Dynamical Gibbs-non-Gibbs transitions and Brownian percolation

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Preface

This thesis deals with two different models in two different contexts.

The first part deals with dynamical Gibbs-non-Gibbs transitions. Gibbs measures are mathematical objects to describe the equilibrium states of a system consisting of a large number of components (for instance, particles with a spin state -1 or $+1$) that interact with each other. As a simplification, particles are placed into a discrete structure, namely, the lattice $\mathbb{Z}^d$ (the particles interact locally) or the complete graph with N sites (the particles interact globally). Due to the large number of particles, it is natural to assume that the state of the system is random. Gibbs measures are probability measures on the state space (e.g. $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$) capturing this randomness. This description was introduced by R.L. Dobrushin, O.E. Lanford and D. Ruelle (DLR) in the late 1960s, through the so-called Maxwell-Boltzmann-Gibbs formula, and involves some particular "regularity" conditions for the conditional probabilities with respect to fixed configurations outside finite volumes. The question of interest is whether this "regularity" condition remains valid after the system is subjected to a stochastic dynamics. In other words, consider the probability measure obtained by sampling the initial condition with a Gibbs measure and running a stochastic dynamics during time t. Is it still possible to describe the evolved measure as a Gibbs measure?

The second part deals with stochastic geometry. The relevant information about the particles is their position. Particles may be placed at random in any region of the space, say $\mathbb{R}^d$. Subsequently, each particle is displaced independently of each other according to a $d-$dimensional Brownian Motion during t time, and the trace produced by that motion is recorded. The question of interest is whether the final set obtained from all the traces has an infinite connected component or not. If so, then is it unique?

Part I (Chapters $1-3$) deals with dynamical Gibbs-non-Gibbs transitions and is based on the articles [FdHM13b] and [FdHM13a]. Part II (Chapters $4-5$) deals with percolation of Brownian paths homogeneously distributed in space and is based on the article [EMP13].

Part I - Dynamical Gibbs-non-Gibbs transitions

1 Introduction to Part I

Gibbs measures are central objects in equilibrium statistical mechanics. Heuristically, macroscopic variables in equilibrium like pressure, temperature, energy, etc., are described in a first step by thermodynamics. Gibbs measures capture how local rules for the interaction of the large number of components of the system at a microscopic level lead to global properties at a macroscopic level. Because of the large number of components it is reasonable to expect that the passage from micro to macro occurs via some Law of Large Numbers. The microscopic laws are used to explain macroscopic features such as the existence of two or more phases of the same substance (vapor, liquid, solid). The phenomenon that the same local rules may lead to different global properties is called phase transition.

Since the Gibbsian formalism was designed for systems in equilibrium it may very well fail whenever this equilibrium is altered, for instance, by subjecting the system to stochastic dynamics. This is one way in which the concept of non-Gibbsianness appears.

In Section 1 we present the general Gibbs formalism. Section 1.1 deals with Gibbs measures, Section 1.2 with non-Gibbs measures. Section 2 lists different approaches to prove non-Gibbsianness and mentions different attempts to overcome the difficulties associated with it.

1.1 Background

The main purpose of this section is to provide a general framework for Gibbs measures without going into technical details. We refer the reader to [Geo11] for a more detailed exposition. The key reference for Gibbs-non-Gibbs is [vEFS93], which also provides physical intuition. The overview papers [Fer06] and [LN08b] summarize the main ideas behind non-Gibbs measures as exposed in [vEFS93].

1.1.1 Lattice models: Generalities on Gibbs measures

Although most of our research focusses on mean-field and Kac-type models, for the sake of exposition we introduce the concept of Gibbs measure in the lattice case, say $\mathbb{Z}^d$.

A spin can be either $+1$ ("up ") or -1 ("down"). $S := \{-1, +1\}$ denotes the single-spin space. More general state spaces can be treated, but we keep the exposition simple.

$\Omega := S^{\mathbb{Z}^d}$ denotes the configuration space. A configuration in Ω is denoted by $\sigma = (\sigma_x)_{x \in \mathbb{Z}^d}$, and its restriction to $\Lambda \subseteq \mathbb{Z}^d$ by σ_Λ. Ω is endowed with the product topology $\mathcal{F}$, in which $\sigma^n \xrightarrow[n\to\infty]{} \sigma$ stands for pointwise convergence in finite sets, i.e., for each $\Lambda \Subset \mathbb{Z}^d$, $\exists\, n_0 \in \mathbb{N}$ such that $\sigma^n_\Lambda \equiv \sigma_\Lambda \;\forall\, n \geq n_0$.

Here, $\Subset$ stands for finite subset, $\sigma_\Lambda \equiv \eta_\Lambda$ for $\sigma_x = \eta_x \ \forall\ x \in \Lambda$. In other words, two configurations σ, η are close in the product topology if they coincide in a large (but finite) region. The larger the region, the closer they are. $\mathcal{F}_\Lambda$ stands for the sub-sigma field of $\mathcal{F}$ corresponding to the events that depend on σ_Λ only.

Observation: If μ is a probability measure on Ω, then $\mu(\cdot|\mathcal{F}^c_\Lambda)(\omega)$ has a version $\pi^\mu_\Lambda(\cdot|\omega)$ that is a probability kernel, i.e.,

- for all $\omega \in \Omega$, $\pi^\mu_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F})$,
- for all $A \in \mathcal{F}$, $\pi^\mu_\Lambda(A|\cdot)$ is $\mathcal{F}$-measurable.

Observables of the system are represented by the space of real-valued bounded measurable functions on Ω. We are interested in those functions that depend weakly on distant spins. More precisely, $f : \Omega \to \mathbb{R}$ is said to be *quasilocal at σ* if

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \sup_{\omega,\eta\in\Omega} |f(\sigma_\Lambda \omega_{\Lambda^c}) - f(\sigma_\Lambda \eta_{\Lambda^c})| = 0, \tag{1.1}$$

and is said to be *quasilocal* if it is quasilocal at σ for all $\sigma \in \Omega$. Here $\sigma_\Lambda\omega_{\Lambda^c}$ is the spin configuration that equals σ in Λ and ω in $\Lambda^c = \mathbb{Z}^d \setminus \Lambda$.

As mentioned before, the goal of statistical mechanics is to describe global properties (macroscopic phenomena) resulting from local rules (microscopic interactions). These local rules are specified through the concept of interaction potential, Hamiltonian and specification, as given in the following definitions.

Definition 1.1.1. *An interaction potential is a family $\Phi = (\Phi_A)_{A\Subset\mathbb{Z}^d}$ of functions $\Phi_A : \Omega \to \mathbb{R}$ such that Φ_A is $\mathcal{F}_A$-measurable. Φ is said to be uniformly absolutely summable if*

$$\sum_{\substack{A\Subset\mathbb{Z}^d \\ A\ni x}} \sup_{\omega\in\Omega} |\Phi_A(\omega)| < \infty \quad \forall\ x \in \mathbb{Z}^d. \tag{1.2}$$

Definition 1.1.2. *Let Φ be an interaction potential.*

- *The Hamiltonian for a set $\Lambda \Subset \mathbb{Z}^d$ with external condition ω is the function defined by*
$$H^\Phi_\Lambda(\sigma_\Lambda|\omega_{\Lambda^c}) := \sum_{\substack{A\Subset\mathbb{Z}^d: \\ A\cap\Lambda\neq\varnothing}} \Phi_A(\sigma_\Lambda\omega_{\Lambda^c}) \tag{1.3}$$
for all $\sigma, \omega \in \Omega$ for which the sum exists.
- *The Boltzmann weight for a set $\Lambda \Subset \mathbb{Z}^d$ with external condition ω for an interaction potential Φ is the function defined for by*
$$\gamma^\Phi_\Lambda(\sigma_\Lambda|\omega_{\Lambda^c}) := \frac{e^{-H^\Phi_\Lambda(\sigma_\Lambda|\omega_{\Lambda^c})}}{Z^\Phi_\Lambda(\omega)}, \tag{1.4}$$
where $Z^\Phi_\Lambda(\omega)$ is the normalization constant, called the partition function.

Some important examples of potentials:

1. *Ising model:*

$$\Phi_A(\sigma) = \begin{cases} -J_{x,y}\sigma_x\sigma_y & \text{if } A = \{x,y\}, \\ -h_x\sigma_x & \text{if } A = \{x\}, \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

   where $J : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ is called the pair potential and $h : \mathbb{Z}^d \to \mathbb{R}$ is called the external magnetic field. In the standard Ising model $J_{x,y} \neq 0$ if only if x, y are nearest-neighbours. When both J and h are constant, the model is called homogeneous, otherwise it is called inhomogeneous. When $J \geq 0$, the model is said to be *ferromagnetic*, when $J < 0$ *anti-ferromagnetic*.

2. *Kac-Ising potentials:* Consider in (1.5) the choice

$$J^{\gamma}_{x,y} := \gamma^d \ \mathcal{J}(\gamma|x-y|) \qquad \forall\ x, y \in \mathbb{Z}^d, \tag{1.6}$$

   where $\mathcal{J}(\cdot)$ is a smooth non-negative function supported by the unit ball and normalized as a probability density. This model has the interesting feature of having a long-range interaction, controlled by the parameter γ.

Definition 1.1.3. *A probability measure μ on $(\Omega, \mathcal{F})$ is called Gibbs if there exists an interaction potential Φ satisfying (1.2) such that, for all $\sigma \in \Omega$ and all $\Lambda \Subset \mathbb{Z}^d$,*

$$\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\omega) = \gamma^{\Phi}_{\Lambda}(\sigma_\Lambda|\omega_{\Lambda^c}) \quad \mu \ a.e. \ \omega. \tag{1.7}$$

The equations in (1.7) are called DLR-equations (Dobrushin, Lanford, Ruelle) and the notation is abbreviated as

$$\mu = \gamma^{\Phi}_{\Lambda}\mu \qquad \forall\ \Lambda \Subset \mathbb{Z}^d. \tag{1.8}$$

Remarks:

1. Condition (1.2) is trivially fulfilled when Φ has finite range, i.e., there exists an $R > 0$ such that $\Phi_A = 0$ when $\mathrm{diam}(A) > R$.

2. If Φ satisfies (1.2), then
   - The summability condition in (1.3) holds for all $\sigma, \omega \in \Omega$.
   - The function $\gamma^{\Phi}_{\Lambda}(\cdot|\omega_{\Lambda^c})$ is a probability measure on Ω with support in $\{\sigma \in \Omega\colon\ \sigma_{\Lambda^c} \equiv \omega_{\Lambda^c}\}$.
   - The function $\omega \mapsto \gamma^{\Phi}_{\Lambda}(\sigma_\Lambda|\omega_{\Lambda^c})$ is quasilocal and hence, by (1.7), there is a representation of $\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})$ that is quasilocal.

3. There exists another important way of describing Gibbs measures, namely, via the so-called variational characterization. This approach is restricted to translation invariant probability measures, and it is related to the concepts of *entropy* and *free energy* also present in the second law of thermodynamics. A good summary of this characterization can be found in Chapter 4 of [LN08b] or in Sections 2.5 – 2.6 of [vEFS93].

Definition 1.1.4. *A probability measure* μ *on* Ω *is called*

- *Quasilocal if, for all* $\sigma \in \Omega$ *and* $\Lambda \Subset \mathbb{Z}^d$, $\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})$ *has a quasilocal version.*
- *Uniformly non-null if, for all* $\Lambda \Subset \mathbb{Z}^d$, *there exist constants* $0 < \alpha_\Lambda \leq \beta_\Lambda < \infty$ *such that*
$$\alpha_\Lambda \mathrm{card}(A) \leq \pi^\mu_\Lambda(A|\omega) \leq \beta_\Lambda \mathrm{card}(A) \quad \forall\, A \in \mathcal{F}_\Lambda\ \forall\, \omega \in \Omega.$$

[Koz74] and [Sul73] gave the following characterization of Gibbsianness.

Theorem 1.1.5. *Let* μ *be a probability measure on* $(\Omega, \mathcal{F})$. *Then* μ *is Gibbs if and only if* μ *is quasilocal and uniformly non-null.*

1.1.2 Non-Gibbsianness

As we have seen in Section 1.1.1, lack of quasilocality is one possible expression of non-Gibbsianness. Negation of quasilocality in Definition 1.1.4 means that there exist $\Lambda \Subset \mathbb{Z}^d$ and $\sigma, \omega \in \Omega$ such that *any version of* $\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})$ is not quasilocal at ω. In other words, *there is no zero-measure modification* of $\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\cdot)$ that makes it quasilocal at ω, and hence ω is an *essential discontinuity* of $\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\cdot)$.

We give the following characterization of essential discontinuity at a configuration ω.

Proposition 1.1.6. *A conditional probability measure of* μ *on* Ω *is essentially discontinuous at* ω *if there exist* $\Lambda \Subset \mathbb{Z}^d$, $\sigma \in \Omega$, $\delta > 0$ *such that for all neighborhoods* $\mathcal{N}_\omega$ *of* ω *there exist disjoint subsets* $\mathcal{N}^+, \mathcal{N}^-$ *of* $\mathcal{N}_\omega$ *with* $\mu(\mathcal{N}^\pm) > 0$ *such that*

$$|\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\omega^+) - \mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\omega^-)| > \delta \qquad \forall\, \omega^\pm \in \mathcal{N}^\pm, \tag{1.9}$$

or equivalently, for all $\Delta \supseteq \Lambda$, *there exists* $\Gamma \supseteq \Delta$ *and* $\eta^\pm \in \Omega$ *such that,*

$$\sup_{\xi,\zeta} |\mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\omega_\Delta \eta^+_{\Gamma\setminus\Delta} \xi_{\Gamma^c}) - \mu(\sigma_\Lambda|\mathcal{F}_{\Lambda^c})(\omega_\Delta \eta^-_{\Gamma\setminus\Delta} \zeta_{\Gamma^c})| > \delta. \tag{1.10}$$

In words, the system inside Λ is not properly shielded by ω from the system inside the distant exterior Δ^c. Sometimes we will refer to a discontinuity point ω as a *bad configuration* for μ.

More details about this notion and further characterizations can be found in [Fer06].

Examples

In what follows we list a few examples of transformations of Gibbs measures that are, respectively, are not Gibbs.

1. *Renormalization Group Theory*: Among the first examples of transformations that can turn a Gibbsian probability measure into a non-Gibbssian probability measure are the ones arising in Renormalization Group Theory. At the critical temperature, fluctuations of all orders occur and hence the system does not have a proper scale. Transformations of scale are therefore useful to study the system at the critical

temperature. Block transformations are key examples. The idea is that the lattice is split into blocks, each of which represents a spin on a larger scale. The value of the block spins is decided via some rule that may involve randomness, e.g. the majority rule with coin tossing when there is a tie.

One of the simplest transformations for which Gibbsianness turns out to fail is decimation. In [Isr81, vEFS93] it is proved that if we take for μ the probability measure of the ferromagnetic Ising model on $\mathbb{Z}^2$ with zero magnetic field at low temperature, and we project μ onto the sub-lattice of even sites, then the resulting probability measure is no longer Gibbs.

2. *Stochastic dynamics*: In [vEFdHR02] a new source of non-Gibbsianness was discovered. Again, let μ be the probability measure of the ferromagnetic Ising model on $\mathbb{Z}^d$, $d \geq 2$, at low temperature. By running independent spin-flip dynamics (similar results are obtained for high-temperature Glauber dynamics) Gibbsianness may be lost and may be recovered, depending on the magnetic field.

   In [LNR02] conservation of Gibbsianness for short times was proved for very general initial Gibbs measures and very general local dynamics.

Other references facing similar questions but in other frameworks are:

- Continuous-spins: Continuous-unbounded-spin Gibbs measures for a double-well pair potential subject to independent spin diffusions were studied in [KR06].
- XY models in external fields: XY spins in $\mathbb{Z}^d$ were studied in [vER08]. It was proved that, starting from an initial Gibbs measure and evolving according to an infinite-temperature stochastic dynamics, Gibbsianness can be lost and recovered.

For a more complete and detailed list of references, see the reviews in [dH04], [vERV08] and [vE12].

Restoration program

After Renormalization Group Theory provided the first examples of non-Gibbsianness, a program of "restoration of Gibbsianness" was started in [DS99]. The main goal of this program was to obtain weaker definitions of Gibbsianness that are stable under renormalization transformations. Some of the main works involve the following definitions:

- *Weakly Gibbs* [DS99]: There exists an interaction potential Φ such that equation (1.7) is fulfilled, but a weaker summability condition than (1.2) is required for Φ, namely, $\mu(\Omega^{\Phi}_{\text{sum}}) = 1$ with

$$\Omega^{\Phi}_{\text{sum}} = \left\{ \sigma \in \Omega \colon H^{\Phi}_{\Lambda}(\sigma) := \sum_{\substack{A \Subset \mathbb{Z}^d: \\ A \cap \Lambda \neq \varnothing}} \Phi_A(\sigma) \text{ exists } \forall\, \Lambda \Subset \mathbb{Z}^d \right\}.$$

- *Almost Gibbs* [MRVM99]: The specification of μ is required to be quasilocal at σ for μ-a.e. σ.
- *Intuitively Gibbs* [vEV04]: There exists a set $\tilde{\Omega}$ with $\mu(\tilde{\Omega}) = 1$ such that the specification of μ satisfies (1.1) for all $\sigma \in \tilde{\Omega}$, with the supremum taken over $\omega, \eta \in \tilde{\Omega}$.

Almost Gibbs was shown to imply Weakly Gibbs (see [MRVM99]). The converse is false (see [Lef99]). Weakly Gibbs seems to be too weak as a notion to work with, due to the fact that the Gibbs Variational Principle may fail (see [KLNR04]). There is also an important example in which Almost Gibbs fails (see [Kül01]). Intuitively Gibbs lies between Weakly Gibbs and Almost Gibbs, and seems to be the most promising notion.

1.2 Criteria for proving Gibbs versus non-Gibbs

1. *Phase transition in the hidden variables*:

   The lack of quasilocality at ω' for the transformed measure μ' can be associated with a phase transition in a constrained system. More precisely, consider the two-layer model, which is a probability measure $\bar{\mu}$ on $(\Omega \times \Omega', \mathcal{F} \times \mathcal{F}')$ with marginals μ and μ', where Ω' is the space in which the transformed configuration takes its values. We will refer to (σ, σ') as *internal spins* and *image spins*, respectively. In what follows, all the quantities related to the transformed measure μ' will be labelled with $'$. Recall that we are looking for $\omega' \in \Omega'$ such that (1.10) occurs for μ'.

   Informally, the way in which the spins outside the box Δ' influence the spins inside the box Λ', even when the annulus in between is frozen at ω', is "through the top floor". Formally, we look at the first layer after conditioning on the second layer to be ω' and ask whether there is phase transition. The key idea is that, under suitable conditions on the renormalization transformation, by choosing different spins outside Δ' we can select different phases for the internal spins, which affects the distribution inside Λ, which in turn affects the distribution inside Λ'.

   Let us explain this scenario in more detail. Consider the example of stochastic dynamics. In this case, at least on a formal level, we can write

   $$\bar{\mu}(\sigma, \sigma') = \mu(\sigma)\; p_t(\sigma, \sigma') = e^{-\beta H_\mu(\sigma) + \log p_t(\sigma, \sigma')}, \tag{1.11}$$

   where H_μ is the Hamiltonian of the Ising model and $p_t(\sigma, \sigma')$ is the probability to move in time t from configuration σ to configuration σ'. In the case of independent spin-flips (infinite-temperature dynamics), the expression for p_t becomes simple. Indeed, an easy computation shows that

   $$\bar{\mu}(\sigma, \sigma') = e^{-\beta H_\mu(\sigma) + \sum_x [h_t \sigma'(x)]\sigma(x)}, \tag{1.12}$$

   where $h_t = \frac{1}{2} \log \frac{1+e^{-2t}}{1-e^{-2t}}$. Thus, the constrained model is again an Ising model, but with an inhomogeneous external magnetic field (which also depends on time).

This external magnetic field has two parameters, h_t and σ'. Note that h_t is strictly decreasing in t, with $h_0 = \infty$ and $h_\infty = 0$.

For certain regimes of the parameters the model is well understood. The following is shown in [vEFdHR02]:

- Short time:

  If t is small enough, then the external magnetic field dominates, and forces σ to be aligned with σ'. Hence, no phase transition occurs for the constrained system, and μ' is Gibbs.

- $h > 0$:

  - Large time: If t is large enough, then the external magnetic field is weak, and hence is not able to compete with the initial magnetic field h. Again, no phase transition occurs for the constrained system, and μ' is Gibbs.
  - $d \geq 3, 0 < h \ll 1$, Intermediate time: For intermediate t, h and h_t are of the same order. That makes possible to tune a configuration σ' in such a way that h_t is able to "compensate" the magnetic field h. If the inverse temperature β is big enough phase transition for the constrained system is shown to occur by the means of the Pirogov-Sinai theory. Therefore, μ' is non-Gibbs.
  - $d = 2$, Intermediate time: In this case, phase transition for the constrained system (under suitable choice of the parameters) occurs. However, there is no machinery to prove it for a continuous time interval as in $d \geq 3$.

- $h = 0$, Large time: Again, following a similar argument, if σ' equals the alternating configuration, then σ has a phase transition. The difference in this case is that, due to the absence of an external magnetic field, μ' is non-Gibbs.

Although the two-layer method is useful and is widely used, conceptually it has its weak points, especially in the case of transformations obtained from a stochastic dynamics: (1) It is a *static* explanation, in the sense that the study of the system a time t is done by observing the system at time 0 only. Hence, the *dynamical* features of the phenomenon are lost. (2) It depends on a detailed understanding of the constrained system (in the example, the Ising model with inhomogeneous magnetic field), which is far from easy to analyze.

2. *Dynamical Large Deviation approach*:

   The following questions are relevant:

   - How does the system evolve towards a bad σ'?
   - $\sigma' = \pm$ is an unlikely configuration for any fixed time t. What is the most likely history prior to time t to arrive at σ' at time t? Is there more than one history?

The last question is formulated with the objective to bring a Large Deviation Principle into the picture. This idea was put forward in [vEFdHR10]. The object of attention is the empirical field rather than the configuration itself,

$$\pi^N(\sigma) = \tfrac{1}{(2N)^d} \sum_{x \in [-N,N]^d} \delta_{\tau_x \sigma}, \tag{1.13}$$

with $(\tau_x \sigma)_y = \sigma_{x+y}$. By using the Feng-Kurtz formalism [FK06], dynamical LDPs can be proved for the path of this quantity, starting at time zero with a Gibbs measure μ and running a stochastic dynamics. The rate function $\varphi \mapsto I(\varphi)$ of the dynamical LDP on the space of paths turns out to be the sum of a *static* part and a *dynamic* part, making explicit the role of the two ingredients of the problem. The concept of *bad measure* is linked to the existence of multiple histories for the same present, i.e., ν' is bad if and only if there are two or more solutions of the variational problem $\inf_{\varphi(t)=\nu'} I(\varphi)$. However, the connection between the conditional distributions of μ' and the solutions of the later variational problem has not yet been established in full generality. Another difficulty of this approach is that the variational problems involved are typically hard to analyze.

1.3 Mean-field models

An important simplification that makes our problem more tractable is:

- Replace $\mathbb{Z}^d$ with its nearest-neighbour edges by the complete graph of size N ($N \in \mathbb{N}$), to be denoted by G_N. Think of G_N as taking over the role of a box $\Lambda \Subset \mathbb{Z}^d$ with N sites.
- Make the strength of the total interaction equal between any pair of spins, such that the interaction per site is bounded in N.

In words, *every site is being influenced by any other site in the same way.*

The Ising model on the complete graph is called the Curie-Weiss model. The state space is $\Omega_N := \{-1,+1\}^N$. The Hamiltonian is

$$H_N(\sigma) := -\tfrac{J}{2N} \sum_{i,j=1}^{N} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i, \qquad \sigma \in \Omega_N, \tag{1.14}$$

and the corresponding Gibbs measure on Ω_N is

$$\mu^N(\sigma) := \tfrac{e^{-H_N(\sigma)}}{Z_N}, \tag{1.15}$$

where Z_N is the normalizing constant.

Key features of this model are:

- The distribution of σ is invariant under permutations of the spins.

- The Hamiltonian is N-dependent.
- Note that H_N can be rewritten as a function of just the magnetization of the configuration, namely,

$$H_N(\sigma) = -N\{\tfrac{J}{2}m_N(\sigma)^2 + h\, m_N(\sigma)\}, \qquad m_N(\sigma) := \tfrac{1}{N}\sum_{i=1}^{N}\sigma_i. \tag{1.16}$$

Detailed results about this model can be found in [OV05],[Bov06] and [Pre09].

1.3.1 Definition of Gibbs measure in the mean-field context

As we saw in Section 1.1.1, on $\mathbb{Z}^d$ (*infinite-volume system*) Gibbsianness of a probability measure μ is characterized by the quasilocality and the uniformly non-nullness of the conditional distributions *in finite boxes.* This approach is no longer possible in the mean-field context due to the lack of geometry. More precisely, the fact that any pair of sites interacts in the same way effectively destroys the role of distance. The permutation invariance for each N gives us, through de Finetti's theorem (see [Geo11]), that the weak limit of μ^N as $N \to \infty$ is exchangeable and therefore is a convex combination of product measures. This limit is trivially non-quasilocal (see [vEL96] and [vEFS93] for further details).

One possibility to overcome this difficulty is to perform limits in a different order:

1. For a *finite-volume system with N spins*, define the single-site conditional probabilities as
$$\gamma^N(\pm 1|\, \alpha_{N-1}) := \mu^N(\sigma_1 = \pm 1|\, \sigma_{[2,\ldots,N]} = \eta^{N-1}), \tag{1.17}$$
where $\alpha_{N-1} \in \{-1, -1+\frac{2}{N-1}, \ldots, +1-\frac{2}{N-1}, +1\}$, $\quad \eta^{N-1} \in \Omega_{N-1}$ and $m_{N-1}(\eta^{N-1}) = \alpha_{N-1}$.
2. Take the *infinite volume limit*
$$\gamma(\pm 1|\, \alpha) := \lim_{N\to\infty} \gamma^N(\pm 1|\, m_{N-1}(\sigma) = \alpha_{N-1}), \qquad \lim_{N\to\infty} \alpha_N = \alpha. \tag{1.18}$$

Remarks: We will refer to γ_1 as the *specification kernel.* Due to permutation invariance, (1.17) is well defined, and it can be shown that γ_1 is independent of how α_N converges to α. Note that γ_1 involves the whole sequence $(\mu^N)_{N\in\mathbb{N}}$ and not just the limit of μ^N as $N \to \infty$.

Definition 1.3.1. [Mean-field model]. *We say that $\mu = (\mu^N)_{N\in\mathbb{N}}$ is a* mean-field model *when μ^N is invariant under permutations of the spins for each N, and has a weak limit $\tilde{\mu}$ as $N \to \infty$.*

Motivated by the characterization of Gibbsianness given by Theorem 1.1.5, the following notion is introduced in [Kül03] :

Definition 1.3.2. [Gibbs (non-Gibbs) mean-field model]. *A mean-field model* $\mu = (\mu^N)_{N \in \mathbb{N}}$ *is said to be:*

- *Gibbs when the limit in* (1.18) *exists and* $\tilde{\alpha} \mapsto \gamma_1(\pm 1 \mid \tilde{\alpha})$ *is continuous at* α *for all* $\alpha \in [-1, +1]$.
- *Non-Gibbs otherwise. The discontinuity points are called* bad magnetizations.

Remarks:

- Comparison of the lattice model and the mean-field model: The spin configuration in the exterior is replaced by the magnetization in the exterior, and continuity w.r.t. the product topology becomes continuity w.r.t. the real-valued variable α.
- The Curie-Weiss model is Gibbs. Its specification kernel is given by

$$\gamma(\pm 1 \mid \alpha) = \tfrac{e^{\pm J\alpha}}{2\cosh(J\alpha)}. \tag{1.19}$$

The link between Definition 1.3.2 in the mean-field case and the standard definition of Gibbs measure in Definition 1.1.3 in the lattice case is not obvious. An informative discussion can be found in [LN08a], Section 5.

1.3.2 Gibbs versus non-Gibbs in the mean-field context

One advantage of the mean-field context is the possibility to obtain explicit formulas that allow for sharp results about the different regimes of Gibbsianness/non-Gibbsianness. This better understanding is useful as a source of new ideas for the lattice case.

We summarize the state of the art:

1. In [Kül03] decimation of the Curie-Weiss model is studied. The scenario shows similarities with the lattice case.
2. In [KLN07] spin-flip dynamics of the Curie-Weiss Ising model is considered following the approach proposed in [vEFdHR02]. Again, the results in [vEFdHR02] for the lattice case were recovered but with sharp transition times between Gibbs and non-Gibbs. The alternating bad configuration mentioned in Section 2 is replaced by $\alpha = 0$. However, new discontinuities arise, which raises the question whether or not the same phenomena occurs in the lattice case.
3. In [vEKOR10] different transformations of mean-field models were studied. In particular, results for planar rotor models subject to a stochastic time evolution are presented.
4. In [EK10] the approach proposed in [vEFdHR10] was pushed forward. The connection between the specification kernel and an underlying Large Deviation Principle was established for spin-flip dynamics. They provided an analysis of the transition between Gibbsian and non-Gibbsian behaviour as a function of time and proved that the time-evolved measure is Gibbs initially and becomes non-Gibbs after a sharp transition time.

5. Similar results were obtained in [RW12]-[dHRvZ13] for a system of real-valued spins interacting with each other through a mean-field Hamiltonian subjected to a stochastic dynamics where the spins perform independent Brownian motions.

1.4 Local mean-field models

A step in between the mean field context and the lattice context is to replace the complete graph by the discrete torus, but still allow all pairs of spins to interact with each other via a long-range interaction.

Formally, let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional unit torus. For $N \in \mathbb{N}$, let $\mathbb{T}_n^d$ be the $(1/N)$-discretization of $\mathbb{T}^d$ defined by $\mathbb{T}_n^d := \Delta_n^d/N$, with $\Delta_n^d := \mathbb{Z}^d/N\mathbb{Z}^d$ the discrete torus of size N. For $N \in \mathbb{N}$, let $\Omega_N := \{-1,+1\}^{\Delta_n^d}$ be the set of Ising-spin configurations on Δ_n^d. The energy of the configuration $\sigma := (\sigma_x)_{x\in\Delta_n^d} \in \Omega_N$ is given by the *Kac-type Hamiltonian*

$$H^N(\sigma) := -\tfrac{1}{2N^d} \sum_{x,y\in\Delta_n^d} J\left(\tfrac{x-y}{N}\right) \sigma_x\sigma_y - \sum_{x\in\Delta_n^d} h(\tfrac{x}{N})\,\sigma_x, \qquad \sigma\in\Omega_N, \tag{1.20}$$

where $J, h \in C(\mathbb{T}^d)$ are continuous functions on $\mathbb{T}^d$, with $J \geq 0$ symmetric and $J \not\equiv 0$. The Gibbs measure associated with H^N is

$$\mu^N(\sigma) := \frac{e^{-H^N(\sigma)}}{Z^N}, \qquad \sigma\in\Omega_N, \tag{1.21}$$

with Z^N the normalizing partition sum.

Thermodynamic variables for this model, such as the free energy, were studied in [EE83]. It is known that if $J \geq 0$ (ferromagnetic), then the thermodynamic behaviour is the same as in the Curie-Weiss model. However, if $J \leq 0$ (antiferromagnetic), then new phenomena appear.

Bifurcation phenomena of the Gibbs Variational Principle associated with the thermodynamic variables are studied in [CES86].

1.4.1 Gibbs versus non-Gibbs in the local mean-field context

The notion of specification kernel in the *mean-field* context relies on the following two facts:

- The Hamiltonian can be written as a function of the magnetization.
- The distribution of a single spin conditional on the exterior only depends on the magnetization of the exterior.

In the *local mean-field* context described above this is no longer true. However, the magnetization can be replaced by the *empirical density* (which contains the same information as the configuration).

For $\Lambda \subseteq \Delta_n^d$, let $\pi_\Lambda^N : \Omega_N \to \mathcal{M}(\mathbb{T}_n^d) \subseteq \mathcal{M}(\mathbb{T}^d)$ be the *empirical density* of σ inside Λ defined by

$$\pi_\Lambda^N(\sigma) := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \sigma_x \delta_{x/N}, \tag{1.22}$$

where $\mathcal{M}(\mathbb{T}_n^d)$ and $\mathcal{M}(\mathbb{T}^d)$ denote the set of signed measures on $\mathbb{T}_n^d$, respectively, $\mathbb{T}^d$ with total variation norm ≤ 1 endowed with the weak topology, and δ_u is the point measure at $u \in \mathbb{T}^d$. Note that $\sigma \in \Omega_N$ determines $\pi_\Lambda^n \in \mathcal{M}(\mathbb{T}_n^d)$ and vice versa.

Abbreviate the function in (1.22) for $\Lambda = \Delta_n^d$ by π^N and for $\Lambda = \Delta_n^d \backslash \{\lfloor nu \rfloor\}$ by $\pi^{u,N}$, $u \in \mathbb{T}^d$, where $\lfloor nu \rfloor$ denotes the component-wise lower-integer part of Nu. The latter is the empirical density *perforated* at $\lfloor nu \rfloor$. Abbreviate

$$\mathcal{M}^N := \pi^N(\Omega_N), \qquad \mathcal{M}^{u,N} := \pi^{u,N}(\Omega_N). \tag{1.23}$$

Note that $\mathcal{M}^N \subseteq \mathcal{M}(\mathbb{T}_n^d)$. Via π^N, the Gibbs measure μ^N on Ω_N in (1.21) induces a probability measure $\breve{\mu}^N$ on $\mathcal{M}^N$ given by

$$\breve{\mu}^N = \mu^N \circ (\pi^N)^{-1}. \tag{1.24}$$

Using (1.22), we can rewrite (1.3) in the form

$$H^N(\sigma) = -N^d H(\pi^N(\sigma)), \tag{1.25}$$

where in the right-hand side we introduce the notation

$$H(\nu) = \left\langle \tfrac{1}{2} J * \nu + h, \nu \right\rangle \tag{1.26}$$

with

$$[f * \nu](u) := \int_{\mathbb{T}^d} J(u - u')\, \nu(du'), \qquad \langle f, \nu \rangle := \int_{\mathbb{T}^d} f(u)\, \nu(du), \tag{1.27}$$

for $f \in C(\mathbb{T}^d)$, $\nu \in \mathcal{M}(\mathbb{T}^d)$.

Let $\lambda^N := \frac{1}{N^d} \sum_{x \in \Lambda} \delta_{x/N}$. We have $w - \lim_{N \to \infty} \lambda^N = \lambda$, where λ is the Lebesgue measure on $\mathbb{T}^d$ and $w-\lim$ stands for weak convergence. In what follows we will represent limit distributions in $\mathcal{M}(\mathbb{T}^d)$ with a Lebesgue density as measures $\alpha\lambda$ with $\alpha \in B$, where

$$B \text{ is the closed unit ball in } L^\infty(\mathbb{T}^d). \tag{1.28}$$

We will refer to α as a *profile*.

The recipe to define Gibbsianness for a local mean-field model will follow similar ideas as in the mean-field case. Given any sequence $(\rho^N)_{N \in \mathbb{N}}$ with ρ^N a probability measure on Ω_N for every $N \in \mathbb{N}$, do the following:

1. Define the single-spin conditional probabilities at site $\lfloor nu \rfloor \in \mathbb{T}^d$ as

$$\gamma^{u,N}\big(\pm 1 \mid \alpha_{N-1}^u \big) := \rho^N\big(\sigma_{\lfloor nu \rfloor} = \pm 1 \mid \pi^{u,N}(\sigma) = \alpha_{N-1}^u \big),\ \alpha_{N-1}^u \in \mathcal{M}^{u,N}. \tag{1.29}$$

2. Take the *infinite volume limit*

$$\gamma^u\big(\pm 1 \mid \tilde{\alpha}\big) := \lim_{N\to\infty} \gamma^{u,N}(\pm 1 \mid \alpha^u_{N-1}) \qquad w - \lim_{N\to\infty} \alpha^u_{N-1} = \tilde{\alpha}\lambda. \tag{1.30}$$

Remark: We will refer to γ^u as the *specification kernel at the point* u. Note that, due to the inhomogeneity of the system, it is necessary to consider the specification kernel for all the "macroscopic" points u in the unit torus.

Definition 1.4.1. *A sequence of probability measures* $\rho = (\rho^N)_{N\in\mathbb{N}}$ *is said to be:*

- *Gibbs when for all* $u \in \mathbb{T}^d$ *the limit in* (1.30) *exists and is independent of how the sequence* $(\alpha^u_{N-1})_{N\in\mathbb{N}}$ *converges to* $\tilde{\alpha}$*, and* $\tilde{\alpha} \mapsto \gamma^u(\pm 1 \mid \tilde{\alpha})$ *is continuous at* α *for all* $\alpha \in B$*.*
- *Non-Gibbs otherwise. The discontinuity points are called* bad profiles.

$\mu = (\mu^N)_{N\in\mathbb{N}}$ with μ^N defined in (1.21) is Gibbs with

$$\gamma^u(\pm 1 \mid \alpha) = \frac{\exp[\pm 1\ \{J * \alpha + h\}(u)]}{2\cosh[\{J * \alpha + h\}(u)]}, \qquad \alpha \in B,\ u \in \mathbb{T}^d. \tag{1.31}$$

In Chapter 3, following the line of [EK10] for the mean-field context, we use the Large Deviation Principle for the Kac-type model subject to Glauber dynamics derived in [Com87] to extend the dynamical approach to the local mean-field context.

1.4.2 Towards the lattice case

In [vEK07] some conjectures about how to establish a link between results in the lattice context and results in the mean-field context are proposed. One way is through the Kac model introduced in (1.6). At least at the level of the free energy, the Kac model gives a connection between the lattice model and the mean-field model (see the Lebowitz-Penrose theorem in [LP66]).

The Kac-type model defined above is different from the Kac model on the lattice. In the Kac-type model, the limit in the size of the box is scaled together with the strength of the interaction. The factor $\frac{1}{N^d}$ in (1.20) is needed because of the compactness of the torus. Without this factor asymptotic observables like the specific free energy may not exist.

1.5 Overview of the main results about dynamical Gibbs-non-Gibbs transitions

In this section we present an overview of the main results to be presented in Chapters 2 and 3. We will denote by γ_t the specification kernel of the system at time t.

1.5.1 Results of Chapter 2: Mean-field context

In Chapter 2 we perform a detailed study of Gibbs-non-Gibbs transitions for the Curie-Weiss model subject to independent spin-flip dynamics ("infinite-temperature" dynamics). Our analysis extends the work of Ermolaev and Külske [EK10], who considered zero magnetic field and finite-temperature spin-flip dynamics. We consider both zero and non-zero magnetic field, but restrict ourselves to infinite-temperature spin-flip dynamics.

We show that the program outlined in [vEFdHR10] can be fully completed, namely, Gibbs-non-Gibbs transitions are *equivalent* to bifurcations in the set of global minima of the large-deviation rate function I^t for the trajectories of the magnetization *conditioned* on their endpoint α_0:

Theorem 1.5.1. [Equivalence of non-Gibbsianness and bifurcation]
$\alpha \mapsto \gamma_t(\sigma \mid \alpha)$ is continuous at α_0 if and only if $\inf_{\varphi:\, \varphi(t)=\alpha_0} I^t(\varphi)$ has a unique minimizing path.

We study in detail the large-deviation rate function for the trajectory of the magnetization in the Curie-Weiss model with pair potential $J > 0$ and magnetic field $h \in \mathbb{R}$. We exploit the fact that, because of the mean-field character of the interaction, and the fact that the dynamics is independent spin-flips, this rate function can be expressed as a function of the initial and the final magnetization only (see Proposition 2.1.2), i.e., the trajectories are uniquely determined by the magnetizations at the beginning and at the end (see Corollary 2.1.3 and Proposition 2.1.5).

Theorem 1.5.2. [Bifurcation analysis]
The main regimes are:

1. *If $0 < J \leq 1$ (supercritical temperature), then the evolved state is Gibbs at all times. On the other hand, if $J > 1$ (subcritical temperature), then there exists some time Ψ_U at which multiple trajectories appear. The associated non-Gibbsianness persists for all later times when $h = 0$ (zero magnetic field).*

2. *For $h \neq 0$ there is a time $\Psi_* > \Psi_U$ at which Gibbsianness is restored for all later times.*

3. *There is a change in behavior at $J = \frac{3}{2}$. For $1 < J \leq \frac{3}{2}$:*
   a) *If $h = 0$, then only the zero magnetization is bad for $t > \Psi_c$.*
   b) *If $h > 0$ ($h < 0$), then there is only one bad magnetization for $\Psi_U < t \leq \Psi_*$. This bad magnetization changes with t but is always strictly negative (strictly positive).*

   For $J > \frac{3}{2}$ (see Fig. 1.1):
   a) *If $h = 0$, then there is a time $\Psi_c > \Psi_U$ such that for $\Psi_U < t < \Psi_c$ there are two non-zero bad magnetizations (equal in absolute value but with opposite signs), while for $t \geq \Psi_c$ only the zero magnetization is bad.*

b) If $h \neq 0$ and small enough, then there are two times $\Psi_T > \Psi_L$ between Ψ_U and Ψ_* such that for $\Psi_U < t \leq \Psi_L$ and $\Psi_T \leq t \leq \Psi_*$ only one bad magnetization occurs, while for $\Psi_L < t < \Psi_T$ two bad magnetizations occur.

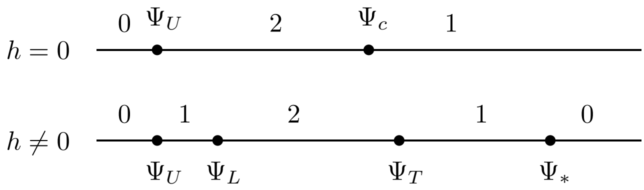

Figure 1.1: Crossover times for $h = 0$ and $h \neq 0$ when $J > \frac{3}{2}$. The numbers on top indicate the number of bad magnetizations at each time.

Some features for $h = 0$ were already found in [EK10].

All the crossover times depend on J, h and are strictly positive and finite. Our analysis gives a detailed picture of the optimal trajectories for different J, h and different conditional magnetizations. Among the novel features we mention:

(1) Presence of *forbidden regions* that cannot be crossed by any optimal trajectory at later times. The boundary of these regions is given by the multiple optimal trajectories when bifurcation sets in. The forbidden regions were predicted in [vEFdHR10] and their existence was proven in [EK10] for $h = 0$.

(2) Existence of *overshoots* and *undershoots* in time of the initial magnetization of the optimal trajectories for $h \neq 0$.

(3) Classification of the bad magnetizations leading to multiple optimal trajectories. These bad magnetizations depend on J, h and change with time.

1.5.2 Results of Chapter 3: Local mean-field context

In Chapter 3 we extend our results for the mean-field context by considering a model with a Kac-type interaction, i.e., Ising spins with a long-range interaction. Non-Gibbsianness is shown to correspond to a discontinuous dependence of the law of the initial profile *conditional* on the final profile.

As in the mean-field context, such discontinuities are expected to arise whenever there is more than one trajectory of the profile that is *compatible* with the bad profile at the end. The actual conditional trajectories are those minimizing the large-deviation rate function I^t on the space of trajectories of profiles in B (Propositions 3.1.2–3.1.3). The time-evolved measure is Gibbs whenever there is a single minimizing trajectory for every final profile, i.e.,

$$\inf_{\substack{\varphi \in C_{[0,t]}(B):\\ \varphi_t \equiv \alpha'}} I^t(\varphi). \tag{1.32}$$

In this case the so-called specification kernel can be computed explicitly (Theorem 3.1.4).

