Uniform infinite and Gibbs causal triangulations

Zohren, S.

Author: Zohren, Stefan
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About the cover: Simulation of the outgrowth along the spine of a size-biased critical Galton-Watson tree with off-spring probability $p(n) = 2^{-n-1}$. The embedding is such that, apart from the edges on the spine, the edge length falls off exponentially with the distance to the root.

UNIFORM INFINITE AND GIBBS CAUSAL TRIANGULATIONS

Stefan Zohren

Universiteit Leiden

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Abstract

We discuss uniform infinite causal triangulations (UICT) and Gibbs causal triangulations which are probabilistic models for the causal dynamical triangulations (CDT) approach to quantum gravity. Since there is a bijection between causal triangulations and planar rooted trees we first discuss some aspects of random trees. In particular, we describe new methods to obtain the fractal and spectral dimension for a large class of random tree ensembles which in the thermodynamic limit have the property that they posses a unique infinite spine. The results are applied to obtain the spectral dimension of generic and non-generic trees, as well as a model of randomly grown trees. In the following, we discuss in detail the relation between UICT and size-biased critical Galton-Watson processes. This relation is used to prove convergence of the joint rescaled length-area-process to a diffusion process and to derive from this the quantum Hamiltonian of CDT. In what follows, in an alternative construction to the branching process, we propose a growth process which samples sections of UICT by elementary moves in which a single triangle is added with a certain probability. This construction is used to show that the fractal dimension of UICT is almost surely 2, in an alternative derivation to the branching process picture. Furthermore, we also derive convergence results for the rescaled length-area-process of the grown triangulation to a diffusion process leading to an interesting duality relation and a mathematically rigorous derivation of the so-called peeling procedure. In the final part, we discuss Gibbs causal triangulations and using the transfer matrix formalism we show convergence of the partition function to a limiting measure. Further, we analyse the transfer matrix of the Ising model coupled to (Gibbs) causal triangulations and derive several properties of the latter.

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CHAPTER 1

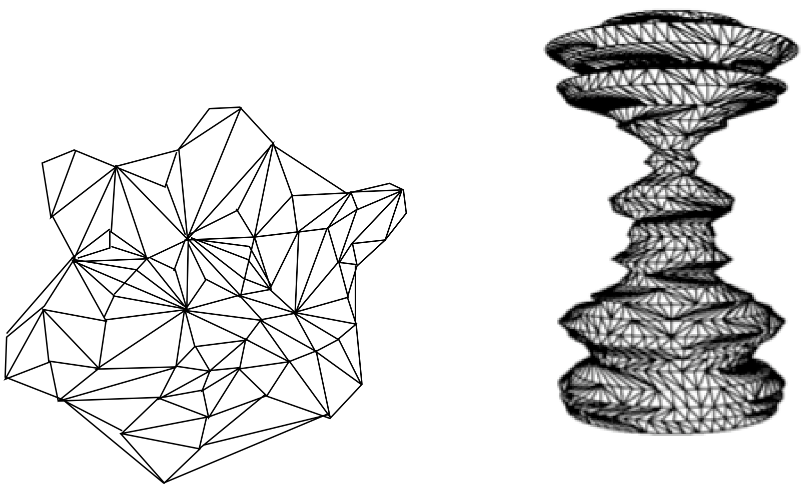
Background and a glimpse of the results

1.1 | Introduction and physical motivation

Models of random graphs are widely studied in the mathematics literature with applications in different areas of physics and sciences in general. In this work we are mainly interested in planar maps which are essentially embedded graphs with the topology of a sphere or a cylinder. The study of planar maps was initiated by Tutte in the 1960s [3, 4, 5, 6] and its first appearance in the physics literature was through the seminal works of 't Hooft [7] and Brezin, Itzykson, Parisi and Zuber [8] in the 1970s which shows the relation to so-called *random matrix models*.

Maybe its most important application in the filed of physics has been as a discretisation of fluctuating surfaces in models of two-dimensional quantum gravity. Firstly, in the 1980s so-called Dynamical Triangulations (DT) were introduced as models of two-dimensional Euclidean quantum gravity (see [9] for an overview), while later, in 1998 Ambjørn and Loll introduced so-called Causal Dynamical Triangulations (CDT) as models of two-dimensional Lorentzian quantum gravity ([10] and see [11, 12] for recent reviews).

For our purposes we can understand both models as statistical mechanical ensembles of planar triangulations (with topology of a sphere or a cylinder). The difference between the two is given in the configuration space: in DT the ensemble is taken over



(a) A typical planar triangulation.

(b) A typical causal triangulation.

Figure 1.1: Illustration of the typical triangulations entering the DT and CDT ensemble.

all planar triangulations (of a given topology), whereas in CDT the ensemble only includes triangulations with certain “time slices”, i.e. strips of topology $S^1 \times [t, t+1]$ as shown in Figure 1.1. The *thermodynamic limit* or *scaling limit* of this model corresponds to the limit when the number of triangles $N \to \infty$.

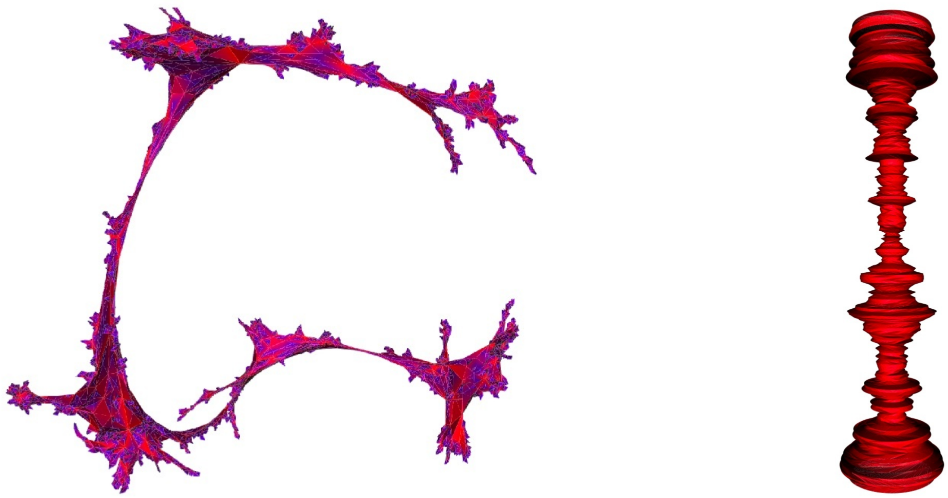
In what follows we are often interested in two “observables”, the *fractal dimension* and the *spectral dimension*. For a fixed graph, the *fractal dimension* d_h is defined from the volume of a ball $\mathcal{B}_R$ of radius R through,

$$|\mathcal{B}_R| \approx R^{d_h} \tag{1.1}$$

as $R \to \infty$ where “$\approx$” means that both sides are asymptotically the same in a sense which will be made more precise in later chapters. In the corresponding physics literature the fractal dimension is often referred to as the *Hausdorff dimension*.[1]

The spectral dimension on the other hand is defined through the return probability $p(t)$ of a simple random walk on a graph, i.e. the probability that the walker is back at

[1]The reason for this is that in the scaling limit one expects the fractal dimension of the discrete graph ensemble to correspond to the Hausdorff dimension of the continuum metric space. In the case of the Brownian map both can be calculated and are indeed equal.



(a) A DT geometry. (b) A CDT geometry.

Figure 1.2: Illustration of the random geometry of DT and CDT in the scaling limit. The figures are generated from data of Monte Carlo simulations [1, 2].

its starting point after (diffusion) time t, through

$$p(t) \approx t^{-d_s/2} \tag{1.2}$$

as $t \to \infty$. For fixed graphs under certain regularity conditions there exists a general bound relating the fractal and spectral dimension, due to Coulhon [13, Theorem 7.7]

$$\frac{2d_h}{1+d_h} \leq d_s \leq d_h. \tag{1.3}$$

We will further comment on this relation later. If one has a statistical mechanical ensemble of random graphs, as in the case of DT or CDT, we will most often be interested in knowing the fractal dimension or spectral dimension *almost surely*, i.e. with probability one with respect to the probability measure associated with the random graph ensemble.

To analyse the fractal and spectral dimension of DT and CDT one uses their mathematical incarnations, the uniform infinite planar triangulation (UIPT) and the uniform infinite causal triangulation (UICT), as to be described below. As we will see for DT one has $d_h = 4$ while for CDT $d_h = 2$. The spectral dimension has been obtained numerically and is believed to be $d_s = 2$ in both cases. One sees that CDT is tight with the right-hand-side of (1.3) whereas DT is close to the left-hand-side of the bound. We say that a typical DT is fractal while a CDT is rather “surface-like”. This is illustrated in Figure 1.2.

1.2 | From uniform infinite planar triangulations to Brownian maps

We now describe the mathematical rigorous model underlying DT. From a statistical physics point of view it is natural to assign a uniform measure to planar triangulations of a fixed size N. It was first shown by Angel and Schramm [14] that the limit $N \to \infty$ of this measure, the so-called uniform infinite planar triangulations (UIPT), exists as a weak limit.

In a later work [15], Angel introduced a growth process which samples UICT. Using this growth process he was able to prove that the area scales as $|\mathcal{B}_R| \approx R^4$ and the boundary length as $|\partial\mathcal{B}_R| \approx R^2$. This proved that the fractal dimension is $d_h = 4$ almost surely (see also [16]). Following this work, Krikun [17] obtained the exact limit theorem for the scaled boundary length $|\partial\mathcal{B}_R|$ (see also [18]).

The result for the spectral dimension is still an unsolved problem. Numerical simulations in the physics literature indicate that $d_s = 2$, but a proof is still missing. A related question is whether random walks on UIPT are almost surely recurrent, which would imply $d_s \leq 2$, or whether they are transient, which would imply $d_s \geq 2$. For planar maps whose vertex degree is bounded from above, it has been shown by Benjamini and Schramm that random walks are recurrent [19] (see also [20] for some recent progress).

Another highly interesting line of research is the study of so-called Brownian maps as pioneered by Le Gall, Miermont and others (see [21] and references therein). At the heart of this work are bijections of certain planar maps with labeled trees. These bijections were introduced by Cori-Vauquelin [22] and Schaeffer [23]. In particular, the Cori-Vauquelin-Schaeffer bijection describes a one-to-one map between (rooted and pointed) quadrangulations of the sphere and well-labelled trees, which are planar rooted trees with integer labels, where the root has label one and labels can only change by an absolute value of one from one vertex to another connected by an edge in the tree. This bijection was extended to more general planar maps, including planar triangulations, by Bouttier, Di Francesco and Guitter [24].

If we take the planar rooted tree and ignore the labels for a moment, we can construct a contour process, the so-called Dyke path, by following the contour of the tree from left to right. In the case of uniform planar trees it can be shown that the rescaled contour process converges to a normalised Brownian excursion. Equipping the Brownian excursion with an equivalence relation which identifies points on the excursion which correspond to equal points in the tree one obtains Aldous' continuum random tree [25]. If we consider the uniform measure on quadrangulations of the sphere of size N, it can be shown that the corresponding well-labeled trees converge to the continuum random tree with Brownian labels as $N \to \infty$. The labelling process determines the equivalence relation and the metric space obtained as the quotient space of the continuum random tree with this equivalence relation is called the *Brownian map* and was first introduced by Marckert and Mokkadem [26] (see [21] for more details). The details of the convergence proofs, in particular, the independence of the limit on a specific sub-sequence, have only been finished recently by Le Gall and Miermont [27, 28].

The above construction can be used to prove a number of interesting properties of the Brownian map. For example it has been shown that the topology of the Brownian map is that of the sphere [29, 30] and that its Hausdorff dimension (in the strict mathematical sense) is 4 [31].

1.3 | Causal triangulations and bijections with planar trees

As for the case of planar triangulations there also exists a bijection between causal triangulations and rooted planar labeled trees. However, in the case of causal triangulations the labelling is trivial and the bijection is thus between rooted causal triangulations and rooted planar trees. This bijection goes back to work by Di Francesco, Guitter and Kristjansen [32] where it was formulated for the dual triangulation. Its formulation in the here presented form was introduced by Malyshev, Yambartsev and Zamyatin [33] and later independently derived by Durhuss, Jonsson and Wheater [34].

Recall that a causal triangulation is composed of slices $S^1 \times [t, t+1]$ as shown in Figure 1.3. The details of the bijection are explained in Chapter 3, in essence it reduces to the following: Starting from a causal triangulation one defines a unique way to cut open the triangulation starting from the root edge and then removes all horizontal as well as every left-most outgoing edge from the triangulation, the result is a planar rooted tree (see Figure 1.3). The way of going from the planar rooted tree to the rooted causal triangulation should now be clear.

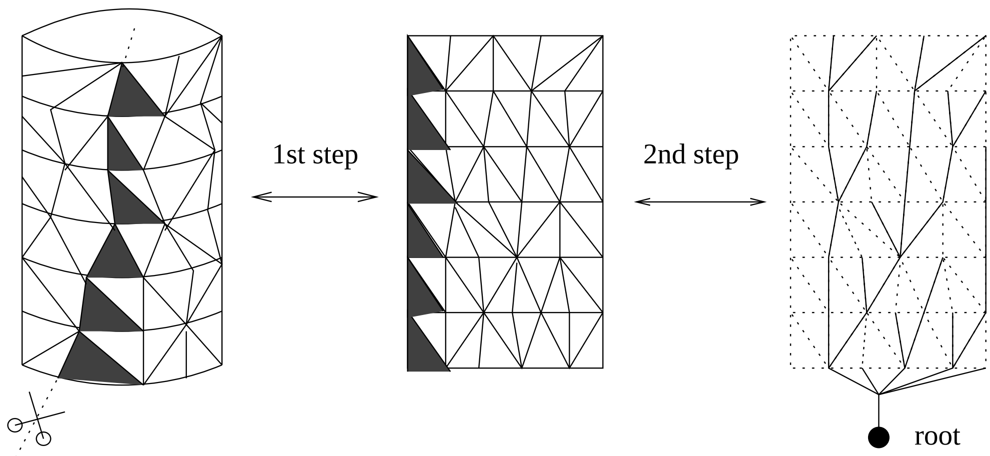


Figure 1.3: Bijection between (rooted) causal triangulations and planar rooted tress.

1.4 | Some aspects of random trees

In the previous section we have seen that causal triangulations are in bijection with planar rooted trees. In this respect it is useful to consider some properties of random planar rooted trees first before moving to the causal triangulations. To do so we associate a probability measure to planar rooted trees to obtain an ensemble of random trees. From a statistical physics point of view the simplest such implementation is via the micro-canonical ensemble. In the micro-canonical ensemble one assigns equal probabilities to all trees with a fixed number N of vertices (or edges) and then takes the thermodynamic limit in which $N \to \infty$. In a rigorous mathematical formulation this corresponds to so-called *uniform infinite planar trees*.

The uniform measure on finite planar rooted trees of size N can be seen to be equivalent to critical Galton Watson trees with offspring distribution $p(n) = 2^{-n-1}$ conditioned to have total progeny N. Recall that a Galton-Watson tree is a ancestor tree where each individual has an independently and identically distributed (i.i.d.) number n of offsprings according to the offspring probability $p(n)$. The process is said to be critical (subcritical or super-critical) if the average number of offsprings $\sum_{n\geq 0} np(n)$ is 1 (less than 1 or bigger than 1). For computational reasons it is often useful to introduce the generating function of the offspring distribution

$$f(s) = \sum_{n=0}^{\infty} p(n)s^n. \tag{1.4}$$

In the generating function language the criticality condition then corresponds to $f'(1) = 1$ and the variance of the offspring distribution is obtain by $f''(1)$.

The generating function for the size of the n-th generation is given by the n-th iterate of the generating function $f(s)$ of the offspring distribution $f_n = f \circ \cdots \circ f$, n-times. A well-known result by Kesten, Ney and Spitzer shows that for $f''(1) < \infty$ and $\eta_0 = 1$ (e.g. see [35])

$$\frac{1}{1-f_n(s)} = \frac{1}{1-s} + \frac{f''(1)}{2}n + o(n), \quad \text{uniformly for } 0 \leq s < 1. \tag{1.5}$$

This shows that the probability that the population size of n-th generation is bigger than zero is given by $1 - f_n(0) = 2/(f''(1)n)(1 + o(1))$.

The existence of the limit $N \to \infty$ as a weak limit of the critical Galton Watson trees with finite variance $f''(1) < \infty$ conditioned to have total progeny N was first proven by Kennedy [36] and later by Aldous and Pitman [37]. It was shown there that the resulting measure, the size-biased critical Galton-Watson tree, is condensed almost surely on trees which have a unique infinite and non-backtracking path to infinity, the so-called *spine*. The (finite) trees attached to the vertices along the spine are (unconditioned) critical Galton-Watson trees. The number of finite trees attached to each vertex on the spine are i.i.d. with generating function $f'(s)$. Alternatively, the same measure can be obtained by taking a critical Galton-Watson tree and condition it on non-extinction.

In the physics literature size-biased Galton-Watson trees fall into the class of so-called *simply generated trees* [38] and the specific model of a size-biased critical Galton-Watson tree with finite variance the ensemble goes under the name of *generic trees* [39]. In the quantum gravity literature those models are also referred to as branched polymers and were used as an effective model to describe the so-called branched polymer phase of quantum gravity [40, 41, 42].

The simple structure of trees make them an easy toy model to study aspects of random geometry. For instance, the fractal dimension and spectral dimension can be computed as $d_h = 2$ and $d_s = 4/3$ almost surely. The former result for d_h is well known and a proof can for instance be found in [34]; the latter result for d_s is due to Barlow, Kumagai and Fujii [43, 44]. Alternative to the techniques developed by Barlow, Coulhon, Croydon, Kumagai and others [45, 43, 46, 44, 47], there is also an approach using generating functions of the return probabilities

$$Q(x) = \sum_t p(t)(1-x)^{t/2} \tag{1.6}$$

due to Duurhus, Jonsson and Wheater [48, 39, 34]. The spectral dimension is then defined through

$$Q(x) \approx x^{-1+d_s/2}, \quad x \to 0, \tag{1.7}$$

provided $Q(x)$ diverges as $x \to 0$. This definition is slightly weaker than the common definition through (1.2).

In Chapter 2, we provide a generalisation of the results of Duurhus, Jonsson and Wheater [39] for a general class of trees with a unique infinite spine. In doing so we introduce what we call the hull dimension. It is defined in a similar manner as the fractal dimension, but using the volume of the *hull* $\bar{\mathcal{B}}_R$, the outgrowths from vertices on the spine within distance R from the root,

$$|\bar{\mathcal{B}}_R| \approx R^{\bar{d}_h}, \quad R \to \infty. \tag{1.8}$$

In Theorem 2.2.1 we prove that for trees with a unique infinite spine one has

$$\frac{2d_h}{1+d_h} \leq d_s \leq \frac{2\bar{d}_h}{1+\bar{d}_h} \tag{1.9}$$

Furthermore, in Theorem 2.2.2 we show that in the case when the outgrowths $\mathcal{A}^i$ from vertices s_i along the spine are i.i.d. and the probability that the size of each outgrowth is bigger than N falls off as $N^{-\alpha}$, then

$$\bar{d}_h \leq \frac{1}{\alpha}, \qquad d_s \leq \frac{2}{1+\alpha}. \tag{1.10}$$

If in addition the probability that the height of an outgrowth from a vertex on the spine exceeds n falls off as n^{-1} then one has equality in (1.10). One can understand this result in the way that if the finite outgrowths die out fast enough then $\mathcal{B}_R$ and $\bar{\mathcal{B}}_R$ are close in size and thus $d_h = \bar{d}_h$.

In a first application, in Section 2.3, we employ these results to determine the fractal and spectral dimension of generic and critical non-generic trees. The case of generic trees was already discussed above. Critical non-generic trees correspond to critical size-biased Galton-Watson trees with infinite variance, where the offspring probability

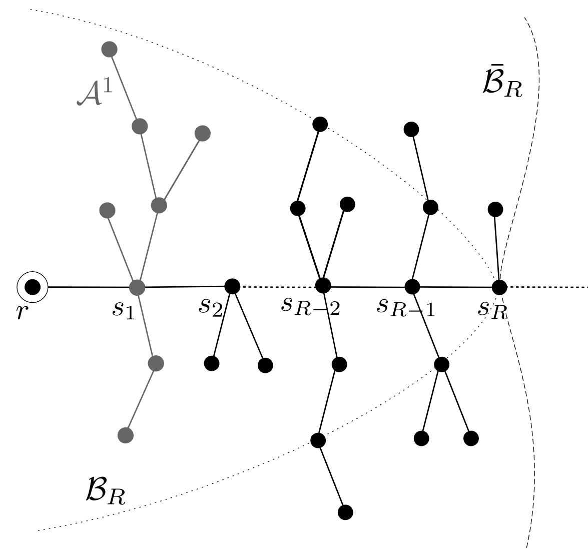


Figure 1.4: An illustration of a tree with a unique infinite spine with vertices $r, s_1, s_2, \ldots$. Shown are also the ball $\mathcal{B}_R$ and the hull $\bar{\mathcal{B}}_R$.

falls off with a power-law behaviour, $p(n) \sim n^{-\beta}$, and $\beta \in (2, 3]$. In the probability literature those random tree ensembles also go under the name of α-stable trees. Their fractal and spectral dimensions are almost surely given by

$$d_h = \frac{\beta - 1}{\beta - 2} \quad \text{and} \quad d_s = \frac{2(\beta - 1)}{2\beta - 3} \tag{1.11}$$

These values were conjectured in the physics literature [49, 50] and first proven by Croydon and Kumagai [46]. In Theorem 2.3.2, we provide an alternative and much simplified proof of these results using Theorem 2.2.2 (though slightly weaker than the proof by Croydon and Kumagai in the sense as described above). Another attractive feature is that our proof has a very intuitive and geometric interpretation.

In a second application, in Section 2.4, we analyse the fractal and spectral dimension of a model of randomly grown trees, the so-called attachment and grafting model [51]. In Theorem 2.4.1, we give an almost surely upper bound on the fractal and spectral dimension for the full parameter range of the model (which is believed to be tight) and determine their values almost surely for a smaller range of parameters.

1.5 | Uniform infinite causal triangulations and their properties

As for the planar triangulations and planar rooted trees one can assign a probability measure to the causal triangulations by introducing the uniform measure on causal triangulations with size (number of triangles) N. The limit $N \to \infty$ then corresponds to what is called *Uniform infinite causal triangulations* (UICT). Since causal triangulations are in bijection with rooted planar trees and since (apart from the boundaries) there are twice as many triangles in the triangulation as vertices in the tree, the UICT measure is equivalent to the uniform measure on infinite planar trees, i.e. the size-biased critical Galton-Watson tree with offspring distribution $p(n) = 2^{-n-1}$. This relation was first noticed by Yambartsev and Krikun [52] and independently proven by Duurhus, Jonsson and Wheater [34] on basis of their earlier work on generic trees [39]. In Chapter 3, we give an alternative proof taking as a starting point the original results by Kennedy [36], Aldous and Pitman [37].

In the remainder of Chapter 3 we are interested in the convergence of the joint boundary length and volume (area) process. Consider a ball $\mathcal{B}_R$ of radius R around the root. From the result of the previous section we know the scaling of the boundary length $|\partial \mathcal{B}_R| \approx R$ and area $|\mathcal{B}_R| \approx R^2$. We then show in Theorem 3.4.1 that the rescaled boundary length process converges to a diffusion process L_τ with Itô's equation

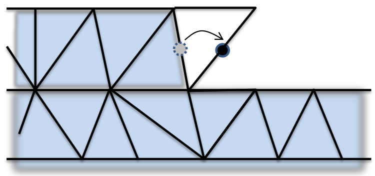
$$dL_\tau = 2d\tau + \sqrt{2L_\tau}dB_\tau \quad L_0 = l, \tag{1.12}$$

where B_τ is standard Brownian motion of variance one. Furthermore, it is shown in Theorem 3.4.2 that jointly the rescaled area process converges to $A_\tau = 2\int_0^\tau L_u du$ as expected. The Feynman-Kac equation for $\phi_{\xi,\lambda}(l,\tau) = \mathbb{E}[\exp(-\xi L_\tau - \lambda A_\tau)|L_0 = l]$ is given by

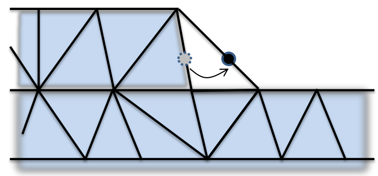
$$-\frac{\partial}{\partial \tau}\phi_{\xi,\lambda}(l,\tau) = \hat{H}\phi_{\xi,\lambda}(l,\tau), \quad \hat{H} = -2\frac{\partial}{\partial l} - l\frac{\partial^2}{\partial l^2} + 2\lambda l, \quad \phi_{\xi,\lambda}(l,0) = e^{-\xi l}. \tag{1.13}$$

Here $\hat{H}$ is called the quantum Hamiltonian with cosmological constant λ.

While many of the results follow rather straightforwardly from the properties of the Markov chain which underlies the Galton-Watson branching process, there are nonetheless very important since they close a gap between the works on causal triangulations in the probability and physics literature. On the one hand, in the probabilistic setting, one proposes a micro-canonical ensemble for the causal triangulations, the



(a) The $(+)$-move



(b) The $(-)$-move

Figure 1.5: The two different moves of the growth process.

UICT and a focus of study have been properties such as the fractal or spectral dimension, as we describe below. On the other hand, in the physics literature, one starts off with a canonical ensemble with cosmological constant λ (the equivalent of $\beta = 1/(k_B T)$ in the thermodynamic language) and is mainly interested in certain boundary length correlators. The importance of the results in Chapter 3 is that it provides first links between the two approaches.

In Chapter 4, analogous to the work of Angel [15], we introduce a growth process which samples UICT by adding a single triangle according to two different moves, denoted "$(+)$" and "$(-)$", with probability

$$\mathbb{P}((\pm) - \text{move} \mid l(T_n) = m) = \frac{1}{2}\frac{m \pm 1}{m}. \tag{1.14}$$

Here $l(T_n)$ denotes the length of the boundary of the triangulation at step n of the growth process. The two possibilities in which a triangle can be added, the $(\pm)$-moves, are shown in Figure 1.5. One observes that the $(+)$-move increases the boundary length by one while the $(-)$-move decreases it by one. We can give an equivalent description of the growth process through the recurrent Markov chain $\{M_n\}_{n\geq 0}$ in terms of the boundary length of the triangulation $M_n = l(T_n)$.

In Theorems 4.2.2 and 4.2.3 we prove that the growth process indeed samples sections of UICT. By introducing stoping times n_t at which the growth process completes the strip $S^1 \times [t, t+1]$ at "height" t of the causal triangulation we prove in Theorem 4.3.1 that the fractal dimension is almost surely $d_s = 2$. This proof is alternative to the corresponding proof by Durhuus, Jonsson and Wheater [34] using the above described relation to Galton-Watson trees which also follows directly from Theorem 2.2.2 of Chapter 2.

In Theorem 4.4.1 we prove convergence of the rescaled Markov chain $\{M_n\}_{n\geq 0}$ to a

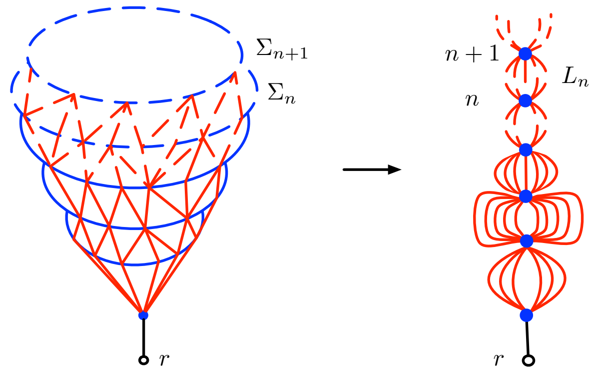


Figure 1.6: Reduction to cut-sets in the recurrence proof of UICT, the so-called multigraph [34].

diffusion process with Itô's equation

$$dM_u = \frac{1}{M_u}du + dB_u. \tag{1.15}$$

It is then shown in Theorem 4.4.2 and 4.4.3 that by a random time change $L_\tau = M_{T_\tau^{-1}}$ one obtains a diffusion process (1.12). This clarifies the so-called peeling procedure as introduced in the physics literature by Watabiki [53] for DT and in [54] for CDT.

One observes that in the calculation using Galton-Watson trees, as presented above, time or geodesic distance τ is fixed while area $A_\tau = \int_0^\tau 2L_u du$ becomes a random variable, while in the growth process one has the opposite situation, growth time u which is equal to area is fixed and geodesic distance $T_u = \int_0^u \frac{1}{2M_v}dv$ is a random variable.

The calculation of the spectral dimension is more involved. Using the Nash-Williams Theorem it can be shown that random walks on the UICT are recurrent and that thus $d_s \leq 2$ almost surely [34]. The proof uses so-called cut sets which correspond to slices of the graph shown on the right-hand side of Figure 1.6 These graphs were introduced in [34] under the name of multi-graphs.

On can obtain an insightful interpretation using the notion of electrical resistance, viewing each edge in the graph as a unit resistor (see for instance [55]). Due to Rayleigh's monotonicity law the resistance between any two points in an electrical network is increased (decreased) if one increases (decreases) the resistance anywhere in the network. Hence, one sees that the multigraph and the tree provide a corresponding lower and upper bound for the resistance of the triangulation. If one introduces an exponent ρ for the resistance growth between two points at distance R, $\mathcal{R}_R \approx R^\rho$, as

$R \to \infty$, one expects the following scaling of the spectral dimension,

$$d_s = \frac{2d_h}{d_h + \rho} \tag{1.16}$$

Indeed, considering that the tree has $\rho = 1$ and that for the multigraph one expects $\rho = 2 - d_h$, one sees that the tree and the multi-graph form the lower and upper bound of (1.3) respectively,

$$\frac{2d_h}{1 + d_h} \leq d_s \leq d_h. \tag{1.17}$$

For the UICT it is conjectured that $d_s = 2$ $(= d_h)$ almost surely. In the absence of a proof, the multi-graphs have been used as toy models to study different aspects of diffusion on causal triangulations ensembles, as was done by Giasemidis, Wheater and Zohren [56, 57] which form part of Giasemidis' PhD thesis [58].

1.6 | Gibbs causal triangulations and coupling to the Ising model

We have so far focused on a formulation through the micro-canonical ensemble of causal triangulations, the UICT. Instead we can also approach the problem from a point of view of a canonical ensemble. This is done by introducing the Gibbs measure $e^{-\mu|T|}/Z$, where each triangulation is weighted by a Boltzmann factor $e^{-\mu|T|}$. Here $|T|$ denotes the size of the triangulation and μ the cosmological constant (the analog of inverse temperature in this context). The action $S(\mu)$ in the Boltzmann factor $e^{-S(\mu)}$ is the discretisation of the so-called Einstein-Hilbert action in two dimensions. Since causal triangulations have an inherent time-sliced structure it is natural to introduce the transfer matrix $U = \{u(n, n')\}_{n,n'=1,2,\ldots}$ for a slice with n vertices on the initial boundary and n' vertices on the final boundary. The partition function for causal triangulations of height N is then given by the trace of the N's product of the transfer matrix

$$Z_N(\mu) = \operatorname{tr} U^N \tag{1.18}$$

From statistical mechanics we expect that when $N \to \infty$ the partition function is governed by the largest eigenvalue $\Lambda(\mu)$ of the transfer matrix U and its transpose U^*,

$$Z_N(\mu) \sim \Lambda(\mu)^N. \tag{1.19}$$

We use so-called Krein-Rutman theory [59] on operators preserving the cone of positive functions to make this statement precise. In particular, following results of Malyshev, Yambartsev, Zamyatin [33], in Theorem 5.3.1 we prove the convergence of the scaled log of the partition function

$$\frac{1}{N}\log Z_N(\mu) = \Lambda(\mu). \tag{1.20}$$

for $\mu > \mu_c = \log 2$. Furthermore, we show that the finite-level Gibbs measure $\mathbb{P}_N$ converges weakly to a limiting measure $\mathbb{P}$. Here μ_c is the critical value of the cosmological constant. One sees that for $\mu \to \mu_c$ the limiting measure corresponds to the UICT, showing equivalence of the micro-canonical and canonical formulation.

In the following, we extend these results to the case of an Ising model coupled to causal triangulations. The transfer matrix is then given by the joint sum over triangulations and spin configurations of a slice. Instead of a critical point $\mu_c = \log 2$ we now expect there to be a critical line $\mu_c(\beta)$, where β is inverse temperature. In fact, $\mu_c(\beta)$ corresponds to the free energy per triangle and its precise knowledge would give us insight into the phase transition of the model. While we are not able to determine $\mu_c(\beta)$ precisely, in Lemma 5.4.1, we use Krein-Rutman Theory to determine a region in the coupling quadrant (μ, β) where the partition function converges. However, our results are not tight and we cannot be sure that the boundary of this region corresponds to the critical line. However, we hope in future work to complement this analysis with numerical results determining the exact shape of the critical line.

1.7 | Structure of the remainder of the thesis

The remaining chapters of this thesis are based on the following articles:

- Chapter 2 on S. Ö. Stefánsson and S. Zohren, “Spectral dimension of trees with a unique infinite spine” *J. Stat. Phys.* **147** (2012) 942, arXiv:1202.5388 [cond-mat.stat-mech];

- Chapter 3 on: V. Sisko, A. Yambartsev and S. Zohren. “A note on weak convergence results for uniform infinite causal triangulations,” to appear in *Markov Proc. Rel. Fields* (2013), arXiv:1201.0264 [math-ph];

- Chapter 4 on: V. Sisko, A. Yambartsev and S. Zohren, “Growth of uniform infinite causal triangulations” *J. Stat. Phys.* OnlineFirst (2012), arXiv: 1203.2869 [math-ph];

- Chapter 5 on: J.H.C. Hernandez, Y. Suhov, A. Yambartsev and S. Zohren "Properties of transfer matrix for Ising model coupled to causal dynamical triangulations" submitted (2012).

The following two articles, completed while at Leiden University, are centred around related applications of random matrix models to CDT, which form the physical foundations behind the growth process described in Chapter 4 and have not been included in the thesis:

- J. Ambjørn, R. Loll, W. Westra, and S. Zohren, "Summing over all topologies in CDT string field theory," *Phys.Lett.* **B678** (2009) 227–232, arXiv:0905.2108 [hep-th];
- J. Ambjørn, R. Loll, W. Westra, and S. Zohren, "Stochastic quantization and the role of time in quantum gravity," *Phys.Lett.* **B680** (2009) 359–364, arXiv:0908.4224 [hep-th].

Furthermore, the following two articles regarding applications to quantum statistics and applied statistics have also not been included in the thesis:

- S. Zohren, P. Reska, R. D. Gill and W. Westra, "A tight Tsirelson inequality for infinitely many outcomes" *Europhysics Letters* **90** (2010) 10002, arXiv:1003.0616 [quant-ph];
- D. Anevski, R. D. Gill and S. Zohren, "Estimating a probability mass function with unknown labels" to appear.

Part I

Simply generated and randomly grown trees

CHAPTER 2

Spectral dimension of trees with unique infinite spine

Using generating function techniques we develop a relation between the fractal and spectral dimension of trees with a unique infinite spine. Furthermore, it is shown that if the outgrowths along the spine are independent and identically distributed, then both the fractal and spectral dimension can easily be determined from the probability generating function of the random variable describing the size of the outgrowths at a given vertex, provided that the probability of the height of the outgrowths exceeding n falls off as the inverse of n. We apply this new method to both critical non-generic trees and the attachment and grafting model, which is a special case of the vertex splitting model, resulting in a simplified proof for the values of the fractal and spectral dimension for the former and novel results for the latter.

Chapter published as: S. Ö. Stefánsson and S. Zohren, "Spectral dimension of trees with a unique infinite spine" *J. Stat. Phys.* **147** (2012) 942.

2.1 | Introduction

Random trees and their properties have been studied in several branches of probability theory, mathematical physics and science in general. Their applications span from phylogenetic trees [60, 61] and random folding of RNA molecules [62] to models of quantum gravity [9, 23, 63, 64], amongst others.

In general terms a random tree ensemble is a set of trees together with a probability measure associated to it. We focus on a class of infinite, rooted trees which have the property that there is a unique, non-backtracking path from the root to infinity. This path will be referred to as *the spine*. In this case the ensemble is characterized by the distribution of the finite outgrowths from the spine. Such ensembles can arise in very different contexts: One example considered in this chapter is the equilibrium statistical mechanics model of *simply generated trees*, first introduced in [38], which describes trees with a local action. These random tree ensembles are related to critical and sub-critial *Galton-Watson processes* [35]. The second example considered is the *attachment and grafting model* [51] which is a specific case of the *vertex splitting model* [65]. This random tree ensemble is very different in nature from the simply generated trees in the sense that it arises from a growth process with non-local action which results in a non-equilibrium statistical mechanics model. An effect similar to the emergence of a unique infinite spine is also observed in triangulation models in quantum gravity where a unique large "universe" emerges with finite baby–universes attached, see e.g. [14].

