L_\tau}dB_\tau \quad L_0 = l, \tag{1.12}$$

where B_τ is standard Brownian motion of variance one. Furthermore, it is shown in Theorem 3.4.2 that jointly the rescaled area process converges to $A_\tau = 2\int_0^\tau L_u du$ as expected. The Feynman-Kac equation for $\phi_{\xi,\lambda}(l,\tau) = \mathbb{E}[\exp(-\xi L_\tau - \lambda A_\tau)|L_0 = l]$ is given by

$$-\frac{\partial}{\partial\tau}\phi_{\xi,\lambda}(l,\tau) = \hat{H}\phi_{\xi,\lambda}(l,\tau), \quad \hat{H} = -2\frac{\partial}{\partial l} - l\frac{\partial^2}{\partial l^2} + 2\lambda l, \quad \phi_{\xi,\lambda}(l,0) = e^{-\xi l}. \tag{1.13}$$

Here $\hat{H}$ is called the quantum Hamiltonian with cosmological constant λ.

While many of the results follow rather straightforwardly from the properties of the Markov chain which underlies the Galton-Watson branching process, there are nonetheless very important since they close a gap between the works on causal triangulations in the probability and physics literature. On the one hand, in the probabilistic setting, one proposes a micro-canonical ensemble for the causal triangulations, the



(a) The $(+)$-move (b) The $(-)$-move

Figure 1.5: The two different moves of the growth process.

UICT and a focus of study have been properties such as the fractal or spectral dimension, as we describe below. On the other hand, in the physics literature, one starts off with a canonical ensemble with cosmological constant λ (the equivalent of $\beta = 1/(k_B T)$ in the thermodynamic language) and is mainly interested in certain boundary length correlators. The importance of the results in Chapter 3 is that it provides first links between the two approaches.

In Chapter 4, analogous to the work of Angel [15], we introduce a growth process which samples UICT by adding a single triangle according to two different moves, denoted "$(+)$" and "$(-)$", with probability

$$\mathbb{P}((\pm) - \text{move} \mid l(T_n) = m) = \frac{1}{2}\frac{m \pm 1}{m}. \tag{1.14}$$

Here $l(T_n)$ denotes the length of the boundary of the triangulation at step n of the growth process. The two possibilities in which a triangle can be added, the $(\pm)$-moves, are shown in Figure 1.5. One observes that the $(+)$-move increases the boundary length by one while the $(-)$-move decreases it by one. We can give an equivalent description of the growth process through the recurrent Markov chain $\{M_n\}_{n\geq 0}$ in terms of the boundary length of the triangulation $M_n = l(T_n)$.

In Theorems 4.2.2 and 4.2.3 we prove that the growth process indeed samples sections of UICT. By introducing stoping times n_t at which the growth process completes the strip $S^1 \times [t, t+1]$ at "height" t of the causal triangulation we prove in Theorem 4.3.1 that the fractal dimension is almost surely $d_s = 2$. This proof is alternative to the corresponding proof by Durhuus, Jonsson and Wheater [34] using the above described relation to Galton-Watson trees which also follows directly from Theorem 2.2.2 of Chapter 2.

In Theorem 4.4.1 we prove convergence of the rescaled Markov chain $\{M_n\}_{n\geq 0}$ to a

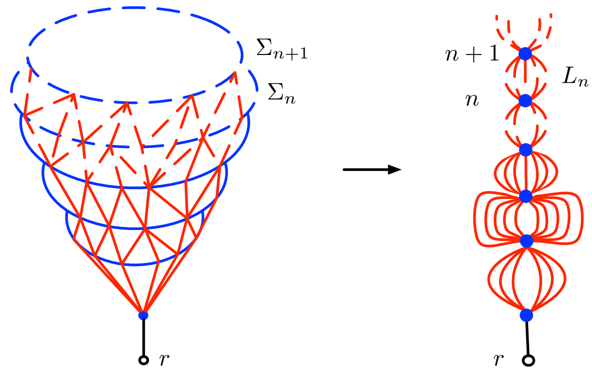


Figure 1.6: Reduction to cut-sets in the recurrence proof of UICT, the so-called multi-graph [34].

diffusion process with Itô's equation

$$dM_u = \frac{1}{M_u} du + dB_u. \tag{1.15}$$

It is then shown in Theorem 4.4.2 and 4.4.3 that by a random time change $L_\tau = M_{T_\tau^{-1}}$ one obtains a diffusion process (1.12). This clarifies the so-called peeling procedure as introduced in the physics literature by Watabiki [53] for DT and in [54] for CDT.

One observes that in the calculation using Galton-Watson trees, as presented above, time or geodesic distance τ is fixed while area $A_\tau = \int_0^\tau 2L_u du$ becomes a random variable, while in the growth process one has the opposite situation, growth time u which is equal to area is fixed and geodesic distance $T_u = \int_0^u \frac{1}{2M_v} dv$ is a random variable.

The calculation of the spectral dimension is more involved. Using the Nash-Williams Theorem it can be shown that random walks on the UICT are recurrent and that thus $d_s \leq 2$ almost surely [34]. The proof uses so-called cut sets which correspond to slices of the graph shown on the right-hand side of Figure 1.6 These graphs were introduced in [34] under the name of multi-graphs.