Theorem 1.5.3. [Equivalence of non-Gibbsianness and bifurcation]
For every $t \geq 0$, $\tilde{\alpha}' \mapsto \gamma_t^u(\,\cdot \mid \tilde{\alpha}')$ *is continuous at* $\alpha' \in B$ *for all* $u \in \mathbb{T}^d$ *if and only if* (1.32) *has a unique minimizing path.*

The rate function for the Kac model contains an action integral whose Lagrangian acts on profiles. This setting constitutes a conceptual step-up from what happens for the Curie-Weiss model, where the Lagrangian acts on magnetizations and is much easier to analyze. However, for infinite-temperature dynamics the Kac Lagrangian can be expressed as an integral of the Curie-Weiss Lagrangian with respect to the profile (Theorem 3.1.5). This link allows us to identify the possible scenarios of bifurcation.

Theorem 1.5.4. *Let* $\langle J \rangle := \int_{\mathbb{T}^d} J(u)du$.
(i) **[Short-time Gibbsianness]** *There exists a* $t_0 = t_0(J, h) \in (0, \infty)$ *such that* (1.32) *has a unique global minimizer for all* $0 \leq t \leq t_0$ *and all* $\alpha' \in B$.
(ii) **[Mean-field behaviour]** *If* $h \equiv c \in [0, \infty)$ *and* $\alpha' \equiv c' \in [-1, +1]$, *then the bifurcation behaviour is the same as for the Curie-Weiss model with parameters* $(J^{\mathrm{CW}}, h^{\mathrm{CW}}) = (\beta\langle J \rangle, \beta c)$ *and final magnetization* c'.

The problem of deciding whether or not there exist multiple global minimizers of (1.32) when α' is not constant presents major difficulties. Similar but easier equations have been studied extensively in [CES86], [DMOPT94] and [BCC05], with partial success. An additional complication in our case is that non-constant α' brings a non-homogeneous parameter into the problem, which makes the analysis even harder. A full analysis of the global minimizers of (1.32) as a function of J and h therefore remains a challenge.

2 Mean-field context

This chapter is based on:
R. Fernández, F. den Hollander, and J. Martínez. *Variational description of Gibbs-non-Gibbs dynamical transitions for the Curie-Weiss model.* *Comm. Math. Phys.*, 319(3):703–730, 2013.

Abstract

We perform a detailed study of Gibbs-non-Gibbs transitions for the Curie-Weiss model subject to independent spin-flip dynamics ("infinite-temperature" dynamics). We show that, in this setup, the program outlined in van Enter, Fernández, den Hollander and Redig [vEFdHR10] can be fully completed, namely, Gibbs-non-Gibbs transitions are *equivalent* to bifurcations in the set of global minima of the large-deviation rate function for the trajectories of the magnetization *conditioned* on their endpoint. As a consequence, we show that the time-evolved model is non-Gibbs if and only if this set is not a singleton for *some* value of the final magnetization. A detailed description of the possible scenarios of bifurcation is given, leading to a full characterization of passages from Gibbs to non-Gibbs and vice versa with sharp transition times (under the dynamics, Gibbsianness can be lost and can be recovered).

Our analysis expands the work of Ermolaev and Külske [EK10], who considered zero magnetic field and finite-temperature spin-flip dynamics. We consider both zero and non-zero magnetic field, but restricted to infinite-temperature spin-flip dynamics. Our results reveal an interesting dependence on the interaction parameters, including the presence of forbidden regions for the optimal trajectories and the possible occurrence of overshoots and undershoots in time of the initial magnetization of the optimal trajectories. The numerical plots provided are obtained with the help of MATHEMATICA.

MSC 2010. 60F10, 60K35, 82C22, 82C27.
Key words and phrases. Curie-Weiss model, spin-flip dynamics, Gibbs vs. non-Gibbs, dynamical transition, large deviations, action integral, bifurcation of rate function.
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2.1 Introduction and main results

Section 2.1.1 provides background and motivation, Section 2.1.2 a preview of the main results. Section 2.1.3 introduces the Curie-Weiss model and the key questions to be explored. Section 2.1.4 recalls a few facts from large-deviation theory for trajectories of the magnetization in the Curie-Weiss model subjected to infinite-temperature spin-flip dynamics and provides the link with the specification kernel of the time-evolved measure when it is Gibbs. Section 2.1.5 states the main results and illustrates these results with numerical pictures. The pictures are made with MATHEMATICA, based on analytical expressions appearing in the text. Proofs are given in Sections 2.2 and 2.3. Section 2.1 takes up half of the chapter.

2.1.1 Background and motivation

Dynamical Gibbs-non-Gibbs transitions represent a relatively novel and surprising phenomenon. The setup is simple: an initial Gibbsian state (e.g. a collection of interacting Ising spins) is subjected to a stochastic dynamics (e.g. a Glauber spin-flip dynamics) at a temperature that is *different* from that of the initial state. For many combinations of initial and dynamical temperature, the time-evolved state is observed to become non-Gibbs after a finite time. Such a state cannot be described by any absolutely summable Hamiltonian and therefore *lacks a well-defined notion of temperature*.

The phenomenon was originally discovered by van Enter, Fernández, den Hollander and Redig [vEFdHR02] for *heating dynamics*, in which a low-temperature Ising model is subjected to an infinite-temperature dynamics (independent spin-flips) or a high-temperature dynamics (weakly-dependent spin-flips). The state remains Gibbs for short times, but becomes non-Gibbs after a finite time. Remarkably, heating in this case does not lead to a succession of states with increasing temperature, but to states where the notion of temperature is lost altogether. Furthermore, it turned out that there is a difference depending on whether the initial Ising model has zero or non-zero magnetic field. In the former case, non-Gibbsianness once lost is never recovered, while in the latter case Gibbsianness is recovered at a later time.

This initial work triggered a decade of developments that led to general results on Gibbsianness for small times (Le Ny and Redig [LNR02], Dereudre and Roelly [DR05]), loss and recovery of Gibbsianness for discrete spins (van Enter, Külske, Opoku and Ruszel [KO08b, vER08, vER09, Opo09, vEKOR10]), and loss and recovery of Gibbsianness for continuous spins (Külske and Redig [KR06], Van Enter and Ruszel [vER08, vER09], Redig, Roelly and Ruszel [RRR10]). A particularly fruitful research direction was initiated by Külske and Le Ny [KLN07], who showed that Gibbs-non-Gibbs transitions can also be defined naturally for *mean-field* models, such as the Curie-Weiss model. Precise results are available for the latter, including sharpness of the transition times and an explicit characterization of the conditional magnetizations leading to non-Gibbsianness (Külske and Opoku [KO08a], Ermolaev and Külske [EK10]). In particular, the work in [EK10] shows that in the mean-field setting Gibbs-non-Gibbs transitions occur for all initial temperatures below criticality, both for *cooling dynamics* and for *heating dynamics*.

The ubiquitousness of the Gibbs-non-Gibbs phenomenon calls for a better understanding of its causes and consequences. Unfortunately, the mathematical approach used in most references is opaque on the intuitive level. Generically, non-Gibsianness is proved by looking at the evolving system at two times, the inital and the final time, and applying techniques from equilibrium statistical mechanics. This is an indirect approach that does not illuminate the relation between the Gibbs-non-Gibbs phenomenon and the dynamical effects responsible for its occurrence. This unsatisfactory situation was addressed in Enter, Fernández, den Hollander and Redig [vEFdHR10], where possible dynamical mechanisms were proposed and a *program* was put forward to develop a theory of Gibbs-non-Gibbs transitions on *purely dynamical grounds.* The present paper shows that this program can be fully carried out for the Curie-Weiss model subject to an infinite-temperature dynamics.

In the mean-field scenario, the key object is the time-evolved single-spin average *conditional* on the final empirical magnetization. Non-Gibbsianness corresponds to a discontinuous dependence of this average on the final magnetization. The discontinuity points are called *bad magnetizations* (see Definition 2.1.1 below). Dynamically, such discontinuities are expected to arise whenever there is more than one possible trajectory *compatible* with the bad magnetization at the end. Indeed, this expectation is confirmed and exploited in the sequel. The actual conditional trajectories are those minimizing the large-deviation rate function on the space of trajectories of magnetizations. The time-evolved measure remains Gibbsian whenever there is a single minimizing trajectory for every final magnetization, in which case the specification kernel can be computed explicitly (see Proposition 2.1.4 below). In contrast, if there are multiple optimal trajectories, then the choice of trajectory can be decided by an infinitesimal perturbation of the final magnetization, and this is responsible for non-Gibbsianness.

2.1.2 Preview of the main results

In the present paper we study in detail the large-deviation rate function for the trajectory of the magnetization in the Curie-Weiss model with pair potential $J > 0$ and magnetic field $h \in \mathbb{R}$ (see (2.1) below). We exploit the fact that, due to the mean-field character of the interaction, and the fact that the dynamics is independent spin-flip, this rate function can be expressed as a function of the initial and the final magnetization only (see Proposition 2.1.2 below), i.e., the trajectories are uniquely determined by the magnetizations at the beginning and at the end (see Corollary 2.1.3 and Proposition 2.1.5 below). Here is a summary of the main results:

1. If $0 < J \leq 1$ (supercritical temperature), then the evolved state is Gibbs at all times. On the other hand, if $J > 1$ (subcritical temperature) there exists some time Ψ_U at which multiple trajectories appear. The associated non-Gibbsianness persists for all later times when $h = 0$ (zero magnetic field). Some features were already found by Ermolaev and Külske [EK10].

2. For $h \neq 0$ there is a time $\Psi_* > \Psi_U$ at which Gibbsianness is restored for all later times.

3. There is a change in behavior at $J = \frac{3}{2}$. For $1 < J \leq \frac{3}{2}$:
   a) If $h = 0$, then only the zero magnetization is bad for $t > \Psi_c$.
   b) If $h > 0$ ($h < 0$), then there is only one bad magnetization for $\Psi_U < t \leq \Psi_*$. This bad magnetization changes with t but is always strictly negative (strictly positive).

   For $J > \frac{3}{2}$ (see Fig. 2.1):
   a) If $h = 0$, then there is a time $\Psi_c > \Psi_U$ such that for $\Psi_U < t < \Psi_c$ there are two non-zero bad magnetizations (equal in absolute value but with opposite signs), while for $t \geq \Psi_c$ only the zero magnetization is bad.
   b) If $h \neq 0$ and small enough, then there are two times $\Psi_T > \Psi_L$ between Ψ_U and Ψ_* such that for $\Psi_U < t \leq \Psi_L$ and $\Psi_T \leq t \leq \Psi_*$ only one bad magnetization occurs, while for $\Psi_L < t < \Psi_T$ two bad magnetizations occur.

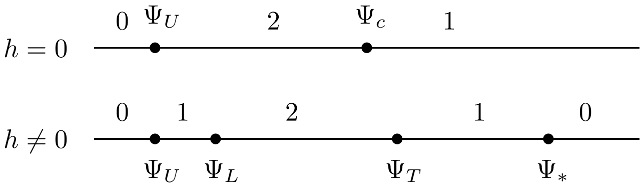


Figure 2.1: Crossover times for $h = 0$ and $h \neq 0$ when $J > \frac{3}{2}$. The numbers on top indicate the number of bad magnetizations at each time.

All the crossover times depend on J, h and are strictly positive and finite. Our analysis gives a detailed picture of the optimal trajectories for different J, h and different conditional magnetizations. Among the novel features we mention:

(1) Presence of *forbidden regions* that cannot be crossed by any optimal trajectory at later times. The boundary of these regions is given by the multiple optimal trajectories when bifurcation sets in. The forbidden regions were predicted in [vEFdHR10] and first found, for $h = 0$, by Ermolaev and Külske [EK10].

(2) Existence of *overshoots* and *undershoots* in time of the initial magnetization of the optimal trajectories for $h \neq 0$.

(3) Classification of the bad magnetizations leading to multiple optimal trajectories. These bad magnetizations depend on J, h and change with time.

2.1.3 The model

Hamiltonian and dynamics

The Curie-Weiss model consists of N Ising spins, labelled $i = 1, \ldots, N$ with $N \in \mathbb{N}$. The spins interact through a mean-field Hamiltonian —that is, a Hamiltonian involving no geometry and no sense of neighborhood, in which each spin interacts equally with all other spins—. The Curie-Weiss Hamiltonian is

$$H^N(\sigma) := -\tfrac{J}{2N}\sum_{i,j=1}^{N}\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i, \qquad \sigma\in\Omega_N, \tag{2.1}$$

where $J > 0$ is the (ferromagnetic) pair potential, $h \in \mathbb{R}$ is the (external) magnetic field, $\Omega_N := \{-1,+1\}^N$ is the spin configuration space, and $\sigma := (\sigma_i)_{i=1}^N$ is the spin configuration. The Gibbs measure associated with H^N is

$$\mu^N(\sigma) := \frac{e^{-H^N(\sigma)}}{Z^N}, \qquad \sigma\in\Omega_N, \tag{2.2}$$

with Z^N the normalizing partition sum.

We allow this model to evolve according to an *independent spin-flip* dynamics, that is, a dynamics defined by the generator L_N given by (see Liggett [Lig85] for more background)

$$(L_N f)(\sigma) := \sum_{i=1}^{N}[f(\sigma^i) - f(\sigma)], \quad f\colon\ \Omega_N\to\mathbb{R}, \tag{2.3}$$

where σ^i denotes the configuration obtained from σ by flipping the spin with label i. The resulting random variables $\sigma(t) := (\sigma_i(t))_{i=1}^N$ constitute a continuous-time Markov chain on Ω_N. We write μ_t^N to denote the measure on Ω_N at time t when the initial measure is μ^N and abbreviate $\mu_t := (\mu_t^N)_{N\in\mathbb{N}}$.

Empirical magnetization

To emphasize its mean-field character, it is convenient to write the Hamiltonian (2.1) in the form

$$H^N(\sigma) = N\bar{H}(m_N(\sigma)) \tag{2.4}$$

where

$$\bar{H}(x) := -\tfrac{1}{2}Jx^2 - hx, \qquad x\in\mathbb{R}. \tag{2.5}$$

and

$$m_N(\sigma) := \frac{1}{N}\sum_{i=1}^{N}\sigma_i \tag{2.6}$$

is the empirical magnetization of $\sigma\in\Omega_N$, which takes values in the set $\mathcal{M}_N := \{-1, -1+2N^{-1}, \dots, +1-2N^{-1}, +1\}$. The Gibbs measure on Ω_N induces a Gibbs measure on $\mathcal{M}_N$ given by

$$\bar{\mu}^N(m) := \binom{N}{\frac{1+m}{2}N}\frac{e^{-N\bar{H}(m)}}{\bar{Z}^N}, \qquad m\in\mathcal{M}_N, \tag{2.7}$$

where $\bar{Z}^N$ is the normalizing partition sum.

The independent (infinite-temperature) dynamics has the simplifying feature of preserving the mean-field character of the model. In fact, the dynamics on Ω_N induces a dynamics on $\mathcal{M}_N$, which is a continuous-time Markov chain $(m_t^N)_{t\geq 0}$ with generator $\bar{L}_N$ given by

$$(\bar{L}_N f)(m) := \frac{1+m}{2} N[f(m-2N^{-1}) - f(m)] + \frac{1-m}{2} N[f(m+2N^{-1}) - f(m)] \ , \quad (2.8)$$

for $f\colon \mathcal{M}_N \to \mathbb{R}$. Adapting our previous notation we denote $\bar{\mu}_t^N$ the measure on $\mathcal{M}_N$ at time t, and abbreviate $\bar{\mu}_t := (\bar{\mu}_t^N)_{N\in\mathbb{N}}$. Due to permutation invariance, μ_t^N characterizes $\bar{\mu}_t^N$ and vice versa, for each N and t. We write P^N to denote the law of $(m_t^N)_{t\geq 0}$, which lives on the space of càdlàg trajectories $D_{[0,\infty)}([-1,+1])$ endowed with the Skorohod topology.

Bad magnetizations

Non-Gibbsianness shows up through discontinuities with respect to boundary conditions of finite-volume conditional probabilities. For the Curie-Weiss model it is enough to consider the single-spin conditional probabilities

$$\gamma_t^N(\sigma_1 \mid \alpha_{N-1}) := \mu_t^N(\sigma_1 \mid \sigma_{N-1}) \ , \quad (2.9)$$

defined for $\sigma_1 \in \{-1,+1\}$ and $\alpha_{N-1} \in \mathcal{M}_{N-1}$, and any spin configuration $\sigma_{N-1} \in \Omega_{N-1}$ such that $m_{N-1}(\sigma_{N-1}) = \alpha_{N-1}$. By permutation invariance, (2.9) does not depend on the choice of σ_{N-1}.

The central definition for our purposes is the following.

Definition 2.1.1. (Külske and Le Ny [KLN07]) *Fix $t \geq 0$.*
(a) A magnetization $\alpha \in [-1,+1]$ is said to be good for μ_t if there exists a neighborhood $\mathcal{N}_\alpha$ of α such that

$$\gamma_t(\cdot \mid \bar{\alpha}) := \lim_{N\to\infty} \gamma_t^N(\cdot \mid \alpha_{N-1}), \quad (2.10)$$

exists for all $\bar{\alpha}$ in $\mathcal{N}_\alpha$ and all $(\alpha_N)_{N\in\mathbb{N}}$ such that $\alpha_N \in \mathcal{M}_N$ for all $N \in \mathbb{N}$ and $\lim_{N\to\infty} \alpha_N = \bar{\alpha}$, *and is independent of the choice of $(\alpha_N)_{N\in\mathbb{N}}$. The limit is called the specification kernel. In particular, $\bar{\alpha} \mapsto \gamma_t(\cdot \mid \bar{\alpha})$ is continuous at $\bar{\alpha} = \alpha$.*
(b) A magnetization $\alpha \in [-1,+1]$ is called bad if it is not good.
(c) μ_t is called Gibbs if it has no bad magnetizations.

2.1.4 Path large deviations and link to specification kernel

The main point of our work is our relation between path large deviations and non-Gibbsianness. For the convenience of the reader, let us recall some basic large deviation results for the Curie-Weiss model. For background on large deviation theory, see e.g. den Hollander [dH00].

Path large deviation principle

Let us recall that a family of measures ν^N on a Borel measure space satisfies a *large deviation principle* with rate function I and speed N if the following two conditions are satisfied:

$$\liminf_{N\to\infty} \frac{1}{N}\log \nu^N(A) \quad \geq \quad -\inf_{x\in A} I(x) \qquad \text{for } A \text{ open} \tag{2.11}$$
$$\limsup_{N\to\infty} \frac{1}{N}\log \nu^N(A) \quad \leq \quad -\sup_{x\in A} I(x) \qquad \text{for } A \text{ closed} \tag{2.12}$$

The proof of the following proposition is elementary and can be found in many references. The indices S and D stand for *static* and *dynamic*.

Proposition 2.1.2. (Ermolaev and Külske [EK10], Enter, Fernández, den Hollander and Redig [vEFdHR10])
(i) $(\bar{\mu}^N)_{N\in\mathbb{N}}$ *satisfies the large deviation principle on* $[-1,+1]$ *with rate* N *and rate function* $I_S - \inf(I_S)$ *given by*

$$I_S(m) := \bar{H}(m) + \bar{I}(m), \qquad \bar{I}(m) := \frac{1+m}{2}\log(1+m) + \frac{1-m}{2}\log(1-m). \tag{2.13}$$

(ii) For every $T > 0$, *the restriction of* $(P^N)_{N\in\mathbb{N}}$ *to the time interval* $[0,T]$ *satisfies the large deviation principle on* $D_{[0,T]}([-1,+1])$ *with rate* N *and rate function* $I^T - \inf(I^T)$ *given by*

$$I^T(\varphi) := I_S(\varphi(0)) + I_D^T(\varphi), \tag{2.14}$$

where

$$I_D^T(\varphi) := \begin{cases} \int_0^T L(\varphi(s),\dot{\varphi}(s))\,ds & \text{if } \dot{\varphi} \text{ exists,} \\ \infty & \text{otherwise,} \end{cases} \tag{2.15}$$

is the action integral with Lagrangian

$$L(m,\dot{m}) := -\frac{1}{2}\sqrt{4\,(1-m^2)+\dot{m}^2} + \frac{1}{2}\dot{m}\log\left(\frac{\sqrt{4\,(1-m^2)+\dot{m}^2}+\dot{m}}{2(1-m)}\right) + 1. \tag{2.16}$$

Let

$$Q_{t,\alpha}^N(m) := P^N\left(m_N(0) = m \mid m_N(t) = \alpha\right), \qquad m \in \mathcal{M}_N \tag{2.17}$$

be the conditional distribution of the magnetization at time 0 given that the magnetization at time t is α. The contraction principle applied to Proposition 2.1.2(ii) implies the following large deviation principle.

Corollary 2.1.3. *For every* $t \geq 0$ *and* $\alpha \in [-1,+1]$, $(Q_{t,\alpha}^N)_{N\in\mathbb{N}}$ *satisfies the large deviation principle on* $[-1,+1]$ *with rate* N *and rate function* $C_{t,\alpha} - \inf(C_{t,\alpha})$ *given by*

$$C_{t,\alpha}(m) := \inf_{\substack{\varphi:\ \varphi(0)=m,\\ \varphi(t)=\alpha}} I^t(\varphi). \tag{2.18}$$

Note that

$$\inf_{m\in[-1,+1]} C_{t,\alpha}(m) = \inf_{m\in[-1,+1]} \inf_{\substack{\varphi:\ \varphi(0)=m,\\ \varphi(t)=\alpha}} I^t(\varphi) = \inf_{\varphi:\ \varphi(t)=\alpha} I^t(\varphi). \tag{2.19}$$

Link to specification kernel

The following proposition – a part of which appears in [EK10]– provides the fundamental link between the specification kernel in (2.10) and the minimizer of (2.19) whenever it is unique, and is a straightforward generalization to an arbitrary magnetic field of a result for zero magnetic field stated and proved in Ermolaev and Külske [EK10].

Proposition 2.1.4. *Fix $t \geq 0$ and $\alpha \in [-1,+1]$. Suppose that (2.19) has a unique minimizing path $(\hat{\varphi}_{t,\alpha}(s))_{0\leq s\leq t}$. Then the specification kernel equals*

$$\gamma_t(z \mid \alpha) = \frac{\sum_{x\in\{-1,+1\}} e^{x[J\hat{\varphi}_{t,\alpha}(0)+h]} p_t(x,z)}{\sum_{x,y\in\{-1,+1\}} e^{x[J\hat{\varphi}_{t,\alpha}(0)+h]} p_t(x,y)}, \qquad z \in \{-1,+1\}, \tag{2.20}$$

where $p_t(\cdot,\cdot)$ is the transition kernel of the continuous-time Markov chain on $\{-1,+1\}$ jumping at rate 1, given by $p_t(1,1) = p_t(-1,-1) = e^{-t}\cosh(t)$ and $p_t(-1,+1) = p_t(1,-1) = e^{-t}\sinh(t)$.

Remark: Note that the expression in the right-hand side of (2.20) depends on the optimal trajectory only via its initial value $\hat{\varphi}_{t,\alpha}(0)$. Thus, (2.20) has the form

$$\gamma_t(z \mid \alpha) = \Gamma_t(z, J\hat{\varphi}_{t,\alpha}(0) + h), \tag{2.21}$$

where $\hat{\varphi}_{t,\alpha}(0)$ is the unique global minimizer of $m \mapsto C_{t,\alpha}(m)$ and $m \mapsto \Gamma_t(z,m)$ is continuous and strictly increasing (strictly decreasing) for $z = 1$ ($z = -1$).

Reduction

The next proposition allows us to reduce (2.19) to a one-dimensional variational problem. Consider the equation

$$k^{J,h}(m) = l_{t,\alpha}(m) \tag{2.22}$$

with

$$\begin{aligned} k^{J,h}(m) &:= a^J(m)\cosh(2h) + b^J(m)\sinh(2h), \\ l_{t,\alpha}(m) &:= m\coth(2t) - \alpha\,\mathrm{csch}(2t), \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} a^J(m) &:= \sinh(2Jm) - m\cosh(2Jm), \\ b^J(m) &:= \cosh(2Jm) - m\sinh(2Jm). \end{aligned} \tag{2.24}$$

Proposition 2.1.5. *Let $C_{t,\alpha}$ be as in (2.18). Then, for every $t \geq 0$ and $\alpha \in [-1,+1]$,*

$$\begin{aligned} C_{t,\alpha}(m) &= I_S(m) \\ &+ \frac{1}{4}\left\{4t + \log\left(\frac{1-\alpha^2}{1-m^2}\right) + \log\left(\left[\frac{1-R-2C_1\alpha e^{-2t}}{1+R-2C_1\alpha e^{-2t}}\right]\left[\frac{1+R-2C_1 m}{1-R-2C_1 m}\right]\right)\right. \\ &\left. +2\left[\alpha\log\left(\frac{R-C_1e^{-2t}+C_2e^{2t}}{1-\alpha}\right) - m\log\left(\frac{R-C_1+C_2}{1-m}\right)\right]\right\} \end{aligned} \tag{2.25}$$

with

$$\begin{array}{rcccl} C_1 & = & C_1(t,\alpha,m) & := & \frac{me^{2t}-\alpha}{e^{2t}-e^{-2t}}, \\ C_2 & = & C_2(t,\alpha,m) & := & \frac{\alpha-me^{-2t}}{e^{2t}-e^{-2t}}, \\ R & = & R(C_1,C_2) & := & \sqrt{1-4C_1C_2}. \end{array} \tag{2.26}$$

Furthermore, the critical points of $C_{t,\alpha}$ are the solutions of (2.22)*. Hence,*

$$\inf_{\varphi:\ \varphi(t)=\alpha} I^t(\varphi) = \min_{m \text{ solves } (2.22)} C_{t,\alpha}(m) \,, \tag{2.27}$$

and the constrained minimizing trajectories are of the form

$$\hat{\varphi}^m_{t,\alpha}(s) \ := \ \operatorname{csch}(2t)\Big\{m\sinh(2(t-s))+\alpha\sinh(2s)\Big\} \qquad 0\le s\le t \tag{2.28}$$

$$\hat{m}=\hat{m}(t,\alpha) \ := \ \operatorname{argmin}\Big[C_{t,\alpha}\big|_{solutions\ of\ (2.22)}\Big] \,. \tag{2.29}$$

The identities

$$k^{J,h}(m) \ = \ 2\cosh^2(Jm+h)\big[\tanh(Jm+h)-m\big]+m \tag{2.30}$$

and

$$\lim_{t\to\infty} l_{t,\alpha}(m) = m \tag{2.31}$$

imply that in the limit $t\to\infty$ (2.22) reduces to $\tanh(Jm+h)=m$. This is the equation for the spontaneous magnetization of the Curie-Weiss model with parameters J,h. This equation has always at least one solution and the value

$$m^\infty = m^\infty(J,h) := \text{ the largest solution of the equation } \tanh(Jm+h)=m \tag{2.32}$$

is well known to be strictly positive if $h>0$ or if $J>1$. In these regimes, the standard Curie-Weiss graphical argument shows that, for $m>0$,

$$k^{J,h}(m) \substack{<\\=\\>} m \quad \Longleftrightarrow \quad m \substack{>\\=\\<} m^\infty \,. \tag{2.33}$$

We also remark that when $t\to 0$ the function $l_{t,\alpha}$ converges to the line defined by the equation $m=\alpha$. This implies that for short times there is a unique solution of (2.22) and it is close to α.

2.1.5 Main results

In Section 2.1.5 we state the equivalence of non-Gibbs and bifurcation that lies at the heart of the program outlined in [vEFdHR10] (Theorem 2.1.6). In Section 2.1.5 we introduce some notation. In Section 2.1.5 we identify the optimal trajectories for $\alpha=0$, $h=0$ (Theorems 2.1.7–2.1.8). In this Section we also extend this identification to $\alpha\in[-1,+1]$, $h\in\mathbb{R}$ (Theorem 2.1.9). Finally we summarize the consequences for Gibbs versus non-Gibbs (Corollary 2.1.10).

Equivalence of non-Gibbs and bifurcation

The following theorem proves the suspected equivalence in the present model between dynamical non-Gibbsianness, i.e., discontinuity of $\alpha \mapsto \gamma_t(\cdot \mid \alpha)$ at α_0, and non-uniqueness of the global minimizer of $m \mapsto C_{t,\alpha_0}(m)$, i.e., the occurrence of more than one possible history for the same α. This equivalence was already mentioned in Ermolaev-Küelske [EK10].

Theorem 2.1.6. *$\alpha \mapsto \gamma_t(\sigma \mid \alpha)$ is continuous at α_0 if and only if $\inf_{\varphi:\ \varphi(t)=\alpha_0} I^t(\varphi)$ has a unique minimizing path or, equivalently, $\inf_{m\in[-1,+1]} C_{t,\alpha_0}(m)$ has a unique minimizing magnetization.*

Notation

Due to relation (2.27), our analysis focusses on the different solutions of (2.22) obtained as t, α are varied. In particular, we must determine which of them are minima of the variational problem in (2.19). We write

$$\Delta_{t,\alpha} := \text{ the set of global minimizers of } C_{t,\alpha}. \tag{2.34}$$

For brevity, when α is kept fixed and $\Delta_{t,\alpha}$ is a singleton $\{\hat{m}(t,\alpha)\}$ for each t, we write $\hat{m}(t)$ instead of $\hat{m}(t,\alpha)$. When $h, \alpha = 0$, by symmetry we have $\Delta_{t,0} = \{0\}$ or $\Delta_{t,0} = \{\pm\hat{m}(t)\}$, where in the last case we denote by $\hat{m}(t)$ the unique positive global minimizer. If both the initial and final magnetizations are fixed, then there is a unique minimizer that we denote as in (2.28). That is,

$$\hat{\varphi}^m_{t,\alpha} := \underset{\substack{\varphi:\ \varphi(0)=m,\\ \varphi(t)=\alpha}}{\operatorname{argmin}} \ I^t_D(\varphi) \tag{2.35}$$

for $m, \alpha \in [-1,+1]$. We emphasize that, by definition, $C_{t,\alpha}(m) = I^t(\hat{\varphi}^m_{t,\alpha})$ and $\hat{\varphi}_{t,\alpha}(s) = \hat{\varphi}^{\hat{m}(t,\alpha)}_{t,\alpha}(s), s \in [0,t]$. In particular $\hat{m}(t,\alpha) = \hat{\varphi}_{t,\alpha}(0)$.

Optimal trajectories for $\alpha = 0$, $h = 0$

The following theorem refers to a critical time

$$\Psi_c = \Psi_c(J) := \begin{cases} \frac{1}{2}\operatorname{arccoth}(2J-1) & \text{if } 1 < J \le \frac{3}{2}, \\ t_* & \text{if } J > \frac{3}{2}, \end{cases} \tag{2.36}$$

where $t_* = t_*(J)$ is implicitly calculable: $t_* = t(m_*)$ where the function $t(m)$ is defined in (2.52) below and $m_* = m_*(J)$ is the solution of (2.59).

Theorem 2.1.7. (See Fig. 2.2.) *Consider $\alpha = 0$ and $h = 0$.*

(i) If $0 < J \le 1$, then

$$\Delta_{t,0} = \{0\}, \quad \forall\, t \ge 0. \tag{2.37}$$

(ii) If $1 < J \leq \frac{3}{2}$*, then*

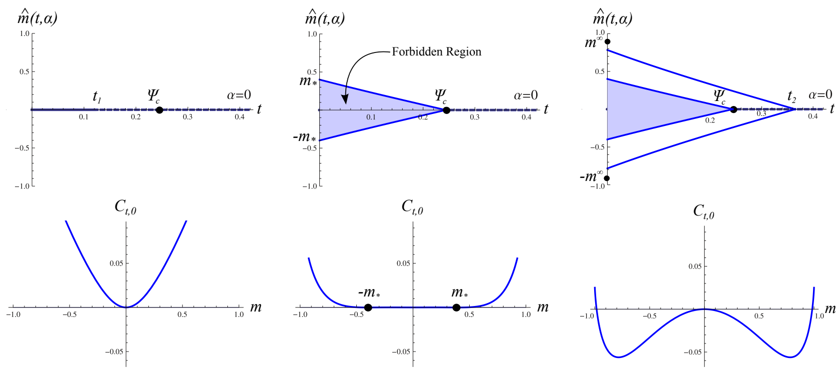
$$\Delta_{t,0} = \begin{cases} \{0\} & \text{if } 0 \leq t \leq \Psi_c, \\ \{\pm \hat{m}(t)\} & \text{if } t > \Psi_c, \end{cases} \tag{2.38}$$

where $t \mapsto \hat{m}(t)$ *is continuous and strictly increasing on* $[\Psi_c, \infty)$ *with* $\hat{m}(\Psi_c) = 0$.

(iii) If $J > \frac{3}{2}$*, then*

$$\Delta_{t,0} = \begin{cases} \{0\} & \text{if } 0 \leq t < \Psi_c, \\ \{\pm \hat{m}(t)\} & \text{if } t \geq \Psi_c, \end{cases} \tag{2.39}$$

where $t \mapsto \hat{m}(t)$ *is continuous and strictly increasing on* $[\Psi_c, \infty)$ *with* $\hat{m}(\Psi_c) =: m_* > 0$.



$t(= t_1) < \Psi_c$ $\quad$ $t = \Psi_c$ $\quad$ $t(= t_2) > \Psi_c$

Figure 2.2: Illustration of Theorem 2.1.7. *First row:* Time evolution of the minimizing trajectories $\pm(\hat{\varphi}_{t,0}(s))_{0 \leq s \leq t}$ for $t < \Psi_c$, $t = \Psi_c$ and $t > \Psi_c$ for an initial Curie-Weiss model with $(J, h) = (1.6, 0)$ [regime (iii) in the Theorem]. The shaded cone is the forbidden region. *Second row:* Plot of $m \mapsto C_{t,0}(m)$ for the same times and parameter values.

Let $\Lambda_{t,0}(J)$ denote the cone between the trajectories $\pm\hat{\varphi}_{t,0}$. As a consequence of the previous theorem, *no* minimal trajectory conditioned in t' with $t' \geq t$ can intersect the interior of this region. Such a cone corresponds, therefore, to a *forbidden region.* Forbidden regions grow, in a nested fashion, as the conditioning time t grows. There is, however, a distinctive difference between regimes (ii) and (iii) in the previous theorem: In Regime (ii) the forbidden region opens up *continuously* after Ψ_c, while in Regime (iii) it opens up *discontinuously*. These facts are summarized in the following theorem.

Theorem 2.1.8. *Suppose that* $\alpha = 0$ *and* $h = 0$.
(i) $J \mapsto m_*(J)$ *is strictly increasing on* $(\frac{3}{2}, \infty)$.

(ii) $J \mapsto \Psi_c(J)$ *is strictly decreasing on* $(1, \infty)$.
(iii) $J \mapsto \Lambda_{t,0}(J)$ *is left-continuous at* $J = \frac{3}{2}$ *for all* $t > \Psi_c(\frac{3}{2})$.
(iv) $J \mapsto \Lambda_{\Psi_c(\bar{J}),0}(J)$ *is right-continuous at* $J = \bar{J}$ *for all* $\bar{J} > \frac{3}{2}$.
(v) For every $J \leq 3/2$ *the map* $t \mapsto \Lambda_{t,0}(J)$ *is continuous.*
(vi) For every $J > 3/2$ *the map* $t \mapsto \Lambda_{t,0}(J)$ *is continuous except at* $t = \Psi_c$ *where it exhibits a right-continuous jump.*

Optimal trajectories for $\alpha \in [-1, +1]$, $h \in \mathbb{R}$

Fore fixed (J, h) and α we say that there is (See Fig. 2.3):

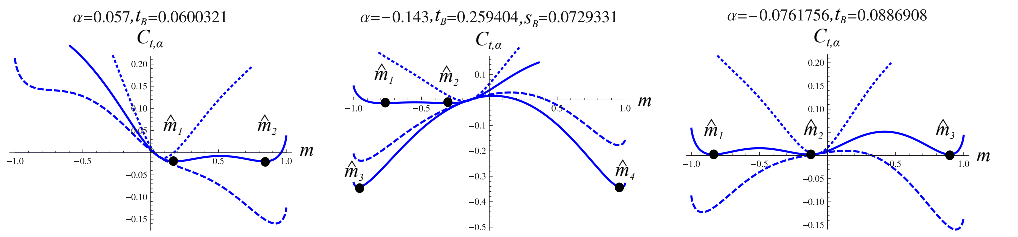
- *No bifurcation* if $\Delta_{t,\alpha} = \{\hat{m}(t, \alpha)\}$, for all $t \geq 0$ and the map $t \mapsto \hat{m}(t, \alpha)$ is continuous on $[0, \infty)$.

- *Bifurcation* when there exists a $0 < t_B < \infty$ such that $t \mapsto \hat{m}(t, \alpha)$ continuous except at $t = t_B$ and $|\Delta_{t_B,\alpha}| = 2$.

- *Double bifurcation* if there exist times $0 < s_B < t_B < \infty$ such that $t \mapsto \hat{m}(t, \alpha)$ continuous except at $t = t_B$ and $t = s_B$, and $|\Delta_{s_B,\alpha}| = |\Delta_{t_B,\alpha}| = 2$.

- *Trifurcation* if there exists a $0 < t_T < \infty$ such that $t \mapsto \hat{m}(t, \alpha)$ is continuous except at $t = t_T$ and $|\Delta_{t_T,\alpha}| = 3$.

The bifurcation times t_B and s_B, the trifurcation time t_T and the trifurcation magnetization M_T (defined below) all depend on J, h.



Bifurcation | Double bifurcation | Trifurcation

Figure 2.3: Different scenarios for the evolution in time of $m \mapsto C_{t,\alpha}(m)$. Drawn lines: $t = t_B$, $t = t_B, s_B$, $t = t_T$ (times at which multiple global minima occur or, equivalently, discontinuity points of $t \mapsto \hat{m}(t, \alpha)$). Dotted lines: earlier time. Dashed lines: later time.

The following theorem summarizes the behaviour of $\Delta_{t,\alpha}$ (and therefore of the minimizing trajectories $\hat{\varphi}_{t,\alpha}$) for different t, α. For $J > \frac{3}{2}$, let

$$\begin{aligned} F(m) &:= \frac{m k'^{J,h}(m) - k^{J,h}(m)}{\operatorname{csch}[\operatorname{arccoth}(k'^{J,h}(m))]}, \\ U_B = U_B(J,h) &:= \max_{m \in [0,1]} F(m), \\ L_B = L_B(J,h) &:= \min_{m \in [-1,0]} F(m). \end{aligned} \tag{2.40}$$

Theorem 2.1.9. (See Figs. 2.3–2.4.)

(1) *Suppose that* $k^{J,h}(\alpha) \neq 0$.
(1a) *If* $k^{J,h}(\alpha) > 0$ *and* $\alpha > 0$, *then there are* $m_R^+ > 0$ *and* $t_R = t_R(m_R^+) > 0$ *(implicitly calculable from* (2.73)*) such that* $t \mapsto \hat{m}(t)$ *is strictly increasing on* $[0, t_R]$ *and strictly decreasing on* $[t_R, \infty)$ *with* $\hat{m}(t_R) = m_R^+ > m^\infty$.
(1b) *If* $k^{J,h}(\alpha) < 0$ *and* $\alpha > 0$, *then* $t \mapsto \hat{m}(t)$ *is strictly decreasing on* $[0, \infty)$.
(1c) *If* $k^{J,h}(\alpha) > 0$ *and* $\alpha < 0$, *then* $t \mapsto \hat{m}(t)$ *is strictly increasing on* $[0, \infty)$.
(1d) *If* $k^{J,h}(\alpha) < 0$ *and* $\alpha < 0$, *then there are* $m_R^- > 0$ *and* $t_R = t_R(m_R^-) > 0$ *(implicitly calculable from* (2.74)*) such that* $t \mapsto \hat{m}(t)$ *is strictly decreasing on* $[0, t_R]$ *and strictly increasing on* $[t_R, \infty)$ *with* $\hat{m}(t_R) = m_R^- < \alpha$.
In all cases $\hat{m}(0) = \alpha$ *and* $\lim_{t\to\infty} \hat{m}(t) = m^\infty$.

(2) *Suppose that* $h = 0$.
(2a) *If* $0 < J \leq 1$, *then there is no bifurcation.*
(2b) *If* $1 < J \leq \frac{3}{2}$, *then there is bifurcation only for* $\alpha = 0$.
(2c) *If* $J > \frac{3}{2}$, *then there is bifurcation if* $\alpha \in (-U_B, U_B)$ *and no bifurcation otherwise.*

(3) *Suppose that* $h > 0$.
(3a) *If* $0 < J \leq 1$, *then there is no bifurcation.*
(3b) *If* $1 < J \leq \frac{3}{2}$, *then there is bifurcation for* $\alpha \in [-1, U_B)$ *and no bifurcation for* $\alpha \in [U_B, 1]$.
(3c) *If* $J > \frac{3}{2}$, *then there exists a* $h_* = h_*(J) > 0$ *such that*

- *for every* $0 < h < h_*$ *there exists a* $M_T \in (L_B, U_B)$ *with* $M_T < 0$ *such that there is*
  - *no bifurcation for* $\alpha \in [U_B, 1]$,
  - *bifurcation for* $\alpha \in (M_T, U_B)$,
  - *trifurcation for* $\alpha = M_T$,
  - *double bifurcation for* $\alpha \in (L_B, M_T)$,
  - *bifurcation for* $\alpha \in [-1, L_B]$.
- *for every* $h \geq h^*$ *the behavior is the same as in* (3b).

In all cases $\alpha \mapsto t_B(\alpha)$ *is continuous and decreasing and* $\alpha \mapsto s_B(\alpha)$ *is continuous and increasing.*

Theorem 2.1.9 gives a complete picture of the bifurcation scenario. Regime (1) —which includes cases with zero and nonzero magnetic field— describes two types of behavior of optimal magnetization trajectories: monotone trajectories [cases (1b) and (1c)] and trajectories with *overshoot* [cases (1a) and (1d)]. In the latter, $\hat{m}(t)$ increases to some magnetization m_R^+ larger (m_R^- smaller) than m^∞ and afterwards decreases (increases) to m^∞. Regimes (2)and (3) refer to the existence of bifurcations and trifurcations. We observe that the different bifurcation behaviors —no bifurcation, single and double bifurcation— hold for whole intervals of the conditioning magnetization. In contrast, trifurcation appears at a single final magnetization for small $h \neq 0$.