A simple observable of a random tree ensemble is its *fractal dimension* defined as d_h provided that the number of edges within a distance R from the root, denoted by $|\mathcal{B}_R|$, scales as

$$|\mathcal{B}_R| \approx R^{d_h} \quad \text{as } R \to \infty. \tag{2.1}$$

The meaning of "$\approx$" is that both sides are asymptotically the same in a sense which will be made precise in Section 2.2. Another notion of dimensionality of the random tree ensemble is the so-called *spectral dimension*. Let $p(t)$ be the probability that a simple random walk on a tree which starts at the root at time 0 is back at the root at time t. If

$$p(t) \approx t^{-d_s/2} \quad \text{as } t \to \infty \tag{2.2}$$

we say that the spectral dimension of the tree is d_s. The spectral dimension of various classes of graphs has been studied by probabilists [45, 43, 46, 44, 47] and by physicists, especially in the quantum gravity literature, see e.g. [40, 41, 42, 34]. Coulhon [13, Theorem 7.7] derived that one has

$$\frac{2d_h}{1+d_h} \leq d_s \leq d_h \tag{2.3}$$

for fixed graphs under certain regularity assumptions. As we will see in the following, random tree ensembles with an infinite spine are tight with the left-hand-side of the inequality and we show below how one can obtain an improved inequality for those tree ensembles which is tight on both sides for many examples.

We develop methods which relate the diffusion of the random walk to properties of the volume distribution of the finite outgrowths. An important concept is the *hull dimension* defined as $\bar{d}_h$, provided that the total volume of the *hull* $\bar{\mathcal{B}}_R$, the outgrowths from vertices on the spine within distance R from the root, scales as

$$|\bar{\mathcal{B}}_R| \approx R^{\bar{d}_h} \quad \text{as } R \to \infty \tag{2.4}$$

see Figure 2.1.

Using simple generating function techniques we begin by proving in Theorem 2.2.1 that the following bounds hold for trees with a unique infinite spine

$$\frac{2d_h}{1+d_h} \leq d_s \leq \frac{2\bar{d}_h}{1+\bar{d}_h} \tag{2.5}$$

which, in many examples of interest, tightens the general bound (2.3) for fixed graphs. Intuitively, if the probability that the outgrowths reach height n decays fast enough with n, d_h and $\bar{d}_h$ should be close in which case (2.5) gives a tight bound on d_s. We make this a precise statement in Theorem 2.2.2 in the case when the outgrowths from different vertices on the spine are independent and identically distributed ($i.i.d.$). There, we prove that if the random variable $|\mathcal{A}^i|$, denoting the size of the outgrowths from a vertex on the spine, has a probability generating function $\langle z^{|\mathcal{A}^i|}\rangle = 1-(1-z)^\alpha l(1-z)$, where $l(x)$ varies slowly at zero, then almost surely

$$d_h \leq \frac{1}{\alpha}, \qquad d_s \leq \frac{2}{1+\alpha}. \tag{2.6}$$

Furthermore, we show that if the probability that the height of the outgrowths from a vertex on the spine exceeds n falls off as the inverse of n then one has equality in (2.6).

Up to that point, our results are very general and we devote the rest of the chapter to applying the results to specific models. In Section 2.3 we calculate the spectral dimension of the so–called non–generic, critical phase of simply generated trees, giving an alternative proof to the recent results by Croydon and Kumagai [46]. We note here that our result is weaker in the sense that we define d_s through the singular behaviour of the generating function of $p(t)$, rather than from the asymptotic behaviour of $p(t)$.

In Section 2.4 we calculate the fractal and spectral dimension of the attachment and grafting model for a certain parameter range, extending results in [51]. We leave the generalisation to the full range of parameters as an open problem.

2.2 | Fractal and spectral dimension of trees with infinite spine

2.2.1 | Trees with infinite spine and their ball and hull dimension

Let us denote by Γ the set of all planar rooted locally finite trees, where the root has valency one.[1] The planarity condition simply means that edges containing the same vertex are ordered around that vertex. One has $\Gamma = \Gamma' \cup \Gamma^{\infty}$, where Γ' is the set of finite such trees and Γ^{∞} the set of infinite such trees. We often denote elements of Γ' by T, T_1 etc., while elements of Γ^{∞} are called τ, τ_1 etc. For $T \in \Gamma'$, let $|T|$ be the size of the tree, i.e. the number of edges in T. Let r denote the root of the tree. We consider in this chapter trees which have a unique, non–backtracking path from the root to infinity and call this path *the spine*. Denote the set of such trees by $\Gamma_S^{\infty} \subset \Gamma^{\infty}$. Furthermore, denote the vertices on the spine, ordered away from the root r, by $s_1, s_2, s_3, \ldots$. We also denote the valency of a vertex v by $\sigma(v)$, i.e. for the root r one has $\sigma(r) = 1$.

For a given $\tau \in \Gamma$ we denote by $\mathcal{B}_R \subset \tau$ the *ball* of radius R around the root, this is the subgraph of τ spanned by the vertices which have a graph distance from the root which is less than or equal to R. Another important concept in the chapter is the notion of the *hull* $\bar{\mathcal{B}}_R$. For a $\tau \in \Gamma$ the hull $\bar{\mathcal{B}}_R$ denotes the subgraph which is the union of the spine from r up to vertex s_R and all finite trees attached to it. The concepts of ball and hull are illustrated in Figure 2.1; it is easy to see that $\mathcal{B}_R \subseteq \bar{\mathcal{B}}_R$.

A random tree ensemble (Γ, ν) is a set of trees Γ equipped with a probability measure ν. An important notion of dimensionality of a random tree ensemble, as discussed in the introduction, is the so-called fractal dimension, which describes the growth of the size of a ball of radius R and can formally be defined as

$$d_h = \lim_{R\to\infty} \frac{\log |\mathcal{B}_R|}{\log R}. \tag{2.7}$$

Throughout the paper we consider the following strong criterion for existence of d_h:

[1] Note that the assumption that the root has valency one is for conventional reasons and the analysis which follows could in principle also been done without it.

There exists an $R_0 > 0$ such that for $R > R_0$

$$R^{d_h}\underline{L}(R) < |\mathcal{B}_R| < R^{d_h}\bar{L}(R), \tag{2.8}$$

where $\underline{L}(R), \bar{L}(R)$ are functions which vary slowly at infinity. A real–valued, positive, measurable function L is said to vary slowly at infinity if for each $\lambda > 0$

$$\lim_{R\to\infty} \frac{L(\lambda R)}{L(R)} = 1. \tag{2.9}$$

We refer to [66] for some properties of slowly varying functions and in particular the fact that (2.8) implies (2.7).

If for a random tree ensemble (Γ, ν) one has that (2.7) holds for ν-almost all $\tau \in \Gamma$ then we call d_h the *quenched fractal dimension*. Furthermore, one can also define the *annealed fractal dimension* through the expected size of the ball of radius R

$$d_H = \lim_{R\to\infty} \frac{\log\langle|\mathcal{B}_R|\rangle_\nu}{\log R}. \tag{2.10}$$

Furthermore, we say that the quenched fractal dimension d_h exists in the sense of (2.8) if the latter is fulfilled for ν-almost all $\tau \in \Gamma$.

In analogy, we also define the *quenched hull dimension*, if for ν-almost all $\tau \in \Gamma$

$$\bar{d}_h = \lim_{R\to\infty} \frac{\log|\bar{\mathcal{B}}_R|}{\log R} \tag{2.11}$$

where existence is again provided through (2.8) analogous to the fractal dimension. From the fact that $\mathcal{B}_R \subseteq \bar{\mathcal{B}}_R$ it is then also clear that one has $d_h \leq \bar{d}_h$.

2.2.2 | Random walks on trees and spectral dimension

In the following, we consider simple, discrete time random walks on trees i.e. walks which, in each discrete time step, jump to a nearest neighbour with uniform probability. For a given $\tau \in \Gamma$ we denote by Ω_τ the set of finite walks on τ. For a walk $\omega \in \Omega_\tau$ we denote by $|\omega|$ the length of the walk and for $t \leq |\omega|$, $\omega(t)$ denotes the vertex where the ω is located after t steps. We now formulate generating functions for the return probabilities of random walks on a given tree τ.

An important quantity is the generating function for the probability of first return to the root r given by

$$P_\tau(x) = \sum_{t=0}^{\infty} \sum_{\omega\in\Omega_\tau:|\omega|=t} \mathbb{P}_\tau\left(\{\omega(t) = r \,|\, \omega(0) = r, \omega(t') \neq r, 0 < t' < t\}\right)(1-x)^{\frac{1}{2}t} \tag{2.12}$$

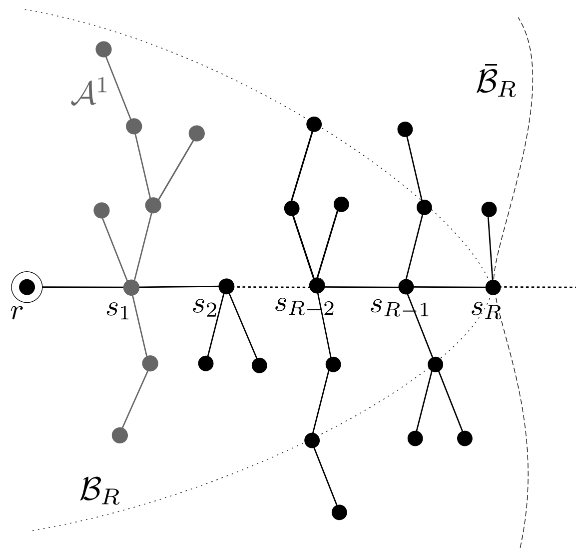


Figure 2.1: An illustration of the ball and the hull in a tree with a unique infinite spine started at the root r and labeled by vertices $s_1, s_2, ...$ away from the root. The ball $\mathcal{B}_R$ is the subgraph of all vertices within distance R from the root. The hull $\bar{\mathcal{B}}_R$ is composed of the spine up to distance R from the root and the finite outgrowths from its vertices $s_1, ..., s_R$. The finite outgrowths from vertex s_i are denoted by $\mathcal{A}^i$.

and for all returns to a starting point which is given by

$$Q_\tau(x) = \sum_{t=0}^{\infty} \sum_{\omega \in \Omega_\tau : |\omega| = t} \mathbb{P}_\tau\left(\{\omega(t) = r \,|\, \omega(0) = r\}\right)(1-x)^{\frac{1}{2}t}. \tag{2.13}$$

By decomposing a walk which returns to the root into a first return, second return etc., the two generating functions can be related through

$$Q_\tau(x) = \sum_{n=0}^{\infty} (P_\tau(x))^n = \frac{1}{1 - P_\tau(x)}. \tag{2.14}$$

The generating function Q_τ encodes all information of the asymptotic behaviour of the return probability and one can extract the spectral dimension from it. In particular, for a given $\tau \in \Gamma$ we define the spectral dimension through

$$d_s = 2\left(1 + \lim_{x \to 0} \frac{\log Q_\tau(x)}{\log x}\right), \tag{2.15}$$

provided that $Q_\tau(x)$ diverges as $x \to 0$. As for the fractal dimension we also consider the strong criterion for existence, that there exists an $x_0 \in (0,1)$ such that for $x < x_0$

$$x^{-1+d_s/2}\underline{l}(x) < Q_\tau(x) < x^{-1+d_s/2}\overline{l}(x) \tag{2.16}$$

where $\underline{l}(x)$ and $\overline{l}(x)$ are functions which vary slowly at $x \to 0^+$. Notice that the definition of d_s through (2.15) is slightly weaker than the more common definition

$$d_s = -2 \lim_{t\to\infty} \frac{\log \mathbb{P}\left(\{\omega(2t) = r \,|\, \omega(0) = r\}\right)}{\log t}. \tag{2.17}$$

For an ensemble (Γ, ν) we say it has *quenched spectral dimension* d_s if (2.15) is fulfilled for ν-almost all trees $\tau \in \Gamma$. Furthermore, we define the *annealed spectral dimension* d_S through

$$d_S = 2\left(1 + \lim_{x\to 0} \frac{\log\langle Q_\tau(x)\rangle_\nu}{\log x}\right), \tag{2.18}$$

provided that $\langle Q_\tau(x)\rangle_\nu$ diverges as $x \to 0$.

To illustrate the concepts introduced in this subsection consider the following:

Example 2.2.1. *The easiest example is the (non-random) tree consisting of a single infinite spine which we denote by $\mathcal{S}$. One has trivially that $|\mathcal{B}_R| = |\bar{\mathcal{B}}_R| = R$ and thus $d_h = \bar{d}_h = 1$. To calculate the first return probability generating function $P_\mathcal{S}(x)$ note that the random walk leaves the root with probability 1, then at vertex s_1 the walker returns to the root with probability $1/2$ or leaves to the right with probability $1/2$ and makes an excursion until its first return to s_1. It then returns to the root with probability $1/2$ or does another excursion with probability $1/2$, and so on. Hence, one has for the generating function that*

$$P_\mathcal{S}(x) = \frac{1}{2}(1-x)\sum_{n=0}^{\infty}\left(\frac{1}{2}P_\mathcal{S}(x)\right)^n = \frac{1-x}{2-P_\mathcal{S}(x)}. \tag{2.19}$$

From (2.19) *it follows that*

$$P_\mathcal{S}(x) = 1 - \sqrt{x} \tag{2.20}$$

and thus from (2.14) *one has*

$$Q_\mathcal{S}(x) = \frac{1}{\sqrt{x}}. \tag{2.21}$$

Hence, using (2.15) *one finds that $d_s = 1$ as expected.*

2.2.3 | Relating spectral and fractal dimension

The main result of this section is the following theorem which we prove in the two subsections below.

Theorem 2.2.1. *For any tree $\tau \in \Gamma_S^\infty$ with a unique infinite spine for which d_h, $\bar{d}_h$ and d_s exist, the following holds*

$$\frac{2d_h}{1+d_h} \leq d_s \leq \frac{2\bar{d}_h}{1+\bar{d}_h}. \tag{2.22}$$

In the proof of this theorem, in the following two subsections, we use results developed in [39] in the context of generic trees and show that they can be employed in a more general context by introducing the hull dimension. When assuming existence of the dimensions we will in practice use the stronger criteria (2.8) and (2.16) although the results also easily follow from assuming existence in the usual sense (2.7) and (2.15).

Upper bound on the spectral dimension

For any tree $\tau \in \Gamma$ one can easily derive the following recursion relation [39] analogous to (2.19) as presented in the example above

$$P_\tau(x) = \frac{1-x}{k_\tau + 1 - \sum_{i=1}^{k_\tau} P_{\tau_i}(x)}, \tag{2.23}$$

where the $\tau_1, ..., \tau_{k_\tau}$ denote the k_τ trees meeting the edge from the root to the unique vertex next to it. In [39] the following simple lemma is proven which follows straightforwardly from (2.23) by induction

Lemma 2.2.1. *For all finite trees $T \in \Gamma'$ one has*

$$P_T(x) \geq 1 - |T|x. \tag{2.24}$$

If the root of T has degree k the above equation is generalized to $P_T(x) \geq 1 - \frac{|T|}{k}x$. To prove the upper bound on the spectral dimension we use the following lemma which is a reformulation of Lemma 7. in [39]

Lemma 2.2.2. *For all $\tau \in \Gamma_S^\infty$ and $R \geq 1$ one has*

$$P_\tau(x) \ \geq \ 1 - \frac{1}{R} - x|\bar{\mathcal{B}}_R|\,. \tag{2.25}$$

Proof. The proof follows directly the proof of Lemma 7 in [39]. Denote by $P_\tau^R(x)$ the generating function for the first return probability of walks which do not visit the vertex s_{R+1} on the spine. An induction proof using the recursion relation (2.23) then easily shows that

$$P_\tau^R(x) \;\geq\; 1 - \frac{1}{R} - xR - \sum_{i=1}^{R}(\sigma(s_i) - 2)(1 - P_{\mathcal{A}^i}(x))\,, \tag{2.26}$$

where $\mathcal{A}^i$ denotes the union of the vertex s_i and the finite trees attached to it. Noting that $P_\tau(x) \geq P_\tau^R(x)$, using Lemma 2.2.1 and observing that $|\bar{\mathcal{B}}_R| = \sum_{i=1}^R |\mathcal{A}^i| + R$ completes the proof. $\square$

To prove the upper bound in the theorem we note that from the existence of the hull dimension one has that $|\bar{\mathcal{B}}_\tau(R)| \leq R^{\bar{d}_h}\bar{L}(R)$, where $\bar{L}(R)$ varies slowly at $R \to \infty$. Thus one has from the proceeding lemma that

$$Q_\tau(x) \geq \frac{R}{1 + xR^{\bar{d}_h+1}\bar{L}(R)} \tag{2.27}$$

Choosing $R = \lfloor x^{-1/(1+\bar{d}_h)} \rfloor$ one has

$$Q_\tau(x) \geq \frac{x^{-1/(1+\bar{d}_h)}\,\underline{l}(x)}{1 + \underline{l}(x)}, \tag{2.28}$$

where $\underline{l}(x) = 1/\bar{L}(\lfloor x^{-1/(1+\bar{d}_h)} \rfloor)$ varies slowly at $x \to 0^+$. Provided that d_s exists and from the fact that $\bar{d}_h \geq 1$, this proves that $d_s \leq 2\bar{d}_h/(1+\bar{d}_h)$.

Lower bound on the spectral dimension

To prove the lower bound on the spectral dimension we exploit the following lemma derived in [39] in the context of generic trees and note that it can be used in a more general situation.

Lemma 2.2.3. *For all $\tau \in \Gamma$ one has*

$$Q_\tau(x) \leq R + \frac{2}{x|\mathcal{B}_\tau(R)|} \tag{2.29}$$

The proof of this lemma can be found in [39]. It uses a decomposition of walks in Ω_1, the set of walks which do not reach further than distance R from the root, and Ω_2, the set of walks which do reach further. Then one can show that $Q_\tau(x) = Q_\tau^{\Omega_1}(x) + Q_\tau^{\Omega_2}(x)$, where $Q_\tau^{\Omega_1}(x) \leq R$ and $Q_\tau^{\Omega_2}(x) \leq \frac{2}{x|\mathcal{B}_\tau(R)|}$.

To prove the lower bound in the theorem we observe that by existence of the fractal dimension one has that $|\mathcal{B}_\tau(R)| \geq R^{d_h}\underline{L}(R)$, where $\underline{L}(R)$ varies slowly at $R \to \infty$. Thus, by Lemma 2.2.3

$$Q_\tau(x) \leq R + x^{-1} R^{-d_h} \underline{L}^{-1}(R) \tag{2.30}$$

Choosing $R = \lceil x^{-1/(1+d_h)} \rceil$ (which also optimises the inequality) proves that $d_s \geq 2d_h/(1+d_h)$.

2.2.4 | Independent and identically distributed outgrowths

If one considers random tree ensembles with a unique infinite spine, where the outgrowths from different vertices along the spine are independent and identically distributed ($i.i.d.$), one can make stronger statements about the dimensions of the tree ensemble. Recall, the definition of the hull $\bar{\mathcal{B}}_R$ as the union of the spine from r up to vertex s_R and all finite trees attached to it. Let us furthermore denote by $\mathcal{A}^i$ the union of the vertex s_i and the finite trees attached to it. Denote by X_n^i the number of vertices in $\mathcal{A}^i$ at a distance n from s_i, e.g. $X_0^i = 1$. Furthermore, denote by $\mathcal{A}_n^i$ the intersection of $\mathcal{A}^i$ with the ball of radius n centered around s_i. We have the following Theorem:

Theorem 2.2.2. *Let (Γ, ν) be a random tree ensemble concentrated on the set of trees with a unique infinite spine Γ_S^∞ and $i.i.d.$ outgrowths $\left(\mathcal{A}^i\right)_{i\geq 1}$ on the spine.*

(i) If for $\alpha \in (0,1)$ and $z \in (0,1)$,

$$\langle z^{|\mathcal{A}^i|} \rangle_\nu = 1 - (1-z)^\alpha l(1-z), \tag{2.31}$$

where $l(x)$ varies slowly at $x \to 0^+$, then almost surely

$$\bar{d}_h \leq \frac{1}{\alpha}, \qquad d_s \leq \frac{2}{1+\alpha} \tag{2.32}$$

provided that $\bar{d}_h$ and d_s exist.

(ii) If furthermore,

$$\nu(\{X_n^i > 0\}) \leq \frac{c}{n}, \tag{2.33}$$

where $c > 0$ is a constant, then almost surely d_h and d_s exist and

$$d_h = \frac{1}{\alpha}, \qquad d_s = \frac{2}{1+\alpha}. \tag{2.34}$$

Proof of first part of Theorem 2.2.2

Since $|\bar{\mathcal{B}}_R| = \sum_{i=1}^{R} |\mathcal{A}^i| + R$ is just a sum of $i.i.d.$ random variables $|\mathcal{A}^i|$ (with possibly infinite variance), one can easily determine a bound on their sum $|\bar{\mathcal{B}}_R|$ and with that a bound on $\bar{d}_h$ almost surely. We begin by observing that by a Tauberian theorem [67], (2.31) is equivalent to

$$\nu(\{|\mathcal{A}^i| > R\}) \sim R^{-\alpha} L(R), \tag{2.35}$$

where $L(R)$ varies slowly at $R \to \infty$. Here, we define "$\sim$" to mean that the ratio of the two sides tends to one as $R \to \infty$. By (2.35) there exists a function $L_1(R)$ which varies slowly at infinity such that

$$\sum_{i=1}^{\infty} \nu(\{|\mathcal{A}^i| > R^{\frac{1}{\alpha}}\, L_1(R)\}) < \infty. \tag{2.36}$$

It then follows from a theorem by Feller [68, Theorem 2] that

$$\nu(\{\tau : \sum_{i=1}^{R} |\mathcal{A}^i| > R^{\frac{1}{\alpha}}\, L_1(R) \quad \text{infinitely often}\}) = 0 \tag{2.37}$$

One then has that for ν-almost all trees there exists a constant $C > 1$ and an R_0 such that for $R > R_0$ one has

$$|\bar{\mathcal{B}}_R| < C R^{\frac{1}{\alpha}}\, L_1(R). \tag{2.38}$$

This proves that $\bar{d}_h \leq 1/\alpha$ almost surely and using Theorem 2.2.1, one has further $d_s \leq 2/(1+\alpha)$ almost surely.

Proof of second part of Theorem 2.2.2

In this section we prove that there exists a constant C such that,

$$\nu(\{\tau : |\mathcal{B}_R| < \lambda R^{\frac{1}{\alpha}}\}) \leq C e^{-\lambda^{-\alpha} l(\lambda^{-1} R^{-1/\alpha})} \tag{2.39}$$

for λ small enough such that $\lambda^{-\alpha} l(\lambda^{-1} R^{-1/\alpha}) > c$ with $l(x)$ from (2.31). Assuming that (2.39) holds we proceed as follows. By [66, Theorem 1.5], for any $\tilde{c} \in \mathbb{R}$, there exists a function $L_2(R)$, slowly varying as $R \to \infty$, which satisfies

$$\lim_{R\to\infty} \frac{L_2(R)^{-\alpha} \ell(L_2(R)^{-1} R^{-1/\alpha})}{\log(R^{\tilde{c}})} = 1. \tag{2.40}$$

To see this, let L^* be a slowly varying function conjugate to $L(R) = 1/l(R^{-1/\alpha})$ i.e.

$$\lim_{R\to\infty} L^*(R)L(RL^*(R)) = 1$$

and choose $L_2(R) = \left(L^*(\frac{R}{\log R^{\tilde{c}}})/\log(R^{\tilde{c}})\right)^{1/\alpha}$.

Therefore, by choosing $\lambda = L_2(R)$ which obeys the requirement on λ given above and $\tilde{c} > 1$ one finds that

$$\sum_{R=1}^{\infty} \nu(\{\tau : |\mathcal{B}_R| < R^{\frac{1}{\alpha}} L_2(R)\}) < \infty. \tag{2.41}$$

Using the Borel-Cantelli lemma one then has that

$$\nu(\{\tau : |\mathcal{B}_R| < R^{\frac{1}{\alpha}} L_2(R) \quad \text{infinitely often}\}) = 0. \tag{2.42}$$

Thus, for ν-almost all trees there exists an R_0 such that for $R > R_0$ one has

$$|\mathcal{B}_R| > R^{\frac{1}{\alpha}} L_2(R). \tag{2.43}$$

This proves that $d_h \geq 1/\alpha$ almost surely. Together with the results of the previous section and Theorem 2.2.1 one has thus almost surely

$$d_h = \frac{1}{\alpha}, \qquad d_s = \frac{2}{1+\alpha}. \tag{2.44}$$

The basic idea to prove (2.39) is the following: if the finite outgrowths along the spine die out fast enough, i.e. if (2.33) holds, then the ball $\mathcal{B}_R$ and hull $\bar{\mathcal{B}}_R$ are close in size and one can use the latter to estimate $\mathcal{B}_R$. Recall that $\mathcal{A}^i_n$ is the intersection of $\mathcal{A}^i$ with the ball of radius n centered around s_i. An essential ingredient to measure how close $\mathcal{B}_R$ and $\bar{\mathcal{B}}_R$ are in size is provided by the following lemma:

Lemma 2.2.4. *For $z \leq 1$ one has for the probability generating functions*

$$\langle z^{|\mathcal{A}^i|}\rangle_\nu \leq \langle z^{|\mathcal{A}^i_n|}\rangle_\nu \leq \langle z^{|\mathcal{A}^i|}\rangle_\nu + \nu(\{X^i_n > 0\}) \tag{2.45}$$

Proof. The proof is inspired by ideas of [39]. Firstly, note that $|\mathcal{A}^i| \geq |\mathcal{A}^i_n|$, hence the lower bound follows. For easy notation let us denote the event $\mathcal{E}_n = \{X^i_n > 0\}$. Then

$$\begin{aligned} \langle z^{|\mathcal{A}^i_n|} - z^{|\mathcal{A}^i|}\rangle_\nu &= \int_{\mathcal{E}_n} (z^{|\mathcal{A}^i_n|} - z^{|\mathcal{A}^i|})d\nu + \int_{\mathcal{E}^c_n} (z^{|\mathcal{A}^i_n|} - z^{|\mathcal{A}^i|})d\nu \\ &= \int_{\mathcal{E}_n} (z^{|\mathcal{A}^i_n|} - z^{|\mathcal{A}^i|})d\nu \leq \int_{\mathcal{E}_n} z^{|\mathcal{A}^i_n|} d\nu \\ &\leq \int_{\mathcal{E}_n} d\nu = \nu(\{X^i_n > 0\}) \end{aligned} \tag{2.46}$$

Thus the upper bound follows. □

To prove (2.39) we first note that since

$$\mathcal{A}^1_{[R/2]} \cup \mathcal{A}^2_{[R/2]} \cup \ldots \cup \mathcal{A}^{[R/2]}_{[R/2]} \subset \mathcal{B}_R \tag{2.47}$$

it suffices to prove that

$$\nu(\{\tau : \sum_{i=1}^{R} |\mathcal{A}^i_R| < \lambda R^{\frac{1}{\alpha}}\}) \leq Ce^{-\lambda^{-\alpha} l(\lambda^{-1}R^{-1/\alpha})}. \tag{2.48}$$

Using Markov's inequality, the independence of the $|\mathcal{A}^i_R|$ and Lemma 2.2.4 we find that for any $t \in (0,1)$ and $\lambda > 0$

$$\begin{aligned}
\nu(\{\tau : \sum_{i=1}^{R} |\mathcal{A}^i_R| < \lambda R^{\frac{1}{\alpha}}\}) &\leq \nu(\{\tau : \exp\big(-t\sum_{i=1}^{R} |\mathcal{A}^i_R|\big) > \exp\big(-t\lambda R^{\frac{1}{\alpha}}\big)\}) \\
&\leq e^{t\lambda R^{\frac{1}{\alpha}}} \langle \prod_{i=1}^{R} e^{-t|\mathcal{A}^i_R|}\rangle_\nu = e^{t\lambda R^{\frac{1}{\alpha}}} \langle e^{-t|\mathcal{A}^i_R|}\rangle_\nu^R \\
&\leq e^{t\lambda R^{\frac{1}{\alpha}}} \left(\langle e^{-t|\mathcal{A}^i|}\rangle_\nu + \nu(\{X^i_R > 0\})\right)^R.
\end{aligned} \tag{2.49}$$

From (2.31) it follows that

$$\langle e^{-t|\mathcal{A}^i|}\rangle_\nu \leq 1 - t^\alpha l(t) + o(t^\alpha l(t)) \tag{2.50}$$

where $l(t)$ varies slowly as $t \to 0^+$. Choose $t = \lambda^{-1}R^{-1/\alpha}$ and λ small enough such that

$$\lambda^{-\alpha} l(\lambda^{-1}R^{-1/\alpha}) > c \tag{2.51}$$

with c from (2.33). Apply (2.50) and (2.33) to (2.49) and get, for a suitable constant C'

$$\begin{aligned}
\nu(\{\tau : \sum_{i=1}^{R} |\mathcal{A}^i_R| < \lambda R^{\frac{1}{\alpha}}\}) &\leq C'\left(1 - \frac{\lambda^{-\alpha} l(\lambda^{-1}R^{-1/\alpha}) - c}{R}\right)^R \\
&\leq C' e^{-\lambda^{-\alpha} l(\lambda^{-1}R^{-1/\alpha}) + c}.
\end{aligned} \tag{2.52}$$

Thus (2.39) follows from (2.52) with $C = C'e^c$.

2.3 | Conditioned Galton-Watson trees

2.3.1 | The model

In this section we apply the results of the previous section to the model of simply generated trees. Simply generated trees can, in most cases of interest, be interpreted as a size conditioned Galton-Watson process. A Galton-Watson process is a discrete time branching process which starts from a single particle. At each time step the particles branch independently to k other particles with the same probability $p(k)$, $k \geq 0$ referred to as *offspring probability*. We denote the generating function of the offspring probabilities (or outdegrees) by $f(z) = \sum_{n=0}^{\infty} p(n) z^n$. The value of $f'(1)$, the mean number of offspring, determines the survival properties of the process. If $f'(1) = 1$ and excluding the trivial case $p(1) = 1$ the process is said to be critical and it dies out with probability one. If $f'(1) < 1$ the process is sub–critical and dies out exponentially fast and if $f'(1) > 0$ the process is super–critical and has a positive probability of surviving forever, see e.g. [69].

The model of simply generated trees has as parameters a sequence of non–negative weights $(w_n)_{n \geq 1}$ referred to as *branching weights* and is defined by a (Gibbs) measure on the set of trees with n edges by

$$\nu_n(T) = Z_n^{-1} \prod_{v \in V(T) \setminus \{r\}} w_{\sigma(v)}, \quad T \in \Gamma_n \tag{2.53}$$

where Z_n is a normalization

$$Z_n = \sum_{T \in \Gamma_n} \prod_{v \in V(T) \setminus \{r\}} w_{\sigma(v)} \tag{2.54}$$

often referred to as the finite volume partition function. It is useful to define the generating functions

$$\mathcal{Z}(\zeta) = \sum_{n=1}^{\infty} Z_n \zeta^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} w_{n+1} z^n \tag{2.55}$$

which are related by the equation

$$\mathcal{Z}(\zeta) = \zeta g(\mathcal{Z}(\zeta)), \tag{2.56}$$

see e.g. [70]. We will denote their radii of convergence by ζ_0 and ρ respectively and we furthermore define $\mathcal{Z}_0 = \lim_{\zeta \to \zeta_0} \mathcal{Z}(\zeta)$. When $\rho > 0$ the measure ν_n can equivalently be defined by a Galton-Watson branching process with offspring probabilities

$$p(k) = \zeta_0 w_{k+1} \mathcal{Z}_0^{k-1} \tag{2.57}$$

conditioned to have n edges, see e.g. [71]. By (2.56), the mean offspring number can be written as $f'(1) = 1 - g(\mathcal{Z}_0)/\mathcal{Z}'(\zeta_0)$ and thus it is clear that the process is either critical or sub–critical. Here we will only consider the critical case i.e. when $\mathcal{Z}'(\zeta_0) = \infty$ in which case the following theorem holds.

Theorem 2.3.1. *For a critical size conditioned Galton-Watson process, the finite volume measures ν_n converge weakly as $n \to \infty$ towards a measure ν which is concentrated on Γ_S^∞. The degrees of vertices on the spine are independently distributed by*

$$\phi(k) = \zeta_0 (k-1) w_k \mathcal{Z}_0^{k-2} \tag{2.58}$$

and the outgrowths from the spine are independent Galton-Watson trees with offspring probabilities (2.57).

The critical model is usually divided into two cases depending on whether $f''(1)$ is finite or infinite. For the case $f''(1) < \infty$, Theorem 2.3.1 is originally due to Kennedy [36] and later to Aldous and Pitman [37], see also [39]. To our best knowledge, the generalisation which includes the case $f''(1) = \infty$ was first proved in the special case $w_n \sim n^{-\beta}$ by Jonsson and Stefánsson [71] and later in full generality by Janson [72]. The same limiting behaviour was also obtained earlier by Kesten [73] for Galton-Watson trees conditioned on their height.

In the case $f''(1) < \infty$ the trees always belong to the same universality class and have been referred to as *generic trees* in the physics literature. In the infinite case there is a range of universality classes depending on the singular behaviour of f and this case has been referred to as *critical non-generic*.

2.3.2 | Dimensions

The following theorem holds for the fractal and spectral dimension in the critical case

Theorem 2.3.2.

(i) The quenched fractal dimension and spectral dimension of generic trees is almost surely

$$d_h = 2 \quad \text{and} \quad d_s = 4/3 \tag{2.59}$$

respectively.

(ii) For critical non-generic trees with $w_n \sim n^{-\beta} L(n)$, where $\beta \in (2,3]$ and L slowly varying at infinity, the quenched fractal and spectral dimensions are almost surely

$$d_h = \frac{\beta - 1}{\beta - 2} \quad \text{and} \quad d_s = \frac{2(\beta-1)}{2\beta - 3} \tag{2.60}$$

respectively.

The result on d_h in (i) is proven for instance in [34] and the result on d_s was originally proven in [43, 44]. The results in part (ii) were conjectured in the physics literature [49, 50] and later proven in the mathematical literature by Croydon and Kumagai [46]. Below we will show how Theorem 2.3.2 easily follows from Theorem 2.2.2. While the proof in [46] is slightly more general, as discussed in the introduction, the proof presented here is more intuitive and hopefully makes the result more accessible to physicists.

Proof. It is a standard result, see e.g. [35], that for generic trees

$$\langle z^{|\mathcal{A}_i|}\rangle = 1 - c(1-\zeta)^{1/2} + o((1-\zeta)^{1/2}) \tag{2.61}$$

and

$$\frac{1}{1-f_n(z)} - \frac{1}{1-z} = \frac{1}{2}f''(1)n + o(n), \tag{2.62}$$

where $f_n = f \circ \cdots \circ f$, n-times, is the n-th iterate of the generating function $f(z)$ of the offspring distribution. Denoting by $X_n^{i,j}$ the size of the n-th generation of the j-th outgrowth from s_i, one thus has using (2.62) that for $n \geq 1$

$$\nu(\{X_n^{i,j} > 0\}) = 1 - f_{n-1}(0) = \frac{2}{f''(1)n}(1 + o(1)), \tag{2.63}$$

Hence, since the outgrowths are independent, one has using Theorem 2.3.1

$$\begin{aligned} \nu(\{X_n^{i} > 0\}) &= 1 - f'(1 - \nu(\{X_n^{i,j} > 0\})) \\ &= \frac{2}{n}(1 + o(1)). \end{aligned} \tag{2.64}$$

The results then follow from Theorem 2.2.2.

Next, consider the case $f''(1) = \infty$. By a Tauberian theorem [67], $w_n \sim n^{-\beta}L(n)$, implies that one can write

$$f(z) = z + (1-z)^{\beta-1} l_1(1-z) \tag{2.65}$$

where l_1 is slowly varying at zero. Let $\mathcal{W}(\zeta) = \mathcal{Z}(\zeta_0\zeta)/\mathcal{Z}_0$. Using generating function arguments one can deduce from Theorem 2.3.1 that

$$\langle z^{|\mathcal{A}_i|}\rangle = f'(\mathcal{W}(z)). \tag{2.66}$$

Write

$$\mathcal{W}(\zeta) = 1 - (1-\zeta)^{\frac{1}{\beta-1}}\chi(1-\zeta). \tag{2.67}$$

By (2.56) and (2.65) we find that χ has to satisfy

$$\lim_{x\to 0} \chi(x)^{\beta-1} l_1\left(x^{\frac{1}{\beta-1}}\chi(x)\right) = 1 \tag{2.68}$$

which, by [66, Theorem 1.5], entails that χ is slowly varying at zero. Therefore, by (2.65), (2.66), (2.67) and Tauberian theorems one can write

$$\langle z^{|\mathcal{A}_i|}\rangle = 1 - (1-z)^{\frac{\beta-2}{\beta-1}} l_2(1-z) \tag{2.69}$$

with l_2 slowly varying at zero. Thus, by part (i) of Theorem 2.2.2 we have established the upper bounds on d_h and d_s.

