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Consistent Supersymmetric Decoupling in Cosmology

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Consistent Supersymmetric Decoupling in Cosmology.

Kepa Sousa Sánchez

On the front cover: artistic representation of a supersymmetric truncation and a spontaneous supersymmetry breaking. Aquarel by *Ana Sánchez Trujillo*.

Consistent Supersymmetric Decoupling in Cosmology.

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*A mis padres,
por estar siempre a mi lado.*

*Para el adulto que padece
una impotencia creativa (...) la única posibilidad que le queda es remontarse a su propia niñez y empezar de nuevo, a partir del momento en que le arrebataron sus sueños. Que no eran sueños, en absoluto, sino el fundamento de su propia vida, las raíces de su existencia, sin la cuales nunca será persona verdaderamente.*

The only cure for the adult incapable of any act of creativity (...) is to return to his own childhood, to the instant where his dreams were taken from him. Which were not just dreams, (...) but the very roots of his existence, without which he will never really be a person.

F. Hundertwasser

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Preface

This thesis contains the results of a PhD project carried out at the University of Leiden, in collaborations with my advisor Ana Achúcarro, with Mboyó Esole and Sjoerd Hardeman. The present work discusses several problems related to the stability of ground states with broken supersymmetry in supergravity, and to the existence and stability of cosmic strings in various supersymmetric models.

In the last decades there has been a lot of interest in building cosmological models within the frameworks of the main candidate theories to explain the physics at very high energy scales, such as Grand Unification Theories, supersymmetric theories and Superstrings.

The main reason is that the phenomena described by these theories take place at energy scales way above those probed by any particle accelerator built on earth, now or in the near future. On the other hand, due to the extremely high energies involved in the very early stages of the evolution of the universe, cosmology provides an excellent framework to test all these theories.

The standard model in cosmology is the so called Λ CDM, where the acronym refers to the main components of the universe, the cosmological constant Λ , which drives the observed accelerated expansion of the universe, and the Cold Dark Matter, an exotic new type of matter which has only been detected due to its gravitational effects. The Λ CDM model has been very successful in explaining, among other things, the element abundances, galaxy formation and the large scale distribution of galaxies. However, there are still many questions

that need to be answered, for instance the microscopic origin of the cosmological constant, or the nature of the Dark Matter particles.

Another issue of the Λ CDM model is related to the initial conditions of the universe which led to the observed galaxy distribution. According to the currently accepted ideas about galaxy formation, the large scale structure of the universe was seeded by primordial density perturbations which grew mainly due to the gravitational effects of Dark Matter, and can be seen directly as temperature fluctuations on the Cosmic Microwave Background (CMB). These irregularities were originated from quantum fluctuations stretched to cosmic size during an early period of extremely fast accelerated expansion called inflation.

The CMB temperature spectrum is consistent with the predictions of inflation, however the physical processes which are responsible for it remain unclear. In the simplest models inflation is driven by the vacuum energy of a scalar field which is slowly rolling down a very flat potential. The development of inflationary models can provide very useful information about the high energy regime. On the one hand the peculiarities of the theory where inflation is implemented would leave their imprint on the CMB, and on the other hand we have at our disposal extremely precise measurements of the CMB spectrum provided by the WMAP mission, which will be improved even further by observations of the Atacama Cosmology Telescope and the PLANCK satellite.

The topics discussed in this thesis are relevant for cosmological models based on supersymmetric theories. Supersymmetry, a symmetry which transforms bosons into fermions and viceversa, is an attractive framework which provides a solution to the hierarchy problem. Its local version, supergravity, was initially proposed as a method to cure the divergences which typically appear in theories of quantum gravity, but this idea did not succeed. Nowadays, supersymmetry and supergravity are mainly used to describe the low energy regime of the most prominent theory of quantum gravity, String Theory.

Cosmological models derived from superstrings typically involve a large number of scalar fields, such as the moduli. Partly with the aim of gaining control on the analysis, and also because inflationary models based on a single field predict very accurately the spectrum of perturbations of the CMB, cosmological models assume most of these scalar fields stabilized at some high energy scale, leaving behind a low energy supersymmetric theory involving the minimum necessary to implement inflation. Moreover, since supersymmetry is known to be broken at low energies both during inflation and at present (the Standard Model particles cannot be fitted in supermultiplets without introducing new particles), the sector surviving at low energies must also provide the mechanism to break it.

In supergravity the couplings between fields are highly constrained, and

since gravity couples to everything, it is non-trivial to integrate out a sector of the theory consistently while preserving supersymmetry. Moreover, since supersymmetry has to be broken in the surviving sector, there is no easy way to guarantee that the truncated fields remain massive. If the truncation is not made in a consistent way the inflaton will possibly couple to the truncated degrees of freedom, leaving distinctive features on the CMB spectrum. The study of such effect could provide us important information about the high energy regime.

In chapters 3 and 4 we study the necessary conditions to truncate consistently a sector of the theory while preserving supersymmetry, and on those models where these conditions are met we will discuss how the breaking of supersymmetry in the light sector affects the stability of the truncated fields. The basis of this work are the journal papers

- ‘*F-term uplifting and moduli stabilization consistent with Kahler invariance.*’ Ana Achúcarro and Keba Sousa. *JHEP* **0803** (2008) 002; [arXiv:0712.3460](#).
- ‘*Consistent decoupling of heavy scalars and moduli in $\mathcal{N} = 1$ supergravity.*’ Ana Achúcarro Sjoerd Hardeman and Keba Sousa. *Phys.Rev.* **D78** (2008) 101901; [arXiv:0806.4364](#).
- ‘*F-term uplifting and the supersymmetric integration of heavy moduli.*’ Ana Achúcarro, Sjoerd Hardeman and Keba Sousa. *JHEP* **0811** (2008) 003; [arXiv:0809.1441](#).

Although inflation is widely accepted as the main mechanism to produce the primordial energy density perturbations, the observations are still compatible with a minor contribution from other sources, such as cosmic strings. Cosmic strings are extremely thin line-like concentrations of energy of cosmic length which move at relativistic speeds. The formation of cosmic strings is a generic prediction of many promising cosmological models based on supersymmetric Grand Unified Theories and superstrings.

Among this type of solutions supersymmetric cosmic strings, those preserving a fraction of the supersymmetries of the original model, are especially interesting because the unbroken supersymmetries protect them from quantum corrections. In particular, their tension is preserved in the quantum theory along the renormalization flow, and therefore they constitute a probe of the high energy regime. In this thesis we will discuss a new type of supersymmetric cosmic string solutions in $\mathcal{N} = 2$ supergravity. Extended supergravity theories $\mathcal{N} > 1$ are not a common framework for cosmology since $\mathcal{N} = 1$ supergravity, which involves chiral fermions, is more suitable to describe phenomenology. However many compactifications of string theory have a description in terms of $\mathcal{N} = 2$ supergravity, and thus these models can be used as a bridge between the more

phenomenological $\mathcal{N} = 1$ models, and String Theory. In particular $\mathcal{N} = 1$ models are much less constrained than those built in $\mathcal{N} = 2$ supergravity and therefore they have less predictability.

In chapter 7 we present a $\mathcal{N} = 2$ model that which admits a family of cosmic string solutions with the same energy but different core radius. The family is parametrized by the expectation value of a field appearing in the model which has similar couplings to the dilaton and the Kähler moduli in string theory compactifications. The research in this chapter can be found in the following article

- ‘*Half-BPS cosmic string in $\mathcal{N} = 2$ supergravity in the presence of a dilaton.*’ Mboyo Esole and Kepa Sousa. *JHEP* **0703** (2007) 079; [hep-th/0610124](#).

We will also discuss the stability of these strings against perturbations. In particular, in a cosmological context the zero mode associated to the expectation value of the dilaton-type field could be excited, leading to the spread of the magnetic flux and the disappearance of the string. Here we will explore this possibility. The arguments used in this discussion are based on a previous publication where we analyze a class of cosmic string solutions appearing on a $\mathcal{N} = 1$ supersymmetric model with a similar zero mode

- ‘*A Note on the stability of axionic D-term strings.*’ Ana Achúcarro and Kepa Sousa. *Phys.Rev.* **D74** (2006) 081701; [hep-th/0601151](#).

From the summary above it is clear that in this thesis we consider two different topics, on the one hand the consistent supersymmetric decoupling of a sector in $\mathcal{N} = 1$ supergravity models and the stability of the decoupled sector, and on the other hand supersymmetric cosmic strings solutions in $\mathcal{N} = 2$ supergravity, and the stability of these configurations in cosmological context. Nevertheless for convenience we have not made an explicit splitting, instead both topics appear together through the thesis, and we discuss them as we introduce the necessary concepts for doing it.

Introduction: cosmological models in field theory and topological defects.

1.1 Cosmology

During the 20th century our knowledge of the history of the universe experienced a spectacular development, which was triggered by important breakthroughs both in theoretical physics and in observations. The discovery of the recession of distant galaxies at the beginning of the 20th century is often referred to as one of the most significant contributions to the birth of modern cosmology. For a review on cosmology see [1].

Between 1925 and 1929 E. Hubble was able to determine the velocities of 18 galaxies by measuring the Doppler shift (*redshift*) of the light they emit. He confirmed that most of the galaxies appeared to move away from us, and moreover, he found that their recession velocity v satisfied a simple relation with the distance x from us:

$$v = H x.$$

This is the so called *Hubble law* and H is the *Hubble parameter*. At first sight,

the fact the galaxies are moving away from us, seems to imply that we live on a very special place in the universe. However an observer at any arbitrary point of an expanding universe which is homogeneous and isotropic would see exactly the same thing. Indeed, the galaxy surveys seem to indicate that the distribution of galaxies is homogeneous and isotropic on scales larger than about 100 Mpc, and, therefore, *there are no privileged places in our universe*. This statement, which is one of the conceptual cornerstones of modern cosmology, is known as the *cosmological principle*.

From the fact that the universe is currently expanding we could anticipate that at very early times the universe should have been very dense and hot and, extrapolating even further, that at some instant in the past everything was together at the same point. This scenario is known as the *hot Big-Bang cosmology*. The first works on the physics at such early times appeared during the 1940's [2, 3]. These early attempts suggested that the universe had undergone a phase where all the matter was ionized, with both matter and radiation in thermal equilibrium. Eventually, the temperature would be low enough for the first atoms to form, and photons would be able to travel freely, leaving a universe filled with relic radiation.

This relic radiation, known as the *Cosmic Microwave Background*, was detected in 1965 by A. Penzias and R. W. Wilson [4], and became one of the crucial observations that supported the Big-Bang cosmology. The most recent measurements of the Cosmic Microwave Background with the *Wilkinson Microwave Anisotropy Probe* (WMAP) indicate that it is extremely isotropic, with temperature fluctuations of one part in 10^5 , and it is known to have a very accurate black-body spectrum with a temperature of about 2.7 K [5]. The fluctuations of the Cosmic Microwave Background reflect the energy density perturbations which, under the action of gravity, grew to form the large scale structures of the universe.

The first rigorous attempts to describe a dynamical universe appeared at the beginning of the 20th century, shortly after Einstein formulated the equations of General Relativity [6, 7, 8, 9]. The fundamental object in General Relativity is the space-time metric, which in the case of a homogeneous and isotropic universe takes the form (we work in the units $c = \hbar = 1$)

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \quad (1.1.1)$$

This is known as the Friedmann-Lemaître-Robertson-Walker metric, and the time dependent function $a(t)$, the *scale factor*, characterizes the evolution of the universe. This metric is written in terms of spherical *comoving coordinates*, so that the galaxies have the same fixed coordinates at all times, provided that we

1.1. Cosmology

neglect small random velocities. The constant k can take the values $+1, -1, 0$ corresponding to an closed, open and flat universe respectively.

It is very easy to check that the scale factor and the Hubble parameter are related to each other. The distance between our galaxy and the observed one can be obtained from the metric, for instance in a nearly flat universe like ours, $k \approx 0$, it is simply $x = a(t)r$. In this language an expanding universe corresponds to a scale factor that grows with time $\dot{a}(t) > 0$. The Hubble law follows immediatly from this since:

$$v = \frac{dx}{dt} = \frac{\dot{a}}{a} x \equiv H x. \quad (1.1.2)$$

The evolution of the scale factor $a(t)$ is governed by the Friedmann equation, which follows from the Einstein equations of General Relativity:

$$H^2 = \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho - \frac{k}{a^2}, \quad (1.1.3)$$

where G is Newton's constant and ρ is the energy density in the universe. In order to determine the cosmological evolution, the Friedmann equation has to be supplemented by the equation for the conservation of energy

$$\dot{\rho} + 3\frac{\dot{a}}{a}(p + \rho) = 0, \quad (1.1.4)$$

and an equation of state relating the pressure p with the energy density, $p = p(\rho)$.

The energy density of the different constituents of the universe, such as matter and radiation, evolve in different ways as the universe expands. For example, in the case of non-relativistic matter (dust) the equation of state is simply $p = 0$, and therefore, from (1.1.4), we conclude that the energy density of non-relativistic matter evolves as $\rho_m \sim a^{-3}$. On the other hand the equation of state for radiation is given by $p = \frac{1}{3}\rho$, which leads to the following relation between its energy density and the scale factor $\rho_r \sim a^{-4}$. Since the energy content of the universe determines its evolution, (1.1.3), this observation implies that it is possible to infer the composition of the universe from the time evolution of the scale factor.

The evolution of the scale factor was characterized accurately for the first time during the 1990s. Two teams, the *Supernova Cosmology Project* and the *High- z Supernova Search Team*, measured the redshift of large samples of distant galaxies, for which the distance could be determined from type Ia supernova observations. The results from these experiments were a surprise for many cosmologists, because they showed that our universe was expanding at an increasing rate instead of decelerating, which was the most extended belief. This discovery earned S. Perlmutter (from Supernova Cosmology Project), A. G. Riess and B.

P. Smith (both from the High-z Supernova Search Team) the 2011 Nobel prize in physics [10]. Their data implied that the most abundant constituent of our universe (73 %) was an exotic form of energy with negative pressure, which has equation of state of the form $\rho_{de} = \omega p$ and $\omega < -1/3$. Nowadays the microscopic origin of this energy, known as *Dark Energy*, is still uncertain. These are some of the ideas which have been proposed to describe cosmic acceleration (see [11]):

- The simplest one is the introduction of the *cosmological constant* in Einsteins equations. When this extra term is added the Friedmann equation reads:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}. \quad (1.1.5)$$

Although this term is permitted by General Relativity it does not provide an explanation of the physics underlying cosmic acceleration.

- The cosmological constant can be interpreted as the *vacuum energy of empty space*. In such a case $\omega = -1$, which is the value preferred by the observations. However particle physics theories predict a cosmological constant which is orders of magnitude too large to be consistent with the data.
- Another possible explanation is that the Dark Energy is related to the vacuum energy of a scalar field. Scalar fields are ubiquitous in theories of high energy physics such as the Standard Model (the Higgs), Grand Unification Theories (GUTs) and superstring theories.
- It has also been suggested that cosmic acceleration could be a result of gravitational physics, such as non-linear effects due to energy density inhomogeneities, or modifications of General Relativity which only become relevant at cosmological scales.

Recently the composition of the universe has been determined very accurately with the observations of the CMB provided by the WMAP satellite [5]. Usually the abundances are given relative to the critical energy density, $\rho_c \equiv 3H^2/8\pi G$, and thus they are represented by the quantities of the form $\Omega = \rho/\rho_c$, and the curvature of the universe is represented by $\Omega_k = -k/a^2 H^2$. The CMB data alone can bound the curvature of the universe between the limits $-0.273 < \Omega_k < 0.013$ [5], indicating that we live in a closed, almost flat universe. Assuming a completely flat universe, $\Omega_k = 0$, the WMAP data alone give the following results for the abundances [5]:

$$\Omega_b = 0.0449 \pm 0.0028 \quad \Omega_{dm} = 0.222 \pm 0.026 \quad \Omega_{de} = 0.734 \pm 0.029 \quad (1.1.6)$$

From these data we can see that the universe is dominated by Dark Energy, Ω_{de} . Another striking feature is that most of the matter content of the universe (22 %) is non-relativistic non-baryonic matter, known as *dark matter* Ω_{dm} , whose

1.1. Cosmology

origin is still undetermined. On the other hand ordinary matter, characterized by Ω_b , only constitutes 4.5 % of the total energy content.

This is often called the Concordance Model or Λ CDM, where the acronym refers to the most abundant constituents of the universe, the dark energy or cosmological constant Λ , and Cold Dark Matter (CDM).

1.1.1 Cosmic Inflation.

Despite the many successes of the standard Big-Bang cosmology, which describes very well the high redshift supernova observations, the Cosmic Microwave Background and the formation of large scale structure, there are several observations that cannot be explained in this framework. These are the *flatness problem*, the *horizon problem* and the *monopole problem*.

The flatness problem is related to the fact that the universe today is close to having flat (Euclidean) geometry $\Omega_k \approx -0.08$. From the Friedmann equation it is possible to see that a Euclidean universe, $\Omega_k = 0$, is an unstable point, meaning that during the expansion Ω_k tends to move away from zero. Actually, according to the Big-Bang cosmology, if we extrapolate backwards to the time when the CMB was formed, we find that Ω_k should have had an extremely small value $|\Omega_k| \sim 10^{-5}$, and as we extrapolate to earlier times it gets even closer to zero, indicating a severe finetuning of this parameter.

The second problem of the standard Big-Bang cosmology is related to the finite age of the universe. Since light only had a finite time to travel, there is a maximum distance beyond which we couldn't have received any information. Light emitted beyond that distance did not have time to reach us yet. The boundary of the observable universe is known as the *horizon*. Therefore two regions separated by a distance larger than the size of the horizon could never have been in causal contact. As we mentioned above the Earth is bathed in a relic radiation, the CMB, which is extremely isotropic with a temperature of about 2.7 K. Since the temperature is the same in every direction of the sky this seems to indicate that all the universe we observe was once in thermal equilibrium, and thus in causal contact. However, according to the standard Big-Bang cosmology, two opposite points of the sky are separated by a distance over a hundred times the horizon size, and could never have been in causal contact.

The last puzzle, the monopole problem, appears in the context of Grand Unified Theories, which we will discuss briefly in a later section. These theories would describe the physics of the universe at energy scales of order 10^{15} GeV, and thus are relevant at very early stages of the universe. A generic prediction of these theories is the formation of extremely massive particle-like objects,

the *magnetic monopoles*. These particles would rapidly dominate the energy density of the universe, leading to an evolution incompatible with the present observations.

The most widely accepted explanation to all these problems was proposed in 1981 by A. Guth, *cosmic inflation* [12]. The basic idea of inflation is that the universe underwent an extremely fast, almost exponential accelerated expansion, at a very early stage of its evolution. During inflation, which typically could last 10^{-34} seconds, the universe was stretched by a factor of at least 10^{28} , enough to solve the flatness, horizon, and monopole problems. In addition, “slow roll” inflation can give an explanation for the adiabatic, near gaussian, and almost scale invariant spectrum of primordial density perturbations that is observed.

The idea of inflation can be implemented in the context of field theory introducing a scalar field, *the inflaton* ϕ . In the basic picture the scalar field is in a homogeneous configuration ϕ_0 , with a very large potential energy, slowly rolling down an extremely flat potential. The slow roll condition is

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \ll 1. \quad (1.1.7)$$

In this setting the scalar field contributes to the energy density of the universe with its potential energy, which acts as an effective cosmological constant:

$$\rho_{infl} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \approx V(\phi_0). \quad (1.1.8)$$

If the potential energy is sufficiently large to dominate the energy density of the universe, from the Friedmann equation it is possible to see that the scale factor evolves exponentially:

$$a(t) = \exp\left(\sqrt{\frac{V(\phi_0)}{3M_p^2}} t\right). \quad (1.1.9)$$

where M_p is the reduced Plank mass¹. In the present example it can be written in terms of the scalar potential $V(\phi)$, leading to so called slow roll conditions:

$$\epsilon_V \equiv \frac{M_p^2}{2} \frac{V'}{V} \ll 1, \quad |\eta_V| \equiv M_p^2 \frac{|V''|}{V} \ll 1. \quad (1.1.10)$$

Since the scalar field is varying in time, the expansion is not exactly exponential. Eventually the scalar field reaches a point where the conditions for inflation are no longer satisfied and the exponential expansion ends. During this process the inflaton decays leading to the initial state of the standard Big-Bang cosmology.

¹The Plank mass is defined as $m_p^2 = \hbar c/G$, but in our current units is $m_p^{-2} = G$. However, for convenience in this chapter we use the reduced Plank mass, which in our units reads $M_p^{-2} \equiv 8\pi G$. In the remaining chapters we will work in units of the reduced Planck mass $M_p = 1$.

1.2. High energy physics.

In contrast with standard Big-Bang evolution, during the exponential expansion the universe approaches a Euclidean geometry with $\Omega_k = 0$:

$$\Omega_k = \frac{k}{a^2 H^2} \propto \exp\left(-\sqrt{\frac{4}{3M_p^2} V(\phi_0)} t\right), \quad (1.1.11)$$

which would explain the finetuning of the parameter Ω_k at the initial stage of Big-Bang Cosmology. Inflation also solves the horizon problem. During inflation a patch of the universe small enough to be in thermal equilibrium, and thus in causal contact, could have been stretched to be much larger than the present observable universe. Therefore the CMB radiation has the same temperature in every direction because all these points of the sky were once in thermal equilibrium. Finally the monopoles, as non relativistic matter, have an energy density that evolves as $\rho_{monop} \sim a^{-3}$, which implies that the cosmic inflation would have diluted them to the point that they cannot be detected.

Inflation also provides an explanation for the origin of structure formation. According to the standard picture the irregularities observed in the CMB originated from quantum fluctuations, which during inflation were stretched to a cosmic size. This mechanism is currently the most accepted one, due to the excellent agreement between its predictions and the WMAP observations of temperature fluctuations in the CMB.

1.2 High energy physics.

1.2.1 Symmetry and spontaneous symmetry breaking.

Two of the most important developments in theoretical physics during the 20th century were Quantum Field Theory and Einstein's theory of General Relativity (see [13, 14]), which have been essential in the understanding of the four fundamental interactions of nature. While General Relativity treats gravity, Quantum Field Theory led to the formulation of the Standard Model of particle physics, which describes the other three fundamental forces: electromagnetic, strong and weak interactions.

The idea of symmetry plays a fundamental role in these two theories. A physical system presents a symmetry when the laws that govern its evolution are invariant under a particular *group of transformations*. In field theory the symmetries can be classified in various types, such as *space-time* and *internal*. The first ones are related to the transformations that involve space and time, while the internal symmetries relate the different fields that characterize the system to each other. Thus the fields must form a representation of the group of transformations associated with the internal symmetry. A symmetry is said to be *global* when the system is invariant under transformations which

act in the same way at every point of space time. If instead the transformations are allowed to vary from point to point then it is called a *local symmetry*. The last type of symmetries are the basis for *gauge theories*. In order to gauge a local internal symmetry it is necessary to introduce a spin-1 field, a gauge boson, which has to be massless for the theory to be renormalizable.

In some situations the ground state of a system is only invariant under a subgroup H of the full symmetry group, G of the equations of motion. In such a case the group G is said to be *spontaneously broken* to its subgroup H . The classical example is the Heisenberg ferromagnet, an array of spin-1/2 magnetic dipoles interacting only with their nearest neighbors. The equations of motion describing this system are invariant under the group of spatial rotations $SO(3)$, the evolution does not depend on the original orientation of the sample. When the system is in its ground state all the spins are aligned in an arbitrary direction, and therefore the different orientations of the sample are physically distinguishable. However there is a set of rotations that can still act in the ferromagnet without producing any physical change: those which have the axis of rotation aligned with the dipoles. Thus the rotation group $SO(3)$ is spontaneously broken by the ground state to its subgroup $SO(2)$.

1.2.2 Spontaneous breaking of local symmetries, the Abelian-Higgs model.

The case of the Heisenberg ferromagnet is an example of the spontaneous breaking of a global symmetry. Let us consider now what happens when the broken symmetry is gauged. A simple gauge theory that exhibits spontaneous symmetry breaking is the *Abelian-Higgs model*. This model is defined in $3 + 1$ dimensions, it contains a complex scalar field ϕ , the *Higgs*, coupled to a $U(1)$ gauge field A_μ , i.e. electro-magnetism. The action is given by:

$$\mathcal{S} = \int d_4x \left[-D_\mu \phi D^\mu \bar{\phi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - V(\phi) \right], \quad V(\phi) = \lambda(|\phi|^2 - \eta^2)^2, \quad (1.2.1)$$

where the covariant derivative is defined as $D_\mu \phi = \partial_\mu \phi - ig A_\mu \phi$, and the field strength of the gauge boson by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Euler-Lagrange equations corresponding to the action (1.2.1) are:

$$D_\mu D^\mu \phi - 2\lambda(|\phi|^2 - \eta^2)\phi = 0 \quad (1.2.2)$$

$$\partial_\nu F^{\mu\nu} + ig(\bar{\phi} D^\mu \phi - \phi D^\mu \bar{\phi}) = 0. \quad (1.2.3)$$

Note that the action and the equations of motions are invariant under the following local $U(1)$ transformations:

$$\phi \rightarrow e^{ig\Lambda} \phi \quad \bar{\phi} \rightarrow e^{-ig\Lambda} \bar{\phi} \quad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad (1.2.4)$$

1.2. High energy physics.

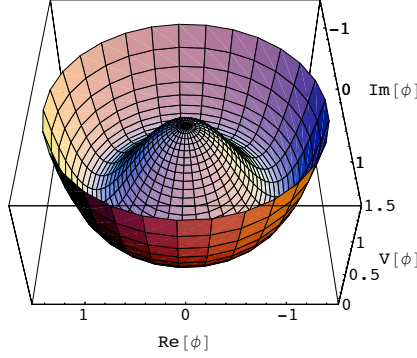


Figure 1.1 – The scalar potential $V(\phi)$ in (1.2.1) with $\lambda = \eta = 1$.

where $\Lambda(x^\mu)$ is a function of the point on space-time. The covariant derivative is defined so that it has the same transformation rule as the Higgs $D_\mu \phi \rightarrow e^{ig\Lambda} D_\mu \phi$.

The energy functional of the Abelian-Higgs model is given by

$$E = \int dx_3 [|D_t \phi|^2 + |D_i \phi|^2 + \frac{1}{2} B^2 + \frac{1}{2} E^2 + \lambda (|\phi|^2 - \eta^2)^2] \geq 0, \quad (1.2.5)$$

where the magnetic and electric field are defined by $B = \partial_1 A_2 - \partial_2 A_1$ and $E_i = \partial_t A_i - \partial_i A_t$ respectively. The ground state of the system is defined by a homogenous configuration with $A_\mu = 0$ and the Higgs fixed at one of the minima of the potential, $|\phi_0|^2 = \eta^2$, such that $V(\phi_0) = 0$. Note that this configuration is stable since it has zero energy and the energy functional is positive definite. The set of minima of the potential can be parametrized as $\phi_0 = \eta e^{i\alpha}$, thus they transform non-trivially under the U(1) symmetry:

$$\phi_0 = \eta e^{i\alpha} \rightarrow \eta e^{i(\alpha + g\Lambda)}. \quad (1.2.6)$$

In other words, when the Higgs field is stabilized at any of these points, the gauge symmetry is spontaneously broken. The fluctuations around the background can be parametrized as:

$$\phi = (\eta + \rho) e^{i\alpha}, \quad A_\mu = \frac{1}{g} \partial_\mu \alpha + a_\mu, \quad (1.2.7)$$

where $\alpha(x^\mu)$ is now a space-time dependent. We can use the gauge freedom to make ϕ real everywhere² (unitary gauge), so that $\phi = \eta + \rho$. Using this parametrization, and after expanding the lagrangian to second order in the fluctuations around the vacuum, it reads:

$$\mathcal{L} = -\partial_\mu \rho \partial^\mu \rho - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} - g^2 \eta^2 a_\mu a^\mu + 4\lambda \eta^2 \rho^2 + \mathcal{O}(3), \quad (1.2.8)$$

²Actually, in the presence of topological defects this is not possible. We discuss topological defects in a later section.

where $f_{\mu\nu}$ is the field strength corresponding to the fluctuation of the gauge field a_μ . Note that the condensation of the Higgs field has led to the appearance of a mass term in the lagrangian for the gauge boson. Thus, in the spontaneously broken theory, the spectrum of fluctuations consists of a real scalar field and a gauge boson with masses given by $m_s = 2\sqrt{\lambda}\eta$, and $m_v = \sqrt{2}g\eta$ respectively. The process where the spontaneous breaking of a symmetry leads to the appearance of mass terms for the gauge boson is known as the *Higgs mechanism*.

The Abelian-Higgs model is the relativistic version of the Landau-Ginzburg model describing a superconductor. Here the complex scalar field is what would play the role of the condensate of Cooper pairs, and A_μ would be the usual vector potential from electro-magnetism. The fact that the vector boson becomes massive reflects in the inability of the magnetic field to penetrate the condensate, this is the so called *Meissner effect*.

The Higgs mechanism acts in a similar way in the case of Yang-Mills theories with a non-abelian gauge group, such as $SU(N)$, $N \geq 2$. In that case only the gauge fields associated to the broken generators of the symmetry group acquire a mass, while the ones associated to the unbroken subgroup remain massless.

1.2.3 The Standard Model and Grand Unification Theories.

The three fundamental forces of the Standard Model can be described by a Yang-Mills theory based on the gauge group $SU(3) \times SU(2) \times U(1)$. In particular the strong interaction is described by the $SU(3)$ factor, and the electro-magnetic and weak interactions are given in terms of the unified theory $SU(2) \times U(1)$, the Weinberg-Salam model.

In spite of being formulated as a unified theory, the electromagnetic and weak interactions have very different properties at low energies. For instance, while the electromagnetic interaction is mediated by a massless gauge boson, the photon, the mediators of the weak force (the W and Z bosons) are massive, and thus the interaction is short range. Therefore, at low energies, the symmetry group $SU(2) \times U(1)$ has to be broken to $U_{em}(1)$, the gauge group of the electromagnetic interactions. The Standard Model contains a spin-0 field, the Higgs, which transforms non-trivially under the electroweak symmetry group. At low energies the Higgs is stabilized at a configuration that minimizes its energy functional, and which breaks the electroweak symmetry group to its subgroup $U_{em}(1)$.

After t'Hooft and Veltman [15] proved the renormalizability of the Weinberg-Salam model in the early 1970's, the idea of unifying all fundamental forces under a single symmetry principle became one of the main quests in theoretical

1.2. High energy physics.

physics. The basic idea behind Grand Unification Theories is that the Standard Model can be formulated as the broken phase of a theory with a larger symmetry group. Before the phase transition that led to the breaking of the unified group there would be no difference between the fundamental interactions. Only after the transition some of the gauge bosons become massive, and the corresponding interactions short range, destroying the symmetry between the various forces.

The Standard Model involves three coupling constants, g_3 , g_2 and g , one for each of the factors of its gauge symmetry group, SU(3), SU(2) and U(1) respectively. These couplings depend logarithmically on the energy. For instance, while g_3 and g_2 decrease for growing energies, g becomes larger. Grand Unified Theories based on simple groups, such as SU(5), involve a single coupling constant. The energy scale where the GUT phase transition takes place can be estimated requiring the three couplings of the Standard Model to be roughly equal. Due to the logarithmic dependence of gauge couplings on energy the obtained symmetry breaking scale (*GUT scale*) is extremely high: $10^{15} - 10^{16}$ GeV.

1.2.4 Supersymmetry and supergravity.

When trying to embed the Standard Model into a Grand Unified Theory we have to confront an important conceptual puzzle, the so called *hierarchy problem*. The hierarchy problem consists of explaining the smallness of the Higgs mass, as compared with the GUT scale.

In the Standard Model the Higgs mass has an upper bound implied by the unitarity of the theory³, $M_H < 10^3$ GeV [17]. However, the mass terms of scalar fields generically receive renormalization corrections which are quadratically divergent $\delta M_H^2 \sim g^2 \Lambda^2$, with g a gauge coupling, and Λ a cutoff scale. When the standard model is embedded in a Grand Unified Theory, the appropriate value for Λ should be of order of the GUT scale or larger $\Lambda \geq 10^{15}$ GeV. Therefore, for the Higgs to have a mass of order of the electroweak scale, the bare mass squared should be of order of $-\Lambda^2$, and cancel the radiative correction with extreme accuracy.

This argument implies that it is not possible to have a large hierarchy of spontaneous symmetry breaking scales without introducing a severe fine tuning of the parameters of the theory. An alternative, more natural, solution to fine tuning would be to invoke a symmetry that protects the Higgs mass from receiving corrections at all orders in perturbation theory. A prominent proposal

³There are studies which suggest stronger bounds on the Higgs mass, $M_H < 225$ GeV [13, 16]. At the time of writing, the Higgs has not been found yet by the LHC but the experiments have bounded its mass to be in the range 116-130 GeV, and there are hints suggesting that the most likely value is around 126 GeV.

for such a symmetry is *supersymmetry* (see [18]).

Supersymmetry is a symmetry that transforms particles of different spin into each other, in particular bosons into fermions and vice versa. In theories invariant under supersymmetry every boson must have a fermionic partner with equal mass. Moreover the mass terms of scalar fields are no longer quadratically divergent, since the contribution from fermions and bosons to the radiative corrections have the same magnitude but opposite sign, leading to an exact cancellation.

The particles in supersymmetric theories are arranged in *supermultiplets*, which are the irreducible representations of the supersymmetry algebra. Each supermultiplet always contains the same number of bosonic and fermionic degrees of freedom, and moreover, all the particles within a supermultiplet have the same mass. The field content of each supermultiplet depends on the number of supersymmetry generators, which is denoted by $\mathcal{N}$. The theories invariant under the action of more than one supersymmetry generator $\mathcal{N} \geq 2$ are called *extended supersymmetry theories*.

The known elementary particles cannot be fitted in supermultiplets, thus supersymmetry cannot be an exact symmetry of nature. An appealing idea is to consider that supersymmetry is only spontaneously broken at low energies. As long as the supersymmetry breaking scale is around 1 TeV, supersymmetry still provides a solution for the hierarchy problem. Interestingly, an analysis of the running of the gauge couplings seems to favor this possibility. In the simplest cases, when the GUT symmetry group breaks down to the Standard Model via a single phase transition, the couplings do not quite meet. However, in the supersymmetric extension of these models, the matching of the gauge couplings at the GUT scale is substantially improved.

Although no evidence of supersymmetry has been found so far in the Large Hadron Collider (LHC), nowadays some of the most prominent candidates to explain the physics above the electroweak scale are supersymmetric theories, such as the Minimal Supersymmetric Standard Model (MSSM), supersymmetric GUTs, and superstrings. These theories provide a framework to investigate important cosmological problems such as the origin of inflation, dark matter or the present day accelerated expansion.

The physical phenomena that occurred during the first stages of the universe involve enormous energies, in particular in the case of inflation they might be as high as 10^{15} GeV. This energy scale is rather close to the Planck scale $M_p \sim 10^{18}$ GeV, where the strength of gravitational interactions becomes comparable to the other three fundamental forces. This suggests that at these energy scales gravity should be treated on an equal footing with the rest of the interactions,

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$\mathcal{N} = 1$ Supersymmetry	$\mathcal{N} = 2$ Supersymmetry
Chiral multiplet $(0, \frac{1}{2})$	Hypermultiplet $(0, \frac{1}{2}, \frac{1}{2}, 0)$
Vector multiplet $(\frac{1}{2}, 1)$	Vector multiplet $(0, \frac{1}{2}, \frac{1}{2}, 1)$
Gravity multiplet $(\frac{3}{2}, 2)$	Gravity multiplet $(1, \frac{3}{2}, \frac{3}{2}, 2)$

Table 1.1 – Massless supermultiplets of globally and locally supersymmetric models with $\mathcal{N} = 1$ and $\mathcal{N} = 2$ in 4 space-time dimensions. We denote the supermultiplets by the helicity s of the particles they involve: $(s_0, s_0 + \frac{1}{2})$ for $\mathcal{N} = 1$, and $(s_0, s_0 + \frac{1}{2}, s_0 + \frac{1}{2}, s_0 + 1)$ for $\mathcal{N} = 2$ supersymmetry.

and in particular should be treated as a quantum theory. Similarly to the other three interactions, gravity can be described as a gauge theory, where the symmetries that are gauged are space-time symmetries. The corresponding gauge group is the Poincaré group, which involves space-time translations, rotations and boosts. In contrast with gauged internal symmetries, the gauge boson is a spin-2 field, the *graviton*, which represents the gravitational field itself.

If all four fundamental interactions are to be treated in the same way, then in supersymmetric theories we should require the gravitational interaction itself to be invariant under supersymmetry. Such theories are called *supergravity theories* (see [18]). Supergravity is the *local version of supersymmetry*, and is obtained by allowing the supersymmetry transformations to vary from point to point in space-time. Actually, supersymmetry is intermediate between internal and space-time symmetries, and any theory invariant under local supersymmetry also contains gravity. Indeed, the supersymmetry algebra contains the Poincaré algebra, and therefore by promoting global supersymmetry to local supersymmetry we are also gauging the Poincaré group. The gauge particle associated to local supersymmetry is the *gravitino*, the spin 3/2 partner of the

graviton. In supergravity theories the number of gravitinos is equal to the number of the supersymmetry generators, $\mathcal{N}$. The different types of massless supermultiplets of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry are summarized in table 1.1.