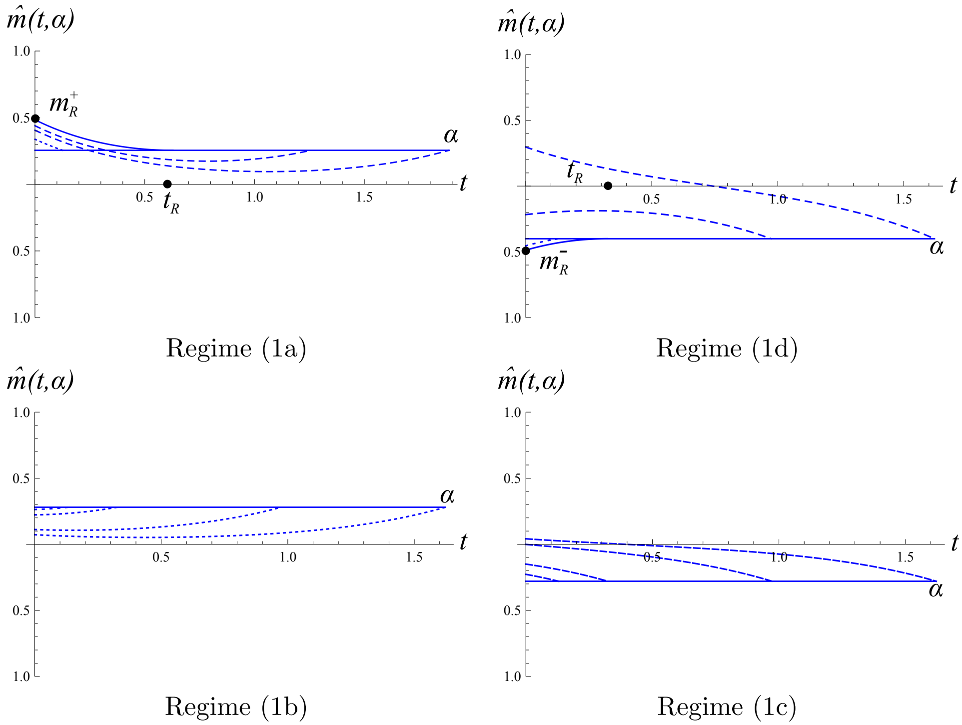


Figure 2.4: Different regimes of Theorem 2.1.9. Evolution in time of the minimizing trajectories $(\hat{\varphi}_{t,\alpha}(s))_{0\leq s\leq t}$ for $t < t_R$ (dotted), $t = t_R$ (drawn), $t > t_R$ (dashed).

Gibbs versus non-Gibbs

Theorem 2.1.6 establishes the equivalence of bifurcation and discontinuity of specifications, as proposed in the program put forward in [vEFdHR10]. Due to this equivalence, the following corollary provides a full characterization of the different Gibbs–nonGibbs scenarios appearing during the infinite-temperature evolution of the Curie-Weiss model.

Let

$$0 < \Psi_U := t_B(U_B) < \Psi_L := t_B(L_B) < \Psi_T := t_B(M_T) < \Psi_* := t_B(-1), \qquad (2.41)$$

and let M_B be the solution of $t_B(M_B) = \Psi_L$. Denote $\mathcal{D}_t \subseteq [-1, +1]$ the set of α-values for which $\alpha \mapsto \gamma_t(\cdot|\alpha)$ is discontinuous.

Corollary 2.1.10. (See Fig. 2.5.)

(1) *Let* $h = 0$.
 (1a) *If* $0 < J \leq 1$, *the evolved measure* μ_t *is Gibbs for all* $t \geq 0$.
 (1b) *If* $1 < J \leq \frac{3}{2}$, *then* μ_t *is*
  - *Gibbs for* $0 \leq t \leq \Psi_c$,
  - *non-Gibbs for* $t > \Psi_c$ *with* $\mathcal{D}_t = \{0\}$.
 (1c) *If* $J > \frac{3}{2}$, *then* μ_t *is*
  - *Gibbs for* $0 \leq t \leq \Psi_U$,
  - *non-Gibbs for* $t > \Psi_U$ *with*
   * $\mathcal{D}_t = \{\pm\alpha\}$ *for some* $\alpha \in (-U_B, U_B)$ *if* $\Psi_U < t < \Psi_c$,
   * $\mathcal{D}_t = \{0\}$ *if* $t \geq \Psi_c$.

(2) *Let* $h > 0$.
 (2a) *If* $0 < J \leq 1$, *then* μ_t *is Gibbs for* $t \geq 0$.
 (2b) *If* $1 < J \leq \frac{3}{2}$, *then* μ_t *is*
  - *Gibbs for* $0 \leq t \leq \Psi_U$,
  - *non-Gibbs for* $\Psi_U < t \leq \Psi_*$ *with* $\mathcal{D}_t = \{\alpha\}$ *for some* $\alpha \in [-1, U_B)$,
  - *Gibbs for* $t > \Psi_*$.
 (2c) *If* $J > \frac{3}{2}$ *and* $h < h^*$ *small enough, then* μ_t *is*
  - *Gibbs for* $0 \leq t \leq \Psi_U$,
  - *non-Gibbs for* $\Psi_U < t \leq \Psi_*$ *with*
   * $\mathcal{D}_t = \{\alpha\}$ *for some* $\alpha \in [M_B, U_B)$ *if* $\Psi_U < t \leq \Psi_L$,
   * $\mathcal{D}_t = \{\alpha^1, \alpha^2\}$ *for some* $\alpha^1, \alpha^2 \in (L_B, M_B)$ *if* $\Psi_L < t < \Psi_T$,
   * $\mathcal{D}_t = \{\alpha\}$ *for some* $\alpha \in [-1, M_T]$ *if* $\Psi_T \leq t \leq \Psi_*$.
  - *Gibbs for* $t > \Psi_*$. *If* $h \geq h^*$, *then the behaviour is as in* (2b).

In all cases $\alpha^1, \alpha^2, \alpha$ *depend on* (t, J, h).

2.2 Proof of Proposition 2.1.5 and Theorems 2.1.6–2.1.8

Proposition 2.1.5 is proven in Section 2.2.1, Theorems 2.1.6–2.1.8 are proven in Sections 2.2.2–2.2.4.

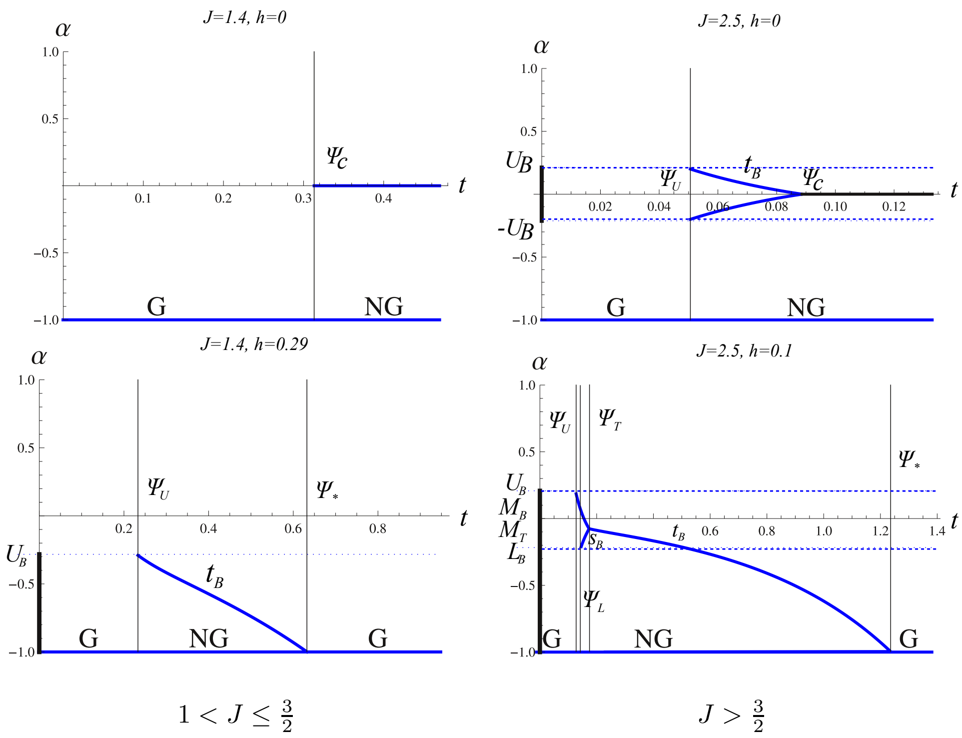


Figure 2.5: Summary of Corollary 2.1.10: Time versus bad magnetizations for different regimes. On the vertical α-axis, indicated by a thick line, is the set of bad magnetizations. G=Gibbs, NG=non-Gibbs.

2.2.1 Proof of Proposition 2.1.5

Proof. First note that, by (2.14),

$$\inf_{\varphi:\ \varphi(t)=\alpha} I^t(\varphi) = \inf_{m\in[-1,+1]} \left\{ I_S(m) + \inf_{\substack{\varphi:\ \varphi(0)=m,\\ \varphi(t)=\alpha}} I_D^t(\varphi) \right\}. \tag{2.42}$$

It follows from (2.14–2.15) and the calculus of variations that the stationary points of the right-hand side of (2.42) are given by the Euler-Lagrange equations, complemented with a free-left-end condition and a fixed-right-end condition:

$$\begin{aligned} \frac{\partial}{\partial s}\frac{\partial L}{\partial \dot{m}}(\varphi(s),\dot{\varphi}(s)) &= \frac{\partial L}{\partial m}(\varphi(s),\dot{\varphi}(s)), \qquad s \in\ (0,t), \\ \frac{\partial L}{\partial \dot{m}}(\varphi(s),\dot{\varphi}(s))\Big|_{s=0} &= \frac{\partial I_S}{\partial m}(\varphi(s))\Big|_{s=0}, \\ \varphi(t) &= \alpha. \end{aligned} \tag{2.43}$$

The first and the third equation in (2.43) come from the third infimum in (2.42) and, together with (2.16), determine the form (2.28) of the stationary trajectory. Inserting this form into (2.14) we identify

$$I^t(\hat{\varphi}^m_{t,\alpha}) \;:=\; C_{t,\alpha}(m), \tag{2.44}$$

as stated in (2.25)–(2.26). This identity reduces (2.19) to a one-dimensional variational problem,

$$\inf_{\varphi:\ \varphi(t)=\alpha} I^t(\varphi) = \inf_{m\in[-1,+1]} I^t(\hat{\varphi}^m_{t,\alpha}) = \inf_{m\in[-1,+1]} C_{t,\alpha}(m) \tag{2.45}$$

The second equation in (2.43) corresponds to the second infimum in (2.42) or, equivalently, to the rightmost infimum in (2.45). It gives a trade-off between the static and the dynamic cost, establishing a relation between the initial magnetization and the initial derivative. After some manipulations this equation can be written in the form

$$-\tfrac{1}{2}\,q = a^J(m)\cosh(2h) + b^J(m)\sinh(2h), \qquad m = \hat{\varphi}^m_{t,\alpha}(0), \quad q = \dot{\hat{\varphi}}^m_{t,\alpha}(0). \tag{2.46}$$

Differentiating (2.28), we get

$$\dot{\hat{\varphi}}^m_{t,\alpha}(s) = 2\,\mathrm{csch}(2t)\Big\{\alpha\cosh(2s) - m\cosh(2(t-s))\Big\}, \tag{2.47}$$

and eliminating q from the last identity in (2.46) in favor of t and α, we conclude that m must be a solution of (2.22). Imposing this restriction to the chain of identities (2.45) we obtain (2.27) and hence (2.28)–(2.29). □

From now on our arguments rely on the study of (2.22), combined with continuity properties of $C_{t,\alpha}$ as a function of t, α.

2.2.2 Proof of Theorem 2.1.6

Equation (2.20) follows in the same way as in the proof of Theorem 2.5 in [EK10]. Having disposed of this identity, we can now proceed to prove the equivalence. The proof relies on the following lemma.

Lemma 2.2.1. *For any $t > 0$ and $\alpha_0 \in [-1,+1]$, there exists an open neighbourhood $\mathcal{N}_{\alpha_0} \neq \varnothing$ of the later, such that for all $\alpha \in \mathcal{N}_{\alpha_0} \setminus \{\alpha_0\}$*

1. *$C_{t,\alpha}$ has only one global minimum, namely, $\hat{m}(t,\alpha)$.*
2. *$\tilde{\alpha} \mapsto \hat{m}(t,\tilde{\alpha})$ is continuous at α. If C_{t,α_0} has a unique global minimum, the continuity is also valid at $\alpha = \alpha_0$.*
3. *If C_{t,α_0} has multiple global minima, for two of them, namely, $\hat{m}_A(t,\alpha_0)$ and $\hat{m}_B(t,\alpha_0)$*

$$\lim_{\alpha\downarrow\alpha_0} \hat{m}(t,\alpha) = \hat{m}_A(t,\alpha_0) \quad \textit{and} \quad \lim_{\alpha\uparrow\alpha_0} \hat{m}(t,\alpha) = \hat{m}_B(t,\alpha_0).$$

Proof. A straightforward study of (2.22) shows that $C_{t,\alpha}$ has a finite number of critical points, for every fixed choice of J, h, t, α.

Clearly, $\alpha \mapsto C_{t,\alpha}$ and $\alpha \mapsto l_{t,\alpha}$ are continuous with respect to the infinity norm in $C([-1,+1],\mathbb{R})$. This, together with the fact that the left-hand side of (2.22) does not depend on α, implies continuity of any critical point with respect to α.

Let $\hat{m}_i(t,\alpha_0),\ i=1,\dots,v$ be the global minima of C_{t,α_0}. By continuity of the critical points, there exists a neighbourhood $\widetilde{\mathcal{N}}_{\alpha_0}$ and smooth functions $\widetilde{\mathcal{N}}_{\alpha_0} \ni \alpha \mapsto \overline{m}_i(t,\alpha)$, $i=1,\dots,v$, such that

i. $\overline{m}_i(t,\alpha)$ are local minima of $C_{t,\alpha}$,

ii. $\lim_{\alpha\to\alpha_0} \overline{m}_i(t,\alpha) = \hat{m}_i(t,\alpha_0)$.

These properties prove the lemma if $v=1$. Otherwise, let

$$B_i(\alpha) := C_{t,\alpha}(\overline{m}_i(t,\alpha)).$$

The minimal cost is attained at the smallest of them:

$$C_{t,\alpha}(\hat{m}(t,\alpha)) = \min_i B_i(\alpha)\ .$$

Note that there is coincidence at α_0 due to the assumed multiplicity of minima:

$$B(\alpha_0)\ :=\ B_i(\alpha_0) \quad , \quad i=1,\dots v\ . \tag{2.48}$$

We expand the functions B_i up to first order order

$$B_i(\alpha) = B(\alpha_0) + B_i'(\alpha_0)(\alpha-\alpha_0) + O(\alpha-\alpha_0)\ , \tag{2.49}$$

and observe that,

$$B_i'(\alpha_0) \neq B_j'(\alpha_0), \qquad i \neq j. \tag{2.50}$$

The latter is due to the strict monotonicity of $\frac{\partial C_{t,\alpha}}{\partial \alpha}$ and the fact that each $\overline{m}_i(t,\alpha)$ is a critical point of the function $C_{t,\alpha}(\,\cdot\,)$. From (2.2.2)–(2.50) we conclude that for α in a possibly smaller neighbourhood $\mathcal{N}_{\alpha_0} \subseteq \widetilde{\mathcal{N}}_{\alpha_0}$ there is a unique global minimum, and that property 3. holds with

$$a = \arg\min_i B_i'(\alpha_0), \qquad b = \arg\max_i B_i'(\alpha_0),$$

and

$$\hat{m}_A(t,\alpha_0) := \hat{m}_a(t,\alpha_0), \qquad \hat{m}_B(t,\alpha_0) := \hat{m}_b(t,\alpha_0).$$

□

We are ready to prove Theorem 2.1.6.

Proof. Suppose that C_{t,α_0} has a unique minimizer, denoted by $\hat{m}(t,\alpha_0)$ and let $\mathcal{N}_{\alpha_0}$ be the neighbourhood of the previous lemma. Then (2.21) holds for every $\alpha \in \mathcal{N}_{\alpha_0}$, and the continuity of $m \mapsto \Gamma_t(z,m)$ for every t, z gives the desired continuity of $\alpha \mapsto \gamma_t(\cdot \mid \alpha)$ at $\alpha = \alpha_0$.

To prove necessity, assume that C_{t,α_0} has multiple global minima. Consider $\hat{m}_A$ and $\hat{m}_B$ as in the previous lemma. Then, we have that there exist sequences $\alpha_n^- < \alpha_0 < \alpha_n^+$ converging to α_0 and such that $\gamma_t(\cdot \mid \alpha_n^{\pm}) = \Gamma_t(\cdot, \hat{m}(t, \alpha_n^{\pm}))$ and

$$\lim_{n\to\infty} \hat{m}(t, \alpha_n^-) = \hat{m}_B(t, \alpha_0) \neq \hat{m}_A(t, \alpha_0) = \lim_{n\to\infty} \hat{m}(t, \alpha_n^+). \tag{2.51}$$

Again using continuity of Γ_t with respect to m, we get

$$\lim_{n\to\infty} \gamma_t(z \mid \alpha_n^-) = \Gamma_t(z, \hat{m}_B(t, \alpha_0)) \neq \Gamma_t(z, \hat{m}_A(t, \alpha_0)) = \lim_{n\to\infty} \gamma_t(z \mid \alpha_n^+).$$

Hence α_0 is a bad magnetization. □

2.2.3 Proof of Theorem 2.1.7

To determine which solutions of (2.22) are global minima of $C_{t,0}$ when $h = 0$, we will pursue the following strategy. Using (2.22) we can write t as a function of m:

$$t(m) := \tfrac{1}{2} \operatorname{arccoth}\left(\frac{a^J(m)}{m}\right). \tag{2.52}$$

This allows us to determine for which time t the magnetization m can be a *possible critical point* (i.e., a solution of (2.22)).

Lemma 2.2.2. *Let $A \subseteq [-1, +1]$ be the set of m-values such that m is the solution of* (2.22) *for some $t > 0$, i.e., $A = \{m \in [-1, +1]\colon\ a^J(m)/m > 1\}$. Then, for every $m \in A$,*

$$C_{t(m),0}(m) = \tfrac{1}{2} J m^2 + \tfrac{1}{2} \log\left[1 - m \tanh(Jm)\right] =: C_M(m). \tag{2.53}$$

In words, (2.53) *is the cost for m at the time at which it is a possible critical point.*

Proof. Insert (2.52) into (2.25) and use (2.43). □

We now start the proof of Theorem 2.1.7.

Proof. First note that $l_{t,0}$ is linear with slope $\coth(2t) \in (1, \infty)$ and $l_{t,0}(0) = 0$, and that a^J is antisymmetric. Hence, if m is a solution of (2.22), then also $-m$ is a solution. Further note that

$$\begin{aligned} {a^J}'(m) &= (2J - 1)\cosh(2Jm) - 2Jm \sinh(2Jm), \\ {a^J}''(m) &= 4J(J - 1)\sinh(2Jm) - 4J^2 m \cosh(2Jm). \end{aligned} \tag{2.54}$$

(i) If $0 < J < \frac{1}{2}$, then ${a^J}'(m) < 0$ for all m, and hence $m = 0$ is the unique solution for all $t > 0$. If $\frac{1}{2} \leq J \leq 1$, then ${a^J}''(m) < 0$ for all m, hence a^J is convex, and so it suffices to compare slopes at 0: ${k^{J,0}}'(0) = {a^J}'(0) = 2J - 1 < 1$ and ${l_{t,0}}'(0) = \coth(2t) > 1$. Again, $m = 0$ is the unique solution for all $t > 0$ (see Fig. 2.6).

(ii) As before, $a^{J''}(m) < 0$ for all m, but now the slopes at 0 can be equal, which occurs when $t = \Psi_c$ with Ψ_c defined in (2.36). This proves that $\Delta_t = \{0\}$ for $0 < t \leq \Psi_c$ and $\Delta_t \subseteq \{-\hat{m}(t), 0, \hat{m}(t)\}$ = the set of solutions of (2.22) for $t > \Psi_c$. It is easily seen from Fig. 2.7(a) that $\hat{m}(t)$ is continuous and strictly increasing on $[\Psi_c, \infty)$ and $\hat{m}(\Psi_c) = 0$. It remains to show that $\{-\hat{m}(t), \hat{m}(t)\}$ are the global minima for all $t > \Psi_c$. This follows from the strategy behind the proof of Lemma 2.2.2. Since $C_{t,0}(0) = 0$ for all $t > 0$, it suffices to prove that $m \mapsto C_M(m)$ is strictly decreasing. From (2.53) we have

$$C'_M(m) = \frac{\partial C_{t(m),0}}{\partial m}(m) + \frac{\partial C_{t(m),0}}{\partial t}(m)\, t'(m). \tag{2.55}$$

The first term is zero by the definition of $t(m)$ (each m is a stationary point of $C_{t,0}$ at time $t = t(m)$). The second term is < 0 because $t'(m) > 0$ and

$$\begin{aligned}
\frac{\partial C_{t,0}}{\partial t}(m) &= L(\hat{\varphi}^m_{t,0}(t), \dot{\hat{\varphi}}^m_{t,0}(t)) \\
&\quad + \int_0^t \left[\frac{\partial L}{\partial m}\big(\hat{\varphi}^m_{t,0}(s), \dot{\hat{\varphi}}^m_{t,0}(s)\big)\, \frac{\partial \hat{\varphi}^m_{t,0}}{\partial t}(s) + \frac{\partial L}{\partial \dot{m}}\big(\hat{\varphi}^m_{t,0}(s), \dot{\hat{\varphi}}^m_{t,0}(s)\big)\, \frac{\partial \dot{\hat{\varphi}}^m_{t,0}}{\partial t}(s) \right] ds \\
&= L(0, \dot{\hat{\varphi}}^m_{t,0}(t)) + \int_0^t \frac{\partial}{\partial s} \left\{ \frac{\partial L}{\partial \dot{m}}\big(\hat{\varphi}^m_{t,0}(s), \dot{\hat{\varphi}}^m_{t,0}(s)\big)\, \frac{\partial \hat{\varphi}^m_{t,0}}{\partial t}(s) \right\} ds \\
&= L(0, r) - \frac{\partial L}{\partial \dot{m}}(0, r)\, r \\
&= -\tfrac{1}{2}\sqrt{4 + r^2} + 1 < 0
\end{aligned} \tag{2.56}$$

with $r = \dot{\hat{\varphi}}^m_{t,0}(t)$, where the second equality uses (2.43). Since $C_M(0) = 0$, this yields the claim (see Figs. 2.7(a),2.7(b),2.7(c)).

(iii) This case is more difficult, because a^J no longer is convex on $(0, 1)$. Let Ψ^1_c be the first time at which a solution different from 0 exists. To identify Ψ^1_c, let

$$T_m(x) := (x - m)a^{J'}(m) + a^J(m), \tag{2.57}$$

and let m_1 be the solution of the equation $T_{m_1}(0) = 0 = -m_1 a^{J'}(m_1) + a^J(m_1)$, i.e.,

$$m_1 = \frac{a^J(m_1)}{a^{J'}(m_1)}. \tag{2.58}$$

From m_1 we get t^1_c by using (2.52): $t^1_c = t(m_1)$. As before, a solution of (2.22) for $t \geq t^1_c$ is not necessarily a minimum. To find out when it is, we follow the same strategy as in case (ii). Again, $C_{t,0}(0) = 0$ for all $t > 0$, and hence we must look for $m_* > 0$ such that

$$C_M(m_*) = 0. \tag{2.59}$$

Knowing m_*, we are able to compute Ψ_c using (2.52),

$$t_* = t(m_*). \tag{2.60}$$

In words, t_* is the first time at which 0 no longer is a global minimum. As in case (ii), it suffices to prove that $m \mapsto C_M(m)$ is strictly decreasing on (m_*, ∞). Again, we have (2.55). Since

$$t'(m) = \frac{1}{2}(\text{arccoth})'\left(\frac{a^J(m)}{m}\right)\left\{a^{J\prime}(m) - \frac{a^J(m)}{m}\right\}\frac{1}{m}, \tag{2.61}$$

it follows that $t'(m) = 0$ if and only of $m = a^J(m)/a^{J\prime}(m)$, which is the same condition as (2.58). This gives us a graphical argument to conclude that $t'(m) < 0$ for $0 < m < m_1$ and $t'(m) > 0$ for $m > m_1$ (see Figs. 2.7(d),2.7(e),2.7(f)). On the other hand, $m_* > m_1$. □

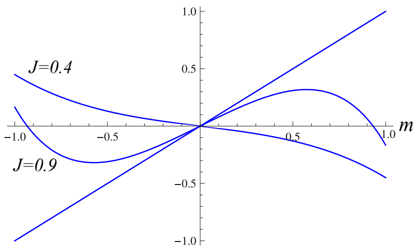


Figure 2.6: $m \mapsto a^J(m)$, $m \mapsto l_{t,0}(m)$ for Regime (i).

2.2.4 Proof of Theorem 2.1.8

Proof. (i) First note that, because $C_M(m_*) = 0$,

$$\frac{\partial m_*}{\partial J} = -\frac{\dfrac{\partial C_M}{\partial J}(m_*)}{\dfrac{\partial C_M}{\partial m}(m_*)} . \tag{2.62}$$

As in Section 2.2.3, case (iii), we have $(\partial C_M/\partial m)(m_*) < 0$. But

$$\frac{\partial C_M}{\partial J}(m_*) > 0 \qquad \Longleftrightarrow \qquad m_* < \tanh(Jm_*), \tag{2.63}$$

which yields the claim because $m_* < m^\infty$.

(ii) The claim is straightforward for $1 < J \leq \frac{3}{2}$. For $J > \frac{3}{2}$ we need to prove that the function $J \to a^J(m_*)/m_*$ is strictly increasing. In fact,

$$\begin{aligned}\frac{\partial}{\partial J}\left[\frac{a^J(m_*)}{m_*}\right] &= \frac{1}{m_*^2}\left[-\frac{\partial m_*}{\partial J}\left\{a^J(m_*) - m_*\frac{\partial a^J}{\partial m}(m_*)\right\} + m_*\frac{\partial a^J}{\partial J}(m_*)\right] \\ &= 1 + \cosh(2Jm_*) - m_*\sinh(2Jm_*) .\end{aligned} \tag{2.64}$$

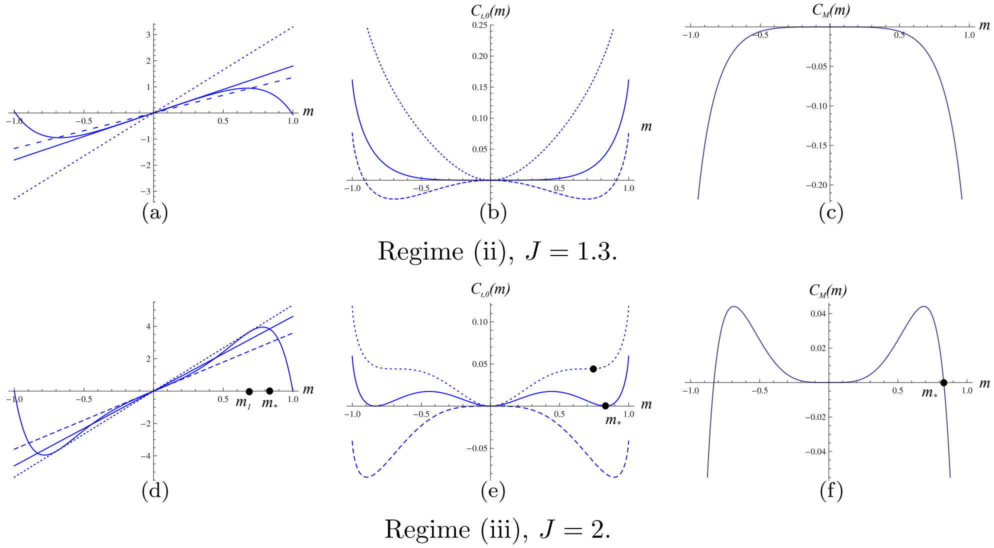


Figure 2.7: (a+d) $m \mapsto a^J(m)$, $m \mapsto l_{t,0}(m)$; (b+e) $m \mapsto C_{t,0}(m)$, for $0 \le t < \Psi_c$ (dotted), $t = \Psi_c$ (drawn), $t > \Psi_c$ (dashed); (c+f) $m \mapsto C_M(m)$.

The strict positivity of the last expression is equivalent to the inequality $m_* < \coth(Jm_*)$, which is satisfied for $J > 1$.
For later use, we have that from (2.28) it follows that

$$\tilde{m} > m,\, \tilde{t} > t \qquad \Longrightarrow \qquad \hat{\varphi}^{\tilde{m}}_{t,0}(s) > \hat{\varphi}^{m}_{t,0}(s) \quad \forall\, 0 \le s \le t. \tag{2.65}$$

(iii) We have $a^J(m) \downarrow a^{\frac{3}{2}}(m)$ as $J \downarrow \frac{3}{2}$ for all $m \in (0,1)$, with a^J and $a^{\frac{3}{2}}$ continuous. By Dini's theorem, the convergence is uniform.

(iv) Since $\Psi_c(\tilde{J}) < \Psi_c(\frac{3}{2})$, the same argument as in case (iii) can be used.

(v) and (vi) are consequences of parts (ii) and (iii) of Theorem 2.1.7. □

2.3 Proof of Theorem 2.1.9

In Sections 2.3.1 and 2.3.2 we prove that overshoots, respectively, bifurcations take place in regime (1), respectively (2)–(3), of Theorem 2.1.9. The analysis of the former regime does not distinguish whether the initial field h is zero or not.

2.3.1 Overshoots

The trick is again to write t as a function of m. From (2.22), we have

$$-k^{J,h}(m) + 2m\coth(2t) = m\coth(2t) + \alpha\,\mathrm{csch}(2t). \tag{2.66}$$

Hence, from (2.23),

$$\begin{aligned} k^{J,h}(m)[-k^{J,h}(m) + 2m\coth(2t)] \\ = [m\coth(2t) - \alpha\,\mathrm{csch}(2t)]\,[m\coth(2t) + \alpha\,\mathrm{csch}(2t)] \end{aligned} \tag{2.67}$$

which implies

$$-k^{J,h}(m)^2 - \alpha^2 = (m^2-\alpha^2)\coth^2(2t) - 2mk^{J,h}(m)\coth(2t)\ . \tag{2.68}$$

Solving for t we find

$$t_{\mathrm{F}}(m) := \begin{cases} \frac{1}{2}\,\mathrm{arccoth}\Big(\frac{mk^{J,h}(m) + |\alpha|\,\sqrt{\Phi(m)}}{m^2-\alpha^2}\Big) & \text{if } m \neq \pm\alpha \\ \frac{1}{2}\,\mathrm{arccoth}\Big(\frac{\left[k^{J,h}(m)\right]^2 + m^2}{2m\,k^{J,h}(m)}\Big) & \text{if } m = \pm\alpha\ ,\ m\,k^{J,h}(m) < 0 \\ 0 & \text{if } m = \pm\alpha\ ,\ m\,k^{J,h}(m) > 0 \end{cases} \tag{2.69}$$

$$t_{\mathrm{L}}(m) := \begin{cases} \frac{1}{2}\,\mathrm{arccoth}\Big(\frac{mk^{J,h}(m) - |\alpha|\,\sqrt{\Phi(m)}}{m^2-\alpha^2}\Big) & \text{if } m \neq \pm\alpha \\ \frac{1}{2}\,\mathrm{arccoth}\Big(\frac{\left[k^{J,h}(m)\right]^2 + m^2}{2m\,k^{J,h}(m)}\Big) & \text{if } m = \pm\alpha\ ,\ m\,k^{J,h}(m) > 0 \\ 0 & \text{if } m = \pm\alpha\ ,\ m\,k^{J,h}(m) < 0 \end{cases} \tag{2.70}$$

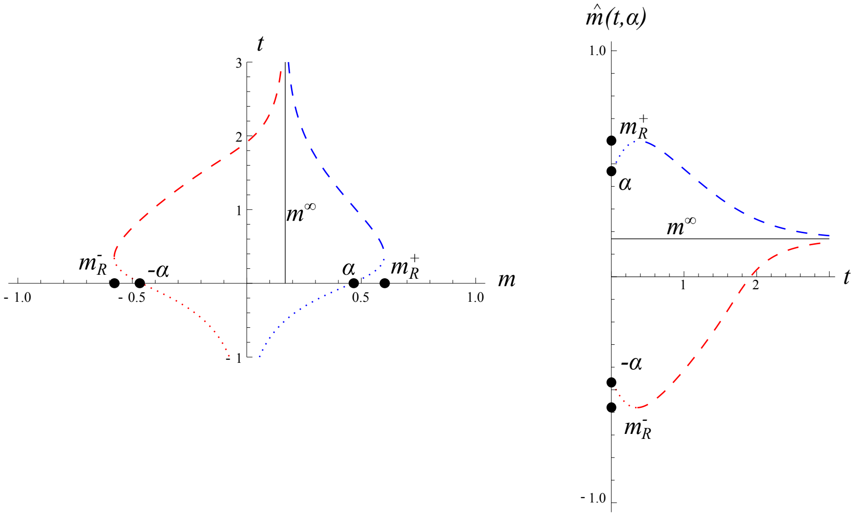
with

$$\Phi(m) := k^{J,h}(m)^2 - m^2 + \alpha^2. \tag{2.71}$$

These are times at which m is a stationary point (not necessarily a minimum) both for $C_{t,\alpha}(m)$ and for $C_{t,-\alpha}(m)$ [Equation (2.67), and hence the solutions (2.69), are insensitive to the sign of α]. A necessary condition for overshoots and undershoots occur for values of m satisfying (i) $t_F(m) > 0$ and $t_L(m) > 0$, and (ii) at these times m is a minimum. The m-dependence of t_F and t_L is depicted in Fig. 2.8 and 2.9 for cases where both functions are injective, i.e., for α for which there is only one critical point for each t. In more complicated cases, for instance, when overshoot and bifurcation occur simultaneously, there are two or more stationary points, only one of which is a minimum.
We divide the analysis into four steps.

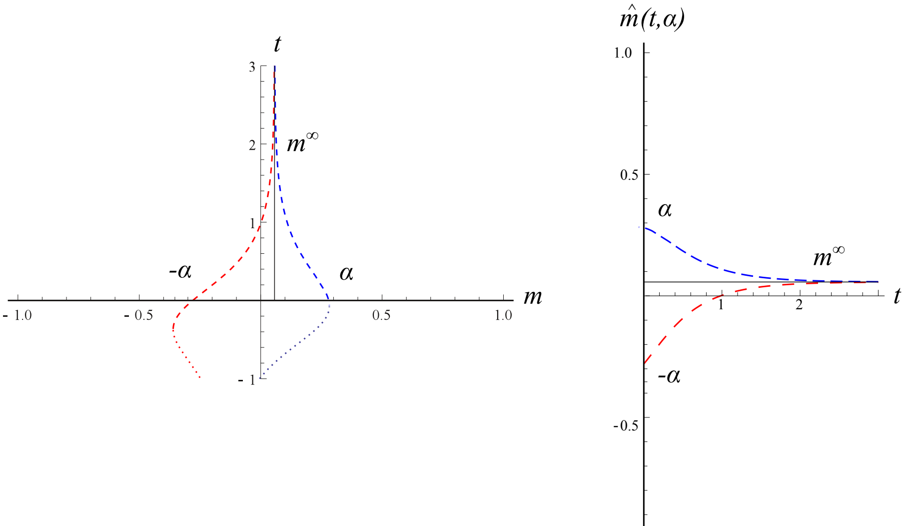
Step A: *Existence of z^+, z^- and z^{-+}.* We observe that there exists a unique $m^\infty > 0$ such that $k^{J,h}(m^\infty) = m^\infty$. Furthermore,

$$k^{J,h}(m) = 0 \qquad \Longleftrightarrow \qquad \tanh(2h) = \frac{-a^J(m)}{b^J(m)} =: A(m). \tag{2.72}$$



Plot of t_F (dotted), t_L (dashed) for α and $-\alpha$. Plot of $t \mapsto \hat{m}(t)$ for α and $-\alpha$.

Figure 2.8: Overshoot for (J, h, α) in Regime (1a) or for $(J, h, -\alpha)$ in Regime (1d). Parameters: $(J, h) = (0.95, 0.01)$, $\alpha = 0.46$.



Plot of t_F (dotted), t_L (dashed) for α and $-\alpha$. Plot of $t \mapsto \hat{m}(t, \alpha)$ for α and $-\alpha$.

Figure 2.9: Absence of overshoot for (J, h, α) in Regime (1b) or for $(J, h, -\alpha)$ in Regime (1c). Parameters: $(J, h) = (0.3, 0.04)$, $\alpha = 0.28$.

This function A has the features depicted in Fig. 2.10: it is odd, satisfies $A(0) = 0$ and $A(1) = 1$, and is convex with only one global minimum between 0 and 1. We conclude that there exists a unique $z^+ = z^+(J, h) > 0$ such that $k^{J,h}(z^+) = 0$ and, in addition, if $\tanh(2h) < \max_{m\in[-1,0]} A(m)$, then there exist $-1 < z^- < z^{-+} < 0$ such that $k^{J,h}(z^-) = k^{J,h}(z^{-+}) = 0$ (see Fig. 2.10).

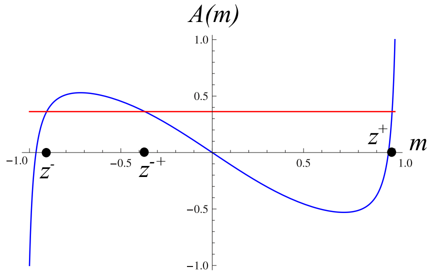


Figure 2.10: Plot of A and intersection with the constant $\tanh(2h)$.

Step B: *Existence of* m_R^+, m_R^- *and relation between* m_∞ *and* α.

(Ba) Existence of m_R^+: $k^{J,h}(\alpha) > 0$ together with $\alpha > 0$ and Step A imply $\alpha < z^+$. Since $\Phi(\alpha) > 0$ and $\Phi(z^+) < 0$, it follows that there exists a m_R^+ such that $0 < \alpha < m_R^+ < z^+$ and

$$\Phi(m_R^+) = 0. \tag{2.73}$$

The latter in turn implies that $k^{J,h}(m_R^+)^2 = {m_R^+}^2 - \alpha^2$. This, together with $k^{J,h}(m_R^+) > 0$, implies that $k^{J,h}(m_R^+) < m_R^+$, which leads to $m^\infty < m_R^+$.

(Bd) Existence of m_R^-: As in (Ba), $\Phi(z^-) < 0$ and $\Phi(\alpha) > 0$ imply that there exists a m_R^- such that $z^- < m_R^- < \alpha < 0$ and

$$\Phi(m_R^-) = 0. \tag{2.74}$$

(Bc): $k^{J,h}(\alpha) > 0$ and $\alpha < 0$ imply $\alpha < m_\infty$. This follows from the fact that $k^{J,h}(\alpha) > \alpha$ implies $\alpha < m_\infty$ by (2.33).

(Bb): $k^{J,h}(\alpha) < 0$ and $\alpha > 0$ imply $\alpha > m_\infty$. Again, this is a consequence of (2.33).

Step C: *Consequence of the positivity of times.* Only positive solutions of Equation (2.69) are of interest. This implies the constraints

$$t_F(m) > 0 \quad \Longleftrightarrow \quad \eta_F(m) := m\, k^{J,h}(m) + (\alpha^2 - m^2) + |\alpha|\sqrt{\Phi(m)} \begin{cases} > 0 & \text{if } m^2 > \alpha^2, \\ < 0 & \text{if } m^2 < \alpha^2, \end{cases} \tag{2.75}$$

and

$$t_L(m) > 0 \quad \Longleftrightarrow \quad \eta_L(m) := m\, k^{J,h}(m) + (\alpha^2 - m^2) - |\alpha|\sqrt{\Phi(m)} \begin{cases} > 0 & \text{if } m^2 > \alpha^2, \\ < 0 & \text{if } m^2 < \alpha^2. \end{cases} \tag{2.76}$$

The functions η_F and η_L satisfy

$$\eta_F(\alpha) = \alpha k^{J,h}(\alpha) + |\alpha||k^{J,h}(\alpha)| = \begin{cases} > 0 & \text{if } \alpha\ k^{J,h}(\alpha) > 0, \\ = 0 & \text{if } \alpha\ k^{J,h}(\alpha) < 0, \end{cases} \tag{2.77}$$

$$\eta_L(\alpha) = \alpha k^{J,h}(\alpha) - |\alpha||k^{J,h}(\alpha)| = \begin{cases} = 0 & \text{if } \alpha\ k^{J,h}(\alpha) > 0, \\ < 0 & \text{if } \alpha\ k^{J,h}(\alpha) < 0. \end{cases} \tag{2.78}$$

Also, from (2.33),

$$\begin{array}{ll} \eta_F(m^\infty) = 2|\alpha|^2, & \eta_F'(m^\infty) = 2m^\infty k^{J,h\prime}(m^\infty), \\ \eta_L(m^\infty) = 0, & \eta_L'(m^\infty) = 0. \end{array} \tag{2.79}$$

The last line implies that m^∞ is a root of η_L, but that there is no change of sign around it. Finally, from expressions (2.75)-(2.79) we conclude that:

- The zeros of $\eta_F,\ \eta_L$ are a subset of $\{m^\infty, \pm\alpha\}$.
- The intervals in which η_F and η_L satisfies the constraints (2.75)-(2.76) are:

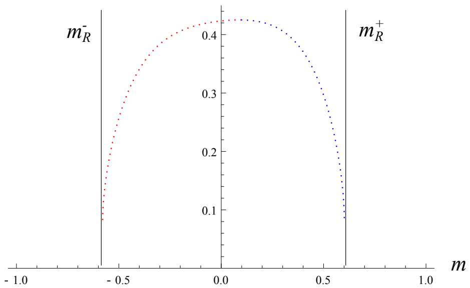
Regime	$(1a)$	$(1b)$	$(1c)$	$(1d)$
Condition	$\alpha > 0$, $k^{J,h}(\alpha) > 0$	$\alpha > 0$, $k^{J,h}(\alpha) < 0$	$\alpha < 0$, $k^{J,h}(\alpha) > 0$	$\alpha < 0$, $k^{J,h}(\alpha) < 0$
$t_F > 0$	$[\alpha, m_R^+]$	$\varnothing$	$\varnothing$	$[m_R^-, \alpha]$
$t_L > 0$	$[m^\infty, m_R^+]$	$[m^\infty, \alpha]$	$[\alpha, m^\infty]$	$[m_R^-, m^\infty]$

We observe that in regime (1a) each value of $m \in [\alpha \wedge m_\infty, m_R^+]$ is attained at two different times t_F ("First") and t_L ("Last"). The same happens in regime (1d) for $m \in [m_R-, \alpha \vee m_\infty]$. These phenomena correspond, respectively, to an overshoot and an undershoot. The proof is completed by showing that the initial condition of the trajectories have the right monotonicity properties.

Step D: *Monotonicity.*
By using implicit differentiation we get

$$\frac{\partial \hat{m}}{\partial t}(t) = \frac{\frac{\partial l_{t,\alpha}}{\partial t}(\hat{m})}{[k^{J,h}]'(\hat{m}) - l_{t,\alpha}'(\hat{m})} = \frac{2\operatorname{csch}(2t)\Big\{\hat{m}\operatorname{csch}(2t) - \alpha\coth(2t)\Big\}}{[k^{J,h}]'(\hat{m}) - \coth(2t)}\ . \tag{2.80}$$

On the other hand, if $\hat{m} = \hat{m}(t)$ is a critical point $\big(k^{J,h}(\hat{m}) = l_{t,\alpha}(\hat{m})\big)$, then $\Phi(\hat{m}) = (\hat{m}\operatorname{csch}(2t) - \alpha\coth(2t))^2$. Hence



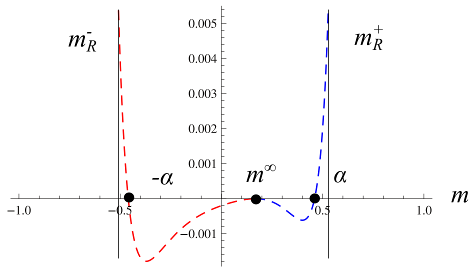


Figure 2.11: η_F (dotted) and η_L (dashed) for (J, h, α) in regime 1(a) and for $(J, h, -\alpha)$ in regime 1(d). Parameters: $(J, h) = (0.95, 0.01)$, $\alpha = 0.46$ as in Fig. 2.8.

$$\frac{\partial \hat{m}}{\partial t}(t) = 0 \quad \Longleftrightarrow \quad \Phi(\hat{m}(t)) = 0.$$

Splitting into cases according to different values of $\hat{m}$, we conclude from (2.73–2.80) the following monotonicity properties of the trajectories:

Regime	$(1a)$	$(1b)$	$(1c)$	$(1d)$
$\hat{m}$ incr.	$0 < t < t(m_R^+)$	$\varnothing$	$0 < t < \infty$	$t(m_R^-) < t < \infty$
$\hat{m}$ decr.	$t(m_R^+) < t < \infty$	$0 < t < \infty$	$\varnothing$	$0 < t < t(m_R^-)$

This concludes the proof of part 1 of Theorem 2.1.9.