For the lower bounds we use part (ii) and consider the survival probability of the outgrowths. Let $X_n^{i,j}$ be the size of the n-th generation of the j-th outgrowth from s_i. It was shown by Slack [74, Lemma 2] that

$$\left(\nu(\{X_n^{i,j}>0\})\right)^{\beta-2}\, l_1(\nu(\{X_n^{i,j}>0\})) = \frac{1}{(\beta-2)n}(1+o(1)), \tag{2.70}$$

where l_1 is the same slowly varying function as in (2.65). Then, since the outgrowths are independent, we find by (2.65) that

$$\begin{aligned}
\nu(\{X_n^i>0\}) &= 1 - f'\left(1-\nu(\{X_n^{i,j}>0\})\right) \\
&= (\beta-1)(\nu(\{X_n^{i,j}>0\}))^{\beta-2}\, l_1(\nu(\{X_n^{i,j}>0\}))(1+o(1)) \\
&= \frac{\beta-1}{\beta-2}\frac{1}{n}(1+o(1)).
\end{aligned} \tag{2.71}$$

which completes the proof. □

2.4 | The attachment and grafting model

2.4.1 | The model

The attachment and grafting (ag) model [51] is a recent model of randomly growing rooted planar trees which is a special case of a very general tree growth model, referred to as the vertex splitting model. The vertex splitting model was introduced in [65] as a modification of a combinatorial model encountered in the theory of random RNA folding [62]. The original motivation for studying the special case of the ag–model is that it has a so–called *Markov branching property* which makes it exactly solvable in a strong

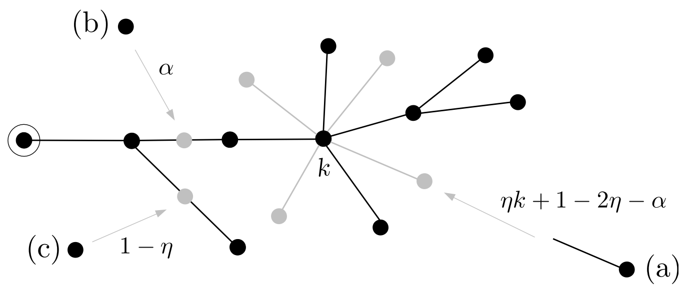


Figure 2.2: Growth rules of the ag–model. The root is indicated by a circled vertex.

sense. It furthermore turns out to have a unique infinite spine which enables us to apply Theorem 2.2.2 to study its fractal and spectral dimension. Using the first part of the theorem we establish what we believe to be tight upper bounds on the dimensions for the full range of parameters. The corresponding lower bounds require information on the extinction probability of the outgrowths and we will provide results on that only for a certain range of parameters.

We give an informal description of the growth rules of the ag–model below but refer to [51] for a more detailed discussion. The model has two parameters $\alpha, \gamma \in [0, 1]$ and a constraint D on the maximum degree of vertices. For easier notation we define

$$\eta = \begin{cases} -\frac{1-\alpha}{D-2} & \text{if } D < \infty, \\ 1-\gamma & \text{if } D = \infty. \end{cases} \tag{2.72}$$

Call the edges which are adjacent to vertices of degree one (besides the root) *leaves* and call the other edges *internal edges*. Starting from the unique tree with two edges, in each time step the number of edges is increased by one by randomly selecting

(a) a vertex of degree $k \geq 2$ with relative probability $\eta k + 1 - 2\eta - \alpha$ and attaching a new edge to it (the k possibilities of attaching chosen uniformly at random) or

(b) an inner edge with relative probability α and dividing it into two edges by grafting a vertex to it or

(c) a leaf with relative probability $1 - \eta$ and dividing it into two edges by grafting a vertex to it,

see Fig. 2.2. The growth rule generates a sequence of probability measures on Γ which we denote by $(\nu_n)_{n\geq 1}$ (not writing explicitly the dependency on α, γ and D). The probability of a tree $T \in \Gamma_{n+1}$ is given by a recursion

$$\nu_{n+1}(T) = \sum_{T' \in \Gamma_n} \nu_n(T')\mathbb{P}(T' \to T) \tag{2.73}$$

where $\mathbb{P}(T' \to T)$ is the probability of growing T from T' according to the above growth rule.

2.4.2 | Markov branching and convergence of the finite volume measures

The ag–model has a property called Markov branching, a concept originally introduced by Aldous [60]. Markov branching means that for any $k \geq 2$ there is a function $q_k : \mathbb{N}^{k-1} \to \mathbb{R}_+$ such that for each finite tree T_0 which branches at the nearest neighbour of the root to subtrees $T_1, \dots, T_{k-1}$ it holds that

$$\nu_{|T_0|}(T_0) = q_k(|T_1|, \dots, |T_{k-1}|) \prod_{i=1}^{k-1} \nu_{|T_i|}(T_i). \tag{2.74}$$

The functions q_k are referred to as *the first split distribution*. In the ag–model, the first split distribution is given by [51]

$$\begin{aligned} q_k(n_1, \dots, n_{k-1}) \quad = \quad & \frac{\Gamma\left(k-2+\frac{1-\alpha}{\eta}\right)}{\Gamma\left(\frac{1-\alpha}{\eta}\right)\Gamma(k)} \frac{\Gamma(1-\eta)\Gamma(n)}{\eta\Gamma(n-\eta)} \prod_{i=1}^{k-1} \frac{\eta\Gamma(n_i-\eta)}{\Gamma(1-\eta)\Gamma(n_i+1)} \\ & \times \left(1-\eta-\alpha+\alpha\sum_{i=1}^{k-1}\frac{n_i}{n-n_i}\right) \end{aligned} \tag{2.75}$$

where $n_1 + \cdots + n_{k-1} = n - 1$.

In [51] it was shown, using (2.74) and (2.75), that the sequence $(\nu_n)_{n\geq 1}$ converges weakly as $n \to \infty$ to a measure ν which is concentrated on the set of infinite trees with exactly one infinite spine having finite outgrowths. Outgrowths from different vertices on the spine are independent and the probability that a vertex on the spine has degree k and that the finite outgrowths attached to it are $T_1, \dots, T_{k-2}$, is given by

$$\mu_k(T_1, \dots, T_{k-2}) = \frac{\alpha\Gamma\left(k-2+\frac{1-\alpha}{\eta}\right)}{\Gamma\left(\frac{1-\alpha}{\eta}\right)\Gamma(k-1)(1+m)} \prod_{i=1}^{k-2} \frac{\eta\Gamma(|T_i|-\eta)\nu_{|T_i|}(T_i)}{\Gamma(1-\eta)\Gamma(|T_i|+1)}, \tag{2.76}$$

where $m = |T_1| + \cdots + |T_{k-2}|$ (with the convention that $\mu_2(\emptyset) = 1$).

2.4.3 | Dimensions

The annealed fractal dimension of the ag–model was calculated in [51] using a generating function argument. It was shown that $d_H = 1/\alpha$ when D is finite and in the case $D = \infty$ and $\alpha > 1 - \gamma$. Note that it is meaningless to consider the annealed fractal dimension when $D = \infty$ and $\alpha \leq 1 - \gamma$ since then it follows from (2.76) that the expected degree of a vertex on the spine is infinite and thus $d_H = \infty$.

Despite this, one might still conjecture that a.s. $d_h = 1/\alpha$. This was confirmed in [51] when $\gamma = 0$ in which case the outgrowths from the spine are single edges and it was furthermore shown that in this case $d_s = 2d_h/(1 + d_h) = 2/(1 + \alpha)$. Below, we extend these results to a wider range of parameters.

Theorem 2.4.1. *1. For the ag–ensemble* (Γ, ν) *it holds for any* α, γ *and* D, *that almost surely*

$$d_h \leq \frac{1}{\alpha} \qquad \text{and} \qquad d_s \leq \frac{2}{1+\alpha}. \tag{2.77}$$

2. Furthermore, for $D = 3$, $\frac{1}{2} \leq \alpha \leq 1$ *and for* $D = \infty$, $\gamma = 0$, *we have almost surely*

$$d_h = \frac{1}{\alpha} \qquad \text{and} \qquad d_s = \frac{2}{1+\alpha}. \tag{2.78}$$

Proof. As before, we denote the outgrowths from vertex s_i on the spine by $\mathcal{A}^i$. The upper bounds in (2.77) follow from the simple observation that

$$\begin{aligned}
\langle z^{|\mathcal{A}^i|}\rangle_\nu &= \sum_{k=2}^{\infty} \sum_{m=k-2}^{\infty} \sum_{n_1+\cdots+n_{k-2}=m} \sum_{T_1\in\Gamma_{n_1},\dots,T_{k-2}\in\Gamma_{n_{k-2}}} \mu_k(T_1,\dots,T_{k-2}) z^m \\
&= \frac{1}{z}\int_0^z dx \sum_{k=2}^{\infty} \frac{\alpha\Gamma\left(k-2+\frac{1-\alpha}{\eta}\right)}{\Gamma\left(\frac{1-\alpha}{\eta}\right)\Gamma(k-1)} \left[1-(1-x)^\eta\right]^{k-2} \\
&= \frac{1-(1-z)^\alpha}{z}.
\end{aligned} \tag{2.79}$$

The second equality in (2.79) is obtained by noting that the innermost sums of $\nu_{n_i}(T_i)$ over T_i, $i = 1, \dots, k-2$, yield 1 and then using standard gamma function identities [75]

$$\sum_{n=1}^{\infty} \frac{\sigma\Gamma(n-\sigma)}{\Gamma(1-\sigma)\Gamma(n+1)} z^n = 1 - (1-z)^\sigma. \tag{2.80}$$

Finally, an application of the first part of Theorem 2.2.2 yields (2.77).

To obtain the equalities in (2.78) we study the height distribution of the outgrowths from the spine and apply the second part of Theorem 2.2.2. We start by noting that the case when $D = \infty$ and $\gamma = 0$, which was already proved in [51], follows immediately from Theorem 2.2.2 since in this case the outgrowths from the spine all have height 1. We now focus on the case $D = 3$. As before, denote by X_n^i the size of the n–th generation of $\mathcal{A}^i$. The result follows from

Lemma 2.4.1. *When $D = 3$ and $\frac{1}{2} \leq \alpha \leq 1$ it holds that*

$$\nu(\{X_n^i > 0\}) \leq \frac{2}{n}(1 + o(1)). \tag{2.81}$$

We devote the following subsection to the proof of this lemma. □

2.4.4 | Proof of Lemma 2.4.1

We start the proof with a general approach and reduce to the specific parameters stated in Lemma 2.4.1 only in the end when needed. In this way we keep the problem of extending the results to all parameters clearly approachable.

For a finite tree T, let $h(T)$ be its height, i.e. the maximum distance from the root to any vertex. We make the following definitions for more compact notation

$$c_k = \frac{\Gamma\left(k - 2 + \frac{1-\alpha}{\eta}\right)}{\Gamma\left(\frac{1-\alpha}{\eta}\right)\Gamma(k)} \quad \text{and} \quad C_{n,R} = \frac{-\eta\Gamma(n-\eta)\nu_n(\{h(T) \leq R\})}{\Gamma(1-\eta)\Gamma(n+1)} \tag{2.82}$$

with the convention that $\nu_0(\{h(T) \leq R\}) = 1$. The main tool that we will use in the proof is the following generating function

$$H_R(\zeta) = \sum_{n=0}^{\infty} C_{n,R}\zeta^n. \tag{2.83}$$

Proposition 2.4.1. *The probability that $\mathcal{A}^i$ is extinct at level $R+1$ is given by*

$$\nu(\{X_{R+1}^i = 0\}) = \alpha \int_0^1 (H_R(\zeta))^{\frac{\alpha-1}{\eta}} d\zeta. \tag{2.84}$$

Proof. Using the distribution of the outgrowths given in (2.76) one can write

$$\begin{aligned} \nu(\{X_{R+1}^i = 0\}) &= \nu(\{h(\mathcal{A}^i) \leq R\}) \\ &= \alpha \int_0^1 \left[\sum_{k=2}^{\infty} (k-1)c_k \sum_{m=k-2}^{\infty} \sum_{n_1+\cdots+n_{k-2}=m, n_i \geq 1} \prod_{i=1}^{k-2} (-C_{n_i,R})\zeta^{n_i} \right] d\zeta. \end{aligned} \tag{2.85}$$

The sum over m is first performed giving $(1 - H_R(\zeta))^{k-2}$ and the sum over k is then performed using standard gamma function identities similar as above giving (2.84). $\square$

Proposition 2.4.1 shows that $H_R(\zeta)$ contains all the information needed to calculate the extinction probability of $\mathcal{A}^i$. Next, using the Markov branching property, we will derive a recursion equation for $H_R(\zeta)$ which enables us to extract enough information.

Proposition 2.4.2. *For $R \geq 1$, it holds that*

$$H_R(\zeta) \;=\; 1 + \alpha H_{R-1}(\zeta) \int_0^\zeta (H_{R-1}(y))^{\frac{\alpha-1}{\eta}} dy - (\alpha+\eta) \int_0^\zeta (H_{R-1}(y))^{\frac{\alpha+\eta-1}{\eta}} dy \tag{2.86}$$

with the convention that $H_0(\zeta) = 1$.

Proof. Using the Markov branching property and (2.75) we write

$$\begin{aligned} -nC_{n,R} \;&=\; \sum_{k=2}^{\infty} c_k \sum_{n_1+\dots+n_{k-1}=n-1, n_i \geq 1} \prod_{i=1}^{k-1} (-C_{n_i,R-1}) \\ &\times \left(1 - \eta - \alpha + \alpha \sum_{j=1}^{k-1} \frac{n_j}{n-n_j}\right). \end{aligned} \tag{2.87}$$

Multiply by ζ^{n-1} and sum from $n = 2, \dots \infty$ to get

$$\begin{aligned} &-(H_R'(\zeta)+\eta) = \sum_{k=2}^{\infty} c_k \sum_{n=k}^{\infty} \sum_{n_1+\dots+n_{k-1}=n-1, n_i \geq 1} \prod_{i=1}^{k-1} (-C_{n_i,R-1})\zeta^{n_i} \\ &\times \left(1 - \eta - \alpha + \alpha \sum_{j=1}^{k-1} \frac{n_j}{n-n_j}\right) \\ &= (1-\eta-\alpha) \sum_{k=2}^{\infty} c_k \left(-\sum_{n=1}^{\infty} C_{n,R-1}\zeta^n\right)^{k-1} \\ &- \alpha \left(\frac{d}{d\zeta} \sum_{n=1}^{\infty} C_{n,R-1}\zeta^n\right) \int_0^\zeta \sum_{k=2}^{\infty} (k-1)c_k \left(-\sum_{n=1}^{\infty} C_{n,R-1} y^n\right)^{k-2} dy. \end{aligned} \tag{2.88}$$

Using the definition of H_R and performing the sum over k yields

$$H_R'(\zeta) = -\eta \left(H_{R-1}(\zeta)\right)^{\frac{\alpha+\eta-1}{\eta}} + \alpha H_{R-1}'(\zeta) \int_0^\zeta \left(H_{R-1}(y)\right)^{\frac{\alpha-1}{\eta}} dy. \tag{2.89}$$

Integrating this equation and using integration by parts on the second term on the right hand side finally yields the result (2.86). $\square$

We now reduce to the case $D = 3$ in which case $\eta = \alpha - 1 \leq 0$. The recursion (2.86) is then greatly simplified. We define $F_R(\zeta) = \alpha \int_0^\zeta H_R(y)dy$ and then $\nu(\{X^i_{R+1} = 0\}) = F_R(1)$. Next we integrate (2.86) to get

$$F_R(\zeta) = \alpha\zeta + \frac{1}{2}F_{R-1}(\zeta)^2 - \frac{2\alpha - 1}{\alpha}\int_0^\zeta \int_0^y \left(F'_{R-1}(x)\right)^2 dxdy. \tag{2.90}$$

One can see directly from the definition of H_R (2.83) that when $\eta \leq 0$, F_R and all its derivatives are increasing in R. One can furthermore deduce from (2.83) that

$$F_\infty(\zeta) := \lim_{R\to\infty} F_R(\zeta) = 1 - (1-\zeta)^\alpha. \tag{2.91}$$

Therefore, under the assumption that $\frac{1}{2} \leq \alpha \leq 1$ we may insert $F_\infty(\zeta)$ into the integral in (2.90) to get the inequality

$$F_R(\zeta) \geq \frac{1}{2} + \frac{1}{2}\left(F_{R-1}(\zeta)\right)^2 - \frac{1}{2}(1-\zeta)^{2\alpha}. \tag{2.92}$$

Evaluated at $\zeta = 1$ this can be written as

$$F_R(1) \geq f(F_{R-1}(1)) \tag{2.93}$$

with $f(x) = \frac{1}{2} + \frac{1}{2}x^2$. We can interpret f as a generating function of the offspring probabilities $p(0) = p(2) = 1/2$ of a critical Galton-Watson process, cf. Section 2.3. Let f_R be the R–th iterate of f i.e. $f_R = f \circ \cdots \circ f$, R times. Now, f is an increasing function and therefore, by repeatedly applying (2.93) one gets

$$\nu(\{X^i_{R+1} = 0\}) = F_R(1) \geq f_R(F_0(1)) \geq f_R(1/2) \tag{2.94}$$

where in the last step we used that $F_0(1) = \alpha \geq 1/2$. It then follows from (2.62) that $f_R(1/2) = 1 - \frac{2}{R}(1 + o(1))$. This concludes the proof of Lemma 2.4.1.

2.5 | Discussion

We developed relatively simple methods, relying on generating function arguments, for calculating the fractal and spectral dimension of trees with a unique infinite spine resulting in Theorems 2.2.1 and 2.2.2, where the latter applies when the outgrowths along the spine are independent and identically distributed. These methods were applied to two models of random trees of very different nature, demonstrating the versatility of our results.

The first application of Theorem 2.2.2 concerns non-generic critical trees which are a special case of simply generated trees. The values of the fractal and spectral dimension of the non–generic, critical, size conditioned Galton–Watson trees were conjectured by mathematical physicists 15 years ago [49, 50] but only recently proved by mathematicians [46]. We used the opportunity here to communicate these results to the physics community as well as providing a simple alternative proof.

We would like to point out that these results complete the study of the dimensionality of the different phases of simply generated trees. Theorems 2.3.1 and 2.3.2 concern critical Galton–Watson processes. As was mentioned in Section 2.3, the simply generated trees can also correspond to sub–critical Galton–Watson processes. In the sub–critical case, the finite volume measures converge weakly to a measure concentrated on the set of trees with a finite spine ending with a vertex of infinite degree. The length of the spine has a geometric distribution and the outgrowths from the spine are finite, independent, sub–critical Galton–Watson processes. This convergence theorem was originally proved in [71] in the case $w_n \sim n^{-\beta}$ and later in full generality in [72].

Due to the presence of a vertex of infinite degree it clearly follows that almost surely $d_s = \infty$ since a random walker will eventually hit the infinite degree vertex and similarly $d_h = d_H = \infty$. However, it was shown in [71] that when $w_n \sim n^{-\beta}$, the annealed spectral dimension is finite and $d_S = 2(\beta - 1)$. This is due to the fact that the fluctuations of the outgrowths from the spine serve to slow the random walker down on the way to the infinite degree vertex. Note that in this case it does not hold that $d_S = 2d_H/(1 + d_H)$ as in the other phases due to the absence of the infinite spine.

We summarize the above discussion in the phase diagram in Figure 2.3 where we consider the case when w_1 and β are the free parameters and $w_n \sim n^{-\beta}$. This choice of parameters allows us to access all phases and explore the full range of dimensions in each phase, see [71] for a more detailed explanation.

The second application of Theorem 2.2.2 concerns the attachment and grafting model which is a special case of the vertex splitting model. We give a novel proof that for the parameter range $D = 3$, $\frac{1}{2} \leq \alpha \leq 1$ and for $D = \infty$, $\gamma = 0$, one has almost surely $d_h = 1/\alpha$ and $d_s = 2/(1 + \alpha)$. This proves part of a previous conjecture [51].

It remains an open problem to prove that the quenched fractal and spectral dimension of the ag–model are almost surely $d_h = 1/\alpha$ and $d_s = 2/(1 + \alpha)$ for the full range of the parameters α, γ and D. The only missing ingredient in the proof is to generalize Lemma 2.4.1 to the full range of parameters. We formulate this as the following:

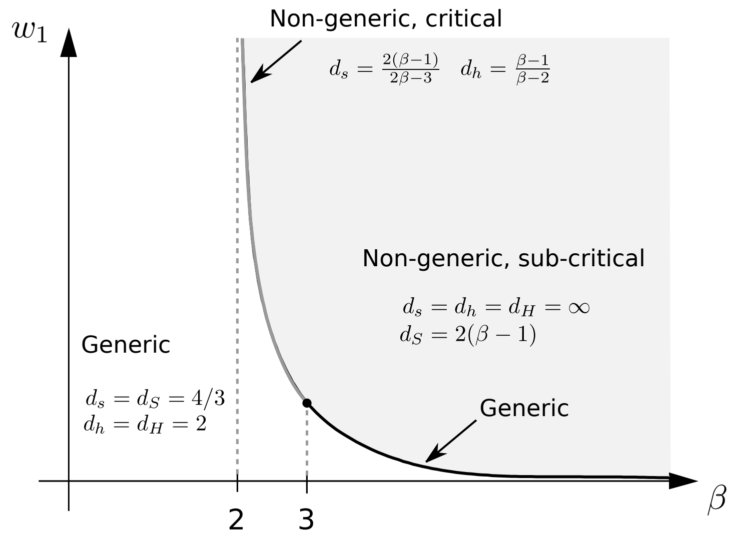


Figure 2.3: A phase diagram for the size conditioned Galton-Watson trees, having free parameters w_1 and β, where $w_n \sim n^{-\beta}$. A critical line separates the generic phase from the non–generic phase. For the values $2 < \beta \leq 3$ on the critical line (grey line) the off-spring probability distribution has infinite variance and therefore this case corresponds to non–generic, critical trees.

Conjecture 2.5.1. *For the ag–ensemble* (Γ, ν) *it holds for any* α, γ *and* D*, that there exists a constant* $c > 0$ *such that*

$$\nu(\{X_n^i > 0\}) \leq \frac{c}{n}(1 + o(1)). \tag{2.95}$$

Furthermore, we expect that one even has the stronger result $\nu(\{X_n^i > 0\}) \sim 1/(\alpha n)$. This conjecture could possibly be proven by analysing the recursion relation (2.86) to find the critical behaviour of the generating function $H_R(\zeta)$ when $\zeta \to 1$ and $R \to \infty$. We hope to return to this proof in the near future.

At this point we would also like to note that all of the results derived for the ag–model can straightforwardly be extended to Ford's α-model [61] and its generalisation, the $\alpha\gamma$–model [76]. We leave a more detailed discussion for future work after the proof of the above conjecture.

There are also different applications of the presented work in the field of quantum gravity. Firstly, the results derived in the chapter for critical non-generic trees are useful to understand in more detail the branched polymer phase of Euclidean quantum gravity and, in particular, in the presence of matter. Furthermore, the critical non-generic trees are described by branching processes whose population size has polynomial growth which is potentially relevant to the analysis of recently introduced multigraph ensembles as models for four-dimensional causal or Lorentizan quantum gravity [57, 56].

Part II

Uniform infinite causal triangulations (UICT)

CHAPTER 3

A note on weak convergence results for uniform infinite causal triangulations

We discuss uniform infinite causal triangulations and equivalence to the size biased branching process measure - the critical Galton-Watson branching process distribution conditioned on non-extinction. Using known results from the theory of branching processes, this relation is used to prove weak convergence of the joint length-area process of a uniform infinite causal triangulations to a limiting diffusion. The diffusion equation enables us to determine the physical Hamiltonian and Green's function from the Feynman-Kac procedure, providing us with a mathematically rigorous proof of certain scaling limits of causal dynamical triangulations.

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3.1 | Introduction

Models of planar random geometry provide a rich field with an interplay between mathematical physics and probability.

On the physics side so-called dynamical triangulations (DT) have been introduced as models for two-dimensional Euclidean quantum gravity and string theory (see e.g. [9] for an overview). The basic idea is to define the gravitational path integral as a sum over triangulated surfaces. Any physical observable is then defined on the ensemble of all such triangulations. At the end, continuum physics is obtained by performing a scaling limit in which one takes the size of the triangulations to infinity keeping the physical area constant.

On the probabilistic side Angel and Schramm [14] first introduced the uniform measure on infinite planar triangulations proving the existence of the above scaling limit as a weak limit. This construction was essential to prove several properties of such uniform infinite triangulations. In particular Angel [15] proved that the volume of a ball $B(R)$ of radius R is of order R^4 and that the length of the boundary is of order R^2. This proved rigorously that the fractal dimension of such triangulations is $d_h = 4$, a result long known to physicists (see e.g. [9]). Later Krikun [17] obtained the exact limit theorem for the scaled boundary length of $B(R)$.

While two-dimensional Euclidean quantum gravity defined through DT definitely has a rich mathematical structure as pointed out above, as a model of quantum gravity it failed to be numerically extended to higher dimensions. This led to the development of a different approach of so-called Causal Dynamical Triangulations (CDT) by Ambjørn and Loll [10]. In contrast to the *Euclidean* model, CDT provides a nonperturbative definition of the *Lorentzian* gravitational path integral. These causal triangulations differ from their Euclidean analogs in the fact that they have a time-sliced structure of fixed spatial topology. Consider for example triangulations of an overall topology of a cylinder. Then the triangulation consists of slices $S^1 \times [t, t+1]$ from time t to time $t+1$ as illustrated in Figure 4.1. Here, edges connecting vertices in slices of equal time are called space-like edges, while edges connecting subsequent slices are called time-like edges.

Note that this class of triangulations forms a causal structure needed to model Lorentzian geometries. In particular, we can think that a vertex v' lies in the future of a vertex v iff there is a path of time-like edges leading from v to v'. For example, the vertices v to v' as illustrated in Figure 4.1 are not causally related.

The physical properties of the ensemble of causal triangulations behaves much more regular than its Euclidean counterpart. For example, it has a fractal dimension of $d_h = 2$ instead of $d_h = 4$ for DT. Also when coupled to simple matter models, such as the Ising model, it behaves much more like a regular lattice such as $\mathbb{Z}^2$ [77].

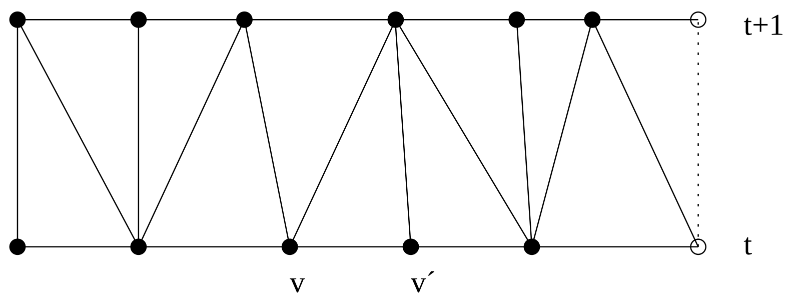

Figure 3.1: A time slice from time t to time $t+1$. The left and right side of the strip should be identified to form a band with topology $S^1 \times [t, t+1]$.

While the approach of CDT has recently led to a number of interesting physical results, especially with respect to higher-dimensional numerical implementations (see [78] for a review), the probabilistic aspects of this model have hardly been studied. In fact, recently Durhuus, Jonsson and Wheater defined the uniform measure on infinite causal triangulations [34] (see also [79] for earlier ideas), proving almost surely (a.s.) recurrence, that a.s. the fractal dimension is $d_h = 2$ and that the spectral dimension is a.s. bounded from above by $d_s \leq 2$. A similar definition of the uniform measure has previously also been used by Yambartsev and M. Krikun to prove the existence of a phase transition for the Ising model coupled to CDT [52].

In this chapter we discuss the existence of the uniform measure on infinite causal triangulations in an alternative presentation to [34]. In the line of [14] and [17] we give weak convergence limits for the distribution of the area and length of the boundary of a ball of radius t, confirming that they scale as t^2 and t. These results follow from a bijection between causal triangulations and certain Galton-Watson branching processes through the size biased branching process measure [80] - the critical branching process distribution conditioned on survival at infinity. Exploiting the relation to conditioned critical Galton-Watson processes, one can go further and obtain weak convergence of the joint length and area process. The process is diffusive and the corresponding Kolmogorov equation enables us to derive the physical Hamiltonian, providing us with a mathematically rigorous formulation of scaling limits of CDT.

In the next section we give basic definitions and introduce infinite causal triangulations and show existence of the uniform measure on infinite causal triangulations in an alternative presentation to [34]. In Section 3.3 we then present the relation to critical Galton-Watson processes conditioned to never die out. In Section 3.4 we exploit this relation to obtain weak convergence of the length process (Theorem 3.4.1) and the joint length-area process (Theorem 3.4.2). Theorem 3.4.1 is proven in Appendix 3.A. These results provide a mathematically rigorous formulation of certain scaling limits of CDT which we discuss in Section 3.5.

3.2 | Uniform infinite causal triangulations

We consider rooted causal triangulations of $\mathcal{C}_h = S^1 \times [0, h]$, $h = 1, 2, \ldots$, and of $\mathcal{C} = S^1 \times [0, \infty)$, where S^1 is a unit circle.

Definition 3.2.1. *Consider a (finite) connected graph G. Its embedding into $\mathcal{C}_h$ (that is, considered as a subset $G \subset \mathcal{C}_h$) is called a causal triangulation T of $\mathcal{C}_h$ if all faces of T are triangles and each face of T that belongs to $\mathcal{C}_h$ belongs to some strip $S^1 \times [j, j+1]$, $j = 0, 1, \ldots, h-1$ and has all vertices and exactly one side on the boundary $(S^1 \times \{j\}) \cup (S^1 \times \{j+1\})$ of the strip $S^1 \times [j, j+1]$.*

Remark 3.2.1. *Some care has to be put into the definition of what is meant by a triangle due to self-loops and multiple edges. Let the size of a face be the number of edges incident to it, with the convention that an edge incident to the same face on both sides counts for two. We then call a face with size 3 (or 3-sided face) a triangle.*

Definition 3.2.2. *A causal triangulation T of $\mathcal{C}_h$ is called rooted if it has a root. The root (x, e) of T consists of vertex x and directed edge e that runs from x, they are called root vertex and root edge correspondingly. The root vertex and the root edge belong to $S^1 \times \{0\}$. The orientation induced by the ordered pair that consists of the root edge and the vector that runs from the root vertex in positive time direction coincides with the fixed orientation of $\mathcal{C}_h$.*

Definition 3.2.3. *Two rooted causal triangulations of $\mathcal{C}^h$, say T and T', which are embeddings of graphs G and G' correspondingly, are equivalent if there exists an orientation preserved homeomorphism $u : \mathcal{C}^h \to \mathcal{C}^h$ which transforms each slice $S^1 \times \{j\}$, $j = 0, \ldots, h$ to itself, sends the root of T to the root of T', and the restriction of u on G is an isomorphism between G and G'.*

For convenience, we usually abbreviate “equivalence class of rooted causal triangulations” to “causal triangulation” or CT.

Cutting off the strip $S^1 \times (h, h+1]$ from $\mathcal{C}^{h+1}$ we obtain a natural map from the set of causal triangulations of $\mathcal{C}^{h+1}$ to the set of causal triangulations of $\mathcal{C}^h$ that we denote by r_h.

Definition 3.2.4. *We say that T is a causal triangulation of $\mathcal{C}$, if $T = (T_1, T_2, \dots)$, where T_h is a causal triangulation of $\mathcal{C}_h$, $h = 1, 2, \dots$, and the sequence is subject to consistency condition $T_h = r_h(T_{h+1})$, $h = 1, 2, \dots$.*

By $\mathbb{CT}_\infty$ denote the set of all causal triangulations of $\mathcal{C}$ and by $\mathbb{CT}_h$ denote the set of all causal triangulations of $\mathcal{C}^h$. Let

$$\mathbb{CT}h = \bigcup_{i=1}^{h} \mathbb{CT}_i,\ h = 1, 2, \dots \quad \text{and} \quad \mathbb{CT}\infty = \mathbb{CT}_\infty \cup \bigcup_{i=1}^{\infty} \mathbb{CT}_i.$$

The restriction map $r_h \colon \mathbb{CT}_{h+1} \to \mathbb{CT}_h$ can be naturally generalized to become the restriction map $r_h \colon \mathbb{CT}\infty \to \mathbb{CT}h$. We see that $T \in \mathbb{CT}\infty$ is identified by the sequence $(T_1, T_2, \dots)$, where $T_h \in \mathbb{CT}h$ are subject only to consistency condition $T_h = r_h(T_{h+1})$, $h = 1, 2, \dots$.

We use the standard formalism for planar trees (see [81] or [37]). Let

$$\hat{\mathcal{U}} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \dots\}$ and by convention $\mathbb{N}^0 = \{\varnothing\}$. The *height* of $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ is $|u| = n$. If $u = (u_1, \dots u_m)$ and $v = (v_1, \dots, v_n)$ belong to $\hat{\mathcal{U}}$, then $uv = (u_1, \dots u_m, v_1, \dots, v_n)$ denotes the concatenation of u and v. In particular $u\varnothing = \varnothing u = u$. If v is of the form $v = uj$ for $u \in \hat{\mathcal{U}}$ and $j \in \mathbb{N}$, we say that u is the *predecessor* of v, or that v is a *successor* of u. More generally, if v is of the form $v = uw$ for $u, w \in \hat{\mathcal{U}}$, we say that u is an *ancestor* of v, or that v is a *descendant* of u.

Definition 3.2.5. *A (finite or infinite) family tree τ is a subset of $\hat{\mathcal{U}}$ such that*

(i) $\varnothing \in \tau$;

(ii) if $u \in \tau$ and $u \neq \varnothing$, the predecessor of u belongs to τ;

(iii) for every $u \in \tau$, there exists an integer $k_u(\tau) \geq 0$ such that $uj \in \tau$ if and only if $1 \leq j \leq k_u(\tau)$.

The *height of a finite tree* is the maximum height of all vertices in the tree. Let $\mathbb{T}^{(\infty)}$ be the set of all family trees and $\mathbb{T}^{(h)}$ the set of all finite family trees of height at most h. There is a natural restriction map $r_h : \mathbb{T}^{(\infty)} \to \mathbb{T}^{(h)}$ such that if τ is a family tree, then $r_h\tau$ is the tree formed by all vertices of τ of height at most h. Let $\mathbb{T}_\infty$ be the set of all infinite family trees and $\mathbb{T}_h$ the set of family trees of height h.

A family tree $\tau \in \mathbb{T}^{(\infty)}$ is identified by the sequence $(r_h\tau, h \geq 1)$. Note that the $r_h\tau \in \mathbb{T}^{(h)}$ are subject only to the consistency condition that $r_h\tau = r_h(r_{h+1}\tau)$.

Theorem 3.2.1. *There is a bijection* $\phi\colon \mathbb{CT}\infty \to \mathbb{T}^{(\infty)}$ *such that*

- $\phi \circ r_h = r_h \circ \phi$*, that is,* ϕ *respects* r_h*,* $h = 1, 2, \ldots$ *;*
- *for* $t = 1, 2, \ldots, \infty$*, restrictions of* ϕ *to* $\mathbb{CT}_t$ *denoted by* $\phi_t\colon \mathbb{CT}_t \to \mathbb{T}_t$ *are also bijections that respect* r_h*,* $h = 1, 2, \ldots$ *.*

This theorem dates back to [32] and a detailed proof can be found in [34] (see also [33]). We refer to Figure 4.3 for an illustration of the proof.