Supergravity plays an important role in the construction of cosmological models, since it describes the low energy regime of the most prominent theory of quantum gravity to date, *String Theory*.

1.2.5 String theory and the integration of heavy moduli.

In contrast with conventional quantum field theory, where the elementary particles are mathematical points, the fundamental entities in string theory are one-dimensional extended objects, namely *the strings* (see [19]). The basic idea is that the elementary particles of the Standard Model would arise as oscillation modes of these fundamental strings. An appealing feature of string theory is that it involves a single fundamental parameter, the characteristic length of the strings l_s , or the corresponding energy scale $E_s = 1/l_s$. Any other quantity, such as the strength of the interactions between strings, is field dependent.

Another remarkable property of superstring theories is that they predict the number of space-time dimensions. Indeed, for these theories to be consistent, the fundamental strings must live in a 10- or 11-dimensional space-time. So far there is no experimental evidence of extra dimensions, thus in order for string theory to describe the 4-dimensional world we experience, we must invoke a mechanism to hide the 6 or 7 extra dimensions. The most accepted solution is that the extra dimensions are *compactified* on some internal manifold, so small that it remains invisible even at the largest energy scales, (the shortest length scales), tested with current accelerators.

Simple compactifications can be characterized by a single length scale⁴, l_c , or its associated energy scale $E_c = 1/l_c$. Many results in string theory have been derived in the limit $E_c \ll E_s$, because, in this regime, the physics at the energy scales $E \ll E_s$ admit an effective description in terms of 10 or 11 dimensional supergravity. However, given the absence of evidence of extra dimensions, most of the proposed cosmological models assume the extra condition $E \ll E_c$, so that they can be described in the framework of 4-dimensional supergravity.

A common problem that has to be treated in all cosmological models derived from string compactifications is the presence of a large number of massless scalar fields, for which so far there is no observational evidence. In particular, this set of fields involves the *moduli fields*, which characterize the size and shape

⁴Such a characteristic length scale might be related, for example, to the volume of the internal manifold $\mathcal{V}_n \sim l_c^n$.

1.3. Cosmic strings and other topological defects.

of the extra dimensions. The mechanisms to generate a potential that can stabilize the moduli have only been developed recently [20, 21]. This requires compactifying string theory in a new type of backgrounds, *flux compactifications*, which require the presence of D-branes⁵ wrapping the internal dimensions, and certain generalizations of the magnetic flux sourced by the branes.

In flux compactifications some of the moduli are stabilized at a high energy scale and decouple from the low energy effective theory. On the other hand, as we have mentioned before, it is phenomenologically appealing that supersymmetry remains unbroken at scales as low as 1 TeV, therefore the stabilization of these heavy fields has to leave supersymmetry unbroken. However, integrating out heavy scalars in such a way that the effective theory is still invariant under supersymmetry is not a trivial problem, the couplings between the light and heavy sectors have to satisfy certain conditions. Thus, it is important to characterize the type of couplings between the two sectors that allow for the supersymmetric integration of the heavy fields. This problem will be discussed in detail in chapter 3.

There are two situations where details of the decoupling of heavy moduli become important. The first one is in inflationary models derived from supergravity or superstrings with moduli stabilization. In these models the inflaton belongs to the light sector of the theory, and the slow roll conditions can be easily spoiled by its couplings to the heavy fields [22]. Thus the predictions of these models can only be trusted after having a precise characterization of the interactions between the inflationary and heavy sectors [23] (see also [24]). The second situation is supersymmetry breaking. Since supersymmetry is not an exact symmetry of nature, any realistic cosmological model should involve the mechanism for supersymmetry breaking. The breaking of supersymmetry in the light sector will affect the stability properties of the heavy fields that were integrated out. In particular, if the heavy fields become unstable due to the supersymmetry breaking effects, the integration of heavy moduli no longer makes sense. This is especially relevant in string theory models describing late time accelerated expansion of the universe. We will come back to this problem in chapter 4.

1.3 Cosmic strings and other topological defects.

We continue this introductory chapter with a discussion about the formation of topological defects in the early universe and their cosmological implications, paying particular attention to cosmic strings.

⁵A D-brane is a hypersurface where open strings can end, which can be extended along several of the ten or eleven space-time dimensions.

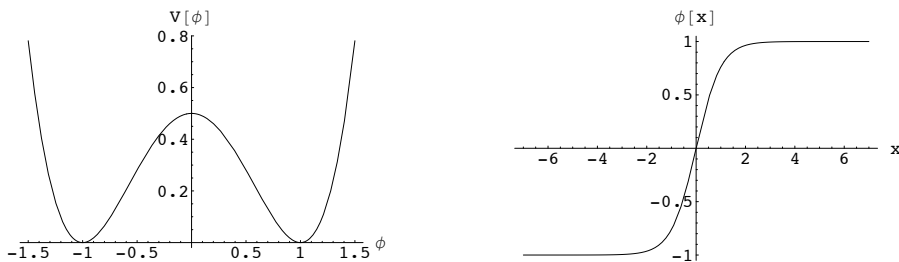


Figure 1.2 – The scalar potential $V(\phi)$ in (1.3.2) (left) with $\lambda = 2$, $\eta = 1$ leads to the kink solution (1.3.4) $\phi(x)$ on the right.

As we have mentioned in the previous sections, phase transitions are a fundamental ingredient of many interesting cosmological scenarios, such as those based on Grand Unified Theories (GUT). An important example are *hybrid inflationary* models, where the exponential expansion ends in a natural way as the inflaton reaches a point of the potential with a tachyonic instability leading to a phase transition. Like in the more familiar condensed matter systems, phase transitions in the early universe also lead to the formation of localized, non-dissipative objects called *topological defects* (see [25, 26]). In particular, it has been proven that cosmological models based on Supersymmetric Grand Unified Theories generically predict the formation of cosmic strings at the end of inflation [27].

In this section we will introduce topological defects without considering the gravitational effects. In a theory without gravity, where the energy density is given by the T_0^0 component of the energy-momentum tensor (see [25]), the energy is bounded below $T_0^0 \geq 0$ and the ground states satisfy $T_0^0 = 0$. We understand by *dissipative solutions* of the classical equations of motion those satisfying:

$$\lim_{t \rightarrow \infty} T_0^0(x^i, t) = 0, \quad (1.3.1)$$

at every point of space x^i .

The simplest example of a topological defect is the kink in 1 + 1 dimensions. It is possible to construct a model that exhibits kink solutions with a single real scalar field ϕ . The corresponding action is given by:

$$\mathcal{S} = \int dt dx \left[-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right], \quad (1.3.2)$$

where the potential is given by $V(\phi) = \frac{1}{4} \lambda (\phi^2 - \eta^2)^2$. The energy functional for this action reads:

$$E = \int dx \left[\frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} (\partial_x \phi)^2 + \frac{1}{4} \lambda (\phi^2 - \eta^2)^2 \right], \quad (1.3.3)$$

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In order for the solution to have a finite energy, the field configuration ϕ describing the kink must satisfy $\phi^2 = \eta^2$ at $x \rightarrow \pm\infty$, it must be in one of the ground states of the potential $V(\phi_0) = 0$. The set of ground states of a given theory is called the *vacuum manifold* $\mathcal{V}$, and in this case it is given by the homogeneous configurations $\phi_0 = \pm\eta$. Suppose now that we choose the boundary conditions such that $\phi(-\infty) = -\eta$ and $\phi(\infty) = \eta$, then by continuity this field configuration must necessarily have a zero for some value $x = x_0 \in \mathbb{R}$. This configuration has a nonzero energy, which is expected to be concentrated around $x = x_0$, where the scalar field is out of the vacuum. The kink solution to the classical equations of motion is given by:

$$\phi(x) = \eta \tanh \left(\sqrt{\frac{\lambda}{2}} \eta (x - x_0) \right), \quad (1.3.4)$$

which is plotted in figure 1.2. This field configuration is non-dissipative in the sense that no classical time evolution can make this object decay into the vacuum $\phi = \pm\eta$. The reason is simple: for the energy to be finite, the value of ϕ at spatial infinity must remain in the vacuum manifold at all times. Since the vacuum manifold consists of two disconnected pieces, and time evolution is continuous, it is not possible for the scalar field to evolve to a constant field configuration $\phi = \pm\eta$.

This model provides an illustrative example of spontaneous breaking of a symmetry. Its lagrangian has a $\mathbb{Z}_2$ symmetry that exchanges ϕ and $-\phi$. When the Higgs condenses and chooses one of the vacua $\phi_0 = \pm\eta$, the $\mathbb{Z}_2$ symmetry, which transforms the two vacua into each other, is spontaneously broken. In general, the vacuum manifold of a theory has a close relation with its underlying symmetry group G . Actually, since the scalar potential is invariant under the symmetry group, if ϕ_0 is a zero of the potential so is $g\phi_0$ for any transformation $g \in G$. Moreover, in the absence of accidental degeneracies, the vacuum manifold $\mathcal{V}$ can be identified with the coset space G/H , where H is the unbroken subgroup. In the present case $G = \mathbb{Z}_2$ is completely broken ($H = \{1\}$), and therefore the vacuum manifold is isomorphic to the symmetry group itself $\mathcal{V} \cong \mathbb{Z}_2$. This discussion suggests that in the case of spontaneously broken GUT theories the vacuum manifold can be highly non-trivial.

In the following two sections we will review the basic features of vortex solutions in non-supersymmetric theories. We will discuss two field theory models: the global U(1) model, and the Abelian-Higgs model. For a review see [26]. We will finish this chapter commenting shortly on supersymmetric vortices, leaving the main discussion for the next chapter, after we have introduced $\mathcal{N} = 1$ supergravity.

1.3.1 Global vortices.

The global U(1) model is the simplest field theory that exhibits cosmic string solutions. It describes the dynamics of a single complex scalar field ϕ evolving in a scalar potential with a mexican hat form, fig. 1.3. The action reads:

$$\mathcal{S} = \int dx_4 [-\partial_\mu \phi \partial^\mu \bar{\phi} - V(\phi)], \quad V = \lambda(|\phi|^2 - \eta^2)^2, \quad (1.3.5)$$

and its corresponding equation of motion is given by:

$$[\partial_\mu \partial^\mu - 2\lambda(|\phi|^2 - \eta^2)]\phi = 0. \quad (1.3.6)$$

This model is invariant under a global U(1) symmetry defined by the transformations:

$$\phi \rightarrow \phi e^{ig\Lambda}, \quad (1.3.7)$$

where Λ is a real constant. The energy functional associated to our action is:

$$E = \int dx_3 [|\partial_t \phi|^2 + |\partial_i \bar{\phi}|^2 + \lambda(|\phi|^2 - \eta^2)^2]. \quad (1.3.8)$$

As in the Abelian-Higgs model, this theory has a stable set of vacua given by all the configurations satisfying $|\phi| = \eta^2$, and therefore the vacuum manifold is isomorphic to a circle:

$$\mathcal{V} = \{\phi \in \mathbb{C} / |\phi|^2 - \eta^2 = 0\} \cong S^1. \quad (1.3.9)$$

Since these vacua are not invariant under the U(1) transformations (1.2.6), the global symmetry is said to be spontaneously broken down to the trivial group $\{1\}$. The mass spectrum around the vacuum consists of two types of particles, one particle has a mass $m_s = \sqrt{2}\lambda\eta$, and the other is massless, the *goldstone boson*. The original U(1) symmetry of the action prevents the goldstone boson from having a mass, since this particle corresponds to the space dependent fluctuations of the phase of the scalar field.

This model admits cosmic string solutions, which are also known as *global vortices*. We will now discuss static cylindrically symmetric cosmic solutions laying along the z-axes. An appropriate ansatz for this configuration is given by:

$$\phi = \eta f(r) e^{in\theta}, \quad (1.3.10)$$

where we have used cylindrical coordinates $\{r, \theta, z\}$. In order for the string configuration to have finite energy, the scalar field ϕ must approach the vacuum manifold asymptotically at spatial infinity, and thus the profile function $f(r)$ must satisfy the boundary condition $f(r \rightarrow \infty) = 1$. Therefore at spatial infinity the field configuration is

$$\phi = e^{in\theta}, \quad (1.3.11)$$

1.3. Cosmic strings and other topological defects.

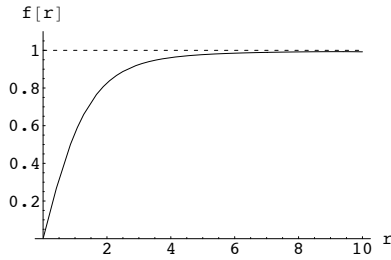


Figure 1.3 – Profile function (1.3.10) $f(r) = |\phi(r)|$ characterizing the field configuration of a global string for $n = \lambda = \eta = 1$.

which represents a mapping between the the circle at infinity in real space and a circle in field space, the vacuum manifold. The parameter n , called the *winding number*, is an integer which counts the number of times the phase of the scalar field winds around the vacuum manifold when we go once around the circle in field space.

Since the field ϕ has to be single valued we have to require the profile function $f(r)$ to vanish at the origin $r = 0$, thus ϕ leaves the vacuum manifold which implies that the field configuration has nonzero energy. Following the same argument we used in the case of the kink, we can see that there is no continuous way to deform the configuration 1.3.11 into a homogeneous solution of the form $\phi = e^{i\alpha}$, and in particular, since time evolution is continuous, the cosmic string solution cannot decay into the vacuum.

Introducing the ansatz (1.3.10) into the equation of motion we obtain a second order ordinary differential equation for the profile function $f(r)$:

$$f'' + \frac{1}{r^2}f' - \frac{n^2}{r^2}f + 2\lambda\eta^2(1 - f^2)f = 0. \quad (1.3.12)$$

The numerical solution of this equation is displayed in fig 2. The profile function $f(r)$ has the following asymptotic behavior:

$$f(r) \approx C_n r^n \quad \text{for } r \rightarrow 0, \quad (1.3.13)$$

$$f(r) \approx 1 - \frac{n^2}{4\lambda\eta^2 r^2} \quad \text{for } r \rightarrow \infty. \quad (1.3.14)$$

This indicates that the string core has a width of the order of the Compton wavelength of the massive scalar $\delta \sim 1/(\sqrt{2\lambda}\eta) = m_s^{-1}$; for r equal to a few δ the Higgs is approximately in the vacuum manifold $|\phi| \approx \eta$ (see Fig. 1.3).

Strings of this type have an infinite energy per unit length. This can be checked inserting the asymptotic value of the Higgs, $\phi(r \rightarrow \infty) = \eta e^{in\theta}$, in the

energy functional (1.3.8). Including a small contribution from the core we obtain the following energy per unit length μ :

$$\mu \approx \eta^2 + 2\pi n^2 \eta^2 \int_{\delta}^R \frac{dr}{r} \approx 2\pi n^2 \eta^2 \log R/\delta, \quad (1.3.15)$$

which is logarithmically divergent for large values of the cutoff R . In a cosmological context this cutoff would be given by the distance to the nearest string with the opposite winding number. A characteristic property of these strings is that they interact with long range forces, which decay as R^{-1} with the distance R between them. Actually, due to the repulsion between the strings, those with winding number $|n| > 1$ are unstable to decay into single $|n| = 1$ strings, which are completely stable [28].

1.3.2 Abrikosov-Nielsen-Olesen vortex.

Let us discuss again the Abelian-Higgs model presented in section 1.2.2. In this case the internal $U(1)$ symmetry has been promoted to a local one. In our previous discussion of the Abelian-Higgs model we mentioned that the ground states of the system, up to gauge transformations, were given by the configurations satisfying $A_\mu = 0$ and $\phi = \eta e^{i\alpha}$, with α some real constant. Thus, as in the global $U(1)$ model, the vacuum manifold is isomorphic to S^1 , implying the existence of cosmic string solutions.

The cosmic strings in the Abelian-Higgs model have different properties from those in the global $U(1)$ model. In particular it is possible to find solutions with finite energy per unit length. For the energy to be finite we have to require, as in the case of global vortices, the Higgs to lay in the vacuum manifold at spatial infinity:

$$\lim_{|\vec{x}| \rightarrow \infty} |\phi|^2 = \eta^2. \quad (1.3.16)$$

Moreover, the covariant derivatives $D_\mu \phi$ must vanish asymptotically. In order to achieve this the gauge fields have to satisfy the condition:

$$\lim_{|\vec{x}| \rightarrow \infty} A_\mu = \frac{1}{ig} \partial_\mu \log \phi, \quad (1.3.17)$$

which also implies that the field strength $F_{\mu\nu}$ vanishes at spatial infinity, since $[D_\mu, D_\nu]\phi = -igF_{\mu\nu}\phi$.

As in the previous section we now consider static cylindrically symmetric cosmic string solutions along the z -axis. We will use in cylindrical coordinates $\{r, \theta, z\}$, and we will impose the gauge $A_0 = A_r = 0$. A cosmic string with a winding number n can be described by the ansatz:

$$\phi = \eta f(r) e^{in\theta} \quad A_\theta = \frac{n}{g} v(r). \quad (1.3.18)$$

1.3. Cosmic strings and other topological defects.

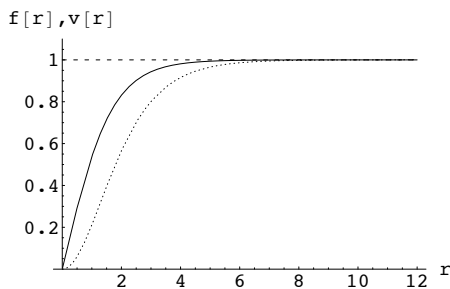


Figure 1.4 – Profile functions (1.3.18) $f(r)$ (solid line) and $v(r)$ (dotted line) characterizing the field configuration of a local string saturating the BPS bound, $2\lambda = g^2$, with $n = g = 2\eta = 1$.

where $f(r)$ and $v(r)$ are real functions satisfying the following boundary conditions:

$$f(r), v(r) \rightarrow 1 \quad \text{for } r \rightarrow \infty, \quad \text{and} \quad f(r), v(r) \rightarrow 0 \quad \text{for } r \rightarrow 0. \quad (1.3.19)$$

Introducing this ansatz in the equations of motion (1.2.3) we find the equations for the profile functions $f(r)$ and $v(r)$:

$$f''(r) + \frac{1}{r}f'(r) - \frac{n^2}{r^2}[1 - v(r)]^2 + 2\lambda\eta^2(1 - f(r)^2)f(r) = 0, \quad (1.3.20)$$

$$v''(r) - \frac{1}{r}v'(r) + g^2\eta^2[1 - v(r)] = 0. \quad (1.3.21)$$

This type of vortices have a tube of magnetic flux in their core which is quantized by the winding number n . Integrating the magnetic flux over the whole plane perpendicular to the string gives:

$$\Phi_m = \int_{R^2} dx^2 B = \int_{S_\infty^1} d\vec{l} \vec{A} = \frac{2\pi n}{g}. \quad (1.3.22)$$

Here we have applied Stokes' theorem with S_∞^1 being the circle at spatial infinity containing the string, and used the asymptotic behavior of the gauge field and the Higgs. The energy per unit length of these strings is finite and has a dependence on the parameters of the form [29]:

$$\mu = \pi\eta^2\epsilon(2\lambda/g^2). \quad (1.3.23)$$

This result can be obtained analytically for the particular choice of couplings $2\lambda = g$, which is called the *BPS limit* (Bogomonlyi-Prasad-Sommerfield), borrowing the name from magnetic monopoles [30], [31]. The energy per unit length for a static cosmic string parallel to the z -axis can be written in the following

way:

$$\begin{aligned} \mu &= \int dx^2 \left[|(D_1 \pm iD_2)\phi|^2 + \frac{1}{2}[B \pm g(|\phi|^2 - \eta^2)]^2 + \frac{1}{2}(2\lambda - g^2)(|\phi|^2 - \eta^2)^2 \right] \\ &\quad \pm g\eta^2 \int d_2x B \geq 0. \end{aligned} \quad (1.3.24)$$

This expression can be obtained from (1.2.5), using the identities:

$$|(D_1 \pm iD_2)\phi|^2 = |\vec{D}\phi|^2 \mp i\bar{\phi}[D_1, D_2]\phi \pm \vec{\nabla} \wedge \vec{J}, \quad (1.3.25)$$

$$[D_1, D_2]\phi = -igB\phi, \quad (1.3.26)$$

and discarding the boundary term associated to the curl of the current $\vec{J} = i\bar{\phi}\vec{\nabla}\phi$. Note that in the limit $2\lambda = g^2$ the first line in (1.3.24) is a sum of positive definite terms, and thus the energy per unit length μ is bounded below⁶:

$$\mu \geq \pm g\eta^2 \int d_2x B = 2\pi\eta^2|n|. \quad (1.3.27)$$

Any field configuration saturating this bound must also be a solution to the static equations of motion, since it extremizes the energy functional. In order to saturate the bound the cosmic string background must satisfy the following first order differential equations:

$$(D_1 \pm iD_2)\phi = 0, \quad B \pm g(|\phi|^2 - \eta^2) = 0, \quad (1.3.28)$$

which are known as the BPS equations. Inserting the ansatz for a cylindrically symmetric straight string (1.3.18) we obtain the following system of equations for the profile functions:

$$f'(r) + \frac{|n|}{r}(v(r) - 1)f(r) = 0 \quad |n|v'(r) + g^2\eta^2r(f(r)^2 - 1) = 0. \quad (1.3.29)$$

These equations only admit solutions with the correct boundary conditions (1.3.19) provided we choose the signs so that $n = \pm|n|$. The functions $f(r)$ and $v(r)$ which solve this set of equations are represented in figure 1.4. These cosmic strings have an energy per unit length $\mu = \pi\eta^2|n|$, which satisfies the expression (1.3.23) with $\epsilon = 1$.

For generic values of the couplings, λ and g , parallel cosmic strings interact with short-range forces. For instance, if $2\lambda > g^2$ the cosmic strings repel, and if $2\lambda < g^2$ they attract [30]. When the Abelian-Higgs model satisfies the BPS limit $2\lambda = g^2$, it is possible to find static multivortex solutions of the equations of motion [32]. In consequence, string configurations with windings $|n| > 1$ are only stable provided that $2\lambda \leq g^2$, otherwise the strings decay into n stable vortices with a unit of magnetic flux [28]. The vortex solutions of the AH model satisfying the BPS limit are especially interesting because their low energy dynamics can be studied with high accuracy using the moduli space approximation [33, 34, 35].

⁶In order for the energy per unit length to be positive definite, as required by (1.2.5), we must choose the signs so that $n = \pm|n|$.

1.4 Topological defects in cosmology.

1.4.1 Defect formation.

The process of formation of topological defects in cosmology is known as the *Kibble mechanism* [36] (for a review see [26]). As the universe undergoes a phase transition, in general the Higgs will condense into different vacua at different points in space time. For example in the model we just discussed, as the two vacua $\phi = \pm\eta$ are completely equivalent from the point of view of the equations of motion, the Higgs could condense into any of them with equal probability. This leads to the formation of separate domains, or regions of the universe where the Higgs has the same value. As the universe cools down, these domains expand and eventually coalesce. In the boundary of the domains the scalar field interpolates between different vacua, as occurs in the previous example. These field configurations interpolating between different vacua at spatial infinity are the defects. The formation of these domains is unavoidable, because the vacuum where the Higgs condenses can only be correlated over finite distances smaller than the size of the horizon (see section 1.1), and therefore it will be different at points of space-time which are not in causal contact.

The type of topological defects that are formed during a phase transition is determined by the topology of the vacuum manifold. The value of the Higgs in a m -sphere S^m at spatial infinity defines a continuous map h between S^m and $\mathcal{V}$:

$$h : S^m \longrightarrow \mathcal{V}. \quad (1.4.1)$$

Since the Higgs has to remain in the vacuum manifold at spatial infinity at all times, and time evolution is a continuous operation, then it is clear that the existence of stable defects can be determined studying the set of continuous deformations of the map h . In particular, non-dissipative solutions are associated with non-contractible m -spheres in the vacuum manifold. Saying that the image of the map h cannot be contracted to a point, is the same as stating that the field cannot evolve to a homogeneous configuration. Note that if the field configuration "wraps around" one of these non-contractible m -spheres of the vacuum manifold, then continuity implies that somewhere in real space the Higgs must leave the vacuum manifold restoring the old (symmetric) phase. As in the case of the kink, the energy density is expected to peak in the region of space where the Higgs leaves the vacuum. These concentrations of energy signal the positions of the defects.

The non-contractible m -spheres of the vacuum manifold are classified by the elements of the m -th homotopy group⁷ $\pi_m(\mathcal{V})$. In particular, if the vacuum manifold is disconnected, $\pi_0(\mathcal{V}) \neq \mathbb{1}$, sheet-like structures are formed: topological *domain walls*. When the first homotopy group of the vacuum manifold is

⁷The set $\pi_m(\mathcal{V})$ has a group structure only for $m \geq 1$.

non-trivial then one-dimensional defects form during the transition: topological *cosmic strings*. If $\pi_2(\mathcal{V})$ is non trivial then the phase transition will lead to the formation of point defects, topological *monopoles*.

For example, in the case of vortices (both global and local), the gauge group $G = \text{U}(1)$ is completely broken by the vacuum, i.e. the unbroken subgroup $H = \mathbb{1}$. Therefore the vacuum manifold has a non-trivial first homotopy group $\pi_1(\text{U}(1)/\mathbb{1}) = \pi_1(S^1) = \mathbb{Z}$, implying that the action should admit cosmic string solutions labeled by an integer $n \in \mathbb{Z}$, which is precisely the winding number of the string.

1.4.2 Cosmological implications.

The formation of defects has many cosmological implications. In particular, in four dimensions, they contribute to the energy density of the universe, affecting its geometry and the evolution of the scale factor. They might source density perturbations in the CMB in addition to those seeded by inflation, and they can have an impact in the amount of observed dark matter. The predictions of cosmological models related to the formation of defects, when contrasted with astronomical data, can help to constrain their parameter space, or even rule them out completely.

Domain walls and magnetic monopoles are very constrained from the observations. Let's consider first the case of a domain wall network. Although the expansion of the universe tends to dilute the density of the network, at the same time these sheet-like defects are stretched. Since their surface energy density remains constant, the energy density of the network evolves with the scale factor as $\rho_{dw} \sim a^{-1}$ rapidly dominating any other contribution, which is in contradiction with the observations (1.1.6). The production of magnetic monopoles during the GUT transition leads to a similar problem. According to these theories magnetic monopoles would be created with high abundances and, moreover, since they are extremely massive, $m_{monop} \sim 10^{16}$ GeV, they would behave as non-relativistic matter $\rho_{monop} \sim a^{-3}$. Although, in principle, monopoles and anti-monopoles can annihilate, the process is not effective enough to reduce their abundance significantly. Therefore in a standard Big-Bang cosmological scenario their formation would lead to an overclosed universe, and would also disrupt the Big-Bang nucleosynthesis. As we saw in section 1.1, this discrepancy with the astronomical data can be cured invoking a period of inflation after the GUT transition. After inflation monopoles become extremely rare, with only one per Hubble volume, and do not have significant cosmological implications.

Local cosmic strings do not suffer from these problems. The main reason is that the string network has a very efficient energy loss mechanism, the produc-

1.5. Cosmic Strings.

tion of small loops⁸. The numerical simulations of cosmic string formation show that generically the networks consist of a small number of infinite strings (~ 10) crossing the Horizon, together with a scale invariant distribution of small loops. These loops are continuously formed during the self-intersections of the long strings. Due to the tension of the string, the loops oscillate and shrink emitting gravitational waves. The network has an additional energy loss mechanism, namely the emission of massive radiation. Full field theory simulations indicate that the later one is the main energy loss mechanism [37, 38], while Nambu-Goto simulations seem to favor loop production [39]. According to a recent work by Blanco-Pillado *et al.* [40] the origin of the discrepancies arises because most loops form in scales too large to fit in the simulation box⁹ of field theory calculations, while Nambu-Goto simulations do not describe the emission of massive radiation.

Eventually, no matter which is the main energy mechanism, the network reaches a scaling regime, where all its relevant length scales grow in proportion with the size of the Horizon. It follows that during the scaling regime the energy density of the network evolves as $\rho_{string} \sim \mu t^{-2}$, with μ the string tension, similarly to the critical density $\rho_c \sim G^{-1}t^{-2}$. Therefore their quotient remains a constant, which for GUT strings is of the order of $\Omega_{string} \sim G\mu \sim 10^{-6}$, and does not contradict the observations.

1.5 Cosmic Strings.

The formation of a cosmic string network after inflation is a common prediction of many promising cosmological models (see [41, 42, 43]). In particular it was shown in [27] that all supersymmetric GUT scenarios below a certain complexity, (based in gauge groups with rank $r \leq 8$), lead to the formation of cosmic strings. More recently, in the context of superstrings, string networks have been proven to appear at the end of brane inflation¹⁰, which is a particular realization of hybrid inflationary models in the context of superstrings.

Cosmic strings were originally proposed in the 80's as a possible candidate to explain the primordial perturbations observed in the CMB. Nowadays inflation, due to its remarkable prediction of the CMB spectrum, has been established as the main mechanism to seed the perturbations, but the CMB data is still compatible with a 4.8% contribution from cosmic strings¹¹ [44].

⁸Global string networks lose energy in a very efficient way too. Due to the coupling to the goldstone boson the strings can radiate their energy into massless particles.

⁹The size of the simulation box is limited by the computational resources available and the complexity of the problem.

¹⁰In brane inflationary models the brane positions along the compact directions play the role of the inflaton.

¹¹This bound was obtained in [44] using field theory simulations of the string network. Other works which use the Nambu-Goto approximation to simulate the network suggest a slightly

Cosmic strings could be detected in several ways. Despite being a subdominant component of the temperature power spectrum, it might be the main contribution to certain modes (*B-modes*) of its polarization power spectrum [46, 47, 48]. Since the energy of the strings eventually is radiated in form of gravitational waves, they also produce a gravitational wave background. This signature could be detected by measurements of pulsar timing [49], or directly with the LIGO and LISA gravitational wave detectors.

It is also possible for a cosmic string to be detected directly. The space-time surrounding an infinite straight string is deformed due to its tension, μ . Although the space-time background of the string is asymptotically flat, it has the topology of a cone. Physically this means that a circle of radius R centered in the string has a length $L = (2\pi - \Delta)R$, which is shorter than in a euclidean background $2\pi R$. The quantity $\Delta = 8\pi G\mu > 0$, known as the *deficit angle*, only depends on the tension of the string. As a result of the conical geometry around it, the cosmic strings act as a gravitational lens producing double images of the objects behind them, such as a galaxies [26]. Due to this lensing effect, strings moving respect to the CMB background would also produce step-like discontinuities in the CMB temperature [50].

Note that all these effects are gravitational, and thus they depend on the string tension μ , which only appears through the dimensionless combination $G\mu$. The current bound on this quantity coming from CMB measurements is of the order of $G\mu < 0.5 \times 10^{-6}$ [44, 45]. Although the measurements on pulsar timing imply a more severe constraint $G\mu < 2 \times 10^{-7}$ [51], the production of gravitational waves by the network is not completely understood yet, and thus this bound is less reliable than the previous one.

1.5.1 Supersymmetric cosmic strings.

During the last decade there has been a lot of interest in understanding cosmic string solutions in both globally and locally supersymmetric theories. This is partly motivated by the large number of cosmological scenarios based on supersymmetric theories, such as supersymmetric extensions of Grand Unified Theories and hybrid inflationary models derived from superstrings [52, 53] (see also [41, 42]). Topological defects arise naturally in most of these models, and among them, cosmic strings are the defects which are most consistent with astronomical observations, therefore it is interesting to search for cosmic string solutions in cosmological models based on supersymmetric theories.

more restrictive bounds [45].

1.5. Cosmic Strings.

Cosmic strings in supersymmetric theories can have interesting new properties, such as fermionic zero modes from partial supersymmetry breaking which are confined to the core of the string [54, 55]. These zero modes result in the string carrying a massless current, which can stabilize string loops (vortons). Vorton formation leads to severe constraints for the cosmological models, since they tend to contribute to the dark matter density causing a cosmological evolution which is in contradiction with observed astrophysical data [56]. Although most authors agree that these modes are not present in supergravity theories [57, 58, 59], there are studies which indicate that some zero modes survive the coupling to gravity [60].

The BPS limit discussed earlier has a close relationship with supersymmetry: when the coupling constants of the Abelian-Higgs model satisfy the BPS limit then the model can be identified as the bosonic sector of a supersymmetric theory, and the cosmic string solutions preserve half of the supersymmetries [54]. Actually, in a supersymmetric theory, solitons saturating a BPS-type of bound typically leave unbroken a fraction of the supersymmetries. These solitons constitute interesting probes of the high energy regime of the theory as they are often protected from quantum corrections by the unbroken supersymmetries.

The interest in *supersymmetric cosmic strings* solutions in $\mathcal{N} = 1$ supergravity models increased after the authors of [61] conjectured that they could represent the low energy manifestation of fundamental objects in string theory called *D-strings*, which are D-branes with one non-compact spatial dimension.¹² In [62] it was shown fundamental strings and D-branes of cosmic size could be formed after inflation. For example, this occurs in the brane anti-brane inflationary model [64, 65], which predicts the formation of a network of fundamental strings and D-strings at the end of inflation. In order to determine precisely the cosmological implications of such a network it is necessary to find effective field theory models that describe the formation and evolution of these objects, and the conjecture proposed in [61] provided a way to construct the corresponding low energy effective actions. If these defects have observable effects on the evolution of the early universe we could obtain information about superstrings and M-theory from cosmological data (see [42, 41]). Since [61] was published several string theory analyses have appeared in the literature that support the conjecture [66, 67, 68, 69, 70], however they are also indications of limitation of the conjecture¹³.

¹² For a review of cosmic strings in superstring theory, see [62, 49, 63].

¹³ The conjecture was based on the observation that *D-term* strings were the only BPS saturated strings available in $\mathcal{N} = 1$ supergravity. By now, other BPS strings have been obtained with different stability behaviors. For example semilocal strings [71, 66] and axionic *D-term* strings [59, 72] have a core radius that can vary in size, a property that is not generally expected for D-strings. Moreover, *D-term* strings are not expected to reproduce the scattering properties of D-strings [62, 49].

The study of supersymmetric cosmic strings has some advantages with respect to their non-supersymmetric counterparts. Usually cosmological models describe the evolution of a large number of particles, (both bosons and fermions), especially those based on superstring theories. Thus the full set of equations of motion can be extremely involved, making the search of cosmic string solutions particularly difficult. However, supersymmetric cosmic strings solutions are simpler to study than the rest, since they obey first order differential equations similar to (1.3.28), in contrast with regular cosmic strings solutions which are obtained solving second order differential equations such as (1.2.3), i.e. the full set of equations of motion. Some aspects of the dynamics of these supersymmetric vortices are also simpler to study than in the general case.

In general cosmological models based on supergravity are $\mathcal{N} = 1$ theories, such as the one discussed in [61], since they are chiral and therefore more suitable for phenomenology than higher $\mathcal{N}$ models. However, in order to understand better the connection between $\mathcal{N} = 1$ supergravity half-BPS solitonic solutions and superstring models it is useful to work in $\mathcal{N} = 2$ supergravity, since many string theory compactifications admit a low energy description in terms of $\mathcal{N} = 2$ supergravity in 4 space-time dimensions. The first known example of a half-BPS cosmic string in $\mathcal{N} = 2$ supergravity was constructed in [73], however the main purpose of this paper was to prove the possibility of obtaining this kind of solutions in $\mathcal{N} = 2$ supergravity, and thus it involved only the minimal necessary matter content. In chapter 6 we present a work which is intended to enlarge the family of $\mathcal{N} = 2$ supergravity models admitting half-BPS cosmic string solutions. The model we discuss there is related to many type-IIB superstring and heterotic string compactifications, and in particular to a well-known brane inflationary model, the D3-D7 model [74].

1.6 Overview

The chapters of this thesis can be arranged in two different groups according to the type of problem that is considered. Chapters from 2 to 5 are mainly concerned with the problem of consistent truncation of heavy moduli in $\mathcal{N} = 1$ supergravity theories and their stability after supersymmetry breaking in the surviving sector, and chapters 6 and 7, together with appendix B, are dedicated to the study of Cosmic Strings in extended $\mathcal{N} = 2$ supergravity and in $\mathcal{N} = 1$ globally supersymmetric models respectively.

Chapter 2.

In this chapter we will introduce the essential elements in $\mathcal{N} = 1$ supergravity theories. In section 2.2 we present the supermultiplets, the supersymmetry

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transformations, and we discuss the most important features of the action following the review in [75].

In section 2.4 we introduce the analytical technique used in chapter 4 to study the stability of the truncated scalar fields in a supersymmetric reduction of an $\mathcal{N} = 1$ supergravity model. As an application we derive the well known result that all supersymmetric critical points are perturbatively stable and, in addition, we find a relation between the curvatures of the scalar potential and the Kähler function at the critical point. Indeed we show that there is a one to one correspondence between the minima of the Kähler function and the supersymmetric AdS maxima of the scalar potential.

The chapter ends with section 2.5, which contains a short review of cosmic string solutions in $\mathcal{N} = 1$ supergravity theories based on the article [61]. The solutions discussed here leave unbroken the supersymmetries of the system, and provide an intermediate step between the Abrikosov-Nielsen-Olesen vortices and the cosmic string solutions discussed in the last chapter of this thesis.