2.3.2 Bifurcation

Bifurcation proofs rely on the following facts.

(B1) For short times there is a unique critical point, close to α.

(B2) Therefore, in order bifurcation to occur, a local maximum and a local minimum must appear in the course of time. Given condition (2.22), standard arguments imply that two (or more) stationary points appear at times larger than $\tilde{t}$ if the curves $l_{\tilde{t},\alpha}$ and $k^{J,h}$ become tangent at a certain magnetization $\tilde{m}$. The pairs $(\tilde{m}, \tilde{t})$ are determined by the following two equations (a similar argument was used in the proof of Theorem 2.1.7(iii)):

$$\begin{aligned} \left[k^{J,h}\right]'(\tilde{m}) &= \coth(2\tilde{t}), \\ k^{J,h}(\tilde{m}) &= \tilde{m}\coth(2\tilde{t}) - \alpha\,\mathrm{csch}(2\tilde{t}). \end{aligned} \tag{2.81}$$

Inserting the first equation into the second, we get

$$F(\tilde{m}) := \frac{\tilde{m}\left[k^{J,h}\right]'(\tilde{m}) - k^{J,h}(\tilde{m})}{\mathrm{csch}\left[\mathrm{arccoth}\left(\left[k^{J,h}\right]'(\tilde{m})\right)\right]} = \alpha. \tag{2.82}$$

We are left with the task of determining whether or not this equation has solutions. Note that

$$F'(m) = 0 \quad \Longleftrightarrow \quad \big[k^{J,h}\big]''(m) = 0 \quad \text{or} \quad m = k^{J,h}(m)\big[k^{J,h}\big]'(m). \tag{2.83}$$

In what follows all the assertions about F can be checked by using the equivalence in (2.83) and doing a straightforward analysis of $k^{J,h}$.

(B3) $t \mapsto C_{t,\alpha}$ is continuous with respect to $\|\cdot\|_\infty$. Hence, when a new minimum appears it cannot be a global one. Both $\alpha \mapsto C_{t,\alpha}$ and $h \mapsto C_{t,\alpha}$ are also continuous.

(B4) When $t \to \infty$ we have two global minima $\pm m^\infty$ if $h = 0$. If $h > 0$, then the symmetry is broken and there is only one global minimum $m^\infty > 0$.

Whenever a local maximum/local minimum appears (disappears), we will refer to this behaviour as LMLMA (LMLMD). We proceed by looking at $h = 0$ and $h \neq 0$ separately.

Part (2) ($h = 0$, see Fig. 2.12). The scenario for $\alpha = 0$ has already been proven in Theorem 2.1.7. We concentrate on $\alpha > 0$; this is no loss of generality due to the antisymmetry of F.

Claim: Whenever $\alpha > 0$, negative solutions of (2.82) cannot cause bifurcations. In fact, let t_{nc} be the time at which the critical points in the negative side emerge. Let also

$$d(t) := C_{t,\alpha}(\overline{m}_-(t,\alpha)) - C_{t,\alpha}(\hat{m}(t,\alpha)), \qquad t \geq t_{nc},$$

where $\overline{m}_-(t,\alpha)$ is the negative *local* [because of **(B3)**] minimum and $\hat{m}(t,\alpha)$ is the global minimum of $C_{t,\alpha}$. The last one is positive due to **(B1)** and the assumption of $\alpha > 0$. By definition, $d(t_{nc}) > 0$ and, by **(B4)**, $\lim_{t\to\infty} d(t) = 0$. Doing calculations similar to (2.56) and using that $\overline{m}_-(t,\alpha)$, $\hat{m}(t,\alpha)$ are both critical points, we get that $d'(t) < 0 \ \forall\, t > t_{nc}$. This proves the claim.

In what follows we focus on (2.82). By the previous claim, in order a bifurcation to occur a positive solution of (2.82) is needed.

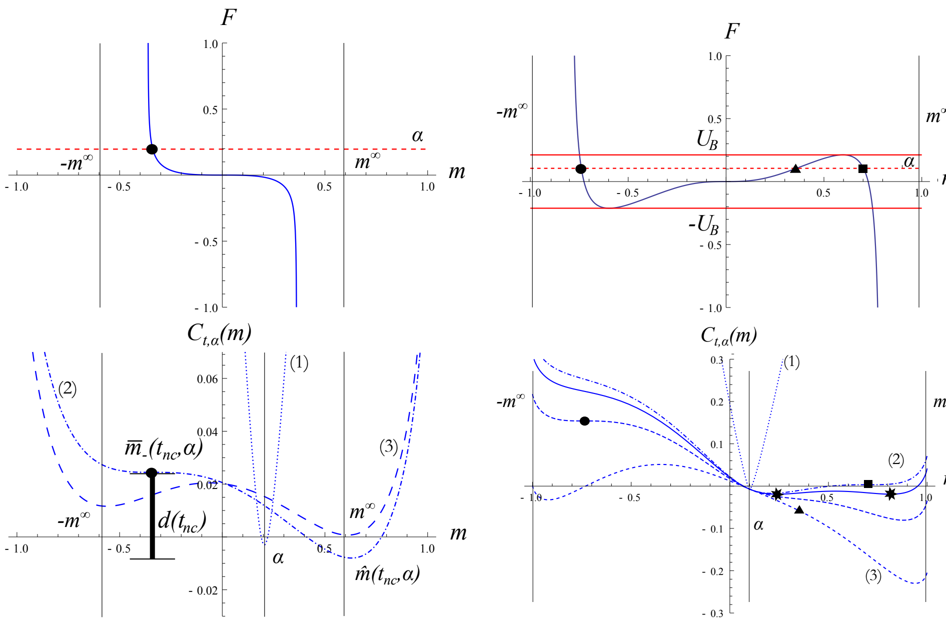
(2a-b) If $0 < J \leq \frac{3}{2}$, then $F'(m) < 0$ for all $m \in (-1,+1)$. Hence, for all $\alpha > 0$ there is only one solution of (2.82). This solution turns out to be negative and hence it cannot correspond to a bifurcation.

(2c) If $J > \frac{3}{2}$, then F has only one global maximum on the positive side, with value $U_B = U_B(J) > 0$. Combining **(B1)-(B4)**, we get that there is bifurcation if and only if $\alpha \in [0, U_B] = \mathrm{Im}(F|_{[0,1]})$.

Part (3) ($h > 0$, see Fig. 2.13).
Remark: If $h > 0$, then **(B4)** holds (symmetry breaking) allows the appearance of negative solutions of (2.82), leading to bifurcations.

Once more, let us study the different scenarios for F when $h > 0$.

(3a) If $0 < J \leq 1$, then $\mathrm{Im}(F) \subseteq [-1,+1]^c$. Therefore (2.82) has no solution for any $|\alpha| \leq 1$ and hence, no bifurcation.



Regime (2a-b), $J = 1.15$, $h = 0$ Regime (2c), $J = 2.5$, $h = 0$.

Figure 2.12: First row: Plot of $m \mapsto F(m)$. ● = LMLMA not leading to bifurcation; ▲ = LMLMD; ■ = LMLMA leading to bifurcation; ★ = bifurcation . Second row: Plot of $m \mapsto C_{t,\alpha}(m)$ for different times. Short time (dotted) bifurcation (solid), long time (dashed).

(3b) If $1 < J \leq \frac{3}{2}$, then F has a unique maximum for $m \in [0,1]$ with value $U_B = U_B(J,h) < 0$, and $[-1, U_B] = \mathrm{Im}(F|_{[0,1]})$. Arguing as in the claim of Part 2 for $\alpha > U_B$, we conclude that there is bifurcation if and only if $\alpha \in [-1, U_B]$.

(3c) Assume $J > \frac{3}{2}$.

1. For $h > 0$ small enough the behaviour is "close" to that for $h = 0$ due to the continuity of $h \mapsto C_{t,\alpha}$ with respect to the infinity norm. Indeed, there exist $L_B := \min_{[-1,0]} F \approx -U_B(J,0)$ and $U_B := \max_{[0,1]} F \approx U_B(J,0)$ with $(-1, U_B] = \mathrm{Im}(F|_{[0,1]})$. There are different regimes for α:

   a) For $\alpha < L_B$, there is a unique solution of (2.82), which is on the positive side, leading to a bifurcation [because of **(B4)**].

   b) For $0 > \alpha \gtrapprox L_B$, there are both a negative and a positive solution of (2.82). Both lead to bifurcations, the negative one by continuity **(B3)** and the positive one due to **(B3)**-**(B4)**. The negative solution appears earlier in time. We write $s_B = s_B(\alpha)$ for the first bifurcation time (negative side) and $t_B = t_B(\alpha)$ for the second bifurcation time (positive side).

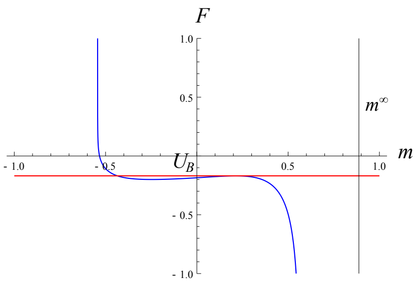
   c) For $0 < \alpha \lessapprox U_B$ there is LMLMA on the positive and the negative side. As in the case $h = 0$, the negative one does not lead to a bifurcation, and thus only one bifurcation occurs, which happens to be on the positive side.
   d) By the continuity property **(B3)** and the monotonicity of $\alpha \mapsto s_B(\alpha)$ (proved below), the two previous regimes merge, leading to an intermediate value $M_T \in (L_B, U_B)$ such that trifurcation occurs at $\alpha = M_T$.
   e) For $\alpha > U_B$ there is no positive solution to (2.82) with $\alpha \in [0, 1]$. Hence, no bifurcation occurs.

2. The limit $h \to \infty$ in (2.82) yields

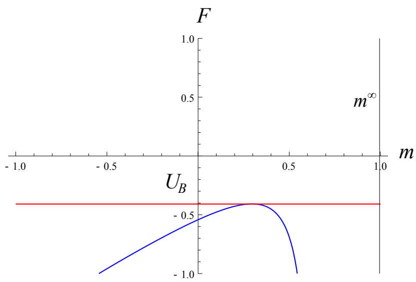
$$\frac{m\big(a^{J\prime}(m) + b^{J\prime}(m)\big) - \big(a^J(m) + b^J(m)\big)}{\big(a^{J\prime}(m) + b^{J\prime}(m)\big)} = \alpha.$$

   Hence, for $h > 0$ large enough we get a behavior similar to (3b), but with $U_B(J, h) > 0$.

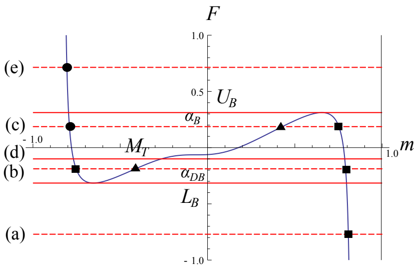
3. The existence of h^* follows from the continuity of the function $h \mapsto C_{t,\alpha}$ commented in **(B3)** with respect to h.



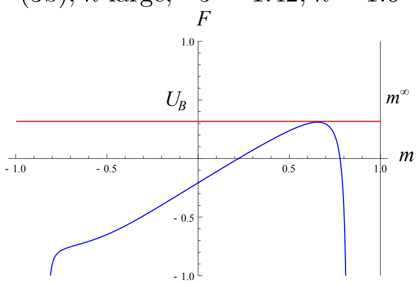
(3b), h small, $J = 1.42$, $h = 0.15$



(3b), h large, $J = 1.42$, $h = 1.6$



(3c), h small, $J = 2.9$, $h = 0.15$



(3c), h large, $J = 2.9$, $h = 1.6$

Figure 2.13: Plot of $m \mapsto F(m)$ for different regimes of J when $h > 0$. ● = LMLMA not causing a bifurcation; ▲ = LMLMD; ■ = LMLMA causing a bifurcation.

Monotonicity of the functions $t_B(\alpha)$ and $s_B(\alpha)$

The bifurcation times are characterized by the following equations:

$$\begin{array}{rcl} k^{J,h}(\hat{m}_1) & = & l_{t_B,\alpha}(\hat{m}_1), \\ k^{J,h}(\hat{m}_2) & = & l_{t_B,\alpha}(\hat{m}_2), \\ C_{t_B,\alpha}(\hat{m}_1) & = & C_{t_B,\alpha}(\hat{m}_2). \end{array} \tag{2.84}$$

The first two equations say that $\hat{m}_1$ and $\hat{m}_2$ are stationary points at the same time t_B, while the third one establishes the equality of costs at time t_B. Taking the derivative with respect to α of the third equation, we get

$$\frac{\partial t_B}{\partial \alpha} = -\frac{\dfrac{\partial C_{t_B,\alpha}}{\partial \alpha}(\hat{m}_2) - \dfrac{\partial C_{t_B,\alpha}}{\partial \alpha}(\hat{m}_1)}{\dfrac{\partial C_{t_B,\alpha}}{\partial t_B}(\hat{m}_2) - \dfrac{\partial C_{t_B,\alpha}}{\partial t_B}(\hat{m}_1)} \,. \tag{2.85}$$

A straightforward computation that uses the first two equations shows that $\frac{\partial t_B}{\partial \alpha} < 0$, which implies that $\alpha \mapsto t_B(\alpha)$ is continuous and decreasing. A similar argument shows that $\alpha \mapsto s_B(\alpha)$ is continuous and increasing.

3 Local mean-field context

This chapter is based on:
R. Fernández, F. den Hollander, and J. Martínez. *Variational description of Gibbs-non-Gibbs dynamical transitions for spin-flip systems with a kac-type interaction. arXiv preprint:1309.3667*, Submitted to Journal of Statistical Physics.

Abstract

We continue our study of Gibbs-non-Gibbs dynamical transitions. In the present chapter we consider a system of Ising spins on a large discrete torus with a Kac-type interaction subject to an independent spin-flip dynamics (infinite-temperature Glauber dynamics). We show that, in accordance with the program outlined in [vEFdHR10], in the thermodynamic limit Gibbs-non-Gibbs dynamical transitions are *equivalent* to bifurcations in the set of global minima of the large-deviation rate function for the trajectories of the empirical density *conditional* on their endpoint. More precisely, the time-evolved measure is non-Gibbs if and only if this set is not a singleton for *some* value of the endpoint. A partial description of the possible scenarios of bifurcation is given, leading to a characterization of passages from Gibbs to non-Gibbs and vice versa, with sharp transition times.

Our analysis provides a conceptual step-up from our earlier work on Gibbs-non-Gibbs dynamical transitions for the Curie-Weiss model, where the mean-field interaction allowed us to focus on trajectories of the empirical magnetization rather than the empirical density.

MSC 2010. 60F10, 60K35, 82C22, 82C27.
Key words and phrases. Curie-Weiss model, Kac model, spin-flip dynamics, Gibbs versus non-Gibbs, dynamical transition, large deviation principles, action integral, bifurcation of rate function.
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3.1 Introduction and main results

3.1.1 Background

Gibbs-non-Gibbs dynamical transitions are a surprising phenomenon. An initial Gibbsian state (e.g. a collection of interacting Ising spins) is subjected to a stochastic dynamics (e.g. a Glauber dynamics) at a temperature that is *different* from that of the initial state. For many combinations of initial and dynamical temperature, the time-evolved state is observed to become non-Gibbs after a finite time. Such a state cannot be described by any absolutely summable Hamiltonian and therefore *lacks a well-defined notion of temperature*.

The phenomenon was originally discovered by van Enter, Fernández, den Hollander and Redig [vEFdHR02] for *heating dynamics*, in which a low-temperature Ising model is subjected to a high-temperature Glauber dynamics. The state remains Gibbs for short times, but becomes non-Gibbs after a finite time. Remarkably, heating in this case does not lead to a succession of states with increasing temperature, but to states where the notion of temperature is *lost altogether*. Moreover, it turned out that there is a difference depending on whether the initial Ising model has zero or non-zero magnetic field. In the former case, non-Gibbsianness once lost is never recovered, while in the latter case Gibbsianness is recovered at a later time.

This initial work triggered a decade of developments. By now, results are available for a variety of interacting particle systems, both for *heating dynamics* and for *cooling dynamics*, including estimates on transition times and characterizations of the so-called *bad configurations* leading to non-Gibbsianness, i.e., the discontinuity points of the conditional probabilities. It has become clear that Gibbs-non-Gibbs transitions are the rule rather than the exception. For references we refer to the recent overview by van Enter [vE12].

3.1.2 Motivation and outline

The ubiquity of the Gibbs-non-Gibbs phenomenon calls for a better understanding of its causes and consequences. Historically, non-Gibbsianness is proved by looking at the evolving system at two times, the inital time and the final time, and applying techniques from equilibrium statistical mechanics. This is an indirect approach that does not illuminate the relation between the Gibbs-non-Gibbs phenomenon and the dynamical effects responsible for its occurrence. This unsatisfactory situation was addressed in van Enter, Fernández, den Hollander and Redig [vEFdHR10], where possible dynamical mechanisms were proposed and a *program* was put forward to develop a theory of Gibbs-non-Gibbs transitions on *purely dynamical grounds*.

In Fernández, den Hollander and Martínez [FdHM13a], building on earlier work by Külske and Le Ny [KLN07] and Ermolaev and Külske [EK10], we showed that this program can be fully carried out for the Curie-Weiss model subject to an infinite-temperature dynamics. The goal of the present paper is to extend this work away from the mean-field setting by considering a model with a Kac-type interaction, i.e., Ising spins with a long-range interaction. Whereas for the Curie-Weiss model the key object was the empirical magnetization in the thermodynamic limit, for the Kac model the key object is the *empirical density* in the thermodynamic limit, which we refer to as the *profile*. Non-Gibbsianness corresponds to a discontinuous dependence of the law of the initial profile *conditional* on the final profile. The discontinuity points are called *bad profiles* (Definition 3.1.1 below).

Dynamically, such discontinuities are expected to arise whenever there is more than one trajectory of the profile that is *compatible* with the bad profile at the end. Indeed, this expectation is confirmed and exploited in the sequel. The actual conditional trajectories are those minimizing the large-deviation rate function on the space of trajectories (Propositions 3.1.2–3.1.3 below), in the spirit of what is behind hydrodynamic scaling. The time-evolved measure is Gibbs whenever there is a single minimizing trajectory for every final profile, in which case the so-called specification kernel can be computed explicitly (Theorem 3.1.4 below). In contrast, if there are multiple optimal trajectories, then the choice of trajectory can be decided by an infinitesimal perturbation of the final profile, and the time-evolved measure is non-Gibbs (Theorem 3.1.6 below).

The rate function for the Kac model contains an action integral whose Lagrangian acts on profiles. This setting constitutes a conceptual step-up from what happens for the Curie-Weiss model, where the Lagrangian acts on magnetizations and is much easier to analyze. However, for infinite-temperature dynamics the Kac Lagrangian can be expressed as an integral of the Curie-Weiss Lagrangian with respect to the profile (Theorem 3.1.5 below). This link allows us to identify the possible scenarios of bifurcation (Theorem 3.1.7 below).

3.1.3 Hamiltonian

Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional unit torus. For $n \in \mathbb{N}$, let $\mathbb{T}_n^d$ be the $(1/n)$-discretization of $\mathbb{T}^d$ defined by $\mathbb{T}_n^d := \Delta_n^d/n$, with $\Delta_n^d := \mathbb{Z}^d/n\mathbb{Z}^d$ the discrete torus of size n. For $n \in \mathbb{N}$, let $\Omega_n := \{-1,+1\}^{\Delta_n^d}$ be the set of Ising-spin configurations on Δ_n^d. The energy of the configuration $\sigma := (\sigma(x))_{x\in\Delta_n^d} \in \Omega_n$ is given by the *Kac-type Hamiltonian*

$$H^n(\sigma) := -\tfrac{1}{2n^d} \sum_{x,y\in\Delta_n^d} J\left(\tfrac{x-y}{n}\right) \sigma(x)\sigma(y) - \sum_{x\in\Delta_n^d} h(\tfrac{x}{n})\,\sigma(x), \qquad \sigma\in\Omega_n, \tag{3.1}$$

where $J,h \in C(\mathbb{T}^d)$ are continuous functions on $\mathbb{T}^d$, with $J \geq 0$ symmetric and $J \not\equiv 0$. The Gibbs measure associated with H^n is

$$\mu^n(\sigma) := \frac{e^{-\beta H^n(\sigma)}}{Z^n}, \qquad \sigma\in\Omega_n, \tag{3.2}$$

with $\beta \in [0,\infty)$ the *static* inverse temperature and Z^n the normalizing partition sum.

3.1.4 Gibbs versus non-Gibbs

For $\Lambda \subseteq \Delta_n^d$, let $\pi_\Lambda^n\colon \Omega_n \to \mathcal{M}(\mathbb{T}_n^d) \subseteq \mathcal{M}(\mathbb{T}^d)$ be the *empirical density* of σ inside Λ defined by

$$\pi_\Lambda^n(\sigma) := \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \sigma(x)\delta_{x/n}, \tag{3.3}$$

where $\mathcal{M}(\mathbb{T}_n^d)$ and $\mathcal{M}(\mathbb{T}^d)$ denote the set of signed measures on $\mathbb{T}_n^d$, respectively, $\mathbb{T}^d$ with total variation norm ≤ 1 endowed with the weak topology, and δ_u is the point measure at $u \in \mathbb{T}^d$. Note that $\sigma \in \Omega_n$ determines $\pi_\Lambda^n \in \mathcal{M}(\mathbb{T}_n^d)$ and vice versa.

Abbreviate (3.3) for $\Lambda = \Delta_n^d$ by π^n and for $\Lambda = \Delta_n^d\backslash\{\lfloor nu \rfloor\}$ by $\pi^{u,n}$, $u \in \mathbb{T}^d$, where $\lfloor nu \rfloor$ denotes the component-wise lower-integer part of nu. The latter is the empirical density *perforated* at $\lfloor nu \rfloor$. Abbreviate

$$\mathcal{M}^n := \pi^n(\Omega_n), \qquad \mathcal{M}^{u,n} := \pi^{u,n}(\Omega_n). \tag{3.4}$$

Note that $\mathcal{M}^n \subseteq \mathcal{M}(\mathbb{T}_n^d)$. Via π^n, the Gibbs measure μ^n on Ω_n in (3.2) induces a probability measure $\breve{\mu}^n$ on $\mathcal{M}^n$ given by

$$\breve{\mu}^n = \mu^n \circ (\pi^n)^{-1}. \tag{3.5}$$

Using (3.3), we can rewrite (3.1) in the form

$$H^n(\sigma) = -n^d H(\pi^n(\sigma)), \tag{3.6}$$

where in the right-hand side we introduce the notation

$$H(\nu) = \left\langle \tfrac{1}{2} J * \nu + h, \nu \right\rangle \tag{3.7}$$

$$[f*\nu](u) := \int_{\mathbb{T}^d} J(u-u')\,\nu(du'), \qquad \langle f,\nu\rangle := \int_{\mathbb{T}^d} f(u)\,\nu(du), \qquad f \in C(\mathbb{T}^d),\ \nu \in \mathcal{M}(\mathbb{T}^d). \tag{3.8}$$

Let $\lambda^n := \frac{1}{n^d}\sum_{x\in\Lambda} \delta_{x/n}$. We have $w-\lim_{n\to\infty} \lambda^n = \lambda$, where λ is the Lebesgue measure on $\mathbb{T}^d$ and $w-\lim$ stands for weak convergence. In what follows we will represent limit distributions in $\mathcal{M}(\mathbb{T}^d)$ with a Lebesgue density as measures $\alpha\lambda$ with $\alpha \in B$, where

$$B \text{ is the closed unit ball in } L^\infty(\mathbb{T}^d). \tag{3.9}$$

We will refer to α as a *profile*.

The definition of Gibbs versus non-Gibbs is the following. Given any sequence $(\rho^n)_{n\in\mathbb{N}}$ with ρ^n a probability measure on Ω_n for every $n \in \mathbb{N}$, define the single-spin conditional probabilities at site $\lfloor nu \rfloor \in \Delta_n^d$ as

$$\gamma^{u,n}\big(\cdot \mid \alpha_{n-1}^u\big) := \rho^n\big(\sigma(\lfloor nu \rfloor) = \cdot \mid \pi^{u,n}(\sigma) = \alpha_{n-1}^u\big), \qquad \alpha_{n-1}^u \in \mathcal{M}^{u,n}. \tag{3.10}$$

Definition 3.1.1. [Good and bad profiles, Gibbs]
(a) A profile $\alpha \in B$ is called good for $(\rho^n)_{n\in\mathbb{N}}$ if there exists a neighborhood $\mathcal{N}_\alpha$ of α in $L^\infty(\mathbb{T}^d)$ such that for all $\tilde\alpha \in \mathcal{N}_\alpha$ and $u \in \mathbb{T}^d$ there exists

$$\gamma^u(\,\cdot\mid\tilde\alpha) := \lim_{n\to\infty} \gamma^{u,n}(\,\cdot\mid\alpha^u_{n-1}) \tag{3.11}$$

where $(\alpha^u_{n-1})_{n\in\mathbb{N}}$ with $\alpha^u_{n-1} \in \mathcal{M}^{u,n}$ is any sequence so that $w-\lim_{n\to\infty}\alpha^u_{n-1} = \tilde\alpha\lambda$, and the limit is independent of the choice of $(\alpha^u_{n-1})_{n\in\mathbb{N}}$.
(b) A profile $\alpha \in B$ is called bad for $(\rho^n)_{n\in\mathbb{N}}$ if it is not good for $(\rho^n)_{n\in\mathbb{N}}$.
(c) $(\rho^n)_{n\in\mathbb{N}}$ is called Gibbs if it has no bad profiles in B.

Remark:
(1) Definition 3.1.1(a) implies continuity of $\alpha \mapsto \gamma^u(\,\cdot\mid\alpha)$ in the $L^\infty(\mathbb{T}^d)$-norm for all $u \in \mathbb{T}^d$ at good profiles. (A proof by contradiction is based on a diagonal argument.)
(2) For $(\mu^n)_{n\in\mathbb{N}}$ with μ^n defined in (3.1–3.2) all profiles $\alpha \in B$ are good with

$$\gamma^u(k\mid\alpha) = \frac{\exp[k\beta\{J*\alpha+h\}(u)]}{2\cosh[\beta\{J*\alpha+h\}(u)]}, \qquad k\in\{-1,+1\},\ \alpha\in B,\ u\in\mathbb{T}^d. \tag{3.12}$$

(The factor $\frac{1}{2}$ in (3.7) drops out because every spin is counted twice in the Hamiltonian but once in the convolution.) In particular, $(\mu^n)_{n\in\mathbb{N}}$ is Gibbs in the sense of Definition 3.1.1(c).
(3) Definition 3.1.1 assigns the notion of Gibbs to a sequence of probability measures that live on different spaces. It is different from the classical notion of Gibbs based on the Dobrushin-Lanford-Ruelle condition, which is used to define Gibbs measures on infinite lattices. Nonetheless, the quantity in (3.12) can be viewed as some sort of *specification kernel*.
(4) Definition 3.1.1 does not consider sequences $(\alpha^u_{n-1})_{n\in\mathbb{N}}$ whose weak limit is singular with respect to λ. In Proposition 3.1.2 below we will see that in the thermodynamic limit we can ignore trajectories that do not lie in the set $\{\alpha\lambda\colon\ \alpha\in B\}$ because they are too costly.

3.1.5 Stochastic dynamics

For fixed n, we let the spin configuration evolve according to a Glauber dynamics with generator L_n given by

$$(L_nf)(\sigma) := \sum_{x\in\Delta^d_n} c_n(x,\sigma)\,[f(\sigma^x)-f(\sigma)], \qquad f\colon\ \Omega_n\to\mathbb{R}, \tag{3.13}$$

where the spin-flip rate takes the form

$$c_n(x,\sigma) := \frac{\exp[-\frac{\beta'}{2}\{H^n(\sigma^x)-H^n(\sigma)\}]}{2\cosh[\frac{\beta'}{2}\{H^n(\sigma^x)-H^n(\sigma)\}]} \tag{3.14}$$

with σ^x the configuration obtained from σ by flipping the spin at site x, and $\beta' \in [0,\infty)$ the *dynamical* inverse temperature. We write $(\sigma_s)_{s\geq 0}$ to denote the trajectory of the spin configuration, which lives on $D_{[0,\infty)}(\Omega_n)$, the space of càdlàg paths on Ω_n endowed with the Skorohod topology.

Abbreviate $\pi^n_s := \pi^n(\sigma_s)$, and let $\bar{\pi}^n = (\pi^n_s)_{s\geq 0}$ denote the trajectory of the empirical density under the Glauber dynamics. For a given probability measure $\check{\rho}^n_0$ on $\mathcal{M}^n$ we define

$$P^n_{\check{\rho}^n_0} := \text{ law of } (\pi^n_s)_{s\geq 0} \text{ conditional on } \pi^n_0 \text{ being drawn according to } \check{\rho}^n_0, \tag{3.15}$$

which lives on $D_{[0,\infty)}(\mathcal{M}^n)$, the space of càdlàg paths on $\mathcal{M}^n$ endowed with the Skorohod topology.

3.1.6 Large deviation principles

For $t \geq 0$, we say that $\varphi = (\varphi_s)_{s\in[0,t]} \in C_{[0,t]}(B)$ is absolutely continuous in time when

$$\exists\, \dot{\varphi} = (\dot{\varphi}_s)_{s\in[0,t]} \in L^1_{[0,t]}(\mathbb{T}^d)\colon \quad \varphi_s(u) - \varphi_0(u) = \int_0^s \dot{\varphi}_r(u)\, dr \quad \forall\, s \in [0,t],\ \lambda - a.e.\ u. \tag{3.16}$$

Let us recall that a family of probability measures $(\nu^n)_{n\in\mathbb{N}}$ on a Polish space $\mathcal{X}$ satisfies a *large deviation principle* (LDP) with rate n and rate function I when $I\colon \mathcal{X} \to [0,\infty]$ has compact level sets, is not identically infinite, and

$$\begin{aligned} \liminf_{n\to\infty} \frac{1}{n} \log \nu^n(O) \geq & \; -\inf_{x\in O} I(x), \qquad O \subseteq \mathcal{X} \text{ open},\\ \limsup_{n\to\infty} \frac{1}{n} \log \nu^n(C) \leq & \; -\inf_{x\in C} I(x), \qquad C \subseteq \mathcal{X} \text{ closed}. \end{aligned} \tag{3.17}$$

(See Dembo and Zeitouni [DZ98, Section 1.2].) The following LDPs can be found in Comets [Com87].

Proposition 3.1.2. *(i)* **[LDP for initial Gibbs measure]** $(\check{\mu}^n)_{n\in\mathbb{N}}$ *satisfies the LDP on* $\mathcal{M}(\mathbb{T}^d)$ *with rate* n^d *and rate function* $I_S - \inf_{\mathcal{M}(\mathbb{T}^d)} I_S$ *given by*

$$I_S(\nu) := \begin{cases} -\beta\big\langle \frac{1}{2} J * \alpha + h, \alpha\lambda\big\rangle + \langle \Phi \circ \alpha, \lambda\rangle, & \text{if } \nu = \alpha\lambda \text{ with } \alpha \in B,\\ \infty, & \text{otherwise}, \end{cases} \tag{3.18}$$

where Φ *is the relative entropy*

$$\Phi(m) := \tfrac{1+m}{2} \log(1+m) + \tfrac{1-m}{2} \log(1-m), \qquad m \in [-1,+1]. \tag{3.19}$$

(ii) **[Dynamical LDP for deterministic initial law]** *Let* $t \geq 0$ *and* $\alpha \in C(\mathbb{T}^d)$*, and let* $(\varphi^n_0)_{n\in\mathbb{N}}$ *be any sequence with* $\varphi^n_0 \in \mathcal{M}^n$ *for every* $n \in \mathbb{N}$ *such that* $w-\lim_{n\to\infty} \varphi^n_0 = \alpha\lambda$. *Then*

$$\left(P^n_{\delta_{\varphi^n_0}}\right)_{n\in\mathbb{N}} \text{ restricted to } [0,t] \tag{3.20}$$

satisfies the LDP on $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ *with rate* n^d *and rate function* $I_D^t - \inf_{D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I_D^t$ *given by*

$$I_D^t(\psi) := \begin{cases} \int_0^t \mathcal{L}\big(\varphi_s, \dot{\varphi}_s\big)\, ds, & \text{if } \psi = \varphi\lambda, \text{ with } \varphi \text{ satisfying property (3.16) and } \varphi_0 \equiv \alpha, \\ \infty, & \text{otherwise,} \end{cases} \tag{3.21}$$

where

$$\mathcal{L}(q,p) := \int_{\mathbb{T}^d} L[q(u), p(u)]\, du, \qquad q \in B,\, p \in L^1(\mathbb{T}^d), \tag{3.22}$$

with

$$\begin{aligned} L[q(u),p(u)] = & \tfrac{p(u)}{2} \log \left[\frac{\frac{p(u)}{2} + \sqrt{1 - q(u)^2 + \left[\frac{p(u)}{2}\right]^2}}{1 - q(u)} \right] - \tfrac{p(u)}{2} \big[\beta'(J * q + h)\big](u) \\ & + \Bigg\{ - \sqrt{1 - q(u)^2 + \left[\tfrac{p(u)}{2}\right]^2} \\ & \quad + \cosh\big[\beta'(J*q+h)\big](u) - q(u) \sinh\big[\beta'(J*q+h)\big](u) \Bigg\}. \end{aligned} \tag{3.23}$$

Note that (3.23) simplifies considerably when $\beta' = 0$ (independent spin-flip dynamics).

To ease notation, we write $I_S(\alpha)$ instead of $I_S(\nu)$ when $\nu = \alpha\lambda$ with $\alpha \in B$, and $I_D^t(\varphi)$ instead of $I_D^t(\psi)$ when $\psi = \varphi\lambda$ with $\varphi \in C_{[0,t]}(B)$, i.e., we henceforth suppress the reference measure λ from the notation.

Let $P^n = P^n_{\tilde{\mu}^n}$. Define

$$Q^n_{t,\alpha'}(\cdot) := P^n\big((\pi^n_s)_{s\in[0,t]} \in \cdot \mid \pi^n_t = \alpha'_n\big), \qquad t \geq 0,\ \alpha' \in B, \tag{3.24}$$

with $\alpha'_n \in \mathcal{M}^n$ the element closest to $\alpha' \in B$ in any metric that metrizes the weak topology. The following LDPs are key to our analysis. In what follows we write $f \equiv g$ when $f(u) = g(u)$ for all $u \in \mathbb{T}^d$.

Proposition 3.1.3. [Dynamical LDP for Gibbs initial law]
(i) For every $t \geq 0$, $(P^n)_{n\in\mathbb{N}}$ *satisfies the LDP on* $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ *with rate* n^d *and rate function* $I^t - \inf_{D_{[0,T]}(\mathcal{M}(\mathbb{T}^d))} I^t$ *given by*

$$I^t(\varphi) := I_S(\varphi_0) + I_D^t(\varphi). \tag{3.25}$$

(ii) For every $t \geq 0$ *and* $\alpha' \in B$, $(Q^n_{t,\alpha'})_{n\in\mathbb{N}}$ *satisfies the LDP on* $D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ *with rate* n^d *and rate function* $I^{t,\alpha'} - \inf_{D_{[0,t](\mathcal{M}(\mathbb{T}^d))}} I^{t,\alpha'}$ *given by*

$$I^{t,\alpha'}(\varphi) := \begin{cases} I^t(\varphi), & \text{if } \varphi_t \equiv \alpha', \\ \infty, & \text{otherwise.} \end{cases} \tag{3.26}$$

The proof of Proposition 3.1.3 is given in Appendix 3.4 and is based on large deviation techniques coming from hydrodynamic scaling. A somewhat delicate issue is the fact that we cannot use Proposition 3.1.2(ii) because this has a deterministic initial condition, while in Proposition 3.1.3(i) the initial condition is random.

Note that, by (3.18), (3.21) and (3.25–3.26),

$$\inf_{\varphi\in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))} I^{t,\alpha'}(\varphi) = \inf_{\alpha\in B} \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} I^t(\varphi) = \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_t\equiv\alpha'}} I^t(\varphi). \tag{3.27}$$

3.1.7 Link to the specification kernel

Henceforth we only consider trajectories $\varphi \in C_{[0,t]}(B)$ satisfying (3.16), because the rate functions are infinite otherwise. The following theorem provides the fundamental link between the specification kernel in (3.11) and the minimizer of (3.27) when it is *unique.*

Theorem 3.1.4. [Specification kernel in absence of bifurcation] *Fix $t \geq 0$ and $\alpha' \in B$. Suppose that (3.27) has a unique minimizing path $\hat\varphi^{t,\alpha'} = (\hat\varphi_s^{t,\alpha'})_{s\in[0,t]}$. Then the specification kernel at time t equals*

$$\gamma_t^u(k' \mid \alpha') := \frac{\sum\limits_{k\in\{-1,+1\}} \exp\left[k\beta\{J*\hat\varphi_0^{t,\alpha'}+h\}(u)\right] p_t^{u,t,\alpha'}(k,k')}{\sum\limits_{j,j'\in\{-1,+1\}} \exp\left[j\beta\{J*\hat\varphi_0^{t,\alpha'}+h\}(u)\right] p_t^{u,t,\alpha'}(j,j')}, \tag{3.28}$$

$k' \in \{-1,+1\}$, $u \in \mathbb{T}^d$, where $p_t^{u,t,\alpha'}(j,j')$ is the probability to go from j at time 0 to j' at time t in the time-inhomogeneous Markov process on $\{-1,+1\}$ with generator $L_s^{u,t,\alpha'}$ at time $s \in [0,t]$ given by

$$\begin{aligned}(L_s^{u,t,\alpha'}f)(k) &= \frac{\exp\left[k\beta'\{J*\hat\varphi_s^{t,\alpha'}+h\}(u)\right]}{2\cosh\left[\beta'\{J*\hat\varphi_s^{t,\alpha'}+h\}(u)\right]}\,[f(-k)-f(k)],\\ &k\in\{-1,+1\},\ f\colon\{-1,+1\}\to\mathbb{R},\ u\in\mathbb{T}^d,\ s\in[0,t].\end{aligned} \tag{3.29}$$

Remark: Note that for $\beta' = 0$ (independent spin-flip dynamics) the right-hand side of (3.29) simplifies to $\frac{1}{2}[f(-k)-f(k)]$ and that, consequently, the right-hand side of (3.28) depends on the optimal trajectory $\hat\varphi^{t,\alpha'}$ only via its initial value $\hat\varphi_0^{t,\alpha'}$, and takes the form

$$\gamma_t^u(k' \mid \alpha') = \Gamma_t\big(k', \beta\{J*\hat\varphi_0^{t,\alpha'}+h\}(u)\big) \tag{3.30}$$

for some $\Gamma_t\colon \{-1,+1\}\times\mathbb{R}\to[0,1]$, with the property that $m \mapsto \Gamma_t(k',m)$ is continuous, strictly increasing for $k' = +1$ and strictly decreasing for $k' = -1$.

3.1.8 Reduction: critical trajectories

In what follows we restrict ourselves to the case of infinite-temperature dynamics, i.e., $\beta' = 0$. Let

$$\begin{aligned}\hat\varphi^{\alpha;t,\alpha'} &:= \mathrm{argmin}_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} I^t(\varphi),\\ C_{t,\alpha'}(\alpha) &:= I^t(\hat\varphi^{\alpha;t,\alpha'}).\end{aligned} \tag{3.31}$$

Remark: Note that
$$\inf_{\alpha\in B} C_{t,\alpha'}(\alpha) = \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_t\equiv\alpha'}} I^t(\varphi). \tag{3.32}$$

The following theorem says that $\hat{\varphi}^{\alpha;t,\alpha'}$ is unique for every $t\geq 0$ and $\alpha,\alpha'\in B$, and can be computed because the Kac model can be linked to the Curie-Weiss model treated in Fernández, den Hollander and Martínez [FdHM13a]. (In the notation of that paper β is absorbed into J,h.)

Theorem 3.1.5. [Critical trajectories] *Let $\beta'=0$. For every $t\geq 0$ and $\alpha,\alpha'\in B$,*
$$\hat{\varphi}_s^{\alpha;t,\alpha'}(u) = \hat{\varphi}_{t,\alpha'(u)}^{\mathrm{CW};\alpha(u)}(s), \qquad u\in\mathbb{T}^d,\ s\in[0,t], \tag{3.33}$$
where $\hat{\varphi}_{t,m'}^{\mathrm{CW};m}(s)$, $s\in[0,t]$, is the unique trajectory in $[-1,+1]$ between magnetization m at time 0 and magnetization m' at time t for the Curie-Weiss model. Accordingly (see (3.21–3.23) *and* 3.25–3.26)*),*
$$C_{t,\alpha'}(\alpha) = I_S(\alpha) + \int_{\mathbb{T}^d} du \int_0^t ds\ L^{\mathrm{CW}}\left[\hat{\varphi}_{t,\alpha'(u)}^{\mathrm{CW};\alpha(u)}(s), \dot{\hat{\varphi}}_{t,\alpha'(u)}^{\mathrm{CW};\alpha(u)}(s)\right], \tag{3.34}$$
where L^{CW} is the Lagrangian of the Curie-Weiss model. The critical points of (3.34) *(i.e., the local minima and the local maxima) satisfy the functional equation*
$$\sinh[2\beta(J*\alpha+h)](u) - \alpha(u)\cosh[2\beta(J*\alpha+h)](u) = \frac{\alpha(u)}{\tanh(2t)} - \frac{\alpha'(u)}{\sinh(2t)} \quad \text{a.e. } u\in\mathbb{T}^d. \tag{3.35}$$

In Theorem 3.1.5, the Lagrangian of the Curie-Weiss model is given by
$$L^{\mathrm{CW}}(m,\dot{m}) := -\tfrac{1}{2}\sqrt{4\,(1-m^2)+\dot{m}^2} + \tfrac{1}{2}\dot{m}\log\left(\frac{\sqrt{4\,(1-m^2)+\dot{m}^2}+\dot{m}}{2(1-m)}\right) + 1, \tag{3.36}$$
which is the same as (3.23) with $\beta'=0$, $p(\cdot)=m$ and $q(\cdot)=\dot{m}$, and the unique trajectory is given by
$$\hat{\varphi}_{t,m'}^{\mathrm{CW};m}(s) := \frac{1}{\sinh(2t)}\Big\{m\sinh(2(t-s)) + m'\sinh(2s)\Big\}, \qquad 0\leq s\leq t. \tag{3.37}$$
(See [FdHM13a, Eqs. (1.16) and (1.28)].) The intuition behind Theorem 3.1.5 is that the dynamics has no spatial interaction. Consequently, we may think of $\alpha(u)$ and $\alpha'(u)$ as the local initial and final magnetization near u, and thereby reduce the minimization problem in (3.26) to that of the Curie-Weiss model.

With the help of Theorem 3.1.5 we are able to prove the equivalence of non-Gibbs and *bifurcation*, the latter meaning that (3.27) has more than one global minimizer. This is in accordance with the program outlined in van Enter, Fernández, den Hollander and Redig [vEFdHR10].