The set $\mathbb{T}^{(\infty)}$ is now identified as a subset of an infinite product of countable sets

$$\mathbb{T}^{(\infty)} \subset \mathbb{T}^{(0)} \times \mathbb{T}^{(1)} \times \mathbb{T}^{(2)} \times \ldots$$

We give $\mathbb{T}^{(\infty)}$ the topology derived by this identification from the product of discrete topologies on $\mathbb{T}^{(h)}$. Therefore, a sequence of family trees τ_n has a limit

$$\lim \tau_n = \tau \in \mathbb{T}^{(\infty)}$$

iff for every h there exists $\tau^{(h)} \in \mathbb{T}^{(h)}$ and $n(h)$ such that $r_h\tau_n = \tau^{(h)}$ for all $n \geq n(h)$; the limit is then the unique $\tau \in \mathbb{T}^{(\infty)}$ with $r_h\tau = \tau^{(h)}$. In particular, for each $\tau \in \mathbb{T}^{(\infty)}$ the sequence $r_h\tau$ has limit τ as $n \to \infty$. The topology is metrizable, e.g., set $d(\tau, \tau') = k^{-1}$, where

$$k = \sup\{h : r_h\tau = r_h\tau'\}.$$

It is easy to see that the metric space is complete and separable.

The topology gives us the Borel σ-algebra to define probability measures on it. Besides, we can define the weak convergence of measures. As usual, a measure μ is the weak limit of the sequence of measures μ_n if

$$\int f d\mu_n \to \int f d\mu, \text{ as } n \to \infty$$

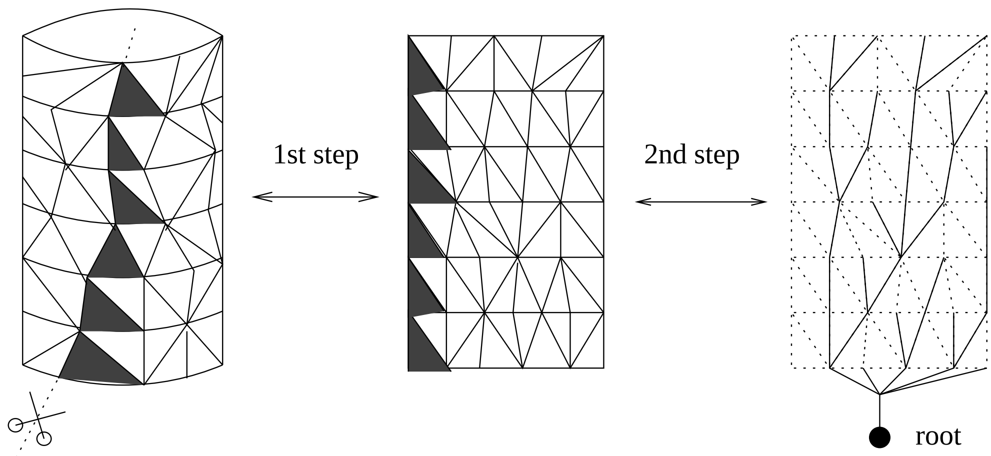

Figure 3.2: Tree parametrization. The following two steps outline how to go from a triangulation to a tree: *Step 1.* The first step is cutting. We construct the sequence of root vertices (or triangles) on each slice $S^1 \times \{i\}$ by the following rule. Let v_1 be vertex in the slice $S^1 \times \{1\}$ which belongs to the rooted triangle containing the root edge $[v_0, v_0']$. One chooses the right most neighbour v_1' of v_1 on the slice $S^1 \times \{1\}$ as the new rooted edge. Following this procedure we can cut open the triangulation along the left side of the root triangles. *Step 2.* We add one vertex in the slice below the initial boundary and connect all vertices on the inital boundary to this vertex. We then remove all spatial (horizontal edges) and each leftmost outgoing time-like edge of every vertex. The lowest vertex is then connected to the root. The resulting graph is a tree. The inverse relation should now be clear from the construction.

for every bounded continuous real-valued function f given on $\mathbb{T}^{(\infty)}$.

Let $\mathbb{T} = \cup_{h=0}^{\infty} \mathbb{T}^{(h)}$. Consider a system of nonnegative numbers

$$\pi = \{p(\tau), \tau \in \mathbb{T}\}$$

such that the following conditions hold:

1. for $h = 0, 1, 2, \ldots$ we have

$$\sum p(\tau_{h+1}) = p(\tau_h) \text{ for any } \tau_h \in \mathbb{T}^{(h)},$$

   where the sum is over $\tau_{h+1} \in \mathbb{T}^{(h+1)}$ such that $r_h \tau_{h+1} = \tau_h$;

2. $p(\tau_0) = 1$ for $\tau_0 \in \mathbb{T}^{(0)}$.

It is easy to see that if μ is a probability measure on $\mathbb{T}^{(\infty)}$, then

$$\pi = \left\{\mu\big(\{\tau \in \mathbb{T}^{(\infty)} : r_h\tau = \tau_h\}\big) : \tau_h \in \mathbb{T}^{(h)}, h = 0, 1, 2, \dots\right\}$$

is a system of numbers that satisfies the both conditions.

The following fact can easily be checked (it is proved in the same way as Kolmogorov extension theorem). It helps to define a measure on $\mathbb{T}^{(\infty)}$. The fact is that for every system of nonnegative numbers π satisfying the two conditions above there is a probability measure μ on $\mathbb{T}^{(\infty)}$ such that

$$p(\tau_h) = \mu\big(\{\tau \in \mathbb{T}^{(\infty)} : r_h\tau = \tau_h\}\big) \text{ for all } \tau_h \in \mathbb{T}^{(h)}, h = 0, 1, 2, \dots .$$

In other words, a random family tree is a random element of $\mathbb{T}^{(\infty)}$, formally specified by its sequence of restrictions, say $\mathcal{T} = (r_h\mathcal{T}, h = 0, 1, \dots)$, where each $r_h\mathcal{T}$ is a random variable with values in the countable set $\mathbb{T}^{(h)}$, and $r_h\mathcal{T} = r_h(r_{h+1}\mathcal{T})$ for all h. The distribution of $\mathcal{T}$ is determined by the sequence of distributions of $r_h\mathcal{T}$ for $h \geq 0$. Such a distribution is determined by a specification of the conditional distributions of $r_{h+1}\mathcal{T}$ given $r_h\mathcal{T}$ for $h \geq 0$. To give a more exact specification of the distribution of $\mathcal{T}$, we need some definitions.

For every $v \in \hat{\mathcal{U}}$, let $c_v\tau$ be the number of successors of v (if $v \notin \tau$, then $c_v\tau = 0$). For every $\tau \in \mathbb{T}^{(\infty)}$ and $g \geq 0$, let the *gth generation of individuals in τ*, denoted by $\mathrm{gen}(g, \tau)$, be the set of $u \in \tau$ such that the height of u is g, also let $Z_g\tau$ be the number of elements of the set $\mathrm{gen}(g, \tau)$ (to simplify notation let $Z = Z_0$). Note that $\mathbb{T}^{(0)}$ contains only one family tree that consists of only one element $\varnothing$, and for any $\tau \in \mathbb{T}^{(\infty)}$, we have $r_0\tau = \{\varnothing\}$. A family tree τ is conveniently specified as the unique $\tau \in \mathbb{T}^{(\infty)}$ such that $r_h\tau = \tau^{(h)}$ for all h for some sequence of trees $\tau^{(h)} \in \mathbb{T}^{(h)}$ determined recursively as follows. Given that $\tau^{(h)} \in \mathbb{T}^{(h)}$ has been defined ($\tau^{(0)}$ is the unique tree from $\mathbb{T}^{(h)}$), the set of vertices $\mathrm{gen}(h, \tau) = \mathrm{gen}(h, \tau^{(h)}) = r_h\tau \setminus r_{h-1}\tau$ is determined, hence so is the size $Z_h\tau = Z_h\tau^{(h)}$ of this set; for each possible choice of $Z_h\tau$ non-negative integers $\big(a_v, v \in \mathrm{gen}(h, \tau)\big)$, there is a unique $\tau^{(h+1)} \in \mathbb{T}^{(h+1)}$ such that $r_h\tau^{(h+1)} = \tau^{(h)}$ and $c_v\tau^{(h+1)} = a_v$ for all $v \in \mathrm{gen}(h, \tau)$. So a unique $\tau \in \mathbb{T}^{(\infty)}$ is determined by specifying for each $h \geq 0$ the way in which these $Z_h\tau$ non-negative integers are chosen given that $r_h\tau = \tau^{(h)}$ for some $\tau^{(h)} \in \mathbb{T}^{(h)}$.

Thus a more exact specification of the distribution of $\mathcal{T}$ is a specification of the joint conditional distribution given $r_h\tau$ of the numbers of children $c_v\tau$ as v ranges over $\mathrm{gen}(h, \tau)$ for $h \geq 0$.

Since the topology on $\mathbb{T}^{(\infty)}$ is a product of discrete topologies on $\mathbb{T}^{(h)}$, $h \geq 0$, the weak convergence of measures on $\mathbb{T}^{(\infty)}$ can be easily reformulated in the following way. For random family trees $\mathcal{T}_n$, $n = 1, 2, \dots$ and $\mathcal{T}$, we say that $\mathcal{T}_n$ converges in distribution to $\mathcal{T}$, and write $\mathrm{dist}(\mathcal{T}_n) \to \mathrm{dist}(\mathcal{T})$ if

$$\mathbb{P}(r_h\mathcal{T}_n = \tau) \to \mathbb{P}(r_h\mathcal{T} = \tau) \quad \forall h \geq 0, \tau \in \mathbb{T}.$$

3.3 | Uniform infinite causal triangulations and critical branching processes

Let $p(\cdot) = (p(0), p(1), \dots)$ be a probability distribution on the non-negative integers with $p(1) < 1$. Call a random family tree $\mathcal{G}$ a *Galton-Watson (GW) tree with offspring distribution* $p(\cdot)$ if the number of children $Z\mathcal{G}$ of the root has distribution $p(\cdot)$:

$$\mathbb{P}(Z\mathcal{G} = n) = p(n) \quad \forall n \geq 0$$

and for each $h = 1, 2, \dots$, conditionally given $r_h\mathcal{G} = t^{(h)}$, the numbers of children $c_v\mathcal{G}$, $v \in \mathrm{gen}(h, t^{(h)})$, are i.i.d. according to $p(\cdot)$.

Suppose that $\sum_{n=1}^{\infty} np(n) = 1$ and $\nu = \sum n^2 p(n) < \infty$. A random family tree $\mathcal{G}^\infty$, which we call $\mathcal{G}$ *conditioned on non-extinction* is derived from the family tree $\mathcal{G}$ in the way described in Theorem 3.3.1 below. The probabilistic description of $\mathcal{G}^\infty$ involves the *size-biased* distribution $p^*(\cdot)$ associated with probability distribution $p(\cdot)$:

$$p^*(n) = \mu^{-1} np(n) \quad \forall n \geq 0.$$

Putting together Proposition 2 and Proposition 5 from [37] (they correspond to reformulation in this family tree language of theorems from [73] and [36]), we have the following

Theorem 3.3.1. *The following statements are valid.*

1.

$$\mathrm{dist}(\mathcal{G}|\#\mathcal{G} = n) \to \mathrm{dist}(\mathcal{G}^\infty) \quad \textit{as } n \to \infty, \tag{3.1}$$

where $\mathrm{dist}(\mathcal{G}^\infty)$ *is the distribution of a random family tree* $\mathcal{G}^\infty$ *specified by*

$$\mathbb{P}(r_h\mathcal{G}^\infty = \tau) = Z_h\tau\mathbb{P}(r_h\mathcal{G} = \tau) \quad \forall \tau \in \mathbb{T}^{(h)}, h \geq 0. \tag{3.2}$$

2. *Almost surely $\mathcal{G}^\infty$ contains a unique infinite path $(V_0, V_1, V_2, \dots)$ such that $V_0 = \varnothing$ and V_{h+1} is a successor of V_h for every $h = 0, 1, 2, \dots$.*

3. *For each h the joint distribution of $r_h\mathcal{G}^\infty$ and V_h is given by*

$$\mathbb{P}(r_h\mathcal{G}^\infty = \tau, V_h = v) = \mathbb{P}(r_h\mathcal{G} = \tau) \quad \forall \tau \in \mathbb{T}^{(h)}, v \in \mathrm{gen}(h, \tau), h \geq 0. \tag{3.3}$$

4. *The joint distribution of $(V_0, V_1, V_2, \dots)$ and $\mathcal{G}^\infty$ is determined recursively as follows: for each $h \geq 0$, giver $(V_0, V_1, \dots, V_h)$ and $r_h\mathcal{G}^\infty$, the numbers of successors $c_v\mathcal{G}^\infty$ are independent as v ranges over $\mathrm{gen}(h, \mathcal{G}^\infty)$, with distribution $p(\cdot)$ for $v \neq V_h$, and with the size-biased distribution $p^*(\cdot)$ for $v = V_h$; given also the numbers of successors $c_v\mathcal{G}^\infty$ for $v \in \mathrm{gen}(h, \mathcal{G}^\infty)$, the vertex V_{h+1} has uniform distribution on the set of $c_{V_h}\mathcal{G}^\infty$ successors of V_h.*

Remark 3.3.1. *The UICT is a special case for which the critical branching process has offspring probability $p(n) = (1/2)^{n+1}$. In this case the conditional probability of the left-hand-side of* (3.1) *provides the same probability for any tree as well as CT with n vertices and thus defines the uniform measure on this set. The measure on the right-hand side of* (3.1) *determines the uniform measure on the set of infinite causal triangulation (UICT).*

Remark 3.3.2. *Another measure of interest is the Gibbs measure on the set of CTs. Its Hamiltonian H is simply the number of triangles multiplied by a coupling μ (the cosmological constant). There is a correspondence between the number of triangles and vertices in a tree: let $\tau^{(h)}$ be some finite tree of height h and $t^{(h)}$ its corresponding triangulation. The number of triangles in $t^{(h)}$ is equal to $1 + 2\sum_{k=1}^{h-1} Z_k\tau^{(h)} + Z_h\tau^{(h)} = H(t^{(h)})$. The probability of $t^{(h)}$ on the set of causal triangulations of the cylinder with height h is given by the Gibbs measure $P_h(t^{(h)}) = Z_h^{-1}e^{-\mu H(t^{(h)})}$, where Z_h^{-1} is the normalisation. Moreover it is not difficult to prove that for $\mu = \ln 2$ the measure P_h also converges to the UICT as $h \to \infty$.*

3.4 | Weak convergence from conditioned critical branching processes

Having established the relation between UICTs and critical Galton-Watson processes conditioned to never die out in the previous section, one can now use several known convergence results for the conditioned branching process to determine the corresponding convergence of several observables of the UICT.

From the point of view of universality one expects that the continuum processes shall be the same for any kind of underlying critical Galton-Watson process. We will see that this is indeed the case. Let us therefore consider an arbitrary critical Galton-Watson process $\mathcal{G}$ with generation function $f(s) = \sum_{n\geq 0} p(n)s^n$ of the offspring distribution $p(\cdot)$. Since the process is critical we have $f'(1) = 1$. Let us further assume that $\nu = f''(1)/2 < \infty$. For shorthand denote the size of the t'th generation by $\eta_t \equiv Z_t\mathcal{G}$. It was shown by Lindvall [82, 83] that if $\eta_0 = \nu tx + o(t)$ with $x > 0$:

$$\frac{\eta_{[t\tau]}}{\nu t} \Rightarrow X_\tau, \quad 0 \leq \tau < \infty,$$

where $\Rightarrow$ denotes weak convergence on the functions space $D[0,\infty)$ and the continuous process solves the following Itô's equation

$$dX_\tau = \sqrt{2X_\tau}dB_\tau, \quad X_0 = x,$$

with B_τ standard Brownian motion of variance 1.

Let us note that the finite-dimensional distributions of η_t can be easily obtained from the following relation due to Kesten, Ney and Spitzer for the generating function of the size of the t'th generation of a critical Galton-Watson process with $\nu = f''(1)/2 < \infty$ and $\eta_0 = 1$ (e.g. see [35])

$$\frac{1}{1-f_t(s)} = \frac{1}{1-s} + \nu t + o(t), \quad \text{uniformly for } 0 \leq s < 1.$$

Tightness can then be obtained by standard techniques (e.g. see [84]). An alternative detailed proof of Lindvall's theorem using convergence of the generator of the Markov process can be found in [85].

We now investigate the convergence of the length of the boundary of an infinite CT as a process of time. Since any Galton-Watson tree conditioned to never die out is in bijection with an infinite CT, we refer to the corresponding probability measure as an infinite CT constructed from a critical Galton-Watson process. The UICT is then a special case for which the critical branching process has offspring probability $p(n) = (1/2)^{n+1}$. In particular, this offspring probability satisfies $f^{(n)}(1) < \infty$ for all $n \in \mathbb{N}$.

By the relation discussed in the previous section, the size of the boundary k_t of an infinite CT constructed from a critical Galton-Watson process at time t corresponds to the size of the t'th generation of the Galton-Watson process conditioned to never die out, denoted by $\hat{\eta}_t \equiv Z_t\mathcal{G}^\infty$. Define the length process

$$k_\tau^{(t)} := \frac{k_{[t\tau]}}{\nu t} \equiv \frac{\hat{\eta}_{[t\tau]}}{\nu t}, \quad 0 \leq \tau < \infty, \tag{3.4}$$

The convergence of the finite-dimensional distributions of the process $\{\hat{\eta}_t\}$ was studied by Lamperti and Ney [80] and we can deduce the following theorem for the length process (3.4):

Theorem 3.4.1. *For an infinite CT constructed from a critical Galton-Watson process with $\nu = f''(1)/2 < \infty$ and $f'''(1) < \infty$, and initial boundary $m_0 \equiv k_0 = \nu t l + o(t)$ with $l \geq 0$ we have*

$$k_\tau^{(t)} \Rightarrow L_\tau, \quad 0 \leq \tau < \infty,$$

in the sense of weak convergence on the functions space $D[0,\infty)$, where the continuous process solves the following Itô's equation

$$dL_\tau = 2d\tau + \sqrt{2L_\tau}dB_\tau \quad L_0 = \ell.$$

The process L_τ is diffusive and the Feynman-Kac equation for $\phi_\xi(\ell,\tau) = \mathbb{E}[\exp(-\xi L_\tau)|L_0 = \ell]$ is given by

$$-\frac{\partial}{\partial\tau}\phi_\xi(\ell,\tau) = \hat{H}\phi_\xi(\ell,\tau), \quad \hat{H} = -2\frac{\partial}{\partial\ell} - \ell\frac{\partial^2}{\partial\ell^2}, \quad \phi_\xi(\ell,0) = e^{-\xi\ell}.$$

Here the operator $\hat{H}$ is known in the physics literature as the Hamiltonian of two-dimensional CDT (having cosmological constant equal zero, see e.g. [10]).

In [80], Theorem 1 convergence of the finite-dimensional distributions of the process $\{\hat{\eta}_t\}$ to those of the above diffusion process was shown. However, to prove convergence of the process one also has to prove tightness. The complete proof of Theorem 3.4.1 is presented in Appendix 3.A.

Corollary 3.4.1. *In the special case of $l = 0$, corresponding to an infinite CT constructed from a critical Galton-Watson process with zero initial boundary, we have*

$$\mathbb{E}\left[e^{-\xi L_\tau}\middle| L_0 = 0\right] = \frac{1}{(1+\xi\tau)^2}$$

which for $\tau = 1$ is random variable with gamma distribution with parameter two, i.e. $\mathbb{P}(\Gamma_n \in dx)/dx = x\,e^{-x}$, $x \geq 0$. (the sum of two independent random variables with exponential distribution with rate 1).

This gives us the distribution of the rescaled upper boundary L_1, i.e. of the random variable k_t/t in the limit $t \to \infty$. It is hence the analog of Theorem 4 of [17] which states the corresponding result for UIPT.

We now want to discuss the convergence of the rescaled area of a neighbourhood of the boundary Γ_t of height t. Let us denote the number of triangles in Γ_t by α_t. Define the area process

$$\alpha_\tau^{(t)} := \frac{\alpha_{[t\tau]}}{\nu t^2}, \quad 0 \leq \tau < \infty, \tag{3.5}$$

We then have the following theorem based on a theorem of Pakes for conditioned critical Galton-Watson processes [86]:

Theorem 3.4.2. *For an infinite CT constructed from a critical Galton-Watson process with $\nu = f''(1)/2 < \infty$ and $f'''(1) < \infty$, and initial boundary $m_0 \equiv k_0 = \nu t l + o(t)$ we have*

$$(k_\tau^{(t)}, \alpha_\tau^{(t)}) \Rightarrow (L_\tau, 2\int_0^\tau L_u du), \quad 0 \leq \tau < \infty,$$

in the sense of weak convergence on the functions space $D[0,\infty) \times D[0,\infty)$, where the continuous process L_τ solves the Itô's equation as in Theorem 3.4.1

$$dL_\tau = 2d\tau + \sqrt{2L_\tau}dB_\tau \quad L_0 = l.$$

The Feynman-Kac equation for $\phi_{\xi,\lambda}(l,\tau) = \mathbb{E}[\exp(-\xi L_\tau - 2\lambda \int_0^\tau L_u du)|L_0 = l]$ is given by

$$-\frac{\partial}{\partial \tau}\phi_{\xi,\lambda}(l,\tau) = \hat{H}\phi_{\xi,\lambda}(l,\tau), \quad \hat{H} = -2\frac{\partial}{\partial l} - l\frac{\partial^2}{\partial l^2} + 2\lambda l, \quad \phi_{\xi,\lambda}(l,0) = e^{-\xi l}.$$

Proof. By construction of the bijection between CTs and Galton-Watson trees we have $\alpha_t = k_0 + 2(k_1 + ... + k_{t-1}) + k_t$, i.e. each internal spatial (horizontal) edge is connected to two triangles while each boundary edge is connected to one triangle (see Figure 4.1). Hence

$$\alpha_\tau^{(t)} = \frac{\alpha_{[t\tau]}}{\nu t^2} = \frac{1}{\nu t^2}\left(2\sum_{i=0}^{[t\tau]} \hat{\eta}_i - \hat{\eta}_0 - \hat{\eta}_{[t\tau]}\right) = 2\int_0^\tau k_u^{(t)} du + o(1). \tag{3.6}$$

Following ideas of [86], Theorem 3.3, the weak convergence of $(k_\tau^{(t)}, \alpha_\tau^{(t)})$ then follows from the weak convergence of

$$(k_\tau^{(t)}, 2\int_0^\tau k_u^{(t)}) \Rightarrow (L_\tau, 2\int_0^\tau L_u du).$$

It is enough to note that by (3.6) we have that $h(k_\tau^{(t)}) := (k_\tau^{(t)}, \alpha_\tau^{(t)})$ is a continuous functional of $k_\tau^{(t)}$ and hence the convergence of $(k_\tau^{(t)}, \alpha_\tau^{(t)}) \Rightarrow (L_\tau, 2\int_0^\tau L_u du)$ follows by the continuous mapping theorem (Theorem 2.7, [84]) applied to Theorem 3.4.1.

Having established the convergence, we can then apply the Feynman-Kac formula to

$$\phi_{\xi,\lambda}(l,\tau) = \mathbb{E}\left[e^{-\xi L_\tau - 2\lambda\int_0^\tau L_u du}\middle| L_0 = l\right]$$

with

$$dL_\tau = 2dt + \sqrt{2L_\tau}dB_\tau \quad L_0 = l.$$

which yields

$$-\frac{\partial}{\partial\tau}\phi_{\xi,\lambda}(l,\tau) = \left(-2\frac{\partial}{\partial l} - l\frac{\partial^2}{\partial l^2} + 2\lambda l\right)\phi_{\xi,\lambda}(l,\tau), \quad \phi_{\xi,\lambda}(l,0) = e^{-\xi l}.$$

□

The last equation is again known from the physics literature in the context of CDT with cosmological constant λ. In fact, one can easily solve the differential equation leading to

$$\phi_{\xi,\lambda}(l,\tau) = \frac{\bar{\xi}^2(\xi,\tau) - 2\lambda}{\xi^2 - 2\lambda}e^{-l\bar{\xi}(\xi,\tau)} \tag{3.7}$$

with

$$\bar{\xi}(\xi,\tau) = \sqrt{2\lambda}\coth(\sqrt{2\lambda}\tau) - \frac{2\lambda}{\sinh^2(\sqrt{2\lambda}\tau)\left[\xi + \sqrt{2\lambda}\coth(\sqrt{2\lambda}\tau)\right]}$$

Corollary 3.4.2. *Setting $\tau = 1$ and $l = 0$ in (3.7) one has*

$$\phi_{\xi,\lambda}(0,1) = \frac{2\lambda}{(\sqrt{2\lambda}\cosh\sqrt{2\lambda} + \xi\sinh\sqrt{2\lambda})^2}$$

which also follows from [86], Theorem 3.3. In particular, for $\lambda = 0$ one recovers

$$\mathbb{E}\left[e^{-\xi L_1}\middle| L_0 = 0\right] = \frac{1}{(1+\xi)^2}$$

as in Corollary 3.4.1 and for $\xi = 0$

$$\mathbb{E}\left[e^{-\lambda A_1}\middle| L_0 = 0\right] = \frac{1}{\cosh^2(\sqrt{2\lambda})} \tag{3.8}$$

with $A_1 = 2\int_0^1 L_u du$.

This gives the distribution of the random variable A_1, i.e. α_t/t^2 in the limit $t \to \infty$. The distribution of A_1 appears at several places related to the study of Brownian motion as has been exposed for example in [87]. Based on the discussion in [87] we can make two remarks:

Remark 3.4.1. *The random variable A_1, as introduced in Remark 3.4.2, can be written in the following series representation*

$$A_1 = \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\Gamma_n}{(n-1/2)^2},$$

where the Γ_n, $n \geq 1$ are i.i.d. random variables with gamma distribution with parameter two, i.e. $\mathbb{P}(\Gamma_n \in dx)/dx = x\, e^{-x}$, $x \geq 0$. The relation can easily be seen by noting that

$$\mathbb{E}(e^{-\lambda \Gamma_n}) = \frac{1}{(1+\lambda)^2} \quad \textit{and} \quad \cosh z = \prod_{n \geq 1} \left(1 + \frac{z^2}{\pi^2 (n-1/2)^2}\right). \tag{3.9}$$

Remark 3.4.2. *In the framework of Lévy-Khintchine representations a distribution is called infinitely divisible iff its Laplace transform $\phi(\lambda)$ admits the following representation*

$$\phi(\lambda) = \exp\left(-c\lambda - \int_0^\infty (1 - e^{-\lambda x}) \nu(dx)\right).$$

for some $c \geq 0$. Here $v(dx)$ is the so-called Lévy measure and for the present application it is sufficient to consider the form of a simple density $v(dx) = \rho(x)dx$. By a straightforward and explicit computation using (3.8) *and* (3.9) *one sees that the distribution of A_1 is infinitely divisible and has a Lévy-Khintchine representations with $c = 0$ and Lévy density*

$$\rho(x) = \frac{2}{x} \sum_{n \geq 1} e^{-\pi^2 (n-1/2)^2 x/2}.$$

3.5 | Discussion

We discussed infinite causal triangulations and the existence of the uniform measure on those, so-called UICT, in an alternative presentation to [34]. One observes that under this measure the probability of a causal triangulation of a cylinder is related to a critical Galton-Watson process conditioned to never die out. We used this relation to prove weak convergence of the joint rescaled length-area process $(k_\tau^{(t)}, \alpha_\tau^{(t)})$ of an infinite CT constructed from an arbitrary critical Galton-Watson process to a limiting diffusion process (L_τ, A_τ), with $A_\tau = 2\int_0^\tau L_u du$, where the Itô's equation for L_τ is given by (e.g. Theorem 3.4.1 and 3.4.2)

$$dL_\tau = 2d\tau + \sqrt{2L_\tau} dB_\tau, \quad L_0 = l.$$

In particular, we show that the Feynman-Kac formula for $\mathbb{E}[\exp(-\xi L_\tau - \lambda A_\tau)|L_0 = l]$ corresponds to a imaginary time Schrödinger equation with the following Hamiltonian

$$\hat{H}(l, \partial_l) = -2\frac{\partial}{\partial l} - l\frac{\partial^2}{\partial l^2} + 2\lambda l.$$

This is the well-known Hamiltonian for two-dimensional CDT with cosmological constant λ (see [10]).[1]

By calculating the inverse Laplace transform of (3.7) one can also obtain the transition amplitude or Green's function

$$\begin{aligned} \phi_\lambda(l_1, l_2, \tau) &= \mathbb{E}\left[\mathbb{I}\{L_\tau \in dl_2\} \cdot e^{-2\lambda \int_0^\tau L_u du} \middle| L_0 = l_1\right] / dl_2 \\ &= \frac{\sqrt{\Lambda} l_2}{\sqrt{l_1 l_2}} \frac{e^{-\sqrt{\Lambda}(l_1+l_2)\coth(\sqrt{\Lambda}\tau)}}{\sinh(\sqrt{\Lambda}\tau)} I_1\left(\frac{2\sqrt{\lambda l_1 l_2}}{\sinh(\sqrt{\Lambda}\tau)}\right) \end{aligned}$$

where $\mathbb{I}\{\cdot\}$ is the indicator function and $I_1(\cdot)$ is the modified Bessel function of first kind. This expression is also known in physics as the CDT propagator. In particular, setting $\lambda = 0$ one obtains the transition amplitude for the length process

$$\phi_0(l_1, l_2, \tau) = \frac{l_2}{\tau\sqrt{l_1 l_2}} e^{-\frac{l_1+l_2}{\tau}} I_1\left(\frac{2\sqrt{l_1 l_2}}{\tau}\right)$$

In conclusion, Theorem 3.4.1 and 3.4.2 provide us with a mathematically rigorous proof of certain scaling limits of two-dimensional causal dynamical triangulations (CDT). In ongoing work, we further show how to obtain these results in a slightly different manner from a certain growth process of UICT. While in this chapter we gave a mathematically rigorous derivation for several correlations functions of CDT from the UICT it would be interesting to obtain the full scaling limit using a framework like in Le Gall's and Mierment's work on the Brownian map in the context of DT [21].

[1] In fact, it is the Hamiltonian acting on an non-rooted boundary. This is due to the fact that by the construction of the Feynman-Kac or Kolmogorov backwards equation we are acting on the upper, non-rooted boundary. Alternatively, one could have also used the Kolmogorov forward equation to obtain the Hamiltonian acting on the rooted, lower boundary.

We hope the discussion in the chapter helps mathematicians and physicists to connect the mathematicians' work on branching processes with the physicists' work on quantum gravity.

3.A | Proof of Theorem 3.4.1

Define $\nu = f''(1)/2 < \infty$ as before and $\mu = f'''(1)/2 < \infty$. Recall that we want to show convergence of

$$k_\tau^{(t)} = \frac{\hat{\eta}_{[t\tau]}}{\nu t} \Rightarrow L_\tau, \quad 0 \leq \tau < \infty, \tag{3.10}$$

on the functions space $D[0,\infty)$. To do so we consider the rescaled process

$$\tilde{k}_\tau^{(t)} = \nu k_\tau^{(t)} \Rightarrow \tilde{L}_\tau = \nu L_\tau \tag{3.11}$$

where then $\tilde{L}_\tau$ is a diffusion process with generator

$$Ag(x) = 2\nu g'(x) + \nu x g''(x), \tag{3.12}$$

where by Theorem 2.1 of Chapter 8 of [85] one has $g \in C_c^\infty([0,\infty))$, i.e. the set of continouse functions $f : [0,\infty) \to \mathbb{R}$ which are infinitely differentiable and have compact support in $[0,\infty)$. To show convergence of the process $\tilde{k}_\tau^{(t)}$ to the diffusion $\tilde{L}_\tau$ with the above generator we follow closely the strategy employed in Theorem 1.3 of Chapter 9 in [85] to prove Lindvall's theorem.

Note that $\hat{\eta}_n/t$ is a Markov chain taking values in $E_t = \{l/t | l = 1,2,3,...\}$. Given $\hat{\eta}_n = tx$ we can then write

$$\hat{\eta}_{n+1} = \sum_{k=1}^{tx-1} \xi_k + \xi_0, \tag{3.13}$$

where ξ_k for $k \geq 0$ are iid random variables with generating function $f(s)$ and ξ_0 is a random variable with generating functions $sf'(s)$. Recall that $f(s)$ is the generating function of the offspring probabilities. The above statement follows directly from Theorem 3.3.1. Indeed, by Theorem 3.3.1 we have

$$\sum_{k\geq 0} \mathbb{P}(\hat{\eta}_{n+1} = k | \hat{\eta}_n = tx) s^k = \frac{1}{tx} s \frac{d}{ds} f^{tx}(s) = f^{tx-1}(s) \cdot sf'(s) \tag{3.14}$$

which is the generating function for (3.13). We have

$$\begin{aligned} \mathbb{E}\xi_k &= 1, \quad \mathbb{E}\xi_k^2 = 1 + 2\nu, \quad \text{for } k \geq 1 \\ \mathbb{E}\xi_0 &= 1 + 2\nu, \quad \mathbb{E}\xi_0^2 = 1 + 6\nu + 2\mu \end{aligned} \tag{3.15}$$

We now define

$$T_t g(x) = \mathbb{E}\left\{ g\left(\frac{1}{t}\left[\sum_{k=1}^{tx-1}\xi_k + \xi_0\right]\right)\right\}. \tag{3.16}$$

By Theorem 6.5 of Chapter 1 and Corollary 8.9 of Chapter 4 of [85], to prove the convergence (3.11) it is enough to show that

$$\lim_{t\to\infty}\sup_{x\in E_t}|t(T_t g(x)-g(x)) - 2\nu g'(x) - \nu x g''(x)| = 0, \quad g\in C_c^\infty([0,\infty)). \tag{3.17}$$

For $x \in E_t$ we define

$$\begin{aligned}
\varepsilon_t(x) &= t(T_t g(x)-g(x)) - 2\nu g'(x) - \nu x g''(x) \\
&= \mathbb{E}\left\{ tg\left(\frac{1}{t}\left[\sum_{k=1}^{tx-1}\xi_k + \xi_0\right]\right) - tg(x) - (\xi_0-1)g'(x) + \right. \\
&\quad \left. -\frac{1}{2}g''(x)\frac{1}{t}\left[\left(\sum_{k=1}^{tx-1}(\xi_k-1)\right)^2 + \xi_0 - 1\right]\right\} \\
&= \Delta_t^1(x) + \Delta_t^2(x)
\end{aligned} \tag{3.18}$$

where

$$\Delta_t^1(x) = \frac{1}{t}\mu g''(x) \tag{3.19}$$

$$\Delta_t^2(x) = \mathbb{E}\left\{\int_0^1 S_{tx}^2 x(1-u)\left[g''(x+u\sqrt{\frac{x}{t}}S_{tx}) - g''(x)\right]du\right\} \tag{3.20}$$

and

$$S_{tx} = \frac{1}{\sqrt{tx}}\left(\sum_{k=1}^{tx-1}(\xi_k-1) + (\xi_0-1)\right). \tag{3.21}$$

Now, since $\mu = f'''(1)/2 < \infty$ and we have

$$\lim_{t\to\infty}\sup_{x\in E_t}|\Delta_t^1(x)| \le \mu\|g''\|\lim_{t\to\infty}\frac{1}{t} = 0. \tag{3.22}$$

Let us suppose that g has support in $[0,c]$. Then, since $\xi_k \ge 0$ for $k \ge 1$ and $\xi_0 \ge 1$ we have

$$x + u\sqrt{\frac{x}{t}}S_{tx} \ge x(1-u). \tag{3.23}$$

hence one gets that

$$\begin{aligned} &\left| \int_0^1 S_{tx}^2 x(1-u) \left[g''(x + u\sqrt{\frac{x}{t}} S_{tx}) - g''(x) \right] du \right| \\ \leq \;& 2xS_{tx}^2 \int_{0 \vee (1-c/x)}^1 (1-u)\|g''\| du = x\|g''\|((c/x) \wedge 1)^2 S_{tx}^2 \end{aligned} \tag{3.24}$$

To show that $\lim_{t\to\infty} \sup_{x\in E_t} |\Delta_t^2(x)| = 0$ it suffices to show that one has $\lim_{t\to\infty} |\Delta_t^2(x_t)| = 0$ for any convergent series x_t, as well as for $x_t \to 0$ and $x_t \to \infty$. Let us first treat the special cases $\lim_{t\to\infty} x_t = 0$ and $\lim_{t\to\infty} x_t = \infty$. Note that

$$\mathbb{E}S_{tx}^2 = 2\nu + \frac{2\mu}{tx} \leq 2(\nu + \mu), \quad \text{for all } t \text{ and } x \in E_t. \tag{3.25}$$

From (3.24) and (3.25) it then follows that $\lim_{t\to\infty} |\Delta_t^2(x_t)| = 0$ if $\lim_{t\to\infty} x_t = 0$ or $\lim_{t\to\infty} x_t = \infty$.