Chapter 3.

An important problem present in cosmological models based on compactifications of superstring theories is the prediction of a large number of scalar fields, named moduli fields, which are not observed in nature. In flux compactifications a fraction of these fields are stabilized at a high energy scale while, for phenomenological reasons, it is assumed that the stabilization leaves supersymmetry unbroken (see section 1.2.5).

In supergravity it is not trivial to decouple a heavy sector of the theory from the low energy fields in a supersymmetric way. Indeed the interactions between the heavy fields, and those surviving the truncation have to satisfy certain constraints. This issue is especially relevant for inflationary models, where the slow roll conditions can be easily spoiled by the interactions between the inflaton and the heavy fields. Moreover, even when these couplings are consistent with an inflationary period, they might produce characteristic features in the CMB spectrum which would give valuable information about the high energy regime. Thus, it is important to have a precise characterization of the interactions between the inflationary and heavy sectors.

This chapter is dedicated to the problem of consistent supersymmetric decoupling of heavy scalars in supergravity theories. Here we will derive a set of necessary and sufficient conditions for truncating heavy fields in $\mathcal{N} = 1$ supergravity theories subject to explicit requirements. First, that the truncated fields should not be sourced due to the interactions with the surviving fields, and second the low energy effective action should be described by $\mathcal{N} = 1$

supergravity. These conditions can be expressed as constraints on the couplings between the truncated fields and those surviving in the reduced theory. After solving these constraints we will present the most general class of models which are compatible with the supersymmetric truncation of the heavy fields. To the best of our knowledge, this analysis has not been published yet.

In view of these results, in section 3.3 we discuss various approaches followed in the literature in order to truncate a heavy sector in supergravity models while preserving supersymmetry, and to obtain the corresponding low energy action. Finally in sections 3.5 and 3.6 we analyze some explicit examples which illustrate the differences between models which allow for the supersymmetric decoupling of the heavy sector, and those where such a decoupling is not possible. Part of the work presented in the last sections of this chapter can be found in the article [76].

Chapter 4.

Since supersymmetry is not a symmetry of nature, any viable cosmological model based on a supersymmetric theory should include a mechanism of supersymmetry breaking. Moreover, in $\mathcal{N} = 1$ supergravity, supersymmetric vacua can not have a positive cosmological constant, and thus they are not suitable to describe the present acceleration of the universe. Therefore, in supergravity models such as those studied in chapter 3, where part of the field content is truncated in a supersymmetric way, the sector surviving the reduction must necessarily break supersymmetry.

The decoupling conditions found in chapter 3 ensure that the truncated fields are fixed at a critical point of the scalar potential, but once supersymmetry is broken in the surviving sector there is nothing to guarantee that the critical point is a minimum of the scalar potential, and thus the perturbative stability of the truncated fields needs to be studied.

In chapter 4 we will discuss the case where supersymmetry is broken in the surviving sector due to the interactions among the chiral fields (F -term mechanism). We will make a detailed stability analysis for the simplest class of models satisfying the decoupling conditions derived in the previous chapter, that is when the Kähler function is separable in the truncated and the surviving sectors. We will show that in these models the supersymmetric configurations of the truncated fields given by the minima of the Kähler function remain stable for arbitrarily high scales of supersymmetry breaking. Interestingly this result can be extended to the most general case provided that we impose a mild restriction on the field dependence of the fermionic masses. The work presented in this chapter is based on the results we published in [77, 78].

This analysis is complementary to the study by Covi *et al.* [79], who

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provided a necessary condition in order to have a (meta)stable de Sitter vacua in supergravity theories. This condition can be expressed as a constraint on the geometry of the Kähler manifold characterizing the supergravity model, but it only applies to the sector of the theory where supersymmetry is broken. Therefore it can not be used constraint the interactions on the sector that is truncated in a supersymmetric way, which is the precisely the aim of the work presented in chapter 4.

As we mentioned above, the idea of supersymmetric decoupling is especially relevant to inflationary models, indeed this stability analysis has already been used to study the viability of *sgoldstino inflation* [80], where the inflaton is precisely the scalar partner of the goldstino.

Chapter 5.

Supersymmetry breaking can also be induced in the surviving sector by gauge interactions (D -term mechanism). An important ingredient of cosmological models which implement the D -term supersymmetry breaking are Fayet-Iliopoulos terms, which are related to the charge of the gravitino under abelian gauge interactions. Fayet-Iliopoulos terms can also be used to describe the present accelerated expansion of the universe, cosmic inflation and to construct cosmic string solutions which partially preserve the supersymmetries of the system.

Recently there has been some discussion in the literature about the viability of constructing consistent string theory models which are described at low energies by a supersymmetric effective action containing a Fayet-Iliopoulos term (see for example [81] and [82]). These discussions have led to a better understanding of the properties of Fayet-Iliopoulos terms, and in particular they have helped to clarify the issue of identifying these terms in non-linear sigma model with general gauge couplings.

In chapter 5 we use these new results to discuss the possibility of producing these terms by integrating out the heavy fields in a $\mathcal{N} = 1$ supergravity theory. We find that the Fayet-Iliopoulos terms cannot be generated during the supersymmetric truncation of a sector of the theory. This conclusion generalizes previous results presented by Binetruy *et al.* in [75], which only apply to models with a particular choice of gauge couplings.

This chapter provides a bridge between the first part of the thesis, which focuses on supersymmetric truncations in $\mathcal{N} = 1$ supergravity, and the second part which discusses supersymmetric cosmic string solutions in $\mathcal{N} = 2$ supergravity, where Fayet-Iliopoulos terms play crucial rôle.

Indeed, in the last section of chapter 5, we consider the properties of supersymmetric cosmic strings in the context of consistent truncations, and we show that if a cosmic string solution leaves unbroken part of the supersymmetries of the system in the reduced theory, the same supersymmetries have to be preserved in the full parent theory. Moreover we will present a $\mathcal{N} = 1$ supergravity model containing a Fayet-Iliopoulos term which, as shown in chapter 6 admits an embedding in $\mathcal{N} = 2$, and can be used to construct supersymmetric cosmic string solutions in extended $\mathcal{N} = 2$ supergravity.

Chapter 6.

In order to provide the basis needed to discuss the supersymmetric cosmic string solutions presented in chapter 7, here we give a short review of $\mathcal{N} = 2$ supergravity theories. We discuss the bosonic sector of the action, and the supersymmetry transformations.

We also explain the difficulties encountered when trying to embed supersymmetric cosmic string solutions in $\mathcal{N} = 2$ supergravity, associated to the strong constraints to the presence of Fayet-Iliopoulos terms in these theories. The solution to these problems, presented by Achúcarro *et al.* in [73], consists in considering the embedding of $\mathcal{N} = 1$ supergravity models with a Fayet-Iliopoulos term in $\mathcal{N} = 2$.

This technique relies on the idea of consistent reductions of a $\mathcal{N} = 2$ supergravity model down to $\mathcal{N} = 1$, reviewed in section 6.8, and is closely related to the supersymmetric truncations from $\mathcal{N} = 1$ to $\mathcal{N} = 1$ discussed in chapter 3.

Chapter 7.

In the final chapter of this thesis we construct an explicit example of a local supersymmetric cosmic string solution in $\mathcal{N} = 2$ supergravity. To the best of our knowledge, this is one of the only two known cosmic string solutions in four dimensional $\mathcal{N} = 2$ supergravity. In the model we discuss here the self coupling of the scalar and vector fields is characteristic of many compactifications of string theory, in particular it is related to type-IIB superstrings compactified in $K3 \times T^2/\mathbb{Z}_2$, which is the framework of one of the best studied inflationary models in string theory, the D3-D7 model [74].

The solution has interesting supersymmetric properties. In particular the cosmic string ansatz is compatible with a consistent truncation of supergravity down to $\mathcal{N} = 1$, and the string background leaves unbroken half of the $\mathcal{N} = 2$ supersymmetry transformations. The string configuration contains a zero mode connecting solutions with different radii and equal energy. In [72] we proved

1.6. Overview

that a similar class of cosmic strings which contain such a zero mode do not usually survive in a cosmological context, since the excitation of the zero mode typically makes the string grow in width until it disappears. However we argue that in this case the zero mode requires an infinite energy to be excited in an infinite volume, and therefore is not dynamical. This study can be found in the article [83].

$\mathcal{N} = 1$ supergravity and supersymmetric cosmic strings.

2.1 Introduction.

In the previous chapter we already introduced the concept of *supersymmetry*, the symmetry that transforms bosons and fermions into each other. In particular, we discussed how global supersymmetry turned out to provide a natural explanation for the Hierarchy problem, justifying the smallness of the Higgs mass as compared with the GUT scale. In supersymmetric theories the quadratically divergent renormalization corrections to the Higgs mass are absent, since the fermionic and the bosonic contributions cancel each other. This particular example illustrates an important property of supersymmetric theories: they have a better ultraviolet behavior than non-supersymmetric ones.

Originally local supersymmetry, or *supergravity*, was conceived as a theory that could cure the non-renormalizability of quantum gravity, providing a natural framework for the unification of the four fundamental forces. However, nowadays it is mainly considered as an effective field theory describing the light degrees of freedom of a more fundamental theory of quantum gravity, string/M-theory.

This chapter will provide an introduction to the basic concepts in $\mathcal{N} = 1$ supergravity. In the first section we will discuss the main features of the bosonic part of the $\mathcal{N} = 1$ supergravity action, and the rest of the chapter is dedicated to the discussion of supersymmetric configurations in $\mathcal{N} = 1$ supergravity, i.e. those configurations which leave unbroken all, or at least some, of the supersymmetry generators. In section 2.4 we study the stability of vacua which fully preserve supersymmetry. This is relevant for the construction of superstring inspired models describing late time cosmology with low energy supersymmetry breaking. In particular we present a derivation of the well known result that all supersymmetric vacua are stable. This analysis will serve us to introduce the techniques required in chapter 4 to study the stability of consistently decoupled heavy fields.

We will close the chapter reviewing another class of supersymmetry preserving configurations: *half-BPS cosmic strings*, which appear in many cosmological scenarios based on superstrings. These cosmic strings solutions break only half of the original supersymmetries of the model. In particular, we will present the method to obtain the cosmic string solutions from the condition of unbroken supersymmetry. This type of calculation is essential for the last two chapters of this thesis where we discuss supersymmetric cosmic strings in $\mathcal{N} = 2$ supergravity.

2.2 Overview of $\mathcal{N} = 1$ supergravity.

A detailed review of $\mathcal{N} = 1$ supergravity can be found in [18], but in this thesis we will follow the notation and conventions of [84, 85]. The *superPoincaré* algebra characterizes the symmetry underlying supergravity theories. The super-Poincaré algebra is the result of extending the regular Poincaré algebra with the $\mathcal{N}$ generators of the supersymmetry transformations¹ Q , which, schematically, act on the fermionic ($|F\rangle$) and bosonic ($|B\rangle$) states as follows:

$$Q|F\rangle = |B\rangle \quad Q|B\rangle = |F\rangle \quad (2.2.1)$$

As supersymmetry relates bosons and fermions, consistency requires the supercharges Q to transform as spinors under the Lorentz group. In particular, in 4-dimensional spacetime with Minkowski signature the supercharges are Majorana spinors, which can be decomposed into two chiral spinors of opposite chirality. The anticommutator of two supercharges yields the generator of translations², P^μ (see [84]):

$$\{Q_\alpha, Q_\beta\} = \frac{1}{2}(\gamma_\mu \mathcal{C}^{-1})_{\alpha\beta} P^\mu. \quad (2.2.2)$$

¹The algebra can also contain extra generators commuting with all the supercharges, the central charges.

²The operator $\mathcal{C}$ is the charge conjugation matrix, and is related to Majorana conjugation: $\bar{\lambda} \equiv \lambda^T \mathcal{C}$, [84].

2.2. Overview of $\mathcal{N} = 1$ supergravity.

As we anticipated in the previous chapter, this implies that any theory of local supersymmetry is also invariant under local space-time translations, and therefore it also describes gravity.

In supersymmetric theories there is always a group of rotations of the $\mathcal{N}$ supercharges which commutes with the Lorentz transformations and leaves invariant the supersymmetry algebra, the *R-symmetry group* H_R . The R-symmetry is the automorphism group of the supersymmetry algebra, which in 4-dimensions with Minkowski signature is given by $H_R = U(\mathcal{N})$. As we shall see, it plays an important role in constraining the type of interactions between the different fields in a theory. In the present case, $\mathcal{N} = 1$ supergravity, the R-symmetry is given by $H_R = U(1)$.

Unlike in Grand Unified Theories where all the particles of a given representation have the same spin, the irreducible representations of the super-Poincare algebra, the *supermultiplets*, involve particles of different spin. For unextended supergravity, $\mathcal{N} = 1$, the supermultiplets consist of only two particles which differ in spin $1/2$. We will consider theories that involve the following multiplets:

- The chiral multiplet: contains one complex real scalar ξ^I and one Weyl fermion, the chiralino χ^I , where the label runs in $I = 1, \dots, n_C$ for n_C chiral multiplets.
- The vector multiplet: it consists of one Weyl fermion, called gaugino λ^a and one gauge field A_μ^a . Here $a = 1, \dots, n_V$ labels n_V different vector multiplets.
- The graviton multiplet : it contains a spin 2 field representing the space-time metric $g_{\mu\nu}$, and one spin $3/2$ gravitino, ψ_μ .

Each chiral and vector multiplet also involves a real auxiliary field, F^I and D^a respectively. These fields have no kinetic terms in the action, and thus they are usually expressed in terms of the scalars ξ^I after solving their algebraic equations of motion.

In supergravity the gravitino plays the role of the gauge field for local supersymmetry. More generally, $\mathcal{N}$ extended supergravity theories contain $\mathcal{N}$ gravitini, all of which belong to the corresponding graviton multiplet.

Some string compactifications appear to be described in a natural way by a type of multiplets not introduced here, the tensor (linear) multiplets (see for example [86]), however it has been proven that these multiplets admit a dual description in terms of chiral multiplets (see [87]). Thus, the type of multiplets considered here is sufficient to obtain a complete description of $\mathcal{N} = 1$ supergravity.

Chiral multiplets $(0, \frac{1}{2})$	scalars chiralini	ξ^I χ^I	$I = 1, \dots, n_C$
Vector multiplet $(\frac{1}{2}, 1)$	gauge fields gaugini	A_μ^a λ^a	$a = 1, \dots, n_V$
Gravity multiplet $(\frac{3}{2}, 2)$	vielbein gravitino	e_μ^m ψ_μ	$\mu, m = 0, \dots, 3$

Table 2.1 – Field content of $\mathcal{N} = 1$ supergravity coupled to n_V vector multiplets and n_C chiral multiplets.

2.2.1 Bosonic sector of the action without gauge couplings.

In this thesis we will focus on the study of bosonic configurations, therefore we only review the features of the bosonic sector of the action. We will first consider actions with no gauge couplings between the vectors and the scalars, and we leave for the next section the discussion on gauged supergravity actions. We will work in units of the reduced Plank mass $M_p^{-2} = 8\pi G = 1$.

The bosonic part of a $\mathcal{N} = 1$ supergravity action with n_C chiral multiplets and n_V vector multiplets reads,

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2}R + \mathcal{T} + \mathcal{L}_{\text{gauge}} - V \right). \quad (2.2.3)$$

The action can be decomposed in four different contributions: the standard gravitational term $\frac{1}{2}R$, the kinetic terms for the scalars $\mathcal{T}$, the kinetic terms for the gauge bosons $\mathcal{L}_{\text{gauge}}$, and the scalar potential V .

In general, the kinetic terms of the scalars in a supergravity theory are characterized by a non-linear sigma model. In the particular case of ungauged $\mathcal{N} = 1$ supergravity with n_C chiral multiplets they are given by:

$$\mathcal{T} = -G_{I\bar{J}} \partial_\mu \xi^I \partial^\mu \bar{\xi}^{\bar{J}}, \quad I, J = 1, \dots, n_C. \quad (2.2.4)$$

The target space of the sigma model $\mathcal{M}$ is called the scalar manifold. In a geometric description of supergravity the scalar fields are usually interpreted as

2.2. Overview of $\mathcal{N} = 1$ supergravity.

coordinates on the scalar manifold, and the quantities $G_{I\bar{J}}$ as the metric on $\mathcal{M}$. In this picture the invariance of the action under supersymmetry transformations leads to constraints in the geometry of the scalar manifold. In the present case, $\mathcal{N} = 1$ supergravity, $\mathcal{M}$ should be a Kähler-Hodge manifold (see [88]), and thus the metric $G_{I\bar{J}}$ can be expressed locally as the derivatives of a real function of the scalar fields, the *Kähler potential* $K(\xi, \bar{\xi})$:

$$G_{I\bar{J}}(\xi, \bar{\xi}) = \partial_I \partial_{\bar{J}} K(\xi, \bar{\xi}). \quad (2.2.5)$$

The kinetic terms of the gauge fields A_μ^a , $a = 1, \dots, n_V$, are given in terms of the (holomorphic) *gauge kinetic functions* $f_{ab}(\xi)$:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4}(\text{Re } f_{ab})F_{\mu\nu}^a F^{b\mu\nu} + \frac{1}{4\sqrt{-g}}(\text{Im } f_{ab})F_{\mu\nu}^a \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}^b, \quad (2.2.6)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a$ is the field strength corresponding to the gauge boson A_μ^a .

The scalar potential has two contributions, the F -term potential, V_F , and the D -term potential, V_D :

$$V = V_F + V_D. \quad (2.2.7)$$

The scalar potential appears in the action after solving the algebraic equations of motion of the auxiliary fields F^I , and D^a in terms of the scalars ξ^I . The form of the F -term potential is very constrained by supersymmetry. Actually, it can be expressed in terms of the Kähler potential $K(\xi, \bar{\xi})$ and holomorphic function $W(\xi)$, known as the *superpotential*:

$$V_F = e^K \left(K^{I\bar{J}} \mathcal{D}_{\bar{I}} W \mathcal{D}_{\bar{J}} W - 3|W|^2 \right). \quad (2.2.8)$$

Here the Kähler covariant derivatives are given by: $\mathcal{D}_I W = \partial_I W + \partial_I K W$. The D -term potential is only non-vanishing in the presence of gauge couplings, and thus we will discuss it in the next section.

2.2.2 Gauging of global symmetries.

As in the case of non-supersymmetric theories we can couple the scalars and gauge bosons by promoting to local the global symmetries of the action. Any global symmetry transformation must leave invariant the kinetic terms of the scalars, and therefore, in the geometrical picture, it can be associated to an isometry of the scalar manifold. The transformation of the scalars under an isometry transformation is characterized by the killing vectors $k_a(\xi) = k_a^I \partial_I$:

$$\delta_{\text{gauge}} \xi^I = k_a^I(\xi) \alpha^a, \quad a = 1, \dots, n_V, \quad (2.2.9)$$

which, in $\mathcal{N} = 1$ supergravity, have a holomorphic dependence on the scalars. Here α^a are the gauge parameters. We say that the configuration ξ_0 is a *fixed point* of k_a when

$$k_a^I(\xi_0) = 0 \quad \text{for all } I = 1 \dots n_G. \quad (2.2.10)$$

The set of killing vectors of a scalar manifold generate a representation of its isometry group $\mathbb{G}$, and they satisfy the following Lie bracket relations:

$$[k_a, k_b]^I = k_{b,J}^I k_a^J - k_{a,J}^I k_b^J = f_{ab}{}^c k_c^I, \quad (2.2.11)$$

where $f_{ab}{}^c$ are the *structure constants* of $\mathbb{G}$.

In order to promote the isometries of the scalar manifold to local symmetries the derivatives of the scalars in (2.2.4) have to be substituted by covariant derivatives, which in the case of non-linear sigma models read:

$$D_\mu \xi^I = \partial_\mu \xi^I - A_\mu^a k_a^I(\xi). \quad (2.2.12)$$

Note that this expression reduces to the usual covariant derivative in the case of $U(1)$ transformations $k(\xi) = i\xi \partial_\xi$:

$$\delta_{\text{gauge}} \xi = i\xi \alpha \quad \implies \quad D_\mu \xi = \partial_\mu \xi - iA_\mu^a \xi. \quad (2.2.13)$$

In order to keep invariance under supersymmetry transformations after the gauging of isometries, the bosonic action has to be supplemented with an extra term: the D -term potential V_D . In terms of the auxiliary fields of the vector multiplets, D^a , it reads:

$$V_D = \frac{1}{2}(\text{Re } f_{ab}) D^a D^b = \frac{1}{2}(\text{Re } f)^{-1|ab} \mathcal{P}_a \mathcal{P}_b, \quad D^a = (\text{Re } f)^{-1|ab} \mathcal{P}_b. \quad (2.2.14)$$

The *moment maps* $\mathcal{P}_a(\xi, \bar{\xi})$ are real functions which determine the killing vector through their derivatives:

$$\partial_{\bar{J}} \mathcal{P}_a(\xi, \bar{\xi}) = i k_a^I G_{I\bar{J}}. \quad (2.2.15)$$

Integrating this equation we can find an expression for the moment map:

$$\mathcal{P}_a = i \left(k_a^I(\xi) \partial_I K(\xi, \bar{\xi}) - 3r_a(\xi) \right), \quad (2.2.16)$$

where holomorphic functions $r_a(\xi)$, which are called the *compensators*. The constant factor multiplying the compensators, $-3i$, is for convenience. Using the previous expression it is easy to see that the compensators characterize the gauge transformations of the Kähler potential $K(\xi, \bar{\xi})$:

$$\delta_a K(\xi, \bar{\xi}) = k_a^I(\xi) \partial_I K(\xi, \bar{\xi}) + k^{\bar{I}}(\bar{\xi}) \partial_{\bar{I}} K(\xi, \bar{\xi}) = 3r_a(\xi) + 3\bar{r}_a(\bar{\xi}). \quad (2.2.17)$$

This relation fixes the compensators up to an imaginary constant, which implies that the moment map itself it is only determined up to a constant shift. In other words, if $\mathcal{P}_a$ is a moment map associated to the killing vector k_a^I , so is

$$\mathcal{P}_a + \eta_a \quad \text{for any constant } \eta_a. \quad (2.2.18)$$

2.2. Overview of $\mathcal{N} = 1$ supergravity.

The constant shift η_a of the moment map is called the *Fayet-Iliopoulos* term (FI). A more detailed analysis shows that the freedom to add a FI term to the moment map is restricted to the case of abelian symmetries (see [18]).

It can also be shown that the superpotential, $W(\xi)$, transforms under the killing vectors $k_a^I(\xi)$ as:

$$\delta_a W(\xi) = k_a^I(\xi) \partial_I W(\xi) = -3r_a(\xi) W(\xi). \quad (2.2.19)$$

When the action involves a non vanishing superpotential, it follows from (2.2.19) that the compensators are completely determined by the superpotential and the killing vectors. Conversely, given the geometry of the scalar manifold and the gauge couplings, (2.2.19) can be seen as a set of constraints on the form of the superpotential.

Global symmetries of the action must also leave invariant the kinetic terms of the gauge bosons, and thus the gauge kinetic functions $f_{ab}(\xi)$ must transform in the following way³:

$$\delta_{gauge} f_{ab}(\xi) = (f_{ab,I} k_c^I + f_{da} f_{bc}^d + f_{db} f_{ac}^d) \alpha^c = i c_{abc} \alpha^c \quad (2.2.20)$$

This means that the full gauge transformation of the gauge kinetic function amounts to a shift on the Peccei-Quinn symmetry. This expression can be interpreted as a set of constraints on the allowed form of the gauge kinetic functions, similar to (2.2.19) for the superpotential, which depends on the gauge group and the gauge couplings.

2.2.3 Supersymmetry transformations

Omitting the transformations of the auxiliary fields, the $\mathcal{N} = 1$ supersymmetry transformations linear in the fermions are given by⁴

$$\delta e_\mu^a = \frac{1}{2} i \bar{\epsilon} \gamma^a \psi_\mu, \quad \delta W_\mu^a = -\frac{1}{2} \bar{\epsilon} \gamma_\mu \lambda^a, \quad \delta \xi^I = \bar{\epsilon}_L \chi^I, \quad (2.2.21)$$

$$\delta \psi_{\mu L} = (\partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} + \frac{i}{2} A_\mu^B) \epsilon_L + \frac{1}{2} e^{K/2} W \gamma_\mu \epsilon_R \quad (2.2.22)$$

$$\delta \chi_L^I = \frac{1}{2} \gamma^\mu \nabla_\mu \xi^I \epsilon_R - \frac{1}{2} e^{\frac{1}{2} K} K^{I\bar{J}} \mathcal{D}_{\bar{J}} \bar{W} \epsilon_L \quad (2.2.23)$$

$$\delta \lambda^a = \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu}^a \epsilon + \frac{1}{2} i D^a \gamma_5 \epsilon. \quad (2.2.24)$$

Here ϵ is the parameter of the supersymmetry transformations, and γ^μ represent the gamma matrices as usual. The subscripts R and L of the fermions stand for right and left chirality respectively:

$$\chi_R^I = \frac{1}{2} (1 - \gamma^5) \chi_R^I \quad \chi_L^I = \frac{1}{2} (1 + \gamma^5) \chi_L^I \quad (2.2.25)$$

³Note that each index of the gauge kinetic functions $f_{ab}(\xi)$ transforms in the adjoint representation of the gauge group.

⁴The complete supersymmetry transformations can be found in [75, 89].

The gauge boson A_μ^B is a composite field called the U(1) connection, and is expressed in terms of the other fields as follows:

$$A_\mu^B = \frac{i}{2} [\partial_{\bar{I}} K \partial_\mu \xi^{\bar{I}} - \partial_I K \partial_\mu \xi^I] + A_\mu^a \mathcal{P}_a. \quad (2.2.26)$$

This field appears in the covariant derivative of the gravitino

$$D_{[\mu} \psi_{\nu]} = \left(\partial_{[\mu} + \frac{1}{4} \omega_{[\mu}^{ab} \gamma_{ab} + \frac{1}{2} i A_{[\mu}^B \gamma_5 \right) \psi_{\nu]}. \quad (2.2.27)$$

In the presence of a constant FI term η_a the moment map $\mathcal{P}_a$ has a constant contribution, and thus the gravitino acquires a U(1) charge under one of the abelian gauge symmetries.

2.2.4 Kähler transformations

The $\mathcal{N} = 1$ supergravity action and the supersymmetry transformations are invariant under Kähler transformations, which act on the Kähler potential and the superpotential in the following way:

$$K(\xi, \bar{\xi}) \longrightarrow K(\xi, \bar{\xi}) + h(\xi) + \bar{h}(\bar{\xi}) \quad W(\xi) \longrightarrow W(\xi) e^{-h(\xi)}, \quad (2.2.28)$$

where $h(\xi)$ is an arbitrary holomorphic function of the scalars. Actually, if the superpotential is non-zero, the full supersymmetry action and the supersymmetry transformations can be expressed in terms of a single Kähler invariant quantity, the *Kähler function* $G(\xi, \bar{\xi})$:

$$G(\xi, \bar{\xi}) = K(\xi, \bar{\xi}) + \log |W(\xi)|^2 \quad \text{for } W(\xi) \neq 0. \quad (2.2.29)$$

In particular, the metric of the scalar manifold can be expressed in terms of the derivatives of the Kähler function, since

$$G_{I\bar{J}} = \partial_I \partial_{\bar{J}} K = \partial_I \partial_{\bar{J}} G. \quad (2.2.30)$$

The F -term and D -term potentials are given by⁵:

$$V_F = G_{I\bar{J}} F^I F^{\bar{J}} - 3e^G = e^G (G^{I\bar{J}} G_I G_{\bar{J}} - 3), \quad (2.2.31)$$

$$V_D = \frac{1}{2} \text{Re}(f_{ab}) D^a D^b = \frac{1}{2} (\text{Re } f)^{-1|ab} k_a^I G_I k_b^{\bar{J}} G_{\bar{J}}, \quad (2.2.32)$$

where the auxiliary fields read:

$$F^I = e^{G/2} G^{I\bar{J}} G_{\bar{J}} \quad (2.2.33)$$

$$D^a = i(\text{Re } f)^{-1ab} k_b^I G_I = -i(\text{Re } f)^{-1ab} k_b^{\bar{I}} G_{\bar{I}}. \quad (2.2.34)$$

⁵We will denote with subscripts the derivatives with respect to the scalars: $\partial_I G = G_I$, $\partial_{\bar{I}} G = G_{\bar{I}}$, etc The matrix $G^{I\bar{J}}$ is defined as the inverse of the metric $G_{I\bar{K}} G^{JK} = \delta_I^{\bar{J}}$, $G_{K\bar{J}} G^{K\bar{I}} = \delta_{\bar{J}}^{\bar{I}}$, and we use the matrix $G_{I\bar{J}}$ and its inverse to lower and raise indices.

2.3. Supersymmetric critical points.

The two expressions given for the D-terms are equivalent because, for the action to be invariant under gauge transformations, the Kähler function $G(\xi, \bar{\xi})$ should also be invariant, and thus:

$$\delta_{\text{gauge}} G = (k_a^I G_I + k_a^{\bar{I}} G_{\bar{I}}) \alpha^a = 0, \quad \text{for all } \alpha^a. \quad (2.2.35)$$

In order for the Kähler transformations to leave invariant the fermionic part of the action, we must also require the fermions to transform in the following way:

$$\psi_\mu \rightarrow \exp[-\frac{i}{2}\gamma_5 \text{Im } h] \psi_\mu, \quad \chi^I \rightarrow \exp[\frac{i}{2}\gamma_5 \text{Im } h] \chi^I, \quad \lambda^a \rightarrow \exp[-\frac{i}{2}\gamma_5 \text{Im } h] \lambda^a. \quad (2.2.36)$$

However, these conditions are not sufficient to ensure the action to be well-defined over the whole scalar manifold $\mathcal{M}$, there are consistency problems that might arise globally. In particular the fermions which transform under the Kähler transformations can only be well-defined globally provided the Kähler form,

$$\Omega = \frac{i}{2\pi} G_{I\bar{J}} d\xi^I \wedge d\xi^{\bar{J}} = \frac{i}{2\pi} \partial_I \partial_{\bar{J}} K d\xi^I \wedge d\xi^{\bar{J}}, \quad (2.2.37)$$

defines an even cohomology, so that

$$c_1 = \frac{1}{2}[\mathcal{K}] \in \mathbb{Z}. \quad (2.2.38)$$

Therefore we can only define consistently an $\mathcal{N} = 1$ supergravity action over Kähler manifolds satisfying this constraint, the so called *Kähler-Hodge* manifolds [90].

2.3 Supersymmetric critical points.

In the present thesis we will deal with supersymmetric configurations, which are those where the supersymmetry transformations of all fields are zero for some non-vanishing value of the supersymmetry parameter ϵ . In particular we will focus in two types of supersymmetric configurations: *supersymmetric vacua of the scalar potential*, and *half-BPS cosmic string solutions*. In the rest of this section and in the following one we will focus in supersymmetry preserving vacua, and we will leave the discussion on half-BPS cosmic strings for section 2.5.

In a bosonic background, where all the fermions are set to zero, the supersymmetry transformations of the bosons are all vanishing, and therefore the conditions of unbroken supersymmetry can be derived from the transformations of the fermions alone. Moreover, in a homogeneous background which preserves Lorentz invariance ($\nabla^\mu \xi^I = F^a_{\mu\nu} = 0$), a set of necessary conditions for unbroken supersymmetry is (2.2.23):

$$\mathcal{D}_I W = 0 \quad \text{for all} \quad I = 1, \dots, n_C. \quad (2.3.1)$$

Equivalently, if $W(\xi) \neq 0$, this condition can be written in terms of the Kähler function as:

$$\partial_I G(\xi, \bar{\xi}) = 0 \quad \text{for all} \quad I = 1, \dots, n_C. \quad (2.3.2)$$

Note that although it is always possible to break supersymmetry spontaneously by non-vanishing F -terms (2.2.23) and zero D -terms (2.2.24), the relations (2.2.33) and (2.2.34) imply that non-vanishing D -terms necessarily require non-vanishing F -terms, and therefore supersymmetry can never be broken by D -terms alone [91].

Supersymmetric configurations are also called *supersymmetric critical points* of the scalar potential, since any configuration satisfying (2.3.2) is always a critical point of the scalar potential, as can be easily checked from (2.2.32). Moreover, the result (2.3.2) implies, together with the expression for the scalar potential (2.2.7) and (2.2.32), that supersymmetric critical points ξ_0^I with non vanishing superpotential $W(\xi_0) \neq 0$ always have a negative vacuum energy, i.e. they are AdS critical points:

$$V(\xi_0) = -3e^{G(\xi_0)} < 0. \quad (2.3.3)$$

Interestingly supersymmetric critical points are always perturbatively stable, regardless of being local minima, maxima or saddle points of the scalar potential. The reason is that in an AdS background a fluctuation with a tachyonic mass might still be stable as long as it satisfies the Breitenlohner-Freedman bound [92]:

$$m^2 \geq \frac{3}{4} V(\xi_0), \quad (2.3.4)$$

and this condition is always fulfilled by supersymmetric critical points, as we shall prove in the next section.

2.4 Stability of supersymmetric critical points

In this section we study the stability properties of a supersymmetric critical point in a completely general setup. We take the action to be characterized by a Kähler potential $G(\xi, \bar{\xi})$, and we allow for an arbitrary gauge coupling defined by the gauge kinetic functions $f_{ab}(\xi)$ and Killing vectors $k(\xi)_a^I$. We will only assume that the superpotential is non-vanishing, $W(\xi) \neq 0$, so that the Kähler function $G(\xi, \bar{\xi})$ is well defined. We will relate the stability of the supersymmetric vacua to the curvature of the Kähler function, and in particular we will show that maxima of the scalar potential always correspond to minima of the Kähler function.

2.4.1 Analysis of the Kähler function

We begin by studying the character of the critical points of the Kähler function, which is a simple calculation and will serve us to introduce the technique we will

2.4. Stability of supersymmetric critical points

use later in the analysis of the scalar potential. The Taylor expansion of the Kähler potential $G(\xi^I, \xi^{\bar{J}})$ around a generic point ξ_0^I , reads:

$$G = G(\xi_0) + G_I(\xi_0)\hat{\xi}^I + G_{\bar{I}}(\xi_0)\hat{\xi}^{\bar{I}} + G_{I\bar{J}}(\xi_0)\hat{\xi}^I\hat{\xi}^{\bar{J}} + \frac{1}{2}G_{IJ}(\xi_0)\hat{\xi}^I\hat{\xi}^J + \frac{1}{2}G_{\bar{I}\bar{J}}(\xi_0)\hat{\xi}^{\bar{I}}\hat{\xi}^{\bar{J}} + \dots, \quad (2.4.1)$$

where we define $\hat{\xi}^I = \xi^I - \xi_0^I$. If the point ξ_0^I is a supersymmetric critical point then the first order terms vanish $G_I(\xi_0) = 0$. In order to know if ξ_0^I corresponds to a minimum, a maximum or a saddle point the Kähler function we need to find the eigenvalues of its Hessian evaluated at the critical point

$$\begin{pmatrix} G_{I\bar{J}} & G_{IJ} \\ G_{\bar{I}J} & G_{\bar{I}\bar{J}} \end{pmatrix}_{\xi_0}. \quad (2.4.2)$$

This expression simplifies considerably by redefining the ξ^I fields so that they have canonical kinetic terms at the critical point, $G_{I\bar{J}}(\xi_0) = \delta_{I\bar{J}}$. With this choice of coordinates the equation for the eigenvalues, g , takes the form:

$$\det \begin{pmatrix} (1-g)\mathbb{1} & X \\ X^\dagger & (1-g)\mathbb{1} \end{pmatrix} = 0 \quad (2.4.3)$$

where we have used the matrix notation $X = X^T = G_{IJ}(\xi_0)$ and $\mathbb{1} = \delta_{I\bar{J}}$. Using a known property of determinants

$$\det \begin{pmatrix} M & P \\ Q & N \end{pmatrix} = \det(MN - QP), \quad \text{provided that } QN = NQ, \quad \det(M) \neq 0, \quad (2.4.4)$$

for square submatrices M, N, P, Q , we can see that g is a solution of (2.4.3) if and only if it also solves:

$$\det((1-g)^2 - X^\dagger X) = 0. \quad (2.4.5)$$

Strictly speaking this equation was derived for $g \neq 1$, but it is not difficult to check that it also gives the correct solution for $g = 1$. In order to solve this equation we use the freedom of field redefinition once more. Requiring that the fields have canonical kinetic terms is not enough to fix the choice of fields completely, we can still redefine the fields by a constant unitary transformation of the form $\tilde{\xi}^I = U^I_J \xi^J$. Under this field redefinition the matrix X and the combination $X^\dagger X$ transform as:

$$X = U^T \tilde{X} U, \quad X^\dagger X = U^\dagger (\tilde{X}^\dagger \tilde{X}) U, \quad \text{where } U = U^I_J, \quad (2.4.6)$$

and therefore we can use this freedom to transform the Hermitian combination $X^\dagger X$ into a real diagonal matrix. The eigenvalues of $X^\dagger X$ are necessarily nonnegative, and we will denote them by $|x_\lambda|^2$, with λ labeling the p different

eigenspaces. Moreover, the symmetry of X implies that $X^\dagger X = (XX^\dagger)^*$, thus in the basis that makes $X^\dagger X$ diagonal we have:

$$X^\dagger X = XX^\dagger = \text{Diag}(|x_1|^2 \mathbb{1}_{n_1}, \dots, |x_p|^2 \mathbb{1}_{n_p}), \quad |x_\lambda|^2 \geq 0, \quad (2.4.7)$$

where n_λ is the dimension of the eigenspace of eigenvalue $|x_\lambda|^2$. Note also that in this particular basis the matrices X and $X^\dagger X$ commute, which implies that X should be block diagonal in each of the eigenspaces of $X^\dagger X$:

$$X = \text{Diag}(X_1, \dots, X_p) \quad \text{with} \quad X_\lambda^\dagger X_\lambda = |x_\lambda|^2 \mathbb{1}_{n_\lambda}. \quad (2.4.8)$$

This means that the eigenvalue problem decouples for the m different eigenspaces of $X^\dagger X$, and therefore the equation (2.4.5) takes a very simple form:

$$\prod_{\lambda=1}^p [(1 - g_\lambda)^2 - |x_\lambda|^2]^{n_\lambda} = 0, \quad (2.4.9)$$

which we can solve easily giving the eigenvalues

$$g_{\pm\lambda} = 1 \pm |x_\lambda|. \quad (2.4.10)$$

which have multiplicity n_λ . The different possibilities for the character of the critical point ξ_0^α are summarized in the following table:

Local minimum	$ x_\lambda < 1$	for all $\lambda = 1, \dots, p$	
Saddle point	$ x_\lambda > 1$	for some or all λ	(2.4.11)

The result (2.4.10) also indicates that, for each eigenvalue of $X^\dagger X$ that satisfies $|x_\lambda|^2 = 1$, the Kähler function has a flat direction and a local minimum (a trough) along one of the complex directions ξ^I .