Theorem 3.1.6. [**Equivalence of non-Gibbsianness and bifurcation**] *Let* $\beta' = 0$. *For every* $t \geq 0$, $\tilde{\alpha}' \mapsto \gamma_t^u(\,\cdot \mid \tilde{\alpha}')$ *is continuous at* $\alpha' \in B$ *for all* $u \in \mathbb{T}^d$ *if and only if* $\inf_{\varphi \in C_{[0,t]}(B)\colon\, \varphi_t \equiv \alpha'} I^t(\varphi)$ *has a unique minimizing path.*

Thus, non-Gibbsianness is equivalent to the occurrence of *more than one* possible history for the *same* α'.

We expect Theorem 3.1.6 to hold for $\beta' > 0$ as well, but the present paper deals with $\beta' = 0$ only.

3.1.9 Bifurcation analysis

In this section we study for which choice of J, h, β and t, α' the variational formula in the right-hand side of (3.27) has a unique global minimizer or has multiple global minimizers. According to Definition 3.1.1 and Theorem 3.1.6, this distinction classifies Gibbsianness versus non-Gibbsianness.

Theorem 3.1.7. *Let* $\beta' = 0$ *and* $\langle J \rangle := \int_{\mathbb{T}^d} J(u)du$.
(i) [**Short-time Gibbsianness**] *There exists a* $t_0 = t_0(J,h) \in (0,\infty)$ *such that* (3.27) *has a unique global minimizer* $\hat{\varphi}^{t,\alpha'}$ *for all* $0 \leq t \leq t_0$ *and all* $\alpha' \in B$.
(ii) [**Mean-field behaviour**] *If* $h \equiv c \in [0,\infty)$ *and* $\alpha' \equiv c' \in [-1,+1]$, *then the bifurcation behaviour is the same as for the Curie-Weiss model with parameters* $(J^{\mathrm{CW}}, h^{\mathrm{CW}}) = (\beta\langle J \rangle, \beta c)$ *and final magnetization* c'*:*

J^{CW}	$h^{\mathrm{CW}} = 0$	$h^{\mathrm{CW}} > 0$
$(0,1]$	*No bad* c' *for all* $t \geq 0$	
$(1,\frac{3}{2}]$	$\varnothing$; $\{0\}$ 0 — Ψ_c —	$\varnothing$; $\{c'\}$; $\varnothing$ 0 — Ψ_U — $[-1,U_B]$ — Ψ_* —
$(\frac{3}{2},\infty)$	$\varnothing$; $\{\pm c'\}$; $\{0\}$ 0 — Ψ_U — $[-U_B,U_B]$ — Ψ_c —	$h < h^*$: $\varnothing$; $\{c'\}$; $\{c'_1,c'_2\}$; $\{c'\}$; $\varnothing$ 0 — Ψ_U — $[M_B,U_B)$ — Ψ_L — (L_B,M_B) — Ψ_T — $[-1,M_T]$ — Ψ_* — $h \geq h^*$: $\varnothing$; $\{c'\}$; $\varnothing$ 0 — Ψ_U — $[-1,U_B]$ — Ψ_* —

The above table summarizes the results for the Curie-Weiss model studied in [FdHM13a]. The center line represents the time axis. In each figure, the symbols on top indicate the set of bad magnetizations (which for the Kac-model correspond to bad constant profiles), the intervals below indicate in which range the bad magnetizations occur. For further details, in particular, a definition of the times $\Psi_U, \Psi_*, \Psi_c, \Psi_L, \Psi_T$ and the magnetizations U_B, M_B, L_B, M_T, see [FdHM13a, Section 1.5.5].
Remarks:
(1) The existence of a solution of (3.27) is guaranteed by the lower semi-continuity of $\alpha \mapsto C_{t,\alpha'}(\alpha)$, which follows from the lower semi-continuity of $\varphi_0 \mapsto I_S(\varphi_0)$ and $\varphi \mapsto I_D^t(\varphi)$, together with the fact that $w-\lim_{n\to\infty} \alpha_n = \alpha$ implies $w-\lim_{n\to\infty} \hat{\varphi}^{\alpha_n;t,\alpha'} = \hat{\varphi}^{\alpha;t,\alpha'}$ in

the Skorohod topology by (3.37).
(2) The claims in Theorem 3.1.7(ii) only concern the case where α' is constant. The problem of deciding whether or not there exist multiple global minimizers of (3.27) when α' is not constant presents major difficulties. Similar but easier equations have been studied extensively in Comets, Eisele and Schatzman [CES86], De Masi, Orlandi, Presutti and Triolo [DMOPT94] and Bates, Chen and Chmaj [BCC05], with partial success. An additional complication in our case is that non-constant α' brings a non-homogeneous parameter into the problem, which makes the analysis even harder. A full analysis of the global minimizers of (3.27) as a function of J and h therefore remains a challenge.

3.2 Proof of Theorems 3.1.4–3.1.6

3.2.1 Proof of Theorem 3.1.4

Proof. Recall that $\pi_t^{u,n} = \pi^{u,n}(\sigma_t)$ defined below (3.3) does not depend on $\sigma_t(\lfloor nu \rfloor)$. Let $\mathbb{P}^n$ denote the law of $(\sigma_s)_{s\geq 0}$ with σ_0 distributed according to μ^n, and abbreviate $\pi_{<t}^{u,n} := (\pi_s^{u,n})_{s\in[0,t)}$ and $\xi_{<t}^{n-1} := (\xi_s^{n-1})_{s\in[0,t)}$. Write (recall (3.10))

$$
\begin{aligned}
\gamma_t^{u,n}\left(k' \mid \alpha'^u_{n-1}\right) &:= \mathbb{P}^n\left(\sigma_t(\lfloor nu \rfloor) = k' \;\middle|\; \pi_t^{u,n} = \alpha'^u_{n-1}\right)\\
&= \int_{D_{[0,t)}(\mathcal{M}^{u,n})} \mathbb{P}^n\left(d\xi_{<t}^{n-1} \;\middle|\; \pi_t^{u,n} = \alpha'^u_{n-1}\right)\mathbb{P}^n\left(\sigma_t(\lfloor nu \rfloor) = k' \;\middle|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\right)\\
&= \int_{D_{[0,t)}(\mathcal{M}^{u,n})} \mathbb{P}^n\left(d\xi_{<t}^{n-1} \;\middle|\; \pi_t^{u,n} = \alpha'^u_{n-1}\right)\\
&\quad\times\left\{\sum_{k=\pm 1} \mathbb{P}^n\left(\sigma_t(\lfloor nu \rfloor) = k' \;\middle|\; \sigma_0(\lfloor nu \rfloor) = k,\, \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\right)\right.\\
&\qquad\qquad \left.\mathbb{P}^n\left(\sigma_0(\lfloor nu \rfloor) = k \;\middle|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\right)\right\}.
\end{aligned}
\tag{3.38}
$$

We proceed by analyzing the three terms under the integral.

(1) The LDP for $(Q^n_{t,\alpha'})_{n\in\mathbb{N}}$ in Proposition 3.1.3(ii), together with the assumption that (3.27) has a unique minimizing path, implies

$$
w-\lim_{n\to\infty} \mathbb{P}^n\left(\;\cdot\;\middle|\; \pi_t^{u,n} = \alpha'^u_{n-1}\right) = \delta_{\hat{\varphi}^{t,\alpha'}_{<t}}(\cdot) \quad \text{on} \quad D_{[0,t)}(\mathcal{M}(\mathbb{T}^d)). \tag{3.39}
$$

(2) Because $(\sigma_s(\lfloor nu \rfloor), \pi_s^{u,n})_{s\geq 0}$ is Markov, we have

$$
\mathbb{P}^n\left(\sigma_t(\lfloor nu \rfloor) = k' \;\middle|\; \sigma_0(\lfloor nu \rfloor) = k,\, \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\right) = p_t^{\xi_{<t}^{n-1}}(k,k'), \tag{3.40}
$$

where $p_t^{\xi_{<t}^{n-1}}(k,k')$ is the probability to go from k at time 0 to k' at time t in the time-inhomogeneous Markov process on $\{-1,+1\}$ with generator at time $s \in [0,t)$ given by

(3.29) with $\hat{\varphi}_s^{t,\alpha'}$ replaced by ξ_s^{n-1}. Note that $\xi_{<t}^{n-1} \mapsto p_t^{\xi_{<t}^{n-1}}(k,k')$ is continuous on $D_{[0,t)}(\mathcal{M}^{u,n})$ for fixed k,k',t and u,n (recall (3.4)), and that $\lim_{n\to\infty} p_t^{\xi_{<t}^{n-1}}(k,k') = p_t^{\hat{\varphi}_{<t}^{t,\alpha'}}(k,k')$ for fixed k,k',t,α' when $\lim_{n\to\infty} \xi_{<t}^{n-1} = \hat{\varphi}_{<t}^{t,\alpha'}$ on $D_{[0,t)}(\mathcal{M}(\mathbb{T}^d))$ (recall (3.29)).

(3) Write

$$\begin{aligned} &\mathbb{P}^n\Big(\sigma_0(\lfloor nu \rfloor) = k \;\Big|\; \pi_{<t}^{u,n} = \xi_{<t}^{n-1}\Big) \\ &\qquad = \Big[1 + c^{u,n}(\xi_{<t}^{n-1},k)\exp\Big(-2\beta k\{\tfrac{1}{2}J * \xi_0^{n-1} + h\}\big(\tfrac{\lfloor nu \rfloor}{n}\big)\Big)\Big]^{-1} \end{aligned} \tag{3.41}$$

with

$$c^{u,n}(\xi_{<t}^{n-1},k) := \frac{d\mathbb{P}^{u,n}_{\xi_0^{n-1},-k}}{d\mathbb{P}^{u,n}_{\xi_0^{n-1},k}}(\xi_{<t}^{n-1}), \tag{3.42}$$

where

$$\mathbb{P}^{u,n}_{\xi_0^{n-1},k}(\cdot) = \mathbb{P}^{u,n}\big(\pi_{<t}^{u,n} \in \cdot \mid \pi_0^{u,n} = \xi_0^{n-1}, \sigma_0(\lfloor nu \rfloor) = k\big) \tag{3.43}$$

and we use (3.1–3.2) to write

$$\frac{\mathbb{P}^n(\pi_0^{u,n} = \xi_0^{n-1}, \sigma_0(\lfloor nu \rfloor) = -k)}{\mathbb{P}^n(\pi_0^{u,n} = \xi_0^{n-1},\, \sigma_0(\lfloor nu \rfloor) = k)} = \exp\Big(-2\beta k\{\tfrac{1}{2}J * \xi_0^{n-1} + h\}\big(\tfrac{\lfloor nu \rfloor}{n}\big)\Big). \tag{3.44}$$

Finally, note that $\lim_{n\to\infty} c^{u,n}(\xi_{<t}^{n-1},k) = 1$ for fixed k,t and u when $\lim_{n\to\infty} \xi_{<t}^{n-1} = \hat{\varphi}_{<t}^{t,\alpha'}$ on $D_{[0,t)}(\mathcal{M}(\mathbb{T}^d))$. Indeed, (3.13–3.14) show that in the thermodynamic limit a single spin has no effect on the dynamics of the empirical density (Feller property). Combine this observation with (3.39–3.41) to get the identity in (3.28) (see Yang [Yan11]). □

3.2.2 Proof of Theorem 3.1.5

Proof. For $\beta' = 0$ (infinite-temperature dynamics), (3.23) reduces to $\int_{\mathbb{T}^d} du\, L^{\mathrm{CW}}[q(u),p(u)]$ with L^{CW} the Curie-Weiss Lagrangian in (3.36). Hence, recalling

(3.26), we have

$$\begin{aligned} C_{t,\alpha'}(\alpha) &= \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} I^t(\varphi) \\ &= I_S(\alpha) + \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} I_D^t(\varphi) \\ &\geq I_S(\alpha) + \int_{\mathbb{T}^d} du \inf_{\substack{\varphi\in C_{[0,t]}(B):\\ \varphi_0\equiv\alpha,\,\varphi_t\equiv\alpha'}} \int_0^t ds\; L^{\mathrm{CW}}\left[\varphi_s(u),\dot\varphi_s(u)\right] \\ &\geq I_S(\alpha) + \int_{\mathbb{T}^d} du \inf_{\substack{\rho\in C_{[0,t]}([-1,+1]):\\ \rho_0=\alpha(u),\,\rho_t=\alpha'(u)}} \int_0^t ds\; L^{\mathrm{CW}}\left[\rho_s,\dot\rho_s\right] \\ &= I_S(\alpha) + \int_{\mathbb{T}^d} du \int_0^t ds\; L^{\mathrm{CW}}\Big[\hat\varphi^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \dot{\hat\varphi}^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s)\Big], \end{aligned} \tag{3.45}$$

which settles half of (3.34). To get equality we pick, as in (3.33),

$$\hat\varphi^{\alpha;t,\alpha'}_s(u) := \hat\varphi^{\mathrm{CW};\alpha(u)}_{t,\alpha'(u)}(s), \quad s\in[0,t],\ ,u\in\mathbb{T}^d. \tag{3.46}$$

Since $(\hat\varphi^{\alpha;t,\alpha'}_s)_{s\in[0,t]}\in C_{[0,t]}(B)$ verifies the restrictions $\varphi_0\equiv\alpha$, $\varphi_t\equiv\alpha'$, it is a minimizer of the variational problem in the left-hand side of (3.45).

The derivation of (3.35) follows in the same way as for the Curie-Weiss model in [FdHM13a, Section 2.1], with the Fréchet derivative replacing the standard derivative. Note that $\alpha\mapsto C_{t,\alpha'}(\alpha)$ is Fréchet differentiable on $\mathrm{int}(B)$, while the argument in Ellis [EE83, Section V, Theorem 5.1] shows that all its critical points lie in $\mathrm{int}(B)$. □

The following way of rewriting $C_{t,\alpha'}$ will be useful later on. Adding and subtracting $\frac14\beta\int_{\mathbb{T}^d}du\int_{\mathbb{T}^d}dv\,J(u-v)[\alpha(u)-\alpha(v)]^2$, we may rewrite (3.18) as

$$I_S(\alpha)=\tfrac14\beta\int_{\mathbb{T}^d}du\int_{\mathbb{T}^d}dv\,J(u-v)[\alpha(u)-\alpha(v)]^2+\int_{\mathbb{T}^d}du\;I_S^{CW}(\alpha(u)), \tag{3.47}$$

where I_S^{CW} is the rate function for the magnetization in the Curie-Weiss model. With this formula, (3.34) reduces to

$$C_{t,\alpha'}(\alpha)=\tfrac14\beta\int_{\mathbb{T}^d}du\int_{\mathbb{T}^d}dv\,J(u-v)[\alpha(u)-\alpha(v)]^2+\int_{\mathbb{T}^d}du\,C^{\mathrm{CW}}_{t,\alpha'}(\alpha(u)). \tag{3.48}$$

This form clarifies the interplay between the non-local interaction and the independent spin-flip dynamics.

3.2.3 Proof of Theorem 3.1.6

As emphasized in (3.30), $\gamma^u_t(k'\mid\alpha')$ depends on α' only through $\hat\varphi^{t,\alpha'}_0$, the starting value of the global minimizer of $C_{t,\alpha'}$. The following lemma is the basis for the proof of

Theorem 3.1.6. It describes the behavior of $\hat{\varphi}_0^{t,\alpha'}$ when the constraint $\alpha' \in B$ at time t is varied. Loosely speaking, it says that global minimizers are isolated, are continuous under variations of α', and can be selected by variation of α'.

Below we fix t and suppress it from the notation. In what follows we write $\hat{\alpha}(\alpha')$ to denote a global minimum of $C_{t,\alpha'}$.

Lemma 3.2.1. *For every $t \geq 0$ and $\alpha_0' \in B$ there exists an open neighborhood $\mathcal{N}_{\alpha_0'}$ of α_0' such that for all $\alpha' \in \mathcal{N}_{\alpha_0'} \setminus \{\alpha_0'\}$ the following hold:*
(a) **[Isolation of global minimizers]** *$\alpha \mapsto C_{t,\alpha'}$ has a unique global minimum at, say, $\hat{\alpha}(\alpha')$.*
(b) **[Continuity of global minimizers]** *$\alpha'' \mapsto \hat{\alpha}(\alpha'')$ is continuous at $\alpha'' = \alpha'$. If $\alpha'' \mapsto C_{t,\alpha_0'}(\alpha'')$ has a unique global minimum, then it is continuous at $\alpha'' = \alpha_0'$.*
(c) **[Selection of global minimizers]** *If $C_{t,\alpha_0'}$ has multiple global minima, then there are two of them, say $\hat{\alpha}_k(\alpha_0')$ and $\hat{\alpha}_l(\alpha_0')$, and a $\gamma' \in B$ such that*

$$\lim_{\varepsilon \downarrow 0} \hat{\alpha}(\alpha_0' + \varepsilon\gamma') \equiv \hat{\alpha}_k(\alpha_0'), \qquad \lim_{\varepsilon \uparrow 0} \hat{\alpha}(\alpha_0' + \varepsilon\gamma') \equiv \hat{\alpha}_l(\alpha_0'). \tag{3.49}$$

Proof. The following 3 steps describe the behavior of the minimizers under small perturbations of α' are around α_0'.

(a) Under the assumption that $\sup_{\alpha \in B} |C_{t,\alpha'} - C_{t,\alpha_0'}| \to 0$ as $\|\alpha' - \alpha_0'\|_\infty \to 0$, whenever a local minimum is emerging as α' is varied this local minimum cannot be a global minimum. Indeed, we have that

$$|C_{t,\alpha'}(\alpha) - C_{t,\alpha_0'}(\alpha)| \leq \int_{\mathbb{T}^d} du\, |C^{CW}_{t,\alpha'(u)}(\alpha(u)) - C^{CW}_{t,\alpha_0'(u)}(\alpha(u))| \leq \int_{\mathbb{T}^d} du\, \|C^{CW}_{t,\alpha'(u)} - C^{CW}_{t,\alpha_0'(u)}\|_\infty.$$

On the other hand, we know from [FdHM13a] that $\|C^{CW}_{t,m'} - C^{CW}_{t,m_0'}\|_\infty \to 0$ when $m' \to m_0'$. Hence the claim follows by dominated convergence.

(b) Let $\hat{\alpha}_i(\alpha_0')$, $i \in \mathcal{I}$, denote the global minima of $C_{t,\alpha_0'}$. Each of these verifies (3.35), which may be written in the form $F(\alpha, \alpha') \equiv 0$ for some functional F. From the implicit function theorem (see e.g. Drábek and Milota [DM13, Theorem 4.2.1]) it follows that there exist a neighborhood $\widetilde{\mathcal{N}}_{\alpha_0'}$ of α_0' and smooth functions $\alpha' \mapsto \overline{\alpha}_i(\alpha')$, $i \in \mathcal{I}$, on this neighborhood such that $\overline{\alpha}_i(\alpha')$, $i \in \mathcal{I}$, are minima of $C_{t,\alpha'}$, and $\lim_{\alpha' \to \alpha_0'} \overline{\alpha}_i(\alpha') \equiv \hat{\alpha}_i(\alpha_0')$.

(c) Let

$$B_i(\alpha') := C_{t,\alpha'}(\overline{\alpha}_i(\alpha')). \tag{3.50}$$

The minimal cost is

$$C_{t,\alpha'}(\hat{\alpha}(\alpha')) = \min_{i \in \mathcal{I}} B_i(\alpha'). \tag{3.51}$$

Because of the assumed multiplicity of minima at α_0', we have

$$B_i(\alpha_0') = B_j(\alpha_0'), \qquad i, j \in \mathcal{I}. \tag{3.52}$$

Expand each B_i up to first order order,

$$B_i(\alpha_0' + \varepsilon\gamma') = B(\alpha_0') + \varepsilon\big\langle [DB_i](\alpha_0'), \gamma' \big\rangle + O\big(\varepsilon\|\gamma'\|_\infty\big), \qquad \varepsilon > 0, \tag{3.53}$$

where $[DB_i](\alpha'_0)$ is the Fréchet derivative. Put $G(\alpha,\alpha') := C_{t,\alpha'}(\alpha)$. Then the chain rule implies that

$$[DB_i](\alpha'_0) \equiv [D_\alpha G]\big(\hat{\alpha}_i(\alpha'_0),\alpha'_0\big) \circ [D_{\alpha'}\overline{\alpha}_i](\alpha'_0) + [D_{\alpha'}G]\big(\hat{\alpha}_i(\alpha'_0),\alpha'_0\big), \tag{3.54}$$

where $\circ$ denotes composition and the lower indices α,α' on the letter D refer to the variable with respect to which the derivative is taken. The first term in (3.54) vanishes due to the criticality of $\hat{\alpha}_i(\alpha'_0)$. Standard calculations with Fréchet derivatives show that

$$[D_{\alpha'}G]\big(\hat{\alpha}_i(\alpha'_0),\alpha'_0\big)(u) = H^{\mathrm{CW}}\big(\hat{\alpha}_i(\alpha'_0)(u),\alpha'_0(u)\big), \quad u \in \mathbb{T}^d, \tag{3.55}$$

with $H^{\mathrm{CW}}(m,m') := (\frac{\partial}{\partial m'}C^{\mathrm{CW}}_{t,m'})(m)$. The identity in (3.55) helps us to select different global minimizers by small variations of α'. Indeed, for $i \neq j$ we have $\|\hat{\alpha}_i(\alpha'_0) - \hat{\alpha}_j(\alpha'_0)\|_\infty > 0$, and hence there exists a $\delta > 0$ such that $\lambda(\{\hat{\alpha}_i(\alpha'_0) - \hat{\alpha}_j(\alpha'_0) > \delta\}) > 0$. Take $I = \{u \in \mathbb{T}^d \;:\; \hat{\alpha}_i(\alpha'_0)(u) - \hat{\alpha}_j(\alpha'_0)(u) > \delta\}$. Then

$$\hat{\alpha}_j(\alpha'_0)(u) + \delta < \hat{\alpha}_i(\alpha'_0)(u) \quad \forall\, u \in I. \tag{3.56}$$

Combining (3.54–3.56) and using the strict monotonicity of $m \mapsto H^{\mathrm{CW}}(m,m')$, we get

$$[DB_j](\alpha'_0)(u) < [DB_i](\alpha'_0)(u) \quad \forall\, u \in I. \tag{3.57}$$

The claim follows by picking $\gamma' \equiv 1_I$ and expressions (3.53), (3.55). □

We are now ready to prove Theorem 3.1.6. We continue to use the same notation as in Lemma 3.2.1.

Proof. Suppose that C_{t,α'_0} has a unique global minimizer, say $\hat{\alpha}(\alpha'_0)$, and let $\mathcal{N}_{\alpha'_0}$ be the neighborhood in Lemma 3.2.1. Then (3.30) holds for every $\alpha' \in \mathcal{N}_{\alpha'_0}$, and the continuity of $m \mapsto \Gamma_t(k',m)$ for all t,k' gives the desired continuity of $\alpha' \mapsto \gamma^u_t(\cdot \mid \alpha')$ at $\alpha' \equiv \alpha'_0$ for all $u \in \mathbb{T}^d$. Hence α'_0 is a good profile.

Conversely, suppose that C_{t,α'_0} has multiple global minimizers. Consider the pair $\hat{\alpha}_k(\alpha'_0)$ and $\hat{\alpha}_l(\alpha'_0)$ and the box I in the proof of Lemma 3.2.1, and put $\alpha'^k_\epsilon := \alpha'_0 + \epsilon\gamma'$ for $\epsilon > 0$ and $\alpha'^l_\epsilon := \alpha'_0 + \epsilon\gamma'$ for $\epsilon < 0$. Then $\gamma^u_t(\cdot \mid \alpha'^i_\epsilon) = \Gamma_t(\cdot, \beta\{J * \hat{\alpha}(\alpha'^i_\epsilon) + h\}(u))$, $i \in \{k,l\}$, and

$$\lim_{\epsilon\downarrow 0} \hat{\alpha}(\alpha'^k_\epsilon)(u) = \hat{\alpha}_k(\alpha'_0)(u) \neq \hat{\alpha}_l(\alpha'_0)(u) = \lim_{\epsilon\uparrow 0} \hat{\alpha}(\alpha'^l_\epsilon)(u) \quad \forall\, u \in I. \tag{3.58}$$

On the other hand, $\hat{\alpha}_k(\alpha'_0)$ and $\hat{\alpha}_l(\alpha'_0)$ are critical points, they satisfy (3.35) with $\alpha' \equiv \alpha'_0$, and so

$$\hat{\alpha}_k(u) \neq \hat{\alpha}_l(u) \quad \Longrightarrow \quad (J * \hat{\alpha}_k)(u) \neq (J * \hat{\alpha}_l)(u). \tag{3.59}$$

This, together with the continuity and the monotonicity of $m \mapsto \Gamma_t(k',m)$ for all t and k', forces the discontinuity

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \gamma^u_t(k' \mid \alpha'^k_\epsilon) &= \Gamma_t\big(k', \beta\{J * \hat{\alpha}_k(\alpha'_0) + h\}(u)\big) \\ &\neq \Gamma_t\big(k', \beta\{J * \hat{\alpha}_l(\alpha'_0) + h\}(u)\big) = \lim_{\epsilon\uparrow 0} \gamma^u_t(k' \mid \alpha'^l_\epsilon) \quad \forall u \in I. \end{aligned} \tag{3.60}$$

Hence α'_0 is a bad profile. □

3.3 Proof of Theorem 3.1.7

Proof. Without loss of generality we may assume that $\langle J \rangle = 1$. For simplicity, we consider only $\alpha' \in C(\mathbb{T}^d)$. In that case, due to the regularization property of the convolution operator, the solutions of (3.35) may be taken to be continuous, and (3.35) must be fulfilled for all $u \in \mathbb{T}^d$. The extension to $\alpha' \notin C(\mathbb{T}^d)$ is straightforward.

(i) Let $\alpha_1, \alpha_2 \in B$ be two different solutions of (3.27). After some algebra with trigonometrical identities, we get from (3.27) that the following equation must be fulfilled:

$$\frac{2\sinh\left(\frac{A_u - B_u}{2}\right)}{a_u - b_u}\left\{ \cosh\left(\tfrac{A_u+B_u}{2}\right) - a_u \sinh\left(\tfrac{A_u+B_u}{2}\right) \right\} - \cosh\left(B_u\right) = \coth(2t) \qquad \forall\, u \in \mathbb{T}^d, \tag{3.61}$$

where $A_u = (\beta J * \alpha_1)(u) + \beta h(u)$ and $a_u = \alpha_1(u)$ (and similarly for B_u, b_u, α_2). Note that the left-hand side depends only on u and the right-hand side only on t, and that $\lim_{t\downarrow 0} \coth(2t) = \infty$. Since $|A_u|, |B_u| \leq \beta(1 + \|h\|_\infty)$ and $|a_u|, |b_u| \leq 1$, the left-hand side of (3.61) is bounded from above by

$$\frac{2\sinh\left(\frac{A_u - B_u}{2}\right)}{a_u - b_u} C_1 + C_2 \tag{3.62}$$

for some constants C_1, C_2. By taking $t > 0$ small enough, we force $a_u - b_u$ to be small for all $u \in \mathbb{T}^d$ (equivalently, $\|\alpha_1 - \alpha_2\|_\infty < \delta$). By choosing v_0 such that $|\alpha_1(v_0) - \alpha_2(v_0)| = V_0$ with $V_0 = \max_{u\in\mathbb{T}^d} |\alpha_1(u) - \alpha_2(u)|$, we get $|A_{v_0} - B_{v_0}| \leq \beta V_0$ which, together with the series expansion of sinh, leads to a contradiction.

(ii) From (3.48), whenever $\alpha' \equiv c'$ we have that

$$\inf_{\alpha\in B} C_{t,c'}(\alpha) \geq \inf_{\alpha\in B} \tfrac{1}{4}\beta \int_{\mathbb{T}^d} du \int_{\mathbb{T}^d} dv\, J(u-v)[\alpha(u) - \alpha(v)]^2 + \inf_{\alpha \in B} \int_{\mathbb{T}^d} du\, C^{\mathrm{CW}}_{t,c'}(\alpha(u)). \tag{3.63}$$

Because $J \geq 0$, the minimizers of the first term are the constant profiles. If we take the constant of the profile equal to a minimizer of $C^{\mathrm{CW}}_{t,c'}$, then the second term is also minimal. □

Appendix

3.4 Proof of Proposition 3.1.3

3.4.1 Outline

In Sections 3.4.2–3.4.4 we sketch the proof of the LDP in Proposition 3.1.3(i) for deterministic initial conditions (as in Proposition 3.1.2(ii)), and explain why it remains true for random initial conditions. We follow the line of argument in Benois, Mourragui, Orlandi, Saada and Triolo [BMO+12] rather than Comets [Com87], and use various results from Kipnis and Landim [KL99]. The strategy of the proof consists in first proving the claim for random initial conditions drawn according to $\vartheta_\kappa^n = \otimes_{x\in\mathbb{T}_n^d}\vartheta_\kappa$ with $\vartheta_\kappa = \mathrm{BER}(\kappa)$, $\kappa \in [0,1]$ (i.e., $\vartheta_\kappa(+1) = \kappa$ and $\vartheta_\kappa(-1) = 1-\kappa$), and afterwards replacing ϑ_κ^n by μ^n in (3.2) with the help of Varadhan's Lemma and Bryc's Lemma. In Section 3.4.5 we indicate how Proposition 3.1.3(ii) follows.

Below we will make frequent reference to formulas in [BMO+12] and [KL99], so our arguments are not self-contained. We begin with the following observation.

Lemma 3.4.1. *Suppose that μ and ν are equivalent probability measures. If P_μ and Q_ν are the laws of equivalent Markov processes with starting measures μ and ν, then*

$$\frac{dP_\mu}{dQ_\nu}(\bar\eta) = \frac{d\mu}{d\nu}(\eta_0)\,\frac{dP_\mu}{dQ_\mu}(\bar\eta) = \frac{d\mu}{d\nu}(\eta_0)\,\frac{dP_\nu}{dQ_\nu}(\bar\eta). \tag{3.64}$$

The general technique to prove an LDP relies on finding a family of mean-one positive martingales that can be written as functions of the empirical density. For Markov processes this is achieved by considering the Radon-Nikodym derivative of the original dynamics w.r.t. a small perturbation of this dynamics. It is here that Lemma 3.64 comes into play: it factorizes the Radon-Nikodym derivative into a *static part* and a *dynamic part*, as in (3.25).

3.4.2 Upper bound

For initial condition $\gamma \in C(\mathbb{T}^d; [-1,+1])$ and potential $V \in C^{1,0}([0,t]\times\mathbb{T}^d)$, we denote by $P^{n,V}_{\vartheta^n_\gamma}$ the law of the (γ, V)-perturbed inhomogeneous Markov process starting at

$$\vartheta^n_\gamma = \otimes_{x\in\mathbb{T}_n^d}\vartheta_{\chi^{-1}(\gamma(\frac{x}{n}))}, \tag{3.65}$$

where $\chi\colon [0,1]\to[-1,+1]$ is the linear map that transforms a profile taking values in $[-1,+1]$ into a profile taking values in $[0,1]$. Details about such a perturbation and its Radon-Nikodym derivative can be found in [BMO+12, Eq. (5.8)].

1. *Large deviation upper bound for compact sets.* Fix $\kappa\in[0,1]$. Let $\mathcal{K} \in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ be compact. By Lemma 3.4.1, we have (recall the notation introduced in Section 3.1.5)

$$\begin{aligned}\tfrac{1}{n^d}\log P^n_{\vartheta^n_\kappa}[\bar\pi^n\in\mathcal{K}] &= \tfrac{1}{n^d}\log E^{n,V}_{\vartheta^n_\gamma}\left[\left(\frac{dP^n_{\vartheta^n_\kappa}}{dP^{n,V}_{\vartheta^n_\gamma}}\,\mathbb{I}_{\mathcal{K}}\right)(\bar\pi^n)\right]\\ &= \tfrac{1}{n^d}\log E^{n,V}_{\vartheta^n_\gamma}\left[\left(\frac{d\vartheta^n_\kappa}{d\vartheta^n_\gamma}\,\frac{dP^n_{\vartheta^n_\kappa}}{dP^{n,V}_{\vartheta^n_\kappa}}\,\mathbb{I}_{\mathcal{K}}\right)(\bar\pi^n)\right]\\ &= \tfrac{1}{n^d}\log E^{n,V}_{\vartheta^n_\gamma}\left[e^{-n^d h_\gamma(\pi^n_0)+O_\gamma(n^{-1})}\,e^{-n^d\{\hat J_V(\bar\pi^n * l^{\varepsilon,n})+r(V,\varepsilon,n)\}}\,\mathbb{I}_{\mathcal{K}}(\bar\pi^n)\right],\end{aligned} \tag{3.66}$$

where h_γ is the analogue of [KL99, Eq. (1.1), Chapter 10], $\hat J_V$ is defined in [BMO+12, Eq. (6.8)], $\varepsilon>0$ is small, $l^{\varepsilon,n}$ is an approximation of the identity for $\varepsilon\downarrow 0$, and $r(V,\varepsilon,n)$

is an error term that vanishes as $n \to \infty$ for fixed V, ε. By letting $n \to \infty$, optimizing over γ, V, ε and using the mini-max lemma, we get

$$\begin{aligned}\limsup_{n\to\infty} \tfrac{1}{n^d} \log P^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{K}] &\le \inf_{\gamma,V,\varepsilon} \sup_{\bar\pi\in\mathcal{K}} \{-h_\gamma(\pi_0) - \hat{J}_V(\bar\pi * l^\varepsilon)\} \\ &\le \sup_{\bar\pi\in\mathcal{K}} \inf_{\gamma,V,\varepsilon} \{-h_\gamma(\pi_0) - \hat{J}_V(\bar\pi * l^\varepsilon)\} \\ &\le - \inf_{\bar\pi\in\mathcal{K}} \{I_S(\pi_0) + I^t_D(\bar\pi)\}.\end{aligned} \tag{3.67}$$

The last inequality uses that $\sup_\gamma h_\gamma(\pi_0) = I_S(\pi_0)$, $\sup_V \hat{J}_V(\tilde\pi) = I^t_D(\tilde\pi)$, and $\sup_\varepsilon I^t_D(\bar\pi * l^\varepsilon) \ge I^t_D(\bar\pi)$ by lower semi-continuity of I^t_D.

2. *Exponential tightness.* While in [KL99, Section 4] the initial condition is drawn from equilibrium, this is immaterial. Indeed, the proof of [BMO+12, Proposition 6.1] uses the same ideas as in [KL99, Section 4] even though the initial condition is deterministic. Hence the same computations apply to our case.

3.4.3 Lower bound

1. *Large deviation lower bound for open sets.* Fix $\kappa \in [0,1]$. Let $\mathcal{O} \in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))$ be open. By Lemma 3.4.1, we have

$$\begin{aligned}\tfrac{1}{n^d} \log P^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{O}] &= \tfrac{1}{n^d} \log \left\{ E^{n,V}_{\vartheta^n_\gamma} \left[\frac{dP^n_{\vartheta^n_\kappa}}{dP^{n,V}_{\vartheta^n_\gamma}}(\bar\pi^n) \,\middle|\, \bar\pi^n \in \mathcal{O} \right] P^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}) \right\} \\ &\ge E^{n,V}_{\vartheta^n_\gamma} \left[\tfrac{1}{n^d} \log \frac{dP^n_{\vartheta^n_\kappa}}{dP^{n,V}_{\vartheta^n_\gamma}}(\bar\pi^n) \,\middle|\, \bar\pi^n \in \mathcal{O} \right] + \tfrac{1}{n^d} \log P^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}),\end{aligned} \tag{3.68}$$

where we use Jensen's inequality. By the law of large numbers for $P^{n,V}_{\vartheta^n_\gamma}$, we have

$$w - \lim_{n\to\infty} P^{n,V}_{\vartheta^n_\gamma} = \delta_{\bar\pi^{\gamma,V}}, \tag{3.69}$$

where $\bar\pi^{\gamma,V}$ is the solution of [BMO+12, Eq. (5.5)] with initial condition γ and potential V. (The proof of (3.69) follows in the same fashion as in [BMO+12]: all that is needed is that the laws of the random initial conditions converge to a law associated with continuous profile.) Hence, if $\bar\pi^{\gamma,V} \in \mathcal{O}$, then $\lim_{n\to\infty} P^{n,V}_{\vartheta^n_\gamma}(\mathcal{O}) = 1$. After some calculations with the Radon-Nikodym derivative, we get

$$\liminf_{n\to\infty} \tfrac{1}{n^d} \log P^n_{\vartheta^n_\kappa}[\bar\pi^n \in \mathcal{O}] \ge -I^t(\bar\pi^{\gamma,V}) \tag{3.70}$$

with $I^t = I_S + I^t_D$.

2. *Density arguments.* It remains to show that

$$\inf_{\substack{\gamma,V \\ \bar\pi^{\gamma,V}\in\mathcal{O}}} I^t(\bar\pi^{\gamma,V}) = \inf_{\bar\pi\in\mathcal{O}} I^t(\bar\pi). \tag{3.71}$$

In other words, $(\bar{\pi}^{\gamma,V})_{\gamma,V}$ is dense with respect to (ϱ_t^w, I), i.e.,

$$\begin{aligned}&\forall\, \bar{\pi} \in D_{[0,t]}(\mathcal{M}(\mathbb{T}^d))\colon\ I(\bar{\pi}) < \infty,\\ &\exists\, (\bar{\pi}^{\gamma_n,V_n})_{n\in\mathbb{N}}\colon\ \lim_{n\to\infty} \varrho_t^w(\bar{\pi}^{\gamma_n,V_n}, \bar{\pi}) = 0,\ \lim_{n\to\infty} I(\bar{\pi}^{\gamma_n,V_n}) = I(\bar{\pi}),\end{aligned} \tag{3.72}$$

where ϱ_t^w is the supremum distance in $[0,t]$ when the marginal distance is ϱ^w (any metric that metrizes the weak topology). A density argument of this type typically exploits the fact that I is lower semi-continuous and convex, but in our case $I = I^t$, which is not convex. However, in [BMO+12] density arguments are given without convexity. In order to extend these to our setting of random initial conditions, minor modifications are needed in [BMO+12, Lemma 7.5]. In particular, the space regularization of the trajectory must be done for all $s \in [0,t]$, and hence [BMO+12, Lemma 7.6] together with the arguments in [KL99, p. 279] prove our assertion.

3.4.4 Replace ϑ_κ^n by μ^n

The observations made in Sections 3.4.2–3.4.3 prove the LDP in Proposition 3.1.3(i), but for starting measures ϑ_κ^n given by (3.65). Note that

$$\frac{d\mu^n}{d\vartheta_\kappa^n} = e^{n^d \beta H(\pi^n)} \tag{3.73}$$

with $\pi^n \mapsto H(\pi^n)$ in (3.6) continuous. Hence, by Lemma (3.4.1), Varadhan's Lemma and Bryc's Lemma, the LDP in Proposition 3.1.3(i) for starting measures μ^n follows.

3.4.5 Contraction principle

Proposition 3.1.3(ii) follows from Proposition 3.1.3(i) via the approximate contraction principle based on exponential approximation estimates. See Dembo and Zeitouni [DZ98, Section 4.2].

Part II - Brownian percolation

4 Introduction to Part II

4.1 Motivation

The model we study in this part of the thesis fits into the class of continuum percolation models, which have been studied by both mathematicians and physicists. Their first appearance can be traced back (at least) to [Gil61] under the name of random plane networks. This paper provides a model for infinite communication networks of stations with finite range: each pair of points of a Poisson point process on $\mathbb{R}^2$ is connected whenever their distance is less than a prescribed threshold $R > 0$. From a percolation point of view this may also be described by a Boolean percolation model (see Section 5.2). Another application mentioned in [Gil61] is the modeling of a contagious infection, where each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: draw a random set of points (a Poisson point process, for instance) and attach to each of the points a geometric object, like a disk with a random radius (Boolean model) or a segment with a random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: we attach to each point of a Poisson point process on $\mathbb{R}^d, d \geq 1$, a Brownian path (a path of a Wiener sausage when $d \geq 4$).

Notations. For $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$. $||\cdot||$ and $||\cdot||_\infty$ stand for the Euclidean norm and sup-norm on $\mathbb{R}^d$, respectively. For any set A, the symbol $A^\complement$ refers to the complement $\mathbb{R}^d \setminus A$. The open ball with center z and radius r with respect to the Euclidean norm is denoted by $\mathcal{B}(z, r)$, while $\mathcal{B}_\infty(z, r)$ stands for the same ball with respect to the sup-norm. Furthermore, for $0 < r < r'$ we denote by $\mathcal{A}(r, r') = \mathcal{B}(0, r') \setminus \overline{\mathcal{B}}(0, r)$ and $\mathcal{A}_\infty(r, r') = \mathcal{B}_\infty(0, r') \setminus \overline{\mathcal{B}}_\infty(0, r)$ the annulus delimited by the balls of radii r and r' with respect to the Euclidean norm and sup-norm, respectively. Moreover, given a d-dimensional Brownian motion $(B_t)_{t\geq 0}$, we denote its i-th component by $(B_{t,i})_{t\geq 0}$, $i \in \{1, 2, \ldots, d\}$. Finally, for $I \subseteq \mathbb{R}^+$ we denote by B_I the set $\{B_t,\, t \in I\}$. The symbol $\mathbb{P}^a$ denotes the law of a standard Brownian motion starting at a. In case of two or more independent copies we add a superscript, i.e., $\mathbb{P}^{a_1,a_2}$.

4.2 Overview

For $\lambda > 0$, let $(\Omega_p, \mathcal{A}_p, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \mathrm{Leb}_d$ is defined. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\, x \in \mathcal{E}\}$ such that $B_0^x = x$ for each $x \in \mathcal{E}$, and such that

$(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$. A more rigorous definition is provided in Section 5.1.3 below, where ergodic properties are obtained along the way. We study for $t, r \geq 0$ the *occupied* set:

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r). \tag{4.1}$$

We write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.

Two points x and y of $\mathbb{R}^d$ are said to be *pairwise connected* in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) = y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a component. A component $\mathcal{C}$ is bounded if and only if there exists an $R > 0$ such that $\mathcal{C} \subseteq \mathcal{B}(0, R)$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component that is maximal in the sense that it is not strictly contained in any other connected component.

We are interested in the percolative properties of the occupied set: Is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: Is there a percolation transition in t?

4.3 Preliminaries on Boolean percolation

The model of Boolean percolation is discussed in detail in [MR96], and we refer to this book for a discussion that goes beyond the description we are giving here.

Let ρ be a probability measure on $[0,\infty)$ and let χ be the Poisson point process on $\mathbb{R}^d \times [0,\infty)$ with intensity $(\lambda \times \mathrm{Leb}_d) \otimes \rho$. We denote the corresponding probability measure by $\mathbb{P}_{\lambda,\rho}$. A point $(x, r(x)) \in \chi$ is interpreted to be the open ball in $\mathbb{R}^d$ with center x and radius $r(x)$. Furthermore, we let $\mathcal{E}$ be the projection of χ onto $\mathbb{R}^d$. Boolean percolation deals with properties of the random set

$$\Sigma = \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r(x)). \tag{4.2}$$

Let $C(y)$, $y \in \mathbb{R}^d$, denote the cluster of Σ that contains y. If $y \notin \Sigma$, then $C(y) = \varnothing$.