We now consider the case $\lim_{t\to\infty} x_t = x$, where $0 < x < \infty$. In this case one has

$$\lim_{t\to\infty} \mathbb{E}e^{rS_{tx_t}} = e^{\nu r^2} \tag{3.26}$$

and hence $S_{tx_t} \Rightarrow \Sigma$ with $\Sigma \sim \mathcal{N}(0, 2\nu)$. Following the steps of Theorem 1.3 in [85] Chapter 9, one then obtains $\lim_{t\to\infty} |\Delta_t^2(x_t)| = 0$ from (3.24) and the dominant convergence theorem.

Hence we showed that

$$\lim_{t\to\infty} \sup_{x\in E_t} |\varepsilon_t(x)| = 0. \tag{3.27}$$

Noting that the initial condition converges $\tilde{k}_0^t \to \nu l$ with $l \geq 0$ one completes the proof. $\square$

CHAPTER 4

A growth process for uniform infinite causal triangulations

We introduce a growth process which samples sections of uniform infinite causal triangulations by elementary moves in which a single triangle is added. A relation to a random walk on the integer half line is shown. This relation is used to estimate the geodesic distance of a given triangle to the rooted boundary in terms of the time of the growth process and to determine from this the fractal dimension. Furthermore, convergence of the boundary process to a diffusion process is shown leading to an interesting duality relation between the growth process and a corresponding branching process.

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4.1 | Introduction

In the field of quantum gravity, models of random geometry and sometimes called quantum geometry have been studied intensively in the search of a non-perturbative definition of the gravitational path integral (see [9] for an overview). In two dimensions one distinguishes between models of Euclidean quantum gravity, so-called dynamical triangulations (DT) [9] and Lorentzian quantum gravity, so-called causal dynamical triangulations (CDT) ([10], and [11] for an overview of recent progress in two dimensions).

In the area of probability theory, the uniform measure on infinite planar triangulations (UIPT) [14] has been introduced as a mathematically rigorous model of DT,[1] while recently uniform infinite causal triangulations (UICT) [34, 88] have been employed as a mathematically rigorous model of CDT. In particular, the formulation of the UICT measure is based on a bijection to planar rooted trees as was first formulated in [33] and independently later in [34]. Both formulations are based on a similar bijection for the dual graphs of CDT which was introduced in [32]. The work in [34] shows convergence of the uniform measure on causal triangulations in the limit where the number of triangles goes to infinity and proves that the fractal dimension is two almost surely (a.s.) as well as that the spectral dimension is bounded above by two a.s. In [88] further convergence properties of the UICT measure are proven, in particular, using the relation to a size-biased critical Galton-Watson process, the convergence of the joint boundary length-area process to a diffusion process is shown from which one can extract the quantum Hamiltonian through the standard Feynman-Kac procedure. In a different work [52], the existence of a phase transition of the quenched Ising model coupled to UICT is shown. All the above mentioned articles rely on the bijection to trees and the relation to branching processes. In this chapter we give an alternative formulation of UICT through a growth process.

In [15] Angel studied a growth process which samples sections of UIPT. This growth process is a mathematically rigorous formulation of the so-called peeling procedure for DT, as introduced by Watabiki in the physics literature [53], where it can also be understood as a time-dependent version of the so-called loop equation, a combinatorial equation derived from random matrix models of DT [9].

In the context of CDT a similar peeling procedure as for DT can be formulated as was shown recently [54]. Furthermore, one can relate it to a random matrix model which itself can be understood as a new continuum limit of the standard matrix model for DT [89, 90].

In this chapter we introduce a growth process which samples sections of UICT by elementary moves in which a single triangle is added. This growth process is based

[1] Lately also much progress has been made in understanding the scaling limit of DT as the Brownian map (see [21] for an overview).

on the peeling procedure of CDT [54] and analogous to the corresponding growth process for UIPT [15]. The growth process is related to a Markov chain $\{M_n\}_{n\geq 0}$ with state space $\mathbb{N} = \{1, 2, 3, ...\}$ which describes the evolution of the boundary length M_n of the triangulation as a function of "growth time" n. Using this relation it is shown how to estimate the stoping times n_t at which the growth process finishes a strip of a fixed geodesic distance t to the rooted boundary. We use this to prove that the fractal dimension is almost surely two, in an alternative manner to the derivation using branching processes as was done in [34]. Further, we prove convergence of the Markov chain $\{M_n\}_{n\geq 0.}$ to a diffusion process. It is then shown how to relate this diffusion process using a random time change to another diffusion process describing the evolution of the generation size of a critical Galton-Watson conditioned on non-extinction. This provides us with an interesting duality picture with the growth process on the one side and the branching process on the other side.

4.2 | A growth process for uniform infinite causal triangulations

4.2.1 | Definitions

We consider rooted causal triangulations of a cylinder $\mathcal{C} = S^1 \times [1, \infty)$, where S^1 is the unit circle.

Consider a connected graph G with a countable number of vertices embedded in $\mathcal{C}$. Suppose that all its faces are triangles (using the convention that an edge incident to the same face on both sides counts twice, see [88] for more details). A triangulation T of $\mathcal{C}$ is the pair of the embedded graph G and the set F of all the faces: $T = (G, F)$.

Definition 4.2.1. *A triangulation T of $\mathcal{C}$ is called an almost causal triangulation (ACT) if the following conditions hold:*

- *each triangular face of T belongs to some strip $S^1 \times [j, j+1], j = 1, 2, ...$, and has all vertices on the boundary $(S^1 \times \{j\}) \cup (S^1 \times \{j+1\})$ of the strip $S^1 \times [j, j+1]$;*
- *let $k_j = k_j(T)$ be the number of edges on $S^1 \times \{j\}$, then we have $0 < k_j < \infty$ for all $j = 1, 2, ...$.*

Definition 4.2.2. *A triangulation T of $\mathcal{C}$ is called a causal triangulation (CT) if it is an almost causal (ACT) and any triangle has exactly one edge on the boundary of the strip to which it belongs.*

Example 4.2.1. *The first two slices from the bottom of the triangulation of Figure 4.4 form a CT while the third strip is an example of an ACT.*

Definition 4.2.3. *A triangulation T of $\mathcal{C}$ is called rooted if it has a root. The root in the triangulation T consists of a triangular face t of T, called the root triangle, with an ordering on its vertices (x, y, z). The vertex x is the root vertex and the directed edge (x, y) is the root edge. The root vertex and the root edge belong to $S^1 \times \{1\}$.*

Definition 4.2.4. *Two almost causal or two causal rooted triangulations of $\mathcal{C}$, say $T = (G, F)$ and $T' = (G', F')$, are equivalent if there exists a self-homeomorphism of $\mathcal{C}$ such that it transforms each slice $S^1 \times \{j\}, j = 1, \dots, M$ to itself preserving its direction, it induces an isomorphism of the graphs G and G' and a bijection between F and F', also the root of T goes to the root of T'.*

We usually abbreviate "equivalence class of embedded rooted (almost) causal triangulations" by "(almost) causal triangulations". In the same way we can define an (almost) causal triangulations of a cylinder $C_t = S^1 \times [1, t]$, where $t = 2, 3, \dots$.

4.2.2 | Uniform infinite causal triangulations

Denote by $\mathcal{T}(N, m_0, m)$ the set of finite causal triangulation with N triangles with a rooted boundary of length m_0 and second boundary of length m. Let $\mathcal{T}(N, m_0)$ be the set of finite causal triangulations with N triangles, with the length of the rooted boundary equal m_0 and the length of the other boundary not fixed, i.e.

$$\mathcal{T}(N, m_0) = \cup_{m=1}^{\infty} \mathcal{T}(N, m_0, m).$$

Let $C(N, m_0) = \#\mathcal{T}(N, m_0)$ be the number of triangulations of a cylinder with N triangles and m_0 boundary edges of the rooted boundary. Define the uniform distribution on the (finite) set $\mathcal{T}(N, m_0)$ by

$$\mathbb{P}_{N,m_0}(T) = \frac{1}{C(N, m_0)}. \tag{4.1}$$

One can now define the limiting measure as the uniform measure on infinite causal triangulations, as done in [34, Theorem 2] which is based on the generic random tree measure [39, Theorem 2] (see also [88, 37]):

Theorem 4.2.1. *There exists the measure π, called the uniform infinite causal triangulation (UICT) measure, on the set of causal triangulations of the cylinder $\mathcal{C}$ such that*

$$\mathbb{P}_{N,m_0} \to \pi_{m_o}, \quad N \to \infty$$

as a weak limit.

There is an interesting relation between UICT and critical Galton-Watson family trees due to a bijection of causal triangulations and rooted planar trees (or forests) (see [33] and [34]) which is illustrated in Figure 4.1. To get from the rooted causal triangulation to the rooted planar forest we remove all horizontal edges from the triangulation, furthermore at each vertex we remove the leftmost up-pointing edge. The result is a planar rooted forest where we chose the sequence of roots to be the sequence of vertices on the initial boundary starting with the root vertex x of the root triangle (x, y, z) and following the boundary in an anti-clockwise direction. Connecting the root vertices of the forests to a single external vertex one obtains a planar rooted tree. The inverse relation should now be clear. A detailed and slightly different formulation of the bijection can be found in [33, 34].

Consider the critical Galton-Watson branching process with one particle type and off-spring distribution $p_k = (1/2)^{k+1},\ k \geq 0$. Let η_j be the number of particles at generation j. One easily sees that η_j is a recurrent Markov chain with transition probability

$$\mathbb{P}_{GW}(\eta_{j+1} = k \mid \eta_j = l) = \frac{1}{2^{k+l}}\binom{k+l-1}{k}.$$

The following lemma clarifies the connection between the Galton-Watson branching process and the UICT (see for instance [39, Lemma 4]):

Lemma 4.2.1. *Recall the definition of k_j in Definition 5.2.1. We have*

$$\pi_{m_0}(k_j = m) = \frac{m}{m_0}\mathbb{P}_{GW}(\eta_j = m \mid \eta_0 = m_0). \tag{4.2}$$

Note that the RHS of the previous expression can also be interpreted as a critical Galton-Watson process conditioned on non-extinction, i.e. $\pi_{m_0}(k_j = m) = \lim_{N\to\infty} \mathbb{P}_{GW}(\eta_j = m \mid \eta_0 = m_0, \eta_N > 0)$.

4.2.3 | Growth process

Here we define the process with discrete time which constructs (samples) a UICT by adding one triangle at each step.

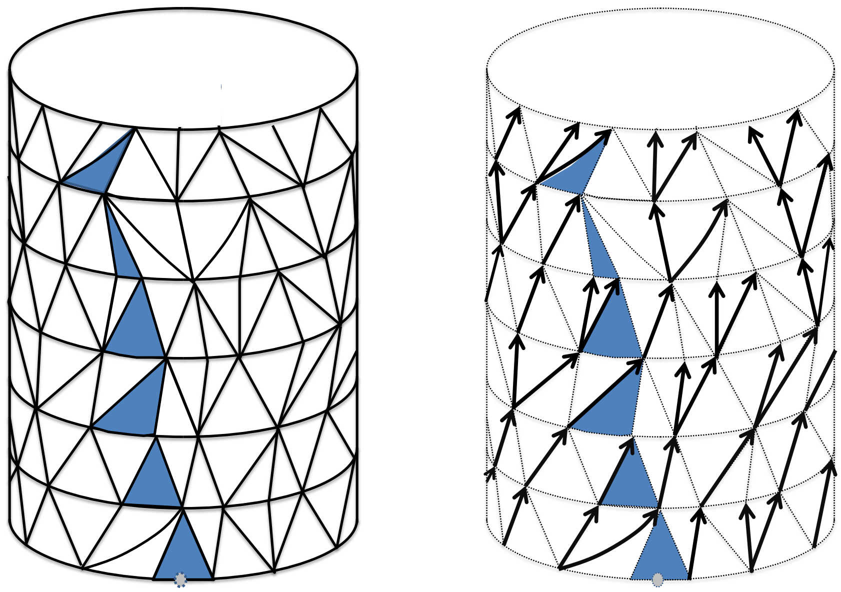
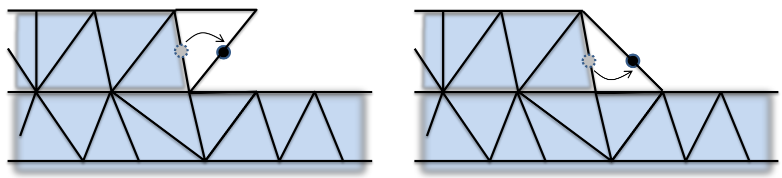


Figure 4.1: Bijection between rooted causal triangulations and (rooted) forests.

Let $\{x_1, \dots, x_m\}$ be the set of boundary vertices for some triangulation of the disc with m boundary edges. We label the vertices following the direction on the boundary where (x_m, x_1) is the marked edge.

In any step, we will add a triangle to the marked edge and after that we put the new mark on another edge. One allows this to be done in two different ways. In particular, one can add a triangle (x_m, y, x_1) to the marked edge (x_m, x_1) where y is either a new vertex, we call this the $(+)$-move, or $y = x_2$, where x_2 is the next vertex after x_1 following the direction on the boundary; we call this the $(-)$-move. If y is a new vertex, then the next marked edge will be (y, x_1). In the case $y = x_2$ the new marked edge is (x_m, x_2). If the boundary consists of only one edge (x_1, x_1), then in the next step one can only add a triangle (x_1, y, x_1) with the marked edge (y, x_1). Note that the marked edge belongs to the boundary at each step of the growth process.

We will consider the following special starting triangulation with $m+1$ vertices ($m \geq 1$): a triangulation of the disc with m edges on the boundary (a m-gon), having m triangles and one vertex in the interior of the disc which is a common vertex of all m triangles. This vertex we call the 0-root or 0 (in contrast with the root triangle). Let



(a) The $(+)$-move (b) The $(-)$-move

Figure 4.2: The two different moves of the growth process.

us denote this triangulation as $\mathcal{S}_m$. Note that any move preserves the topology of the triangulation as a disc. Denote by T_n the triangulation of the disc after n moves and let $l(T_n)$ be the length of the boundary of the triangulation T_n. Further, let $e_n = (v_n^1, v_n^2)$ be the marked edge of T_n with vertices $v_n^1, v_n^2 \in T_n$.

We now assign probabilities to the growth process: Conditioning on the length of the boundary of the triangulation T_n, we can add another triangle to it by choosing one of the above two moves randomly according to the probabilities

$$\mathbb{P}((\pm) - \text{move} \mid l(T_n) = m) = \frac{1}{2}\frac{m \pm 1}{m}. \tag{4.3}$$

Denote by $\mathcal{T}_m(n)$ the set of all possible triangulations of the disc obtained by applying all possible (permitted) sequences of length n of the $(+)$ and $(-)$ moves starting with $\mathcal{S}_m$. Given the transition probabilities (4.3) one has that T_n is a Markov chain with state space $\cup_{n \geq 0} \mathcal{T}_m(n)$.

Note that in any move one adds one triangle to the triangulation and changes the length of the boundary of the triangulation by one: the $(+)$-move increases the boundary by one, while the $(-)$-move decreases the boundary by one. This process of growing the triangulation is basically the time reversal of the so-called "peeling" process, which is related to so-called loop equations for matrix models in the physics literature (see [9] in the context of DT and [54, 89, 90, 91] in the context of CDT).

The process T_n determines a process which describes the evolution of the length of the boundary of the triangulation T_n. Denote the length $M_n = l(T_n)$. Define $\xi_n = M_{n+1} - M_n$. The probabilities (4.3) one can rewrite as

$$\mathbb{P}(\xi_n = \pm 1 \mid M_n = m) = \frac{1}{2}\frac{m \pm 1}{m}. \tag{4.4}$$

It is clear that M_n is a Markov chain with state space $\mathbb{N} = \{1, 2, 3, \dots\}$ and transition probabilities (4.4).

We now describe the relationship between the process T_n and the process M_n. If we know the sequence of $\{M_n\}_{n=0,1,\dots,s}$ then we know the sequence of $(\pm)$-moves and consequently we know the triangulation T_s. Inversely, if we fix the triangulation T_s from $\mathcal{T}_m(s)$ we can reconstruct the sequence $\{M_n\}_{n=0,1,\dots,s}$. Hence, one has:

Remark 4.2.1. *For any $s \in \mathbb{N}$, there is a one-to-one correspondence g between $\mathcal{T}_m(s)$ and the set of sequences $\{M_n\}_{n=0,1,\dots,s}$, with $M_0 = m$, fulfilling*

$$\mathbb{P}(T_s = T \mid T_0 = \mathcal{S}_m) = \mathbb{P}(g(T) \mid M_0 = m).$$

Due to this relation we also call the Markov chain $\{M_n\}_{n=0,1,\dots,s}$ the growth process.

We will now make the link to almost causal triangulations. For any $s \in \mathbb{N}$ and any triangulation T_s from the set $\mathcal{T}_m(s)$ there exists a number $h(T_s)$ (to be defined as the "height" of the triangulation) such that the set of all vertices of the triangulation $V(T_s)$ can be divided into the disjoint sets corresponding to the distances between the vertices and the 0-root: $V(T_s) = \cup_{i=0}^{h(T_s)} V_i$, where $V_i = V_i(T_s)$ is the set of vertices of T_s which have distance to the 0-root equal to i, where V_0 contains only the 0-root vertex. Thus $h(T_s)$ is the maximal distance to the 0-root.

Definition 4.2.5. *For the growth process T_n let us define the following moments n_t, $t = 1, 2, \dots$.*

$$n_t := \min\{s > 0 : \; dist(v_s^1, 0-root) = dist(v_s^2, 0-root) = t\}$$

where we recall that $(v_s^1, v_s^2) = e_s$ are the vertices adjacent to the root edge.

Denote by $\hat{T}_n$ the triangulation T_n without the 0-root and the edges attached to it, then we have:

Theorem 4.2.2. *$\hat{T}_{n_t}$ is an almost causal triangulation of C_t.*

This means that between the moments n_t and n_{t-1} the growth process T_t constructs an almost causal triangulation of the strip $S^1 \times [t-1, t]$.

Proof. The proof follows directly from the detailed description of the growth process: It is obvious that $\hat{T}_{n_t}$ is a triangulation of the cylinder C_t, because it is a triangulation of the disc without the faces of an initial m-gon $\mathcal{S}_m$. Removing $\mathbb{S}_m = \mathcal{S}_m \setminus \partial\mathcal{S}_m$ from the disc adds a hole in the disc and makes $\hat{T}_{n_t}$ homeomorphic to the cylinder.

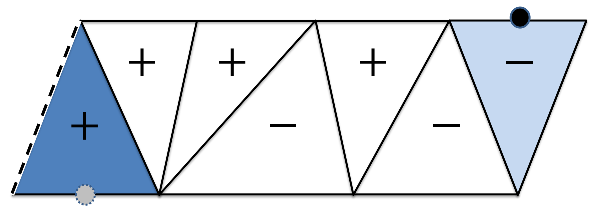


Figure 4.3: Construction of a slice of an almost causal triangulation using the following sequence of moves: $+, +, +, -, +, -, -$. The left hand side and the right hand side of the strip are periodically identified.

The theorem states that there exists a homeomorphism f of the disc with a hole with the embedded graph $\hat{T}_{n_t}$ into the cylinder which maps the set V_i, $i = 1, \dots, t$ into the slice $S^1 \times \{i\}$ of the cylinder C_t such that any triangle will belong to some strip $S^1 \times [i, i+1]$. For that, firstly, we prove that for any i, $i = 1, \dots, t$, there exists a Hamilton path (circle) consisting of all vertices V_i: suppose $|V_i| = l_i$, ordering the vertices of $V_i = \{x_1, x_2, \dots, x_{l_i}\}$ in order of their appearance we will show that there exists the circle $(x_j, x_{j+1}), j = 1, \dots, l_i - 1$ and (x_{l_i}, x_1) in $\hat{T}_{n_t}$. Secondly, we will show that the homeomorphism f that maps the vertices V_i with its Hamilton path into $S^1 \times \{i\}$ maps any triangular face into one strip.

In the following we describe the three phases of the construction of a triangulation of a strip $S^1 \times [i-1, i]$: starting, filling and finishing off the strip.

(i) *Starting a strip.* We start from l_{i-1} edges

$$E_{i-1} = \{(x_1, x_2), \dots, (x_{l_{i-1}-1}, x_{l_{i-1}}), (x_{l_{i-1}}, x_1)\}$$

and l_{i-1} vertices $x_1, x_2, \dots, x_{l_{i-1}} \in S^1 \times \{i-1\}$ with distance $i-1$ from the 0-root: $V_{i-1} = \{x_1, x_2, \dots, x_{l_{i-1}}\}$. Let $e = (x_{l_{i-1}}, x_1)$ be the marked edge. The first phase continues until the first $(+)$-move.

Suppose $l_{i-1} > 1$. If the first move is a $(+)$-move, then the process adds a new triangle $(x_{l_{i-1}}, y_1, x_1)$ and $y_1 \in S^1 \times \{i\}$ has distance i to the 0-root ($V_i = \{y_1\}$). The new marked edge $e = (y_1, x_1)$ connects two vertices with different distances to the 0-root and we continue to the next phase. If, on the other hand, the first move is a $(-)$-move, it adds a new edge $(x_{l_{i-1}}, x_2)$ which will be the new marked edge. One observes that all points $x_1, \dots, x_{l_i}$ have the same distance to the 0-root and the $(-)$-move does not change their distances. Moreover, a sequence of k $(-)$-moves, $k < l_{i-1}$, maintains the set V_i, and the next $(+)$-move starts the next level. Thus starting with k $(-)$-moves $(k < l_{i-1})$ before the first $(+)$-move one obtains the following boundary of the triangulation $(x_{l_{i-1}}, y_1, x_{k+1}, \dots, x_{l_{i-1}-1})$, with $y_1 \in S^1 \times \{i\}$. Note that the case $x_{k+1} = x_{l_{i-1}}$ is allowed. In this particular case, after $l_{i-1} - 1$ $(-)$-moves the length of the current boundary of the triangulation is equal 1, and the next step has to be a $(+)$-move.

In the case $l_{i-1} = 1$, $V_{i-1} = \{x_1\}$, and $e = (x_1, x_1)$ the boundary of the triangulation is equal to 1, and the first step can be only a $(+)$-move.

(ii) *Filling a strip*. During this phase we have a set of vertices $V_i = V_i(T_n) = \{y_1, \dots, y_k\} \neq \emptyset$ with distance $i, i \geq 2$ to the 0-root in the graph T_n and $h(T_n) = i$. Starting with $V_i = \{y_1\}$, let $E^0_{i-1} \subseteq E_{i-1}$ be the set of edges which belong to the boundary of the triangulation (see Figure 4.4). The set E^0_{i-1} decreases by one element with any $(-)$-move and once it becomes empty this phase stops and we proceed to (*iii*). Furthermore, any $(+)$-move adds a new vertex to the set V_i connected by an edge to the previous vertex from V_i.

(iii) *Finishing off a strip*. This phase continues until the first $(-)$-move, which finishes a triangulation of the strip. Suppose that we have l_i vertices in the set V_i before the $(-)$-move. The marked edge connects y_{l_i} (the last vertex in V_i) and $x_{l_{i-1}}$ (the last vertex in V_{i-1}): $e = (y_{l_i}, x_{l_{i-1}})$. The following edge on the boundary is $(x_{l_{i-1}}, y_1)$, thus the $(-)$-move will connect the vertices y_{l_i} and y_1. Note also that this is the first moment when the next marked edge $e = (y_{l_i}, y_1)$ will connect two points with the same distance i to the 0-root, i.e. it defines the moment n_i as given in Definition 4.2.5.

From the description given above it is clear that any set $V_i = \{v_1, \dots, v_{l_i}\}$ of T_n with $n \geq n_i$ has a circle connecting a sequence of vertices from V_i: the edges $(v_i, v_{i+1}), i = 1, \dots, l_i - 1$ are created by $(+)$-moves and (v_{l_i}, v_1) is created by a $(-)$-move defining the moment n_i. Denote this circle graph by G_i. Moreover, any $(+)$-move during the filling stage will create a “down” triangle of which exactly one edge will connect vertices with distance i to the 0-root and two edges will connect these two vertices to one vertex from V_{i-1}; further any $(-)$-move creates an “up” triangle consisting of two vertices from V_{i-1} and one vertex from V_i.

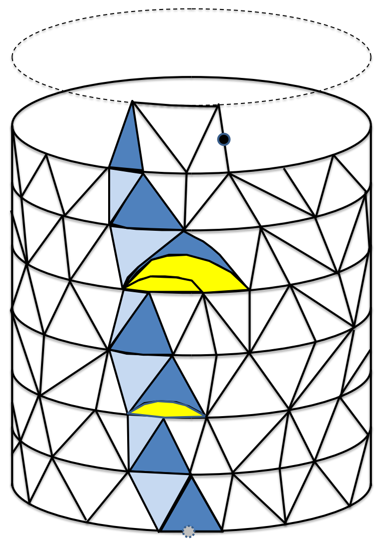


Figure 4.4: An example of a triangulation created by the growth process, where the first six slices are an example of an almost causal triangulation.

Thus the description of the construction of a strip provides the existence of a homeomorphism f of the disc without $\mathbb{S}_m$ into C_t such that the image of the G_i in the disc maps into the circle $S^1 \times \{i\}$ of C_t for any $i = 1, \dots, t$. Moreover any such homeomorphism maps the triangular faces of T_n created between the time n_{i-1} and n_i into the strip $S^1 \times [i-1, i]$ for $i = 2, \dots, t$. $\square$

Remark 4.2.2. *From the proof of the proceeding theorem it follows directly that $n_t < \infty$ $\mathbb{P}$-a.s.*

Example 4.2.2. *Figure 4.3 shows an example of a sequence of moves of size $n = 7$: Starting with $\mathcal{S}_3$, i.e. $l(T_0) = M_0 = 3$ and creating a strip with final boundary of length $l(T_7) = M_7 = 4$. Here the last move completes the first strip of the triangulation, hence, $n_1 = 0$ and $n_2 = 7$. We observe that the result is a causal triangulation (with a specific, so-called staircase boundary condition). However, as one can observe in Figure 4.4, if one starts a new strip with a $(-)$-move one can create certain outgrowths. Therefore the name almost causal triangulations.*

In this section we have presented a growth process which samples almost causal triangulations by adding one triangle at a time using two different moves with probabilities (4.3).

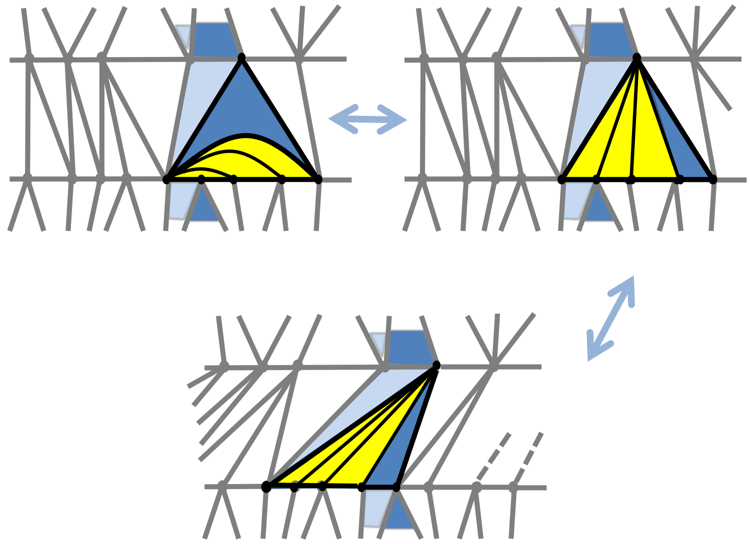


Figure 4.5: Removing the “defects”: Bijection between the set of almost causal triangulations created by the growth process and causal triangulations.

The “defects”, where the triangulation generated by the growth process differs from a causal triangulation, can only occur at the moments where one starts a new strip and in particular if the process starts a strip by a sequence of $(-)$-moves. The defects can never occur during the filling and finishing of a strip (see Figure 4.4). For example starting the strip with a sequence of k $(-)$-moves and then a $(+)$-move we obtain a configuration like in the left-up-side picture in Figure 4.5. In fact, one can transform any such almost causal triangulation to a causal triangulation.

Lemma 4.2.2. *There is a one-to-one map between $\mathcal{C}^g_{m_0}(t)$, the set of all possible triangulations T_{n_t} formed by the subset of almost causal triangulations created by the growth process started from $\mathcal{S}_{m_0}$ and stopped at n_t, i.e. where all vertices of the boundary are at distance t to the 0-root, and the set $\mathcal{C}^c_{m_0}(t)$ of rooted causal triangulations of height t with initial boundary of length m_0.*

Proof. We defined $\mathcal{C}^g_{m_0}(t)$ to be the set of all possible rooted triangulations of the disc with all vertices of the boundary at distance t to the 0-root obtained by applying permitted sequences of the $(+)$ and $(-)$ moves starting from $\mathcal{S}_{m_0}$. Note that the set $\mathcal{C}^g_{m_0}(t)$ is only a subset of the set of all possible almost causal triangulations allowed by its definition. We denoted $\mathcal{C}^c_{m_0}(t)$ to be the set of all rooted causal triangulation of the cylinder

C_t, with m vertices on the zero-slice. Lemma 4.2.2 then states that there exists an one-to-one correspondence between $\mathcal{C}^g_{m_0}(t)$ and $\mathcal{C}^c_{m_0}(t)$. To prove the Lemma we give an explicit construction of this correspondence:

The construction is divided in two steps as illustrated in Figure 4.5. The first step is to "correct" the sequence of moves that creates "defects" in the triangulations. Suppose that we have defects in the i-th strip $S^1 \times [i, i+1]$. The move that describes the transformation of such an almost causal triangulation into a causal triangulation is presented in the upper line of Figure 4.5, where the sequence of moves

$$\underbrace{(-), \dots, (-)}_{l}(+)$$

is substituted by the sequence of moves

$$(+)\underbrace{(-), \dots, (-)}_{l}$$

or in terms of the M_n process:

$$\{\xi_{n_i+1} = -1, \dots, \xi_{n_i+l} = -1, \xi_{n_i+l+1} = +1\} \to \\ \{\xi_{n_i+1} = +1, \xi_{n_t+2} = -1, \dots, \xi_{n_i+l+1} = -1\}.$$

One can further verify that

$$\mathbb{P}(\xi_{n_i+1} = -1, \dots, \xi_{n_i+l} = -1, \xi_{n_i+l+1} = +1 | M_{n_i} = m) = \\ \mathbb{P}(\xi_{n_i+1} = +1, \xi_{n_i+2} = -1, \dots, \xi_{n_i+l+1} = -1 | M_{n_i} = m). \tag{4.5}$$

The second step is a l-shift of the i-th slice of the strip $S \times \{i\}$. Suppose $v^i_1, v^i_2, \dots, v^i_m$ is the sequence of vertices in i-th slice and $v^{i+1}_1, v^{i+1}_2, \dots, v^{i+1}_n$ on the $(i+1)$-th slice of the strip. After the first step the triangulation on $S \times \{i\}$ is a causal triangulation. The l-th shift is defined by the following transformation of the strip: any vertex v^i_j in the i-th slice is shifted to the vertex v^i_{j-l}, i.e. any "up" triangle $(v^i_j, v^i_{j+1}, v^{i+1}_k)$ becomes a $(v^i_{j-l}, v^i_{j+1-l}, v^{i+1}_k)$ triangle and any "down" triangle $(v^i_{j'}, v^{i+1}_{k'}, v^{i+1}_{k'+1})$ becomes a $(v^i_{j'-l}, v^{i+1}_{k'}, v^{i+1}_{k'+1})$ triangle, where the sum in indices is the cyclic sum modulo m.

Note that the first step gives us already a causal triangulation. However, the marked edges can be anywhere in the slice. The second step deals with this problem. After the second step all dark blue triangles are connected to each other as marked triangles on the causal triangulations.

The inverse transformation is clear now. Consider the marked triangle of a causal triangulation in a strip $S \times [i, i+1]$. If the left-neighbor triangle is a "down" triangle we do not change anything in this strip. Otherwise we find the first left "down" triangle Δ, thus all "up" triangles between the marked triangle and Δ are yellow triangles in the Figure 5. After that the transformation is obvious. This shows that there is an one-to-one correspondence between $\mathcal{C}^g_{m_0}(t)$ and $\mathcal{C}^c_{m_0}(t)$ which completes the proof. □

We now have the following theorem:

Theorem 4.2.3. *The growth process M_n with $M_0 = m_0$ stopped at times n_t samples a section $T \in \mathcal{C}^c_{m_0}(t)$ of a UICT of height t with initial boundary of length m_0, where $\mathcal{C}^c_{m_0}(t)$ is the image of $\mathcal{C}^g_{m_0}(t)$ under the transformation described in the proof of Lemma 4.2.2.*

Proof. By the bijection of Lemma 4.2.2, the growth process M_n with $M_0 = m_0$ stopped at times n_t creates every possible causal triangulation of height t with initial boundary of length m. Further, by (4.5) removing the defects does not affect the probabilities. One has that the probability of each vertex at height i having k down triangles in $[i, i+1]$ attached to it is $p_k = 1/2^{k+1}$. One can also show that the probability to create a new slice with boundary length $m + k$ given that the preceding boundary is of length m, is equal to the corresponding probability for the UICT. Indeed, we calculate the probability $\mathbb{P}(M_{n_{i+1}} = m + k \mid M_{n_i} = m)$ in Lemma 4.4.1 later, yielding

$$P(M_{n_{i+1}} = m + k \mid M_{n_i} = m) = \frac{m+k}{m} \frac{1}{2^{2m+k}} \binom{2m+k-1}{m-1}.$$

This completes the proof. □

Remark 4.2.3. *One observes that in*

$$\mathbb{P}((\pm) - \textit{move} \mid l(T_n) = m) = \frac{1}{2} \frac{m \pm 1}{m}.$$

the factor $1/2$ directly relates to the off-spring distribution $p_k = 2^{-k-1}$, while the pre-factor, $(m \pm 1)/m$ results in the conditioning of the branching process on non-extinction.

4.3 | Growth rate and fractal dimension

In this section we are interested in determining the growth rate of the process $\{M_n\}_{n \geq 0}$ and from this the fractal dimension d_h of the (infinite) causal triangulation generated by this process.

Recall the definition of the moments $n_t, t = 1, 2, ...$ in Definition 4.2.5. By Remark 4.2.1 we can also obtain the moments n_t from the Markov chain $\{M_n\}_{n\geq 0}$.

Lemma 4.3.1. *Let $n_1 = 0$. Suppose that n_{t-1} is defined, then by Definition 4.2.5 one has*

$$n_t = \min\{s : s - n_{t-1} = M_s + M_{n_{t-1}}\}, \tag{4.6}$$

$$n_t = \min\Big\{s : s > n_{t-1} \text{ and } \sum_{k=n_{t-1}}^{s-1} \mathrm{I}\{\xi_k = -1\} = M_{n_{t-1}}\Big\}. \tag{4.7}$$

Proof. The proof is a direct consequence of Definition 4.2.5 of the stopping times $\{n_t\}_{t=1,2,...}$. Let us suppose that $M_{n_{t-1}} = m$. Hence, to complete the next slice, we need to put m "up" triangles, considering that only the first "up" triangle we add using the $(+)$-move, while the remaining "up" triangles are added using $(-)$-moves. Note also that we complete the slice with the last $(-)$-move which adds a "down" triangle. Thus, to fill the slice we need exactly m $(-)$-moves, and some (random) number of $(+)$-moves. This is provided exactly by (4.7): n_t is the growth time of the m-th $(-)$-move after n_{t-1}. This shows that Definition 4.2.5 implies (4.7).