2.4.2 Analysis of the scalar potential with vanishing D -terms

We will now analyze how the maxima and saddle points of the Kähler function relate to the different types of supersymmetric critical points of the scalar potential. This is especially interesting because the Kähler function is much easier to study than the scalar potential. We will demonstrate a remarkable result: the minima of the Kähler function are in one to one correspondence with the supersymmetric AdS maxima of the scalar potential. We start assuming that there are no gauge interactions, and in the next section we will prove this result in full generality.

The stability analysis of a supersymmetric critical point of the scalar potential can be done using the same techniques of the previous subsection. Consider first its Taylor expansion around a supersymmetric critical point ξ_0^I :

$$V = V(\xi_0) + \frac{1}{2} V_{IJ}(\xi_0) \hat{\xi}^I \hat{\xi}^J + \frac{1}{2} V_{I\bar{J}}(\xi_0) \hat{\xi}^I \hat{\xi}^{\bar{J}} + V_{I\bar{J}}(\xi_0) \hat{\xi}^I \hat{\xi}^{\bar{J}} + \dots, \quad (2.4.12)$$

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where the second derivatives of the potential evaluated at the point $\xi^I = \xi_0^I$ can be calculated from (2.2.32) using that $G_I(\xi_0) = 0$:

$$V_{IJ}(\xi_0) = -G_{IJ}(\xi_0)e^{G(\xi_0)} \quad (2.4.13)$$

$$V_{I\bar{J}}(\xi_0) = e^{G(\xi_0)} \left[G^{R\bar{S}} G_{RI} G_{\bar{S}\bar{J}} - 2G_{I\bar{J}} \right]_{\xi_0}. \quad (2.4.14)$$

In order to determine the character of the critical point ξ_0^I we need to find the eigenvalues of the Hessian of the potential, which gives the mass spectrum of the fluctuations around ξ_0^I . As in the previous subsection we define the fields ξ^I so that they have canonical kinetic terms, $G_{I\bar{J}}(\xi_0) = \delta_{I\bar{J}}$, and the Hermitian matrix $X^\dagger X$ becomes diagonal. With this choice the Hessian has the simple form:

$$\begin{pmatrix} V_{I\bar{J}} & V_{IJ} \\ V_{\bar{I}\bar{J}} & V_{\bar{I}J} \end{pmatrix}_{\xi_0} = e^{G(\xi_0)} \begin{pmatrix} XX^\dagger - 2\mathbb{1} & -X \\ -X^\dagger & X^\dagger X - 2\mathbb{1} \end{pmatrix}. \quad (2.4.15)$$

Since in the basis we have chosen $X^\dagger X = XX^\dagger$ it is easy to check that this matrix also satisfies the first of the conditions necessary to apply (2.4.4), and we find that the equation for the spectrum of masses m^2 reads:

$$\det((X^\dagger X - (2 + e^{-G(\xi_0)} m^2)\mathbb{1})^2 - X^\dagger X) = 0. \quad (2.4.16)$$

In order to use (2.4.4) we also need to assume that the following matrix is non singular,

$$\det(X^\dagger X - (2 + e^{-G(\xi_0)} m^2)\mathbb{1}) \neq 0, \quad (2.4.17)$$

but after some algebra it is possible to prove that (2.4.16) also gives the right result in the singular case. As in the previous section we can use that X has the block diagonal form (2.4.8) to show that the eigenvalue problem can be decomposed in each of the eigenspaces of $X^\dagger X$. Using this fact the eigenvalue equation (2.4.16) can be written as

$$\prod_{\lambda=1}^p \left[(|x_\lambda|^2 - 2 - e^{-G(\xi_0)} m^2)^2 - |x_\lambda|^2 \right]^{n_\lambda} = 0. \quad (2.4.18)$$

Therefore the spectrum of masses at the supersymmetric critical point is given by:

$$m_{\pm\lambda}^2 = e^{G(\xi_0)} (|x_\lambda|^2 - 2 \pm |x_\lambda|) = e^{G(\xi_0)} \left((|x_\lambda|^2 \pm \frac{1}{2})^2 - \frac{9}{4} \right). \quad (2.4.19)$$

Each of these energy levels contains n_λ different excitations with the same mass. The set of quantities $|x_\lambda|$ determine which type of extremum the supersymmetric critical point ξ_0^I is:

$$\begin{aligned} |x_\lambda| > 2 \quad \text{for all } \lambda &\Rightarrow \text{local AdS minimum,} \\ |x_\lambda| < 1 \quad \text{for all } \lambda &\Rightarrow \text{local AdS maximum,} \end{aligned} \quad (2.4.20)$$

and any other combination corresponds to AdS saddle points ($|x_\lambda| = 1, 2$ give flat directions). The result (2.4.19) provides a proof of the stability of all supersymmetric critical points, regardless of the possible negative curvature of the potential. Since all these critical points are AdS, the perturbative stability is determined by the Breitenlohner-Freedman bound (2.3.4), which is always satisfied as is clear from (2.4.19):

$$m^2 \geq -\frac{9}{4}e^{G(\xi_0)} = \frac{3}{4}V(\xi_0), \quad (2.4.21)$$

Now we already have at hand all the results we need to check the claim we made at the beginning of this subsection. Comparing equations (2.4.11) and (2.4.20) we see immediately that the supersymmetric AdS maxima of the potential always coincide with the minima of the Kähler function.

2.4.3 Analysis of the scalar potential with non-vanishing D -terms

Let us now generalize the result of the previous subsection to the case where the gauge couplings are turned on. In this case we have to add to the scalar potential the contribution from D -terms (2.2.34). In order to calculate the new contributions to the Hessian of the scalar potential around the critical point ξ_0^I we must find the derivatives of the D -term potential at this point. Using that $G_I(\xi_0) = 0$ we find:

$$\begin{aligned} V_{D|IJ}(\xi_0) &= \frac{1}{2}(\text{Re}f(\xi_0))^{-1ab}k_a^R(\xi_0)k_b^{\bar{S}}(\bar{\xi}_0)[G_{IR}G_{J\bar{S}} + G_{JR}G_{I\bar{S}}]_{\xi_0}, \\ V_{D|IJ}(\xi_0) &= \frac{1}{2}(\text{Re}f(\xi_0))^{-1ab}k_a^R(\xi_0)k_b^{\bar{S}}(\bar{\xi}_0)[G_{IR}G_{J\bar{S}} + G_{JR}G_{I\bar{S}}]_{\xi_0} \end{aligned} \quad (2.4.22)$$

As we have done previously we will define the scalar fields ξ^I so that they have trivial kinetic terms $G_{IJ} = \delta_{IJ}$ and the Hermitian matrix $X^\dagger X$ is diagonal. In the case of the D -term potential we can simplify the calculations even further making use of the freedom we have to define the gauge fields A_μ^a . Actually, the action is invariant under constant linear transformations of the gauge fields $A_\mu^b = O_a^b \tilde{A}_\mu^a$, with O_a^b any non-singular real matrix, provided that the gauge kinetic functions f_{ab} and the Killing vectors k_a^I transform as follows:

$$\tilde{f}_{cd} = O_c^a O_d^b f_{ab} \quad \tilde{k}_b^I = O_b^a k_a^I. \quad (2.4.23)$$

In particular note that the gauge covariant derivatives and the Yang-Mills terms do not transform under these redefinitions since:

$$A_\mu^a k_a^I = \tilde{A}_\mu^b \tilde{k}_b^I, \quad (\text{Re}f_{ab})F_{\mu\nu}^a F^{a\mu\nu} = (\text{Re}\tilde{f}_{cd})\tilde{F}_{\mu\nu}^c \tilde{F}^{d\mu\nu}. \quad (2.4.24)$$

Then we can always use this freedom to turn $\text{Re}f_{ab}$ into a matrix proportional to the identity

$$\text{Re}f_{ab} = e^{G(\xi_0)} \delta_{ab}, \quad (2.4.25)$$

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where the overall factor has been chosen for convenience. Using these conventions, and defining the matrix $k = k_a^I(\xi_0)$, we can write the Hessian of the D -term potential as:

$$\begin{pmatrix} V_{D|I\bar{J}} & V_{D|IJ} \\ V_{D|\bar{I}\bar{J}} & V_{D|\bar{I}J} \end{pmatrix}_{\xi_0} = \frac{1}{2} e^{G(\xi_0)} \begin{pmatrix} kk^\dagger + Xkk^\dagger X^\dagger & Xkk^\dagger + k^*k^T X \\ X^\dagger k^*k^T + kk^\dagger X^\dagger & k^*k^T + X^\dagger k^*k^T X \end{pmatrix}, \quad (2.4.26)$$

which has to be added to (2.4.15) in order to get the Hessian of the full scalar potential.

Before we continue the calculation let us derive a useful property of the Killing vectors k . We mentioned in section (2.2.4) that the Kähler function $G(\xi^I)$ has to be invariant under gauge transformations. In particular in a Taylor expansion of the Kähler function around ξ_0 (2.4.1) all the terms have to be invariant under gauge transformations order by order in $\hat{\xi} = \xi - \xi_0$. From the gauge transformation of the order one terms in the expansion we find the condition:

$$\left(G_{IJ}(\xi_0) k_a^J(\xi_0) + G_{I\bar{J}}(\xi_0) k_a^{\bar{J}}(\xi_0) + G_J(\xi_0) \partial_I k_a^J \right) \hat{\xi}^I \alpha^a = 0, \quad (2.4.27)$$

which has to be satisfied for all values of the gauge parameters α^a , and the fluctuations $\hat{\xi}^I$. Since $G_I(\xi_0) = 0$, then with our field definitions and in matrix notation this condition reads simply:

$$k^* = -Xk. \quad (2.4.28)$$

An immediate consequence of this requirement is that the Killing vectors are eigenvectors of the matrix $X^\dagger X$ with eigenvalue $g_\lambda = 1$:

$$X^\dagger X k = -X^\dagger k^* = k, \quad (2.4.29)$$

since $X^T = X$. This means that the matrices $kk^\dagger$ and kk^T have all the entries zero except in the block that corresponds to the eigenspace of eigenvalue $|x_\lambda|^2 = 1$ of $X^\dagger X$. As we saw in the previous section the eigenvalue problem of the Hessian decouples in the different eigenspaces of the matrix $X^\dagger X$. We have just proven that the corrections introduced by the D -term potential respect this decoupling and moreover, that the corrections only affect the eigenspace with eigenvalue $|x_\lambda|^2 = 1$. Therefore we can use the results of the previous section for all the eigenspaces with $|x_\lambda| \neq 1$ to find the corresponding eigenvalues. In the remaining of this section we will just focus on solving the eigenvalue problem in the eigenspace where $|x_\lambda|^2 = 1$, which we label by $\lambda = 1$. In order to keep notation simple, we will use the matrices X_1 and k_1 to represent the submatrices corresponding to the eigenspace $\lambda = 1$, thus:

$$X_1^\dagger X_1 = X_1 X_1^\dagger = \mathbb{1}_{n_1}. \quad (2.4.30)$$

Notice that, since the hermitian matrix $k_1 k_1^\dagger$ transforms under scalar field redefinitions in the same way as $X_1 X_1^\dagger$, we can use the residual freedom to choose the

eigenvectors in the eigenspace with $|x_\lambda|^2 = 1$ to turn $k_1 k_1^\dagger$ into a real diagonal matrix:

$$k_1 k_1^\dagger = \text{Diag}(|k_1|^2, \dots, |k_{n_1}|^2), \quad (2.4.31)$$

where n_1 is the dimension of the eigenspace with $|x_\lambda|^2 = 1$. The final expression for the Hessian of the full potential restricted to this eigenspace can be obtained from (2.4.26) and (2.4.15)

$$\begin{pmatrix} V_{I\bar{J}} & V_{IJ} \\ V_{\bar{I}\bar{J}} & V_{\bar{I}J} \end{pmatrix}_{\lambda=1} = e^{G(\xi_0)} \begin{pmatrix} -\mathbb{1} + k_1 k_1^\dagger & (-\mathbb{1} + k_1 k_1^\dagger) X_1 \\ (-\mathbb{1} + k_1 k_1^\dagger) X_1^\dagger & -\mathbb{1} + k_1 k_1^\dagger \end{pmatrix}, \quad (2.4.32)$$

where we have used the properties (2.4.28), (2.4.31) and (2.4.30) in order to simplify (2.4.26). It is easy to check that the matrices X_1 and $k_1 k_1^\dagger$ commute, therefore we can use equation (2.4.4) in order to find the equation for the mass spectrum m^2 , which reads:

$$\prod_{i=1}^{n_1} \left(\left(|k_i|^2 - 1 - e^{-G(\xi_0)} m^2 \right)^2 - (|k_i|^2 - 1)^2 \right) = 0, \quad (2.4.33)$$

after having substituting the diagonal form of $k_1 k_1^\dagger$ (2.4.31). The solutions to this equations, together with the results we found in the previous section, which apply for $|x_\lambda| \neq 1$ are summarized below:

$$\begin{aligned} m_{\pm\lambda}^2 &= e^{G(\xi_0)} \left(\left(|x_\lambda \pm \tfrac{1}{2}| \right)^2 - \tfrac{9}{4} \right) && \text{if } |x_\lambda|^2 \neq 1, \\ m_{+1i}^2 &= 2e^{G(\xi_0)} (|k_i|^2 - 1) && \text{if } |x_\lambda|^2 = 1, \\ m_{-1i}^2 &= 0 && \text{if } |x_\lambda|^2 = 1. \end{aligned} \quad (2.4.34)$$

The quantities $|k_i|^2$ determine the mass of the gauge bosons at the supersymmetric critical point ξ_0 , therefore we can see that the breaking of gauge symmetries can only improve the stability of vacuum. This result agrees with the analysis of Gomez-Reino and Scruc a of the stability of uplifted vacua in [93].

The fact that the Killing vectors are associated to the eigenvalues $|x_\lambda|^2 = 1$ should not be surprising. On the one hand the eigenvalues $|x_\lambda|^2 = 1$ are always related to marginally stable directions $m^2 = 0$. And on the other hand we know that the potential has to be invariant under gauge transformations, thus each Killing vector has to be naturally associated with a flat direction of the potential, which appear in the spectrum as massless fluctuations (the would-be Goldstone bosons that disappear due to the Higgs mechanism).

In view of the result (2.4.34) we can argue that the presence of non-vanishing gauge couplings does not modify the conclusion of the previous section, the minima of the K ahler function $G(\xi^I, \xi^{\bar{I}})$ are always in one to one correspondence with the supersymmetric AdS maxima of the scalar potential.

2.5 Supersymmetric Cosmic Strings and FI-terms

In this section we will leave aside supersymmetric vacua, and we will focus on a different type of supersymmetry preserving configurations: *supersymmetric cosmic strings*.

We are going to discuss these solutions in the context of a supersymmetric extension of the Abelian-Higgs model, and to show that they preserve half of the supersymmetries of the system. This review is based on the work by Dvali *et al.* [61], Binetruy *et al.* [75], and Becker *et al.* [57].

The model we review here describes the dynamics of one chiral multiplet, which involves the scalar field acting as a Higgs ϕ and its fermionic partner ζ , one vector multiplet containing the gauge boson A_μ and a gaugino λ , and the graviton multiplet consisting of the gravitino ψ_μ and the graviton itself, represented by the vielbein e_μ^m .

As we discussed in section 2.2, in $\mathcal{N} = 1$ supergravity the action is defined in terms of the Kähler potential, the superpotential, the gauge kinetic functions and the gauge couplings (the killing vectors). We choose the Kähler potential $K = \phi\bar{\phi}$, so that the scalar has trivial kinetic terms as in the Abelian-Higgs model, and we set the superpotential to zero. For simplicity, we will take the gauge kinetic function to be a real constant, which can be set to one via field redefinitions, $f(\phi) = 1$. In order to couple the gauge boson to the scalar field as in the Abelian-Higgs model we gauge the U(1) symmetry of the scalar manifold:

$$\delta_{gauge}\phi = ig\phi\alpha, \quad (2.5.1)$$

which is associated to the usual covariant derivatives $D_\mu\phi = \partial_\mu\phi - ig\phi A_\mu$. Since this symmetry is abelian we can include a FI term $g\eta$ in the theory, and thus the D -term potential is given by:

$$V_D = \frac{1}{2}D^2 \quad \text{where} \quad D = g\eta - g|\phi|^2. \quad (2.5.2)$$

The bosonic sector of the resulting supergravity action reads:

$$e^{-1/2}\mathcal{L} = -\frac{1}{2}R - D_\mu\phi D^\mu\bar{\phi} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g^2(|\phi|^2 - \eta)^2, \quad (2.5.3)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Note that, ignoring the gravitational term, this action is a particular case of the one considered in section 1.2.2, with the parameters of the theory satisfying $2\lambda = g^2$, i.e. the BPS bound. Therefore the vacuum manifold has the same topology $\pi(\mathcal{V}) = \mathbb{Z}$, and there must be topological cosmic strings solutions.

2.5.1 The BPS-equations.

We are interested in cosmic string solutions which leave unbroken a fraction of the supersymmetries, therefore we are looking for a field configuration such that the supersymmetry transformations are zero for some non-vanishing value of the supersymmetry parameter ϵ .

Since we focus on purely bosonic solutions, with all the fermions set to zero, such as those found in section 1.2.2, it is sufficient to study the supersymmetry transformations of the fermions (the rest are vanishing). In our particular model the supersymmetry transformations reduce to:

$$\delta\psi_{\mu L} = \left(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}(e)\gamma_{mn} + \frac{1}{2}iA_\mu^B\right)\epsilon_L, \quad (2.5.4)$$

$$\delta\zeta_L = \frac{1}{2}\not{D}\phi\epsilon_R, \quad (2.5.5)$$

$$\delta\lambda = \frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\epsilon + \frac{1}{2}i\gamma_5 D\epsilon, \quad (2.5.6)$$

where the gravitino U(1) connection A_μ^B is given by:

$$A_\mu^B = \frac{1}{2}i\left[\phi\partial_\mu\bar{\phi} - \bar{\phi}\partial_\mu\phi\right] + gA_\mu\eta. \quad (2.5.7)$$

The condition to have a vanishing supersymmetry transformation of the chiral field (2.5.5) on the string background reads:

$$(\gamma^1 D_1 + \gamma^2 D_2)\phi\epsilon_R = 0. \quad (2.5.8)$$

If we multiply both sides of the equation from the left by $\not{D}\phi$ we obtain:

$$((D_1\phi)^2 + (D_2\phi)^2)\epsilon_R = 0 \iff (D_1 \pm iD_2)\phi = 0, \quad \text{with } \epsilon_R \neq 0. \quad (2.5.9)$$

The condition we just obtained for the chiral field configuration is precisely the first of the BPS equations (1.3.28) derived in section 1.2.2. Note that, although space-time will be curved in general, we are working on a locally Lorentzian frame where the metric is just η_{ab} in order to simplify the manipulations of the gamma matrices.

Substituting the last result back into equation (2.5.8) we obtain a projector condition for the supersymmetry generator:

$$(\gamma^1 \pm i\gamma^2)\epsilon_R = 0, \quad \text{provided that } D_1\phi \neq 0. \quad (2.5.10)$$

The constraint for the left handed supersymmetry generator can be obtained acting with the charge conjugation operator C on the previous expression [94]:

$$(\gamma^1 \mp i\gamma^2)\epsilon_L = 0. \quad (2.5.11)$$

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After some manipulations of the gamma matrices we can write the projector conditions for ϵ_L and ϵ_R together in a single expression:

$$\gamma^{12}\epsilon = \mp i\gamma^5\epsilon. \quad (2.5.12)$$

This condition implies that only half of the supersymmetries will be left unbroken in the string background. To understand this we decompose ϵ into ϵ_+ and ϵ_- , which satisfy the conditions:

$$\gamma^{12}\epsilon_+ = i\gamma^5\epsilon_+, \quad \gamma^{12}\epsilon_- = -i\gamma^5\epsilon_-. \quad (2.5.13)$$

If we choose the upper sign in (2.5.9), then ϵ_- leaves the supersymmetries unbroken while the transformations generated by ϵ_+ are broken, and viceversa.

Let's turn to the supersymmetry transformation of the gaugino (2.5.6). If we consider a static straight cosmic string solution along the z-axis ($\mu = 3$) the only non-vanishing component of the field strength $F_{\mu\nu}$ is F_{12} , therefore the supersymmetry transformation of the gaugino (2.5.6) is simply:

$$\delta\lambda = \frac{1}{2}(\gamma^{12}F_{12} + i\gamma_5 D)\epsilon. \quad (2.5.14)$$

This expression should be equal to zero for any supersymmetry transformation satisfying the projection condition (2.5.12):

$$(\gamma^{12}F_{12} + i\gamma_5 D)\epsilon = i(\mp F_{12} + D)\gamma_5\epsilon = 0, \quad (2.5.15)$$

and thus, for $\epsilon \neq 0$, the field configuration must satisfy:

$$F_{12} \mp D = 0, \quad (2.5.16)$$

which is identical to the second of the BPS equations (1.3.28) found in section 1.2.2.

In order to characterize the space time metric in the background of the cosmic string we need an extra equation, which can be obtained requiring the supersymmetry transformation of the gravitino to be vanishing:

$$(\partial_\mu + \frac{1}{4}\omega_\mu^{mn}(e)\gamma_{mn} + \frac{1}{2}iA_\mu^B)\epsilon_L = 0 \quad (2.5.17)$$

The last equation together with (2.5.9), (2.5.16) and the boundary conditions on the fields (which are specified below) determines the cosmic string solution. To solve them we proceed as in section 1.3.28 and we use the following ansatz for static straight cylindrically symmetric cosmic string:

$$\phi(r, \theta) = \sqrt{\eta}f(r)e^{in\theta} \quad A_\theta = \frac{n}{gr}v(r), \quad (2.5.18)$$

where we are using polar coordinates $\{r, \theta\}$, and $f(r)$ and $v(r)$ are real functions. The most general cylindrically symmetric ansatz for the metric written in polar coordinates is given by [26]:

$$ds^2 = -dt^2 + dz^2 + dr^2 + C^2(r)d\theta^2. \quad (2.5.19)$$

The conditions we obtained above for the chiral field (2.5.9) and the gauge field (2.5.16) are written in a locally lorentzian frame, such as the one defined by the vielbein

$$e^1 = dr \quad \text{and} \quad e^2 = C(r)d\theta. \quad (2.5.20)$$

In these coordinates equations (2.5.9) and (2.5.16) read:

$$(D_r \pm iC(r)^{-1}D_\theta)\phi = 0 \quad (2.5.21)$$

$$C(r)^{-1}F_{r\theta} \mp D = 0, \quad (2.5.22)$$

which after substituting the ansatz (2.5.18) lead to

$$\begin{aligned} C(r)f'(r) &= \pm n f(r) [1 - v(r)], \\ v'(r) &= \pm \frac{g^2 \eta}{n} C(r) [1 - f^2(r)]. \end{aligned} \quad (2.5.23)$$

For the solution to have finite energy the functions $f(r)$ and $v(r)$ must approach to 1 at spatial infinity $r \rightarrow \infty$, and if we require the solution to be regular at the origin then both functions have to vanish for $r \rightarrow 0$.

The gravitino equation (2.5.17) characterizes the profile function of the metric $C(r)$. For the choice of space-time vielbein given in (2.5.20) the only non-vanishing components of the spin connection ω_μ^{mn} are⁶:

$$\omega_r^{12} = 0, \quad \omega_\theta^{12} = -C'(r), \quad (2.5.24)$$

and therefore we write the gravitino equation in polar coordinates reads:

$$\partial_r \epsilon = 0, \quad \left[\partial_\theta - \frac{1}{2} C'(r) \gamma_{12} + \frac{1}{2} i A_\theta^B \right] \epsilon_L = 0, \quad (2.5.25)$$

where we have used that ω_r^{mn} and A_r^B are both vanishing. In order to solve this equation we choose the supersymmetry generator to have the following angular dependence⁷:

$$\epsilon_L(\theta) = e^{\mp \frac{1}{2} i \theta} \epsilon_{0L}. \quad (2.5.26)$$

⁶The components of the spin connection $\omega_\mu^{mn} \equiv \omega_\mu^m{}_p \eta^{pn}$ can be obtained solving the torsion-free condition $de^m + \omega_\mu^m{}_n \wedge e^n = 0$, and $\omega_\mu^{mn} = -\omega_\mu^{nm}$ [88].

⁷This ansatz will ensure that in the absence of strings the metric reduces to $\eta_{\mu\nu}$ which represents a Minkowsky space-time.

2.5. Supersymmetric Cosmic Strings and FI-terms

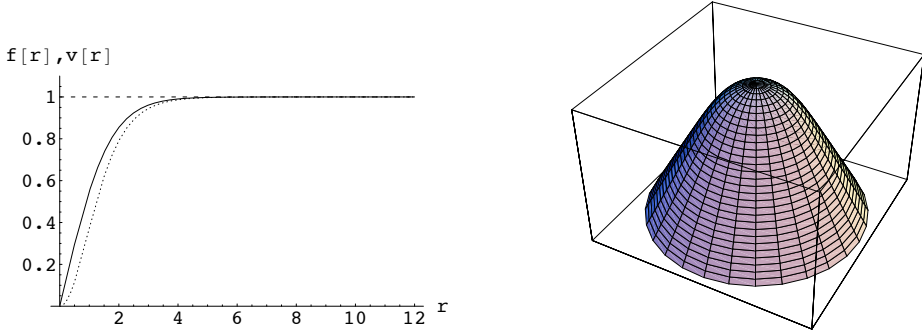


Figure 2.1 – *LEFT: Profile functions (2.5.18) $f(r)$ (solid line) and $v(r)$ (dotted line) characterizing the field configuration of a supersymmetric D-term string, with $n = g = 2\eta = 1$. RIGHT: Embedding of the metric on the plane orthogonal to the string in three dimensions. Far away from the string, which is located at the tip of surface, the metric approaches that of a cone with deficit angle $\Delta = 2\pi n\eta$.*

where ϵ_{0L} is some constant spinor which satisfies the condition (2.5.12). Substituting the ansatz in (2.5.25), we can see that it will vanish provided that the metric profile function $C(r)$ satisfies

$$1 - C'(r) = \pm A_\theta^B, \quad (2.5.27)$$

where the expression for the azimuthal component of the gravitino U(1) connection, A_θ , can be obtained from (2.5.7) after using the ansatz (2.5.18).

$$A_\theta^B = n\eta f^2(r) + \frac{n}{g}v(r)D = n\eta v(r) + n\eta f^2[1 - v(r)]. \quad (2.5.28)$$

The supersymmetric cosmic string configurations are the solutions of the system of ordinary differential equations (2.5.23) and (2.5.27). Any solution of these equations is also a solution of the full set of equations of motion, and in particular t-t component of the Einstein equations can be derived from the gravitino equation (2.5.17) [57].

Regular solutions can only be found provided that we choose the upper sign for $n > 0$ and the lower sign for $n < 0$. In figure 2.1 we have plotted a solution to these equation for the particular values of the parameters $n = g = 2\eta = 1$.

Far away from the core of the string $r \rightarrow \infty$ equation (2.5.27) reduces to

$$C'(r) = 1 - n\eta \implies C(r) \sim (1 - n\eta)r, \quad (2.5.29)$$

and therefore the metric on the plane orthogonal to the string approaches that of a cone

$$ds^2 \approx -dt^2 + dz^2 + dr^2 + r(1 - n\eta)d\theta^2, \quad \text{for } r \rightarrow \infty, \quad (2.5.30)$$

with a deficit angle $2\pi n\eta$. With the Plank masses back the deficit angle reads

$$\Delta = 2\pi n\eta M_p^{-2} = 16\pi^2 nG\eta. \quad (2.5.31)$$

2.5.2 String tension

In a theory including gravity, for time independent configurations, we can use the following definition of the energy:

$$E = \int_M \sqrt{\det g} \left(\frac{M_p^2}{2} R - L_{\text{matter}} \right) + M_p^2 \int_{\partial M} \sqrt{\det h} K. \quad (2.5.32)$$

The last term is the Gibbons-Hawking surface term [95], and K is the Gaussian curvature at the boundary (on which the metric is h). Thus, in the Abelian-Higgs model, the energy of the cosmic string is:

$$\begin{aligned} \mu_{\text{string}} &= \int \sqrt{\det g} \, dr d\theta \left[D_\mu \phi D^\mu \bar{\phi} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^2 + \frac{1}{2} R \right] \\ &\quad + \int d\theta \sqrt{\det h} K \Big|_{r=\infty} - \int d\theta \sqrt{\det h} K \Big|_{r=0}, \end{aligned} \quad (2.5.33)$$

where the sums over μ, ν run only over r, θ . Here the surface term has been evaluated at the boundaries, which are at $r = \infty$ and $r = 0$. Following [96] we choose a highly symmetric ansatz for the metric (2.5.19):

$$\sqrt{\det g} = C(r), \quad \sqrt{\det g} R = 2C'', \quad \sqrt{\det h} K = -C'. \quad (2.5.34)$$

The symmetries of the metric allow us to rearrange the expression for string tension in similar way as we did in section 1.3.2 for a non-gravitational model. Indeed, using the identities (1.3.26) and substituting the ansatz for the metric (2.5.19), the equation (2.5.33) can be rewritten as follows

$$\begin{aligned} \mu_{\text{string}} &= \int dr d\theta C(r) \left\{ |(D_r \phi \pm iC^{-1} D_\theta) \phi|^2 + \frac{1}{2} [F_{12} \mp D]^2 \right\} + \\ &\quad + \int dr d\theta [\partial_r (C' \pm A_\theta^B) \mp \partial_\theta A_r^B] - \int d\theta C' \Big|_{r=\infty} + \int d\theta C' \Big|_{r=0}, \end{aligned} \quad (2.5.35)$$

In the first line, we can recognise the terms between brackets as the l.h.s. of the BPS equations (2.5.22), and therefore they are vanishing. Moreover, the first term in the second line also vanishes in the string background, due to the gravitino BPS equation (2.5.27). The only surviving contribution to the energy are the two boundary terms, which after using (2.5.27) and (2.5.28) read:

$$\mu_{\text{string}} = 2\pi (C'|_{r=0} - C'|_{r=\infty}) = \pm 2\pi n\eta. \quad (2.5.36)$$

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As mentioned in the previous sub section the BPS equations can only be solved if we choose the signs in the BPS equations so that $\pm n = |n|$. Thus all the solutions have positive energy according to (2.5.36). Note that we just obtained the same string tension as in the non-supersymmetric model considered in 1.3.2, and thus these supersymmetric strings saturate the same BPS energy bound. If we write the deficit angle in terms of the string tension we also recover the expression we anticipated in section 1.5

$$\Delta = 8\pi G \mu_{\text{string}}. \quad (2.5.37)$$

The approach to obtain the string tension that we have reviewed here relies on the possibility to rewrite the energy as a sum of squares, as in the non-gravitational case. However, in order to do it we had to choose a highly symmetric ansatz for the metric, i.e. (2.5.19). There are more sophisticated techniques which make use of a spinorial definition for the total energy, and do not require imposing symmetries on the space-time metric. This technique, developed by Nester [97] and Witten [98], has been used in order to study the stability of cosmic strings solutions in supergravity models in three [99] and four space-time dimensions [100], [101].

2.5.3 F -term strings.

For completeness, let us consider another possibility to construct cosmic string solutions in $\mathcal{N} = 1$ supergravity. The scalar potential of the supersymmetric Abelian-Higgs model discussed in the last section is a pure D -term potential, it has no contribution from the F -terms. We have seen that the Fayet-Iliopoulos term plays an essential role because it is responsible for driving the spontaneous breaking of the gauge symmetry, and the tension of the string is proportional to its value.

If we are to construct local cosmic string solutions we cannot avoid the D -term contribution to the potential, due to the coupling between the Higgs and the gauge boson, however it is possible to construct a model which admits cosmic strings solutions in $\mathcal{N} = 1$ supergravity in the absence of a FI term. This can be done including a F -term contribution to the potential to break the gauge symmetry. However, it was proven in [61] that this type of strings, which are called F -term strings, necessarily break all the supersymmetries of the original model. Following [61] we discuss this statement using an example.

Consider a theory with three chiral multiplets involving the scalar fields ϕ_0 , ϕ_+ and ϕ_- , with the Kähler potential given by:

$$K = \phi_0 \bar{\phi}_0 + \phi_+ \bar{\phi}_+ + \phi_- \bar{\phi}_-, \quad (2.5.38)$$

thus with trivial kinetic terms, and minimally coupled to a $U(1)$ gauge field A_μ

with charges $g_0 = 0$, $g_+ = g$, and $g_- = -g$ respectively:

$$\delta_{gauge}\phi_0 = 0, \quad \delta_{gauge}\phi_{\pm} = \pm ig\phi_{\pm}\alpha, \quad D = -g|\phi|^2, \quad (2.5.39)$$

where we have set the FI-term to zero, and we have chosen a constant gauge kinetic function $f(\phi_0, \phi_{\pm}) = 1$ as in the previous section. Moreover, if we use the superpotential

$$W(\phi_0, \phi_+, \phi_-) = \sqrt{\lambda}\phi_0(\phi_+\phi_- - \eta), \quad (2.5.40)$$

the total scalar potential has a vacuum manifold defined by the conditions

$$\phi_0 = 0, \quad |\phi_+|^2 = |\phi_-|^2, \quad \phi_+\phi_- - \eta = 0, \quad (2.5.41)$$

which is isomorphic to S^1 , and thus the theory admits topological string solutions. In this case the constant η is what plays the role of the FI-term.

For the special choice of parameters $2\lambda = g^2$ it is possible to find a cosmic string solution with the same tension as the D -term string (2.5.36), and thus saturating the BPS bound. In that case the string configuration must satisfy $|\phi_+|^2 = |\phi_-|^2 = \eta^2$ and $\phi_0 = 0$.

To check that these strings break all the supersymmetries of the model we can look at the gaugino supersymmetry transformations (2.2.24). In our example at the center of the string the fields are in the configuration $\phi_0 = \phi_+ = \phi_- = 0$, and therefore the D auxiliary field is also vanishing. This implies that in order to preserve supersymmetry the following condition must be satisfied at the center of the string:

$$\delta\lambda|_{core} = \frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}|_{core}\epsilon = 0. \quad (2.5.42)$$

It can be seen that, since there is nothing to compensate the magnetic term, this expression cannot be zero for any non zero value of the supersymmetry parameter, and therefore the string solution breaks all the supersymmetries.

This conclusion is also true for the globally supersymmetric Abelian-Higgs model [54] since the vacuum manifold of the scalar potential has the same topology, and the supersymmetry transformation of the gaugino contains the same terms.

It was shown in [102] that, for certain values of the coupling constants, F -term strings can also be BPS saturated objects in the sense that they preserve half of the supersymmetries of the model. For that particular choice of the parameters the model becomes invariant under a second supersymmetry, and actually the F -term string breaks only half of the $\mathcal{N} = 2$ supersymmetries.

Consistent supersymmetric decoupling of heavy scalars in $\mathcal{N} = 1$ supergravity.

3.1 Introduction

As we mentioned in the first chapter, the problem of supersymmetric decoupling of heavy fields is specially relevant when considering moduli stabilization in flux compactifications of superstring/M-theory. In 2003 Kallosh, Kachru, Linde and Trivedi (KKLT) [21] provided the first example of a mechanism to stabilize all the moduli of a compactification. In their approach all moduli were stabilized at a supersymmetric critical point of the potential, which was generated by a combination of background fluxes and non-perturbative effects. A sector of these fields, the complex structure moduli, were stabilized at some high energy scale and integrated out in a supersymmetric way, leaving behind an effective theory described by $\mathcal{N} = 1$ supergravity.