Theorem 4.3.1 ([Gou08], Theorem 2.1)**.** *For all probability measures ρ on $(0,\infty)$ the following assertions are equivalent:*
(a)

$$\int_0^\infty x^d\,\rho(dx) < \infty. \tag{4.3}$$

(b) *There exists a $\lambda_0 \in (0,\infty)$ such that for all $\lambda < \lambda_0$,*

$$\mathbb{P}_{\lambda,\rho}\big(\exists y \in \mathbb{R}^d \ :\ \mathrm{Leb}_d(C(y)) = \infty\big) = 0. \tag{4.4}$$

Moreover, if (a) *holds, then there exists a* $C = C(d) > 0$ *such that* (4.4) *is satisfied for all*

$$\lambda < C\biggl(\int_0^\infty x^d \rho(dx)\biggr)^{-1}. \tag{4.5}$$

It is immediate from Theorem 4.3.1 that

$$\lambda_c(\rho) := \inf\bigl\{\lambda > 0 : \mathbb{P}_{\lambda,\rho}\bigl(\exists y \in \mathbb{R}^d \ : \ \mathrm{Leb}_d(C(y)) = \infty\bigr) > 0\bigr\} > 0. \tag{4.6}$$

Moreover, from the remark on page 52 of [MR96] it also follows that $\lambda_c(\rho) < \infty$ as soon as $\rho((0,\infty)) > 0$. A more geometric way to characterize (4.6) is via crossing probabilities. For that, fix $N_1, N_2, \ldots, N_d > 0$ and let $\mathrm{CROSS}(N_1, N_2, \ldots, N_d)$ be the event that the set $[0, N_1] \times [0, N_2] \times \cdots \times [0, N_d]$ contains a component $\mathcal{C}$ such that $\mathcal{C} \cap \{0\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \varnothing$ and $\mathcal{C} \cap \{N_1\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \varnothing$. The critical value λ_{CROSS} with respect to this event is defined by

$$\lambda_{\mathrm{CROSS}}(\rho) = \inf\Bigl\{\lambda > 0 : \lim_{N\to\infty} \mathbb{P}_{\lambda,\rho}\left(\mathrm{CROSS}(N, 3N, \ldots, 3N)\right) > 0\Bigr\}. \tag{4.7}$$

It is proved in [MMS86] that

$$\lambda_c(\rho) = \lambda_{\mathrm{CROSS}}(\rho), \tag{4.8}$$

provided ρ has compact support.

4.4 Main results on Brownian percolation

Fix $\lambda > 0$.

Theorem 4.4.1. [No percolation for $d = 1$**]** *Let* $d = 1$*. Then for all* $t \geq 0$ *the set* $\mathcal{O}_t$ *has almost surely no unbounded cluster.*

Theorem 4.4.2. [Percolation phase transition and uniqueness for $d \in \{2,3\}$**]** *Let* $d \in \{2,3\}$*. Then there exists a* $t_c = t_c(\lambda, d) > 0$ *such that, for* $t < t_c$*,* $\mathcal{O}_t$ *has almost surely no unbounded cluster, but, for* $t > t_c$*,* $\mathcal{O}_t$ *has a unique unbounded cluster.*

Let $d \geq 4$, $r > 0$ and let δ_r be the Dirac measure at r. Denote by $\lambda_c(\delta_r)$ the critical value for $\mathcal{O}_{0,r}$, i.e., for all $\lambda < \lambda_c(\delta_r)$ the set $\mathcal{O}_{0,r}$ has almost surely no unbounded cluster, but for $\lambda > \lambda_c(\delta_r)$ $\mathcal{O}_{0,r}$ has a unique unbounded cluster (see also (4.6)). It follows from Theorem 4.3.1 that $\lambda_c(\delta_r) > 0$ and $\lim_{r\to 0} \lambda_c(\delta_r) = \infty$.

Theorem 4.4.3. [Percolation phase transition and uniqueness for $d \geq 4$**]** *Let* $d \geq 4$ *and let* $r > 0$ *be such that* $\lambda < \lambda_c(\delta_r)$*. Then there exists a* $t_c = t_c(\lambda, d, r) > 0$ *such that, for* $t < t_c$*,* $\mathcal{O}_{t,r}$ *has almost surely no unbounded cluster, but, for* $t > t_c$*,* $\mathcal{O}_t$ *has a unique unbounded cluster.*

5 Brownian paths homogeneously distributed in space - Percolation

This chapter is based on:
D. Erhard, J. Martínez, and J. Poisat. *Brownian paths homogeneously distributed in space: Percolation phase transition and uniqueness of the unbounded cluster.* *arXiv preprint:1311.2907*, Submitted to Annals of Applied Probability, 2013.

Abstract

We consider a continuum percolation model on $\mathbb{R}^d$, $d \geq 1$. For $t, \lambda \in (0, \infty)$ and $d \in \{1, 2, 3\}$, the occupied set is given by the union of independent Brownian paths running up to time t whose initial points form a Poisson point process with intensity $\lambda > 0$. When $d \geq 4$, the Brownian paths are replaced by Wiener sausages with radius $r > 0$.
We establish that, for $d = 1$ and all choices of t, no percolation occurs, whereas for $d \geq 2$, there is a non-trivial percolation transition in t, provided λ and r are chosen properly. The last statement means that λ has to be chosen to be strictly smaller than the critical percolation parameter for the occupied set at time zero (which is infinite when $d \in \{2, 3\}$, but finite and dependent on r when $d \geq 4$). We further show that for all $d \geq 2$, the unbounded cluster in the supercritical phase is unique.
Along the line a finite box criterion for non-percolation in the Boolean model is extended to radius distributions with an exponential tails. This may be of independent interest.

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5.1 Introduction

Notation. For every $d \geq 1$, we denote by Leb_d the Lebesgue measure on $\mathbb{R}^d$. $||\cdot||$ and $||\cdot||_\infty$ stand for the Euclidean norm and supremum norm on $\mathbb{R}^d$, respectively. For any set A, the symbol $A^{\complement}$ refers to the complement set of A. The open ball with center z and radius r with respect to the Euclidean norm is denoted by $\mathcal{B}(z,r)$, whereas $\mathcal{B}_\infty(z,r)$ stands for the same ball with respect to the supremum norm. Furthermore, for every $0<r<r'$, we denote by $\mathcal{A}(r,r') = \mathcal{B}(0,r') \setminus \overline{\mathcal{B}}(0,r)$ and $\mathcal{A}_\infty(r,r') = \mathcal{B}_\infty(0,r') \setminus \overline{\mathcal{B}}_\infty(0,r)$ the annulus delimited by the balls of radii r and r' with respect to the Euclidean norm and supremum norm, respectively. Given a d-dimensional Brownian motion $(B_t)_{t\geq 0}$, we denote its i-th component by $(B_{t,i})_{t\geq 0}$, for $i \in \{1,2,\dots,d\}$. For all $I \subseteq \mathbb{R}^+$, we denote by B_I the set $\{B_t,\, t\in I\}$. The symbol $\mathbb{P}^a$ denotes the law of a Brownian motion starting in a. Finally, $\mathbb{P}^{a_1,a_2}$ denotes the law of two independent Brownian motions starting in a_1 and a_2, respectively.

5.1.1 Overview and motivation

For $\lambda > 0$, let $(\Omega_p, \mathcal{A}_p, \mathbb{P}_\lambda)$ be a probability space on which a Poisson point process $\mathcal{E}$ with intensity $\lambda \times \text{Leb}_d$ is defined. Conditionally on $\mathcal{E}$, we fix a collection of independent Brownian motions $\{(B_t^x)_{t\geq 0},\, x\in\mathcal{E}\}$ such that for each $x\in\mathcal{E}$, $B_0^x = x$ and such that $(B_t^x - x)_{t\geq 0}$ is independent of $\mathcal{E}$. A more rigorous definition is provided in Section 5.1.3 below, where ergodic properties are obtained along. We study for $t,r\geq 0$ the *occupied* set (see Figure 5.1 below):

$$\mathcal{O}_{t,r} := \bigcup_{x\in\mathcal{E}} \bigcup_{0\leq s\leq t} \mathcal{B}(B_s^x, r). \tag{5.1}$$

In the rest of the paper, we write $\mathcal{O}_t$ instead of $\mathcal{O}_{t,0}$.

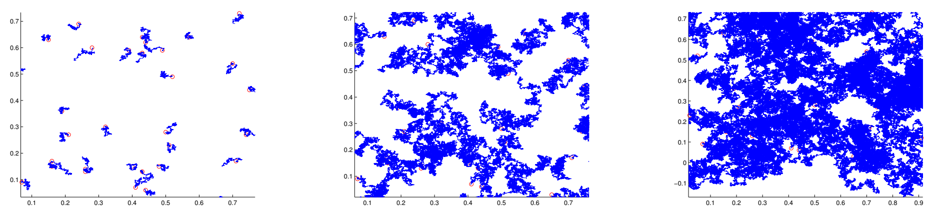

Figure 5.1: Simulations of $\mathcal{O}_t$ in the case $d=2$, at a small time, intermediate and large time.

Two points x and y of $\mathbb{R}^d$ are said to be *connected* in $\mathcal{O}_{t,r}$ if and only if there exists a continuous function $\gamma : [0,1] \mapsto \mathcal{O}_{t,r}$ such that $\gamma(0)=x$ and $\gamma(1)=y$. A subset of $\mathcal{O}_{t,r}$ is connected if and only if all of its points are pairwise connected. In the following a connected subset of $\mathcal{O}_{t,r}$ is called a component. A component $\mathcal{C}$ is bounded if there exists $R > 0$ such that $\mathcal{C} \subseteq \mathcal{B}(0, R)$. Otherwise, the component is said to be unbounded. A *cluster* is a connected component which is maximal in the sense that it is not strictly contained in another connected component.

We are interested in the percolative properties of the occupied set: is there an unbounded cluster for large t? Is it unique? What happens for small t? Since an elementary monotonicity argument shows that $t \mapsto \mathcal{O}_{t,r}$ is non-decreasing, the first and the third question may be rephrased as follows: is there a percolation transition in t?

5.1.2 Results

We fix $\lambda > 0$.

Theorem 5.1.1. **[No percolation for $d = 1$]** *Let $d = 1$. Then, for all $t \geq 0$, the set $\mathcal{O}_t$ has almost surely no unbounded cluster.*

Theorem 5.1.2. **[Percolation phase transition and uniqueness for $d \in \{2, 3\}$]** *Suppose that $d \in \{2, 3\}$. There exists $t_c = t_c(\lambda, d) > 0$ such that for $t < t_c$, $\mathcal{O}_t$ has almost surely no unbounded cluster, whereas for $t > t_c$, $\mathcal{O}_t$ has almost surely a unique unbounded cluster.*

Let $d \geq 4$, $r > 0$ and let δ_r be the Dirac measure concentrated on r. We denote by $\lambda_c(\delta_r)$ the critical value for $\mathcal{O}_{0,r}$ such that for all $\lambda < \lambda_c(\delta_r)$ the set $\mathcal{O}_{0,r}$ almost surely does not contain an unbounded cluster, and such that for $\lambda > \lambda_c(\delta_r)$ it does, see also (5.9). It follows from Theorem 5.2.1, that $\lambda_c(\delta_r) > 0$ and $\lim_{r \to 0} \lambda_c(\delta_r) = \infty$.

Theorem 5.1.3. **[Percolation phase transition and uniqueness for $d \geq 4$]** *Suppose that $d \geq 4$ and let $r > 0$ be such that $\lambda < \lambda_c(\delta_r)$. Then, there exists $t_c = t_c(\lambda, d, r) > 0$ such that for $t < t_c$, $\mathcal{O}_{t,r}$ has almost surely no unbounded cluster, whereas for $t > t_c$, it has almost surely a unique unbounded cluster.*

5.1.3 Construction and an ergodic property.

In this section we briefly outline how to construct the model described in Section 5.1.1 and we state an ergodic theorem. The construction is very close to the construction of the Boolean percolation model, in which balls of random radii are placed around each point of a Poisson point process. We refer the reader to Section 1.4 of [MR96], where a more detailed description of the Boolean percolation model is given (see also Section 5.2 in the present work).

Construction. Let $\mathcal{E}$ be a Poisson point process with intensity $\lambda \times \mathrm{Leb}_d$ defined on $(\Omega_p, \mathcal{A}_p, \mathbb{P}_\lambda)$. Consider the family of binary cubes

$$K(n, z) = \prod_{i=1}^{d} (z_i 2^{-n}, (z_i + 1) 2^{-n}], \qquad \forall n \in \mathbb{N},\ z = (z_i)_{1 \leq i \leq d} \in \mathbb{Z}^d, \tag{5.2}$$

so that for each $n \in \mathbb{N}$, $\{K(n,z), z \in \mathbb{Z}^d\}$ is a partition of $\mathbb{R}^d$. In particular, for each $x \in \mathcal{E}$ and $n \in \mathbb{N}$, there exists a unique $z(n,x)$ such that $x \in K(n, z(n,x))$. Consequently, $\mathbb{P}_\lambda$-a.s., for each $x \in \mathcal{E}$,

$$n_0(x) := \inf\{n \geq 1 \ : \ K(n, z(n,x)) \cap \mathcal{E} = \{x\}\} \tag{5.3}$$

is well defined. Let $\mathcal{B}(C([0,\infty),\mathbb{R}^d))$ be the Borel σ-algebra on $C([0,\infty),\mathbb{R}^d)$ with respect to the supremum norm. To continue define $\Omega_B = C([0,\infty),\mathbb{R}^d)^{\mathbb{N}\times\mathbb{Z}^d}$, equip Ω_B with the product σ-algebra $\mathcal{A}_B = \mathcal{B}(C([0,\infty),\mathbb{R}^d))^{\mathbb{N}\times\mathbb{Z}^d}$ and let $\mathbb{P}_B = W_B^{\otimes\mathbb{N}\times\mathbb{Z}^d}$, where W_B is the Wiener measure on $C([0,\infty),\mathbb{R}^d)$. The Brownian path associated to $x \in \mathcal{E}$ is defined to be

$$w_B(n_0(x), z(n_0(x), x)), \quad w_B \in \Omega_B. \tag{5.4}$$

Finally, we set $\Omega = \Omega_p \times \Omega_B$, $\mathcal{A} = \mathcal{A}_p \times \mathcal{A}_B$ and $\mathbb{P} = \mathbb{P}_\lambda \times \mathbb{P}_B$, so that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ corresponds to the model described in Section 5.1.1.

Ergodicity. For $x \in \mathbb{Z}^d$ let $T_x : \mathbb{R}^d \to \mathbb{R}^d$ be the translation defined by $T_x(y) = y + x$, $y \in \mathbb{R}^d$. This induces a translation S_x on Ω_p via the equation $(S_x\omega_p)(A) = \omega_p(T_x^{-1}A)$, $A \in \mathcal{A}_p$. A translation on Ω_B is given by the formula $(U_x\omega_B)(n,z) = \omega_B(n, z-x)$, so that we finally can define the translation $\widetilde{T}_x$ on the product space Ω as $\widetilde{T}_x\omega = (S_x\omega_p, U_x\omega_B)$. A simple adaption of the proof of Proposition 2.8 in [MR96] yield the following result.

Proposition 5.1.4. *For all $t, r \geq 0$ the set $\mathcal{O}_{t,r}$ defined in (5.1) is ergodic with respect to the family of translations $\{\widetilde{T}_x,\ x \in \mathbb{Z}^d\}$.*

5.1.4 Discussion

Motivation and related models. Our model fits into the class of continuum percolation models, which have been studied by both mathematicians and physicists. Their first appearance can be traced back (at least) to Gilbert [Gil61] under the name of random plane networks. Gilbert was interested in modeling infinite communication networks of stations with range $R > 0$. He did this by connecting each two points of a Poisson point process on $\mathbb{R}^2$, whenever their distance is less than R. Another application, which is mentioned in his work is the modeling of a contagious infection. Here, each individual gets infected when it has distance less than R to an infected individual.

A subclass of continuum percolation models follows the following recipe: first throw a point process (e.g. Poisson point process) and attach to each of its points a geometric object, like a disk of random radius (Boolean model) or a segment of random length and random orientation (Poisson sticks model or needle percolation). Our model also falls into this class: we attach to each point of a Poisson point process a Brownian path (a path of a Wiener sausage when $d \geq 4$). It could actually be seen as a model of defects randomly distributed in a material and propagating at random (see also Menshikov, Molchanov and Sidorenko [MMS86] for other physical motivations of continuum percolation). One can think for example of an (infinite) piece of wood containing (homogeneously distributed) worms, where each worm tunnels through the piece of wood at random, and we wonder when the latter "breaks". The informal description above is reminiscent of (and actually, borrowed from) the problem of the disconnection of a cylinder by a random walk, which itself is linked to interlacement percolation [Szn10]. The latter is given by the random subset obtained when looking at the trace of a simple random walk on the torus $(\mathbb{Z}/N\mathbb{Z})^d$, when started from the uniform distribution and running up to time uN^d, as $N \uparrow \infty$. Here u plays the role of an intensity parameter for the interlacements set. However, even though the model of random interlacements and our model seem to share some similarities, there is an important difference: in the interlacement model, the number of trajectories which enter a ball of radius R scales like cR^{d-2} for some $c > 0$, whereas in our case it is at least of order R^d.

Another motivation for studying such a model is that it should arise as the scaling limit of a certain class of discrete dependent percolation models. More precisely, percolation models for a system of independent finite-time random walks homogeneously distributed on $\mathbb{Z}^d$. This could also be seen as a system of non-interacting ideal polymer chains.

Comments on the results. First of all notice that we investigated a phase transition in t. It would also be possible to play with the intensity λ instead. Indeed, multiplying the intensity λ by a factor η changes the typical distance between two Poisson points by a factor $\eta^{-1/d}$. Thus, by scale invariance of Brownian motion, the percolative behaviour of the model is the same when we consider the Brownian paths up to time $\eta^{-2/d}t$ instead. Hence, tuning λ boils down to tuning t.

Moreover, it is worthwhile mentioning that Theorem 5.1.2 is stated only in the case $r = 0$, which is the case of interest to us. The result is the same when $r > 0$, up to minor modifications. However, if $d \geq 4$ the paths of two independent d-dimensional Brownian motions starting at different points do not intersect. Hence, in this case r has to be chosen positive, otherwise no percolation phase transition occurs.

Besides, we draw the reader's attention to Lemma 5.2.3, which is useful in proving the continuity result in Proposition 5.2.2. This lemma provides a finite-box criterion for non-percolation for the Boolean model. It is stated in the case of radius distributions with exponential tail. To our knowledge such a criterion was only proved for bounded radii.

To sum up, the results proven in this article answer the first questions typically asked when studying a new percolation model. However, there are still many challenges left open. One may wonder for instance how fast is the decay of the probability (in the supercritical regime) that there is a ball of a certain size, centered in the origin, which is contained in the vacant set. Moreover, it would be interesting to investigate the scaling behaviour of t_c in dimension $d \geq 4$ as r tends to zero. In the same line one could ask for sharp upper and lower bounds for t_c. Finally, it is not clear whether percolation occurs at t_c.

5.1.5 Outline of the paper

We shortly describe the organization of the article. In Section 5.2 we introduce the Boolean percolation model and list and prove some of its properties. In Section 5.3 we prove Theorem 5.1.1. The proofs of Theorems 5.1.2 and 5.1.3 are given in Sections 5.4–5.6. Section 5.4 (resp. 5.5) deals with the existence of a non-percolation (resp. percolation) phase. In Section 5.6 the uniqueness of the unbounded cluster is established. The appendix provides proofs of technical lemmas, which are needed in Sections 5.2 and 5.6.

5.2 Preliminaries on Boolean percolation

The model of Boolean percolation has been discussed in great detail in Meester and Roy [MR96] and we refer to this source for a discussion which goes beyond the description we are giving here.

5.2.1 Introduction of the model

Let ρ be a probability measure on $[0,\infty)$ and let χ be the Poisson point process on $\mathbb{R}^d \times [0,\infty)$ with intensity $(\lambda \times \mathrm{Leb}_d) \otimes \rho$. We denote the corresponding probability measure by $\mathbb{P}_{\lambda,\rho}$. A point $(x, r(x)) \in \chi$ is interpreted to be the open ball in $\mathbb{R}^d$ with center x and radius $r(x)$. Furthermore, we let $\mathcal{E}$ be the projection of χ onto $\mathbb{R}^d$. Boolean percolation deals with properties of the random set

$$\Sigma = \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r(x)). \tag{5.5}$$

Moreover, $C(y)$, $y \in \mathbb{R}^d$, denotes the cluster of Σ which contains y. If $y \notin \Sigma$, then $C(y) = \varnothing$.

Theorem 5.2.1 (Gouéré, [Gou08], Theorem 2.1)**.** *For all probability measures ρ on $(0,\infty)$ the following assertions are equivalent:*
(a)

$$\int_0^\infty x^d \, \rho(dx) < \infty. \tag{5.6}$$

(b) *There exists $\lambda_0 \in (0,\infty)$ such that for all $\lambda < \lambda_0$,*

$$\mathbb{P}_{\lambda,\rho}\big(C(0) \text{ is unbounded}\big) = 0. \tag{5.7}$$

Moreover, if (a) *holds, then, for some $C = C(d) > 0$, (5.7) is satisfied for all*

$$\lambda < C\Big(\int_0^\infty x^d \rho(dx)\Big)^{-1}. \tag{5.8}$$

It is immediate from Theorem 5.2.1, that

$$\lambda_c(\rho) := \inf\big\{\lambda > 0 : \mathbb{P}_{\lambda,\rho}\big(C(0) \text{ is unbounded}\big) > 0\big\} > 0. \tag{5.9}$$

Moreover, from the remark on page 52 of [MR96] it also follows that $\lambda_c(\rho) < \infty$ if $\rho((0,\infty)) > 0$. A more geometric fashion to characterize (5.9) is via crossing probabilities. For that fix $N_1, N_2, \ldots, N_d > 0$ and let $\text{CROSS}(N_1, N_2, \ldots, N_d)$ be the event that the set $[0, N_1] \times [0, N_2] \times \cdots \times [0, N_d]$ contains a component $\mathcal{C}$ such that $\mathcal{C} \cap \{0\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \varnothing$ and $\mathcal{C} \cap \{N_1\} \times [0, N_2] \times \cdots \times [0, N_d] \neq \varnothing$. The critical value λ_{CROSS} with respect to this event is defined by

$$\lambda_{\text{CROSS}}(\rho) = \inf\left\{\lambda > 0 : \limsup_{N\to\infty} \mathbb{P}_{\lambda,\rho}\left(\text{CROSS}(N, 3N, \ldots, 3N)\right) > 0\right\}. \tag{5.10}$$

Assuming that ρ has compact support, Menshikov, Molchanov and Sidorenko [MMS86] proved that

$$\lambda_c(\rho) = \lambda_{\text{CROSS}}(\rho). \tag{5.11}$$

5.2.2 Continuity of $\lambda_c(\rho)$

Given two probability measures ν and μ on a predefined probability space we write $\nu \preceq \mu$, if μ stochastically dominates ν.

Proposition 5.2.2. *Let ρ be a probability measure on $[0,\infty)$ with bounded support and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $[0,\infty)$ such that $\rho_n \to \rho$ weakly as $n \to \infty$ and such that $\rho \preceq \rho_n$ for each $n \in \mathbb{N}$. Moreover, assume that*

- *there are $C > 0$ and $R_0 > 0$ such that for all $n \in \mathbb{N}$, $\rho_n([R,\infty)) \leq e^{-CR}$ for all $R \geq R_0$;*
- *there is a probability measure ρ' on $[0,\infty)$ with finite moments of order d such that $\rho_n \preceq \rho'$ for all $n \in \mathbb{N}$.*

Then,

$$\lim_{n\to\infty} \lambda_c(\rho_n) = \lambda_c(\rho). \tag{5.12}$$

The proof of Proposition 5.2.2 relies on the following two lemmas whose proofs are given in the appendix and at the end of this section, respectively.

Lemma 5.2.3. *Let $N \in \mathbb{N}$, $\lambda > 0$ and let ρ be a probability measure on $[0,\infty)$ such that there are constants $C = C(\rho) > 0$ and $R_0 > 0$ such that $\rho([R,\infty)) \leq e^{-CR}$ for all $R \geq R_0$. There is an $\varepsilon = \varepsilon(C,d) > 0$ such that if*

$$\mathbb{P}_{\lambda,\rho}(\text{CROSS}(N, 3N, \ldots, 3N)) \leq \varepsilon, \tag{5.13}$$

then $\mathbb{P}_{\lambda,\rho}(\exists\, y \in \mathbb{R}^d : \text{Leb}_d(C(y)) = \infty) = 0$.

Lemma 5.2.4. *Choose $\eta > 0$ and ρ' according to Proposition 5.2.2, then for all $N \in \mathbb{N}$*

$$\lim_{M\to\infty} \mathbb{P}_{\lambda,\rho'}\left(\exists\; y \in \mathcal{B}_\infty(0,M)^{\complement} \cap \mathcal{E} \text{ s.t. } \mathcal{B}(y,r(y)) \cap [0,N]\times[0,3N]^{d-1} \neq \varnothing\right) = 0. \tag{5.14}$$

Remark 5.2.5. *We expect that our proof of Lemma 5.2.3 still works when ρ has a polynomial tail (of sufficiently large order) instead of an exponential tail. However, since we do not need Lemma 5.2.3 in this stronger version, we did not verify all the details needed for that.*

We start with the proof of Proposition 5.2.2 subject to Lemmas 5.2.3–5.2.4.

Proof of Proposition 5.2.2. The idea of the proof is due to Penrose [Pen95]. First, note that

$$\limsup_{n\to\infty} \lambda_c(\rho_n) \le \lambda_c(\rho), \tag{5.15}$$

since $\rho \preceq \rho_n$ for all $n \in \mathbb{N}$. Thus, we may focus on the reversed direction in (5.15). Second, fix $\lambda < \lambda_c(\rho)$ and let $\varepsilon > 0$ be chosen according to Lemma 5.2.3. By (5.11) there is a $N \in \mathbb{N}$ such that

$$\mathbb{P}_{\lambda,\rho}\left(\mathrm{CROSS}(N, 3N, \dots, 3N)\right) \le \varepsilon/3. \tag{5.16}$$

We consider $(\hat\Omega, \hat{\mathbb{P}})$ the following coupling of $\{\mathbb{P}_{\lambda,\rho_n}\}_{n\in\mathbb{N}}$ and $\mathbb{P}_{\lambda,\rho}$:

- the points of $\mathcal{E}$ are sampled according to $\mathbb{P}_\lambda$;
- for each point $x \in \mathcal{E}$, by Skorokhod's embedding theorem, the radii $\{r_n(x)\}_{n\in\mathbb{N}}$ and $r(x)$ can be chosen such that they have respective distributions $\{\rho_n\}_{n\in\mathbb{N}}$ and ρ and are coupled such that $r_n(x) \xrightarrow[n\to\infty]{} r(x)$ a.s.

The configurations obtained via this coupling are denoted by

$$\Sigma_n := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r_n(x)),\; n \in \mathbb{N}, \quad \text{and} \quad \Sigma_\infty := \bigcup_{x\in\mathcal{E}} \mathcal{B}(x, r(x)). \tag{5.17}$$

Let $M > 0$ and consider the events

$$E_n = \{\hat\Sigma := (\Sigma_k)_{k\in\mathbb{N}\cup\{\infty\}} \;:\; \Sigma_n \in \mathrm{CROSS}^M\}, \quad n \in \mathbb{N}\cup\{\infty\},$$

where

$$\mathrm{CROSS}^M = \left\{ \begin{array}{c} \mathrm{CROSS}(N, 3N, \dots, 3N) \text{ happens by open balls} \\ \text{whose centers are in } \mathcal{B}_\infty(0,M) \end{array} \right\}.$$

Since the number of points in $\mathcal{B}_\infty(0,M) \cap \mathcal{E}$ is finite a.s., we may conclude that

$$\lim_{n\to\infty} \mathbb{I}_{E_n} = \mathbb{I}_{E_\infty} \qquad a.s. \tag{5.18}$$

(Note that the convergence in (5.18) is not true for every possible realization, but indeed on a set of probability one.) Hence, by the dominated convergence theorem,

$$\lim_{n\to\infty} \hat{\mathbb{P}}(E_n) = \hat{\mathbb{P}}(E_\infty).$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}_{\lambda,\rho_n}(\mathrm{CROSS}^M) = \mathbb{P}_{\lambda,\rho}(\mathrm{CROSS}^M),$$

so that for all $n \in \mathbb{N}$ large enough,

$$\mathbb{P}_{\lambda,\rho_n}(\mathrm{CROSS}^M) \leq 2\varepsilon/3. \tag{5.19}$$

Whence, Lemma 5.2.4 and the fact that $\rho_n \preceq \rho'$ for all $n \in \mathbb{N}$, yields that there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\mathbb{P}_{\lambda,\rho_n}\big(\mathrm{CROSS}(N, 3N, \dots, 3N)\big) \leq \varepsilon. \tag{5.20}$$

Thus, an application of Lemma 5.2.3 yields that there is no unbounded component under $\mathbb{P}_{\lambda,\rho_n}$ for all $n \geq n_0$. Consequently, $\lambda < \lambda_c(\rho_n)$ for all $n \geq n_0$, from which Proposition 5.2.2 follows. □

The proof of Lemma 5.2.3 is given in Appendix 5.7.

Proof of Lemma 5.2.4. Recall that $\mathcal{A}_\infty(K, K+1)$ denotes the annulus $\mathcal{B}_\infty(0, K+1) \setminus \overline{\mathcal{B}}_\infty(0, K)$. Then, by summing over the positions of all Poisson points,

$$\begin{aligned}
&\mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{B}_\infty(0,M)^{\complement} \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0,N] \times [0,3N]^{d-1} \neq \varnothing\Big)\\
&= \sum_{K=M}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{A}_\infty(K,K+1) \cap \mathcal{E}\colon\ \mathcal{B}(y, r(y)) \cap [0,N] \times [0,3N]^{d-1} \neq \varnothing\Big)\\
&\leq \sum_{K=M}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(\exists\, y \in \mathcal{A}_\infty(K,K+1) \cap \mathcal{E}\colon\ r(y) \geq K - 3N\Big)\\
&= \sum_{K=M}^{\infty}\sum_{\ell=1}^{\infty} \mathbb{P}_{\lambda,\rho'}\Big(|\mathcal{A}_\infty(K,K+1) \cap \mathcal{E}| = \ell,\ \exists\, y \in \mathcal{A}_\infty(K,K+1) \cap \mathcal{E} : r(y) \geq K - 3N\Big).
\end{aligned} \tag{5.21}$$

Using that for some constant $c = c(d) > 0$ and all $K \in \mathbb{N}$, $\mathrm{Leb}_d(\mathcal{A}_\infty(K+1, K)) = cK^{d-1}$, the last term in (5.21) may be estimated from above by

$$\sum_{K=M}^{\infty}\sum_{\ell=1}^{\infty} e^{-\lambda c K^{d-1}} \frac{(\lambda c K^{d-1})^\ell}{\ell!} \ell \rho'([K-3N, \infty)) \leq \mathrm{Cst} \sum_{K=M-3N}^{\infty} K^{d-1}\rho'([K,\infty)), \tag{5.22}$$

which goes to 0 as M goes to infinity since ρ' has moments of order d. □

5.3 Proof of Theorem 5.1.1

Let $t > 0$. Note that

$$\Sigma_t := \bigcup_{x \in \mathcal{E}} \mathcal{B}\left(x, \sup_{0 \le s \le t} \|B_s^x - x\|\right) \tag{5.23}$$

has the same law as the occupied set in the Boolean percolation model with radius distribution

$$\rho_t([L, \infty)) = \mathbb{P}^0\left(\sup_{0 \le s \le t} ||B_s|| \ge L\right). \tag{5.24}$$

Note that ρ_t has finite moments of order d. Indeed, for all $L > 0$,

$$\rho_t([L, \infty)) \le 2\mathbb{P}^0\left(\sup_{0 \le s \le t} B_s \ge L\right) \le 4\mathbb{P}^0\left(B_t \ge L\right) \le \frac{4}{L}\sqrt{\frac{t}{2\pi}} e^{-L^2/2t}, \tag{5.25}$$

where we used the reflection principle in the second inequality. Thus, by Theorem 3.1 in [MR96], almost-surely, the set Σ_t does not contain an unbounded cluster. Finally, the relation $\mathcal{O}_t \subseteq \Sigma_t$ yields the result.

5.4 Theorems 5.1.2-5.1.3: no percolation for small times

In this section we show that there is a $t_c = t_c(\lambda, d) > 0$ ($t_c = t_c(\lambda, d, r) > 0$ when $d \ge 4$) such that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \ge 4$) does not percolate when $t < t_c$. The proof for $d \in \{2, 3\}$ comes in Section 5.4.1, whereas the proof for $d \ge 4$ comes in Section 5.4.2. Both proofs heavily rely on the results of Section 5.2.

5.4.1 No percolation for $d \in \{2, 3\}$

Let $t > 0$ and define Σ_t and ρ_t as in Section 5.3, but with the one-dimensional Brownian motions of Section 5.3 replaced by its d-dimensional counterparts. As in Section 5.3 it is sufficient to show the existence of a $t_c > 0$ such that for all $t < t_c$ the set Σ_t almost surely does not have an unbounded component. For that we intend to apply Theorem 5.2.1. For all $\varepsilon > 0$,

$$\int_0^\infty x^d \rho_t(dx) \le \varepsilon^d \int_0^\varepsilon \rho_t(dx) + \int_\varepsilon^\infty x^d \rho_t(dx) = \varepsilon^d + \int_{\varepsilon^d}^\infty \rho_t([y^{1/d}, \infty))dy. \tag{5.26}$$

A calculation similar to the one in (5.25) shows that the second term on the right-hand side of (5.26) is bounded by

$$4d\sqrt{\frac{td}{2\pi}} \int_{\varepsilon^d}^\infty \frac{1}{y^{1/d}} e^{-y^{2/d}/2td}\, dy, \tag{5.27}$$

which tends towards zero, as $t \to 0$. Thus, by (5.26)–(5.27) we see that

$$\lim_{t \to 0} \int_0^\infty x^d \, \rho_t(dx) = 0. \tag{5.28}$$

An application of equation (5.8) in Theorem 5.2.1 yields the claim.

5.4.2 No percolation for $d \geq 4$

Let $t > 0$ and let $\rho_{r,t}$ be the probability measure on $[r, \infty)$ defined via

$$\rho_{t,r}([a,b]) = \mathbb{P}^0\left(\sup_{0 \leq s \leq t} \|B_s\| \in [a-r, b-r] \right), \qquad r \leq a \leq b. \tag{5.29}$$

Note that $\rho_{t,r} \to \delta_r$ weakly as $t \to 0$. Thus, by similar calculations as in (5.25) and Proposition 5.2.2 (with $\rho' = \rho_{1,r}$), $\lambda_c(\rho_{t,r}) \to \lambda_c(\delta_r)$ as $t \to 0$. Hence, there is a $t_0 > 0$ such that $\lambda < \lambda_c(\rho_{t,r})$ holds for all $t < t_0$. Finally, observe that the set

$$\Sigma_{t,r} = \bigcup_{x \in \mathcal{E}} \mathcal{B}\left(x, \sup_{0 \leq s \leq t} \|B_s^x - x\| + r \right), \qquad \forall t \geq 0, \tag{5.30}$$

is generated by the Poisson point process with intensity measure $(\lambda \times \mathrm{Leb}_d) \otimes \rho_{t,r}$ and contains $\mathcal{O}_{t,r}$, see (5.1). This is enough to conclude the claim.

5.5 Theorems 5.1.2–5.1.3: percolation for large times

In this section we establish that $\mathcal{O}_t$ ($\mathcal{O}_{t,r}$ when $d \geq 4$) percolates, when t is sufficiently large. The proof for $d \in \{2,3\}$ comes in Section 5.5.1, whereas the proof for $d \geq 4$ comes in Section 5.5.2.

5.5.1 Proof of the percolation phase in $d \in \{2,3\}$

We use a coarse-graining argument to prove existence of a percolation phase. More precisely, we divide $\mathbb{R}^d$ into boxes which are indexed by $\mathbb{Z}^d$ and we consider an edge percolation model on the coarse-grained graph whose vertices are identified with the centers of the boxes and the edges connect nearest-neighbours. An edge connecting nearest-neighbours, say x and x', in $\mathbb{Z}^d$, is said to be open if (i) both boxes associated to x and x' contain at least one point of the Poisson point process, and (ii) the Brownian motions which correspond to the point of the Poisson point process which are the closest to the centers of their respective boxes, intersect each other. Some technical computations and a domination result by Liggett, Schonmann and Stacey [LSS97] finally show that percolation in that coarse-grained model occurs if one suitably chooses the size of the boxes and lets time run long enough. This implies percolation of our original model.

We now define this coarse-grained model more rigorously. Let $R > 0$ and $t > 0$ to be chosen later. For $x \in \mathbb{Z}^d$, we define

$$\mathcal{B}_x^{(R)} := \mathcal{B}_\infty(2Rx, R) \tag{5.31}$$

and the random variable

$$N^{(R)}(x) := |\, \mathcal{E} \cap \mathcal{B}_x^{(R)} \,| \,. \tag{5.32}$$

When $N^{(R)}(x) \geq 1$, we define the point $z^{(R,x)}$, which is almost surely uniquely determined, via

$$\|z^{(R,x)} - 2Rx\| = \inf_{z \in \mathcal{E} \cap \mathcal{B}_x^{(R)}} \|z - 2Rx\|. \tag{5.33}$$

Note that $z^{(R,x)}$ is the point which is the closest to the center of the box $\mathcal{B}_x^{(R)}$ among all Poisson points of $\mathcal{B}_x^{(R)}$. We denote by $B^{(R,x)}$ the Brownian motion starting from $z^{(R,x)}$. For all couples of nearest-neighbours $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we say that the edge (x, y), which connects x and y, is open if

$$\text{(i)} \quad N^{(R)}(x) \geq 1, \tag{5.34}$$

$$\text{(ii)} \quad N^{(R)}(y) \geq 1, \tag{5.35}$$

$$\text{(iii)} \quad B_{[0,t]}^{(R,x)} \cap B_{[0,t]}^{(R,y)} \neq \varnothing. \tag{5.36}$$

We let $X_{(x,y)}^{R,t}$ be the random variable which takes value 1 if the edge (x, y) is open, and 0 otherwise. In what follows, to not burden the notation, we write $X_{(x,y)}$ instead of $X_{(x,y)}^{R,t}$.

Lemma 5.5.1. *Let $\epsilon > 0$. There exists $R > 0$ and $t > 0$ such that for any couple of nearest-neigbours $(x, y) \in \mathbb{Z}^d \times \mathbb{Z}^d$, $\mathbb{P}(X_{(x,y)} = 1) \geq 1 - \epsilon$.*

The proof of Lemma 5.5.1 is deferred to the end of this section. We first show how one may deduce the existence of a percolation phase from it.

Proof of the existence of a percolation phase. Note that if (x, x') and (y, y') is a couple of nearest-neighbour points in $\mathbb{Z}^d$ such that $\{x, x'\} \cap \{y, y'\} = \varnothing$, then $X_{(x,x')}$ and $X_{(y,y')}$ are independent. Therefore, the coarse-grained percolation model is a 2-dependent percolation model. Thus, Theorem 0.0 of Liggett, Schonmann and Stacey [LSS97] yields that we may bound the coarse-grained percolation model from below by Bernoulli bond percolation, whose parameter, say p^*, can be chosen to be arbitrarily close to 1, when $\mathbb{P}(X_{(x,y)} = 1)$ is sufficiently close to 1. Let $p_c(\mathbb{Z}^d)$ be the critical percolation parameter for Bernoulli bond percolation. Then, by Lemma 5.5.1, there are $R_0 > 0$ and $t_0 > 0$ such that $p^* > p_c(\mathbb{Z}^d)$ for all $R \geq R_0$ and $t \geq t_0$. In that case, the coarse-grained model percolates, and so does $\mathcal{O}_t$. □

Consequently, it remains to prove Lemma 5.5.1. For that we need an additional lemma. It states that the probability that two independent Brownian motions, starting at points $x, y \in \mathbb{R}^d$ have a non-empty intersection up to time t increases, when we move the starting points towards each other.

Lemma 5.5.2. *Let $t > 0$. Then,*

$$(x, y) \mapsto \mathbb{P}^{x,y}\Big(B_{[0,t]}^{(1)} \cap B_{[0,t]}^{(2)} \neq \varnothing\Big), \qquad (x, y) \in \mathbb{R}^d \times \mathbb{R}^d, \tag{5.37}$$

is a non-increasing function of $\|x - y\|$.

We first prove Lemma 5.5.1 subject to Lemma 5.5.2. The proof of Lemma 5.5.2 comes afterwards.

Proof of Lemma 5.5.1. By independence of the events in (i)–(iii), we have

$$\mathbb{P}(X_{(x,y)} = 1) = \mathbb{P}(N^{(R)}(x) \geq 1)^2 \times \mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \varnothing\Big). \tag{5.38}$$

To proceed, we fix $R > 0$ large enough such that

$$\mathbb{P}(N^{(R)}(x) \geq 1) = 1 - e^{-\lambda(2R)^d} \geq 1 - \epsilon. \tag{5.39}$$

Furthermore, by Lemma 5.5.2, $\mathbb{P}(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \varnothing)$ decreases, when $\|z^{(R,x)} - z^{(R,y)}\|$ increases. However, note that $\|z^{(R,x)} - z^{(R,y)}\| \leq R\sqrt{4(d-1)+16}$, when $\|x - y\| = 1$. Thus,

$$\mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \varnothing\Big) \geq \mathbb{P}\Big(B^{(R,x)}_{[0,t]} \cap B^{(R,y)}_{[0,t]} \neq \varnothing \Big| \|z^{(R,x)} - z^{(R,y)}\| = R\sqrt{4(d-1)+16}\Big) \tag{5.40}$$

$$= \mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big), \tag{5.41}$$

for any choice of z_1 and z_2 such that $\|z_1 - z_2\| = R\sqrt{4(d-1)+16}$. Using Theorem 9.1 (b) in Mörters and Peres [MP10], there exists t large enough such that for all such choices of z_1 and z_2,

$$\mathbb{P}^{z_1,z_2}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big) \geq 1 - \epsilon, \tag{5.42}$$

which is enough to deduce the claim. □

We now prove Lemma 5.5.2.

Proof of Lemma 5.5.2. Note that it is enough to prove the claim for the function

$$y \mapsto \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big). \tag{5.43}$$

We fix $R' > R > 0$ and $y, y' \in \mathbb{R}^d$ such that $||y|| = R$ and $||y'|| = R'$, respectively. Using rotational invariance of Brownian motion in the first equality and scale invariance of Brownian motion in the last equality, we may write

$$\mathbb{P}^{0,y'}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big) = \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big) \tag{5.44}$$

$$\leq \mathbb{P}^{0,(R'/R)y}\Big(B^{(1)}_{[0,(R'/R)^2t]} \cap B^{(2)}_{[0,(R'/R)^2t]} \neq \varnothing\Big) \tag{5.45}$$

$$= \mathbb{P}^{0,y}\Big(B^{(1)}_{[0,t]} \cap B^{(2)}_{[0,t]} \neq \varnothing\Big). \tag{5.46}$$

This yields the claim. □

5.5.2 Proof of the percolation phase for $d \geq 4$

Throughout the proof, z always denotes the d-th coordinate of $x = (\xi, z) \in \mathbb{R}^d$. We further define

$$\mathcal{H}_0 = \{(\xi, z) \in \mathbb{R}^d : z = 0\}. \tag{5.47}$$

The main idea is to reduce the problem to a Boolean percolation problem on $\mathcal{H}_0$. More precisely, we use that for each $x \in \mathcal{E}$, B^x will eventually hit $\mathcal{H}_0$. From this we deduce that for t large enough, the traces of the Wiener sausages which hit $\mathcal{H}_0$ dominate a supercritical $(d-1)$-dimensional Boolean percolation model, and therefore percolate.