On can obtain an insightful interpretation using the notion of electrical resistance, viewing each edge in the graph as a unit resistor (see for instance [55]). Due to Rayleigh's monotonicity law the resistance between any two points in an electrical network is increased (decreased) if one increases (decreases) the resistance anywhere in the network. Hence, one sees that the multigraph and the tree provide a corresponding lower and upper bound for the resistance of the triangulation. If one introduces an exponent ρ for the resistance growth between two points at distance R, $\mathcal{R}_R \approx R^\rho$, as

$R \to \infty$, one expects the following scaling of the spectral dimension,

$$d_s = \frac{2d_h}{d_h + \rho} \tag{1.16}$$

Indeed, considering that the tree has $\rho = 1$ and that for the multigraph one expects $\rho = 2 - d_h$, one sees that the tree and the multi-graph form the lower and upper bound of (1.3) respectively,

$$\frac{2d_h}{1 + d_h} \leq d_s \leq d_h. \tag{1.17}$$

For the UICT it is conjectured that $d_s = 2\ (= d_h)$ almost surely. In the absence of a proof, the multi-graphs have been used as toy models to study different aspects of diffusion on causal triangulations ensembles, as was done by Giasemidis, Wheater and Zohren [56, 57] which form part of Giasemidis' PhD thesis [58].

1.6 | Gibbs causal triangulations and coupling to the Ising model

We have so far focused on a formulation through the micro-canonical ensemble of causal triangulations, the UICT. Instead we can also approach the problem from a point of view of a canonical ensemble. This is done by introducing the Gibbs measure $e^{-\mu|T|}/Z$, where each triangulation is weighted by a Boltzmann factor $e^{-\mu|T|}$. Here $|T|$ denotes the size of the triangulation and μ the cosmological constant (the analog of inverse temperature in this context). The action $S(\mu)$ in the Boltzmann factor $e^{-S(\mu)}$ is the discretisation of the so-called Einstein-Hilbert action in two dimensions. Since causal triangulations have an inherent time-sliced structure it is natural to introduce the transfer matrix $U = \{u(n, n')\}_{n,n'=1,2,\ldots}$ for a slice with n vertices on the initial boundary and n' vertices on the final boundary. The partition function for causal triangulations of height N is then given by the trace of the N's product of the transfer matrix

$$Z_N(\mu) = \mathrm{tr}\, U^N \tag{1.18}$$

From statistical mechanics we expect that when $N \to \infty$ the partition function is governed by the largest eigenvalue $\Lambda(\mu)$ of the transfer matrix U and its transpose U^*,

$$Z_N(\mu) \sim \Lambda(\mu)^N. \tag{1.19}$$

We use so-called Krein-Rutman theory [59] on operators preserving the cone of positive functions to make this statement precise. In particular, following results of Malyshev, Yambartsev, Zamyatin [33], in Theorem 5.3.1 we prove the convergence of the scaled log of the partition function

$$\frac{1}{N} \log Z_N(\mu) = \Lambda(\mu). \tag{1.20}$$

for $\mu > \mu_c = \log 2$. Furthermore, we show that the finite-level Gibbs measure $\mathbb{P}_N$ converges weakly to a limiting measure $\mathbb{P}$. Here μ_c is the critical value of the cosmological constant. One sees that for $\mu \to \mu_c$ the limiting measure corresponds to the UICT, showing equivalence of the micro-canonical and canonical formulation.

In the following, we extend these results to the case of an Ising model coupled to causal triangulations. The transfer matrix is then given by the joint sum over triangulations and spin configurations of a slice. Instead of a critical point $\mu_c = \log 2$ we now expect there to be a critical line $\mu_c(\beta)$, where β is inverse temperature. In fact, $\mu_c(\beta)$ corresponds to the free energy per triangle and its precise knowledge would give us insight into the phase transition of the model. While we are not able to determine $\mu_c(\beta)$ precisely, in Lemma 5.4.1, we use Krein-Rutman Theory to determine a region in the coupling quadrant (μ, β) where the partition function converges. However, our results are not tight and we cannot be sure that the boundary of this region corresponds to the critical line. However, we hope in future work to complement this analysis with numerical results determining the exact shape of the critical line.

1.7 | Structure of the remainder of the thesis

The remaining chapters of this thesis are based on the following articles:

- Chapter 2 on S. Ö. Stefánsson and S. Zohren, "Spectral dimension of trees with a unique infinite spine" *J. Stat. Phys.* **147** (2012) 942, arXiv:1202.5388 [cond-mat.stat-mech];

- Chapter 3 on: V. Sisko, A. Yambartsev and S. Zohren. "A note on weak convergence results for uniform infinite causal triangulations," to appear in *Markov Proc. Rel. Fields* (2013), arXiv:1201.0264 [math-ph];

- Chapter 4 on: V. Sisko, A. Yambartsev and S. Zohren, "Growth of uniform infinite causal triangulations" *J. Stat. Phys.* OnlineFirst (2012), arXiv: 1203.2869 [math-ph];

- Chapter 5 on: J.H.C. Hernandez, Y. Suhov, A. Yambartsev and S. Zohren “Properties of transfer matrix for Ising model coupled to causal dynamical triangulations” submitted (2012).

The following two articles, completed while at Leiden University, are centred around related applications of random matrix models to CDT, which form the physical foundations behind the growth process described in Chapter 4 and have not been included in the thesis:

- J. Ambjørn, R. Loll, W. Westra, and S. Zohren, “Summing over all topologies in CDT string field theory,” *Phys.Lett.* **B678** (2009) 227–232, arXiv:0905.2108 [hep-th];

- J. Ambjørn, R. Loll, W. Westra, and S. Zohren, “Stochastic quantization and the role of time in quantum gravity,” *Phys.Lett.* **B680** (2009) 359–364, arXiv:0908.4224 [hep-th].

Furthermore, the following two articles regarding applications to quantum statistics and applied statistics have also not been included in the thesis:

- S. Zohren, P. Reska, R. D. Gill and W. Westra, “A tight Tsirelson inequality for infinitely many outcomes” *Europhysics Letters* **90** (2010) 10002, arXiv:1003.0616 [quant-ph];

- D. Anevski, R. D. Gill and S. Zohren, “Estimating a probability mass function with unknown labels” to appear.