In general the solutions of these effective theories are only approximate solutions to the full theory, and are valid for energies much lower than the mass scale

Supersymmetric decoupling of heavy scalars in $\mathcal{N} = 1$ supergravity.

of the fields which are integrated out $E \ll M_H$,

$$\frac{\delta S_{eff}}{\delta L^i} = 0 \quad \implies \quad \frac{\delta S}{\delta L^i} = 0 + \mathcal{O}(E/M_H). \quad (3.1.1)$$

Similarly, supersymmetry is only preserved approximately in the low energy effective theory, and the supersymmetry breaking terms are typically assumed to be suppressed by the masses of the heavy fields. In order to know to what extent are these approximations valid it is important to understand precisely which type of couplings ensure that the solutions of the effective theory are *exact* solutions of the full theory, and that supersymmetry is *exactly* preserved. In particular this analysis is useful to characterize the interactions between the fields appearing in the low energy effective action and the heavy sector. Such interactions have been proven to be specially relevant in inflationary models in supergravity theories [103, 104, 105], where they might leave an observable imprint on the Cosmic Microwave Background.

The aim of the present chapter is to study the consistency conditions for the consistent supersymmetric truncation of part of the fields in a $\mathcal{N} = 1$ supergravity theory. The truncation is defined fixing a fraction of the fields, denoted by H^α , at homogeneous and static configuration, i.e. a covariantly constant configuration:

$$\nabla_\mu H^\alpha = 0, \quad (3.1.2)$$

so that the Lorentz symmetry is preserved. Moreover, the condition of unbroken Lorentz symmetry also implies that the field strengths corresponding to truncated gauge bosons $A_\mu^{\tilde{a}}$ must vanish too:

$$F_{\mu\nu}^{\tilde{a}} = 0 \quad (3.1.3)$$

In section 3.1 we derive the necessary conditions for the corresponding reduced theory to preserve $\mathcal{N} = 1$ supersymmetry which, as we shall see, also ensure that every solution of the reduced model is also a solution of the full equations of motion. Thus the conditions we find can be classified in two blocks: those which guarantee that the equations of motion respect the truncation of any theory (not necessarily supersymmetric), and those which are specific to supersymmetric theories.

We begin reviewing the first set of conditions. The vacuum expectation value of the truncated scalar fields might break some of the gauge symmetries, therefore the corresponding gauge bosons become massive and their field strengths must vanish. The time evolution of the fields has to be consistent with the truncation:

- The kinetic terms of the truncated scalar fields must be decoupled from the kinetic terms of the surviving ones in the reduced theory.

3.1. Introduction

- The kinetic terms of the truncated gauge bosons should be decoupled from kinetic terms of the surviving ones, in particular the gauge kinetic functions should have a block diagonal structure $f_{\tilde{a}b} = 0$, where the indices with a tilde correspond to the gauge fields associated to broken symmetries and the other ones to the surviving gauge fields.

The truncated fields should not reappear due to gauge interactions:

- If the gauge group of the mother theory is non-abelian, then the truncated gauge bosons should not be sourced by the surviving ones by their interactions. In particular the full gauge group should have a cross product structure of the form $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where $\mathbb{G}_h$ is the subgroup associated to the truncated gauge fields, and $\mathbb{G}_l$ corresponds to the surviving subgroup.
- The gauge fields associated to broken symmetries, $\mathbb{G}_h$, should not couple to the scalar fields surviving the truncation.
- The truncated scalar fields should not couple to the gauge fields associated to unbroken symmetries, $\mathbb{G}_l$.

The conditions which are specific to consistent supersymmetric truncations from $\mathcal{N} = 1 \rightarrow \mathcal{N} = 1$ are quite obvious: each time we truncate a field we must also truncate the whole supermultiplet which contains it. In particular if we truncate a scalar field we must truncate the whole chiral multiplet, including the auxiliary fields, leading to the usual condition on the F -terms:

$$D_H W = 0, \quad (3.1.4)$$

In the case of the gauge fields which become massive we must truncate the whole vector multiplet, including auxiliary field, and thus the moment map associated to the broken gauge symmetry should be zero

$$\mathcal{P}_h = 0. \quad (3.1.5)$$

The condition (3.1.4) can be expressed as a constraint on the Kähler function. In section 3.3 and 3.4 we discuss the form of the Kähler functions satisfying this constraint, and we study explicit examples. In section 3.5 we consider the consistency of the truncation when the truncated and surviving sectors are coupled using the following ansatz:

$$\widehat{K} = K_1(H, \bar{H}) + K_2(L, \bar{L}) \quad \widehat{W} = W_1(H) + W_2(L), \quad (3.1.6)$$

which in *global supersymmetry* describes non-interacting sectors. Here the fields surviving the truncation have been denoted by L .

The problem of truncating a sector of a theory while preserving a fraction of the supersymmetries has been studied thoroughly in the context of reductions

of $\mathcal{N}$ extended supergravity theories, to a lower $\mathcal{N}'$ supergravity theories [106], [107]. The study we present here is done in the context of $\mathcal{N} = 1$ supergravity models requiring supersymmetry to be fully preserved $\mathcal{N} = \mathcal{N}'$, and has not been published to the best of our knowledge.

This analysis is intended to be used to study the consistency of truncating a heavy sector in cosmological models, thus we use the notation H to denote the truncated scalar fields, which represent the would-be heavy scalar fields, and L to denote the surviving scalar fields, which would correspond to the light scalars. However, as we will see in chapter 4, even when all the consistency conditions are met, there are unavoidable interactions between the two sectors which might turn the ansatz for the truncation unstable, and thus the names “heavy” and “light” are no longer appropriate for the truncated sector and surviving sectors respectively.

3.2 Consistency conditions for supersymmetric integration.

Consider a general $\mathcal{N} = 1$ supergravity theory as (2.2.3),

$$\mathcal{S} = \int dx_4 \sqrt{-g} \left(-\frac{1}{2} R + \mathcal{T} + \mathcal{L}_{gauge} - V \right),$$

where the fields ξ^I can be split into two sets, H^α and L^i , which represent the heavy and light degrees of freedom respectively, with $\alpha = 1, \dots, n_h$ and $i = 1, \dots, n_l$. The kinetic terms for the scalars $\mathcal{T}$ and the gauge bosons $\mathcal{L}_{gauge}$ were defined in (2.2.4) and (2.2.6) respectively, and the scalar potential V is given by the expressions (2.2.7), (2.2.8) and (2.2.14). We will assume that the masses of the heavy fields are large compared with the energy scale of the physics we are interested in, so that they can be stabilized at an extremum of the scalar potential with a vacuum expectation value¹ H_0^α . After truncating the heavy moduli we are left with an effective field theory describing the dynamics of the light fields only. From now on, we will use hatted quantities in the full theory and unhatted quantities for the effective theory:

$$S(L, \bar{L}) = \hat{S}(H_0, \bar{H}_0, L, \bar{L}). \quad (3.2.1)$$

For the truncation to be consistent, the equations of motion of the light fields derived from the effective theory must coincide with the equations of motion obtained from the full action. To zeroth order in the fluctuations of the heavy

¹Note that the vacuum expectation value of the heavy scalars H_0^α is only determined up to gauge transformations.

3.2. Supersymmetric integration

fields the effective action S should satisfy:

$$\frac{\delta \widehat{S}}{\delta L} \Big|_{H_0} = \frac{\delta \widehat{S}|_{H_0}}{\delta L} = \frac{\delta S}{\delta L}, \quad \text{where} \quad \frac{\delta \widehat{S}}{\delta H} \Big|_{H_0} = 0, \quad (3.2.2)$$

ensuring that the fluctuations of H are not sourced by the light fields. Actually the truncation $H = H_0^\alpha$ can be seen as an ansatz to solve the full equations of motion. As we discussed in the introduction, we are interested in the case where the low energy effective theory preserves the invariance under $\mathcal{N} = 1$ local supersymmetry. Therefore we must also require the supersymmetry transformations to be compatible with the truncation of the heavy fields, which leads to conditions on the couplings and matter content of the effective theory. Since these conditions are more easily obtained than those derived from the equations of motion (3.2.2), we will start with the analysis of the supersymmetry transformations, and then we will find out the extra constraints imposed by (3.2.2).

3.2.1 Reduction of the chiral multiplets.

The supersymmetry transformations must respect the ansatz of the truncation $H = H_0^\alpha$, and therefore from (2.2.21) we find that:

$$\delta_\epsilon H^\alpha = \bar{\epsilon}_L \chi_L^\alpha = 0, \quad (3.2.3)$$

which implies that the fermionic partners of the heavy fields, χ^α , must also be truncated:

$$\chi^\alpha = 0 \quad \implies \quad \delta_\epsilon \chi_L^\alpha = \frac{1}{2} \gamma^\mu \nabla_\mu H^\alpha \epsilon_R - \frac{1}{2} e^{\frac{1}{2} \hat{K}} \widehat{G}^{\alpha \bar{J}} \mathcal{D}_{\bar{J}} \widehat{W} \Big|_{H_0} \epsilon_L = 0. \quad (3.2.4)$$

In order for the supersymmetry transformations to be consistent with the truncation of the heavy chiral fermions we have to set to zero the following quantities in the reduced theory :

$$\widehat{G}^{\alpha \bar{J}} \mathcal{D}_{\bar{J}} \widehat{W} \Big|_{H_0} = [\widehat{G}^{\alpha \bar{\beta}} \mathcal{D}_{\bar{\beta}} \widehat{W} + \widehat{G}^{\alpha \bar{j}} \mathcal{D}_{\bar{j}} \widehat{W}]_{H_0} = 0, \quad (3.2.5)$$

$$\nabla_\mu H^\alpha = \partial_\mu H^\alpha - k_a^\alpha(H_0, L) A_\mu^a = 0, \quad (3.2.6)$$

which have to be satisfied for any value of the light fields L^i . Since we want to allow the possibility that supersymmetry is broken by the light fields, in principle the quantity $\mathcal{D}_{\bar{i}} \widehat{W}$, which is related to the auxiliary field of the light chiral multiplets, could have any arbitrary value. Therefore, in order to solve (3.2.5) we must require the Kähler metric to be block diagonal in the two sectors at $H^\alpha = H_0^\alpha$:

$$\widehat{G}_{\alpha \bar{i}} \Big|_{H_0} = \widehat{G}_{\bar{\alpha} i} \Big|_{H_0} = 0. \quad (3.2.7)$$

This condition ensures that the kinetic terms of the light and heavy scalar fields are decoupled. This is required for the equations of motion to respect the truncation, since otherwise the propagators of the scalars would mix the light and

Supersymmetric decoupling of heavy scalars in $\mathcal{N} = 1$ supergravity.

heavy fields. Using (3.2.7) we can see that the condition (3.2.5) reduces to:

$$\mathcal{D}_\alpha \widehat{W}|_{H_0} = (\partial_\alpha \widehat{W} + \partial_\alpha \widehat{K} \widehat{W})_{H_0} = 0 \quad (3.2.8)$$

When the superpotential is not vanishing, $W(H_0, L) \neq 0$, we can write this constraint in terms of the Kähler function $\widehat{G}(H, \bar{H}, L, \bar{L})$:

$$\widehat{G}_\alpha(H_0, \bar{H}_0, L, \bar{L}) = 0 \quad \text{for all } L^i. \quad (3.2.9)$$

This also implies that all the higher order derivatives of $G_\alpha|_{H_0}$ with respect to the light fields are zero, such as the components of the Kähler metric mixing light and heavy fields $G_{\alpha\bar{i}}$, which is precisely the constraint (3.2.7). The condition (3.2.7), when combined with (3.2.6), leads to a simplification of the kinetic terms of the scalars $\mathcal{T}$ (2.2.4)

$$\begin{aligned} \mathcal{T}|_{H_0} &= \left[\widehat{G}_{\alpha\bar{\beta}} \nabla_\mu H^\alpha \nabla^\mu H^{\bar{\beta}} + \widehat{G}_{i\bar{j}} \nabla_\mu L^i \nabla^\mu L^{\bar{j}} + \widehat{G}_{\alpha\bar{j}} \nabla_\mu H^\alpha \nabla^\mu L^{\bar{j}} + \right. \\ &\quad \left. \widehat{G}_{i\bar{\beta}} \nabla_\mu H^i \nabla^\mu L^{\bar{\beta}} \right]_{H_0} = \widehat{G}_{i\bar{j}}|_{H_0} \nabla_\mu L^i \nabla^\mu L^{\bar{j}}. \end{aligned} \quad (3.2.10)$$

According to the expression we obtained for the kinetic terms of the reduced theory (3.2.10), the truncation $H^\alpha = H_0^\alpha$ must define a Kähler submanifold $\mathcal{M} \subseteq \widehat{\mathcal{M}}$ of the original scalar manifold. The metric of $\mathcal{M}$ is given by the second derivatives of the Kähler potential of the reduced theory $K(L, \bar{L})$:

$$G_{i\bar{j}} = \partial_{i\bar{j}}^2 K(L, \bar{L}) \quad \text{with} \quad K(L, \bar{L}) \equiv \widehat{K}(H_0, \bar{H}_0, L, \bar{L}), \quad (3.2.11)$$

where the Kähler potential is determined only up to Kähler transformations (2.2.28). On the other hand, looking at the supersymmetry transformation of the gravitino (2.2.22),

$$\delta\psi_{\mu L} = (\partial_\mu + \frac{1}{4}\omega_\mu^{ab}\gamma_{ab} + i\hat{A}_\mu^B)_{H_0}\epsilon_L + \frac{1}{2}[e^{\hat{K}/2}\widehat{W}]_{H_0}\gamma_\mu\epsilon_R \quad (3.2.12)$$

we can identify the superpotential of the reduced theory, which is given by

$$W(L) = \widehat{W}(H_0, L). \quad (3.2.13)$$

Therefore we can express the reduced Kähler function $G(L, \bar{L})$ in terms of the Kähler function of the full theory as follows:

$$G(L, \bar{L}) = K(L, \bar{L}) + \log |W(L, \bar{L})|^2 = \widehat{G}(H_0, \bar{H}_0, L, \bar{L}). \quad (3.2.14)$$

Before concluding with the chiral multiplets we have to check that the condition (3.2.9), and thus also (3.2.7), is respected by the supersymmetry transformations. If this were not the case we would have to impose additional constraints to ensure consistency. Note that the equation (3.2.9) only depends on bosonic quantities,

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and then it could only vary under supersymmetry transformations due to the shifts of the scalar fields $\delta_\epsilon H^\alpha$ and $\delta_\epsilon L^i$

$$\delta_\epsilon G_\alpha|_{H_0} = [\nabla_\beta G_\alpha \delta_\epsilon H^\beta + \nabla_{\bar\beta} G_\alpha \delta_\epsilon H^{\bar\beta} + \nabla_i G_\alpha \delta_\epsilon L^i + \nabla_{\bar i} G_\alpha \delta_\epsilon L^{\bar i}]_{H_0}. \quad (3.2.15)$$

Here ∇_I denotes the covariant derivative involving the Levi-Civita connection of the Kähler manifold. As we have discussed at the beginning of the section, the heavy fields do not shift $\delta_\epsilon H^\alpha = 0$ in a supersymmetric background, and thus they cannot induce a change in (3.2.9). On the other hand, due to the condition (3.2.9), which is valid for any value of the light fields L^i , the covariant derivatives appearing in the last two terms of (3.2.15) vanish in the configuration $H^\alpha = H_0^\alpha$

$$\nabla_i G_\alpha|_{H_0} = [G_{i\alpha} - \Gamma_{i\alpha}^\beta G_\beta - \Gamma_{i\alpha}^j G_j]_{H_0} = 0 \quad (3.2.16)$$

$$\nabla_{\bar i} G_\alpha|_{H_0} = G_{\bar i\alpha}|_{H_0} = 0. \quad (3.2.17)$$

In particular, it is possible to prove that the components of the Kähler connection $\Gamma_{i\alpha}^j$ are zero using their relation to the metric $G_{I\bar J}$

$$\Gamma_{i\alpha}^j|_{H_0} = [G^{j\bar k} G_{\bar k i\alpha} + G^{j\bar\beta} G_{\bar\beta i\alpha}]_{H_0} = 0. \quad (3.2.18)$$

This condition ensures that the submanifold $\mathcal{M}$ that defines the reduced theory is totally geodesic (see [108]), which is required for the equations of motion to be consistent with the truncation. Gathering these results we can conclude that supersymmetry preserves the condition (3.2.9), i.e. its transformation under supersymmetry (3.2.15) is zero, and thus there is no need to impose additional constraints.

3.2.2 Reduction of the vector multiplets.

We start the discussion by studying the variation of (3.2.6) under supersymmetry transformations. Using the same argument as in the previous paragraph it is possible to show that the shifts of the chiral fields, $\delta_\epsilon L^i$ and $\delta_\epsilon H^\alpha$, do not induce variations on the condition² (3.2.6). Therefore, for (3.2.6) to be respected by supersymmetry transformations, we only have to impose the constraints:

$$\delta_\epsilon (\partial_\mu H^\alpha - k_a^\alpha(H, L) A_\mu^a)|_{H_0} = -k_a^\alpha(H_0, L) \delta_\epsilon A_\mu^a = 0, \quad (3.2.19)$$

which should hold for every $\alpha = 1, \dots, n_h$ and for any configuration of the light fields L^i . Squaring the previous equation we can express the constraint as

$$M_{ab} \delta_\epsilon A_\mu^a \delta_\epsilon A^{b\mu} = 0, \quad M_{ab} \equiv (G_{\alpha\bar\beta} k_a^\alpha k_b^{\bar\beta})_{H_0}, \quad (3.2.20)$$

²In this reasoning we must also use the fact that the supersymmetry transformations and spatial translations commute $[P_\mu, \delta_\epsilon] = 0$.

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where the symmetric matrix M_{ab} is proportional to the mass matrix of the gauge bosons. The matrix M_{ab} can always be diagonalized redefining the gauge fields conveniently (2.4.23)

$$A^b = O_a^b \tilde{A}^a \quad \tilde{k}_b^\alpha = O_b^a k_a^\alpha, \quad \tilde{M}_{cd} = M_{ab} O_c^a O_d^b \quad (3.2.21)$$

where O_a^b is a non-singular real matrix. If the mass matrix M_{ab} is diagonal then, choosing the Coulomb gauge $A_0^a = 0$, the condition (3.2.20) reads

$$\sum_a M_{aa} (\delta_\epsilon \vec{A}^a)^2 = \sum_a (G_{\alpha\bar{\beta}} k_a^\alpha k_a^{\bar{\beta}})_{H_0} (\delta_\epsilon \vec{A}^a)^2 = 0. \quad (3.2.22)$$

Since the expressions in parentheses are the square of the killing vectors, all the terms in the sum are positive semidefinite. Thus, for the sum to be zero all the terms have to vanish

$$(G_{\alpha\bar{\beta}} k_a^\alpha k_a^{\bar{\beta}})_{H_0} (\delta_\epsilon \vec{A}^a)^2 = 0 \quad \text{for all } a = 1, \dots, n_V. \quad (3.2.23)$$

Each of these equations admit two different solutions. If the gauge symmetry associated to A_μ^a remains unbroken after the stabilization of the heavy fields, i.e. $k_a^\alpha(H_0, L) = 0$ for all L^i , then we do not have to impose any more constraints. Moreover, the equation (3.2.6) does not impose any condition on the gauge field A_μ^a and we can keep the full vector multiplet in the low energy theory. Note that the massless gauge bosons are decoupled from the heavy fields in the reduced theory $k_a^\alpha(H_0, L) = 0$, as required by consistency, since otherwise the heavy fields could be sourced due to gauge interactions.

If, on the contrary, the gauge symmetry is spontaneously broken by the vacuum expectation values of the heavy fields, i.e. $k_a^\alpha(H_0, L) \neq 0$, then the supersymmetry transformations $\delta_\epsilon A_\mu^{\tilde{a}}$ must vanish:

$$\delta_\epsilon A_\mu^{\tilde{a}} = -\frac{1}{2} \bar{\epsilon} \gamma_\mu \lambda^{\tilde{a}} = 0. \quad (3.2.24)$$

In the following we will indicate with a tilde, i.e. $\lambda^{\tilde{a}}$, those indices that correspond to the heavy gauge bosons, while the indices without a tilde are associated to the massless bosons. This constraint implies that the fermionic partner of the massive gauge boson A_μ^a has to be truncated:

$$\lambda^{\tilde{a}} = 0 \quad \implies \quad \delta_\epsilon \lambda^{\tilde{a}} = \frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu}^{\tilde{a}} \epsilon + \frac{1}{2} i D^{\tilde{a}} \gamma_5 \epsilon = 0. \quad (3.2.25)$$

Therefore, in order for the supersymmetry transformations to respect the truncation of the gaugino, we have to require the following quantities to vanish in the low energy theory:

$$F_{\mu\nu}^{\tilde{a}} = \partial_{[\mu} A_{\nu]}^{\tilde{a}} + f_{\tilde{b}\tilde{c}}^{\tilde{a}} A_\mu^{\tilde{b}} A_\nu^{\tilde{c}} + f_{bc}^{\tilde{a}} A_\mu^b A_\nu^c + f_{\tilde{b}\tilde{c}}^{\tilde{a}} A_{[\mu}^b A_{\nu]}^{\tilde{c}} = 0, \quad (3.2.26)$$

$$D^{\tilde{a}} = [(\text{Re } f)^{-1|\tilde{a}\tilde{b}} \mathcal{P}_{\tilde{b}} + (\text{Re } f)^{-1|\tilde{a}b} \mathcal{P}_b]_{H_0} = 0. \quad (3.2.27)$$

3.2. Supersymmetric integration

If these conditions are to be satisfied for any configuration of the massless gauge bosons A_μ^a , the following components of the structure constants have to vanish

$$f_{bc}^{\tilde{a}} = f_{b\tilde{c}}^{\tilde{a}} = 0. \quad (3.2.28)$$

If the original gauge symmetry group $\mathbb{G}$ is semi-simple these conditions imply that it must have a cross product structure $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where subgroups $\mathbb{G}_h$ and $\mathbb{G}_l$ act on the heavy and light fields respectively at the point H_0^α . Then the equation (3.2.26) reduces to

$$F_{\mu\nu}^{\tilde{a}} = \partial_{[\mu} A_{\nu]}^{\tilde{a}} + f_{b\tilde{c}}^{\tilde{a}} A_\mu^b A_\nu^{\tilde{c}} = 0, \quad (3.2.29)$$

which implies that the heavy gauge bosons have to be truncated also. The killing vectors of the light sector k_a^i , and thus also $\mathcal{P}_a$, could have any arbitrary value. Therefore, in order to satisfy the condition (3.2.27) for any value of the light fields we have to set to zero the real part of the components $f_{ab}(H, L)$ mixing the massive and massless gauge fields:

$$\text{Re } f_{a\tilde{b}}(H_0, L) = 0 \quad \text{for any } L^i. \quad (3.2.30)$$

This condition is necessary to ensure that the kinetic terms of the massive and massless gauge bosons are decoupled, so that the dynamical evolution also respects the truncation³. Using the last equation we can see that the condition (3.2.27) implies that the moment maps associated to the killing vectors of the broken symmetries have to vanish in the reduced theory

$$\mathcal{P}_{\tilde{a}}(H_0, L) = 0. \quad (3.2.31)$$

Since the moment map determines the killing vectors (2.2.15), in the reduced theory the killing vectors of the broken symmetries cannot have non-vanishing components along the light directions:

$$k_{\tilde{a}}^i(H_0, L) = -iG^{i\bar{j}}\partial_{\bar{j}}\mathcal{P}_{\tilde{a}}|_{H_0} = 0. \quad (3.2.32)$$

The last constraints are consistent with the intuition that the heavy gauge bosons should decouple from the light fields, otherwise the light fields could source the massive gauge bosons that were integrated out. Finally, using the conditions we just found for the killing vectors and the complex structure constants, it is possible to show that in order to satisfy the lie algebra (2.2.11) the killing vectors have to satisfy:

$$k_{a,i}^\alpha(H_0, L) = k_{a,\alpha}^i(H_0, L) = 0 \quad \text{for all } L^i \quad (3.2.33)$$

It is straightforward to check that no more constraints are necessary to ensure that supersymmetry respects the conditions derived in this section.

³Since the gauge kinetic functions are holomorphic and $\text{Re } f_{a\tilde{b}}(H_0, L) = 0$, then the imaginary part of these components at the configuration $H^\alpha = H_0^\alpha$ has to be independent of the values of the light fields, in other words $\text{Im } f_{a\tilde{b}}(H_0, L) = \text{Im } f_{a\tilde{b}}(H_0)$.

3.2.3 Two fermion terms in the supersymmetry variations.

For simplicity, the supersymmetry transformations presented in the last chapter (2.2.21-2.2.24) only contained terms linear in the fermions, but the full supersymmetry transformations include higher order terms in the fermions. In particular, a set of supersymmetry transformations of the bosons which also contain two fermion terms can be found in references [89]. These terms are of the form:

$$\delta_\epsilon b \sim f f \epsilon, \quad (3.2.34)$$

where b and f denote generic boson and fermion fields. By not considering these higher order terms we are overlooking important consistency conditions for the supersymmetric truncation of the heavy fields. Indeed, we will now prove that the conditions we have derived so far are insufficient to ensure that the heavy field configuration H_0^α is an extremum of the action (2.2.3). This implies that the dynamics of the system are not consistent with the truncation, or in other words, equation (3.2.2) is not satisfied.

Since the action must be extremized with respect to variations of the heavy fields for any configuration of the light fields, it is sufficient to consider each of the terms of the action independently. The gravitational term does not need to be considered because it is independent of the heavy sector. It is easy to check that, due to the constraints (3.2.6), (3.2.9) and (3.2.33), the kinetic terms of the scalars, $\mathcal{T}$, are extremized when the heavy fields are integrated out in a supersymmetric way at $H^\alpha = H_0^\alpha$. On the other hand, neither the kinetic terms of the gauge fields $\mathcal{L}_{\text{gauge}}$, nor the scalar potential V , are extremized by the heavy field configuration. For instance, after using the condition $k_a^\alpha(H_0, L) = 0$ and the equations (3.2.28-3.2.31), we find the variation of the action along H^α is of the form:

$$\delta_\alpha \mathcal{S} \sim \int \frac{1}{4} \text{Re } f_{ab,\alpha}(H_0, L) F_{\mu\nu}^a F^{b\mu\nu} + \frac{1}{2} [(\text{Re } f^{-1ab})_{,\alpha} \mathcal{P}_a \mathcal{P}_b]_{H_0}, \quad (3.2.35)$$

where, as in the previous section the indices a and b run only over the massless gauge bosons. The first term arises from the variation of the kinetic terms of the gauge fields and the second one from the D -term potential V_D . Since, a priori, the field strengths and moment maps associated to the light gauge bosons are arbitrary, a consistent truncation requires that the gauge kinetic function satisfies:

$$\text{Re } f_{ab,\alpha}(H_0, L) = (\text{Re } f(H_0, L)^{-1ab})_{,\alpha} = 0 \quad \text{for all } L^i. \quad (3.2.36)$$

This condition is also necessary in order for the truncation to preserve $\mathcal{N} = 1$ supergravity. The reason we did not find it in the previous two sections is that it only appears when the two-fermion terms of the supersymmetry transformations are taken into account. For instance, after using the conditions found in the

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previous two sections, the variations of the heavy chiral fermions χ^α still contain a non-vanishing two fermion term of the form [89]

$$\delta_\epsilon \chi_L^\alpha \sim \epsilon_L \bar{\lambda}_R^a \lambda_R^b [G^{\alpha\bar{\beta}} \bar{f}_{ab,\bar{\beta}}]_{H_0}. \quad (3.2.37)$$

For the truncation to preserve supersymmetry, these variations have to be zero for any configuration of the light gaugini λ^a , and therefore we have to require (3.2.36) to be satisfied.

With some algebra it is possible to prove that, the conditions we found in the last sections together with (3.2.36), are sufficient to guarantee that the truncation is consistent with the dynamics of the system, i.e. that the solutions of the reduced theory are also solutions of the full equations of motion (3.2.2). In particular, in order to freeze the heavy fields consistently the mass matrix of the fluctuations around any configuration of the form (H_0^α, L^i) should be block diagonal in the heavy and light sectors:

$$\partial_{i\alpha}^2 V(H_0, L) = 0, \quad \partial_{i\bar{\alpha}}^2 V(H_0, L) = 0 \quad \text{for all } L^i. \quad (3.2.38)$$

Indeed, to integrate out the fluctuations with large masses around a given vacuum first we have to find their mass spectrum, which requires diagonalizing the mass matrix, and only after having identified the heavy modes can we set them consistently to zero. Proceeding in this way, by construction, the mass matrix at (H_0^α, L^i) is always block diagonal in the massive and light modes. If the heavy fields are truncated in a supersymmetric way, it is not hard to check that the scalar potential satisfies these requirements.

3.3 Consistency of the effective action

In the literature there are several studies about how to obtain the low energy effective action left after the supersymmetric integration of the heavy fields such as [109, 110, 111, 91]. We will now describe the general ideas involved in these approaches.

Suppose we are given a particular theory, defined by a Kähler potential $\widehat{K}$ and a superpotential $\widehat{W}$, which contains a heavy scalar sector H^α stabilized at some large energy scale. In order to integrate out the heavy fields while preserving supersymmetry the heavy field configuration has to satisfy equation (3.2.8)

$$\left[\partial_\alpha \widehat{W}(H, L) + \partial_\alpha \widehat{K}(H, \bar{H}, L, \bar{L}) \widehat{W}(H, L) \right]_{H_0} \equiv \Phi(H, \bar{H}, L, \bar{L}) = 0. \quad (3.3.1)$$

The left hand side is some function of both the heavy and the light fields, let us call it $\Phi(H, \bar{H}, L, \bar{L})$. In general, the condition $\Phi = 0$ relate the heavy and light

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fields. Therefore, in general, if we solve for H^α we obtain an expression of H_0^α as a function of the light fields,

$$H^\alpha = H_0^\alpha(L, \bar{L}). \quad (3.3.2)$$

The most naive approach would be to substitute this expression back into $\widehat{K}, \widehat{W}$, and identify the resulting functions as the Kähler potential and superpotential of the effective theory

$$K \equiv K(H(L, \bar{L}), \bar{H}(L, \bar{L}), L, \bar{L}), \quad W = W(H(L, \bar{L}), L). \quad (3.3.3)$$

These two quantities would then define an effective action for the light fields

$$S(L, \bar{L}) = \widehat{S}(H_0(L, \bar{L}), \bar{H}_0(L, \bar{L}), L, \bar{L}). \quad (3.3.4)$$

An immediate concern with the consistency of this procedure, pointed out in [109], is that in general this leads to a non-holomorphic expression for the would-be effective superpotential W . A possible solution to this problem was suggested in [109, 76]: there is no conflict with the holomorphicity of the effective superpotential if $\widehat{W}$ is independent of H . The case $\widehat{W} = 0$ is obvious, so let us consider $\widehat{W} \neq 0$. Then, it is always possible to perform a Kähler transformation that makes $\widehat{W}$ constant

$$\widehat{K} \rightarrow \widehat{K} + \log \widehat{W} + \log \bar{\widehat{W}} \equiv \widehat{G} \quad \widehat{W} \rightarrow 1. \quad (3.3.5)$$

In this so called Kähler gauge, eq. (3.2.8) becomes equivalent to (3.2.9)

$$\widehat{G}_\alpha(H, \bar{H}, L, \bar{L}) = 0, \quad (3.3.6)$$

from which we can extract $H^\alpha = H_0^\alpha(L, \bar{L})$. Then we make the previous substitution directly into the Kähler invariant function without having to deal with non-holomorphic superpotentials:

$$G = \widehat{G}(H_0(L, \bar{L}), \bar{H}_0(L, \bar{L}), L, \bar{L}). \quad (3.3.7)$$

Although this approach is apparently satisfactory, it is not entirely consistent yet. The origin of these difficulties can be traced back to the expression (3.3.2). The whole approach can only be consistent with our original assumptions provided the heavy fields H^α have no dependence on the light fields. Indeed, in section 3.2.1 the condition (3.2.8) was derived supposing that the heavy fields H^α were stabilized at the *constant* configuration H_0^α . If the constraint (3.2.9) requires the heavy fields to depend on the light fields, then the approach is not self-consistent. This means that either the heavy fields have not been identified correctly, or the Kähler function $\widehat{G}$ is not suitable to truncate the heavy fields in a supersymmetric way.

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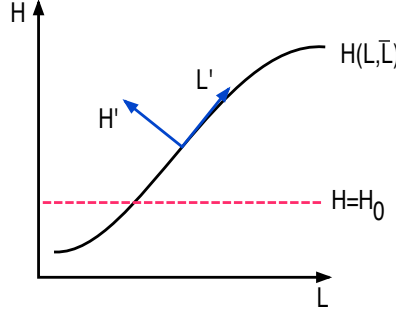


Figure 3.1 – The solid line represents schematically a general solution of (3.3.1), i.e. (3.3.2), which defines the scalar manifold of the reduced theory. Only solutions where H^α is independent of L^i (dashed line) are consistent with a supersymmetric integration.

To clarify the situation, let us consider the meaning of equation (3.3.2). This equation defines the scalar manifold of the reduced theory, $\mathcal{M} \subseteq \widehat{\mathcal{M}}$, which characterizes the kinetic terms in the truncated action. The submanifold $\mathcal{M}$ can be parametrized in the following way

$$\xi^I = (L^i, H^\alpha(L, \bar{L})), \quad (3.3.8)$$

where the fields ξ^I are a parametrization of the full target space $\widehat{\mathcal{M}}$. In section 3.2.1 we proved that $\mathcal{M}$ must be a Kähler submanifold of the full scalar manifold. However, the previous expression only defines a complex submanifold of $\widehat{\mathcal{M}}$ provided that, at least locally, the fields H^α can be expressed as holomorphic functions of L^i [112], so that we have $H^\alpha = H^\alpha(L)$.

This field dependence implies that the *true* light directions on field space are no longer characterized by L^i , but instead they are given by a combination of L^i and H^α . Similarly, the heavy fields can no longer be identified with H^α , since these fields are still allowed to fluctuate as long as they satisfy (3.3.2), and thus they are not the ones to be truncated. This situation is represented schematically in figure 3.1. The solid line of the figure represents the scalar manifold of the reduced theory, which is defined by the solution (3.3.2). The “light directions” in field space are those along the solid curve, such as L'^i . At a particular point of the curve, the “heavy directions” are those orthogonal to the curve with respect to the Kähler metric (3.2.7), such as H'^α . In general, unless the expression for H^α in (3.3.2) is independent from L^i , the fields to be integrated out H'^α do not coincide with H^α , and therefore the condition (3.3.1) is not the one that should be solved.

3.4 Constraints in the sgoldstino direction

In this section we would like to discuss the problem of finding a consistent supersymmetric truncation from a more geometrical perspective. In particular, we would like to study the following problem: given a particular Kähler manifold, characterized by a Kähler potential $K(\xi, \bar{\xi})$, we want to determine the class of superpotentials $W(\xi)$ that allow the supersymmetric integration of *some* sector of the theory.

In order to have a valid $\mathcal{N} = 1$ supergravity low energy theory, the corresponding reduced scalar manifold $\mathcal{M} \subseteq \widehat{\mathcal{M}}$ has to be a Kähler manifold, and moreover, it must be totally geodesic in order to preserve supersymmetry and be consistent with the equations of motion. Actually a simple consistency test to see if a theory admits a consistent supersymmetric reduction, is to check whether the vacuum manifold of the scalar potential, or any of its submanifolds, is a totally geodesic Kähler submanifold of $\widehat{\mathcal{M}}$.

On the other hand, $\mathcal{M}$ is a Kähler submanifold iff it is given locally by the set of zeros of a collection of n_H holomorphic functions $f^1, \dots, f^{n_H}$ defined in $\widehat{\mathcal{M}}$ [112]:

$$f^\alpha(\xi^I) = 0, \quad \text{for all } \alpha = 1, \dots, n_H. \quad (3.4.1)$$

Equivalently it can be defined locally as the image of a holomorphic map $f : U \subseteq \mathbb{C}^{n-n_H} \rightarrow \widehat{\mathcal{M}}$, which is precisely the type of description we used in the previous section:

$$\xi^I = (\xi^i, \xi^\alpha(\xi^i)), \quad (3.4.2)$$

where ξ^i parametrize the open set $U \subseteq \mathbb{C}^{n-n_H}$, and ξ^I are a set of coordinates in $\widehat{\mathcal{M}}$. According to the expression (3.4.1), locally it is possible to define the heavy fields by the change of coordinates

$$H^\alpha = f^\alpha(\xi^I), \quad (3.4.3)$$

since the truncation of these fields is precisely what defines the reduced manifold.