We now formalize this strategy. For each $k \in \mathbb{N}$, let

$$\mathcal{S}_k := \{(\xi, z) \in \mathbb{R}^d \ : \ k-1 < z \leq k\}, \tag{5.48}$$

so that $(\mathcal{S}_k)_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}^{d-1} \times (0, \infty)$. We fix $k \in \mathbb{N}$ and consider

$$\mathcal{E}_k = \{\xi : \exists\ z \in \mathbb{R} \text{ s.t. } (\xi, z) \in \mathcal{S}_k \cap \mathcal{E}\}. \tag{5.49}$$

Note that $(\mathcal{E}_k)_{k\geq 0}$ are i.i.d. Poisson point processes with parameter $\lambda \times \mathrm{Leb}_{d-1}$. Given $\mathcal{E}_k$, we construct a random set C_t^k in the following way:

- Thinning: each $\xi \in \mathcal{E}_k$ is kept if $\tau_0(z^\xi) \leq t$, where z^ξ is such that $(\xi, z^\xi) \in \mathcal{S}_k \cap \mathcal{E}$ (there is almost-surely only one choice), and $\tau_0(z)$ is the first hitting time of the origin of an one-dimensional Brownian motion starting at z. We choose all Brownian motions, which are associated to some $\xi \in \mathcal{E}_k$, to be independent. Otherwise ξ is discarded.

- Translation: each $\xi \in \mathcal{E}_k$ that was not removed after the previous step is translated by $\bar{B}(\tau_0(z^\xi))$, where $\bar{B}$ is $(d-1)$-dimensional Brownian motion starting at the origin, which is independent of all the previous variables.

Note that z^ξ is uniformly distributed in $(k-1, k)$. Moreover, z^ξ, $\tau_0(z^\xi)$ and $\bar{B}$ are independent of ξ. Thus, C_t^k is the result of a thinning and a translation of $\mathcal{E}_k$, and both operations depend on random variables, which are independent of $\mathcal{E}_k$. Therefore, $(C_t^k)_{k\geq 0}$ is a collection of i.i.d. Poisson point processes with parameter $\lambda p_t^k \times \mathrm{Leb}_{d-1}$, where

$$p_t^k = \int_{k-1}^{k} \mathbb{P}^z\Big(\inf_{0\leq s\leq t} B_s \leq 0\Big)\, dz \geq \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq k\Big). \tag{5.50}$$

By independence of the C_t^k's, the set $\mathcal{C}_t := \bigcup_{k=1}^{\infty} C_t^k$ is thus a Poisson point process with parameter $\lambda \sum_{k\geq 1} p_t^k \times \mathrm{Leb}_{d-1}$.

Let us now consider the Boolean model generated by $\mathcal{C}_t$ with deterministic radius r. Observe that,

$$\sum_{k=1}^{\infty} p_t^k \geq \sum_{k=0}^{\infty} \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq k\Big) - \mathbb{P}^0\Big(\sup_{0\leq s\leq t} B_s \geq 0\Big) \geq \mathbb{E}^0\Big[\sup_{0\leq s\leq t} B_s\Big] - 1. \tag{5.51}$$

Note that the right-hand side of (5.51) tends to infinity as $t \to \infty$. Thus, by the remark on page 52 in [MR96], there exists $t_0 > 0$ large enough such that the Boolean model generated by $\mathcal{C}_t$ percolates for all $t \geq t_0$. Finally, note that $\mathcal{C}_t$ is stochastically dominated by $\mathcal{O}_t \cap \mathcal{H}_0$, in the sense that $\mathcal{C}_t$ has the same distribution as a subset of $\mathcal{O}_t \cap \mathcal{H}_0$. This completes the proof.

5.6 Theorems 5.1.2–5.1.3: uniqueness of the unbounded cluster

We fix $t, r, \lambda \geq 0$ such that $t > t_c(\lambda, d, r)$. In the following we denote by N_∞ the number of unbounded clusters in $\mathcal{O}_{t,r}$, which is almost-surely a constant as a consequence of Proposition 5.1.4. For all $d \geq 2$, the proof of uniqueness consists of (i) excluding the case $N_\infty = k$ with $k \in \mathbb{N} \setminus \{1\}$ and of (ii) excluding the case $N_\infty = \infty$. This section is organized as follows. In Section 5.6.1, we give a short heuristic of (i) in the case $d = 2$, which we use as a guideline for the proofs in all other cases. Section 5.6.2 contains the proof of uniqueness for Wiener sausages ($r > 0$) in $d \geq 4$, which is also on a technical level close to the heuristics in Section 5.6.1. This is not true anymore in dimension $d = 3$, which is due to the fact that there is no simple way under which the paths of two independent three-dimensional Brownian motions intersect each other. Therefore, when $d = 3$, the strategy described in Section 5.6.1 needs to be adapted, which requires a certain number of technical steps. Since the proof for $d = 3$ works for $d = 2$ as well, we decided to give a unified proof for both cases in Section 5.6.3.

5.6.1 Heuristics

Let $d = 2$ and $r = 0$. We proceed by contradiction and assume that almost-surely, $N_\infty = k$ with $k \in \mathbb{N}\setminus\{1\}$. For $R_2 > R_1 > 0$, we introduce the event (see Fig. 5.2 below):

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects all } k \text{ unbounded clusters} \\ \text{without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \tag{5.52}$$

We fix $R_1 > 0$. First, note that by monotonicity in R_2,

$$\mathbb{P}(E_{R_1,R_2}) \geq \mathbb{P}(E_{R_1,R_2} \cap \{\mathcal{E} \cap \mathcal{B}(0, R_1) = \varnothing\}) \overset{R_2 \to \infty}{\longrightarrow} \mathbb{P}(\mathcal{E} \cap \mathcal{B}(0, R_1) = \varnothing) > 0. \tag{5.53}$$

Therefore, we can find a $R_2 > 0$ such that $\mathbb{P}(E_{R_1,R_2}) > 0$. Let us fix such a R_2 and observe that E_{R_1,R_2} is independent from the points in $\mathcal{E} \cap \mathcal{B}(0, R_1)$ and the Brownian motions starting from them. Next, one can show that the event

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{E} \cap \mathcal{B}(0, R_1), \\ B^x_{[0,t]} \text{ contains a "loop" in } \mathcal{A}(R_2, R_2 + 1) \end{array} \right\} \tag{5.54}$$

has positive probability. Finally, the contradiction is a consequence of

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(E_{R_1,R_2} \cap L_{R_1,R_2}) = \mathbb{P}(E_{R_1,R_2})\mathbb{P}(L_{R_1,R_2}) > 0, \tag{5.55}$$

since we assumed that $\mathbb{P}(N_\infty = k) = 1$, $k \in \mathbb{N} \setminus \{1\}$.

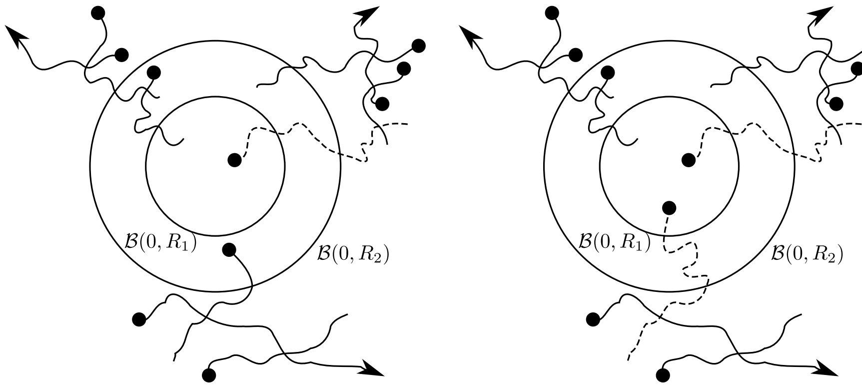

Figure 5.2: The plot on the left hand side represents a configuration of the event E_{R_1,R_2} with $k = 3$. The symbol • represents the points of $\mathcal{E}$, whereas ▶ represents connectivity with infinity. Finally, the dashed line emphasizes the fact that points starting inside $\mathcal{B}(0, R_1)$ are not considered for the intersection condition in (5.52). Because of that, the configuration represented on the right hand side does not belong to E_{R_1,R_2}.

Remark 5.6.1. *The above heuristics also shows how to create trifurcation points. In combination with Lemma 5.6.3, the strategy alluded to above will be used to exclude the possibility of having infinitely many unbounded clusters.*

5.6.2 Uniqueness in $d \geq 4$

Excluding $2 \leq N_\infty < \infty$

Again we proceed by contradiction. Let us assume that N_∞ is almost-surely equal to a constant $k \in \mathbb{N} \setminus \{1\}$. For simplicity, we further assume that $k = 2$, the extension of the argument to other values of k being straightforward.

For $R_2 > R_1 > 0$, let us define E_{R_1,R_2} as follows

$$E_{R_1,R_2} = \left\{ \begin{array}{c} \mathcal{B}(0, R_2) \text{ intersects at least one path of each of the two} \\ \text{unbounded clusters, without using paths starting in } \mathcal{B}(0, R_1) \end{array} \right\}. \quad (5.56)$$

First, we note that there exist R_1 and R_2 such that

$$\mathbb{P}(E_{R_1,R_2}) > 0, \quad (5.57)$$

which can be seen as in the lines following (5.53). Next, we consider the event analogous to (5.54),

$$L_{R_1,R_2} = \left\{ \begin{array}{c} |\mathcal{B}(0, R_1) \cap \mathcal{E}| = 1 \text{ and for } x \in \mathcal{B}(0, R_1) \cap \mathcal{E}, \\ \overline{\mathcal{A}}(R_2 - 3r/2, R_2 - r/2) \subseteq \cup_{0 \leq s \leq t} \mathcal{B}(B^x_s, r) \subseteq \mathcal{B}(0, R_2) \end{array} \right\}, \quad (5.58)$$

which is independent of E_{R_1,R_2} and has positive probability, see Remark 5.53 below. The independence is due to the fact that E_{R_1,R_2} and L_{R_1,R_2} depend on different points of $\mathcal{E}$ and on different Brownian paths. Note that on $E_{R_1,R_2} \cap L_{R_1,R_2}$ the two unbounded clusters, are only connected inside $\mathcal{B}(0, R_2)$.
The contradiction now follows as in (5.55).

Remark 5.6.2. *A sketch of the proof that L_{R_1,R_2} has positive probability goes as follows. Let $\epsilon \in (0, r/8)$. By compactness, $\overline{\mathcal{A}}(R - 3r/2, R_2 - 3r/2 + \epsilon)$ can be covered by a finite number of balls of radius ϵ. Moreover, a Brownian motion starting in $\mathcal{B}(0, R_1)$ has a positive probability of visiting all these balls before time t and before leaving $\mathcal{B}(0, R_2 - r)$. Consequently, on the aforementioned event, L_{R_1,R_2} is satisfied.*

Excluding $N_\infty = \infty$

We assume that $N_\infty = \infty$. We show that this assumption leads to a contradiction. The proof is based on ideas of Meester and Roy [MR94, Theorem 2.1], who extended a technique developed by Burton and Keane [BK89] to a continuous percolation model. In the proof we use the following counting lemma, which is due to Gandolfi, Keane and Newman [GKN92]. It will yield a contradiction to the existence of trifurcation points, which will be constructed in the first step of the proof.

Lemma 5.6.3 (Lemma 4.2 in [GKN92])**.** *Let S be a set, R be a non-empty finite subset of S and $K > 0$. Suppose that*
(a) *for all $z \in R$, there is a family $(C_z^1, C_z^2, \dots, C_z^{n_z})$, $n_z \geq 3$, of disjoint non-empty subsets of S, which do not contain z and are such that $|C_z^i| \geq K$, for all z and for all $i \in \{1, 2, \dots, n_z\}$,*
(b) *for all $z, z' \in R$ one of the following cases occurs (where we abbreviate $C_z = \cup_{i=1}^{n_z} C_z^i$ for all $z \in R$):*
(i) *$(\{z\} \cup C_z) \cap (\{z'\} \cup C_{z'}) = \varnothing$;*
(ii) *there are $i, j \in \{1, 2, \dots, n_z\}$ such that $\{z'\} \cup C_{z'} \setminus C_{z'}^j \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^j$;*
(iii) *there is $i \in \{1, 2, \dots, n_z\}$ such that $\{z'\} \cup C_{z'} \subseteq C_z^i$;*
(iv) *there is $j \in \{1, 2, \dots, n_{z'}\}$ such that $\{z\} \cup C_z \subseteq C_{z'}^j$.*
Then $|S| \geq K(|R| + 2)$.

STEP 1. Balls containing a trifurcation point. Again, we define $E_{R_1,R_2}(0)$ and L_{R_1,R_2} as in (5.56) ,(5.58), respectively. By means of these events, in the same manner as in Subsection 5.6.2, one can show that there are $\delta > 0$ and $R \in \mathbb{N}$ such that the event

$$E_R(0) := \left\{ \begin{array}{l} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0,R)^{\complement} \text{ contains at least} \\ \text{three unbounded clusters, } |C \cap \mathcal{B}_\infty(0,R) \cap \mathcal{E}| \geq 1 \text{ and each cluster which} \\ \text{intersects } \mathcal{B}_\infty(0,R) \text{ belongs to } C. \end{array} \right\}, \tag{5.59}$$

has probability at least δ. Note that $E_R(0)$ implies that each $x \in \mathcal{B}_\infty(0, R)$ which belongs to an infinite cluster also belongs to C. We call each unbounded cluster in $C \cap \mathcal{B}_\infty(0, R)^{\complement}$

a branch. To proceed, we fix $K > 0$ and choose $M > 0$ such that the event

$$E_{R,M}(0) = E_R(0) \cap \left\{ \begin{array}{l} \text{there are at least three different branches of } \mathcal{B}_\infty(0,R) \\ \text{which contain at least } K \text{ points in } \mathcal{E} \cap (\mathcal{B}_\infty(0,RM) \setminus \\ \mathcal{B}_\infty(0,R)) \end{array} \right\}, \tag{5.60}$$

has probability at least $\delta/2$ (see Fig. 5.3 below). For $z \in \mathbb{Z}^d$, the events $E_{R,M}(2Rz)$ and $E_R(2Rz)$ are defined in a similar manner as $E_{R,M}(0)$ and $E_R(0)$, except that the balls in the definitions are centered around $2Rz$.

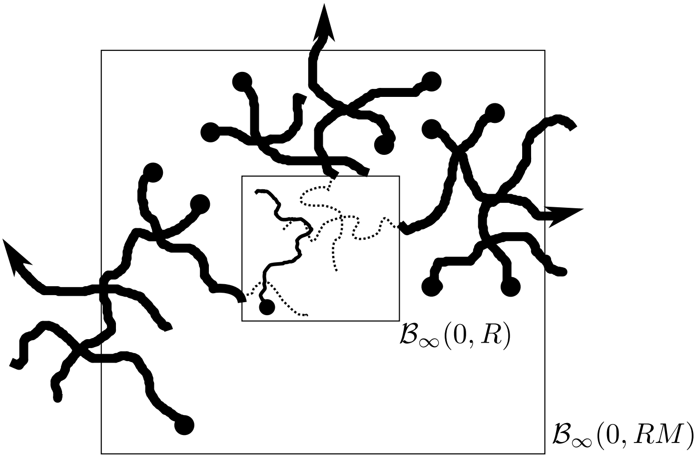

Figure 5.3: The plot represents a configuration in $E_{R,M}(0)$ with $K = 3$ (see (5.59)-(5.60)). The thick lines belong to the branches. As in the previous figure, ▶ represents connection to infinity.

Let $L > M + 2$ and define the set

$$\mathcal{R} = \{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR),\ E_{R,M}(2Rz) \text{ occurs}\}. \tag{5.61}$$

Note that

$$|\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, RM) \subseteq \mathcal{B}_\infty(0, LR)\}| \geq (L - M - 2)^d, \tag{5.62}$$

so that we obtain by stationarity

$$\mathbb{E}(|\mathcal{R}|) \geq \frac{(L-M-2)^d \delta}{2}. \tag{5.63}$$

STEP 2. Application of Lemma 5.6.3 and contradiction. We identify each $z \in \mathcal{R}$ with a Poisson point in $\mathcal{B}_\infty(2Rz, R) \cap C$, which is contained in the corresponding infinite cluster. In what follows we write Λ_z instead of $\mathcal{B}_\infty(2Rz, R)$, for simplicity of notation. Let n_z be the total number of branches of Λ_z, which contain at least K Poisson points in $\mathcal{B}_\infty(2Rz, R)$. For $i \in \{1, \dots, n_z\}$, let $\mathbf{B}_z^i$ be the branch which is the ith-closest to $2Rz$ among all branches of $\mathcal{B}_\infty(2Rz, R)$, see Equation (5.60).

A point x is said to be connected to a set A *through* the set Λ if there exists a continuous function $\gamma : [0,1] \mapsto \Lambda \cap \mathcal{O}_{t,r}$ such that $\gamma(0) = x$ and $\gamma(1) \in A$. We denote it briefly by $x \overset{\Lambda}{\longleftrightarrow} A$. Finally, we define

$$C_z^i = \mathcal{E} \cap \mathcal{B}(0, LR) \cap \mathbf{B}_z^i = \left\{ x \in \mathcal{E} \cap \mathcal{B}_\infty(0, LR) \colon \; x \overset{\Lambda_z^c}{\longleftrightarrow} \mathbf{B}_z^i \right\} \qquad \forall i \in \{1, \dots, n_z\}. \tag{5.64}$$

Now we proceed to check that the conditions of Lemma 5.6.3 are fulfilled. Here $S = \mathcal{B}_\infty(0, LR) \cap \mathcal{E}$. First note that, by definition of a branch, we have that for all $z \in \mathcal{R}$:

- $|C_z^i| \geq K$,
- $C_z^i \cap C_z^j = \varnothing$ for all $i, j \in \{1, \dots, n_z\}$ with $i \neq j$,
- $z \notin C_z$.

Hence, assumption (a) of Lemma 5.6.3 is met.

We now claim that the collection $\{C_z^i\}_{z \in R, i \in \{1, \dots, n_z\}}$ satisfies also assumption (b) of Lemma 5.6.3. At this point we would like to stress some facts to be used later:

a. Due to (5.59), $z \overset{\Lambda_z}{\longleftrightarrow} C_z^i$ for all $i \in \{1, \dots, n_z\}$.

b. If $\tilde{C}$ is an unbounded cluster such that $\tilde{C} \cap \Lambda_z \neq \varnothing$, then $z \overset{\Lambda_z}{\longleftrightarrow} \tilde{C}$.

Suppose that $(\{z\} \cup C_z) \cap (\{z'\} \cup C_{z'}) \neq \varnothing$. We consider three different cases:

1. If $z' \in C_z$ then there exists a unique $i \in \{1, \dots, n_z\}$ such that $z' \in C_z^i$. We consider two sub-cases:
   - If $z \in C_{z'}$, then there exists a unique $i' \in \{1, \dots, n_{z'}\}$ such that $z \in C_{z'}^{i'}$, and we claim that $\{z'\} \cup C_{z'} \setminus C_{z'}^{i'} \subseteq C_z^i$ and $\{z\} \cup C_z \setminus C_z^i \subseteq C_{z'}^{i'}$. Indeed, pick $x' \in C_{z'} \setminus C_{z'}^{i'}$. Then there exists a unique $j' \neq i'$ such that $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$. It is crucial to note that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ since otherwise, due to **b.**, $z \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$ (by first connecting z to x' in $\Lambda_{z'}^c$ and then x' to $C_{z'}^{j'}$ in $\Lambda_{z'}^c$), which contradicts the uniqueness of i'.

     Finally, we have that $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$, $z' \overset{\Lambda_{z'} \subseteq \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$, $z' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$. A concatenation of all these paths gives $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$, that is $x' \in C_z^i$. This proves the first inclusion that we claimed. The second inclusion follows by symmetry.
   - If $z \notin C_{z'}$, then we claim: $\{z'\} \cup C_{z'} \subseteq C_z^i$.

     Indeed, take $x' \in C_{z'}$, then there exists a unique j' such that $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{j'}$. As before we have that $x' \overset{\Lambda_{z'}^c \cap \Lambda_z^c}{\longleftrightarrow} C_{z'}^{j'}$ (this time the contradiction follows from $z \notin C_{z'}$). The conclusion follows in the same way as in the previous case.

2. If $z \in C_{z'}$, then one may conclude as in (1).
3. Suppose that there exist i, i'such that $C_z^i \cap C_{z'}^{i'} \neq \varnothing$. Take $x' \in C_z^i \cap C_{z'}^{i'}$, then we have that $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ and $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$. There are two cases:
   - The path $x' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$ intersects $\Lambda_{z'}$: Due to **b.** we have that $z' \overset{\Lambda_z^c}{\longleftrightarrow} C_z^i$. Hence $z' \in C_z$, which reduces this case to a previous one.
   - In the second case, $x' \overset{\Lambda_z^c \cap \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$: Due to **a.**, we have $z \overset{\Lambda_z \subseteq \Lambda_{z'}^c}{\longleftrightarrow} C_z^i$. Finally, a concatenation of the previous two paths with $x' \overset{\Lambda_{z'}^c}{\longleftrightarrow} C_{z'}^{i'}$ yields that $z \in C_{z'}$, which reduces this case again to a previous one.

Hence, by Lemma 5.6.3

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K(\mathbb{E}(|\mathcal{R}|) + 2), \tag{5.65}$$

so that, by (5.63),

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) \geq K((L - M - 2)^d \delta/2 + 2). \tag{5.66}$$

On the other hand, since $\mathcal{E}$ is a Poisson point process with intensity measure $\lambda \times \mathrm{Leb}_d$,

$$\mathbb{E}\big(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|\big) = \lambda(2LR)^d. \tag{5.67}$$

Thus, combining (5.66) and (5.67), yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \leq \lambda(2LR)^d. \tag{5.68}$$

Note that M depends on K, so in order to get a contradiction one can choose $L = 2M$ and let K go to ∞ in the inequality above.

5.6.3 Uniqueness in $d \in \{2, 3\}$

Excluding $\{2 \leq N_\infty < \infty\}$

As in the heuristic of Section 5.6.1, we proceed by contradiction: we assume that $\mathbb{P}(N_\infty = k) = 1$ for some $k \in \mathbb{N} \setminus \{1\}$ and prove that $\mathbb{P}(N_\infty = 1) > 0$, which is absurd. To make the proof more accessible, we assume that $k = 2$ (see Remark 5.6.7 below).

Remark: The previous heuristic does not work verbatim for $d = 3$ because of clear geometrical reasons: a three-dimensional Brownian motion travelling around an annulus, which is crossed by the two unbounded clusters, does not necessarily connect them. Let us first briefly describe how we adapt this strategy. For R large enough and ϵ small enough, we show that with positive probability, both unbounded clusters intersect $\mathcal{B}(0, R)$ in such a way that each of them contains a Brownian path crossing $\mathcal{A}(R - \epsilon, R + \epsilon)$. Afterwards, we show that, still with positive probability, we can reroute the (let us say first) excursions inside $\mathcal{A}(R-\epsilon, R+\epsilon)$ of each of these two Brownian paths such that they intersect each other and, as a consequence, merge the two unbounded clusters into a single one. This leads to the desired contradiction, since our construction provides a set of configurations of positive probability on which $N_\infty = 1$.

We now give the proof in full detail. Let $R > 0$ and denote by N_∞^R the number of unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$. In the case that N_∞^R is not empty, we denote those by $\{\mathcal{C}_i(R), 1 \le i \le N_\infty^R\}$ (though it has little relevance, let us agree that clusters are indexed according to the order in which one finds them by radially exploring the occupied set from 0). We also consider the 'extended' clusters, defined by

$$\mathcal{C}_i^{\text{ext}}(R) = \bigcup_{x \in \mathcal{E} \,:\, B^x_{[0,t]} \cap \mathcal{C}_i(R) \neq \varnothing} B^x_{[0,t]}, \tag{5.69}$$

i.e. $\mathcal{C}_i^{\text{ext}}(R)$ is the union of all Brownian paths up to time t, which have a non-empty intersection with $\mathcal{C}_i(R)$ (see Fig. 5.4 below).

We further define in five steps a notion of good extended clusters and prove that those occur with positive probability.

Good extended clusters in five steps.

STEP 1. Intersection with a large ball. We use the abbreviations $\mathcal{C}_1^{\text{ext}} := \mathcal{C}_1^{\text{ext}}(R)$ and $\mathcal{C}_2^{\text{ext}} := \mathcal{C}_2^{\text{ext}}(R)$ for the two extended unbounded clusters and define

$$E_R := \{N_\infty^R = 2\} \cap \{\mathcal{C}_1^{\text{ext}} \cap \mathcal{B}(0,R) \neq \varnothing\} \cap \{\mathcal{C}_2^{\text{ext}} \cap \mathcal{B}(0,R) \neq \varnothing\}. \tag{5.70}$$

One way of having exactly two unbounded clusters in $\mathcal{O}_t \setminus \mathcal{B}(0,R)$ is to have exactly two unbounded clusters in total (i.e. on the whole configuration), hence

$$\mathbb{P}(E_R) \ge \mathbb{P}(N_\infty = 2, \mathcal{C}_1^{\text{ext}} \cap \mathcal{B}(0,R) \neq \varnothing, \mathcal{C}_2^{\text{ext}} \cap \mathcal{B}(0,R) \neq \varnothing). \tag{5.71}$$

Since the event on the right-hand side of (5.71) is increasing in R, its probability converges, as R tends to ∞, to $\mathbb{P}(N_\infty = 2)$, which equals 1 by our initial assumption. Therefore, we may choose R large enough such that $\mathbb{P}(E_R) \ge 1/2$.

STEP 2. Choice of a path in each cluster. For $i \in \{1,2\}$, define

$$\text{Cross}(i) = \{x \in \mathcal{E} \cap \mathcal{C}_i^{\text{ext}} \,:\, \exists s \in [0,t], (\|x\| - R)(\|B_s^x\| - R) < 0\}, \tag{5.72}$$

that is the set of points in $\mathcal{E} \cap \mathcal{C}_i^{\text{ext}}$, whose associated Brownian motion crosses $\partial\mathcal{B}(0,R)$. Note that $\text{Cross}(i) \neq \varnothing$ on E_R. For $i \in \{1,2\}$ we denote by x_i the almost-surely uniquely defined $x_i \in \text{Cross}(i)$, such that

$$\|x_i\| = \inf_{y \in \text{Cross}(i)} \|y\|. \tag{5.73}$$

Note that this way of picking x_i is arbitrary. Any other way would serve our purpose as well.

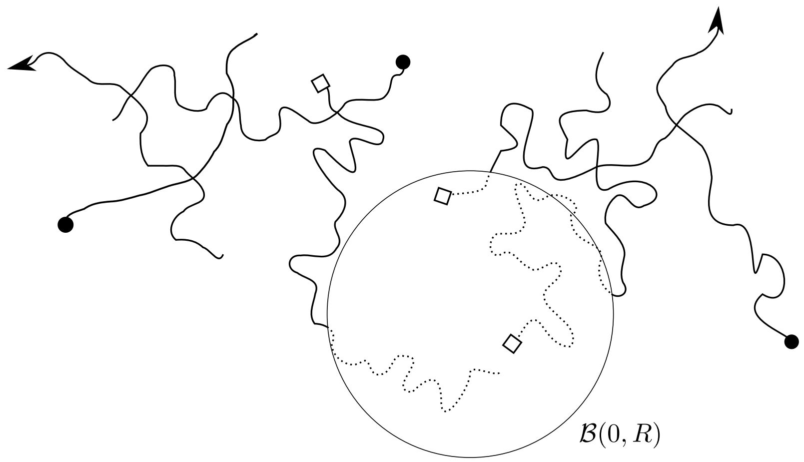

Figure 5.4: The regular lines are a realization of $\mathcal{C}_i,\ i=1,2$. In addition with the dotted lines they form the extended clusters $\mathcal{C}_i^{ext},\ i=1,2$. The points marked with $\Box$ are the ones in Cross$(i),\ i=1,2$.

STEP 3. First excursion through an annulus centered around $\mathcal{B}(0,R)$. For some $\epsilon>0$ to be determined, let us consider the annulus $\mathcal{A}_{R,\epsilon}:=\mathcal{A}(R-\epsilon,R+\epsilon)$. Further, define for each $x\in\mathcal{E}$,

$$I(x):=\mathbb{I}\{\inf\{s\geq 0:\|B_s^x\|=R+\epsilon\}<\inf\{s\geq 0:\|B_s^x\|=R-\epsilon\}\}, \tag{5.74}$$

in the case when at least one of the infima is finite. Otherwise, we set $I(x)=0$. We will see later that the latter case is of no importance. For $i\in\{1,2\}$, we introduce the following entrance and exit times:

$$\begin{aligned}\sigma_i^{\text{out}}&=\inf\{s\geq 0:\|B_s^{x_i}\|=R+(-1)^{I(x_i)}\epsilon\},\\ \sigma_i^{\text{in}}&=\sup\{s\leq\sigma_i^{\text{out}}:\|B_s^{x_i}\|=R-(-1)^{I(x_i)}\epsilon\},\end{aligned} \tag{5.75}$$

i.e. $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ is the first excursion through $\mathcal{A}_{R,\epsilon}$ of B^{x_i} (see Fig. 5.5 below). The reason for this at a first glance strange definition is, that we do not want to exclude the possibility that x_1 or x_2 is located inside $\mathcal{B}(0,R)$. By choosing ϵ small enough we guarantee that the Brownian motions started at x_1 and x_2 cross $\mathcal{A}_{R,\epsilon}$, that is, $\sigma_i^{\text{in}}\leq\sigma_i^{\text{out}}\leq t$ for $i\in\{1,2\}$. Later in the proof we will merge $\mathcal{C}_1^{\text{ext}}$ and $\mathcal{C}_2^{\text{ext}}$ into a single unbounded cluster by "replacing" $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ with suitable excursions. However, this operation should not disconnect $B^{x_i}_{[0,t]}$ from $\mathcal{C}_i^{\text{ext}}$. For that reason, we consider the event on which $B^{x_i}_{[0,\sigma_i^{\text{in}})}$ or $B^{x_i}_{(\sigma_i^{\text{out}},t]}$ is already connected to $\mathcal{C}_i^{\text{ext}}$, i.e. we introduce for $i\in\{1,2\}$

$$E_{\epsilon,i}^{\text{conn}}:=\left\{\left(B^{x_i}_{[0,\sigma_i^{\text{in}})}\cup B^{x_i}_{(\sigma_i^{\text{out}},t]}\right)\cap\mathcal{C}_i^{\text{ext}}\neq\varnothing\right\}. \tag{5.76}$$

Summing everything up, we restrict ourselves to configurations in the set

$$E_{R,\epsilon} = E_R \bigcap_{i=1,2} \{\sigma_i^{\text{in}} \le \sigma_i^{\text{out}} \le t\} \cap E_{\epsilon,i}^{\text{conn}}. \tag{5.77}$$

By monotonicity in ϵ, $\mathbb{P}(E_{R,\epsilon})$ converges to $\mathbb{P}(E_R) > 1/2$ as ϵ tends to 0. Therefore, we may fix for the rest of the proof $\epsilon > 0$ such that $\mathbb{P}(E_{R,\epsilon}) \ge 1/4$.

STEP 4. Restriction on the time spent to cross the annulus. As has been explained above, our goal is to restrict ourselves to some specific excursions of $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$. The probability of those turn out to be easier to control when we have a deterministic lower bound on the random time lengths $\sigma_i^{\text{out}} - \sigma_i^{\text{in}}$. Therefore, we introduce for $T \in (0,t)$ the following event:

$$E_{R,\varepsilon,T} = E_{R,\epsilon} \bigcap_{i=1,2} \{\sigma_i^{\text{out}} - \sigma_i^{\text{in}} \ge T\}. \tag{5.78}$$

Again, by monotonicity in T, we can choose the latter small enough such that $\mathbb{P}(E_{R,\epsilon,T}) \ge \mathbb{P}(E_{R,\epsilon})/2 \ge 1/8$.

STEP 5. Staying away from the boundary of the annulus during the excursion. To obtain a configuration with a unique unbounded cluster, we restrict ourselves to configurations in the set $E_{R,\epsilon,T}$ and we reroute $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ and $B^{x_2}_{[\sigma_2^{\text{in}},\sigma_2^{\text{out}}]}$ such that they intersect each other. Since σ_i^{in} is not a stopping time, the law of $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ is not the one of a Brownian motion. Conditioned on both endpoints, $\big(B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}\big)$, $i \in \{1,2\}$, are instead Brownian excursions, the law of which is not absolutely continuous with respect to the one of a Brownian motion. As a consequence, we cannot directly use our knowledge on the intersection probabilities of two Brownian motions. This is why we will work with $(B^{x_i}_{[\sigma_i^{\text{in}}+\delta,\sigma_i^{\text{out}}-\delta]})$, $i \in \{1,2\}$, for some $\delta \in (0, T/8)$ instead (the restriction to consider the Brownian motions only up to time $\sigma_i^{\text{out}} - \delta$ is just for esthetic reasons) . These subpaths, when conditioned on both endpoints, are Brownian bridges conditioned to stay in $\mathcal{A}_{R,\epsilon}$, and indeed the density of a Brownian bridge with respect to a Brownian motion is explicit and tractable. To be more precise, the latter property holds only on time intervals excluding neighbourhoods of the endpoints, so we need to work with $B^{x_i}_{[\sigma_i^{\text{in}}+2\delta,\sigma_i^{\text{out}}-2\delta]}$ instead. To get a uniform lower bound on the intersection probability (see (5.88)), we consider for some $\overline{\epsilon} \in (0,\epsilon)$ in addition the events

$$\tilde{E}_{R,\epsilon,T,\overline{\epsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{ B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{in}}+2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-2\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\epsilon}} \right\}, \quad \text{and} \tag{5.79}$$

$$E_{R,\epsilon,T,\overline{\epsilon}} := E_{R,\varepsilon,T} \bigcap_{i=1,2} \left\{ B^{x_i}_{\sigma_i^{\text{in}}+\delta}, B^{x_i}_{\sigma_i^{\text{out}}-\delta} \in \mathcal{A}_{R,\overline{\epsilon}} \right\}. \tag{5.80}$$

Again, by monotonicity of $E_{R,\epsilon,T,\overline{\epsilon}}$ w.r.t. $\overline{\epsilon}$, as $\overline{\epsilon}$ converges to ε, $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}})$ converges to $\mathbb{P}(E_{R,\epsilon,T}) = 1/8$. Hence, we may choose $\overline{\epsilon}$ such that $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) \ge 1/16 > 0$. Finally, we call a configuration which lies in $E_{R,\epsilon,T,\bar{\epsilon}}$ a configuration of good extended clusters.

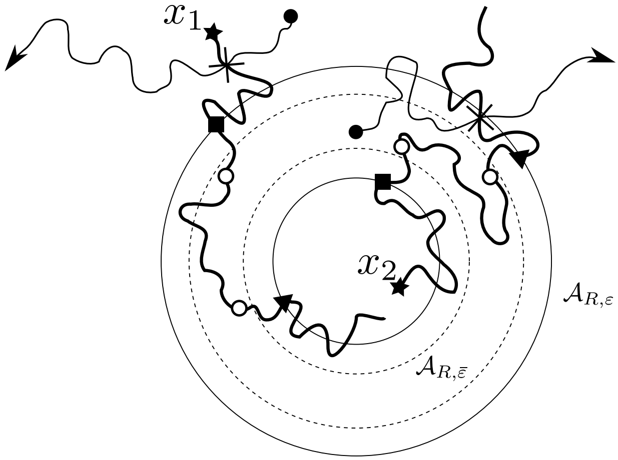

Figure 5.5: In this picture the points marked with ★ are $x_i,\ i=1,2$. The symbols ■, ▲ refer to the times σ^{in} and σ^{out}, respectively. The symbol ∘ represents the times $\sigma^{\text{in}}+\delta$ and $\sigma^{\text{out}}-\delta$, respectively. Finally, the symbol × stresses the fact that condition (5.76) is fulfilled.

Additional notation. At this point we would like to introduce some notation in order to avoid repetitions of complicated expressions.
First, let us introduce the events of interest. Let $s > r \geq 0$. For a set $D \subseteq \mathbb{R}^d$, we denote by

$$\mathcal{S}_{[r,s]}(D) := \{\Pi \in C([0,\infty),\mathbb{R}^d) : \Pi_{[r,s]} \subseteq D\}, \tag{5.81}$$

the set of all continuous paths, which *stay in the set* D during the whole time interval $[r,s]$, and by

$$\mathcal{L}_{r,s}(D) := \{\Pi \in C([0,\infty),\mathbb{R}^d) : \Pi_r, \Pi_s \in D\}, \tag{5.82}$$

the set of all continuous paths , which *belong to the set* D at times r,s.
In the same fashion we also define for $s_1 > r_1 \geq 0$ and $s_2 > r_2 \geq 0$

$$\mathcal{I}_{[s_1,r_1],[s_2,r_2]} := \{\Pi^{(1)},\Pi^{(2)} \in C([0,\infty),\mathbb{R}^d) : \Pi^{(1)}_{[s_1,r_1]} \bigcap \Pi^{(2)}_{[s_2,r_2]} \neq \varnothing\}, \tag{5.83}$$

the set of all pairs of continuous paths $\Pi^{(1)}$ and $\Pi^{(2)}$ whose traces, when restricted to the time intervals $[r_1,s_1]$ and $[r_2,s_2]$, respectively, have a non-empty intersection.
Secondly, we modify our previous notation a bit: $\mathbb{P}^a_t$ now denotes the law of Brownian motion starting at a and running from time 0 up to time t. If we consider Brownian bridges instead of Brownian motions we substitute the letter a by $\mathbf{a} = (\underline{a};\overline{a})$ containing the starting and ending position of the Brownian bridge. In case of considering two independent copies of a Brownian motion (Brownian bridge) we will add a superscript/subscript, ie. $\mathbb{P}^{a_1,a_2}_{t_1,t_2}$ ($\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{t_1,t_2}$). Finally, we will refer to a Brownian bridge as W.

Connecting $\mathcal{C}_1^{\mathrm{ext}}$ and $\mathcal{C}_2^{\mathrm{ext}}$ inside the annulus. Step 1–Step 5 translates into the following lower bound:

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(\tilde{E}_{R,\epsilon,T,\overline{\epsilon}} \bigcap \{B^{x_1}_{[\sigma_1^{\mathrm{in}}+\delta,\sigma_1^{\mathrm{out}}-\delta]} \cap B^{x_2}_{[\sigma_2^{\mathrm{in}}+\delta,\sigma_2^{\mathrm{out}}-\delta]} \neq \varnothing\}), \tag{5.84}$$

which equals

$$\mathbb{P}\Big(\Big\{B^{x_1}_{[\sigma_1^{\mathrm{in}}+\delta,\sigma_1^{\mathrm{out}}-\delta]} \cap B^{x_2}_{[\sigma_2^{\mathrm{in}}+\delta,\sigma_2^{\mathrm{out}}-\delta]} \neq \varnothing\Big\} \bigcap_{i=1,2} \Big\{B^{x_i}_{\sigma_i^{\mathrm{in}}+2\delta}, B^{x_i}_{\sigma_i^{\mathrm{out}}-2\delta} \in \mathcal{A}_{R,\overline{\epsilon}}\Big\} \mid E_{R,\epsilon,T,\overline{\epsilon}}\Big)$$
$$\times\, \mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}). \tag{5.85}$$

Observation: For $i \in \{1,2\}$, conditionally on $T_i := \sigma_i^{\mathrm{out}} - \sigma_i^{\mathrm{in}}$ and the endpoints $(B^{x_i}_{\sigma_i^{\mathrm{in}}+\delta}, B^{x_i}_{\sigma_i^{\mathrm{out}}-\delta}) = (a_i, b_i)$, $B^{x_i}_{[\sigma_i^{\mathrm{in}}+\delta,\sigma_i^{\mathrm{out}}-\delta]}$ is a Brownian bridge running from a_i to b_i in a time interval of length $\tau_i := T_i - 2\delta \geq \frac{3T}{4}$, conditioned to stay in $\mathcal{A}_{R,\epsilon}$ (recall the definitions of σ_i^{in} and σ_i^{out}, $i \in \{1,2\}$).

The observation above yields,

$$\mathbb{P}(N_\infty = 1) \geq \mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) \inf_{\substack{\tau_1,\tau_2 \geq 3T/4 \\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\epsilon}}}} \mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2), \tag{5.86}$$

where

$$\mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2) := \mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ ,\ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1],[0,\tau_2]}\Big) \tag{5.87}$$

and the superscript i, $i \in \{1,2\}$, on the events in (5.87) refers to the i-th copy of the corresponding processes. Since $\mathbb{P}(E_{R,\epsilon,T,\overline{\epsilon}}) > 0$, by Step 1–Step 5, it is enough to prove that

$$\inf_{\substack{\tau_1,\tau_2 \geq 3T/4 \\ \mathbf{a_1},\mathbf{a_2} \in \mathcal{A}^2_{R,\overline{\epsilon}}}} \mathcal{P}_\cap(\mathbf{a_1}, \mathbf{a_2}, \tau_1, \tau_2) > 0. \tag{5.88}$$

Proof of Equation (5.88). We fix $\mathbf{a_1}, \mathbf{a_2} \in \mathcal{A}_{R,\overline{\epsilon}}$ and $\tau_1, \tau_2 \geq 3T/4$. The right-hand side of (5.87) may be bounded from below by

$$\mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ ,\ \mathcal{S}^i_{[0,\tau_i]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{5.89}$$

which equals, by the Markov property applied at time $\tau_i - \delta$, $i \in \{1,2\}$,

$$\mathbb{E}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\prod_{i=1,2} \mathbb{I}\big\{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}}) \ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\big\} \ .\ \mathbb{I}\big\{\mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\big\}\Phi_\delta((W^{(i)}_{\tau_i-\delta}; \overline{a}_i))\Big) \tag{5.90}$$

where

$$\Phi_\delta(\mathbf{a}) := \mathbb{P}^{\mathbf{a}}_\delta(\mathcal{S}_{[0,\delta]}(\mathcal{A}_{R,\epsilon})), \qquad \mathbf{a} = (\underline{a}, \overline{a}) \in (\mathbb{R}^d)^2. \tag{5.91}$$

is the probability that a Brownian bridge, going from $\underline{a}$ to $\overline{a}$ within the time interval $[0,\delta]$, stays in $\mathcal{A}_{R,\epsilon}$. To bound (5.90) from below we use the following three lemmas, whose proofs are postponed to the appendix.