We now show that (4.6) is equivalent to (4.7). Equation (4.6) reflects the fact that the moment n_t is the hitting time of the straight line $M_s = (s - n_{t-1}) - M_{n_t}$ (the line bdf in Figure 4.6). To hit this line the process needs exactly m steps with $\xi = -1$. This proves the equivalence of the two equations. □

Let us consider a causal triangulation generated by the growth process $\{M_n\}_{n\geq 0}$ with initial boundary of length $M_0 = 1$ using the probabilities (4.4). Let $\Gamma(t)$ denote the set of triangles of the corresponding triangulation with all vertices having graph distance less or equal than t from the initial boundary. The fractal dimension d_h describes the growth of $|\Gamma(t)| \sim t^{d_h}$ as $t \to \infty$. Observing that at instance n_t of the growth process we have $|\Gamma(t)| = n_t$, we define the fractal dimension as the limit (if it exists)

$$d_h = \lim_{t\to\infty} \frac{\log n_t}{\log t}.$$

We now want to prove that the fractal dimension of an (infinite) causal triangulation generated by the growth process $\{M_n\}_{n\geq 0}$ with probabilities (4.4) is almost surely 2. To do so we first prove the following slightly stronger statement:

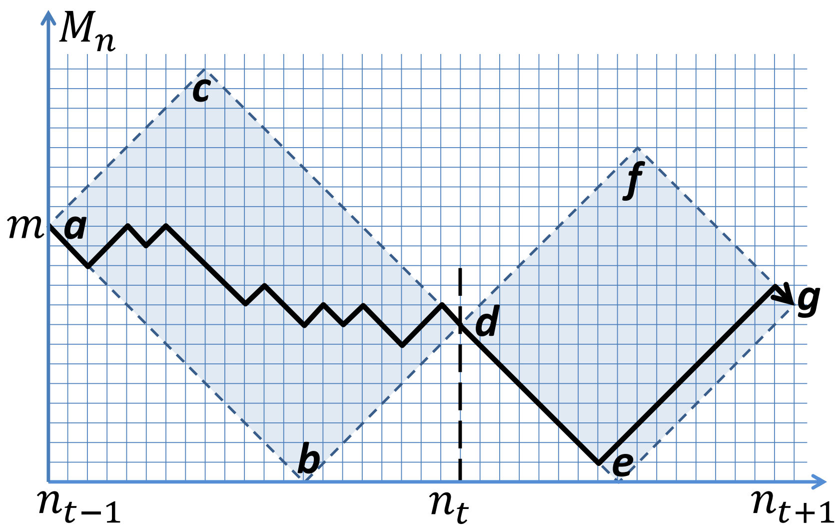


Figure 4.6: Illustration of a path of the Markov chain $\{M\}_{n=0,1,2,...}$.

Proposition 4.3.1. *For almost every trajectory ω of the growth process $\{M_n\}_{n\geq 0}$ with probabilities* (4.4) *and with initial boundary of length $M_0 = 1$, there exist two constants $\gamma_1 = \gamma_1(\omega) > 0$ and $\gamma_2 = \gamma_2(\omega) > 0$ such that*

$$\gamma_1 \frac{t^2}{\log^2 t} \leq n_t \leq \gamma_2 t^2 \log^2 t \tag{4.8}$$

Note that in the proposition the constants γ_1 and γ_2 depend on the whole trajectory ω of the process, but not on t. Using the definition of the fractal dimension we then have the following Theorem:

Theorem 4.3.1. *An infinite causal triangulation generated by the growth process $\{M_n\}_{n\geq 0}$ with probabilities* (4.4) *with initial boundary of length $M_0 = 1$ has fractal dimension $d_h = 2$ almost surely.*

Proof. The proof follows immediately from the previous Proposition 4.3.1. □

This Theorem is analogous to a result by Durhuus, Jonsson and Wheater (Theorem 3 in [34]) which is derived for UICT using the bijection to critical Galton-Watson processes conditioned to never die out. In this construction, subsequent generations in the branching process correspond to vertices of subsequent slices of fixed minimal graph distance, i.e. geodesic distance, from the initial boundary. This is in contrast to the construction through the growth process where the triangulation is grown triangle by triangle. One can think of both constructions as being dual to each other in the sense that in the branching process picture geodesic distance is fixed and area growth is estimated while in the growth process area is fixed and geodesic distance is estimated. We will comment further on this duality in Section 4.5.

To prove Proposition 4.3.1 we need three lemmas of which the first two are proven in Appendix 4.A.

Lemma 4.3.2. *We have*

$$\frac{M_n}{\sqrt{n \log n}} \to 0 \quad a.s.$$

Lemma 4.3.3. *We have*

$$\frac{M_n - \sum_{i=0}^{n} \frac{1}{M_i}}{\sqrt{n} \log n} \to 0 \quad a.s.$$

Lemma 4.3.4. *For any t, we have*

$$\max_{n_t < i \leq n_{t+1}} M_i \leq n_{t+1} - n_t.$$

Proof of Lemma 4.3.4. The proof follows directly from the Figure 4.6. For any given M_{n_t}, $M_{n_{t+1}}$, n_t and n_{t+1} the trajectory of the growth process belongs to the rectangle $abcd$ in Figure 4.6. This means that the maximal accessible point is c. Its y-coordinate is equal to $M_{n_t} + M_{n_{t+1}}$. Thus

$$\max_{n_t < i \leq n_{t+1}} M_i \leq M_{n_t} + M_{n_{t+1}},$$

for any trajectory of the growth process. By (4.6) of Lemma 4.3.1, the RHS is equal to $n_{t+1} - n_t$. This proves the Lemma. □

Proof of Proposition 4.3.1. First let us prove the RHS of (4.8). From Lemma 4.3.2 it follows that for almost every trajectory ω of the growth process there exists a constant $C_1 = C_1(\omega) > 0$ (in the following we will omit ω from our notations) such that, for any n,

$$M_n \leq C_1 \sqrt{n \log n} \tag{4.9}$$

Using (4.6) and (4.9), we get

$$\begin{aligned} n_t &= n_t - n_0 = \sum_{k=0}^{t-1}(n_{k+1} - n_k) = \sum_{k=0}^{t-1}(M_{n_{k+1}} + M_{n_k}) \\ &\leq 2\sum_{k=1}^{t} M_{n_k} \leq 2C_1 \sum_{k=1}^{t} \sqrt{n_k \log n_k} \leq 2C_1 t\sqrt{n_t \log n_t}. \end{aligned} \tag{4.10}$$

From this and using the inequality $\log n < \sqrt{n}$ we also have

$$4C_1^2 t^2 \geq \frac{n_t}{\log n_t} \geq \frac{n_t}{\sqrt{n_t}} = \sqrt{n_t}$$

and thus,

$$n_t \leq 16 C_1^4 t^4. \tag{4.11}$$

From (4.10) and (4.11) it follows that there exists a constant $\gamma > 0$ such that

$$n_t \leq 4C_1^2 t^2 \log n_t \leq \gamma t^2 \log t,$$

where we use (4.10) for the first inequality and (4.11) for the second one. Note also that the constant γ depends on the whole trajectory of the process, but does not depend on t. Thus, the RHS of (4.8) is proved.

Now let us prove the LHS of (4.8). On the one hand, using Lemma 4.3.4, one has

$$\begin{aligned} \sum_{i=n_1}^{n_t} \frac{1}{M_i} &\geq \sum_{k=1}^{t-1} \sum_{i=n_k+1}^{n_{k+1}} \frac{1}{M_i} \\ &\geq \sum_{k=1}^{t-1} \frac{n_{k+1} - n_k}{\max_{n_k < i \leq n_{k+1}} M_i} \geq \sum_{k=1}^{t-1} \frac{n_{k+1} - n_k}{n_{k+1} - n_k} \geq t. \end{aligned} \tag{4.12}$$

On the other hand, using Lemma 4.3.3 it follows that there exists a $C_2 = C_2(\omega) > 0$ such that

$$\sum_{i=n_1}^{n_t} \frac{1}{M_i} - M_{n_t} \leq C_2 \sqrt{n_t} \log n_t,$$

and from (4.9) it follows that there exists a $C_3 > 0$ such that

$$\sum_{i=n_1}^{n_t} \frac{1}{M_i} \leq C_3 \sqrt{n_t} \log n_t. \tag{4.13}$$

Combining (4.12) and (4.13), we get

$$t \leq C_3 \sqrt{n_t} \log n_t. \tag{4.14}$$

Finally, lifting the square in (4.14) and using (4.11), we get the LHS of (4.8). □

Remark 4.3.1. *From the proof of Proposition 4.3.1 we observe that*

$$t \leq \sum_{i=1}^{n_t} \frac{1}{M_i}.$$

In fact, as we will see in the next section, the properly rescaled inverse of n_t, i.e. $t_n := \min\{s : n_s \geq n\}$ and $\sum_{i=1}^{n} 1/M_i$ converge to the same limiting process $T_u = \int_0^u \frac{1}{2M_s} ds$.

4.4 | Weak convergence results

Consider the rescaled process

$$M_u^{(n)} = \frac{M_{[un]}}{\sqrt{n}}, \quad 0 \leq u < \infty.$$

Theorem 4.4.1. *For an infinite causal triangulation generated by the growth process $\{M_n\}_{n\geq 1}$ with initial boundary of length $M_0/\sqrt{n} \to l \geq 0$ we have*

$$M_u^{(n)} \Rightarrow M_u, \quad 0 \leq u < \infty,$$

as $n \to \infty$ in the sense of weak convergence on the functions space $D[0,\infty)$, where the continuous process is diffusive and solves the following Itô's equation

$$dM_u = \frac{1}{M_u} du + dB_u, \quad M_0 = l,$$

with B_u being standard Brownian motion of variance one.

Proof. Consider the process $M_u^{(n)} = M_{[un]}/\sqrt{n}$ as defined above. Using the notation of Appendix 4.B with $h = 1/n$, we now set $Y_{m/n}^{(n)} = M_m/\sqrt{n}$, where the state space is $\mathcal{X}_{1/n} = \{k/\sqrt{n}, 0 < k \leq n\}$. From the transition probability of the Markov chain M_n as discussed in Section 4.2 one has

$$p^{1/n}(x, x+1/\sqrt{n}) = \frac{1}{2}\frac{x+1/\sqrt{n}}{x}, \quad p^{1/n}(x, x-1/\sqrt{n}) = \frac{1}{2}\frac{x-1/\sqrt{n}}{x},$$

for any $x \in \mathcal{X}_{1/n}$. We now want to apply Theorem 4.B.1. It is easy to first check condition (3). From the transition probabilities one immediately sees that for $\epsilon > 1/\sqrt{n}$, $\Delta_{1/n}^{\epsilon}(x) = 0$ for all x and thus condition (3) holds. Further,

$$\begin{aligned} b_{1/n}(x) &= \frac{n}{2}\left(\frac{1}{\sqrt{n}}\frac{x+1/\sqrt{n}}{x} - \frac{1}{\sqrt{n}}\frac{x-1/\sqrt{n}}{x}\right) = 1/x \\ \sigma_{1/n}^2(x) &= \frac{n}{2}\left(\frac{1}{n}\frac{x+1/\sqrt{n}}{x} + \frac{1}{n}\frac{x-1/\sqrt{n}}{x}\right) = 1 \end{aligned}$$

This shows that the conditions (1) and (2) hold for $b(x) = 1/x$ and $\sigma^2(x) = 1$. Further, one observes that $MP(1/x, 1)$ is a well-posed martingale problem (see Definition 4.B.1). Hence by Theorem 4.B.1, setting $X_t^{(n)} = Y_{[nt]/n}^{(n)} = M_{[tn]}/\sqrt{n}$ one obtains the desired result. $\square$

We have shown convergence of the rescaled boundary of our growth process to a limiting diffusion. Here, the diffusion time u is the time associated to the growth process. In the following we would like to relate this to the corresponding diffusion process, where the diffusion time is geodesic distance. Consider therefore the rescaled process

$$L_s^{(t)} = \frac{M_{n_{[st]}}}{t}, \quad 0 \le s < \infty,$$

where n_t is defined as above. We then have the following theorem:

Theorem 4.4.2. *For an infinite causal triangulation generated by the growth process $\{M_n\}_{n\ge1}$ with initial boundary of length $M_1 \to l \ge 0$ we have*

$$L_s^{(t)} = \frac{M_{n_{[st]}}}{t} \Rightarrow L_s, \quad 0 \le s < \infty,$$

as $t \to \infty$ in the sense of weak convergence on the functions space $D[0,\infty)$, where the continuous process is diffusive and solves the following Itô's equation

$$dL_s = 2ds + \sqrt{2L_s}dB_s \quad L_0 = l,$$

with B_s being standard Brownian motion of variance one.

The proof is based on the following lemma:

Lemma 4.4.1. *We have*

$$\mathbb{P}\left(M_{n_{t+1}} = m+k | M_{n_t} = m\right) = \frac{m+k}{m}\frac{1}{2^{2m+k}}\binom{2m+k-1}{m-1},$$

where $k = -m+1, -m+2, \ldots$.

Proof. By Lemma 4.3.1 we have that $n_{t+1} - n_t = 2m+k$. Recall that any trajectory of the growth process needs a "-1" step (i.e. $(-)$-move) to finish a slice, i.e. in Figure 4.6 the trajectory hits the point d "from above". For given $M_{n_{t+1}} = m+k$ and $M_{n_t} = m$ we have $\binom{2m+k-1}{m-1}$ possible trajectories of the growth process. The last observation is that any such trajectory, starting at the point m and finishing at $m+k$ has the same probability $\frac{m+k}{m}\frac{1}{2^{2m+k}}$ (including the trajectories with reflection). This completes the proof. $\square$

Proof of Theorem 4.4.2. Using Lemma 4.4.1 the transition probability from M_{n_t} to $M_{n_{t+1}}$ is exactly equal the corresponding expression for the UICT, i.e. (4.2). The proof then follows directly from Theorem 4.1 of [88]. □

Theorem 4.4.3. *The diffusion process M_u, given in Theorem 4.4.1, and the diffusion process L_s, given in Theorem 4.4.2, are related by a random time change $L_s = M_{T_s^{-1}}$ with*

$$T_s^{-1} = \inf\{t : T_t > s\}$$

and

$$T_u = \int_0^u \frac{1}{2M_s} ds.$$

Proof. The proof is a consequence of Theorem 4.B.2, a well-known theorem from stochastic calculus, presented in Appendix 4.B with $g(x) = 1/(2x)$. □

4.5 | Discussion

In this chapter we present a growth process which samples sections of uniform infinite causal triangulations (UICT). In particular, a triangulation is grown by adding a single triangle according to two different moves, denoted "$(+)$" and "$(-)$", with probability

$$\mathbb{P}((\pm) - \text{move} \mid l(T_n) = m) = \frac{1}{2}\frac{m \pm 1}{m},$$

where $l(T_n)$ denotes the length of the boundary of the triangulation at step n of the growth process. The $(\pm)$-moves are illustrated in Figure 5.4, the $(+)$-move increases the boundary length by one while the $(-)$-move decreases it by one. This growth process can equivalently be described by a recurrent Markov chain $\{M_n\}_{n\geq 0}$ for the boundary length of the triangulation $M_n = l(T_n)$ as noted in Remark 4.2.1.

It is shown in Theorem 4.2.2 that the growth process constructs so-called almost causal triangulations which are causal triangulations with certain defects as shown in Figure 4.4. Defining the stoping times n_t when the growth process completes the strip $S^1 \times [t, t+1]$ at "height" t, it is shown in Theorem 4.2.3 that there is a bijection between the almost causal triangulations created by the growth process and (regular) causal triangulations which furthermore preserves the probability of the corresponding triangulation. Hence, the growth process $\{M_n\}_{n\geq 0}$ with $M_0 = m_0$ when stopped at n_t indeed samples sections of UICT of height t with initial boundary equal to m_0 and final boundary arbitrary.

	boundary length	area	geodesic distance
Growth process	M_u	$0 \leq u < \infty$	$T_u = \int_0^u \frac{1}{2M_v} dv$
Branching process	L_s	$A_s = \int_0^s 2L_v dv$	$0 \leq s < \infty$

Table 4.1: Duality of growth process and branching process.

Using the growth process, as described above, we show in Proposition 4.3.1 that for almost every trajectory of the growth process one can find two constants $\gamma_1, \gamma_2 > 0$ such that $\gamma_1\, t^2/(\log^2 t) \leq n_t \leq \gamma_2 t^2 \log^2 t$. This implies that the fractal dimension is given by $d_h = 2$ almost surely as stated in Theorem 4.3.1. This derivation is dual to previous results in [34] which employs the relation to branching processes: In the branching process geodesic distance is fixed while area growth is estimated, whereas in the growth process area (which is equal to growth time) is fixed and geodesic distance is estimated.

In Theorem 4.4.1 we discuss convergence of the rescaled Markov chain $\{M_n\}_{n\geq 0}$ to a diffusion process given by the following Itô's equation

$$dM_u = \frac{1}{M_u} du + dB_u.$$

It is then shown in Theorem 4.4.2 and 4.4.3 that by a random time change $L_s = M_{T_s^{-1}}$ one obtains a diffusion process with Itô's equation

$$dL_s = 2ds + \sqrt{2L_s} dB_s,$$

which describes the behaviour of the boundary length of completed slices and precisely agrees with the corresponding results obtained from the branching process picture [88]. While Theorem 4.4.1 follows rather straightforwardly from the properties of the Markov chain, Theorem 4.4.2 together with Theorem 4.4.3 result in a physically interesting duality relation which is illustrated in Table 4.1: In the growth process growth time u which is equal to area is fixed and geodesic distance $T_u = \int_0^u \frac{1}{2M_v} dv$ is random, whereas, in the branching process picture geodesic distance s is fixed while area $A_s = \int_0^s 2L_v dv$ is random. This duality relation also clarifies the so-called peeling procedure as introduced by Watabiki [53] in the context of Euclidean quantum gravity and derived in [54] for CDT.

As a continuation of the presented work it would be interesting to extend the work of Angel [14] and investigate the convergence of the boundary length process coming from the growth process of DT to a Lévy process. Furthermore, one could extend the here developed techniques to multi-critical DT [53, 92] as well as to a recently introduced model of multi-critical CDT [93, 94]. The convergence of the boundary length process should hopefully shed light on the failure of the peeling procedure in the context of multi-critical DT [53, 92].

4.A | Proofs of basic Lemmas

4.A.1 | Proof of Lemma 4.3.2

Let $\mathcal{F}_n = \sigma(M_0, M_1, \dots, M_n)$. Recall that $\xi_n = M_{n+1} - M_n$. We thus have $M_n \in \mathcal{F}_n$ and $\xi_n \in \mathcal{F}_{n+1}$. On $\{M_n \geq 1\}$ one has

$$\mathbb{E}(\xi_n|\mathcal{F}_n) = 1 \cdot \frac{1}{2}\Big(1 + \frac{1}{M_n}\Big) - 1 \cdot \frac{1}{2}\Big(1 - \frac{1}{M_n}\Big) = \frac{1}{M_n}. \tag{4.15}$$

Note also that

$$\xi_n^2 = 1. \tag{4.16}$$

Consider $X_n = M_n^2 - 3n$. Let us prove that X_n is a martingale adapted to $\mathcal{F}_n$. Evidently $X_n \in \mathcal{F}_n$ and $\mathbb{E}|X_n| < \infty$. Therefore, we only need to check that $\mathbb{E}(X_{n+1}|\mathcal{F}_n) = X_n$. Using (4.15)–(4.16) we have

$$\begin{aligned}
&\mathbb{E}(X_{n+1} - X_n|\mathcal{F}_n) = \mathbb{E}(M_{n+1}^2 - M_n^2 - 3|\mathcal{F}_n)\\
&\quad = \mathbb{E}[(\xi_n + M_n)^2 - M_n^2 - 3|\mathcal{F}_n] = \mathbb{E}[\xi_n^2|\mathcal{F}_n] + 2M_n\mathbb{E}[\xi_n|\mathcal{F}_n] - 3 = 0.
\end{aligned}$$

Thus one gets

$$\begin{aligned}
\mathbb{E}((X_{n+1} - X_n)^2|\mathcal{F}_n) &= \mathbb{E}[(\xi_n^2 + 2M_n\xi_n - 3)^2|\mathcal{F}_n]\\
&= 4\mathbb{E}[(M_n\xi_n - 1)^2|\mathcal{F}_n]\\
&= 4(M_n^2 + 1 - 2) = 4(M_n^2 - 1),
\end{aligned}$$

and therefore,

$$\mathbb{E}(X_{n+1} - X_n)^2 = 4\mathbb{E}M_n^2 - 4 = 4\mathbb{E}(X_n + 3n) - 4 = 4\mathbb{E}(X_0) + 12n - 4. \tag{4.17}$$

With any sequence of positive numbers $\{a_m\}$ consider

$$B_n = \sum_{m=1}^{n} \frac{X_m - X_{m-1}}{a_m}.$$

Using the fact that X_n is a martingale, it is easy to check that B_n is also a martingale. One has

$$\begin{aligned}
\mathbb{E}B_n^2 &= \mathbb{E}\Big(\mathbb{E}\Big(\Big(\sum_{m=1}^{n}\frac{X_m - X_{m-1}}{a_m}\Big)^2\Big|\mathcal{F}_{n-1}\Big)\Big) \\
&= \mathbb{E}\Big(\sum_{m=1}^{n-1}\frac{X_m - X_{m-1}}{a_m}\Big)^2 + \mathbb{E}\Big(\frac{X_n - X_{n-1}}{a_n}\Big)^2 \\
&\quad +2\mathbb{E}\Big(\sum_{m=1}^{n-1}\frac{X_m - X_{m-1}}{a_m}\mathbb{E}\Big(\frac{X_n - X_{n-1}}{a_n}\Big|\mathcal{F}_{n-1}\Big)\Big) \\
&= \mathbb{E}\Big(\sum_{m=1}^{n-1}\frac{X_m - X_{m-1}}{a_m}\Big)^2 + \mathbb{E}\Big(\frac{X_n - X_{n-1}}{a_n}\Big)^2 = \cdots \\
&= \sum_{m=1}^{n}\frac{\mathbb{E}(X_m - X_{m-1})^2}{a_m^2}.
\end{aligned}$$

From (4.17), we get that if $a_m = m\log m$, then there exists a constant $C > 0$ such that $\mathbb{E}B_n^2 < C < \infty$. Therefore, using the L^p convergence theorem (see e.g. Theorem (4.5) from Chapter 4 of [95]), one has that B_n converges a.s. Using Kroneker's Lemma (see e.g. Lemma (8.5) from Chapter 1 of [95]), we see that $\frac{X_n}{n\log n} \to 0$ a.s. and recalling the definition of X_n, one gets Lemma 4.3.2.

4.A.2 | Proof of Lemma 4.3.3

This proof proceeds using a similar strategy as the previous proof. Consider

$$X_n = M_n - \sum_{i=1}^{n}\frac{1}{M_i}.$$

From (4.15) it follows that X_n is a martingale adapted to $\mathcal{F}_n$. Using the fact that $|\xi_n| = 1$ and $M_n \geq 1$ for any n, we have

$$\mathbb{E}(X_{n+1} - X_n)^2 = \mathbb{E}\Big(\xi_n - \frac{1}{M_{n+1}}\Big)^2 \leq 4. \tag{4.18}$$

Consider further

$$B_n = \sum_{m=1}^{n}\frac{X_m - X_{m-1}}{a_m}.$$

Using the fact that X_n is a martingale, one has again that B_n is also a martingale and we have

$$\mathbb{E}B_n^2 \;=\; \sum_{m=1}^{n} \frac{\mathbb{E}(X_m - X_{m-1})^2}{a_m^2}.$$

Using (4.18), we get that if $a_m = \sqrt{m}\log m$, then $\sup_n \mathbb{E}B_n^2 < \infty$. Using the L^p convergence theorem, we observe that B_n converges a.s. Finally, using Kroneker's Lemma as before, we see that $\frac{X_n}{\sqrt{n}\log n} \to 0$ a.s. Recalling the definition of X_n, we get Lemma 4.3.3.

4.B | Convergence of Markov chains to diffusion processes

To prove Theorem 4.4.1 we need a little background on stochastic differential equations and convergence to diffusion. The following definition and theorem can for instance be found in [96] Chapter 5 and 8, where the latter is a rather good introduction to the topic which itself is based on [84, 85, 97].

Definition 4.B.1. *We say that X_t is a solution to the martingale problem for b and σ^2, or simply X solves MP(b, σ^2) if*

$$X_t - \int_0^t b(X_s)ds \quad \textit{and} \quad X_t^2 - \int_0^t \sigma(X_s)^2 ds$$

are local martingales. Further, we say that the martingale problem is well-possed if there is uniqueness in distribution and no explosion.

Let us now consider a Markov chain $Y_{mh}^{(h)}$, $m \geq 0$, taking values in a set $\mathcal{X}_h \subset \mathbb{R}$ and having transition probabilities

$$p^h(x, A) := \Pr\left[Y_{(m+1)h}^{(h)} \in A \middle| Y_{mh}^{(h)} = x \right], \quad x \in \mathcal{X}_h, \quad A \subseteq \mathbb{R}.$$

Further, set $X_t^{(h)} = Y_{h[t/h]}^{(h)}$ and define

$$\begin{aligned}
\sigma_h^2(x) &= h^{-1} \int_{|y-x|\leq 1} (y-x)^2 p^h(x, dy) \\
b_h(x) &= h^{-1} \int_{|y-x|\leq 1} (y-x) p^h(x, dy) \\
\Delta_h^{\epsilon}(x) &= h^{-1}\, p^h(x, B(x,\epsilon)^c)
\end{aligned}$$

with $B(x,\epsilon) = \{y : |y - x| < \epsilon\}$.

The following Theorem (see e.g. [96], Theorem 8.7.1) proves convergence of the Markov chain to a limiting diffusion:

Theorem 4.B.1. *Suppose that b and σ are continuous functions for which the martingale problem is well-possed and for $R < \infty$ and $\epsilon > 0$*

(1) $\lim_{h\to 0} \sup_{|x|\le R} |\sigma_h^2(x) - \sigma^2(x)| = 0$

(2) $\lim_{h\to 0} \sup_{|x|\le R} |b_h(x) - b(x)| = 0$

(3) $\lim_{h\to 0} \sup_{|x|\le R} \Delta_h^\epsilon(x) = 0$

If $X_0^{(h)} \to x$ then we have $X_t^{(h)} \Rightarrow X_t$, in the sense of weak convergence on the functions space $D[0,\infty)$, where the continuous process X_t is diffusive and solves the following Itô's equation

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x.$$

The following is a well-known theorem from stochastic calculus (see e.g. [96], Theorem 5.6.1) regarding random time changes of a stochastic process:

Theorem 4.B.2. *Let X_u be a solution of the martingale problem $MP(b,\sigma^2)$ for $u \in [0,\infty)$, let g be a positive function and suppose that for all $u \in [0,\infty)$*

$$\tau_u = \int_0^u g(X_s)ds < \infty.$$

Define the inverse of τ_u by $\tau_s^{-1} = \inf\{t : \tau_t > s\}$ and let $Y_s = X_{\tau_s^{-1}}$, then Y_s is a solution of $MP(b/g, \sigma^2/g)$.

Part III

Gibbs causal triangulations

CHAPTER 5

Properties of transfer matrix for Ising model coupled to causal dynamical triangulations

We introduce a transfer matrix formalism for the (annealed) Ising model coupled to two-dimensional causal dynamical triangulations. Using the Krein-Rutman theory of positivity preserving operators we study several properties of the emerging transfer matrix. In particular, we determine regions in the quadrant of parameters $\beta, \mu > 0$ where the infinite-volume free energy converges yielding results on the convergence and asymptotic properties of the partition function and the Gibbs measure.

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5.1 | Introduction. A review of related results

In the study of two-dimensional quantum gravity and non-critical string theory, models of discrete random surfaces play an essential role.

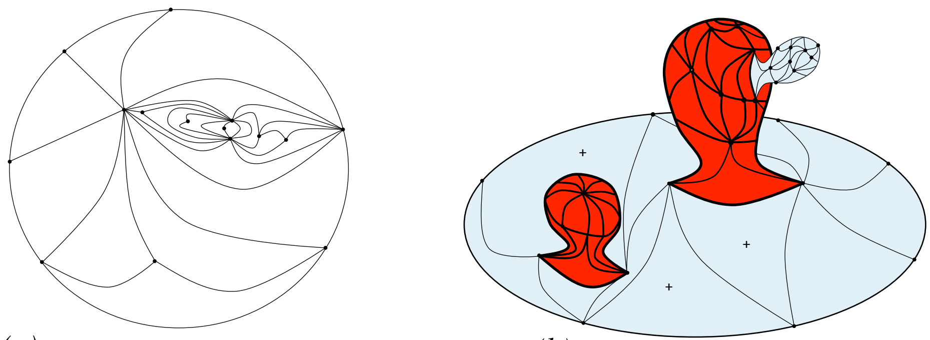
In the 1980s, so-called *dynamical triangulations* (DT) were introduced to define a Euclidean path integral for two-dimensional quantum gravity (see [9] for an overview). In particular, the partition function has been determined as a sum over all possible triangulations of a sphere where each configuration is weighted by a Boltzmann factor $e^{-\mu|T|}$, with $|T|$ standing for the size of the triangulation and μ being the cosmological constant. The evaluation of the partition function was reduced to a purely combinatorial problem that can be solved with the help of the early work of Tutte [4, 3]; alternatively, more powerful techniques were proposed, based on random matrix models (see, e.g., [98]) and bijections to well-labelled trees (see [99, 100]). One can then pass to a continuum limit by taking the number of triangles to infinity. An interesting property of the resulting "quantum geometry" is its fractal structure as illustrated in Figure 5.4 (a). In the physical literature such fractal structures are called "baby universes", and they completely dominate the continuum limit leading to a fractal dimension $d = 4$.

From a probabilistic point of view there has recently been an increasing interest in DT, most notably through the work of Angel and Schramm on a uniform measure on infinite planar triangulations [14], as well as through the work of Le Gall, Miermont and collaborators on Brownian maps (see [21] for a recent review).

From a physical point of view it is interesting to study various models of matter, such as the Ising model, coupled to the DT. The calculation of the partition function in this case also reduces to a combinatorial problem. It was first solved in [101, 102] by using random matrix models and later by using a bijection to well-labelled trees [103]. It is interesting that the solution here is much simpler than in the case of a flat triangular or square lattice as given by Onsager [104]. Further, one can see that the critical exponents in the case where the model is coupled to DT differ from the Onsager values. This is related to the strong back-reaction of the Ising model with the quantum geometry. In particular, the spin clusters energetically prefer to sit within baby universes since those are connected to the main universe through a very short so-called bottleneck boundary (see Figure 5.4 (b)). The spins increase the fractal structure leading to a change in the values of the critical exponents at the critical temperature.

In a continuum framework one attempts to understand the resulting theory as Liouville theory coupled to a conformal field theory with central charge $c = 1/2$. Furthermore, this leads to a simple algebraic identity (the KPZ-relation) between the critical exponents of the Ising model on a flat lattice and the critical exponents of the Ising model coupled to DT [105].

While DT has a very rich mathematical structure which very recently has been related to the SLE (the Schramm-Loewner evolution) and level curves of a Gaussian free field [106], from the point of view of quantum gravity its fractal structure leads to



(a) *(b)*

Figure 5.1: (a) A section of a typical planar triangulation of the DT ensemble illustrating the fractal structure of the quantum geometry. (b) Illustration of the baby universes and the formation of spin clusters within them. Each baby universe corresponds to a fractal structure as in part (a) drawn out of the plane.

causality-violating geometries that are arguably non-physical. This led to the development of so-called *causal dynamical triangulations* (CDT) by Ambjørn and Loll [10], to define the *Lorentzian* gravitational path integral. A causal triangulation is formed by triangulations of spatial strips as illustrated in Figure 5.2. Note that the left and right boundaries of the spatial strip are periodically identified.

A first analytical solution of two-dimensional (pure) CDT was obtained in [10] where it was shown that the resulting quantum geometry, while still random, is much more regular than in the case of a DT, leading to a fractal dimension of $d = 2$. From a probabilistic point of view, we would like to note that a uniform measure on infinite causal triangulations UICT has been recently introduced in [34, 88].

An interesting question is: What are the properties of the Ising model coupled to a CDT ensemble? As was said above, the CDT ensemble is more regular than that of the DT, but it is still random and allows for a back-reaction of the spin system with the quantum geometry. Monte Carlo simulations [77] (see also [107, 108]) give a strong evidence that critical exponents of the Ising model coupled to CDT are identical to the Onsager values.

While recently much progress has been made in the development of analytical tech-

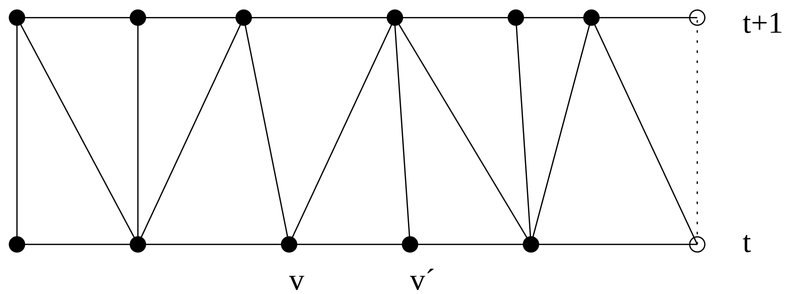


Figure 5.2: A causal triangulation.

niques for CDT [109, 110], particularly random matrix models [90, 54, 89], and their application to multi-critical CDT [93, 94, 111], the causality constraints still makes it difficult to find an analytical solution of the Ising model coupled to CDT. For the quenched Ising model coupled to two-dimensional CDT some progress has been made in proving the existence of a phase transition [52].

In this chapter we develop a transfer matrix formalism for the annealed Ising model coupled to two-dimensional CDT. Spectral properties of the transfer matrix are rigorously analysed by using the Krein-Rutman theorem [59] on operators preserving the cone of positive functions. This yields results on convergence and asymptotic properties of the partition function and the Gibbs measure and allows us to determine regions in the parameter quarter-plane where the partition function converges.

Outline: The chapter is organised as follows. Section 5.2 contains basic definitions. Section 5.3 gives a summary of the transfer-matrix formalism for CDTs. In Section 5.4 we introduce the transfer matrix for the Ising model coupled with a CDT and state and comment on our main results. This is followed with concluding remarks in Section 5.5. An Appendix contains the proofs of the stated results.

5.2 | Definitions

We will work with rooted causal dynamic triangulations of the cylinder $C_N = \mathcal{S} \times [0, N]$, $N = 1, 2, \ldots$, which have N bonds (strips) $\mathcal{S} \times [j, j+1]$. Here $\mathcal{S}$ stands for a unit circle. The definition of a causal triangulation starts by considering a connected graph G embedded in C_N with the property that all faces of G are triangles (using the convention that an edge incident to the same face on two sides counts twice, see Chapter 3 for more details). A triangulation $\underline{\mathbf{t}}$ of C_N is a pair formed by a graph G with the above propetry and the set F of all its (triangular) faces: $\underline{\mathbf{t}} = (G, F)$.

Definition 5.2.1. *A triangulation $\underline{\mathbf{t}}$ of C_N is called a causal triangulation (CT) if the following conditions hold:*

- *each triangular face of $\underline{\mathbf{t}}$ belongs to some strip $\mathcal{S} \times [j, j+1]$, $j = 1, \ldots, N-1$, and has all vertices and exactly one edge on the boundary $(\mathcal{S} \times \{j\}) \cup (\mathcal{S} \times \{j+1\})$ of the strip $\mathcal{S} \times [j, j+1]$;*
- *if $k_j = k_j(\underline{\mathbf{t}})$ is the number of edges on $\mathcal{S} \times \{j\}$, then we have $0 < k_j < \infty$ for all $j = 0, 1, \ldots, N-1$.*

Definition 5.2.2. *A triangulation $\underline{\mathbf{t}}$ of C_N is called rooted if it has a root. The root in the triangulation $\underline{\mathbf{t}}$ is represented by a triangular face t of $\underline{\mathbf{t}}$, called the root triangle, with an anticlock-wise ordering on its vertices (x, y, z) where x and y belong to $S^1 \times \{0\}$. The vertex x is identified as the root vertex and the (directed) edge from x to y as the root edge.*

Definition 5.2.3. *Two causal rooted triangulations of C_N, say $\underline{\mathbf{t}} = (G, F)$ and $\underline{\mathbf{t}}' = (G', F')$, are equivalent if there exists a self-homeomorphism of C_N which (i) transforms each slice $S^1 \times \{j\}$, $j = 0, \ldots, N-1$ to itself and preserves its direction, (ii) induces an isomorphism of the graphs G and G' and a bijection between F and F', and (iii) takes the root of $\underline{\mathbf{t}}$ to the root of $\underline{\mathbf{t}}'$.*

A triangulation $\underline{\mathbf{t}}$ of C_N is identified as a consistent sequence

$$\underline{\mathbf{t}} = (\mathbf{t}(0), \mathbf{t}(1), \ldots, \mathbf{t}(N-1)),$$

where $\mathbf{t}(i)$ is a causal triangulation of the strip $\mathcal{S} \times [i, i+1]$. The latter means that each $\mathbf{t}(i)$ is described by a partition of $\mathcal{S} \times [i, i+1]$ into triangles where each triangle has one vertex on one of the slices $\mathcal{S} \times \{i\}$, $\mathcal{S} \times \{i+1\}$ and two on the other, together with the edge joining these two vertices. The property of consistency means that each pair $(\mathbf{t}(i), \mathbf{t}(i+1))$ is consistent, i.e., every side of a triangle from $\mathbf{t}(i)$ lying in $\mathcal{S} \times \{i+1\}$ serves as a side of a triangle from $\mathbf{t}(i+1)$, and vice versa.