At each point of the scalar manifold $\widehat{\mathcal{M}}$ the derivatives of the Kähler function G_I define a direction in field space, which is known as the *sgoldstino*. The condition for unbroken supersymmetry (3.2.9) can be written as

$$G^\alpha = G^{I\bar{J}} G_{\bar{J}}(\xi, \bar{\xi}) \partial_I f^\alpha(\xi) = 0 \quad \text{for all } \xi^I \text{ such that } f^\alpha(\xi) = 0. \quad (3.4.4)$$

This condition simply means that the functions $f^\alpha(\xi)$ have to remain constant and thus, without loss of generality, zero along the flow defined by the sgoldstino direction. In other words, the sgoldstino direction should be parallel to the reduced manifold in all its points. Therefore, in order to define a consistent supersymmetric truncation we just have to choose a superpotential such that

3.4. Constraints in the sgoldstino direction

the vector field defined by the sgoldstino is parallel to one of the totally geodesic submanifolds of $\widehat{\mathcal{M}}$. Moreover, from our discussion in section 3.2.1 it follows that any Kähler submanifold defined by (3.4.1) that satisfies (3.4.4), is completely geodesic.

Actually, given a theory with a scalar manifold characterized by $K(\xi, \bar{\xi})$, and any totally geodesic submanifold $\mathcal{M}$, *locally* it is always possible to find a superpotential consistent with the supersymmetric reduction $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$. If the reduced manifold is defined in a particular patch by the functions $f^\alpha(\xi)$ as in (3.4.1) then, as we will show, the most general expression for the superpotential that allows the supersymmetric integration of the heavy sector is given by

$$W(\xi) = W_0(\xi) e^{-\gamma_\alpha f^\alpha}, \quad (3.4.5)$$

where $W_0(\xi)$ is an arbitrary holomorphic function, and $\gamma_\alpha(\xi)$ are a set of *holomorphic* functions determined by the Kähler potential, the reduced manifold $\mathcal{M}$, and W_0 .

Let us begin the proof showing that we can always find a set of holomorphic functions $\gamma(\xi)$ so that (3.4.5) satisfies the constraint (3.4.4). First note that one can locally choose the independent functions $f^\alpha(\xi)$ to be the first coordinates of a regular coordinate system $\xi^I = (H^\alpha, L^i)$ (3.4.3). With this parametrization the Kähler invariant function reads

$$G(H, \bar{H}, L, \bar{L}) = K(H, \bar{H}, L, \bar{L}) + \log |W_0(H, L)|^2 - \gamma_\alpha H^\alpha - \bar{\gamma}_\alpha \bar{H}^\alpha. \quad (3.4.6)$$

If we choose the functions $\gamma_\alpha(H, L)$ to be independent of the heavy fields, the constraint on the sgoldstino direction (3.4.4) is simply

$$\partial_\alpha G(H, \bar{H}, L, \bar{L})|_{H=0} = \left[\partial_\alpha K(H, \bar{H}, L, \bar{L}) + \frac{\partial_\alpha W_0(H, L)}{W_0(H, L)} \right]_{H=0} - \gamma_\alpha(L) = 0. \quad (3.4.7)$$

For the superpotential (3.4.5) to be holomorphic, the functions $\gamma_\alpha(L)$ must be holomorphic themselves. An immediate concern is the apparent dependence of the first term on $\bar{L}^i$. However, since the reduced manifold is totally geodesic, the terms mixing light and heavy fields on the metric must vanish over the reduced manifold, what implies that $K_{\alpha\bar{i}}|_{H=0} = 0$ for every light field configuration L^i . Therefore all the terms of the previous equation depend holomorphically on the light fields, and we can solve consistently for $\gamma_\alpha(L)$

$$\gamma_\alpha(L) = \partial_\alpha K|_{H=0} + \frac{\partial_\alpha W_0(0, L)}{W_0(0, L)}. \quad (3.4.8)$$

Now we shall prove that the ansatz (3.4.5) is the most general solution to the constraint on the sgoldstino direction (3.4.4). Consider the Taylor expansion of

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$\widehat{G}$ on the heavy fields around a configuration $H^\alpha = 0$:

$$\begin{aligned} \widehat{G}(H, \bar{H}, L, \bar{L}) = & g_0(L, \bar{L}) + g_\alpha(L, \bar{L}) H^\alpha + g_{\bar{\alpha}}(L, \bar{L}) H^{\bar{\alpha}} + g_{\alpha\beta}(L, \bar{L}) H^\alpha H^\beta + \\ & g_{\bar{\alpha}\bar{\beta}}(L, \bar{L}) H^{\bar{\alpha}} H^{\bar{\beta}} + g_{\alpha\bar{\beta}}(L, \bar{L}) H^\alpha H^{\bar{\beta}} + \mathcal{O}(3) \end{aligned} \quad (3.4.9)$$

The most general Kähler function which solves the condition (3.4.4), or equivalently (3.2.9), satisfies $g_\alpha = g_{\bar{\alpha}} = 0$, and has an expansion of the form

$$\begin{aligned} \widehat{G}(H, \bar{H}, L, \bar{L}) = & g_0(L, \bar{L}) + g_{\alpha\beta}(L, \bar{L}) H^\alpha H^\beta + g_{\bar{\alpha}\bar{\beta}}(L, \bar{L}) H^{\bar{\alpha}} H^{\bar{\beta}} + \\ & g_{\alpha\bar{\beta}}(L, \bar{L}) H^\alpha H^{\bar{\beta}} + \mathcal{O}(3). \end{aligned} \quad (3.4.10)$$

It is straightforward to check that the Taylor expansion of the Kähler function associated to the ansatz for the superpotential (3.4.6) is of this form and, moreover, the coefficients of this expansion are unconstrained, since they depend only on the functions $K(\xi, \bar{\xi})$ and $W_0(\xi)$ which are arbitrary, and not on $\gamma_\alpha(\xi)$ or $f^\alpha(\xi)$. In particular, this implies that (3.4.6) and (3.4.10) are equivalent and thus, as we already anticipated in the previous section, the superpotential (3.4.5) characterizes the most general theory consistent with the supersymmetric integration of a heavy sector.

In this discussion we have only investigated the problem locally, but for a complete analysis we should also consider global issues, and in particular whether the superpotential (3.4.5) is well defined over the whole Kähler manifold. The complete characterization of this solution would also require the understanding of the type of holomorphic functions $f^\alpha(\xi)$ that define the totally geodesic submanifolds of a generic Kähler manifold $\widehat{\mathcal{M}}$, but that is beyond the scope of this thesis.

An example: consistent reduction with $\widehat{\mathcal{M}} = \mathbb{C}^2$.

Let us consider an example, the complex euclidean space $\mathbb{C}^2$. We choose the Kähler potential to be given by:

$$K(\xi, \bar{\xi}) = \xi^1 \bar{\xi}^1 + \xi^2 \bar{\xi}^2. \quad (3.4.11)$$

Since this space has complex dimension 2 the totally geodesic submanifolds coincide with the geodesics. The geodesics are straight lines in $\mathbb{C}^2$, and those defining a Kähler submanifold are configurations of the form (3.4.1)

$$\alpha_1 \xi^1 + \alpha_2 \xi^2 + \beta = 0. \quad (3.4.12)$$

We will show how to find superpotentials W compatible with supersymmetric reductions to these submanifolds.

The condition that the goldstino should be parallel to the reduced manifold (3.4.4) translates into a constraint for the superpotential

$$\bar{\alpha}_1 \partial_1 W + \bar{\alpha}_2 \partial_2 W - \bar{\beta} W = 0, \quad (3.4.13)$$

3.5. Solving the constraints.

which must be satisfied for configurations satisfying (3.4.12). Defining the new coordinates H and L

$$H = \alpha_1 \xi^1 + \alpha^2 \xi^2 + \beta, \quad L = \bar{\alpha}_2 \xi^1 - \bar{\alpha}_1 \xi^2. \quad (3.4.14)$$

we see that we can rewrite the equation which defines the reduced theory (3.4.12) as $H = 0$, and thus H can be identified as the field to be truncated. The field L has been chosen so that it parametrizes the orthogonal direction to the reduced manifold. In terms of these variables the constraint (3.4.13) reduces to

$$\partial_H W(L, H) = \gamma_0 W(L, H) \quad \text{where} \quad \gamma_0 = \bar{\beta} / (|\alpha_1|^2 + |\alpha_2|^2). \quad (3.4.15)$$

The most general superpotential consistent with the reduction defined in (3.4.12) is given by

$$W(L, H) = W_0(L, H) e^{\gamma(L)H} \quad \text{with} \quad \gamma(L) = \gamma_0 + \partial_H \log W_0|_{H=0}. \quad (3.4.16)$$

Here $W_0(L, H)$ is an arbitrary holomorphic function. In particular, a large class of superpotentials are those where W_0 only depends on the field L

$$W(L, H) = W_0(L) e^{\gamma_0 H}. \quad (3.4.17)$$

3.5 Solving the constraints.

Now we would like to discuss some specific situations where the Kähler potential and the superpotential, $\widehat{K}$ and $\widehat{W}$, allow for a consistent supersymmetric decoupling of the heavy fields. As we have discussed in section 3.3 the only consistent solution to the condition of vanishing F-terms (3.3.1) is of the form

$$H^\alpha = H_0^\alpha(L, \bar{L}) = \text{const}, \quad (3.5.1)$$

where the heavy fields have no dependence on the light sector. Although this condition might seem obvious, it is not empty and, for instance, we will see in the next section that it *is not satisfied* generically for couplings of the form $K = K_1 + K_2$ and $W = W_1 + W_2$.

In the two following situations the decoupling condition (3.5.1) does hold:

1. The consistency condition is trivially satisfied if the function $\Phi(H, \bar{H}, L, \bar{L})$ defined in equation (3.3.1) has no explicit dependence on the light fields. If that is the case integrating this equation we recover the condition found in [75]

$$\partial_\alpha \widehat{G} = \Phi(H, \bar{H}) \rightarrow \widehat{G} = \widehat{G}_1(H, \bar{H}) + \widehat{G}_2(L, \bar{L}). \quad (3.5.2)$$

In the Taylor expansion of this Kähler function (3.4.10), except the zero order term $g_0(L, \bar{L})$, all the other coefficients are independent of the light

fields. This ansatz and is particularly interesting because it allows a detailed stability analysis of the heavy fields, even when the light sector is in a supersymmetry breaking vacuum. We will come back to this problem in the next chapter.

2. On the other hand, this requirement is too restrictive. A different possibility is when the function $\Phi(H, \bar{H}, L, \bar{L})$ factorizes:

$$\Phi(H, \bar{H}, L, \bar{L}) = \Phi_1(H, \bar{H}, L, \bar{L}) \Phi_2(H, \bar{H}) = 0 \quad (3.5.3)$$

in which case we just solve $\Phi_2 = 0$ to get constant H_0^α . For example, this condition holds if $\hat{G}$ has the following functional form:

$$\hat{G} = f(L, \bar{L}, h(H, \bar{H})) \quad (3.5.4)$$

since in that case eq. (3.2.9) is replaced by

$$\partial_\alpha h(H, \bar{H}) = 0. \quad (3.5.5)$$

As the heavy fields only appear in $\hat{G}$ through the function $h(H, \bar{H})$, the coefficients of its Taylor expansion must satisfy certain constraints. For instance, to second order the expansion is of the form

$$\begin{aligned} \hat{G}(H, \bar{H}, L, \bar{L}) &= g_0(L, \bar{L}) + \partial_h^2 f|_{H_0}(L, \bar{L}) \left[h_{\alpha\beta} \hat{H}^\alpha \hat{H}^\beta + \right. \\ &\quad \left. h_{\bar{\alpha}\bar{\beta}} \hat{H}^{\bar{\alpha}} \hat{H}^{\bar{\beta}} + h_{\alpha\bar{\beta}} \hat{H}^\alpha \hat{H}^{\bar{\beta}} \right] + \mathcal{O}(3). \end{aligned} \quad (3.5.6)$$

where the coefficients $h_{\alpha\beta}$, $h_{\bar{\alpha}\bar{\beta}}$ and $h_{\alpha\bar{\beta}}$ are independent of L^i .

The first situation, eq. (3.5.2), is a special case of eq. (3.5.5), with Φ_1 constant. In both cases, the same condition that makes $\hat{G}_H|_{H_0} = 0$ also implies that the Kähler metric and the Hessian of V are block diagonal for any Φ_1 . Indeed, from equation (3.5.5) we find that

$$\hat{G}_{LH}|_{H_0} = \partial_L \partial_g f(L, \bar{L}, g(H, \bar{H})) \partial_H g(H, \bar{H})|_{H_0} = 0 \quad (3.5.7)$$

and further all mixed derivatives with only one derivative with respect to the heavy field vanish. As V_{LH} always contains terms $\propto \hat{G}_H$ or $\propto (\partial_L)^n \hat{G}_H$, which vanish, the Hessian of V is block diagonal⁴.

⁴Note that it is always possible to diagonalize the Kähler metric or the Hessian of V at one point, but it is not necessarily the case that both diagonalizations are compatible, as we have here.

3.6 An example of non-decoupling: separable K and W .

In supergravity is impossible to describe completely decoupled systems, since gravity couples to everything. Weakly coupled systems are often described by combining two sectors using separable Kähler potential and superpotential. Given two systems characterized by the Kähler potentials $K_1(H, \bar{H})$ and $K_2(L, \bar{L})$ and the superpotentials $W_1(H)$ and $W_2(L)$, the ansatz reads

$$\widehat{K} = K_1(H, \bar{H}) + K_2(L, \bar{L}) \quad (3.6.1)$$

$$\widehat{W} = W_1(H) + W_2(L) \quad (3.6.2)$$

This way of combining systems ensures that, in the flat limit $M_p \rightarrow \infty$, the two systems appear completely decoupled in the F -term potential of the full theory [113]. The case of the D -term potential is more involved since it depends on the details of the gauging. If we write the Planck masses explicitly in the expression of the F -term potential the equation (2.2.8) becomes:

$$V_F = e^{K/M_p^2} \left(K^{I\bar{J}} \mathcal{D}_I W \mathcal{D}_{\bar{J}} \bar{W} - M_p^{-2} |W|^2 \right), \quad (3.6.3)$$

where the Kahler covariant derivative is given by

$$\mathcal{D}_I W = \partial_I W - M_p^{-2} \partial_I K W. \quad (3.6.4)$$

Then, substituting the ansatz (3.6.2) in the previous expression, and taking the limit $M_p \rightarrow \infty$, the F -term potential becomes

$$V_F = V_1 + V_2 \quad \text{where} \quad V_i = K_i^{I\bar{J}} \partial_I W_i \partial_{\bar{J}} \bar{W}_i. \quad (3.6.5)$$

The ansatz (3.6.2) is not suitable to describe the couplings to a sector that is integrated out while preserving supersymmetry. For instance, the ansatz (3.6.2) does not satisfy the decoupling condition in general. Suppose equation (3.2.9) admits a constant solution $H^\alpha = H_0^\alpha$. Then

$$0 = \partial_\alpha W_1|_{H_0} + \partial_\alpha K_1|_{H_0} [W_1(H_0) + W_2(L)], \quad (3.6.6)$$

which only holds if

$$\begin{aligned} \partial_\alpha K_1|_{H_0} &= 0 \Rightarrow \partial_\alpha W_1|_{H_0} = 0 \\ \partial_\alpha K_1|_{H_0} &\neq 0 \Rightarrow W_2(L) = -\frac{\partial_\alpha W_1|_{H_0}}{\partial_\alpha K_1|_{H_0}} - W_1(H_0). \end{aligned} \quad (3.6.7)$$

Another way to see this: since $\widehat{\mathcal{D}}_\alpha \widehat{W} = 0$ does not factorize, the (Kähler-gauge covariant) requirement that it is independent of the light fields is (see also [114])

$$\widehat{\mathcal{D}}_i(\widehat{\mathcal{D}}_\alpha \widehat{W}) = 0. \quad (3.6.8)$$

Inserting the ansatz (eq. 3.6.1) then gives

$$\partial_\alpha K_1|_{H_0} \partial_i W_2 = 0. \quad (3.6.9)$$

Unless $K_1(H, \overline{H})$ has no linear terms or $W_2(L) = \text{constant}$, the condition will not be met. However, if $W_2(L) = \text{constant}$ (e.g. no scale models [115, 116]) then equation (3.5.2) holds and $\widehat{W}$ is trivially a product. On the other hand, we can always expand $K_1(H, \overline{H})$ around H_0 and remove the linear terms by a Kähler transformation (2.2.28), but this spoils the separability of the superpotential (3.6.2).

In other words, if two sets of fields have separable Kähler functions $K = K_1(\text{heavy}) + K_2(\text{light})$, the addition of their superpotentials does not respect the decoupling condition except in special cases.

3.7 Discussion

In this chapter we have considered the circumstances that allow for a consistent supersymmetric truncation a set of scalar fields H^α in $\mathcal{N} = 1$ supergravity theories. The truncation is defined fixing the fields H^α at a covariantly constant configuration $H^\alpha = H_0^\alpha$:

$$\nabla_\mu H^\alpha = [\partial_\mu H^\alpha - k_{\tilde{a}}^\alpha A_{\mu}^{\tilde{a}}]_{H_0} = 0, \quad (3.7.1)$$

so that the Lorentz symmetry is unbroken in the reduced theory. Some of the gauge symmetries might be broken by the vacuum expectation value of the truncated fields ($k_{\tilde{a}}|_{H_0} \neq 0$), we denote the associated gauge boson by $A_{\mu}^{\tilde{a}}$.

Our discussion is based on two crucial assumptions about the reduced theory describing the dynamics of the fields surviving the truncation L^i . First we have required that solutions of the reduced action should be solutions of the full action. And second we asked the truncated theory to preserve exactly the invariance under $\mathcal{N} = 1$ local supersymmetry. These two conditions constitute the definition of a consistent supersymmetric truncation.

In order to preserve supersymmetry each truncated field has to be removed from the reduced theory together with the whole supermultiplet which contains it. Otherwise the truncated fields could be sourced by supersymmetry transformations.

- The scalar fields H^α have to be truncated together with their fermionic partners, χ^α , which must be set to zero. In order to truncate fully the chiral

3.7. Discussion

multiplets the F -terms must also be set to zero in the reduced theory, and thus the first derivative of the Kähler function $\widehat{G}$ along the directions of truncated fields must vanish too:

$$H^\alpha = H_0^\alpha, \quad \chi^\alpha = 0, \quad \widehat{G}_\alpha|_{H_0} = 0 \quad \text{for all } L^i, \quad (3.7.2)$$

- The gauge bosons $A_\mu^{\tilde{a}}$ associated to the symmetries broken by the vacuum expectation value of the truncated fields $H^\alpha = H_0^\alpha$ have to be truncated together with the whole vector multiplet. These gauge bosons should be at a pure gauge configuration and therefore the field strengths have to vanish. The gaugini $\lambda^{\tilde{a}}$ and the D -terms (or equivalently the moment-maps $\mathcal{P}_{\tilde{a}}$) must also be set to zero in the reduced theory:

$$F_{\mu\nu}^{\tilde{a}} = 0, \quad \lambda^{\tilde{a}} = 0, \quad \mathcal{P}_{\tilde{a}}|_{H_0} = 0 \quad \text{for all } L^i, \quad (3.7.3)$$

We now turn to the conditions which ensure that every solution of the reduced theory must be a solution of the complete model. In particular this must be true for very large momenta, where the interactions can be neglected and the model behaves as a free theory. In that limit we find that the kinetic terms of the truncated fields have to be decoupled from the kinetic terms of the surviving ones:

- If the evolution of the scalar fields is described by a of non linear sigma model, the target space of the reduced theory has to be totally a geodesic submanifold of the complete target space. In particular the sigma model metric has to be block diagonal in the two sectors:

$$G_{i\alpha}(H_0, \bar{H}_0, L, \bar{L}) = 0. \quad (3.7.4)$$

Moreover, the condition (3.7.2) implies that the sgoldstino direction must be parallel to the reduced manifold at all of its points.

- The kinetic terms of the surviving gauge bosons have to be decoupled from the kinetic terms and of those which become massive at the configuration $H^\alpha = H_0^\alpha$. This implies that the real part of the gauge kinetic functions should not have components mixing truncated and surviving gauge fields gauge fields in the reduced theory:

$$\text{Ref}_{a\tilde{b}}(H_0, L) = 0, \quad \text{and} \quad \text{Ref}_{ab,\alpha}(H_0, L) = 0 \quad \text{for all } L^i. \quad (3.7.5)$$

The last condition needs to be imposed for the dynamics to be consistent with the truncation. If this condition is not met a gaugino condensation in the reduced theory could induce supersymmetry breaking in the truncated sector (see [113]).

The truncated fields should not be sourced due to gauge interactions of the surviving fields, which leads to the following conditions:

Supersymmetric decoupling of heavy scalars in $\mathcal{N} = 1$ supergravity.

- The non abelian gauge interactions between gauge bosons should respect the truncation. Then, if the gauge symmetry group $\mathbb{G}$ is semi-simple, it must have a cross product structure $\mathbb{G} = \mathbb{G}_h \times \mathbb{G}_l$, where subgroup $\mathbb{G}_h$ is broken at the configuration $H^\alpha = H_0^\alpha$, while the subgroup $\mathbb{G}_l$ remains unbroken. Therefore the structure constants of the group satisfy:

$$f_{bc}^{\tilde{a}} = f_{\tilde{b}c}^a = f_{b\tilde{c}}^{\tilde{a}} = f_{\tilde{b}\tilde{c}}^a = 0, \quad (3.7.6)$$

where the indices with a tilde run over the generators of $\mathbb{G}_h$, and those without it run over the generators $\mathbb{G}_l$.

- In the reduced theory the fields in the chiral multiplets surviving the truncation should not couple to the truncated gauge bosons. Thus the killing vectors of the broken gauge symmetries, those associated to $\mathbb{G}_h$, should not have components along L^i in the reduced theory:

$$k_a^i(H_0, L) = 0, \quad \text{and they satisfy} \quad k_{a,i}^\alpha(H_0, L) = 0 \quad \text{for all } L^i. \quad (3.7.7)$$

The second condition ensures the decoupling of the surviving chiralini from the massive gauge bosons.

- The fields in the truncated chiral multiplets should be decoupled from the gauge bosons which survive the truncation A_μ^a . Thus the killing vectors of the surviving gauge symmetries, i.e. those associated to $\mathbb{G}_l$, should not have components along H^α in the reduced theory:

$$k_a^\alpha(H_0, L) = 0, \quad \text{and they satisfy} \quad k_{a,\alpha}^i(H_0, L) = 0 \quad \text{for all } L^i. \quad (3.7.8)$$

The second condition ensures the decoupling of the truncated chiralini from the gauge fields in the reduced theory.

These conditions guarantee that the Hessian of the scalar potential of the full theory $\widehat{V}$, is block diagonal in the two sectors, which is required for the dynamics of the system to be consistent with the truncation of the heavy fields. Interestingly, the conditions for supersymmetric decoupling also provide a mechanism to embed BPS solutions of the reduced theory into the full theory without destroying their BPS character.

The condition (3.7.2) restricts the type of couplings, defined by the Kähler potential $\widehat{K}$ and the superpotential $\widehat{W}$, which are compatible with the supersymmetric truncation of a sector of the theory. We have shown that given a theory with a scalar manifold $\widehat{\mathcal{M}}$ and any totally geodesic submanifold $\mathcal{M}$, *locally* the most general superpotential consistent with the supersymmetric reduction $\widehat{\mathcal{M}} \rightarrow \mathcal{M}$ is of the form:

$$W(\xi) = W_0(\xi) e^{-\gamma_\alpha f^\alpha}. \quad (3.7.9)$$

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Here the holomorphic functions f^α characterize the reduced manifold by the solution to the set of equations $f^\alpha(\xi) = 0$, $W_0(\xi)$ is an arbitrary holomorphic function, and $\gamma_\alpha(\xi)$ are a set of holomorphic functions determined by the Kähler potential and W_0 . We have also presented a broad class of Kähler functions $\widehat{G}$ that allow for a consistent supersymmetric truncation of the would-be heavy fields,

$$\widehat{G}(H, \bar{H}, L, \bar{L}) = f(L, \bar{L}, h(H, \bar{H})).$$

From the point of view of model building, the decoupling condition provides a simple consistency test. For instance, we have shown that when the truncated and surviving sectors are coupled using the ansatz,

$$\widehat{K}(H, \bar{H}, L, \bar{L}) = K_h(H, \bar{H}) + K_l(L, \bar{L}), \quad \widehat{W}(H, L) = W_h(H) + W_l(L)$$

in general the would-be heavy fields cannot be truncated while preserving supersymmetry. There are stringy scenarios which approximately satisfy the decoupling condition in the form (3.5.4), such as some LARGE volume compactifications [117] [118] [119].

Despite our notation, it is important to stress that the conditions we have presented here *do not* ensure the stability of the truncated sector. Actually, in the next chapter we discuss the stability of the truncated sector for a particular class of couplings between the truncated and surviving sectors, and we show that the field configuration that defines the truncation might become unstable depending on the values of the surviving fields. In that case the truncated sector can no longer be considered the “heavy” sector.

Stability of consistently decoupled sectors.

4.1 Introduction

Supersymmetric vacua are unsuitable to describe the present day cosmology. First, we know supersymmetry to be broken at low energies, and therefore any cosmologically viable model should include a mechanism of supersymmetry breaking. And second, supersymmetric vacua can never have a positive cosmological constant (2.3.3), and thus they cannot describe the present day acceleration of the universe. The different methods to break supersymmetry and lift the energy of these vacua to de Sitter are known as *uplifting mechanisms*. Many uplifting mechanisms have been proposed, for example in the original KKLT construction this was achieved by including anti-D3 branes in the compactification. Subsequently the work was concentrated on F -term [120, 121, 93], and D -term uplifting [122, 123, 124], where supersymmetry is only broken spontaneously in the light sector, which gives better calculational control. In F -term and D -term uplifting supersymmetry breaks due to the expectation value of the auxiliary fields of the chiral F^I and vector multiplets D^a respectively. Other approaches are discussed in [125, 126], where supersymmetry is broken due to Kähler corrections.

The problem in flux compactifications involving D -term and F -term uplifting mechanisms is that, in general, it cannot be taken for granted that the presence of the light sector is compatible with the supersymmetric integration of the heavy moduli [22, 76, 127, 77, 78]. As we discussed in the previous chapter, in order to be able to stabilize the heavy moduli while preserving supersymmetry the couplings between the light and heavy sectors have to satisfy certain constraints. Moreover, even when these conditions are satisfied, if the light sector breaks supersymmetry nothing guarantees the stability of the heavy field configuration. If the breaking of supersymmetry by the light sector renders this configuration unstable, any perturbation could make the heavy fields evolve far away from it, and thus integrating out this sector would be inconsistent. In this chapter we study the stability of the would-be heavy sector for two classes of models which satisfy the conditions for the supersymmetric integration derived in the last chapter.

4.2 Uplifting with a separable Kähler function.

We now study the perturbative stability of vacua in theories where the heavy sector is integrated out in a supersymmetric way, i.e. the action satisfies the conditions derived in chapter 3, and the light sector is in supersymmetry breaking configuration. In this chapter we will assume that the Kähler invariant function is separable in the heavy and light sectors, eq. (3.5.2). To simplify notation we use the definitions $G = \hat{G}(H, \bar{H}, L, \bar{L})$, $A = \hat{G}_1(H, \bar{H})$, and $B = \hat{G}_2(L, \bar{L})$:

$$G(H, \bar{H}, L, \bar{L}) = A(H, \bar{H}) + B(L, \bar{L}). \quad (4.2.1)$$

The F -term potential derived from this ansatz can be seen to be, using (2.2.32):

$$V_F = e^{A+B} \left(A^{\alpha\bar{\beta}} A_{\alpha} A_{\bar{\beta}} + B^{i\bar{j}} B_i B_{\bar{j}} - 3 \right), \quad (4.2.2)$$

and the D -term potential reads:

$$\begin{aligned} V_D = & \frac{1}{2} (\text{Re } f)^{-1\bar{a}\bar{b}} k_a^{\alpha} k_b^{\bar{\beta}} A_{\alpha} A_{\bar{\beta}} + \frac{1}{2} (\text{Re } f)^{-1ab} k_a^{\alpha} k_b^{\bar{j}} B_i B_{\bar{j}} + \\ & (\text{Re } f)^{-1\bar{a}b} k_a^{\alpha} k_b^{\bar{j}} A_{\alpha} B_{\bar{j}}. \end{aligned} \quad (4.2.3)$$

Where we are using the same notation as in chapter 3. In the reduced theory, with the heavy fields stabilized at the supersymmetric configuration $H^{\alpha} = H_0^{\alpha}$, the scalar potential reads:

$$V = \hat{V}|_{H_0} = \left[e^{A(H)} V_F^{light}(L) + \frac{1}{2} (\text{Re } f)^{-1ab} k_a^{\alpha} k_b^{\bar{j}} B_i B_{\bar{j}} \right]_{H_0} \quad (4.2.4)$$

where $V_F^{light} = e^B (B^{i\bar{j}} B_i B_{\bar{j}} - 3)$ is the F -term potential of the uplifting sector when considered alone. Since heavy fields are integrated out in a supersymmetric

4.2. Uplifting with a separable Kähler function.

way, if H_0^α is a supersymmetric critical point of the heavy sector, and L_0^i is a critical point the reduced potential V , then the field configuration (H_0^α, L_0^i) is a critical point of the full potential. Moreover, the scalar potential is block diagonal in the two sectors (3.2.38), meaning that there are no quadratic interactions between the fluctuations of the heavy and light fields. Therefore the stability of the two sectors can be studied separately. In the next two sections we will study the perturbative stability of the critical point (H_0^α, L_0^i) along the heavy directions. First in section 4.2.1 we will study the situation where all the fields are uncharged, and in section 4.2.2 we will consider the most general case.

In the absence of gauge couplings it is possible to derive a few general results without a detailed stability analysis. In this case, since $V_D = 0$, the scalar potential of the reduced theory is simply

$$\widehat{V}|_{H_0} = e^{A(H_0)} V_F^{light}(L). \quad (4.2.5)$$

This expression implies that the critical points of the reduced potential L_0^i coincide with the critical points of the F -term potential of light sector, $V_F^{light}(L)$. Moreover, the value of the potential of the light sector at the critical point L_0^i determines whether the supersymmetric vacuum is lifted to dS, Minkowski or remains AdS:

$$\begin{aligned} V_F^{light}(L) > 0 &\implies (H_0^\alpha, L_0^i) \text{ is a dS vacuum} \\ V_F^{light}(L) = 0 &\implies (H_0^\alpha, L_0^i) \text{ is a Minkowski vacuum} \\ V_F^{light}(L) < 0 &\implies (H_0^\alpha, L_0^i) \text{ is an AdS vacuum.} \end{aligned}$$

In particular, when there is more than one supersymmetric configuration of the heavy sector, all of them become degenerate when uplifted to Minkowski (note that this makes the possibility of topological inflation quite natural). As the stability along the heavy and light direction can be studied independently, the relation (4.2.5) leads to a remarkably simple result. Namely, the vacuum (H_0^α, L_0^i) is perturbatively stable with respect to fluctuations of the light fields as long as L_0^i is a minimum of the potential of the light sector $V_F^{light}(L)$.

Although the stability analysis along the heavy directions is more involved, it is possible to derive some general results. Due to the block diagonal structure of the Hessian of the potential at the point (H_0^α, L_0^i) , in order to analyze the stability of this configuration along the heavy directions it is enough to study the potential evaluated at L_0^i

$$V|_{L_0^i} = e^{B(L_0)} e^A \left(A^{\alpha\bar{\beta}} A_\alpha A_{\bar{\beta}} + (b-3) \right) \quad (4.2.6)$$

Note that the stability analysis for fluctuations of the heavy fields depends on the light sector only through a single parameter $b = B^{i\bar{j}} B_i B_{\bar{j}}|_{L_0}$ that controls

the amount of uplifting,

$$b - 3 = e^{-G} V|_{H_0, L_0} = \left(\frac{3H}{m_{3/2}} \right)^2. \quad (4.2.7)$$

A remarkable property of this model is that all configurations (H_0^α, L_0^i) are stable or marginally stable after uplifting to Minkowski vacuum ($b = 3$). To see this, we set $b = 3$ in the expression (4.2.6), then the full potential evaluated at L_0^i reads

$$V_{L_0} = e^{B(L_0)} e^A A^{\alpha\bar{\beta}} A_\alpha A_{\bar{\beta}} \geq 0 \quad \text{for all } L^i. \quad (4.2.8)$$

Since, by assumption, $V(H_0, L_0) = 0$, the condition (4.2.8) implies that no fluctuation of the fields on the heavy sector can decrease the energy, and therefore the point (H_0^α, L_0^i) is either a local minimum or a plateau along the heavy directions. For large final values of the cosmological constant the stability analysis simplifies considerably. In this limit the full potential (4.2.6) becomes approximately:

$$V(H, L) \approx b e^{A+B}, \quad (4.2.9)$$

and therefore the stable configurations of the heavy sector are those minimizing the Kähler function $A(H, \bar{H})$.

4.2.1 Stability of uplifted vacua with zero D -term potential

In order to study the stability of the configuration (H_0^α, L_0^i) along the heavy directions we will apply the same technique we used in the previous chapter for supersymmetric critical points. First we need to calculate the derivatives of the potential $V_{\alpha\bar{\beta}}(H_0, L_0)$ and $V_{\alpha\beta}(H_0, L_0)$ from (4.2.2):

$$\begin{aligned} V_{\alpha\bar{\beta}}(H_0, L_0) &= e^{A+B}|_{H_0, L_0} \left[A^{\gamma\bar{\delta}} A_{\alpha\gamma} A_{\bar{\beta}\bar{\delta}} + (b-2) A_{\alpha\bar{\beta}} \right]_{H_0} \\ V_{\alpha\beta}(H_0, L_0) &= e^{A+B}|_{H_0, L_0} (b-1) A_{\alpha\beta}(H_0), \end{aligned} \quad (4.2.10)$$

In the following we will use the notation $X = A_{\alpha\beta}(H_0)$. As in the previous chapter we will assume the fields H^α to be canonically normalized at H_0^α . Moreover, we will use the residual freedom to define the fields to make the matrix $X^\dagger X$ real and diagonal

$$X^\dagger X = \text{Diag}(|x_1|^2 \mathbb{1}_{n_1}, \dots, |x_p|^2 \mathbb{1}_{n_p}), \quad |x_\lambda| \geq 0. \quad (4.2.11)$$

Here the index $\lambda = 1, \dots, p$, labels the subspace corresponding to the eigenvalue $|x_\lambda|^2$, which has dimension n_λ . With this choice we obtain the following expression for the Hessian of the potential at (H_0^α, L_0^i) :

$$\begin{pmatrix} V_{\alpha\bar{\beta}} & V_{\alpha\beta} \\ V_{\bar{\alpha}\bar{\beta}} & V_{\bar{\alpha}\beta} \end{pmatrix}_{H_0, L_0} = e^{A+B}|_{H_0, L_0} \begin{pmatrix} X X^\dagger + (b-2) \mathbb{1} & (b-1) X \\ (b-1) X^\dagger & X^\dagger X + (b-2) \mathbb{1} \end{pmatrix}. \quad (4.2.12)$$

4.2. Uplifting with a separable Kähler function.

The mass spectrum can be calculated along the same lines of section 2.4, what leads to our final result:

$$m_{\pm\lambda}^2 = e^{A+B}|_{H_0, L_0} \left[|x_\lambda|^2 + (b-2) \pm |(b-1)x_\lambda| \right] = e^{A+B}|_{H_0, L_0} \left[\left(|x_\lambda| \pm \frac{1}{2}(b-1) \right)^2 - \frac{1}{4}(b-3)^2 \right]. \quad (4.2.13)$$

To obtain the last expression we assumed that $b > 1$, but in the case $b < 1$ then $m_{+\lambda}^2$ and $m_{-\lambda}^2$ have to be exchanged. For each energy level characterized by $m_{\pm\lambda}^2$ there are n_λ different excitations with the same mass. The stability condition after uplifting the minimum of the potential to Minkowski or de Sitter, $b \geq 3$, reduces to $m_{\pm\lambda}^2 > 0$ for all $\lambda = 1, \dots, p$, but if the minimum remains AdS after the uplifting, $b < 3$, the masses have to satisfy the Breitenlohner-Freedman bound (2.3.4):

$$\begin{aligned} \text{for } b < 3 & \implies \left[\left(|x_\lambda| \pm \frac{1}{2}(b-1) \right)^2 - \frac{1}{4}(b-3)^2 \right] \geq \frac{3}{4}(b-3), \\ \text{for } b \geq 3 & \implies \left[\left(|x_\lambda| \pm \frac{1}{2}(b-1) \right)^2 - \frac{1}{4}(b-3)^2 \right] \geq 0. \end{aligned} \quad (4.2.14)$$

Recalling that $b \geq 0$, and after a little bit of algebra, it is possible to see that the first of the two inequalities is always satisfied. The second shows that there are no instabilities when the minimum is uplifted to Minkowski $b = 3$, although zero modes are possible if any $|x_\lambda| = 1$. Thus, the instabilities can only arise for upliftings to dS. These results are summarized in figure 4.1.

Before the uplifting, i.e. if the light sector does not break supersymmetry, the results we obtained in section 1.3 also apply here, since (H_0^α, L_0^i) is a supersymmetric configuration. In particular, the maxima of the scalar potential before the uplifting coincide with the minima of the total Kähler function $G(H, \bar{H}, L, \bar{L})$, which are the minima of $A(H, \bar{H})$ due to the ansatz (3.5.2). According to the result (2.4.11) the configuration H_0^α is a minimum of $A(H, \bar{H})$ provided that all the eigenvalues of $X^\dagger X$ satisfy $|x_\lambda| < 1$. Therefore the results (4.2.14) imply that local AdS minima and saddle points before the uplifting are only stable for small values of the cosmological constant, while local AdS maxima of the potential, which coincide with the local minima of the Kähler function, are always stable.

