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Uniform infinite and Gibbs causal triangulations

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Uniform Infinite and Gibbs Causal Triangulations

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About the cover: Simulation of the outgrowth along the spine of a size-biased critical Galton-Watson tree with off-spring probability $p(n) = 2^{-n-1}$. The embedding is such that, apart from the edges on the spine, the edge length falls off exponentially with the distance to the root.

UNIFORM INFINITE AND GIBBS CAUSAL TRIANGULATIONS

Stefan Zohren



Universiteit Leiden

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Abstract

We discuss uniform infinite causal triangulations (UIC_T) and Gibbs causal triangulations which are probabilistic models for the causal dynamical triangulations (CDT) approach to quantum gravity. Since there is a bijection between causal triangulations and planar rooted trees we first discuss some aspects of random trees. In particular, we describe new methods to obtain the fractal and spectral dimension for a large class of random tree ensembles which in the thermodynamic limit have the property that they possess a unique infinite spine. The results are applied to obtain the spectral dimension of generic and non-generic trees, as well as a model of randomly grown trees. In the following, we discuss in detail the relation between UIC_T and size-biased critical Galton-Watson processes. This relation is used to prove convergence of the joint rescaled length-area-process to a diffusion process and to derive from this the quantum Hamiltonian of CDT. In what follows, in an alternative construction to the branching process, we propose a growth process which samples sections of UIC_T by elementary moves in which a single triangle is added with a certain probability. This construction is used to show that the fractal dimension of UIC_T is almost surely 2, in an alternative derivation to the branching process picture. Furthermore, we also derive convergence results for the rescaled length-area-process of the grown triangulation to a diffusion process leading to an interesting duality relation and a mathematically rigorous derivation of the so-called peeling procedure. In the final part, we discuss Gibbs causal triangulations and using the transfer matrix formalism we show convergence of the partition function to a limiting measure. Further, we analyse the transfer matrix of the Ising model coupled to (Gibbs) causal triangulations and derive several properties of the latter.

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