Lemma 5.6.4. [Positive probability for a Brownian bridge to stay inside the annulus] *There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}^2_{R,\overline{\epsilon}}$, $\Phi_\delta(\mathbf{a}) \geq c$.*

Lemma 5.6.5. [Substitution of the Brownian bridge by a Brownian motion] *Let $\tau > 0$ and $\delta \in (0,\tau)$. There exists $c > 0$ such that for all $\mathbf{a} \in \mathcal{A}^2_{R,\overline{\epsilon}}$, $\mathbf{a} = (\underline{a}, \overline{a})$,*

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_\tau(W_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))}{\mathrm{d}\mathbb{P}^{\underline{a}}_\tau(B_{[0,\tau-\delta]} \in \cdot\ ,\ \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))} \geq c. \tag{5.92}$$

Lemma 5.6.6. [Two Brownian motions restricted to be inside the annulus do intersect] *Let $\tau_1, \tau_2 > 0$ and $0 < \delta < \frac{\tau_1 \wedge \tau_2}{2}$. There exists $c > 0$ such that for all $a_1, a_2 \in \mathcal{A}_{R,\overline{\epsilon}}$*

$$\mathbb{P}^{a_1,a_2}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big) \geq c. \tag{5.93}$$

We now explain how to get (5.88) by applying Lemmas 5.6.4–5.6.6 to (5.90). Since the $W_{\tau_i-\delta}$, $i \in \{1,2\}$, appearing in (5.90) are in $\mathcal{A}_{R,\overline{\epsilon}}$, Lemma 5.6.4 yields that, for some $c > 0$, (5.90) is not smaller than

$$c^2\ .\ \mathbb{P}^{\mathbf{a_1},\mathbf{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big). \tag{5.94}$$

Next, a change of measure argument together with the bound on the Radon-Nikodym derivative as provided in Lemma 5.6.5 yields, for a possibly different constant $c > 0$, that (5.94) is at least

$$c\ .\ \mathbb{P}^{\underline{a_1},\underline{a_2}}_{\tau_1,\tau_2}\Big(\bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]}\Big), \tag{5.95}$$

which is positive by Lemma 5.6.6. To deduce Equation (5.88) from it, it is enough to note that all the previous estimates were uniform in $\mathbf{a_1}, \mathbf{a_2} \in \mathcal{A}_{R,\overline{\epsilon}}$. This finally yields the claim.

Remark 5.6.7. *If $k > 2$, then one follows the same scheme. The notion of good extended clusters is easily generalized and one ends up connecting k excursions in an annulus. Using the same proof as for two excursions, one can connect $B^{x_1}_{[\sigma_1^{\text{in}},\sigma_1^{\text{out}}]}$ to $B^{x_i}_{[\sigma_i^{\text{in}},\sigma_i^{\text{out}}]}$ during the time interval $[\sigma_1^{\text{in}} + (i-1)\delta/k, \sigma_1^{\text{in}} + i\delta/k]$, where $\delta \in (0,T)$, for all $1 < i \leq k$.*

Excluding $N_\infty = \infty$

Let us assume that the number N_∞ of unbounded clusters in $\mathcal{O}_t$ is almost-surely equal to infinity. In the same fashion as in Subsection 5.6.2 we show that this leads to a contradiction. We define the event

$$E_R(0) := \left\{ \begin{array}{l} \exists \text{ an unbounded cluster } C \text{ such that } C \cap \mathcal{B}_\infty(0,R)^{\complement} \text{ contains at least three} \\ \text{unbounded clusters and each unbounded cluster which has a non-empty} \\ \text{intersection with } \mathcal{B}_\infty(0,R) \text{ equals } C. \end{array} \right\}, \tag{5.96}$$

The fact that there is R large enough such that $E_R(0)$ has positive probability can be seen as follows. First, note that for R large enough, with positive probability the event

$$E^1_R(0) = \bigcup_{k\geq 3} \left\{ \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement} \text{ which intersect } \mathcal{B}(0,R) \right\} \tag{5.97}$$

happens. As a consequence, there is $k^* \geq 3$ such that the event inside the union in (5.97) occurs for $k = k^*$ with positive probability. Moreover, we may write

$$\begin{aligned} E_R(0) &= \bigcup_{k\geq 3} \left\{ \begin{array}{c} \exists\, k \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement}, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{array} \right\} \\ &\supseteq \left\{ \begin{array}{c} \exists\, k^* \text{ unbounded clusters in } \mathcal{B}_\infty(0,R)^{\complement}, \text{ which intersect} \\ \mathcal{B}_\infty(0,R) \text{ and all of them are connected inside } \mathcal{B}_\infty(0,R) \end{array} \right\}. \end{aligned} \tag{5.98}$$

Remark 5.6.7 and the lines preceding (5.98) yield that the last event in (5.98) has positive probability and consequently, so does $E_R(0)$. From now on, the proof works similarly as the proof in Section 5.6.2. Thus, to avoid repetitions we just point out the differences with the proof in Section 5.6.2.

The identification done in **STEP 2.** of Section 5.6.2 has to be changed. For each $z \in \mathbb{Z}^d$, we replace the Poisson point inside $\mathcal{B}_\infty(2Rz, R)$ that was used to connect the "external" clusters by what we call an *intersection point*, which is just an arbitrarily chosen point $\tilde{z} \in \mathcal{B}_\infty(2Rz, R)$ contained in all the clusters. Finally, at the moment of applying Lemma 5.6.3, we consider

$$C^i_z = \left\{ x \in \{\mathcal{E} \cap \mathcal{B}_\infty(0, LR)\} \cup \{\text{intersection points}\} \colon\ x \overset{\Lambda^c_z}{\longleftrightarrow} \mathbf{B}^i_z \right\} \qquad i = 1, \ldots, n_z,$$

and

$$S = \mathcal{B}_\infty(0, LR) \cap (\mathcal{E} \cup \{\text{intersection points}\}).$$

This choice generates a minor difference at the moment of getting the contradiction in (5.68). Indeed, we have that

$$\mathbb{E}\big(|S|\big) \geq K((L-M-2)^d\delta/2 + 2) \tag{5.99}$$

but, taking into account the intersection points we have that,

$$\mathbb{E}\big(|S|\big) \leq \mathbb{E}(|\mathcal{B}_\infty(0, LR) \cap \mathcal{E}|) + \mathbb{E}(|\mathcal{R}|) \leq \lambda(2LR)^d + (L-M+2)^d. \tag{5.100}$$

In the last inequality we used that

$$|\mathcal{R}| \le |\{z \in \mathbb{Z}^d : \mathcal{B}_\infty(2Rz, R) \subseteq \mathcal{B}_\infty(0, LR)\}| \le (L - M + 2)^d. \tag{5.101}$$

Thus, combining (5.99) and (5.100) yields

$$\forall L > M + 2, \quad K((L - M - 2)^d \delta/2 + 2) \le \lambda(2LR)^d + (L - M + 2)^d, \tag{5.102}$$

from which we obtain the desired contradiction in the same way as in the case $d \ge 4$.

Appendix

5.7 Proof of Lemma 5.2.3

The proof consists of two steps. In the first step a coarse-graining procedure is introduced, which reduces the problem of showing subcriticality of a continuous percolation model to showing subcriticality of an infinite range site percolation model on $\mathbb{Z}^d$. This coarse-graining was essentially already introduced in [MR96, Lemma 3.3], where ρ was supposed to have a compact support. To overcome the additional difficulties arising from the long range dependencies in the coarse-grained model we use a renormalization scheme, which is very similar to the one in Sznitman [Szn10, Theorem 3.5].

STEP 1. Coarse-graining.
We fix $N \in \mathbb{N}$. For $n \in \mathbb{N}$, a sequence of vertices $z_0, z_1, \dots, z_{n-1}$ in $\mathbb{Z}^d$ is called a $*$-path, when $\|z_i - z_{i-1}\|_\infty = 1$ for all $i \in \{1, 2, \dots, n-1\}$. Furthermore, a site $z = (z(j), 1 \le j \le d) \in \mathbb{Z}^d$ is called open when there is an occupied cluster Λ of Σ such that

$$\begin{aligned} &(i)\ \Lambda \cap \prod_{j=1}^{d} [z(j)N, (z(j)+1)N) \ne \varnothing \text{ and} \\ &(ii)\ \Lambda \cap \left(\prod_{j=1}^{d} [(z(j)-1)N, (z(j)+2)N) \right)^{\mathsf{C}} \ne \varnothing. \end{aligned} \tag{5.103}$$

Otherwise z is called closed. It was shown in [MR96, Lemma 3.3] that to obtain Lemma 5.2.3 it suffices to show that

$$\mathbb{P}_{\lambda,\rho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) = 0. \tag{5.104}$$

To prove (5.104) we introduce a renormalization scheme.

STEP 2. Renormalization.
• **New notation and a first bound.** We start by introducing a fair amount of new notation. We fix integers $R > 1$ and $L_0 > 1$, both to be determined and we introduce an increasing sequence of scales via

$$\forall n \in \mathbb{N}_0, \quad L_{n+1} = R^{n+1} L_n. \tag{5.105}$$

Moreover, for $i \in \mathbb{Z}^d$, we introduce a sequence of increasing boxes via

$$\begin{aligned} C_n(i) &= \prod_{j=1}^{d} [i(j)L_n, (i(j)+1)L_n) \cap \mathbb{Z}^d \quad \text{and} \\ \widetilde{C}_n(i) &= \prod_{j=1}^{d} [(i(j)-1)L_n, (i(j)+2)L_n) \cap \mathbb{Z}^d. \end{aligned} \tag{5.106}$$

We further abbreviate $C_n = C_n(0)$ and $\widetilde{C}_n = \widetilde{C}_n(0)$. Thus, $\widetilde{C}_n(i)$ is the union of boxes $C_n(j)$ such that $\|j - i\|_\infty \le 1$. Moreover, for $n \in \mathbb{N}$, we introduce the events

$$A_n(i) = \Big\{ \text{There is a } *\text{-path of open sites from } C_n(i) \text{ to } \partial_{\text{int}} \widetilde{C}_n(i). \Big\}, \tag{5.107}$$

and we write A_n instead of $A_n(0)$. Here, $\partial_{\text{int}} B$ refers to the inner boundary of a set $B \subseteq \mathbb{Z}^d$ with respect to the $\|\cdot\|_\infty$-norm. The idea of the renormalization scheme is to bound the probability of A_{n+1} in terms of the probability of the intersection of events $A_n(i)$ and $A_n(k)$, where $i \in \mathbb{Z}^d$ and $k \in \mathbb{Z}^d$ are far apart. By our assumption on the radius distribution ρ, the events $A_n(i)$ and $A_n(k)$ can then be treated as being basically independent. This will result in a recursion inequality, which relates the events A_n, $n \in \mathbb{N}$, at different scales to each other. For that, we fix $n \in \mathbb{N}$ and let

$$\begin{aligned} \mathcal{H}_1 &= \Big\{ i \in \mathbb{Z}^d \,:\, C_n(i) \subseteq C_{n+1}, C_n(i) \cap \partial_{\text{int}} C_{n+1} \neq \varnothing \Big\} \quad \text{and} \\ \mathcal{H}_2 &= \Big\{ k \in \mathbb{Z}^d \,:\, C_n(k) \cap \Big\{ z \in \mathbb{Z}^d \,:\, \text{dist}(z, C_{n+1}) = \frac{L_{n+1}}{2} \Big\} \neq \varnothing \Big\}. \end{aligned} \tag{5.108}$$

Here, $\text{dist}(z, C_{n+1})$ denotes the distance of z from the set C_{n+1} with respect to the supremum norm. Note that here and in the rest of the proof, for notational convenience, we pretend that expressions like $L_{n+1}/2$ are integers. Observe that if A_{n+1} occurs, then there are $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$ such that both $A_n(i)$ and $A_n(k)$ occur. Hence,

$$\begin{aligned} \mathbb{P}_{\lambda,\rho}(A_{n+1}) &\le \sum_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathbb{P}_{\lambda,\rho}(A_n(i) \cap A_n(k)) \\ &\le c_1 R^{2(d-1)(n+1)} \sup_{i \in \mathcal{H}_1, k \in \mathcal{H}_2} \mathbb{P}_{\lambda,\rho}(A_n(i) \cap A_n(k)), \end{aligned} \tag{5.109}$$

where $c_1 = c_1(d) > 0$ is a constant which only depends on the dimension.

•Partition of $A_n(i) \cap A_n(k)$. We fix $i \in \mathcal{H}_1$ and $k \in \mathcal{H}_2$. Let $z \in \widetilde{C}_n(i)$ and note that to decide if z is open, it suffices to look at the trace of the Boolean percolation model on

$$\prod_{j=1}^{d} [(z(j)-1)N, (z(j)+2)N). \tag{5.110}$$

In a similar fashion one sees that the area which determines if $A_n(i)$ occurs is given by

$$\begin{aligned}&\prod_{j=1}^{d}[((i(j)-1)L_n-1)N,((i(j)+2)L_n+2)N]\\&\subseteq\prod_{j=1}^{d}[(i(j)-2)L_nN,(i(j)+3)L_nN)\stackrel{\text{def}}{=}\mathrm{DET}(\widetilde{C}_n(i))\end{aligned}\tag{5.111}$$

and likewise for $A_n(k)$ with i replaced by k. Here, we used that by our choice of R and L_0 the relation $L_n \geq 2$ holds for all $n \in \mathbb{N}$. We introduce

$$\mathcal{D}(x,r(x)) := \{\mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(i))\neq\varnothing,\,\mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(k))\neq\varnothing\}\tag{5.112}$$

and

$$B_n(i,k) := \bigcup_{x\in\mathcal{E}}\mathcal{D}(x,r(x))\tag{5.113}$$

so that,

$$\begin{aligned}\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)) &= \mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\&\quad+\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k))\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)).\end{aligned}\tag{5.114}$$

•Analysis of the first term on the right hand side of (5.114). We claim that under $\mathbb{P}_{\lambda,\rho}(\cdot\big|B_n(i,k)^{\complement})$ the events $A_n(i)$ and $A_n(k)$ are independent. To see that, note that the Poisson point process χ on $\mathbb{R}^d\times[0,\infty)$ with intensity measure $\nu=(\lambda\times\mathrm{Leb}_d)\otimes\rho$ (see Section 5.2.1) is a Poisson point process under $\mathbb{P}_{\lambda,\rho}(\cdot|B_n(i,k)^{\complement})$, with intensity measure

$$\mathbb{I}\{\text{there is no } (x,r(x))\in\chi \text{ such that } \mathcal{D}(x,r(x)) \text{ occurs}\}\times\nu.\tag{5.115}$$

However, on $B_n(i,k)^{\complement}$, the events $A_n(i)$ and $A_n(k)$ depend on disjoint subsets of $\mathbb{R}^d\times[0,\infty)$. Consequently, they are independent under $\mathbb{P}_{\lambda,\rho}(\cdot\big|B_n(i,k)^{\complement})$. Hence,

$$\begin{aligned}&\mathbb{P}_{\lambda,\rho}(A_n(i)\cap A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\&\quad=\mathbb{P}_{\lambda,\rho}(A_n(i)\big|B_n(i,k)^{\complement})\mathbb{P}_{\lambda,\rho}(A_n(k)\big|B_n(i,k)^{\complement})\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})\\&\quad\leq\mathbb{P}_{\lambda,\rho}(A_n)^2\times\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement})^{-1}.\end{aligned}\tag{5.116}$$

For the last inequality in (5.116) we also used the fact that $\mathbb{P}_{\lambda,\rho}(A_n(i))$ does not depend on $i\in\mathbb{Z}^d$.

•Analysis of the second term on the right hand side of (5.114). To bound the second term on the right hand side of (5.114) it will be enough to bound $\mathbb{P}_{\lambda,\rho}(B_n(i,k))$, since the other term may be bounded by one. Note that

$$\mathbb{P}_{\lambda,\rho}(B_n(i,k))\leq\sum_{\ell\in3\mathbb{Z}^d}\mathbb{P}_{\lambda,\rho}\begin{pmatrix}\exists x\in\mathcal{E}\cap\widetilde{C}_{n+1}(\ell)N: & \mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(i))\neq\varnothing\\ \text{and} & \mathcal{B}(x,r(x))\cap\mathrm{DET}(\widetilde{C}_n(k))\neq\varnothing\end{pmatrix}.\tag{5.117}$$

Here, the set $\widetilde{C}_{n+1}(\ell)N$ is the set $\{x \in \mathbb{R}^d : x = zN, z \in \widetilde{C}_{n+1}(\ell)\}$. To warm up, we first treat the term $\ell = 0$ in the sum (5.117). Note that,

$$\operatorname{dist}(\mathrm{DET}(\widetilde{C}_n(i)), \mathrm{DET}(\widetilde{C}_n(k)) \geq \Big(\frac{L_{n+1}}{2} - 8L_n\Big)N \geq \frac{L_{n+1}}{3}N, \tag{5.118}$$

where the last inequality holds for all $n \in \mathbb{N}$, provided R and L_0 are chosen accordingly. Thus, if there is a Poisson point whose corresponding ball intersects $\mathrm{DET}(\widetilde{C}_n(i))$ and $\mathrm{DET}(\widetilde{C}_n(k))$, then its radius is at least $L_{n+1}N/6$. This yields

$$\begin{aligned} &\mathbb{P}_{\lambda,\rho}\begin{pmatrix} \exists x \in \mathcal{E} \cap \widetilde{C}_{n+1}N: & \mathcal{B}(x, r(x)) \cap \mathrm{DET}(\widetilde{C}_n(i)) \neq \varnothing \\ \text{and} & \mathcal{B}(x, r(x)) \cap \mathrm{DET}(\widetilde{C}_n(k)) \neq \varnothing \end{pmatrix} \\ &\qquad \leq \mathbb{P}_{\lambda,\rho}\Big(\exists x \in \mathcal{E} \cap \widetilde{C}_{n+1}N : r(x) \geq L_{n+1}N/6\Big). \end{aligned} \tag{5.119}$$

We may rewrite (5.119) as

$$\begin{aligned} &1 - \sum_{m=0}^{\infty} \mathbb{P}_{\lambda,\rho}\Big(\forall x \in \mathcal{E} \cap \widetilde{C}_{n+1}, r(x) < L_{n+1}N/6 \Big| |\mathcal{E} \cap \widetilde{C}_{n+1}N| = m\Big) \\ &\qquad \times \mathbb{P}_{\lambda,\rho}(|\mathcal{E} \cap \widetilde{C}_{n+1}N| = m) \\ &= 1 - \sum_{m=0}^{\infty} [1 - \rho([L_{n+1}N/6, \infty))]^m \times \frac{(\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N))^m}{m!} \times e^{-\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)} \\ &= 1 - \exp\big\{ - \lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)\rho([L_{n+1}N/6, \infty))\big\}, \end{aligned} \tag{5.120}$$

which is at most $\lambda \mathrm{Leb}_d(\widetilde{C}_{n+1}N)\rho([L_{n+1}N/6, \infty))$. By our assumption on the radius distribution, for R and L_0 large enough, there is a constant $c_2 = c_2(\rho) > 0$ such that the last term may be bounded by $\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}$. Similar arguments show that the left hand side of (5.117) is at most

$$\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6} + \sum_{m=1}^{\infty} \sum_{\substack{\ell \in 3\mathbb{Z}^d \\ \|\ell\|_\infty = m}} \lambda(3L_{n+1}N)^d \times e^{-c_2(3(m-1)+1/2)L_{n+1}N}. \tag{5.121}$$

This may be bounded by

$$c_3 \lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}, \tag{5.122}$$

for some constant $c_3 > 0$ which is independent of R, L_0 and N. Hence, we have bounded the second term on the right hand side of (5.114). In particular, we may deduce that for all $n \in \mathbb{N}$, again for a suitable choice of R and L_0, $\mathbb{P}_{\lambda,\rho}(B_n(i,k)^{\complement}) \geq 1/2$.

•**Analysis of the recursion scheme.** Equation (5.109) in combination with (5.114) and the arguments following it show that

$$\mathbb{P}_{\lambda,\rho}(A_{n+1}) \leq 2c_1 R^{2(d-1)(n+1)}\Big(\mathbb{P}_{\lambda,\rho}(A_n)^2 + c_3\lambda(3L_{n+1}N)^d e^{-c_2 L_{n+1}N/6}\Big). \tag{5.123}$$

To deduce the desired result, we first show with the help of (5.123) that $\mathbb{P}_{\lambda,\rho}(A_n)$ being small implies that $\mathbb{P}_{\lambda,\rho}(A_{n+1})$ is small as well. As a final step it then remains to show that $\mathbb{P}_{\lambda,\rho}(A_0)$ is already small. We now make this idea more precise. We put

$$\forall n \in \mathbb{N}, \quad a_n = 2c_1 R^{2(d-1)n} \mathbb{P}_{\lambda,\rho}(A_n). \tag{5.124}$$

Claim 5.7.1. *For R large enough, for all $n \in \mathbb{N}$ and for all $L_0 \geq 2R^{4(d-1)+1}$, the inequality $a_n \leq L_n^{-1}$ implies that $a_{n+1} \leq L_{n+1}^{-1}$.*

Proof. To prove the claim, let $n \in \mathbb{N}$ and assume that $a_n \leq L_n^{-1}$. Then,

$$\begin{aligned} a_{n+1} &= 2c_1 R^{2(d-1)(n+1)} \mathbb{P}_{\lambda,\rho}(A_{n+1}) \\ &\leq 4c_1^2 R^{4(d-1)(n+1)} \Big[\mathbb{P}_{\lambda,\rho}(A_n)^2 + c_3 \lambda (3L_{n+1}N)^d e^{-c_2 L_{n+1} N/6} \Big] \\ &= a_n^2 R^{4(d-1)} + 4c_1^2 c_3 R^{4(d-1)(n+1)} \lambda (3L_{n+1}N)^d e^{-c_2 L_{n+1} N/6}. \end{aligned} \tag{5.125}$$

Thus, it is enough to show that

$$a_n^2 R^{4(d-1)} \leq (2L_{n+1})^{-1} \quad \text{and} \quad 4c_1^2 c_3 R^{4(d-1)(n+1)} (3L_{n+1}N)^d e^{-c_2 L_{n+1} N/6} \leq (2L_{n+1})^{-1}. \tag{5.126}$$

For that, note that by our assumption on a_n

$$a_n^2 R^{4(d-1)} 2L_{n+1} \leq 2L_n^{-2} R^{4(d-1)} L_{n+1} = 2R^{4(d-1)} \frac{R^{n+1}}{R^n L_{n-1}} \leq 2R^{4(d-1)+1} L_0^{-1}. \tag{5.127}$$

Thus, choosing $L_0 \geq 2R^{4(d-1)+1}$ yields the first desired inequality. The second term on the right hand side of (5.125) may be bounded using similar considerations. This yields Claim 5.7.1. □

Hence, to use the claim, we need that $\mathbb{P}_{\lambda,\rho}(A_0) \leq L_0^{-1}$. For that observe that

$$\begin{aligned} \mathbb{P}_{\lambda,\rho}(A_0) &= \mathbb{P}_{\lambda,\rho}\Big(\text{There is a } *\text{-path of open sites from } [0, L_0)^d \text{ to } \partial_{\text{int}}[-L_0, 2L_0)^d.\Big) \\ &\leq \mathbb{P}_{\lambda,\rho}\Big(\text{There is } z \in \partial_{\text{int}}[-L_0, 2L_0)^d, \text{ which is open.}\Big) \\ &\leq c_4 L_0^{d-1} \mathbb{P}_{\lambda,\rho}(0 \text{ is open}), \end{aligned} \tag{5.128}$$

where $c_4 = c_4(d) > 0$ does only depend on the dimension. Equation (3.64) of [MR96] shows that

$$\mathbb{P}_{\lambda,\rho}(0 \text{ is open}) \leq 2d \mathbb{P}_{\lambda,\rho}(\text{CROSS}(N, 3N, \ldots, 3N)). \tag{5.129}$$

Therefore, if the right hand side of (5.129) is smaller than $(4dc_1 c_4 L_0^d)^{-1}$, we get from (5.128) that $\mathbb{P}_{\lambda,\rho}(A_0) \leq (2c_1 L_0)^{-1}$, which is the same as saying that $a_0 \leq L_0^{-1}$. This, in combination with Claim 5.7.1 and the observation that an infinite $*$-path of open sites containing zero implies the events A_n for all $n \in \mathbb{N}$, finally yields

$$\mathbb{P}_{\lambda,\rho}\Big(0 \text{ is contained in an infinite } *\text{-path of open sites}\Big) \leq \lim_{n\to\infty} \mathbb{P}_{\lambda,\rho}(A_n) = 0. \tag{5.130}$$

Consequently, we have shown that Lemma 5.2.3 is satisfied for $\varepsilon \leq (4dc_1 c_4 L_0^{d+1})^{-1}$.

5.8 Proofs of Lemmas 5.6.4–5.6.6

5.8.1 Proof of Lemma 5.6.4

Proof. Let $\mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}$. First, note that

$$\Phi_\delta(\mathbf{a}) > 0. \tag{5.131}$$

Indeed, since for all $\bar{\delta} < \delta$ the path of a Brownian motion $B_{[0,\bar{\delta}]}$ starting in $\underline{a}$ is absolutely continuous with respect to that of the Brownian bridge $W_{[0,\bar{\delta}]}$,

$$\mathbb{P}^{\mathbf{a}}_\delta(W_{[0,\delta/2]} \subseteq \mathcal{A}_{R,\epsilon},\, W_{\delta/2} \in \mathcal{B}(\overline{a}, \epsilon')) > 0, \tag{5.132}$$

where $\epsilon' > 0$ is chosen so small that $\mathcal{B}(\overline{a}, \epsilon') \subseteq \mathcal{A}_{R,\epsilon}$. From the representation

$$\forall s \in [0, \delta], \quad W_s = B_s - \frac{s}{\delta}(B_\delta - \overline{a}) \tag{5.133}$$

and the fact that a Brownian motion stays with a positive probability in an arbitrary small ball around its starting point within finite time intervals, we have the following:

$$\forall a' \in \mathcal{B}(\overline{a}, \epsilon'), \quad \mathbb{P}^{\mathbf{a}}_\delta(W_{[\delta/2,\delta]} \subseteq \mathcal{A}_{R,\epsilon} \mid W_{\delta/2} = a') > 0. \tag{5.134}$$

Equation (5.131) then follows from (5.132), the Markov property applied at time $\delta/2$ and (5.134). Second, the representation in (5.133) shows that the map

$$\mathbf{a} \mapsto \mathbb{P}^{\mathbf{a}}_\delta\Big(W_{[0,\delta]} \in \cdot\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}, \tag{5.135}$$

is weakly continuous. Moreover, the probability for a Brownian bridge to hit the boundary of $\mathcal{A}_{R,\varepsilon}$ but to stay inside $\mathcal{A}_{R,\varepsilon}$ is zero. Thus, an application of the Portmanteau Theorem yields that the function

$$\mathbf{a} \mapsto \mathbb{P}^{\mathbf{a}}_\delta\Big(W_{[0,\delta]} \in \mathcal{A}_{R,\varepsilon}\Big), \qquad \mathbf{a} \in \overline{\mathcal{A}}^2_{R,\overline{\epsilon}}, \tag{5.136}$$

is continuous. This fact together with (5.131) is enough to conclude the claim. □

5.8.2 Proof of Lemma 5.6.5

Proof. First, for $\mathbf{a} = (\underline{a}, \overline{a}) \in \mathcal{A}_{R,\overline{\epsilon}}$ we have that (see Exercise 1.5 in [MP10])

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_\tau(W_{[0,\tau-\delta]} \in \cdot\,)}{\mathrm{d}\mathbb{P}^{\underline{a}}_\tau(B_{[0,\tau-\delta]} \in \cdot\,)} = \frac{p(\delta, W_{\tau-\delta}, \overline{a})}{p(\tau, \underline{a}, \overline{a})}, \tag{5.137}$$

where

$$p(s, x, y) := \frac{1}{(2\pi s)^{d/2}} \exp\left(-\frac{\|x - y\|^2}{2s}\right). \tag{5.138}$$

Moreover, there exist constants c_1 and c_2 such that

$$0 < c_1 \leq \inf_{\substack{\delta \leq s \leq \tau \\ x,y \in \overline{\mathcal{A}}_{R,\overline{\epsilon}}}} p(s,x,y) \leq \sup_{\substack{\delta \leq s \leq \tau \\ x,y \in \overline{\mathcal{A}}_{R,\overline{\epsilon}}}} p(s,x,y) \leq c_2 < \infty. \tag{5.139}$$

Therefore,

$$\frac{\mathrm{d}\mathbb{P}^{\mathbf{a}}_{\tau}(W_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))}{\mathrm{d}\mathbb{P}^{a}_{\tau}(B_{[0,\tau-\delta]} \in \cdot, \mathcal{L}_{\delta,\tau-\delta}(\mathcal{A}_{R,\overline{\epsilon}}))} \geq \left(\frac{c_1}{c_2}\right) > 0. \tag{5.140}$$

□

5.8.3 Proof of Lemma 5.6.6

Proof. To achieve the intersection event, we use the following strategy:

- before time δ, both paths enter a ball inside $\mathcal{A}_{R,\overline{\epsilon}}$, and from this moment, stay in a slightly bigger ball;
- the two paths intersect each other between time δ and $\tau_1 \wedge \tau_2 - \delta$, while staying in a larger ball contained in $\mathcal{A}_{R,\overline{\epsilon}}$.

More precisely, let us choose arbitrarily $d \in \mathcal{A}_{R,\overline{\epsilon}}$. Let $\epsilon_4 > \epsilon_3 > \epsilon_2 > \epsilon_1 > 0$ to be determined later. For the moment we only assume that $\mathcal{B}(d,\epsilon_4) \subseteq \mathcal{A}_{R,\overline{\epsilon}}$. For $i \in \{1,2\}$, let us define

$$\sigma_1^{(i)} = \inf\{s \geq 0 : B_s^{a_i} \in \overline{\mathcal{B}}(d,\epsilon_1)\} \tag{5.141}$$

$$\sigma_2^{(i)} = \inf\{s \geq \sigma_1^{(i)} : B_s^{a_i} \notin \mathcal{B}(d,\epsilon_2)\}. \tag{5.142}$$

First note that with $\hat{\tau} := \tau_1 \wedge \tau_2$ and $\check{\tau} := \tau_1 \vee \tau_2$

$$\Big\{ \bigcap_{i=1,2} \{\mathcal{L}^i_{\delta,\tau_i-\delta}(\mathcal{A}_{R,\bar{\varepsilon}})\ ,\ \mathcal{S}^i_{[0,\tau_i-\delta]}(\mathcal{A}_{R,\varepsilon})\}, \quad \mathcal{I}_{[0,\tau_1-\delta],[0,\tau_2-\delta]} \Big\} \supseteq \tag{5.143}$$

$$\left\{ \begin{array}{l} \sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta,\ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta,\ \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\epsilon}), \\ \bigcap_{i=1,2} \Big\{ \mathcal{S}^i_{[\delta,\hat{\tau}-\delta]}(\mathcal{B}(d,\epsilon_3)), \mathcal{S}^i_{[\hat{\tau}-\delta,\check{\tau}-\delta]}(\mathcal{B}(d,\epsilon_4)) \Big\},\ \mathcal{I}_{[\delta,\hat{\tau}-\delta],[\delta,\hat{\tau}-\delta]} \end{array} \right\}.$$

An application of the Markov property at time δ shows that it is enough to establish that

$$\inf_{a_1,a_2 \in \mathcal{A}_{R,\overline{\epsilon}}} \mathbb{P}^{a_1,a_2}_{\delta,\delta} \Big(\sigma_1^{(1)} \vee \sigma_1^{(2)} < \delta,\ \sigma_2^{(1)} \wedge \sigma_2^{(2)} > \delta,\ \bigcap_{i=1,2} \mathcal{S}^i_{[0,\delta]}(\mathcal{A}_{R,\epsilon}) \Big) > 0, \tag{5.144}$$

and

$$\inf_{x,y \in \mathcal{B}(d,\epsilon_2)} \mathbb{P}^{x,y}_{\check{\tau}-2\delta,\check{\tau}-2\delta} \Big(\mathcal{I}_{[0,\hat{\tau}-2\delta],[0,\hat{\tau}-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat{\tau}-2\delta]}(\mathcal{B}(d,\epsilon_3)), \mathcal{S}^i_{[\hat{\tau}-2\delta,\check{\tau}-2\delta]}(\mathcal{B}(d,\epsilon_4)) \Big) > 0. \tag{5.145}$$

Let us first prove (5.144). The probability in the infimum is clearly positive for all a_1, a_2 in the compact set $\overline{\mathcal{A}}_{R,\overline{\epsilon}}$. Furthermore, one can use the same arguments as in the proof of Lemma 5.6.4 to show that it is continuous in (a_1, a_2) on $\overline{\mathcal{A}}_{R,\overline{\epsilon}} \times \overline{\mathcal{A}}_{R,\overline{\epsilon}}$, hence the infimum is also positive.

Now we proceed to prove (5.145). Again, an application of the Markov property at time $\hat{\tau} - 2\delta$ shows that it is enough to prove that

$$\inf_{x,y\in\mathcal{B}(d,\epsilon_2)} \mathbb{P}^{x,y}_{\hat{\tau}-2\delta,\hat{\tau}-2\delta}\Big(\mathcal{I}_{[0,\hat{\tau}-2\delta],[0,\hat{\tau}-2\delta]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat{\tau}-2\delta]}(\mathcal{B}(d,\epsilon_3))\Big) > 0, \tag{5.146}$$

and

$$\inf_{x,y\in\mathcal{B}(d,\epsilon_3)} \mathbb{P}^{x,y}_{\check{\tau}-\hat{\tau},\check{\tau}-\hat{\tau}}\Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\check{\tau}-\hat{\tau}]}(\mathcal{B}(d,\epsilon_4))\Big) > 0. \tag{5.147}$$

Now we focus on (5.146). For all $\tau_0 > 0$ and $R_0 > 1$, let us consider

$$\begin{aligned}\rho(\tau_0, R_0) &:= \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,\tau_0]}(\mathcal{B}(0,R_0))\Big) \\ &\geq \inf_{x,y\in\mathcal{B}(0,1)} \mathbb{P}^{x,y}_{\tau_0,\tau_0}\Big(\mathcal{I}_{[0,\tau_0],[0,\tau_0]}\Big) - 2 \sup_{x\in\mathcal{B}(0,1)} \mathbb{P}^{x}_{\tau_0}\Big(\sup_{s\in[0,\tau_0]} \|B_s\| > R_0\Big).\end{aligned} \tag{5.148}$$

By using the monotonicity argument in Lemma 5.5.2 and Theorem 9.1 in [MP10], the last infimum can be made arbitrarily close to 1 by choosing τ_0 large enough, whereas standard estimates yield that the supremum goes to 0 as R_0 goes to infinity. Therefore, there is a choice of τ_0 and R_0 leading to $\rho(\tau_0, R_0) > 0$. By the scale invariance of Brownian motion,

$$\forall u > 0, \quad \inf_{x,y\in\mathcal{B}(0,u)} \mathbb{P}^{x,y}_{u^2\tau_0,u^2\tau_0}\Big(\mathcal{I}_{[0,u^2\tau_0],[0,u^2\tau_0]} \bigcap_{i=1,2} \mathcal{S}^i_{[0,u^2\tau_0]}(\mathcal{B}(0,uR_0))\Big) = \rho(\tau_0, R_0) > 0. \tag{5.149}$$

We may now choose $u_0 > 0$ such that

$$\tau' := u_0^2\tau_0 < \hat{\tau} - 2\delta, \quad 2u_0R_0 < \mathrm{dist}(d, \mathcal{A}_{R,\epsilon}), \tag{5.150}$$

and we set

$$\epsilon_2 := u_0, \quad \epsilon_3 := 2u_0R_0. \tag{5.151}$$

Note that we may choose R_0 such that $\epsilon_3/2 > \epsilon_2$. Hence, an application of the Markov property at time τ' to the left hand side of (5.146) yields,

$$\text{l.h.s. of (5.146)} \geq \rho(\tau_0, R_0) \inf_{x,y\in\mathcal{B}(0,\epsilon_3/2)} \mathbb{P}^{x,y}_{\hat{\tau}-2\delta-\tau',\hat{\tau}-2\delta-\tau'}\Big(\bigcap_{i=1,2} \mathcal{S}^i_{[0,\hat{\tau}-2\delta-\tau']}(\mathcal{B}(0,\epsilon_3))\Big) > 0. \tag{5.152}$$

The positivity of the second factor of (5.152) and of (5.147) may be shown by using similar arguments as in the proof of Lemma 5.6.4. This finally yields the claim. □

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Samenvatting

Dit proefschrift behandelt twee verschillende modellen in twee verschillende contexten.

Deel 1: Dynamische Gibbs-niet-Gibbs overgangen

Gibbs-maten zijn wiskundige objecten die gebruikt worden om evenwichtstoestanden van systemen met een groot aantal interagerende componenten (bijvoorbeeld deeltjes met 'spin'-toestand -1 of 1) te modelleren. Vanwege dit grote aantal ligt het voor de hand om te veronderstellen dat de toestand van het systeem stochastisch (kansafhankelijk) is. Gibbs-maten zijn kansmaten op de toestandsruimte (bijvoorbeeld $\{-1,+1\}^{\mathbb{Z}^d}$) die deze stochastiek beschrijven. De beschrijving is gebaseerd op de zogenaamde Maxwell-Boltzmann-Gibbs formule en bevat een zekere "regulariteitsconditie" voor conditionele kansen waarbij de configuratie buiten een eindig volume gegeven is. Een van de belangrijke vragen is of deze "regulariteitsconditie" behouden blijft indien het systeem onderhevig is aan een stochastische dynamica. Met andere woorden, beschouw de kansmaat van de initiële verdeling en laat een stochastische dynamica hierop werken gedurende tijd t. Kan men de tijdsgeëvolueerde maat nog steeds beschrijven als een Gibbs-maat? Indien niet, waarom wordt er dan niet aan de "regulariteitsconditie" voldaan? Het proefschrift behandelt deze vragen voor twee verschillende modellen.

In hoofdstuk 1 geven we zowel een introductie tot het Gibbs-formalisme en de stand van zaken m.b.t. het fenomeen Gibbs-niet-Gibbs voor rooster-modellen, als een beschrijving van de modellen die behandeld worden in het proefschrift.

Hoofdstuk 2 behandelt het 'mean-field' model, hetgeen een versimpeling is van het rooster-model. Deze verkrijgt men door $\mathbb{Z}^d$ samen met zijn naaste-buren bindingen te vervangen door de volledige graaf van grootte $N \in \mathbb{N}$ en door de sterkte van de interactie gelijk te maken tussen ieder paar van spins. Meer in het bijzonder beschouwen we het Curie-Weiss model met paar-potentiaal $J > 0$ en magnetisch veld $h \in \mathbb{R}$ (i.e., het Ising model op de volledige graaf) dat onderhevig is aan onafhankelijke spin-flip dynamica. De definitie van Gibbs-maat in deze context is in essentie gerelateerd aan de continuïteit van een zekere limietfunctie. Met deze definitie bevestigen we in Stelling 2.1.6 het scenario dat voorgesteld is in eerder werk voor rooster-modellen, namelijk, Gibbs-niet-Gibbs overgangen zijn *equivalent* aan bifurcaties (splitsingen) in de verzameling van globale minima van de grote-afwijkingen entropie-functie van de paden van de magnetisatie *geconditioneerd* op hun eindwaarde. De globale minimizers van deze entropie-functie worden onderzocht. Daarna kunnen we met behulp van de vorige karakterisatie een gedetailleerd beeld krijgen van de Gibbs-niet-Gibbs overgangstijden en de bijbehorende discontinuïteiten als functie van de parameters J, h en t (zie Stelling 2.1.9). Tot de gegeven nieuwe resultaten behoren: (1) het optreden van *verboden gebieden* die niet gekruist kunnen worden door optimale paden op latere tijden; (2) het optreden van niet-monotoon gedrag in de tijd voor optimale paden voor $h \neq 0$.

Een stap van het mean-field model in de richting van het rooster-model wordt gemaakt door het vervangen van de volledige graaf door de discrete torus, waarbij echter nog steeds wordt toegestaan dat paren van spins interageren met elkaar via een lange-dracht interactie. In hoofdstuk 3 breiden we onze resultaten voor mean-field modellen uit door een model met een Kac-type interactie te beschouwen, namelijk, Ising spins met een lange-dracht interactie die afhangt van twee functies $J, h \in C(\mathbb{T}^d)$. Sommige resultaten worden bewezen voor algemene Glauber spin-flip dynamica, maar de meest belangrijke resultaten worden alleen bewezen voor onafhankelijke spin-flips. Niet-Gibbs stemt overeen met een discontinue afhankelijkheid van de wet van het initiële magnetisatie-profiel *geconditioneerd* op het uiteindelijke magnetisatie-profiel. Net als in de mean-field context, komen zulke discontinuïteiten voor als er meerdere paden van het profiel zijn die *compitabel* zijn met het slechte profiel aan het eind (zie Stelling 1.5.1). De meest waarschijnlijke geconditioneerde paden zijn de paden die de entropie-functie op de ruimte van paden van profielen minimaliseren (Proposities 3.1.2–3.1.3).

De entropie-functie voor het Kac model bevat een actie-integraal waarvan de Lagrangiaan werkt op profielen van magnetisaties in plaats van alleen magnetisaties. Dit model is conceptueel moeilijker dan het Curie-Weiss model, waar de Lagrangiaan werkt op magnetisaties en gemakkelijker te analyseren valt. Toch, voor een oneindige- temperatuur dynamica kan de Kac-Lagrangiaan uitgedrukt worden als een integraal van de Curie-Weiss Lagrangiaan (Stelling 3.1.5). Met behulp van dit verband kunnen we de mogelijke scenarios van bifurcaties identificeren. Net als in de mean-field context kan korte-tijd Gibbs bewezen worden. We bewijzen ook mean-field gedrag: als alle parameters van het systeem constante functies zijn dan is het bifurcatiegedrag hetzelfde als voor het Curie-Weiss model met deze constanten.

Deel 2: Stochastische meetkunde

Deeltjes worden stochastisch in $\mathbb{R}^d$ geplaatst volgens een Poisson punt-process met intensiteit $\lambda > 0$. Vervolgens volgt ieder deeltje (onafhankelijk van de andere deeltjes) een d-dimensionale Brownse beweging gedurende tijd t en volgen we het net van paden dat ontstaat door deze beweging. Een van de belangrijke vragen is of dit net, genoteerd door $\mathcal{O}_t$, een onbegrensde cluster heeft of niet. En zo ja, is deze cluster dan uniek?

In hoofdstuk 4 geven we een korte introductie tot het model (evenals als een inleiding over een gerelateerd model). In hoofdstuk 5 bewijzen we de volgende resultaten:

- Voor $d = 1$ en voor alle $t \geq 0$ is het bijna zeker zo dat de verzameling van paden van de Brownse bewegingen geen onbegrensd cluster bevat.
- Voor $d \in \{2, 3\}$ bestaat er een $t_c = t_c(\lambda, d) > 0$ zodat voor $t < t_c$ het bijna zeker zo is dat $\mathcal{O}_t$ geen onbegrensde cluster bevat, terwijl voor $t > t_c$ de verzameling $\mathcal{O}_t$ een uniek oneindig cluster bevat.
- Voor $d \geq 2$ snijden twee Brownse bewegingen elkaar nooit. Daarom beschouwen

we een bol van straal r rond de startposities van de deeltjes. Ook hier volgt ieder deeltje een d-dimensionale Brownse beweging, maar nu wordt het pad beschouwd dat de (bewogen) bol achter laat. Vergelijkbare resultaten worden behaald voor $d \in \{2, 3\}$.

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Curriculum Vitae

Julián Martínez was born in Ciudad Autónoma de Buenos Aires on 06 January 1983. He attended the Universidad de Buenos Aires (UBA), completing a licenciate in mathematics in 2008. During the months of November and December 2008 he was visiting student at the Núcleo de Apoio á Pesquisa em Modelagem Estocástica e Complexidade, of the Universidade de So Paulo under the supervision of Pablo Ferrari. From March 2009 until August 2010 he was working as a PhD student at the Facultad de Ciencias Exactas y Naturales (UBA) under the supervision of Pablo Ferrari. In May 2010 he received an Erasmus Mundus-BAPE scholarship and restarted his PhD at Leiden University. From September 2010 until December 2013 he pursued a PhD in Mathematics at Leiden University under the supervision of Frank den Hollander and Roberto Fernández. In April 2014 he will start post-doctoral research at the Mathematics Institute at Universidad de Buenos Aires under the supervision of Prof. Dr. P. Groisman.