The triangles forming the causal triangulation $\mathbf{t}(i)$ are denoted by $t(i, j)$, $1 \leq j \leq n(\mathbf{t}(i))$ where, $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$. The enumeration of these triangles starts with what we call the root triangle in $\mathbf{t}(i)$; it is determined recursively as follows. First, we have the root triangle $t(0,1)$ in $\mathbf{t}(0)$ (see Definition 2.2). Take the vertex of the triangle $t(0,1)$ which lies on the slice $\mathcal{S}\times\{1\}$ and denote it by x'. This vertex is declared the root vertex for $\mathbf{t}(1)$. Next, the root edge for $\mathbf{t}(1)$ is the one incident to x' and lying on $\mathcal{S}\times\{1\}$, so that if y' is its other end and z' is the third vertex of the corresponding triangle then x', y', z' lists the three vertices anticlock-wise. Accordingly, the triangle with the vertices x', y', z' is called the root triangle for $\mathbf{t}(1)$. This construction can be iterated, determining the root vertices, root edges and root triangles for $\mathbf{t}(i)$, $0\leq i\leq N-1$.

It is convenient to introduce the notion of "up" and "down" triangles (see Figure 5.2). We call a triangle $t\in\mathbf{t}(i)$ an up-triangle if it has an edge on the slice $\mathcal{S}\times\{i\}$ and a down-triangle if it has an edge on the slice $\mathcal{S}\times\{i+1\}$. By Definition 2.1, every triangle is either of type up or down. Let $n_{\mathrm{up}}(\mathbf{t}(i))$ and $n_{\mathrm{do}}(\mathbf{t}(i))$ stand for the number of up- and down-triangles in the triangulation $\mathbf{t}(i)$.

Note that for any edge lying on the slice $\mathcal{S}\times\{i\}$ belongs to exactly two triangles: one up-triangle from $\mathbf{t}(i)$ and one down-triangle from $\mathbf{t}(i-1)$. This provides the following relation: the number of triangles in the triangulation $\underline{\mathbf{t}}$ is twice the total number of edges on the slices. More precisely, let n^i be the number of edges on slice $\mathcal{S}\times\{i\}$. Then, for any $i=0,1,\ldots,N-1$,

$$n(\mathbf{t}(i))=n_{up}(\mathbf{t}(i))+n_{do}(\mathbf{t}(i))=n^i+n^{i+1}, \tag{5.1}$$

implying that

$$\sum_{i=0}^{N-1}n(\mathbf{t}(i))=2\sum_{i=0}^{N-1}n^i. \tag{5.2}$$

There is another useful property regarding the counting of triangulations. Let us fix the number of edges n^i and n^{i+1} in the slices $\mathcal{S}\times\{i\}$ and $\mathcal{S}\times\{i+1\}$. The number of possible rooted CTs of the slice $\mathcal{S}\times[i,i+1]$ with n^i up- and n^{i+1} down-triangles is equal to

$$\binom{n^i+n^{i+1}-1}{n^i-1}=\binom{n(\mathbf{t}(i))-1}{n_{up}(\mathbf{t}(i))-1}. \tag{5.3}$$

5.3 | Transfer matrix formalism for pure CDT

We begin by discussing the case of pure causal dynamical triangulations, as was first introduced in [10] (see also [33] for a mathematically more rigorous account).

The partition function for rooted CTs in the cylinder C_N with periodical spatial boundary conditions (where $\mathbf{t}(0)$ is consistent with $\mathbf{t}(N-1)$) and for the value of the cosmological constant μ is given by

$$Z_N(\mu) = \sum_{\underline{\mathbf{t}}} e^{-\mu n(\underline{\mathbf{t}})} = \sum_{(\mathbf{t}(0),\dots,\underline{\mathbf{t}}(N-1))} \exp\Big\{-\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i))\Big\}. \tag{5.4}$$

Using the properties (5.2) and (5.3) we can represent the partition function (5.4) in the following way

$$Z_N(\mu) = \sum_{n^0\geq 1,\dots,n^{N-1}\geq 1} \exp\Big\{-2\mu \sum_{i=0}^{N-1} n^i\Big\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1}. \tag{5.5}$$

Moreover, $Z_N(\mu)$ admits a trace-related representation

$$Z_N(\mu) = \mathrm{tr}\,\left(U^N\right). \tag{5.6}$$

This gives rise to a transfer matrix $U = \{u(n,n')\}_{n,n'=1,2,\dots}$ describing the transition from one spatial strip to the next one. It is an infinite matrix with strictly positive entries

$$u(n,n') = \binom{n+n'-1}{n-1} g^{n+n'}. \tag{5.7}$$

For notational convenience we use the parameter $g = e^{-\mu}$ (a single-triangle fugacity). The entry $u(n,n')$ yields the number of possible triangluations of a single strip (say, $\mathcal{S}\times[0,1]$) with n lower boundary edges (on $\mathcal{S}\times\{0\}$) and n' upper boundary edges (on $\mathcal{S}\times\{1\}$). See Figure 5.2. The asymmetry in n and n' is due to the fact that the lower boundary is marked while the upper one is not. However, a symmetric transfer matrix $\widetilde{U} = \{\widetilde{u}(n,n')\}$ can be introduced, associated with a strip where both boundaries are kept unmarked:

$$\widetilde{u}(n,n') = n^{-1} u(n,n'). \tag{5.8}$$

The N-strip Gibbs distribution $\mathcal{P}_N$ assigns the following probabilities to strings $(n^0,\dots,n^{N-1})$ with the number of triangles $n^i \geq 1$ for all $i = 0,\dots,N-1$:

$$\mathcal{P}_N(n^0,\dots,n^{N-1}) = \frac{1}{Z_N(\mu)} \exp\Big\{-2\mu \sum_{i=0}^{N-1} n^i\Big\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1}. \tag{5.9}$$

We state two lemmas featuring properties of matrix U:

Lemma 5.3.1. *For any $g > 0$ the matrix U and its transpose U^{T} have an eigenvalue $\Lambda = \Lambda(g)$ given by*

$$\Lambda(g) = \left[(1 - \sqrt{1-4g^2})/(2g)\right]^2. \tag{5.10}$$

The corresponding eigenvectors

$$\phi = \{\phi(n)\}_{n=1,2,\ldots} \ \text{ and } \ \phi^* = \{\phi^*(n)\}_{n=1,2,\ldots}$$

have entries

$$\phi(n) = n\big(\Lambda(g)\big)^n, \quad \phi^*(n) = (\Lambda(g))^n. \tag{5.11}$$

Proof. A direct verification shows that

$$\sum_{n'} u(n,n')n'\Lambda^{n'}(g) = n\Lambda^{n+1}(g) \ \text{ and } \ \sum_{n} \Lambda^n(g)u(n,n') = \Lambda^{n'+1}(g).$$

(In fact, each of these relations implies the other.) See Theorem 1 in [33]. □

Lemma 5.3.2. *For any fixed n and any $g < 1$ (equivalently, $\mu > 0$) one has*

$$\sum_{n'} u(n,n') = \Big(\frac{g}{1-g}\Big)^n \big(1-(1-g)^n\big). \tag{5.12}$$

Proof. The proof again follows from a straightforward verification. □

A transfer-matrix formalism of Statistical Mechanics predicts that, as $N \to \infty$, the partition function is governed by the largest eigenvalue Λ of the transfer matrix:

$$Z_N(g) = \operatorname{tr}\, U^N \sim \Lambda^N \tag{5.13}$$

We make this statement more precise in the statements of Lemma 5.3.3 and Theorem 5.3.1 below. Here the symbol ℓ^2 stands for the Hilbert space of square-summable complex sequences (infinite-dimensional vectors) $\psi = \{\psi(n)\}_{n=1,2,\ldots}$ equipped with the standard scalar product $\langle\psi',\psi''\rangle = \sum_n \psi'(n)\overline{\psi}''(n)$. Accordingly, the matrices U and U^{T} are treated as operators in ℓ^2.

Lemma 5.3.3. *For any $g < 1/2$ (equivalently $\mu > \ln 2$) the following statements hold true:*

1. *U and U^{T} are bounded operators in ℓ^2 preserving the cone of positive vectors;*

2. *The sum* $\sum_{n,n'} u(n,n') < \infty$*. Consequently,* U *and* U^{T} *have*

$$\mathrm{tr}\,\left(UU^{\mathsf{T}}\right) = \mathrm{tr}\,\left(U^{\mathsf{T}}U\right) < \infty,$$

i.e., U *and* U^{T} *are Hilbert-Schmidt operators. Therefore,* $\forall\ N \geq 2$*,* U^N *and* $\left(U^{\mathsf{T}}\right)^N$ *are trace-class operators.*

3. *The maximal eigenvalue* $\Lambda = \Lambda(g)$ *of* U *in* ℓ^2 *is positive, coincides with the maximal eigenvalue of* U^{T} *and is given by Eqn (5.10). The corresponding eigenvectors* $\phi, \phi^* \in \ell^2$ *are unique up to multiplication by a constant factor and given in Eqn (5.11).*

4. *The following asymptotical formulas hold as* $N \to \infty$*:*

$$\frac{1}{\Lambda^N}\,\mathrm{tr}\,\left(U^N\right),\quad \frac{1}{\Lambda^N}\,\mathrm{tr}\,\left((U^{\mathsf{T}})^N\right) \to 1,$$

and, $\forall$ *vectors* $\psi', \psi'' \in \ell^2$*,*

$$\frac{1}{\Lambda^N}\langle \psi', U^N \psi''\rangle = \langle \psi', \phi\rangle\langle \phi^*, \psi''\rangle,$$

where the eigenvectors ϕ *and* ϕ^* *are normalized so that* $\langle \phi, \phi^*\rangle = 1$*.*

Theorem 5.3.1. *For any* $g < 1/2$ *the following relation holds true:*

$$\lim_{N\to\infty} \frac{1}{N} \log\, Z_N(g) = \log\, \Lambda \tag{5.14}$$

with $\Lambda = \Lambda(g)$ *given in (5.10). Further, the* N*-strip Gibbs measure* $\mathcal{P}_N$ *converges weakly to a limiting measure* $\mathcal{P}$ *which is represented by a positive recurrent Markov chain on* $\mathbb{Z}_+ = \{1, 2, ...\}$*, with the transition matrix* $P = \{P(n, n')\}_{n=1,2,...}$ *and the invariant distribution* π*. Here*

$$P(n, n') = \frac{u(n, n')\phi(n')}{\Lambda\phi(n)}$$

and

$$\pi(n) = \frac{\phi^*(n)\phi(n)}{\langle \phi^*, \phi\rangle}.$$

where $\phi(n)$ *and* $\phi^*(n)$ *are as in (5.11).*

Proof. The proof is a consequence of Lemma 5.3.1 and 5.3.3 and the Krein-Rutman theory [59]. $\square$

5.4 | An Ising model coupled to the CDT: statement of results

5.4.1 | The model

With any triangle from a triangulation $\underline{\mathbf{t}}$ we associate a spin taking values ± 1. An N-strip configuration of spins is represented by a collection

$$\underline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}(0), \boldsymbol{\sigma}(1), \dots, \boldsymbol{\sigma}(N-1))$$

where $\boldsymbol{\sigma}(i) = \boldsymbol{\sigma}(\mathbf{t}(i))$ is a configuration of spins $\sigma(i,j)$ over triangles $t(i,j)$ forming a triangulation $\mathbf{t}(i)$, $1 \leq j \leq n(\mathbf{t}(i))$. We will say that a single-strip configuration of spins $\boldsymbol{\sigma}(i)$ is supported by a triangulation $\mathbf{t}(i)$ of strip $\mathcal{S} \times [i, i+1]$. We consider a usual (ferromagnetic) Ising-type energy where two spins $\sigma(i,j)$ and $\sigma(i',j')$ interact if their supporting triangles $t(i,j)$, $t(i',j')$ share a common edge; such triangles are called nearest neighbors, and this property is reflected in the notation $\langle \sigma(i,j), \sigma(i',j') \rangle$, where we require $0 \leq i \leq i' \leq N-1$. Thus, in our model each spin has three neighbors. Moreover, a pair $\langle \sigma(i,j), \sigma(i',j') \rangle$ can only occur for $i' - i \leq 1$ or $i = 0$, $i' = N-1$. Formally, the Hamiltonian of the model reads:

$$\mathbb{H}(\underline{\boldsymbol{\sigma}}) = - \sum_{\langle \sigma(i,j), \sigma(i',j') \rangle} \sigma(i,j)\sigma(i',j'). \tag{5.15}$$

We will use the following decomposition:

$$\mathbb{H}(\underline{\boldsymbol{\sigma}}) = \sum_{i=0}^{N-1} H(\boldsymbol{\sigma}(i)) + \sum_{i=0}^{N-1} V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1)), \tag{5.16}$$

where we assume that $\boldsymbol{\sigma}(0) \equiv \boldsymbol{\sigma}(N)$ (the periodic spatial boundary condition). Here $H(\boldsymbol{\sigma}(i))$ represents the energy of the configuration $\boldsymbol{\sigma}(i)$:

$$H(\boldsymbol{\sigma}(i)) = - \sum_{\langle \sigma(i,j), \sigma(i,j') \rangle} \sigma(i,j)\sigma(i,j'). \tag{5.17}$$

Further, $V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1))$ is the energy of interaction between neighboring triangles belonging to the adjacent strips $\mathcal{S} \times [i, i+1]$ and $\mathcal{S} \times [i+1, i+2]$:

$$V(\boldsymbol{\sigma}(i), \boldsymbol{\sigma}(i+1)) = - \sum_{\langle \sigma(i,j), \sigma(i+1,j') \rangle} \sigma(i,j)\sigma(i+1,j'). \tag{5.18}$$

The partition function for the (annealed) N-strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ and for the cosmological constant μ, is given by

$$\Xi_N(\mu,\beta) = \sum_{(\mathbf{t}(0),\ldots,\mathbf{t}(N-1))} \exp\Big\{-\mu\sum_{i=0}^{N-1} n(\mathbf{t}(i))\Big\} \times \sum_{(\boldsymbol{\sigma}(0),\ldots,\boldsymbol{\sigma}(N-1))} \prod_{i=0}^{N-1} \exp\Big\{-\beta H(\boldsymbol{\sigma}(i)) - \beta V(\boldsymbol{\sigma}(i),\boldsymbol{\sigma}(i+1))\Big\}. \tag{5.19}$$

Here $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$. Like before, the formula

$$\Xi_N(\mu,\beta) = \mathrm{tr}\ \mathbf{K}^N \tag{5.20}$$

gives rise to a transfer matrix $\mathbf{K}$ with entries $K((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}'))$ labelled by pairs $(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')$ representing triangulations of a single strip (say, $\mathcal{S}\times[0,1]$) and their supported spin configurations which are positioned next to each other. Formally,

$$K((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')) = \mathbf{1}_{\mathbf{t}\sim\mathbf{t}'} \exp\Big\{-\frac{\mu}{2}(n(\mathbf{t})+n(\mathbf{t}'))\Big\} \times \exp\Big\{-\frac{\beta}{2}\big(H(\boldsymbol{\sigma})+H(\boldsymbol{\sigma}')\big) - \beta V(\boldsymbol{\sigma},\boldsymbol{\sigma}')\Big\}. \tag{5.21}$$

As earlier, $n(\mathbf{t})$ and $n(\mathbf{t}')$ are the numbers of triangles in the triangulations $\mathbf{t}$ and $\mathbf{t}'$. The indicator $\mathbf{1}_{\mathbf{t}\sim\mathbf{t}'}$ means that the triangulations $\mathbf{t},\mathbf{t}'$ have to be consistent with each other in the above sense: the number of down-triangles in $\mathbf{t}$ should equal the number of up-triangles in $\mathbf{t}'$, and an upper-marked edge in $\mathbf{t}$ should coincide with a lower-marked edge in triangulation $\mathbf{t}'$. It means that the pair $(\mathbf{t},\mathbf{t}')$ forms a CDT for the strip $\mathcal{S}\times[0,2]$.

We would like to stress that the trace $\mathrm{tr}\ \mathbf{K}^N$ in (5.20) is understood as the *matrix trace*, i.e., as the sum $\sum_{\mathbf{t},\boldsymbol{\sigma}} K^{(N)}((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t},\boldsymbol{\sigma}))$ of the diagonal entries $K^{(N)}((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t},\boldsymbol{\sigma}))$ of the matrix $\mathbf{K}^N$. (Indeed, in what follows, the notation "tr" is used for the matrix trace only.) Our aim will be to verify that the matrix trace in (5.20) can be replaced with an *operator trace* invoking the eigenvalues of $\mathbf{K}$ in a suitable linear space.

As before, we can introduce the N-strip Gibbs probability distribution associated with formula (5.19):

$$\mathbb{P}_N\big((\mathbf{t}(0),\boldsymbol{\sigma}(0)),\ldots,(\mathbf{t}(N-1),\boldsymbol{\sigma}(N-1))\big) = \frac{1}{\Xi(\mu,\beta)} \prod_{i=0}^{N-1} \exp\Big\{-\mu n(\mathbf{t}(i)) - \beta H(\boldsymbol{\sigma}(i)) - \beta V(\boldsymbol{\sigma}(i),\boldsymbol{\sigma}(i+1))\Big\}. \tag{5.22}$$

Consider several special cases of interest.

The case $\beta \approx 0$. This is the first term of the so-called high temperature expansion [77]. Here one has

$$\begin{aligned}
\Xi(\mu, 0) &= \sum_{(\mathbf{t}(0),\dots,\mathbf{t}(N-1))} \exp\Big\{-\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i))\Big\} \sum_{(\boldsymbol{\sigma}(0),\dots,\boldsymbol{\sigma}(N-1))} 1 \\
&= \sum_{n^0 \geq 1,\dots,n^{N-1}\geq 1} \exp\Big\{-2(\mu - \ln 2)\sum_{i=0}^{N-1} n^i\Big\} \prod_{i=0}^{N-1} \binom{n^i + n^{i+1} - 1}{n^i - 1} \\
&= Z_N(\mu - \ln 2); \quad \text{cf. (5.4).}
\end{aligned}$$

The condition $\mu - \ln 2 > \ln 2$ which guarantees properties listed in Lemma 5.3.3 and Theorem 5.3.1 resuls in

$$\mu > 2 \ln 2. \tag{5.23}$$

Thus, Eqn. (5.23) yields a sub-criticality condition when $\beta = 0$.

The case $\beta \approx \infty$. Observe that for any triangulation $\underline{\mathbf{t}} = (\mathbf{t}(0), \dots, \mathbf{t}(N-1))$ there are two ground states: all spins $+1$ and all spins -1, with the overall energy equals minus three half times the total number of triangles: $-3/2 \sum_{i=0}^{N-1} n(\mathbf{t}(i))$. Discarding all other spin configurations, we obtain that

$$\Xi(\mu, \beta) > \Xi_*(\mu, \beta)$$

where

$$\begin{aligned}
\Xi_*(\mu, \beta) &= \sum_{\mathbf{t}(0),\dots,\mathbf{t}(N-1)} 2\exp\Big\{\big(-\mu + \frac{3}{2}\beta\big)\sum_{i=0}^{N-1} n(\mathbf{t}(i))\Big\} \\
&= 2 \sum_{n^0 \geq 1,\dots,n^{N-1}\geq 1} \exp\Big\{-2\big(\mu - \frac{3}{2}\beta\big)\sum_{i=0}^{N-1} n^i\Big\} \binom{n^i + n^{i+1} - 1}{n^i - 1} \\
&= 2 Z_N\left(\mu - \frac{3}{2}\beta\right)
\end{aligned}$$

where $\exp\left[\frac{3}{2}\beta \sum_i n(\mathbf{t}(i))\right]$ is the energy of the $(+)$-configuration (or, equivalently, the $(-)$-configuration). For β large, we can expect that $\Xi(\mu, \beta) \sim \Xi_*(\mu, \beta)$. Then the critical inequality

$$\mu - \frac{3}{2}\beta > \ln 2$$

yields

$$\mu > \ln 2 + \frac{3}{2}\beta. \tag{5.24}$$

Equation (5.24) gives a necessary (and probably tight) criticality condition for the Ising model under consideration for large values of β. A similar result was obtained in [77].

The case $0 < \beta < \infty$. Firstly, we note that for any fixed triangulation $\underline{\mathbf{t}}$ the energy of any spin configuration $\underline{\boldsymbol{\sigma}}$ on $\underline{\mathbf{t}}$ will be bigger or equal than the energy of a pure configuration (all $+$s or all $-$s):

$$\begin{aligned} H(\underline{\boldsymbol{\sigma}}) &= \sum_j H(\boldsymbol{\sigma}(j)) + \sum_j V(\boldsymbol{\sigma}(j), \boldsymbol{\sigma}(j+1)) \\ &\geq -\frac{3}{2}\#(\text{of all triangles in } \underline{t}) = -3\sum_{i=0}^{N-1} n^i, \end{aligned}$$

where n^i is the number of edges in the ith level $S \times \{i\}$, $i = 0, 1 \ldots, N-1$. Thus, for any $\beta > 0$ the inequality $\Xi(\mu, \beta) < \Xi^*(\mu, \beta)$ holds true, where

$$\begin{aligned} \Xi^*(\mu, \beta) &= \sum_{(\mathbf{t}(0), \ldots, \mathbf{t}(N-1))} \exp\Big\{\big(-\mu + \frac{3}{2}\beta + \ln 2\big)\sum_{i=0}^{N-1} n(\mathbf{t}(i))\Big\} \\ &= \sum_{n^0 \geq 1, \ldots, n^{N-1} \geq 1} \exp\Big\{-2\big(\mu - \frac{3}{2}\beta - \ln 2\big)\sum_{i=0}^{N-1} n^i\Big\} \\ &= Z_N\big(\mu - \frac{3}{2}\beta - \ln 2\big) \quad . \end{aligned}$$

Hence, the inequality

$$\mu - \frac{3}{2}\beta - \ln 2 > \ln 2 \quad \text{or} \quad \mu > 2\ln 2 + \frac{3}{2}\beta \tag{5.25}$$

provides a sufficient condition for subcriticality of the Ising model under consideration.

5.4.2 | The transfer-matrix $\mathbf{K}$ and its powers $\mathbf{K}^N$

The main results of this chapter are summarized in Lemma 5.4.1 and Theorems 5.4.1 and 5.4.2 below. Let us start with a statement (see Proposition 5.4.1 below) which merely re-phrases standard definitions and explains our interest in the matrices $\mathbf{K}$, $\mathbf{K}^{\mathsf{T}}$, $\mathbf{K}^{\mathsf{T}}\mathbf{K}$, $\mathbf{K}\mathbf{K}^{\mathsf{T}}$ and their powers. Cf. Definition 2.2.2 on p.83, Definition 2.4.1 on p.101, Lemma 2.3.1 on p.85 and Theorem 3.3.13 on p.139 in [112]).

We treat the transfer-matrix $\mathbf{K}$ and its transpose $\mathbf{K}^{\mathsf{T}}$ as linear operators in the Hilbert space $\ell^2_{\mathsf{T-C}}$ (the subscript T-C refers to triangulations and spin-configurations). The space $\ell^2_{\mathsf{T-C}}$ is formed by functions $\psi = \{\psi(\mathbf{t}, \boldsymbol{\sigma})\}$ with the argument $(\mathbf{t}, \boldsymbol{\sigma})$ running over single-strip triangulations and supported configurations of spins, with the scalar product $\langle \psi', \psi'' \rangle_{\mathsf{T-C}} = \sum_{\mathbf{t},\boldsymbol{\sigma}} \psi'(\mathbf{t}, \boldsymbol{\sigma})\overline{\psi''}(\mathbf{t}, \boldsymbol{\sigma})$ and the induced norm $\|\psi\|_{\mathsf{T-C}}$. The action of $\mathbf{K}$ in $\ell^2_{\mathsf{T-C}}$, in the basis formed by Dirac's delta-vectors $\delta_{(\mathbf{t},\boldsymbol{\sigma})}$, is determined by

$$\big(\mathbf{K}\psi\big)(\mathbf{t}, \boldsymbol{\sigma}) = \sum_{\mathbf{t}',\boldsymbol{\sigma}'} K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))\psi(\mathbf{t}', \boldsymbol{\sigma}'); \tag{5.26}$$

in following we use the notation $\mathbf{K}$, $\mathbf{K}^{\mathsf{T}}$, etc., for the matrices and the corresponding operators in $\ell^2_{\mathsf{T-C}}$. Accordingly, the symbols $\|\mathbf{K}\|_{\mathsf{T-C}}$, $\|\mathbf{K}^{\mathsf{T}}\|_{\mathsf{T-C}}$ etc. refer to norms in $\ell^2_{\mathsf{T-C}}$.

Given $n = 1, 2, ...$, suppose that the operator $\mathbf{K}^n$ (respectively, $\big(\mathbf{K}^{\mathsf{T}}\big)^n$) is of trace class . Then the following series absolutely converges:

$$\sum_j \mathbf{\Lambda}_j^{(n)} \quad \left(\text{respectively, } \sum_j \mathbf{\Lambda}^{*(n)}_j\right), \tag{5.27}$$

where $\mathbf{\Lambda}_j^{(n)}$ ($\mathbf{\Lambda}^{*(n)}_j$) runs through the eigenvalues of $\mathbf{K}^n$ ($(\mathbf{K}^{\mathsf{T}})^n$), counted with their multiplicities. In this case the sum (5.27) is called the operator trace of $\mathbf{K}^n$ (respectively, $(\mathbf{K}^{\mathsf{T}})^n$) in $\ell^2_{\mathsf{T-C}}$. We adopt an agreement that the eigenvalues in (5.27) are listed in the decreasing order of their moduli, beginning with $\mathbf{\Lambda}_0^{(n)}$ ($\mathbf{\Lambda}^{*(n)}_0$) .

Set $|\mathbf{K}^n| = \sqrt{(\mathbf{K}^{\mathsf{T}})^n\,\mathbf{K}^n}$ and $\big|\big(\mathbf{K}^{\mathsf{T}}\big)^n\big| = \sqrt{\mathbf{K}^n\,(\mathbf{K}^{\mathsf{T}})^n}$.

Proposition 5.4.1. *For any positve integer r, the following inequalities are equivalent:*

$$\begin{aligned}&\mathrm{tr}\left(\left(\mathbf{K}^r(\mathbf{K}^{\mathsf{T}}\right)^r\right) = \mathrm{tr}\left(\left(\mathbf{K}^{\mathsf{T}}\right)^r\mathbf{K}^r\right) < \infty \ \ \textit{and}\\ &\mathrm{tr}|\mathbf{K}^{2r}| = \mathrm{tr}|(\mathbf{K}^{\mathsf{T}})^{2r}| < \infty.\end{aligned} \tag{5.28}$$

Moreover, each of the inequalities in (5.28) *implies that* $\forall\ N \geq 2r$*, the operators* $\mathbf{K}^N$ *and* $(\mathbf{K}^{\mathsf{T}})^N$ *are of trace class in* $\ell^2_{\mathsf{T-C}}$*. Hence, for* $N \geq 2r$*, the matrix traces* $\mathrm{tr}\left(\mathbf{K}^N\right)$ *and* $\mathrm{tr}((\mathbf{K}^{\mathsf{T}})^N)$ *are finite and coincide with the corresponding operator traces in* $\ell^2_{\mathsf{T-C}}$*.*

Theorem 5.4.1. *Suppose that the condition (5.28) is satisfied with $r = 1$. Then the following properties of transfer matrix $\mathbf{K}$ are fullfilled.*

1. *The square* $\mathbf{K}^2$ *and its transpose* $(\mathbf{K}^\mathsf{T})^2$ *are trace-class operators in* $\ell^2_{\mathsf{T-C}}$*.*

2. $\mathbf{K}$ *and* $\mathbf{K}^\mathsf{T}$ *have a common eigenvalue,* $\mathbf{\Lambda} = \mathbf{\Lambda}_0(\beta, \mu) > 0$ *such that the norms* $\|\mathbf{K}\|_{\mathsf{T-C}} = \|\mathbf{K}^\mathsf{T}\|_{\mathsf{T-C}} = \mathbf{\Lambda}$*. Furthermore,* $\mathbf{K}^2$ *and* $(\mathbf{K}^\mathsf{T})^2$ *have the common eigenvalue* $\mathbf{\Lambda}^2 = \mathbf{\Lambda}_0^{(2)} = \mathbf{\Lambda}^{*(2)}_0$ *such that the norms* $\|\mathbf{K}^2\|_{\mathsf{T-C}} = \|(\mathbf{K}^\mathsf{T})^2\|_{\mathsf{T-C}} = \mathbf{\Lambda}^2$ *.*

3. $\mathbf{\Lambda}$ *is a simple eigenvalue of* $\mathbf{K}$ *and* $\mathbf{K}^\mathsf{T}$*, i.e., the corresponding eigenvectors* $\phi = \{\phi(\mathbf{t}, \boldsymbol{\sigma})\}$ *and* $\phi^* = \{\phi^*(\mathbf{t}, \boldsymbol{\sigma})\}$ *are unique up to multiplicative constants. Moreover,* ϕ *and* ϕ^T *can be made strictly positive:* $\phi(\mathbf{t}, \boldsymbol{\sigma}), \phi^\mathsf{T}(\mathbf{t}, \boldsymbol{\sigma}) > 0 \;\forall\; (\mathbf{t}, \boldsymbol{\sigma})$*. Furthermore,* $\mathbf{\Lambda}$ *is separated from the remaining singular values and the remaining eigenvalues of* $\mathbf{K}$ *and* $\mathbf{K}^\mathsf{T}$ *by a positive gap. The same is true for* $\mathbf{\Lambda}^2$ *and* $\mathbf{K}^2$ *and* $\left(\mathbf{K}^\mathsf{T}\right)^2$*.*

Proof of Theorem 5.4.1. Because the entries $K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ are non-negative, the condition (5.28) with $r = 1$ means that

$$\sum_{(\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) < \infty, \tag{5.29}$$

that is, $\mathbf{K}$ and $\mathbf{K}^\mathsf{T}$ are Hilbert-Schmidt operators. It means that the operator $\mathbf{K}\mathbf{K}^\mathsf{T}$ has an orthonormal basis of eigenvectors and the series of squares of its eigenvalues (counted with multiplicities) converges and gives the trace $\mathrm{tr}_{\mathsf{T-C}}(\mathbf{K}\mathbf{K}^\mathsf{T})$. Consequently, the operators $\mathbf{K}$ and $\mathbf{K}^\mathsf{T}$ are bounded (and even completely bounded) and $\mathbf{K}^2$ and $(\mathbf{K}^\mathsf{T})^2$ are of trace class. The latter fact means that the matrix trace of the operator $\mathbf{K}^2$ coincides with its operator trace in $\ell^2_{\mathsf{T-C}}$, and the same is true of $(\mathbf{K}^\mathsf{T})^2$. In addition, the operator $\mathbf{K}^2$ has the property that its matrix entries $K^{(2)}((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ are strictly positive:

$$K^{(2)}((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) = \sum_{(\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\sigma}})} K((\mathbf{t}, \boldsymbol{\sigma}), (\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\sigma}}))K((\widetilde{\mathbf{t}}, \widetilde{\boldsymbol{\sigma}}), (\mathbf{t}', \boldsymbol{\sigma}')) > 0. \tag{5.30}$$

The Krein–Rutman theory (see [59], Proposition VII′) guarantees that both $\mathbf{K}$ and $\mathbf{K}^\mathsf{T}$ have a maximal eigenvalue $\mathbf{\Lambda}$ that is positive and non-degenerate, or simple. That is, the eigenvector ϕ of $\mathbf{K}$ and the eigenvector ϕ^* of $\mathbf{K}^\mathsf{T}$ corresponding with $\mathbf{\Lambda}$ are unique up to multiplication by a constant, and all entries $\phi(\mathbf{t}, \boldsymbol{\sigma})$ and $\phi^*(\mathbf{t}, \boldsymbol{\sigma})$ are non-zero and have the same sign. In other words, the entries $\phi(\mathbf{t}, \boldsymbol{\sigma})$ and $\phi^*(\mathbf{t}, \boldsymbol{\sigma})$ can be made positive. The spectral gaps are also consequences of the above properties. $\square$

Set:

$$\lambda(\mu,\beta) = c^2\,(m^2+1)\,(\cosh 2\beta)\left(1+\sqrt{1-\frac{1}{(\cosh 2\beta)^2}\frac{(m^2-1)^2}{(m^2+1)^2}}\right) \tag{5.31}$$

where c and m are determined by

$$\begin{aligned} c &= \frac{\exp(\beta-\mu)}{e^{2\beta}(1-\exp(\beta-\mu))^2 - e^{-2\mu}} && (5.32)\\ m &= e^{2\beta} + (1-e^{4\beta})\exp\left(-(\beta+\mu)\right). && (5.33)\end{aligned}$$

Lemma 5.4.1. *For any $\beta,\mu>0$ such that*

$$\lambda(\mu,\beta) < 1, \tag{5.34}$$

the condition (5.28) *is satisfied for $r=1$:*

$$\mathrm{tr}(\mathbf{K}\mathbf{K}^\mathsf{T}) = \mathrm{tr}(\mathbf{K}^\mathsf{T}\mathbf{K}) < \infty \;\; \textit{and} \;\; \mathrm{tr}|\mathbf{K}^2| = \mathrm{tr}|(\mathbf{K}^\mathsf{T})^2| < \infty, \tag{5.35}$$

implying the assertions of Proposition 5.4.1 and Theorem 5.4.1. Moreover, the condition (5.28) *implies* (5.34)

The proof of Lemma 5.4.1 is given in the next section. Here we only remark that the proof is based on the following representation of the trace (5.35): there exists matrices Q, Q_m (see formulas (5.47) and (5.48)) with positive entries and of size 4×4 such that (see formula (5.51))

$$\mathrm{tr}(\mathbf{K}\mathbf{K}^\mathsf{T}) = \mathrm{tr}\Big(\Big(\sum_{k\geq 1} Q^k\Big)\,Q_m\Big) + \dots\,.$$

The convergence of the matrix series $\sum_{k\geq1} Q^k$ is equivalent to the condition that the maximal eigenvalue of the matrix Q is less then 1. This is exactly the condition (5.34).

Theorem 5.4.2. *Under condition (5.34), the following limit holds:*

$$\lim_{N\to\infty}\frac{1}{N}\log\,\Xi_N(\beta,\mu) = \log\,\mathbf{\Lambda}. \tag{5.36}$$

Moreover, as $N\to\infty$, the N-strip Gibbs measure $\mathbb{P}_N$ (see Eqn (5.22)) converges weakly to a limiting probability distribution $\mathbb{P}$ that is represented by a positive recurrent Markov chain with states $(\mathbf{t},\boldsymbol{\sigma})$, the transition matrix

$\mathbf{P} = \{P((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))\}$ *and the invariant distribution* $\boldsymbol{\pi} = \{\pi(\mathbf{t}, \boldsymbol{\sigma})\}$ *where*

$$\begin{aligned} P((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) &= \frac{K((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))\phi(\mathbf{t}', \boldsymbol{\sigma}')}{\boldsymbol{\Lambda}\phi(\mathbf{t}, \boldsymbol{\sigma})} \\ \pi(\mathbf{t}, \boldsymbol{\sigma}) &= \phi(\mathbf{t}, \boldsymbol{\sigma})\phi^{\mathsf{T}}(\mathbf{t}, \boldsymbol{\sigma}) \Big/ \left\langle \phi, \phi^{\mathsf{T}} \right\rangle_{\mathsf{T-C}} \end{aligned}$$

with the norm $\|\phi\|^2_{\mathsf{T-C}} = \sum_{\mathbf{t},\boldsymbol{\sigma}} \phi(\mathbf{t}, \boldsymbol{\sigma})^2$.