4.2.2 Stability of uplifted vacua with non-zero D -term potential

Now we study the stability of uplifted vacua when the gauge couplings are turned on. Including gauge interactions is specially relevant in the case of the light sector, since it includes the visible sector. As we discussed in section 3.1.3, the conditions necessary for the supersymmetric decoupling of the heavy sector ensure that the Hessian of the D -term potential is block diagonal in the heavy and

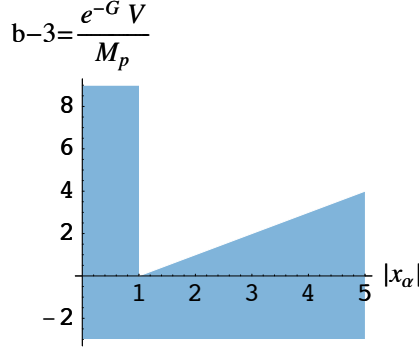


Figure 4.1 – Stability of supersymmetric critical points after the uplifting. The quantity on the vertical axes $b-3$ is proportional to the cosmological constant (or the Hubble parameter during inflation). The horizontal axes represents the curvature of the Kähler function at the critical point along one of the heavy field directions H^α : $|x_\lambda| < 1$ corresponds to local minima and $|x_\lambda| > 1$ to saddle points. The shaded region represents stable configurations under perturbations of the heavy fields. For $b < 3$ and $b = 3$ the uplifted vacua, which are AdS and Minkowski respectively, are always stable. Local AdS minima of the scalar potential at zero uplifting, $|x_\lambda| > 2$, are always destabilized for large uplifting. Local AdS maxima, $|x_\lambda| < 1$, remain stable for arbitrary large uplifting.

light sectors, $V_{D|\alpha i}(H_0, L_0) = V_{D|\alpha \bar{i}}(H_0, L_0) = 0$. Therefore in order to calculate the contribution of the D -term potential to the Hessian of $\hat{V}$ at (H_0^α, L_0^i) we just have to calculate the second derivatives of V_D along the heavy directions:

$$\begin{aligned} V_{D|\alpha\beta}(H_0, L_0) &= \frac{1}{2} (\text{Re } f(H_0, L_0))^{-1ab} k_a^\gamma(H_0, L_0) k_b^{\bar{\delta}}(\bar{H}_0, \bar{L}_0) A_{\gamma\alpha} A_{\bar{\delta}\beta}, \\ V_{D|\alpha\bar{\beta}}(H_0, L_0) &= \frac{1}{2} (\text{Re } f(H_0, L_0))^{-1ab} k_a^\gamma(H_0, L_0) k_b^{\bar{\delta}}(\bar{H}_0, \bar{L}_0) A_{\gamma\alpha} A_{\bar{\delta}\bar{\beta}} \end{aligned} \quad (4.2.15)$$

The mass matrix of the heavy fields at (H_0^α, L_0^i) , is given by the sum of the Hessian of the D -term potential with respect to the heavy fields and the one we found for the F -term potential (4.2.12). Following the discussion in section 2.4.1, we choose our heavy scalar fields so that they have trivial kinetic terms $G_{\alpha\bar{\beta}} = 1$ and that the matrices $X^\dagger X$ ($X = A_{\alpha\beta}$) and $k_a^\alpha k_a^{\bar{\beta}}$ are both real and diagonal. We also define the gauge fields $A_\mu^{(h)a}$ such that the real parts of the gauge kinetic functions of the heavy sector are proportional to the identity matrix:

$$\text{Re } f_{ab}^{(h)} = e^G|_{H_0, L_0} \delta_{ab}. \quad (4.2.16)$$

The calculation of the D -term contribution to the Hessian can be done along the same lines as in section 2.4.3. Is not difficult to check that the properties (2.4.28) and (2.4.29) still hold, thus here again the matrices $kk^\dagger$ and kk^T , with

4.3. Uplifting with more general couplings

$k = k_a^\alpha$, have non-vanishing components only in the eigenspace corresponding to the eigenvalue $|x_1|^2 = 1$. In particular, the block $k_1^\alpha k_1^{\bar{\beta}}$ is of the form:

$$k_1 k_1^\dagger = \text{Diag}(|k_1|^2, \dots, |k_{n_1}|^2). \quad (4.2.17)$$

Thus, after some simplifications, the Hessian of the total scalar potential reads:

$$\begin{pmatrix} V_{\alpha\bar{\beta}} & V_{\alpha\beta} \\ V_{\bar{\alpha}\bar{\beta}} & V_{\bar{\alpha}\beta} \end{pmatrix}_{H_0, L_0} = e^{A+B}|_{H_0, L_0} \begin{pmatrix} XX^\dagger + kk^\dagger + (b-2)\mathbb{1} & (b-1+kk^\dagger)X \\ (b-1+kk^\dagger)X^\dagger & X^\dagger X + kk^\dagger + (b-2)\mathbb{1} \end{pmatrix}. \quad (4.2.18)$$

From this expression it is straightforward to find the mass spectrum of fluctuations of the heavy sector along the heavy directions:

$$\begin{aligned} m_{\pm\lambda}^2 &= e^{G(\xi_0)} \left((|x_\lambda| \pm \tfrac{1}{2}(b-1))^2 - \tfrac{1}{4}(b-3)^2 \right) && \text{if } |x_\lambda|^2 \neq 1, \\ m_{+1i}^2 &= 2 e^{G(\xi_0)} (|k_i|^2 + b - 1) && \text{if } |x_\lambda|^2 = 1, \\ m_{-1i}^2 &= 0 && \text{if } |x_\lambda|^2 = 1 \end{aligned} \quad (4.2.19)$$

We can see that figure 4.1 is still valid when we include the gauge interactions. The only difference with the result in the previous section is that if some of the gauge symmetries are spontaneously broken the mass degeneracy in the eigenspace with $|x_\lambda| = 1$ is destroyed. From (4.2.19) we can see that the presence of gauge interactions only increases the stability of the critical point.

4.3 Uplifting with more general couplings

An interesting question to consider is whether it is possible apply our results to other systems where heavy moduli are supersymmetrically decoupled, but its interactions with the heavy are given by an ansatz more general than (4.2.1). As we discussed in section 3.2, the condition that the potential has to be block diagonal on the light and heavy fields is necessary in order to integrate out the heavy fields consistently. Therefore, in any scenario where part of the moduli are going to be integrated out the stability of these fields can be studied independently considering only the "heavy" directions in field space.

Let us assume only the mild condition that the Kähler potentials are separable in the light and heavy sectors:

$$K = K^h(H, \bar{H}) + K^l(L, \bar{L}).$$

This condition ensures that mixed derivatives of the Kähler function of the form $G_{i\bar{\alpha}}(H_0, L_0)$, $G_{i\bar{\alpha}\beta}(H_0, L_0)$ et cetera... involving both holomorphic and antiholomorphic indices from the two sectors must vanish. The condition (3.2.9) also implies that the derivatives of the form $G_{\alpha i}|_{H_0} = 0$. Since the Hessian of the

Kähler function is block diagonal in the light and heavy sectors, it makes sense to study the curvature of $G(H, L)$ at the critical point H_0^α only along the heavy directions. Thus, we can repeat the analysis of section 2.4.1 arriving at similar conclusions:

- The Kähler function $G(L, H)$ has a local minimum at H_0^α along the heavy directions if the eigenvalues of the matrix $X^\dagger X$ satisfy the conditions $|x_\lambda| < 1$ for all $\lambda = 1, \dots, p$, with $X = G_{\alpha\beta}(H_0, L_0)$.
- If any of the eigenvalues of $X^\dagger X$ satisfies $|x_\lambda| > 1$ the function $G(H, L)$ has a saddle point at H_0 .
- For each eigenvalue of $X^\dagger X$ satisfying $|x_\lambda| = 1$ the Kähler function has a neutrally stable direction and a minimum along some complex direction H^α .

Using all these results, we can now study the stability of the scalar potential along the heavy directions as in sections 2.4.2 and 2.4.3. The second derivatives of the scalar potential are given by

$$V_{\alpha\beta}(H_0, L_0) = e^G|_{H_0, L_0} \left[(b-1)G_{\alpha\beta} + G^{i\bar{j}}G_{i\alpha\beta}G_{\bar{j}} \right]_{H_0, L_0}, \quad (4.3.1)$$

$$V_{\alpha\bar{\beta}}(H_0, L_0) = e^G|_{H_0, L_0} \left[G^{\gamma\bar{\delta}}G_{\alpha\gamma}G_{\bar{\beta}\bar{\delta}} + (b-2)G_{\alpha\bar{\beta}} \right]_{H_0, L_0}, \quad (4.3.2)$$

where we have used the notation $b = G^{i\bar{j}}G_i G_{\bar{j}}|_{H_0, L_0}$.

Note that, apart from the second term in the equation (4.3.1), the result we have obtained is of the same form as (4.2.10). If the quantity $G_{i\alpha\beta}$ stays of order $\mathcal{O}(1)$, the extra term that we have obtained is roughly of order $\mathcal{O}(b^{1/2})$, which means that for large values of the uplifting, $b \gg 3$, it will become subdominant. Therefore, in this limit, the mass matrix becomes proportional to the Hessian of the Kähler function at H_0^α :

$$\begin{pmatrix} V_{\alpha\bar{\beta}} & V_{\alpha\beta} \\ V_{\bar{\alpha}\bar{\beta}} & V_{\bar{\alpha}\beta} \end{pmatrix}_{H_0, L_0} = b \begin{pmatrix} \mathbb{1} & X \\ X^\dagger & \mathbb{1} \end{pmatrix} e^{A+B}|_{H_0, L_0} = b \begin{pmatrix} G_{\alpha\bar{\beta}} & G_{\alpha\beta} \\ G_{\bar{\alpha}\bar{\beta}} & G_{\bar{\alpha}\beta} \end{pmatrix}_{H_0, L_0} e^{A+B}|_{H_0, L_0}, \quad (4.3.3)$$

indicating that the minima of the Kähler function along the heavy directions will always survive uplifting to an arbitrary large value of the cosmological constant. Note also that before uplifting, $G_i(H_0, L_0) = 0$, the mass matrix given by (4.3.2) coincides with (2.4.14), so we can again identify the AdS maxima of the scalar potential with the local minima of the Kähler function along the heavy directions.

We would like to emphasize that in order to obtain this result we have made very mild assumptions. We have required that the Kähler potential is separable in the two sectors, we have also imposed the condition that the effective action

left after integrating out the heavy moduli is invariant under supersymmetry, and finally we asked the quantity $G_{i\alpha\beta}$ to stay of order $\mathcal{O}(1)$ for large values of the uplifting. In this scenario we have proved that the AdS maxima of the potential along the heavy directions at zero uplifting ($G_i(H_0, L_0) = 0$), which are perturbatively stable configurations, remain stable after the uplifting for arbitrary large values of the cosmological constant.

4.4 Summary

We have studied the perturbative stability of the heavy sector in two different classes of models that allow for the supersymmetric integration of the heavy moduli. In the first case the couplings between the heavy and light sectors are characterized by a separable Kähler function

$$\widehat{G}(H, \bar{H}, L, \bar{L}) = G_h(H, \bar{H}) + G_l(L, \bar{L}), \quad (4.4.1)$$

which can be expressed in terms of the Kähler potential and the superpotential as follows

$$\widehat{K}(H, \bar{H}, L, \bar{L}) = K_h(H, \bar{H}) + K_l(L, \bar{L}), \quad \widehat{W}(H, L) = W_h(H)W_l(L).$$

An interesting property of this type of couplings is that there is always choice of fields such that the mass matrix and the Kähler metric can be diagonalized simultaneously. This allows expressing the stability requirement of having a positive definite mass matrix as a constraint on the curvature of the Kähler function at the vacua of the full theory (H_0^α, L_0^i). Our results, which are displayed in fig. 4.1, show that if the heavy fields are fixed at a minimum of the Kähler function, an AdS maximum of the scalar potential, the configuration remains stable for any final value of the cosmological constant. However, if the heavy fields are fixed at a saddle point of the Kähler function, (the Kähler function cannot have maxima), the configuration always becomes unstable for large enough values of the cosmological constant.

This analysis complements that of Covi *et al.* [79], who formulated a necessary (and in most practical situations sufficient) condition for the existence of (meta)stable de Sitter vacua, following earlier work by Gomez Reino and Scrucce [120, 121, 93]. The constraint restricts the Kähler geometry of the non-linear sigma model associated to the chiral multiplets. Expressed in terms of the metric $G_{I\bar{J}}$ and the Riemann tensor $R_{I\bar{J}M\bar{N}}$ of the Kähler manifold it reads:

$$\sigma \equiv \left[\frac{1}{3} \left(G_{I\bar{J}} G_{M\bar{N}} + G_{I\bar{N}} G_{M\bar{J}} \right) - R_{I\bar{J}M\bar{N}} \right] G^I G^{\bar{J}} G^M G^{\bar{N}} > 0. \quad (4.4.2)$$

This condition, they point out, is e.g. not satisfied by moduli with no-scale Kähler functions of the form $K = -3\log(\xi + \bar{\xi})$, or more generally

$K = -\sum_I n_I \log(\xi^I + \bar{\xi}^{\bar{I}})$, $\sum_I n_I = 3$. Clearly, the constraint (4.4.2) is only sensitive to the geometry of the Kähler manifold along the direction of the goldstino vector G_I , and therefore it can say nothing about the perturbative stability of moduli with zero F-terms, $G_I = 0$. In particular, it cannot be used to restrict the interactions of those fields that are supersymmetrically decoupled from the sector that breaks supersymmetry. Our work provides necessary and sufficient conditions for the perturbative stability of these $G^I = 0$ fields in the class of models where the Kähler function is of the form (4.4.1).

Finally, we have been able to prove that even in more general scenarios where the integrated heavy moduli do not satisfy (4.4.1), the supersymmetric AdS maxima are always stable for large values of the cosmological constant.

Fayet-Iliopoulos terms and supersymmetric decoupling.

5.1 Introduction

Fayet-Iliopoulos terms, which appear as a constant contribution to the moment map (2.2.18), have many important applications. The presence of FI-terms can provide a mechanism of spontaneous supersymmetry breaking, and also, as the D-term potential is positive definite, they can be used to construct field theories with de Sitter solutions, i.e. with a positive vacuum energy (see [113]). These field theories are suitable to describe an inflationary period, or the present accelerated expansion of the universe.

In the last section of chapter 2, we discussed how Fayet-Iliopoulos terms also play an essential role in the construction of supersymmetric cosmic string solutions. The scalar potential of the supersymmetric Abelian-Higgs model is a pure D -term potential, it has no contribution from the F -terms. The existence of supersymmetric cosmic string solutions in this model is closely related to the presence of a FI term. Actually the tension of the string is proportional to the magnitude of the FI term (2.5.36).

So far it is not known how to derive supergravity theories with a constant FI-term from superstrings/M-theory. One of the first attempts was made in [128], where Dine *et al.* proposed a mechanism to generate such FI-terms in string theory models with pseudo-anomalous $U(1)$'s. In these models the shift symmetry associated to an axio-dilaton, $S = e^\rho + ia$, is gauged under the $U(1)$ as a result of the Green-Schwarz mechanism of anomaly cancellation [129]. This gauging leads to a field dependent contribution in the moment-map of the 4-dimensional theory:

$$\mathcal{P} \sim i(e^{-\rho} + \dots). \quad (5.1.1)$$

The authors of [128] suggested that, provided the dilaton gets an expectation value, such a contribution could act as an effective Fayet-Iliopoulos term in the low energy effective theory. However, as was later shown in [75], the field dependence of the dilaton cannot be integrated out in a supersymmetric way at a high energy scale without integrating out the full vector multiplet too. Thus, if the dilaton is truncated consistently then the D -term potential receives no contribution from the pseudo-anomalous $U(1)$, and in particular no effective FI-term is generated.

More recently the debate was reactivated due to the results presented by Komargodski *et al.* [81, 130], where it was shown that rigid $\mathcal{N} = 1$ supersymmetric theories with a FI-term cannot be the effective low energy theory of any consistent theory of quantum gravity. This conclusion would explain part of the difficulties found when trying to construct string theory models where the FI-term is dynamically generated at low energies. However later works [131, 82] have shown that it is still possible to construct intrinsically supergravity models which do not suffer from the same problems as the ones discussed by Komargodski *et al.* In any case the value of the FI-terms is very constrained, since they have to satisfy quantization conditions [131, 132], and they are also subject to anomaly cancellation conditions in order to ensure the consistency at the quantum level (see [82]).

In view of these discussions, we would like to address the question of how do the FI-terms of a low energy effective theory arise after the integration of the heavy fields of a given model. In particular in this chapter we will discuss the connection between the FI terms of the full $\mathcal{N} = 1$ theory and those appearing in the model obtained after truncating some of the fields while preserving supersymmetry.

In section 5.3 we will show that the size of the FI-terms is unaffected by the supersymmetric truncation, or in other words, that FI-terms cannot be generated during the stabilization of heavy fields without breaking supersymmetry. This study extends the conclusions presented in [75] about supersymmetric truncations and effective FI-terms to more general gauge couplings. For the

5.2. FI-terms in non-linear sigma models.

discussion we have used the recent work by Catino *et al.* [82] which clarifies the issue of identifying the FI-term in non-linear sigma models with general gauge couplings.

In section 5.4 we analyze the gauged axio-dilaton system proposed by Dine *et al.* [128] to construct effective FI-terms. We recover the results by Binetruiy *et al.* [75] which show that with the gauge couplings considered by Dine *et al.* it is not possible to obtain an FI-term after the supersymmetric truncation of the axio-dilaton. We also present an alternative gauging of the sigma-model which allows the supersymmetric integration of the axio-dilaton leaving a reduced theory with a non-vanishing FI-term. As we will see in the next chapter the axio-dilaton system gauged in this way can be embedded in $\mathcal{N} = 2$ supergravity, giving models with a effective FI-terms which can be used to construct de Sitter vacua, to break gauge symmetries and to obtain cosmic string solutions.

Finally in section 5.4.2 we couple the gauged axio-dilaton system to the supersymmetric Abelian-Higgs model discussed in section 2.5. The coupling allows for the supersymmetric truncation of the axio-dilaton leaving the Abelian-Higgs model as the reduced theory. This construction illustrates an important property of consistent truncations, BPS solutions of the reduced theory, i.e. those preserving part of the supersymmetries, are also BPS solutions of the parent theory.

5.2 FI-terms in non-linear sigma models.

In chapter 2 we introduced the compensators $r_a(\xi)$, which characterize the gauge transformations of the Kähler potential $K(\xi, \bar{\xi})$ and the superpotential $W(\xi)$ under the killing vectors $k_a^I(\xi)$

$$\delta_a K(\xi, \bar{\xi}) = 3r_a(\xi) + 3\bar{r}_a(\bar{\xi}), \quad \delta_a W(\xi) = -3r_a(\xi)W(\xi). \quad (5.2.1)$$

Given the Kähler potential and the killing vectors the compensators are only determined up to a constant shift, the Fayet-Iliopoulos term, which also appears as a constant contribution to the moment-map $\mathcal{P}_a$ (2.2.16).

From (5.2.1) we can see that gauge transformations induce a Kähler transformation (2.2.28) with the holomorphic function given by $h(\xi) = 3r_a(\xi)$. Actually, the invariance of the action under (abelian) gauge transformations also requires that the fermions transform as:

$$\begin{aligned} \delta_a \psi_\mu &= -\frac{i3}{2}\gamma_5 \text{Im}(r_a) \psi_\mu, \\ \delta_a \chi^I &= \partial_J k_a^I \chi^J + \frac{i3}{2}\gamma_5 \text{Im}(r_a) \chi^I, \\ \delta_a \lambda^b &= -\frac{i3}{2}\gamma_5 \text{Im}(r_a) \lambda^b. \end{aligned} \quad (5.2.2)$$

If the compensator $r_a(\xi)$ has a constant contribution, i.e. an FI-term

$$r_a(\xi) = \dots + \frac{i\eta_a}{3}, \quad (5.2.3)$$

then the gravitino and the gaugini become charged under the abelian gauge symmetry generated by the killing vector k_a^I , and the gauge couplings of the chiralini are modified. These new couplings introduced by the FI-terms are responsible for the quantization and the anomaly cancellation conditions which constrain their allowed values.

However in non-linear sigma models it is not obvious how to identify such constant contribution of the compensators, which are field dependent in general. Moreover, the gauge couplings of the fermions induced by the compensator (5.2.2) are not well defined since they are both Kähler dependent and U(1) gauge dependent. Indeed, the compensator transforms in the following way under Kähler and U(1) gauge transformations:

$$\delta_K r_a = \frac{1}{3} k_a^I \partial_I h, \quad \delta_a r_b = k_a^I \partial_I r_b, \quad (5.2.4)$$

where $h(\xi^I)$ is the holomorphic function characterizing the Kähler transformation. Catino *et al.* proved in [82] that FI-terms are only present in those theories where the compensator cannot be ‘gauged away’ everywhere in field space. As a matter of fact, Kähler transformations relate equivalent theories, and therefore if we can find a Kähler gauge where the compensator is zero everywhere, in particular the U(1) gauge couplings of the gravitino and the gaugino become also vanishing, and therefore the FI-term must be vanishing too.

In order to find the appropriate Kähler transformation to gauge away the FI-term we have to solve the following differential equation for the holomorphic function $h(\xi)$

$$k_a^I(\xi) \partial_I h(\xi) = -3r_a(\xi). \quad (5.2.5)$$

Assuming $h(\xi)$ to be regular, this can only be solved provided the compensator vanishes at the fixed point of the killing vector¹ ξ_0 , i.e. where $k_a^I(\xi_0) = 0$. Then, if the killing vector has no fixed points then the compensator can always be gauged away and no FI-terms can be present [82]. This equation has an obvious solution whenever the theory has a superpotential $W(\xi)$ which is non-vanishing everywhere in field space. Indeed, from the gauge transformation of the superpotential (2.2.19) we see that the appropriate Kähler transformation is given by

$$h(\xi^I) = \log W(\xi^I). \quad (5.2.6)$$

¹We can always choose a local coordinate system where $k_a(\xi) = iq_a \xi^1 \partial_1$ (see [87]). In these coordinates the integration of equation (5.2.5) is trivial provided the compensator vanishes at the fixed point of the killing vector.

5.2. FI-terms in non-linear sigma models.

Conversely, suppose that the compensator has a finite value at ξ_0

$$r_a(\xi_0) = \frac{i}{3}\eta_a. \quad (5.2.7)$$

At the point $\xi = \xi_0$ the Kähler potential should not transform, and therefore from (5.2.1) we see that η_a must be a real constant. In such a case it is still possible to find a regular Kähler transformation which sets the compensator to a constant everywhere solving the equation

$$k_a^I(\xi)\partial_I h(\xi) = -3r_a(\xi) + i\eta_a. \quad (5.2.8)$$

After this transformation the compensator is simply $r_a(\xi) = i\eta_a/3$, and thus the Kähler potential becomes invariant under gauge transformations. Moreover, recalling the expression for the moment map (2.2.16) we find

$$\mathcal{P}_a|_{\xi_0} = -i3r_a(\xi_0) = \eta_a, \quad (5.2.9)$$

and therefore we can identify the FI-term with the value of the moment map at the fixed point of the killing vector k_a^I . The previous expression is both Kähler invariant and gauge invariant. Notice that, if the scalar manifold is simply connected and the gauged isometry is compact, then the corresponding killing vector has a fixed point and thus it is possible to have a non-zero FI-term.

In situations where the killing vector has more than one fixed point $\xi_0, \xi_1, \dots$, we can repeat the procedure locally in patches $U_i \subseteq \mathcal{M}$ of the Kähler manifold small enough to contain a single fixed point ξ_i . In each patch we will have to change the Kähler gauge in order to transform the compensator into a constant. In general the resulting Kähler potential and the superpotential will be different in different patches, but on the overlap regions $U_i \cap U_j$ they will be related to each other by Kähler transformations [90]:

$$K_i - K_j = h_{ij} + \bar{h}_{ij}, \quad (5.2.10)$$

where h_{ij} are holomorphic functions. These transition functions will determine the relation between the values of the FI-term in different patches, and are constrained by the topology of the Kähler-Hodge manifold [90].

Example: FI-terms on the sphere

Consider the supersymmetric CP^1 model coupled to a $U(1)$ gauge field A_μ . This model leads to an anomalous quantum theory which can be cured adding extra matter fields [133, 134], but in order to keep this discussion simple we will not consider this issue. The target space $\mathcal{M}$ is a Kähler manifold, the two-sphere $S^2 \cong \frac{SU(2)}{U(1)}$. In order to parametrize the sphere completely we divide it into two patches, the northern and the southern hemispheres.

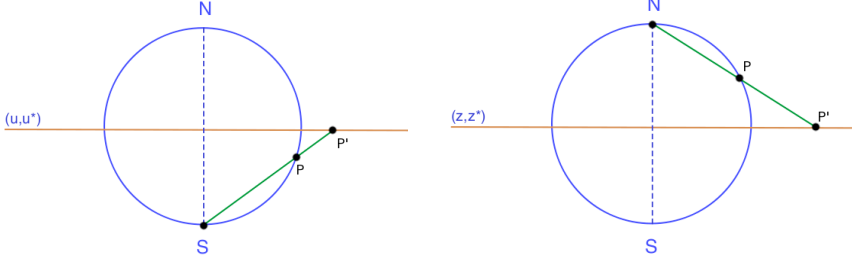


Figure 5.1 – LEFT: Stereographic projection from the South pole on the complex plane parametrized by $\{u, \bar{u}\}$. RIGHT: Stereographic projection from the North pole on the complex plane parametrized by $\{z, \bar{z}\}$.

The northern hemisphere can be parametrized using the stereographic projection from the South pole onto the complex plane with coordinates $\{u, \bar{u}\}$, see Fig. 5.1. This parametrization excludes the south pole, which is projected to the point at infinity on the complex plane. For the southern hemisphere we use the stereographic projection from the north pole on the complex plane with coordinates $\{z, \bar{z}\}$. Both parametrizations are related by the holomorphic change of coordinates

$$z = \frac{1}{u}, \quad \bar{z} = \frac{1}{\bar{u}}, \quad (5.2.11)$$

An appropriate Kähler potential in the northern hemisphere is $K^{(N)}(u, \bar{u}) = n \log(1 + u\bar{u})$, with n being an even integer [90], leading to the following kinetic terms for the scalars

$$\mathcal{T} = -n \frac{\partial_\mu u \partial^\mu \bar{u}}{(1 + u\bar{u})^2}. \quad (5.2.12)$$

On the southern hemisphere we choose the Kähler potential $K^{(S)}(z, \bar{z}) = n \log(1 + z\bar{z})$, which is related to $K^{(N)}$ on the overlap of both patches, i.e. the equator, by the Kähler transformation

$$K^{(S)} - K^{(N)} = h_{NS} + \bar{h}_{NS}, \quad \text{with} \quad h_{NS}(u) = n \log u. \quad (5.2.13)$$

Suppose we gauge the isometry associated to rotations of the sphere around the North-South axis. The corresponding gauge transformations of the coordinates on the northern and southern hemispheres are respectively

$$\delta u = i g u \alpha, \quad \delta z = -i g z \alpha, \quad (5.2.14)$$

Since the Kähler potentials $K^{(N)}$ and $K^{(S)}$ are invariant under the gauge symmetry the compensators must be a constant proportional to the FI-terms, and therefore we do not have to perform any additional Kähler transformation in

5.3. FI-terms and supersymmetric truncations

order to gauge away the field dependence of the compensator. Thus the moment maps are given by

$$\mathcal{P}^{(N)} = -g \left(\frac{n u \bar{u}}{(1 + u \bar{u})} - \eta^{(N)} \right), \quad \mathcal{P}^{(S)} = g \left(\frac{n z \bar{z}}{(1 + z \bar{z})} - \eta^{(S)} \right). \quad (5.2.15)$$

Where $\eta^{(N)}$ and $\eta^{(S)}$ are the FI-terms on the northern and the southern hemisphere respectively. Note that, since the gauged isometry has fixed points at the South pole, $u = 0$, and the North pole, $z = 0$, the FI-terms cannot be gauged away by a Kähler transformation and thus, according to the results in [131, 132], the FI-terms have to be quantized

$$\eta^{(N)} = p \quad \text{and} \quad \eta^{(S)} = q, \quad \text{with } p \text{ and } q \text{ being even integers.} \quad (5.2.16)$$

Moreover, according to (5.2.4) the compensators on the two hemispheres are related by the Kähler transformation (5.2.13)

$$r^{(S)} - r^{(N)} = i g u \partial_u h_{NS} = i g n \quad \Longleftrightarrow \quad q - p = n. \quad (5.2.17)$$

Therefore we can see that, since in this model the values of the compensator at the two fixed points of the gauged isometry are related to each other, and we are only free to choose one of them.

5.3 FI-terms and supersymmetric truncations

As we discussed in the introduction it is interesting to understand whether it is possible to generate effective FI-terms by the stabilization of part of the fields of a theory. Here we will show that the size of the FI-term cannot change if the truncation leaves supersymmetry unbroken. In other words, the size of the FI-term in the reduced theory is the same as the size of the FI-term in the full parent theory.

Suppose we perform a supersymmetric truncation of a theory with target space in the Kähler-Hodge manifold $\widehat{\mathcal{M}}$ where we have gauged a U(1) isometry. The manifold $\widehat{\mathcal{M}}$ is parametrized by the fields $\{H^\alpha, L^i\}$, and the truncation is done fixing a sector of the fields H^α at the point $H^\alpha = H_0^\alpha$. Let us denote by $\hat{k}$ the killing vector associated to the gauged U(1) symmetry in the full theory, we call $\widehat{\mathcal{P}}$ the corresponding moment map, and $\hat{\eta}$ the FI-term.

According to our discussion in the previous section, in order to have a non vanishing FI-term the killing vector $\hat{k}$ must have at least one fixed point. For simplicity we restrict ourselves to a patch $U \subseteq \widehat{\mathcal{M}}$ containing a single fixed point of $\hat{k}$. Then, if $\xi_0^I = \{H_0^\alpha, L_0^i\}$ is the fixed point of the U(1) isometry, $\hat{k}(\xi_0^I) = 0$, in the full theory we have the following relation

$$\widehat{\mathcal{P}}(\xi_0^I, \xi_0^{\bar{I}}) = \hat{\eta}. \quad (5.3.1)$$

The most interesting situation is when the corresponding gauge field survives in the low energy theory, so that the FI-term in the reduced theory can be used for the spontaneous breaking of gauge symmetries. In such a case the moment map in the reduced theory is given by:

$$\mathcal{P}(L^i, L^{\bar{i}}) = \widehat{\mathcal{P}}(H_0^\alpha, H_0^{\bar{\alpha}}, L^i, L^{\bar{i}}). \quad (5.3.2)$$

In the discussion of section 3.2.2 we saw that if we want the gauge symmetry to survive in the reduced theory, the components of the killing vector $\hat{k}$ along the H^α directions must vanish in the reduced theory, $\hat{k}^\alpha|_{H_0^\alpha} = 0$. Actually the killing vector in the reduced theory is given by

$$k(L^i) = \hat{k}(H_0^\alpha, L^i) = \hat{k}^i(H_0^\alpha, L^j) \partial_i, \quad (5.3.3)$$

which implies that the fixed point of the killing vector k of the truncated theory is L_0^i , since the killing vector $\hat{k}$ of the parent theory vanishes at the configuration $\xi_0^I = \{H_0^\alpha, L_0^i\}$. From this result, together with equation (5.3.2), it follows that the FI-term of the reduced theory η must be equal to the FI-term before the truncation, that is, $\hat{\eta}$:

$$\eta = \mathcal{P}(L_0^i, L_0^{\bar{i}}) = \widehat{\mathcal{P}}(H_0^\alpha, H_0^{\bar{\alpha}}, L_0^i, L_0^{\bar{i}}) = \hat{\eta}. \quad (5.3.4)$$

Therefore the size of the FI-term cannot change by the supersymmetric truncation of part of the field content of the theory. In other words, in order to generate an FI term by the stabilization of some fields supersymmetry must be broken necessarily.

5.4 The axio-dilaton system

In this section we study the *axio-dilaton* system, a sigma model with target space on the hyperboloid $H^2 \cong \frac{\text{SU}(1,1)}{\text{U}(1)}$. As we mentioned in the introduction, this model was considered by Dine *et al.* in [128], where they proposed a mechanism to obtain low energy effective supergravity lagrangians with FI-terms from string theory. In their construction the shift symmetry of the axio-dilaton, $S = e^\rho + ia$, was coupled to a U(1) gauge field as a result of the Green-Schwarz mechanism of anomaly cancellation.

Using arguments from our discussion in the previous two sections we recover the results by Binetruy *et al.* [75]. We show that with the gauge coupling given by the Green-Schwarz mechanism it is not possible to obtain an FI-term after the supersymmetric truncation of the axio-dilaton. We also present an alternative gauging of the sigma-model which allows the supersymmetric integration of the axio-dilaton leaving a reduced theory with a non-vanishing FI-term.

5.4. The axio-dilaton system

The idea of generating effective FI-terms from field dependent moment maps is also especially relevant in the context of $\mathcal{N} = 2$ supergravity. As we will see in the next chapter, $\mathcal{N} = 2$ supergravity theories are highly constrained. In particular, in $\mathcal{N} = 1$ supergravity models which admit an embedding in $\mathcal{N} = 2$ it is no longer allowed to introduce FI terms as shifts of the moment map (2.2.18). In this type of models the compensator $r_a(\xi)$ is completely determined by the gauging of isometries. Then, in order to find constant FI-terms in $\mathcal{N} = 2$ supergravity the only freedom we have is the gauging of isometries, i.e. the choice of the killing vectors.

In the next chapter we will present a technique to obtain $\mathcal{N} = 1$ models with an effective FI-term from the truncation of $\mathcal{N} = 2$ parent theory [73]. This technique is also based on the idea of truncating a sector of the theory in a supersymmetric way, and therefore is rather similar to our discussion here. The axio-dilaton system is an important building block in this construction. Indeed, the gauged axio-dilaton model we present here can be embedded in $\mathcal{N} = 2$ supergravity, and the truncation of the axio-dilaton in the $\mathcal{N} = 2$ theory leaves behind a reduced $\mathcal{N} = 1$ theory with an FI-term.

5.4.1 Gauging of isometries and FI-term

We consider a sigma model with target space $\mathcal{M} = \frac{\text{SU}(1,1)}{\text{U}(1)}$, i.e. an axio-dilaton system, coupled to a $\text{U}(1)$ gauge field A_μ . An appropriate Kähler potential for the axio-dilaton system would be:

$$K(S, \bar{S}) = -\log(S + \bar{S}), \quad \text{with} \quad S = e^\rho + ia, \quad (5.4.1)$$

For simplicity we choose a vanishing superpotential and the gauge kinetic function to be a constant $f(S) = 1$. The scalar manifold $\frac{\text{SU}(1,1)}{\text{U}(1)}$ has three different isometries characterized by the killing vectors:

$$k_1 = 2S \partial_S \quad k_2 = i \partial_S \quad k_3 = -iS^2 \partial_S. \quad (5.4.2)$$

The isometry defined by k_2 corresponds to the invariance under shifts of the axion a which is manifest in the Kähler potential. We can see that none of the killing vectors has fixed points within the Kähler manifold, since the point $S = 0$ does not belong to H^2 .

The compensators corresponding to the killing vectors can be determined, up to an arbitrary imaginary constant, by the equation (2.2.17). In order to illustrate the situation in $\mathcal{N} = 2$ supergravity, where the only freedom we have to obtain non vanishing FI-term is the choice of gauging, we will fix completely the value of the compensators from the beginning, thus removing our freedom to shift them. One possible choice is

$$r_1 = -\frac{1}{3} \quad r_2 = 0 \quad r_3 = \frac{1}{3}iS, \quad (5.4.3)$$

and therefore the moment maps are given by²

$$\mathcal{P}_1 = \frac{S - \bar{S}}{i(S + \bar{S})}, \quad \mathcal{P}_2 = \frac{1}{S + \bar{S}}, \quad \mathcal{P}_3 = \frac{S\bar{S}}{S + \bar{S}}. \quad (5.4.4)$$

Since none of the three killing vectors in (5.4.2) has a fixed point, gauging any one of them would result into a theory with vanishing FI-term.

In the construction by Dine *et al.* [128] the gauge coupling appeared as a result of the Green-Schwarz mechanism of anomaly cancellation. This gauge coupling is obtained promoting to a local symmetry the shift generated by k_2

$$\delta S = i g \alpha, \quad D_\mu S = \partial_\mu S - i g A_\mu, \quad (5.4.5)$$

where α is the gauge parameter. The gauge symmetry is broken everywhere in field space and thus this coupling leads to a theory with a vanishing FI-term. Moreover, since there is no configuration where the gauge boson and the axion decouple, from the discussion in chapter 3 it follows that it is not possible to truncate S consistently and preserve supersymmetry.