Proof of Theorem 5.4.2. The spectral gap for $\mathbf{K}$ implies that $\forall\ \psi \in \ell^2_{\mathsf{T-C}}$, we have the convergence

$$\lim_{N\to\infty} \frac{1}{\boldsymbol{\Lambda}^N} \mathbf{K}^N \psi = \left(\langle \psi, \phi \rangle_{\mathsf{T-C}}\right) \phi$$

in the norm of space $\ell^2_{\mathsf{T-C}}$. Moreover, let $\boldsymbol{\Pi}$ denote the operator of projection to the subspace spanned by the eigenvectors of $\mathbf{K}$ different from ϕ. Then

$$\frac{1}{\boldsymbol{\Lambda}} \|\boldsymbol{\Pi}\mathbf{K}\mathbf{P}\|_{\mathsf{T-C}} < 1 \implies \lim_{N\to\infty} \frac{1}{\boldsymbol{\Lambda}^N} \left\| \left(\boldsymbol{\Pi}\mathbf{K}\mathbf{P}\right)^N \right\|_{\mathsf{T-C}} = 0.$$

In turn, this implies that

$$\frac{1}{N} \log\ \Xi_N(\mu, \beta) = \frac{1}{N} \log\ \mathrm{tr}_{\mathsf{T-C}} \mathbf{K}^N \to \log \boldsymbol{\Lambda}.$$

Convergence of the Gibbs measure $\mathbb{P}_N$ follows as a corollary. □

5.5 | Concluding remarks

This chapter makes a step towards determining the subcriticality domain for an Ising-type model coupled to two-dimensional causal dynamical triangulations (CDT). In doing so we employ transfer-matrix techniques and in particular the Krein-Rutman theorem. We complement the discussion of the previous sections with the following two concluding remarks:

Remark 5.5.1. *It is instructive to summarise the logical structure of the argument establishing Lemma 5.4.1 and Theorems 5.4.1 and 5.4.2:*

- *First,* (5.35) *holds iff condition* (5.34) *holds: see the proof of Lemma 5.4.1.*

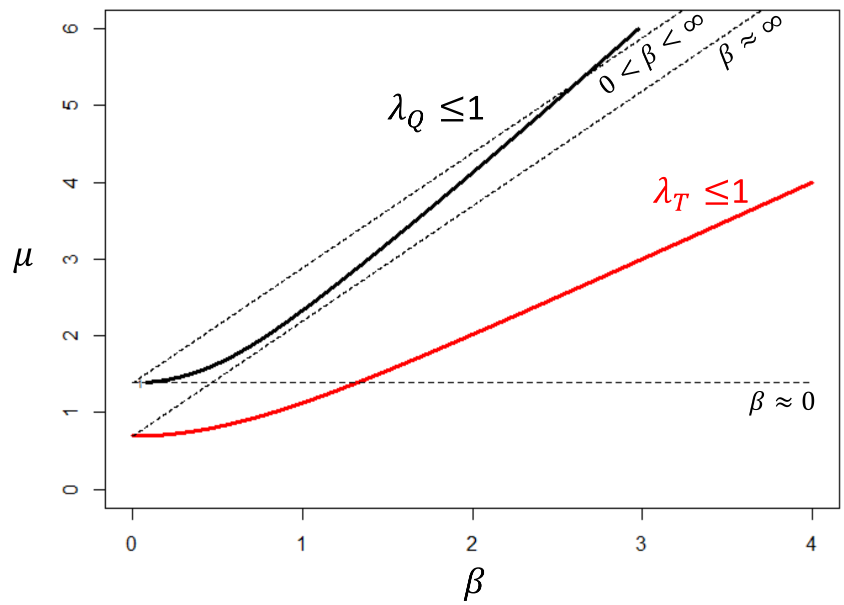


Figure 5.3: $\lambda_Q = \lambda$ and λ_T are the maximal eigenvalues of the matrix Q and a related matrix T respectively (see Appendix 5.A). The area above the black curve is where the condition (5.34) holds true.

- *Next,* (5.35) *implies that* $\mathbf{K}$ *is a Hilbert–Schmidt operator and* $\mathbf{K}^2$ *is a trace class operator in* $\ell^2_{\mathsf{T-C}}$.
- *The last fact, together with the property of positivity* (5.30)*, allow us to use the Krein–Rutman theory, deriving all assertions of Theorems 5.4.1 and 5.4.2.*

On the other hand, if (5.34) *fails (and therefore* (5.35) *fails), it does not necessarily mean that the assertions Theorems 5.4.1 and 5.4.2 fail. In other words, we do not claim that the boundary of the domain of parameters* β *and* μ *where the model exhibits uncritical behavior is given by Eqn.* (5.34)*. Moreover, Figure 5.3 shows the result of a numerical calculation indicating that the condition* (5.34) *is worse than* (5.25) *for (moderately) large values of* β*.*

An apparent condition closer to necessity is the pair of inequalities (5.28) *for some (possibly) large* r*. This issue needs a further study.*

Remark 5.5.2. *Physical considerations suggest that the critical curve in the* (β, g) *quarter-plane would have some predictable patterns of behavior: as a function of* β*, it would decay and exhibit a first-order singularity at a unique point* $\beta = \beta_{\mathsf{cr}} \in (0, \infty)$*.*

A plausible conjecture is that the boundary of the critical domain coincides with the locus of points (β, μ) where $\mathbf{\Lambda}$ looses either the property of positivity or the property of being a simple eigenvalue. This direction also requires further research.

5.A | Proof of Lemma 5.4.1.

By definition the trace (5.35) we need to calculate the series

$$\begin{aligned}\mathrm{tr}(\mathbf{K}^{\mathsf{T}}\mathbf{K}) &= \sum_{(\mathbf{t},\boldsymbol{\sigma})} \mathbf{K}^{\mathsf{T}}\mathbf{K}((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t},\boldsymbol{\sigma}))\\ &= \sum_{(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')} K((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}'))K((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}'))\\ &= \sum_{(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')} K^2((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')).\end{aligned} \tag{5.37}$$

A single-strip triangulation $\mathbf{t}$ consists of up- and down-triangles. Accordingly, it is convenient to employ new labels for spins: if a triangle $t(l)$ is an lth up-triangle then we denote it by t^l_{up}; the corresponding spin $\sigma(j)$ will be denoted by σ^l_{up}. Similarly, if $t(j)$ is an lth down-triangle then we denote it by t^l_{do}; the spin $\sigma(j)$ will be denoted by σ^l_{do}. Consequently, the triangulation $\mathbf{t}$ and its supported spin-configuration $\boldsymbol{\sigma}$ are represented as

$$\mathbf{t} := (\mathbf{t}_{\text{up}}, \mathbf{t}_{\text{do}}) \text{ and } \boldsymbol{\sigma} := (\boldsymbol{\sigma}_{\text{up}}, \boldsymbol{\sigma}_{\text{do}}).$$

Here

$$\mathbf{t}_{\text{up}} = (t^1_{\text{up}}, \dots, t^n_{\text{up}}), \quad \mathbf{t}_{\text{do}} = (t^1_{\text{do}}, \dots, t^m_{\text{do}}),$$

and

$$\boldsymbol{\sigma}_{\text{up}} = (\sigma^1_{\text{up}}, \dots, \sigma^n_{\text{up}}), \quad \boldsymbol{\sigma}_{\text{do}} = (\sigma^1_{\text{do}}, \dots, \sigma^m_{\text{do}}),$$

assuming that the supporting single-strip triangulation $\mathbf{t}$ contains n up-triangles and m down-triangles. (The actual order of up- and down-triangles and supported spins does not matter.)

The same can be done for the pair $(\mathbf{t}', \boldsymbol{\sigma}')$ as illustrated in (5.37). Let recall that the triangulations $\mathbf{t}$ and $\mathbf{t}'$ are consistent ($\mathbf{t} \sim \mathbf{t}'$) iff number of the down-triangles in $\mathbf{t}$ equals that of up-triangles in $\mathbf{t}'$.

To calculate the sum (5.37) we divide the summation over $(\mathbf{t}', \boldsymbol{\sigma}')$ into a summation over $(\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}})$ and $(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})$. Firstly, fix a pair $(\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}})$ and make the sum over

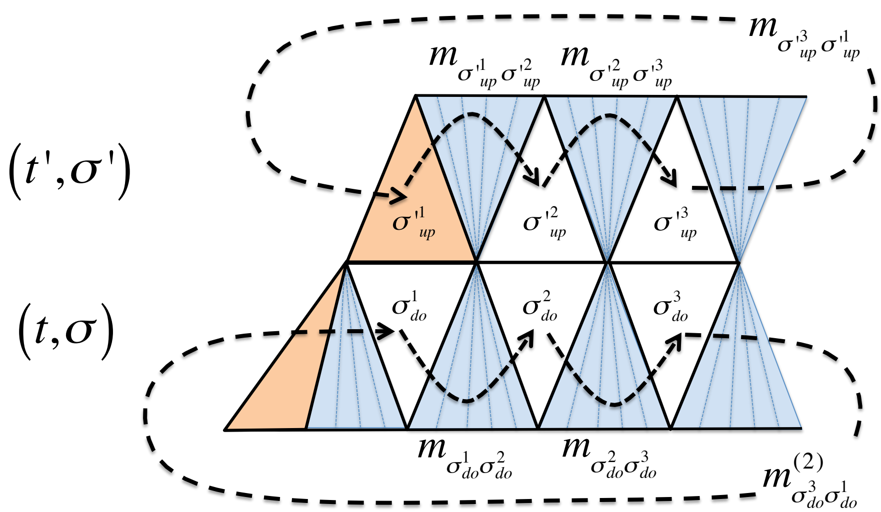


Figure 5.4: Interaction betwen the elements z_i, η_i

$(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})$. Note that the term $V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))$ depends only on $\boldsymbol{\sigma}_{\text{do}}$ and $\boldsymbol{\sigma}'_{\text{up}}$. Consequently,

$$\begin{aligned}&\sum_{\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) \\ &= e^{-\beta H(\boldsymbol{\sigma})} e^{-2\beta V((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}'))} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')}.\end{aligned} \tag{5.38}$$

The sum in the right-hand side of (5.38) can be represented in a matrix form. Denote by $e_{\pm 1}$ the standard spin-$1/2$ unit vectors in $\mathbb{R}^2$:

$$e_{+1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Next, let us introduce a 2×2 matrix T where

$$T = e^{-\mu} \begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix} := \begin{pmatrix} t_{++} & t_{+-} \\ t_{-+} & t_{--} \end{pmatrix}. \tag{5.39}$$

Denote by $n(i)$, $i = 1, \dots, n_{up}(\mathbf{t}')$ the number of down-triangles in $\mathbf{t}'$ which are between the ith and $(i+1)$th up-triangles in $\mathbf{t}'$. Let $n_{up}(\mathbf{t}') = k$ then

$$\sum_{\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}}} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} = \sum_{n(i)\geq 0:\ \sum_i n(i)\geq 1} \prod_{l=1}^{k} \left(e^{\mathsf{T}}_{\sigma'^{l}_{\text{up}}} T^{n(l)+1} e_{\sigma'^{l+1}_{\text{up}}} \right)$$
$$= \prod_{l=1}^{k} \left(e^{\mathsf{T}}_{\sigma'^{l}_{\text{up}}} M e_{\sigma'^{l+1}_{\text{up}}} \right) - \prod_{l=1}^{k} \left(e^{\mathsf{T}}_{\sigma'^{l}_{\text{up}}} T e_{\sigma'^{l+1}_{\text{up}}} \right) \tag{5.40}$$

where the matrix M is the sum of the geometric progression

$$M = \sum_{n=1}^{\infty} T^n := \begin{pmatrix} m_{++} & m_{+-} \\ m_{-+} & m_{--} \end{pmatrix}. \tag{5.41}$$

Using the same procedure we can obtain the sum over all up-triangles into the triangulation $\mathbf{t}$. The only difference is the existence of marked up-triangle in the strip: let as before $n_{up}(\mathbf{t}') = n_{do}(\mathbf{t}) = k$ then

$$\sum_{\mathbf{t}_{\text{up}}, \boldsymbol{\sigma}_{\text{up}}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} = \prod_{l=1}^{k-1} \left(e^{\mathsf{T}}_{\sigma^{l}_{\text{up}}} M e_{\sigma^{l+1}_{\text{up}}} \right) \left(e^{\mathsf{T}}_{\sigma^{k}_{\text{up}}} M^2 e_{\sigma^{1}_{\text{up}}} \right) \tag{5.42}$$

See Figure 5.4 for illustration of these calculations (5.40) and (5.42). Further, supposing the existence of the matrix M and using (5.40) and (5.42) we obtain the following:

$$\begin{aligned}
&\sum_{\mathbf{t}_{\text{up}}, \boldsymbol{\sigma}_{\text{up}}} \sum_{\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}}} K^2((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{t}', \boldsymbol{\sigma}')) = e^{-2\beta V((\mathbf{t}_{\text{do}}, \boldsymbol{\sigma}_{\text{do}}), (\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}}))} \\
&\quad \times \sum_{\mathbf{t}_{\text{up}}, \boldsymbol{\sigma}_{\text{up}}} e^{-\beta H(\boldsymbol{\sigma})} e^{-\mu n(\mathbf{t})} \sum_{(\mathbf{t}'_{\text{do}}, \boldsymbol{\sigma}'_{\text{do}})} e^{-\beta H(\boldsymbol{\sigma}')} e^{-\mu n(\mathbf{t}')} \\
&= e^{-2\beta V((\mathbf{t}_{\text{do}}, \boldsymbol{\sigma}_{\text{do}}), (\mathbf{t}'_{\text{up}}, \boldsymbol{\sigma}'_{\text{up}}))} \\
&\times \Bigg[\prod_{l=1}^{k} \left(e^{\mathsf{T}}_{\sigma'^{l}_{\text{up}}} M e_{\sigma'^{l+1}_{\text{up}}} \right) \prod_{l=1}^{k-1} \left(e^{\mathsf{T}}_{\sigma^{l}_{\text{do}}} M e_{\sigma^{l+1}_{\text{do}}} \right) \left(e^{\mathsf{T}}_{\sigma^{k}_{\text{up}}} M^2 e_{\sigma^{1}_{\text{up}}} \right) \\
&\quad - \prod_{l=1}^{k} \left(e^{\mathsf{T}}_{\sigma'^{l}_{\text{up}}} T e_{\sigma'^{l+1}_{\text{up}}} \right) \prod_{l=1}^{k-1} \left(e^{\mathsf{T}}_{\sigma^{l}_{\text{do}}} M e_{\sigma^{l+1}_{\text{do}}} \right) \left(e^{\mathsf{T}}_{\sigma^{k}_{\text{up}}} M^2 e_{\sigma^{1}_{\text{up}}} \right) \Bigg].
\end{aligned} \tag{5.43}$$

Necessary and sufficient condition for the convergence of the matrix series for M is that the maximal eigenvalue of matrix T is less then 1. The eigenvalues of T are

$$\lambda_{\pm} = e^{(\beta-\mu)} \pm e^{-(\beta+\mu)}, \tag{5.44}$$

and the above condition means that $\lambda_+ < 1$ or, equivalently,

$$\mu > \ln\bigl(2\cosh(\beta)\bigr). \tag{5.45}$$

Under this condition (5.45), the matrix M is calculated explicitly:

$$\begin{aligned} M = & \frac{e^{(\beta-\mu)}}{e^{2\beta}(1-e^{(\beta-\mu)})^2 - e^{-2\mu}} \\ & \times \begin{pmatrix} e^{2\beta} + (1-e^{4\beta})e^{-(\beta+\mu)} & 1 \\ 1 & e^{2\beta} + (1-e^{4\beta})e^{-(\beta+\mu)} \end{pmatrix} . \end{aligned} \tag{5.46}$$

We are now in a position to calculate the sum in (5.37). To this end, we again represent it through the product of transfer matrices. Pictorially, we express the above sum as the partition function of a one-dimensional Ising-type model where states are pairs of spins $(\sigma^l_{\text{do}}, \sigma^l_{\text{up}})$ and the interaction is via the matrix T between the members of the pair and via matrix M between neighboring pairs. More precisely, define the following 4×4 matrices:

$$Q = \begin{pmatrix} e^{2\beta} m_{++} m_{++} & m_{++} m_{+-} & m_{+-} m_{++} & e^{2\beta} m_{+-} m_{+-} \\ m_{++} m_{-+} & e^{-2\beta} m_{++} m_{--} & e^{-2\beta} m_{+-} m_{-+} & m_{+-} m_{--} \\ m_{-+} m_{++} & e^{-2\beta} m_{-+} m_{+-} & e^{-2\beta} m_{--} m_{++} & m_{--} m_{+-} \\ e^{2\beta} m_{-+} m_{-+} & m_{-+} m_{--} & m_{--} m_{-+} & e^{2\beta} m_{--} m_{--} \end{pmatrix} \tag{5.47}$$

$$Q_m = \begin{pmatrix} e^{2\beta} m_{++} m^{(2)}_{++} & m_{++} m^{(2)}_{+-} & m_{+-} m^{(2)}_{++} & e^{2\beta} m_{+-} m^{(2)}_{+-} \\ m_{++} m^{(2)}_{-+} & e^{-2\beta} m_{++} m^{(2)}_{++} & e^{-2\beta} m_{+-} m^{(2)}_{-+} & m_{+-} m^{(2)}_{--} \\ m_{-+} m^{(2)}_{++} & e^{-2\beta} m_{-+} m^{(2)}_{+-} & e^{-2\beta} m_{--} m^{(2)}_{++} & m_{++} m^{(2)}_{++} \\ e^{2\beta} m_{-+} m^{(2)}_{-+} & m_{-+} m^{(2)}_{--} & m_{--} m^{(2)}_{-+} & e^{2\beta} m_{--} m^{(2)}_{--} \end{pmatrix} \tag{5.48}$$

$$Q_t = \begin{pmatrix} e^{2\beta} t_{++} m_{++} & t_{++} m_{+-} & t_{+-} m_{++} & e^{2\beta} t_{+-} m_{+-} \\ t_{++} m_{-+} & e^{-2\beta} t_{++} m_{--} & e^{-2\beta} t_{+-} m_{-+} & t_{+-} m_{--} \\ t_{-+} m_{++} & e^{-2\beta} t_{-+} m_{+-} & e^{-2\beta} t_{--} m_{++} & t_{--} m_{+-} \\ e^{2\beta} t_{-+} m_{-+} & t_{-+} m_{--} & t_{--} m_{-+} & e^{2\beta} t_{--} m_{--} \end{pmatrix} \tag{5.49}$$

$$Q_{tm} = \begin{pmatrix} e^{2\beta} t_{++} m^{(2)}_{++} & t_{++} m^{(2)}_{+-} & t_{+-} m^{(2)}_{++} & e^{2\beta} t_{+-} m^{(2)}_{+-} \\ t_{++} m^{(2)}_{-+} & e^{-2\beta} t_{++} m^{(2)}_{++} & e^{-2\beta} t_{+-} m^{(2)}_{-+} & t_{+-} m^{(2)}_{--} \\ t_{-+} m^{(2)}_{++} & e^{-2\beta} t_{-+} m^{(2)}_{+-} & e^{-2\beta} t_{--} m^{(2)}_{++} & t_{++} m^{(2)}_{++} \\ e^{2\beta} t_{-+} m^{(2)}_{-+} & t_{-+} m^{(2)}_{--} & t_{--} m^{(2)}_{-+} & e^{2\beta} t_{--} m^{(2)}_{--} \end{pmatrix} \tag{5.50}$$

where $m_{ij}, m^{(2)}_{ij}$ and $t_{i,j}$ $(i, j \in \{-, +\})$ are elements of the matrices M, M^2, and T respectively.

Now for the sum under consideration (5.37) we obtain using representation (5.43)

$$
\begin{aligned}
\sum_{(\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')} K^2((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{t}',\boldsymbol{\sigma}')) = & \sum_{(\mathbf{t}_{\text{do}},\boldsymbol{\sigma}_{\text{do}}),(\mathbf{t}'_{\text{up}},\boldsymbol{\sigma}'_{\text{up}})} e^{-2\beta V((\mathbf{t}_{\text{do}},\boldsymbol{\sigma}_{\text{do}}),(\mathbf{t}'_{\text{up}},\boldsymbol{\sigma}'_{\text{up}}))} \\
& \times \left[\prod_{l=1}^{k}\left(e^{\mathsf{T}}_{{\sigma'}^{l}_{\text{up}}} M e_{{\sigma'}^{l+1}_{\text{up}}}\right)\prod_{l=1}^{k-1}\left(e^{\mathsf{T}}_{\sigma^{l}_{\text{do}}} M e_{\sigma^{l+1}_{\text{do}}}\right)\left(e^{\mathsf{T}}_{\sigma^{k}_{\text{up}}} M^2 e_{\sigma^{1}_{\text{up}}}\right)\right. \\
& \left. - \prod_{l=1}^{k}\left(e^{\mathsf{T}}_{{\sigma'}^{l}_{\text{up}}} T e_{{\sigma'}^{l+1}_{\text{up}}}\right)\prod_{l=1}^{k-1}\left(e^{\mathsf{T}}_{\sigma^{l}_{\text{do}}} M e_{\sigma^{l+1}_{\text{do}}}\right)\left(e^{\mathsf{T}}_{\sigma^{k}_{\text{up}}} M^2 e_{\sigma^{1}_{\text{up}}}\right)\right] \\
= \operatorname{tr}\Big(\Big(\sum_{k=0}^{\infty} Q^k\Big) Q_m\Big) - \operatorname{tr}\Big(\Big(\sum_{k=1}^{\infty} Q_t^k\Big) Q_{tm}\Big). &
\end{aligned}
\tag{5.51}
$$

By the construction the matrix Q is greater then Q_t elementwise. Thus the eigenvalue of matrix Q is greater than the eigenvalue of the matrix Q_t (it follows from the Perron-Frobenius theorem). Therefore the necessary and sufficient condition for the convergence in (5.37) is that the largest eigenvalue of Q is less than 1. It is possible to calculate its eigenvalue analytically. In order to express the eigenvalues of Q it is convinient to use notations (5.32) and (5.33). In this notations the matrix M, i.e. (5.46), is represented as following

$$M = c\begin{pmatrix} m & 1 \\ 1 & m \end{pmatrix}.$$

The equations for the eigenvalues of Q are:

$$
\begin{aligned}
\lambda_1 &= c^2 e^{\beta}(m^2-1) \\
\lambda_2 &= c^2 e^{-\beta}(m^2-1) \\
\lambda_3 &= c^2(m^2+1)(\cosh\beta)\left(1-\sqrt{1-\frac{(m^2-1)^2}{(\cosh\beta)^2(m^2+1)^2}}\right) \\
\lambda_4 &= c^2(m^2+1)(\cosh\beta)\left(1+\sqrt{1-\frac{(m^2-1)^2}{(\cosh\beta)^2(m^2+1)^2}}\right)
\end{aligned}
$$

A straightforward inspection confirms that the largest eigenvalue is given by λ_4. The condition $\lambda_4 < 1$ coincides with (5.37). This completes the proof of Lemma 5.4.1. □

CHAPTER 6

Discussion and outlook

In this thesis we investigated aspects of uniform infinite causal triangulations (UICT) and Gibbs causal triangulations which are probabilistic formulations of so-called causal dynamical triangulation (CDT) models of quantum gravity.

Since causal triangulations are in bijections with planar rooted trees and thus the UICT measure is equivalent to the uniform measure on planar rooted trees - a size-biased critical Galton-Watson process -, we first focused our studies on random trees. In Chapter 2 we investigated the fractal and spectral dimension of trees which have a unique infinite spine. There are many random tree ensembles where in the continuum limit the measure is almost surely condensed on trees which have this property; examples include generic trees and non-generic critical trees, which are essentially size-biased critical Galton-Watson trees with finite and infinite variance respectively. We obtain an upper and lower bound on the spectral dimension in terms of the fractal dimension and what we call the hull dimension. In the case where the outgrowths along the spine are $i.i.d.$ and the outgrowths die out fast enough we obtain simple expressions for the fractal and spectral dimension. We apply these methods to generic and non-generic critical trees as well as a model of randomly grown trees.

In Chapter 3 we discuss in detail the relation between the UICT and size-biased critical Galton-Watson processes. Besides a detailed proof of the existence of the measure we provide results on the convergence of the rescaled boundary length to a diffusion process. Furthermore, it is shown that joint rescaled length-area processe converges to a stochastic integral over the length process. This relation enables us to extract from the Feynman-Kac procedure what is known in the physics literature as the effective quantum Hamiltonian. This provides a mathematically rigorous formulation of certain scaling limits of CDT and provides us with a first connection between previously disconnected works on CDT in the corresponding physics and probability literature.

Chapter 4 is devoted to the investigation of a growth process which samples sections of UICT by two elementary moves in which a single triangle is added with certain probabilities. We show a correspondence of the growth process to a Markov chain $\{M_n\}_{n\geq 0}$ on the positive integers. Using the properties of this Markov chain we estimate the geodesic distance t_n to the initial boundary of a triangle added at after a certain growth time n. This relation is used to prove that the fractal dimension is almost surely $d_h = 2$ in an alternative manner to the corresponding branching process picture. Furthermore, it is shown that the rescaled length process as a function of growth time also converges to a diffusion process. This process can be related to the corresponding length process as a function of geodesic distance via a random time change. This shows a certain duality relation between the growth process and the branching process: in the former, area is fixed (this is simply growth time), while geodesic distance is estimated, whereas in the latter geodesic distance is fixed (this is the generation number) while area is estimated. This duality relation provides us with a rigorous formulation of the so-called peeling procedure for CDT.

In Chapter 5 we discuss a canonical formulation of the causal triangulations ensemble through introduction of a Gibbs measure with Boltzmann factor $e^{-\mu|T|}$, where $|T|$ is the size of the triangulation and μ the cosmological constant. We introduce the transfer matrix formalism and using Krein-Rutman theory, we show the convergence of the free energy to the largest eigenvalue of the transfer matrix. This determines the limiting measure which at the critical value μ_c for the cosmological constant corresponds to UICT. We extend this analysis to the Ising model coupled to causal triangulations. Using Krein-Rutman theory, we determine the regions in the parameter space for which the partition function converges. This is a first step towards determining the critical line and thus towards being able to analyse the phase transition of the model.

In the final sections of chapters 2 - 5 we have already discussed possible extensions of the here developed results. Most interestingly, are, firstly, further application of the results on the spectral dimension of random tree ensembles with a unique infinite spine to obtain a proof for the full parameter range of the attachment and grafting model as well as other models of randomly grown trees such as Ford's α-model [61] and its generalisation, the $\alpha\gamma$–model [76]. Furthermore, there so far only exists an upper bound on the spectral dimension of CDT, $d_s \leq 2$. It would thus also be interesting to extend the previous results to prove that this bound is tight and one has almost surely $d_s = 2$. Secondly, it would be interesting to extend our results on the growth process to generalise previous work by Angel [14] on a growth process for uniform infinite planar triangulations (UIPT) to show convergence of the boundary process to a Lévy process, as well as to extend them to so-called multi-critical models of DT [53, 92] and CDT [93, 94]. These lines of research are currently under investigation.

Another highly interesting direction which we currently follow in collaboration with Stefánsson is the formulation of the scaling limit of CDT in an analogous manner to the Brownian map as described above, in Section 1.2. On the one hand, as we have seen, causal triangulations are in bijection with planar rooted trees and thus the labelling process which enters the formulation of the Brownian map is trivial. This simplifies the formulation of the scaling limit greatly. On the other hand, due to the time-sliced structure of the causal triangulations it is harder to estimate the volume of balls around a randomly chosen point of the finite causal triangulation. Once the convergence to the scaling limit of the causal triangulation is shown, we expect that several properties of the scaling limit, such as its Hausdorff dimension and topology should follow from simple properties of the Brownian excursion.

In summary, we hope to have convinced the reader that probabilistic aspects of CDT and in particular the UICT as well as Gibbs causal triangulations are interesting models to be investigated. These formulations can be seen as interesting probabilistic models on their own right, but are also of essential importance for the physical understanding of CDT, most notably, in context of the study of the spectral dimension of CDT.

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Samenvatting

In deze dissertatie onderzoeken wij aspecten van uniforme oneindige causale triangulaties (UICT) en Gibbs causale triangulaties. Beide zijn kansformuleringen van zogenaamde causale dynamische triangulatiemodellen (CDT) voor kwantumgravitatie.

Omdat causale triangulaties in bijectie zijn met planaire gewortelde bomen, is de UICT maat equivalent met de uniforme maat voor planaire gewortelde bomen. Dit wordt ook wel een grootte geconditioneerd kritisch Galton-Watson proces[1] genoemd. Als eerste focussen wij onze aandacht op kansbomen. In hoofdstuk 2 onderzoeken wij de fractale en spectrale dimensie van bomen met een uniek oneindig pad - de *spine*. Er zijn vele kansboomensembles waar in de continuümlimiet de maat vrijwel zeker condenseert op bomen met deze eigenschap. Voorbeelden zijn generieke kritische bomen en niet generieke kritische bomen. Dit zijn eigenlijk grootte geconditioneerde kritische Galton-Watson bomen met eindige en oneindige variantie. Wij verkrijgen een boven en beneden grens voor de spectrale dimensie in termen van de fractale dimensie en wat wij noemen de *hull* dimensie. In het geval waar de uitgroeiingen vanuit de spine onafhankelijk en identiek gedistribueerd ($i.i.d.$) zijn en als de uitgroeiingen snel genoeg afvallen verkrijgen wij een simpele expressie voor de fractale en spectrale dimensie. Wij passen deze methoden toe op generieke en niet generieke kritische bomen en ook op modellen van willekeurig groeiende bomen.

In hoofdstuk 3 behandelen wij in detail de relatie tussen de UICT maat en de grootte geconditioneerd kritisch Galton-Watson proces maat. Behalve een gedetailleerd bewijs van het bestaan van de maat geven wij resultaten over de convergentie van de herschaalde randlengte naar een diffusieproces. Verder laten wij zien dat het stochastische proces van de herschaalde oppervlakte convergeert naar een stochastische in-

[1]Grootte conditionering betekent dat de grote van de boom vast wordt genomen, en gelijk aan N, waarbij men de limiet neemt $N \to \infty$.

tegraal over het lengteproces. Deze relatie stelt ons in staat om de Feynman-Kac procedure te gebruiken, dit staat in de natuurkundeliteratuur als de effectieve Kwantumhamiltoniaan. Dit geeft een wiskundig precieze formulering van bepaalde schalingslimieten van CDT en geeft een connectie met eerdere publicaties in het gebied van CDT in de natuur- en wiskundeliteratuur.

Hoofdstuk 4 is gewijd aan de bestudering van een groeiproces voor UICT waarbij er twee elementaire bewerkingen zijn waarbij een driehoek wordt toegevoegd met een bepaalde waarschijnlijkheid. Wij laten een overeenkomst zijn met het groeiproces van een Markov ketting $\{M_n\}_{n\geq 0}$ voor de positieve gehele getallen. Gebruikmakende van de eigenschappen van deze Markov ketting schatten wij de geodetische afstand t_n van een driehoek tot de initiële rand na een bepaalde groeitijd n. Deze relatie gebruiken wij om te bewijzen dat de fractale dimensie vrijwel zeker gelijk is aan twee, $d_h = 2$. Wij gebruiken dus het groeiproces een plaats van het vertakkingsprocesbeeld voor een alternatief bewijs. Verder laten wij zien dat het stochastische proces van de herschaalde lengte ook convergeert naar een diffusieproces. Dit proces kan gerelateerd worden aan de geodetische afstand via een willekeurige tijdsverandering. Dit laat zien dat er een dualiteitsrelatie bestaat tussen het groeiproces en het vertakkingsproces. Bij het groeiproces is de oppervlakte vast en de geodetische afstand geschat, terwijl bij de vertakkingsprocesbeeld de geodetische afstand vast is terwijl de oppervlakte wordt geschat. Deze dualiteitsrelatie geeft een precieze formulering van het zogenoemde afpelproces voor CDT.

In hoofdstuk 5 behandelen wij een canonieke formulering van het causale triangulatieensemble door introductie van een Gibbs maat met Boltzmann gewicht $e^{-\mu|T|}$, waarbij $|T|$ de grootte is van de triangulatie en waarbij μ de kosmologische constante is. Wij introduceren het overgangsmatrixformalisme en gebruikmakende van Krein-Rutman theorie laten wij zien dat de vrije energie convergeert naar de grootste eigenwaarde van de overgangsmatrix. Dit bepaalt de maat in de continuümlimiet waar de kritische waarde μ_c voor de kosmologische constante correspondeert met UICT. Wij veralgemenen deze analyse naar het Ising model gekoppeld aan causale triangulaties. Gebruikmakende van Krein-Rutman theorie bepalen wij de gebieden in de parameteruimte waarbij de partitiefunctie convergeert. Dit is een eerste stap naar het bepalen van de kritische curve en daarmee dus een eerste stap naar een analyse van de faseovergang van het model.

In de laatste paragrafen van hoofdstukken 2 - 5 bespreken wij mogelijke uitbreidingen van de door ons ontwikkelde resultaten. Het meest interessante is wellicht om de spectrale dimensies te bepalen voor de volledige parameterruimte van het *attachment and grafting* model alsmede Ford's α-model [61] en de bijbehorende generalisatie, het $\alpha\gamma$–model [76]. Verder is er tot zover slechts een bovenlimiet voor de spectrale dimensie van CDT, $d_s \leq 2$. Het zou daarom dus interessant zijn als wij de hier behaalde resultaten kunnen gebruiken om te bewijzen dat deze grens strikt is, en dat deze vrijwel zeker gelijk is aan, $d_s = 2$. Ook zou het interessant zijn om onze resultaten voor het groeiproces te gebruiken om eerder werk van Angel [14] over het groeiproces voor uniforme oneindelijke planaire triangulaties (UIPT) uit te breiden naar de convergentie van een Lévy proces. In deze context is het ook interessant om dit door te trekken naar zogenoemde multikritische modellen van DT [53, 92] en CDT [93, 94]. Dit zijn op het moment actieve onderzoeksgebieden.

Een andere zeer interessante onderzoeksrichting met Stefánsson is om de schalingslimiet van CDT te formuleren als een variant van een Brownse afbeelding, zie paragraaf 1.2. Wij hebben gezien dat causale triangulaties in bijectie zijn met planaire gewortelde bomen en dat daarbij het labelingproces triviaal is. Dit versimpelt de formulering van de schalingslimiet aanzienlijk. Aan de andere kant is het door de gelaagde structuur van de causale triangulaties moeilijker om het volume van ballen rondom een willekeurig gekozen punt van de triangulatie te bepalen. Gegeven dat de convergentie van de schalingslimiet van de causale triangulatie bewezen is, verwachten wij dat verschillende eigenschappen van de schalingslimiet volgen uit simpele eigenschappen van de Brownse excursie.

Samenvattend, hopen wij de lezer overtuigd te hebben dat kansaspecten van CDT, specifiek de UICT en Gibbs causale triangulaties, interessant zijn om te onderzoeken. Deze formuleringen kunnen gezien worden als kansmodellen op zichzelf, maar zijn ook van essentieel belang voor de het fysische begrip van CDT. Hierbij is wellicht de bestudering van de spectrale dimensie het meest belangrijk.

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Curriculum Vitae

Stefan Zohren was born in Mönchengladbach, Germany on Dec 17, 1980. From 1986 to 1990 he attended the primary school in Birgelen and from 1990 to 2000 the Cusanus Gymnasium grammar school in Erkelenz, followed by civil service at the German Red Cross with training as a rescue medic.

From 2001 to 2005 Stefan studied physics with a mathematics extension at the RWTH Aachen University of Technology and Utrecht University with exchanges to Queen Mary, University of London and Universidad Autónoma de Madrid. He graduated as Dipl.-Phys. (cum laude) from RWTH Aachen, MSc. in theoretical physics (cum laude) and MSc. in mathematical sciences (cum laude) from Utrecht University. For his diploma thesis he received the Sprigorum medal and the Schöneborn Prize of the RWTH Aachen. During the time of his undergraduate studies Stefan worked as system administrator at the Department of Economics, RWTH Aachen, and teaching assistant at the Department of Mathematics and the Department of Theoretical Physics, RWTH Aachen, as well as the Department of Physics at Queen Mary, University of London.

From 2005 to 2008 Stefan was employed as an early stage researcher of the European Research Network on Random Geometries and Random Matrices at the Blackett Laboratory, Imperial College London following his PhD studies in theoretical physics during which he was a JSPS Fellow at Tokyo Institute of Technology and Ochanomizu University from October to December 2007. He finished his PhD at Imperial College, University of London and Diploma of Membership of Imperial College in 2008. On basis of his thesis he was awarded an affiliation as Honorary Research Associate at the Department of Physics at Imperial College from 2009 to 2012.

From 2008 to beginning of 2011 Stefan was employed as Research Assistant at the Mathematical Institute at the University of Leiden where he did postdoctoral research in physics and started to work on a doctoral degree in mathematics. He had several teaching duties in probability and statistics. During his time in Leiden he was a Visiting Postdoctoral Researcher, as well as Visiting Professor at the Department of Statistics at São Paulo University and the Brazilian National Institute for Pure and Applied Mathematics. Further, he was Visiting Scholar at Corpus Christi College at the University of Oxford for Michaelmas term 2010.

From beginning of 2011 to 2012 Stefan worked as a Postdoctoral Research Assistant at the Rudolf Peierls Centre for Theoretical Physics at the University of Oxford and from mid 2011 for two terms as a Lecturer at Christ Church, University of Oxford.

Since 2012, Stefan holds a permanent position as Assistant Professor in Physics at the Pontifica Universidade Católica in Rio de Janeiro. He participates in a Thematic Grant on Probabilistic Aspects of Causal Triangulations with Suhov, Yambartsev and Perchersky at São Paulo University, financed by FAPESP. Further, he recently received several Brazilian grants and was elected Visiting Fellow at Mansfield College, University of Oxford for Hilary Term 2013.