An alternative gauging

Instead we choose to gauge the linear combination $k_{FI} = k_2 + k_3$, which has a fixed point at $S = 1$. The killing vector k_{FI} has a very simple geometrical interpretation, actually it generates the isometry associated to rotations of the hyperboloid around its symmetry axis, and the fixed point $S = 1$ corresponds to the vertex of the hyperboloid. The moment map associated to k_{FI} is given by:

$$\mathcal{P}_{FI} = \mathcal{P}_1 + \mathcal{P}_2 = \frac{(1 + S\bar{S})}{S + \bar{S}} = \frac{1}{2} [e^{2h} + (a^2 + 1)e^{-2h}] \geq 1, \quad (5.4.6)$$

where we have assumed that the moment maps combine in the same way as the killing vectors, linearly, as occurs in $\mathcal{N} = 2$ supergravity models. In particular, in $\mathcal{N} = 1$ supergravity models which admit an embedding in $\mathcal{N} = 2$ the moment maps combine in this way. The scalar potential can be obtained from (2.2.14)

$$V = V_D = \frac{1}{2} \mathcal{P}^2 = \frac{1}{2} \frac{(1 + |S|^2)^2}{(S + \bar{S})^2}. \quad (5.4.7)$$

It is now easy to check that with this choice of gauging our model has a non vanishing FI-term. Its size is given by the value of the moment map at the fixed point of the killing vector, $S = 1$, where the moment map saturates its lower bound

$$\eta_{FI} = \mathcal{P}_{FI}|_{S=1} = 1. \quad (5.4.8)$$

²In order to fix the compensators completely we have chosen the moment maps to be consistent with the gauging of the full $SU(1, 1)$ group [18].

5.4. The axio-dilaton system

Using equations (2.2.5) and (2.2.12) to calculate the kinetic terms of the axio-dilaton we arrive at the following expression for the bosonic sector of the lagrangian

$$e^{-1/2}\mathcal{L} = -\frac{1}{2}R - \frac{D_\mu S D^\mu \bar{S}}{(S + \bar{S})^2} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}\frac{(1 + |S|^2)^2}{(S + \bar{S})^2}, \quad (5.4.9)$$

where we have defined the covariant derivative as

$$D_\mu S = \partial_\mu S - i\eta(1 - S^2)A_\mu. \quad (5.4.10)$$

The axio-dilaton system gauged in this way can be embedded in $\mathcal{N} = 2$ supergravity, giving a theory with an effective FI-term which can be used to obtain de Sitter vacua, to break gauge symmetries, and to construct cosmic string solutions.

5.4.2 Supersymmetric cosmic strings and supersymmetric truncations

An important property of supersymmetric truncations is that they respect BPS configurations, i.e. those which leave unbroken some of the supersymmetries. In other words, if a field configuration is BPS in the reduced theory, it must be BPS in the parent theory. This can be understood noting that the supersymmetric truncation can be considered as an ansatz which ensures that the supersymmetry transformations of the truncated fields are zero, and therefore supersymmetries which survive on the BPS configuration of the reduced theory must also be unbroken in the full theory.

In this section we present a simple model obtained coupling the axio-dilaton system described above with the supersymmetric Abelian-Higgs model discussed in section 2.5. The coupling is done in such a way that the axio-dilaton can be truncated supersymmetrically leaving the Abelian-Higgs model as the reduced theory. Thus, the half-BPS cosmic string solutions found in section 2.5, will also be solutions of the full model which preserve half of the supersymmetries.

Let us consider a sigma model defined on the target space $\widehat{\mathcal{M}} = \frac{\text{SU}(1,1)}{\text{U}(1)} \times \mathbb{C}$, containing an axio-dilaton S and a complex scalar field ϕ , both coupled to a U(1) gauge boson A_μ . We choose the Kähler potential to be

$$K(\phi, \bar{\phi}, S, \bar{S}) = -\log(S + \bar{S}) + \phi\bar{\phi}. \quad (5.4.11)$$

As in the previous section we will assume that the superpotential is vanishing $W = 0$, and that the kinetic function is a constant $f(\phi, S) = 1$. In order to couple the gauge boson A_μ to the fields ϕ and S we promote to local the isometry associated to the killing vector

$$k = gk_4 + g\eta k_{FI}. \quad (5.4.12)$$

Here k_4 is killing vector associated to the standard $U(1)$ gauge coupling, and k_{FI} characterizes the isometry of the axio-dilaton system we gauged in the previous section

$$k_4 = i\phi \partial_\phi, \quad k_{FI} = i(1 - S^2)\partial_S. \quad (5.4.13)$$

The compensator and the moment map associated to the killing vector k can be chosen to be

$$r = i\eta S, \quad \mathcal{P} = -g|\phi|^2 + g\eta \frac{1 + |S|^2}{S + \bar{S}}. \quad (5.4.14)$$

With a vanishing superpotential the full scalar potential coincides with the D -term potential, and therefore it reads

$$V = V_D = \frac{1}{2}\mathcal{P}^2 = \frac{g^2}{2} \left(|\phi|^2 - \eta \frac{1 + |S|^2}{S + \bar{S}} \right)^2, \quad (5.4.15)$$

Gathering everything together we find that the bosonic sector of the lagrangian is given by

$$\begin{aligned} e^{-1/2}\mathcal{L} = & -\frac{1}{2}R - D_\mu\phi D^\mu\bar{\phi} - \frac{D_\mu S D^\mu \bar{S}}{(S + \bar{S})^2} + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ & - \frac{g^2}{2} \left(|\phi|^2 - \eta \frac{1 + |S|^2}{S + \bar{S}} \right)^2, \end{aligned} \quad (5.4.16)$$

where the covariant derivatives are defined as

$$D_\mu\phi = \partial_\mu\phi - i\eta\phi A_\mu \quad \text{and} \quad D_\mu S = \partial_\mu S - i\eta(1 - S^2)A_\mu. \quad (5.4.17)$$

The field configuration $S = 1$ is consistent with the supersymmetric integration of the axio-dilaton. With the axio-dilaton fixed at this point the scalar manifold reduces to

$$\frac{SU(1,1)}{U(1)} \times \mathbb{C} \longrightarrow \mathbb{C} \quad (5.4.18)$$

which, due to the product structure of $\widehat{\mathcal{M}}$, is a totally geodesic Kähler submanifold of the full target space. Moreover at $S = 1$ the gauge boson A_μ and the axio-dilaton decouple, since the component of the gauged killing vector along S becomes zero. From our discussion in chapter 3 it follows that these conditions are enough to ensure the supersymmetric decoupling of the axio-dilaton. In particular it is easy to check that $S = 1$ is an extremum of the scalar potential along the direction S .

In the reduced theory, i.e. with $S = 1$, the compensator reduces to an imaginary constant $r|_{S=1} = i\eta$, an FI-term

$$\mathcal{P}|_{S=1} = -g|\phi|^2 + g\eta \implies V_{S=1} = \frac{g^2}{2} (|\phi|^2 - \eta)^2. \quad (5.4.19)$$

5.5. Summary

Note that the reduced theory coincides with the abelian-Higgs model studied in section 2.5. This reduced model admits half-BPS cosmic string solutions, which preserve a fraction of the supersymmetries of the system. Since the reduced theory is obtained after a supersymmetric truncation, the conditions that ensure the BPS condition in the reduced theory are sufficient to guarantee that the cosmic strings are also BPS in the full theory.

5.5 Summary

In this chapter we have considered the possibility of dynamically generating FI-terms by the supersymmetric truncation of a sector of a $\mathcal{N} = 1$ supergravity theory. We have shown that the size of the FI-terms is unaffected by the supersymmetric truncation, or in other words, that FI-terms cannot be generated during the stabilization of heavy fields without breaking supersymmetry.

We have discussed the proposal by Dine *et al.* [128] to construct effective FI-terms from an axio-dilaton system with a U(1) gauge coupling induced the Green-Schwarz mechanism of anomaly cancellation. This study recovers the results by Binetruy *et al.* [75] which show that in this model it is not possible to truncate the axio-dilaton in a supersymmetric way giving an effective theory with an FI-term. We have shown that it is still possible to find a coupling between the axio-dilaton and the U(1) gauge field which is consistent with truncating the axio-dilaton in a supersymmetric way, so that the reduced theory contains a non-vanishing FI-term. Although the isometry associated to this gauging has an easy geometrical interpretation, it is not so clear what would be its physical meaning within the framework of string theory.

Finally we have discussed a model where we couple the version of the gauged axio-dilaton system presented here to the supersymmetric Abelian-Higgs model discussed in section 2.5. The coupling allows for the supersymmetric truncation of the axio-dilaton leaving the Abelian-Higgs model as the reduced theory. With this model we have illustrated an important property of consistent truncations, that BPS solutions of the reduced theory are also BPS solutions of the full theory. That is, if a particular solution of the reduced theory leaves unbroken part of the supersymmetries, these supersymmetries remain unbroken in the parent theory, i.e. if we consider again the effects of all the truncated fields.

$\mathcal{N} = 2$ supergravity and effective Fayet-Iliopoulos terms.

6.1 Introduction

This chapter is intended to provide the basis to discuss our work about supersymmetric cosmic strings in $\mathcal{N} = 2$ supergravity, thus we will review here some aspects of extended $\mathcal{N} = 2$ locally supersymmetric theories.

In $\mathcal{N} = 2$ supergravity the couplings of the vector multiplets and of the hypermultiplets are governed by *special geometry* and *quaternionic geometry* respectively. In sections 6.3 and 6.4 we will present the basic features of the kinetic terms of the bosons in both sectors, and then in section 6.5 we will discuss the form of the scalar potential. Contrary to what happens in $\mathcal{N} = 1$ supergravity, in $\mathcal{N} = 2$ there is no analogue of the superpotential, and thus the only contribution to the scalar potential resembles the $\mathcal{N} = 1$ D -term potential, which is non-zero provided we promote to local some of the global symmetries of the action.

According to our discussion in chapter 2, Fayet-Iliopoulos terms play a crucial rôle in the construction of supersymmetric cosmic string solutions. In

particular, they induce the spontaneous gauge symmetry breaking which leads to the formation of the string, and the string tension is proportional to the value of the FI-term. In $\mathcal{N} = 1$ supergravity, if the gauged isometry has a fixed point, then we can obtain a model with a non-zero FI-term using the freedom to add a constant to the moment map (2.2.18). However, as we will show in section 6.5.1, in $\mathcal{N} = 2$ supergravity a killing vector has associated a triplet of moment maps, $\mathcal{P}^x$ with $x = 1, 2, 3$, which are completely determined by the killing vector, and thus we are not free to shift them. This statement can also be understood in terms of the gauge couplings of the gravitino. In the last chapter we identified the $\mathcal{N} = 1$ FI-term with the charge of the gravitino under the gauge symmetry, then the freedom to shift the moment map (2.2.18) ensures that we are free to choose this charge. However in $\mathcal{N} = 2$ supergravity the coupling of the gravitino to the corresponding gauge boson is completely determined by the choice of the killing vectors, i.e. the gauge couplings of the scalar fields.

Identifying a constant contribution of the $\mathcal{N} = 2$ moment maps $\mathcal{P}^x$ which could act as an FI-term is not obvious. Since FI-terms are better understood in $\mathcal{N} = 1$ supergravity, in order to solve this problem we will discuss consistent truncations of $\mathcal{N} = 2$ supergravity models with partial reduction of supersymmetry down to $\mathcal{N} = 1$ [106, 107, 135]. This technique, reviewed in section 6.8, is similar to the one we presented in chapter 3, and consists in truncating part of the field content of the theory in such a way that the reduced model is only invariant under $\mathcal{N} = 1$ local supersymmetry transformations. As for $\mathcal{N} = 1 \rightarrow \mathcal{N} = 1$ truncations, this method also ensures that any solution of the equations of motion of the reduced theory is also a solution of the full equations of motion. Within the reduced $\mathcal{N} = 1$ theory the identification the FI-term is straightforward using the arguments of the previous chapter, indeed its magnitude is given by the value of the moment map of the reduced theory at the fixed point of the killing vector.

In section 6.9 we will present the method to construct $\mathcal{N} = 2$ supergravity actions leading $\mathcal{N} = 1$ supergravity models with a constant FI-term after a consistent reduction of supersymmetry. We will end the chapter with a couple of explicit examples, and in particular we discuss the embedding in $\mathcal{N} = 2$ supergravity of the gauged axio-dilaton system which was studied in the previous chapter.

6.2 Overview

The $\mathcal{N} = 2$ supersymmetry algebra involves two supersymmetry generators represented by Majorana spinors, which can be decomposed into two chiral spinors. We shall work with the corresponding four chiral spinors that we denote (ϵ^i, ϵ_i) . The index $i = 1, 2$ labels the original Majorana spinors and the position of that

6.2. Overview

index represents the chirality. ϵ^i is a left-handed spinors while ϵ_i is right-handed:

$$\epsilon^i = \frac{1}{2}(1 + \gamma_5)\epsilon^i, \quad \epsilon_i = \frac{1}{2}(1 - \gamma_5)\epsilon_i. \quad (6.2.1)$$

We follow the notation and conventions of [94, 84]. Charge conjugation relates the two chiral projections of a given Majorana spinor. We shall use the same convention for other chiral spinors.

In four space-time dimensions with a Minkowski signature the automorphism group of the $\mathcal{N} = 2$ supersymmetry algebra, the R-symmetry group, is $H_R = \text{U}(2)$. Under the $\text{SU}(2)$ part of the R-symmetry, the supersymmetry generators ϵ^i transform as a doublet. The $\text{U}(1)$ part of the R-symmetry acts on the generators by a change of phase.

In $\mathcal{N} = 2$ supergravity, we shall consider three type of supermultiplets, the *graviton multiplet*, the *vector multiplet*, and the *hypermultiplet*. Inside each multiplet, the fields are arranged into representations of the R-symmetry group:

- The graviton multiplet: it contains the vielbein of the spacetime metric, two gravitini ψ_μ^i and one graviphoton A_μ^0 . The label $i = 1, 2$ is associated to the $\text{SU}(2)$ R-symmetry transformations.
- The vector multiplet: it contains one complex scalar z^α , two gaugini λ_i^α and one gauge field A_μ^α . Here $\alpha = 1, \dots, n_V$ labels n_V different vector multiplets.
- The hypermultiplet: it contains four real scalars q^X and two hyperini, ζ^A , where the labels are $X = 1, \dots, 4n_H$ and $A = 1, \dots, 2n_H$ for n_H hypermultiplets.

For convenience, the $\mathcal{N} = 2$ supersymmetry transformations will be reviewed later on, after discussing the bosonic part of the action

$$S_{bos} = \int d^4x \sqrt{-g} \left[-\frac{1}{2}R + \mathcal{L}_{hyper} + \mathcal{L}_{vector} - \mathcal{V} \right], \quad (6.2.2)$$

where $\mathcal{L}_{hyper}$ are the kinetic terms of the hyperscalars, $\mathcal{L}_{vector}$ are the kinetic terms of the bosonic fields in the vector multiplets and $\mathcal{V}$ is the scalar potential.

In $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets, the scalar manifold that characterizes the kinetic terms of the scalars is a direct product

$$\mathcal{M} = \mathcal{M}_{S\mathcal{K}} \times \mathcal{M}_{\mathcal{Q}}, \quad (6.2.3)$$

where $\mathcal{M}_{S\mathcal{K}}$ and $\mathcal{M}_{\mathcal{Q}}$ are respectively the scalar manifold of vector and hypermultiplets. The restrictions coming from supersymmetry impose that $\mathcal{M}_{S\mathcal{K}}$ is a so-called *special manifold*, whereas $\mathcal{M}_{\mathcal{Q}}$ is a *quaternionic manifold*. We will pay particular attention to the geometry of the scalar manifold and the gauging of isometries.

Gravity multiplet	vielbein	e_μ^m	$i = 1, 2$ $\mu, m = 0, \dots, 3$
	gravitini	$\psi_\mu^i, \psi_{i\mu}$	
	graviphoton	$“A_\mu^0”$	
Vector multiplet	gauge fields	A_μ^α	$\alpha = 1, \dots, n_V$
	gaugini	$\lambda_i^\alpha, \lambda_\alpha^i$	
	scalars	z^α	
Hypermultiplets	hyperscalars	q^X	$X = 1, \dots, 4n_H$ $A = 1, \dots, 2n_H$
	hyperini	ζ^A, ζ_A	

Table 6.1 – Field content of $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets and n_H hypermultiplets. The physical graviphoton is not necessarily A_μ^0 but whatever field appears in the supersymmetric transformation of the gravitini through its field strength $T_{\mu\nu}$. The latter is a linear combination of field strengths of all the gauge fields A_μ^Λ present in the theory ($\Lambda = 0, \dots, n_V$) with coefficients that depend on the scalar fields z^α of vector multiplets. The couplings of all the gauge fields A_μ^Λ and the scalar fields z^α is controlled by special geometry.

6.3 Vector multiplets and special geometry

We consider $\mathcal{N} = 2$ supergravity coupled to n_V vector multiplets [136, 137, 138]. For a modern review, see [139, 140, 141].

Since the gravity multiplet contains a vector field, the *graviphoton*, this theory admits $n_V + 1$ vector fields A_μ^Λ where $\Lambda = 0, \dots, n_V$. The n_V vector multiplets contain as well n_V complex scalar fields z^α , $\alpha = 1, \dots, n_V$, parametrizing a Special Kähler manifold. The kinetic terms of the scalar and vector fields are :

$$\begin{aligned}
 \mathcal{L}_{vector} &= \frac{1}{4} \text{Im}(\mathcal{N})_{\Lambda\Sigma} F^{\Lambda|\mu\nu} F_{\mu\nu}^\Sigma - \frac{1}{8} e^{-1} \text{Re}(\mathcal{N})_{\Lambda\Sigma} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Sigma - g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}, \\
 &= \frac{1}{2} \text{Im} \left(\mathcal{N}_{\Lambda\Sigma} F_{\mu\nu}^{+\Lambda} F^{+\Sigma|\mu\nu} \right) - g_{\alpha\bar{\beta}} \partial_\mu z^\alpha \partial^\mu \bar{z}^{\bar{\beta}}, \tag{6.3.1}
 \end{aligned}$$

6.3. Vector multiplets and special geometry

where $F_{\mu\nu}^{\pm\Lambda}$ is the self-dual combination¹:

$$F_{\mu\nu}^{\pm\Lambda} = \frac{1}{2} \left(F_{\mu\nu}^{\Lambda} \mp \frac{1}{2} i \varepsilon_{\mu\nu\rho\sigma} F^{\Lambda|\rho\sigma} \right), \quad F_{\mu\nu}^{\Lambda} = \partial_{\mu} A_{\nu}^{\Lambda} - \partial_{\nu} A_{\mu}^{\Lambda}. \quad (6.3.2)$$

The couplings of vector multiplets to $\mathcal{N} = 2$ supergravity are characterized by *special geometry*. The latter relies heavily on the existence of *duality transformations* for vector fields in supersymmetric theories. Duality transformations generalize the electric-magnetic duality of Maxwell's equations without sources. Defining the magnetic duals of the field strengths $F_{\mu\nu}^{+\Lambda}$ as

$$G_{+\Lambda}^{\mu\nu} = 2i \frac{\partial \mathcal{L}}{\partial F_{\mu\nu}^{+\Lambda}} = \mathcal{N}_{\Lambda\Sigma} F^{+\Sigma|\mu\nu}, \quad (6.3.3)$$

the equations of motion and Bianchi identities read

$$\begin{aligned} \partial^{\mu} \text{Im } F_{\mu\nu}^{+\Lambda} &= 0, & \text{Bianchi Identity,} \\ \partial_{\mu} \text{Im } G_{+\Sigma}^{\mu\nu} &= 0, & \text{Equations of motion.} \end{aligned} \quad (6.3.4)$$

Then, the duality transformations are linear transformations $\mathcal{S} \in \text{G}\ell(2n_V + 2, \mathbb{R})$ of the field strengths and their magnetic duals

$$\begin{pmatrix} \tilde{F}^+ \\ \tilde{G}_+ \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^+ \\ G_+ \end{pmatrix}, \quad (6.3.5)$$

that preserve the Bianchi identities and equations of motion for vector fields. The relation between the field strengths and their magnetic duals (6.3.3) determines the transformation rule of the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ under (6.3.5):

$$\mathcal{N} \implies (C + D\mathcal{N})(A + B\mathcal{N})^{-1}, \quad (6.3.6)$$

where A, B, C, D are $(n_V + 1) \times (n_V + 1)$ real matrices that characterize the duality transformation

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{G}\ell(2n_V + 2, \mathbb{R}). \quad (6.3.7)$$

If equations (6.3.3) are derived from a Lagrangian $\mathcal{L}$, the matrix $\mathcal{N}_{\Lambda\Sigma}$ should be symmetric. Asking the symmetry of $\mathcal{N}_{\Lambda\Sigma}$ to be preserved under a fractional transformation (6.3.6) restricts the duality transformations to be given by symplectic matrices $\mathcal{S}$:

$$\mathcal{S} \in \text{Sp}(2n_V + 2, \mathbb{R}). \quad (6.3.8)$$

As the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ transforms under duality transformations, the scalar fields z^{α} should also transform in a specific way. The action of the duality

¹In our convention the Levi-Civita tensor satisfies $\varepsilon_{0123} = 1$.

transformations on the scalar fields is much more transparent once we introduce a *symplectic section* which depends on the scalar fields of vector multiplets and transforms linearly under symplectic rotations. Special geometry can be completely defined in terms of this symplectic section.

The symplectic section is given by:

$$v = \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}, \quad \Lambda = 0, \dots, n_V \quad (6.3.9)$$

and is endowed with a symplectic scalar product

$$\langle v | \bar{v} \rangle = -v^T \begin{pmatrix} \mathbf{0}_{n_V} & -\mathbb{1}_{n_V} \\ \mathbb{1}_{n_V} & \mathbf{0}_{n_V} \end{pmatrix} \bar{v}. \quad (6.3.10)$$

Here Z^Λ and F_Λ are functions of the coordinates z^α ($\alpha = 1, \dots, n_V$) of the scalar fields of the vector multiplets. Recall that the index Λ runs from 0 to n_V where n_V is the number of vector multiplets whereas $\alpha = 1, \dots, n_V$ because the graviphoton that appears in the graviton multiplet is not related to any scalar fields. This is compensated by the freedom to re-scale the symplectic section: the symplectic section is a projective section.

Symplectic rotations act linearly on the symplectic section as:

$$\begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix} \Rightarrow \mathcal{S} \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}. \quad (6.3.11)$$

We see that the upper part Z^Λ and the lower part F_Λ of the symplectic section transform respectively as the field strengths $F_{\mu\nu}^{\pm\Lambda}$ and their magnetic duals $G_{\mu\nu}^{\pm\Lambda}$. This can be understood from the following remark: Z^Λ and F_Λ are the fermi-fermi components of the superspace generalization of electric and magnetic field strenghts $\hat{F}_{\mu\nu}^\Lambda$ and $\hat{G}_{\Lambda|\mu\nu}$:

$$\begin{aligned} \hat{F}_{\mu\nu}^\Lambda &= F_{\mu\nu}^\Lambda + \bar{Z}^\Lambda \bar{\psi}_\mu^i \psi_\nu^j \varepsilon_{ji} + Z^\Lambda \varepsilon^{ji} \bar{\psi}_{i\mu} \psi_{j\nu}, \\ \hat{G}_{\Lambda|\mu\nu} &= G_{\Lambda|\mu\nu} + \bar{F}_\Lambda \bar{\psi}_\mu^i \psi_\nu^j \varepsilon_{ji} + F_\Lambda \varepsilon^{ji} \bar{\psi}_{i\mu} \psi_{j\nu}. \end{aligned} \quad (6.3.12)$$

A *special manifold* is a Kähler manifold in which the Kähler potential is not a fundamental quantity but is given by the following symplectic invariant expression:

$$\mathcal{K} = -\log(-i\langle v | \bar{v} \rangle) = -\log[-i(Z^\Lambda \bar{F}_\Lambda - F_\Lambda \bar{Z}^\Lambda)]. \quad (6.3.13)$$

In special geometry, the kinetic terms of both scalar and vector fields are computed from the symplectic section as follow

$$g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}(z, \bar{z}) = i \langle \mathcal{D}_\alpha v | \mathcal{D}_{\bar{\beta}} \bar{v} \rangle, \quad \mathcal{N}_\Lambda \equiv \begin{pmatrix} F_\Lambda & \bar{\mathcal{D}}_{\bar{\alpha}} \bar{F}_\Lambda \end{pmatrix} \begin{pmatrix} Z^\Sigma & \bar{\mathcal{D}}_{\bar{\alpha}} \bar{Z}^\Sigma \end{pmatrix}^{-1}. \quad (6.3.14)$$

6.3. Vector multiplets and special geometry

Here the covariant derivatives are defined by

$$\mathcal{D}_\alpha v = \partial_\alpha v + (\partial_\alpha \mathcal{K})v, \quad \mathcal{D}_{\bar{\alpha}} \bar{v} = \partial_{\bar{\alpha}} \bar{v} + (\partial_{\bar{\alpha}} \mathcal{K})\bar{v}. \quad (6.3.15)$$

In contrast to the metric $\mathcal{N}_{\Lambda\Sigma}$ of the vector fields, the metric of the scalar fields is a symplectic invariant quantity. When the theory is gauged, the electric-magnetic duality is explicitly broken by the introduction of electric charges. In particular, the scalar potential generated by the gauging is not symplectic invariant.

Special geometry and prepotentials

A special geometry is said to *admit a prepotential* when the lower component of the symplectic section (the variable F_Λ) can be expressed as derivative of a scalar function $F(Z)$ depending only on the upper part of the symplectic section (Z^Λ):

$$F_\Lambda = \frac{\partial}{\partial Z^\Lambda} F(Z). \quad (6.3.16)$$

$F(Z)$ is restricted to be an homogeneous function of second degree in the Z^Λ fields and is called the *prepotential*. Interestingly, any symplectic section can be rotated to a section admitting a prepotential [140]. In the presence of a prepotential the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ can be written in the form²

$$\mathcal{N}_{\Lambda\Sigma} = \bar{F}_{\Lambda\Sigma} + 2i \frac{\text{Im}(F_{\Lambda\Delta}) \text{Im}(F_{\Sigma\Gamma}) Z^\Delta Z^\Gamma}{\text{Im}(F_{\Delta\Gamma}) Z^\Delta Z^\Gamma}. \quad (6.3.17)$$

For phenomenological reasons the most interesting sections are those related to a prepotential only after a symplectic rotation. For instance, in $\mathcal{N} = 2$ compactifications of string theory it is not possible to describe chiral fermions, and partial supersymmetry breaking to $\mathcal{N} = 1$ is only possible with symplectic sections that do not admit a prepotential [142]. Nevertheless, prepotentials are still interesting since they provide a handy way to classify special manifolds, and they simplify the study of symplectic invariant properties of the action.

Minimal special geometries correspond to quadratic prepotential defined with a metric $\eta_{\Lambda\Sigma}$ of signature $(1, n)$:

$$F(Z) = -i Z^\Lambda \eta_{\Lambda\Sigma} Z^\Sigma, \quad \eta_{\Lambda\Sigma} = \begin{pmatrix} 1 & 0_{1 \times n} \\ 0_{n \times 1} & -\mathbb{1}_n \end{pmatrix}. \quad (6.3.18)$$

The corresponding scalar manifold of the vector multiplets is $\frac{\text{SU}(1, n)}{\text{U}(1) \text{SU}(n)}$. Although unusual, minimal special geometry is not incompatible with string theory. To the best of our knowledge, there is so far only one case in which it

² $F_{\Lambda \dots \Sigma} = \partial_\Lambda \dots \partial_\Sigma F$.

occurs in string theory [143]. This are the $\mathcal{N} = 2$ vacua coming from the $\mathcal{N} = 3$ flux compactification on $T^6/\mathbb{Z}_2$ studied by Frey and Polchinski [144].

Very special Kähler geometries are characterized by cubic prepotentials

$$F(Z) = \text{id}_{\Lambda\Sigma\Gamma} \frac{Z^\Lambda Z^\Sigma Z^\Gamma}{Z^0}, \quad (6.3.19)$$

where $d_{\Lambda\Sigma\Gamma}$ is a real symmetric tensor. Very special Kähler geometries are familiar in string theory where they occur in many different compactifications as for example in toroidal compactifications of the heterotic string with possible Wilson lines [139], in compactifications of type *II* string theories on Calabi-Yau threefolds, in compactification of type *II* string theories on orientifolds like $K_3 \times T^2/\mathbb{Z}_2$ in the presence of *D3* and *D7* branes [145, 146].

6.4 Hypermultiplets and quaternionic-Kähler geometry

In four dimensional spacetime with the usual Minkowski signature, a hypermultiplet is composed of four real scalar fields and two Majorana spinors. As a Majorana spinor can be decomposed into two chiral spinors of opposite chirality, one can describe n_H hypermultiplets in terms of $4n_H$ real scalar fields q^X ($X = 1, \dots, 4n_H$) and $2n_H$ chiral spinors ζ^A ($A = 1, \dots, 2n_H$) of positive chirality and $2n_H$ chiral spinors ζ_A of negative chirality. The spinors (ζ^A, ζ_A) of hypermultiplets are called the *hyperini* and the $4n_H$ real scalar fields q^X are the *hyperscalars*. The kinetic terms of the scalar fields q^X are characterized by a sigma model with target space the manifold $\mathcal{M}_Q$, which is constrained by $\mathcal{N} = 2$ supersymmetry to be a quaternionic manifold [147]³

$$\mathcal{L}_{\text{hyper}} = -\frac{1}{2} g_{XY} \partial_\mu q^X \partial^\mu q^Y. \quad (6.4.1)$$

The metric g_{XY} is computed from the vielbein f_X^{iA} as

$$g_{XY} = f_X^{iA} f_{YiA}, \quad (6.4.2)$$

where $f_{XiA} = (f_X^{iA})^*$. We denote by f_{iA}^X and $f^{XiA} = (f_{iA}^X)^*$ the inverse of the vielbein as a $4n_H \times 4n_H$ matrix and its complex conjugate, thus

$$f_Y^{iA} f_{iA}^X = \delta_Y^X. \quad (6.4.3)$$

It can be shown that the vielbein and its inverse satisfy the following reality conditions:

$$f_{iA}^X = f^{XjB} \mathbb{C}_{BA} \epsilon_{ji}, \quad f^{XiA} = \varepsilon^{ij} \mathbb{C}^{AB} f_{jB}^X. \quad (6.4.4)$$

³ For a review of quaternionic geometry see [148, 139, 149]. In particular, we shall use the conventions of appendix B of [149].

6.4. Hypermultiplets and quaternionic-Kähler geometry

In our conventions, the matrices ϵ^{ij} and $\mathbb{C}^{AB}$ read:

$$\epsilon^{ij} = i\sigma^2, \quad \mathbb{C}^{AB} = \epsilon^{ij} \otimes \mathbb{1}_{st}, \quad i, j = 1, 2, \quad s, t = 1, \dots, n_H, \quad (6.4.5)$$

and $\mathbb{C}_{AB}$ the inverse of $\mathbb{C}^{AB}$. In the previous equation, the indices $A, B = 1, \dots, 2n_H$ has been decomposed into $A \equiv (i, t), B \equiv (j, s)$ where $i, j = 1, 2$ and $t, s = 1, \dots, n_H$.

The reality condition for the vielbein f_X^{iA} can be translated into the following property of the 2×2 matrices $f_X^t \equiv (f_X^t)^i_j$

$$(f_X^t)^* = \sigma^2 f_X^t \sigma^2. \quad (6.4.6)$$

This implies that f^t can be seen as n_H one-forms with quaternion entries⁴ written in the representation where the quaternionic units are $(\mathbb{1}_2, -i\sigma^x)$ where $x = 1, 2, 3$ ⁵. We can then say that f_X^t is a *quaternionic vielbein* as at each point of the scalar manifold $\mathcal{M}_Q$, the $4n_H$ real scalar fields q^X can be organized into n_H quaternions q^t :

$$q^t = f_X^t q^X. \quad (6.4.7)$$

In a quaternionic manifold it is possible to define a triplet J^x ($x = 1, 2, 3$) of complex structures:

$$(J^x)_X^Y = -i f_X^{iA} (\sigma^x)_i^j f_{jA}^Y \quad (6.4.8)$$

which satisfy the multiplication table of quaternionic units

$$J^x J^y = -\delta^{xy} \mathbb{1}_{4n_H} + \epsilon^{xyz} J^z. \quad (6.4.9)$$

Any linear combination of the form $\tilde{J} = a_x J^x$ also defines a complex structure ($J^2 = -\mathbb{1}$) provided that

$$\|\vec{a}\|^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 = 1. \quad (6.4.10)$$

It follows that at each point of the manifold there is a sphere of complex structures which are related to each other by $SU(2)$ rotations. Note that the quaternionic vielbein f_X^{iA} contains an index i associated to the $SU(2)$ R-symmetry. Therefore, from the definition of the complex structures (6.4.8), it is easy to see to check that the $SU(2)$ R-symmetry can be identified with the $SU(2)$ that rotates the complex structures.

The complex structures are covariantly constant with respect to an $SU(2)$ connection $\omega_{Xi}{}^j = i\omega_X^x (\sigma^x)_i^j$

$$\nabla_X \vec{J} \equiv \nabla_X^{\text{LC}} \vec{J} + 2\vec{\omega}_X \times \vec{J} = 0, \quad (6.4.11)$$

⁴The set of quaternions is defined by $\mathbb{H} = \{q_0 \mathbb{1} + q_1 i + q_2 j + q_3 k | q_i \in \mathbb{R}\}$, with the elements of the basis satisfying $i \cdot j = k$, together with all cyclic permutations and $i^2 = j^2 = k^2 = -1$.

⁵Any 2×2 matrix q satisfying the condition $q^* = \sigma^2 q \sigma^2$ can be written as a quaternion $q = q^0 \mathbb{1} - i q^x \sigma^x$ with $q^0, q^x \in \mathbb{R}$ and σ^x are the Pauli matrices.

where ∇^{LC} is the Levi-Civita covariant derivative on the quaternionic manifold. In quaternionic manifolds the $\text{SU}(2)$ curvature

$$\vec{\mathcal{R}}_{XY} \equiv 2\partial_{[X}\vec{\omega}_{Y]} + 2\vec{\omega}_X \times \vec{\omega}_Y \quad (6.4.12)$$

is proportional to the quaternionic structure $\vec{J}_{XY}$:

$$R_{XY} = \frac{1}{4n_H} g_{XY} R, \quad \vec{\mathcal{R}}_{XY} = \frac{1}{2} \nu \vec{J}_{XY}, \quad \nu = \frac{1}{4n_H(n_H + 2)} R, \quad (6.4.13)$$

with $R_{XY} = R_{XY}^Z$. Here the constant ν is proportional to the gravity coupling constant $\nu = -\kappa^2$. Since we work in units in which $\kappa = 1$, that is $\nu = -1$.

6.5 Isometries, gauging and scalar potential

In $\mathcal{N} = 2$ supergravity coupled to vector multiplets and hypermultiplets, the only way to generate a scalar potential is to promote some of the symmetries of the scalar manifold to be local symmetries. This implies a choice of the Killing vectors of the scalar manifolds and a choice of vector fields that will be used as gauge fields in the covariant derivatives.

In this thesis we will only consider abelian gauging of the symmetries of $\mathcal{M}_{\mathcal{Q}}$. The gauged symmetry is defined by the transformation with parameters α^Λ :

$$\delta q^X = k_\Lambda^X \alpha^\Lambda, \quad (6.5.1)$$

where k_Λ^X are the Killing vectors that we will gauge with the vector fields A_μ^Λ . To gauge a symmetry all the derivatives of the hyperscalars have to be extended to covariant derivatives. The gauge field is taken from the vector multiplets:

$$\nabla_\mu q^X = \partial_\mu q^X - k_\Lambda^X A_\mu^\Lambda. \quad (6.5.2)$$

In order to preserve supersymmetry the action has to be corrected adding the following scalar potential

$$\mathcal{V} = (g_{\alpha\bar{\beta}} k_\Lambda^\alpha k_\Sigma^{\bar{\beta}} + 2g_{XY} k_\Lambda^X k_\Sigma^Y) e^{\mathcal{K}} \bar{Z}^\Lambda Z^\Sigma + 4(U^{\Lambda\Sigma} - 3e^{\mathcal{K}} \bar{Z}^\Lambda Z^\Sigma) \mathcal{P}_\Lambda^x \mathcal{P}_\Sigma^x. \quad (6.5.3)$$

where $\mathcal{P}_\Lambda^{ij} = \mathcal{P}_\Lambda^x (i\sigma^x)^{ij}$ and $\mathcal{P}_{\Lambda|ij} = \mathcal{P}_\Lambda^x (i\sigma^x)_{ij}$ are the *moment maps* [148, 139, 149] related to the Killing vectors k_Λ^X of the quaternionic-Kähler manifold, and k_Λ^α are the Killing vectors of the special manifold. We also have used the definition

$$U^{\Lambda\Sigma} \equiv e^{\mathcal{K}} g^{\alpha\bar{\beta}} \mathcal{D}_\alpha Z^\Lambda \mathcal{D}_{\bar{\beta}} \bar{Z}^\Sigma. \quad (6.5.4)$$

As the scalar fields of vector multiplets transform in the adjoint representation of the gauge group, the Killing vectors $k_\Lambda^\alpha, k_\Lambda^{\bar{\alpha}}$ of the special manifold vanish for Abelian gauging. In particular, the sector proportional to $g_{\alpha\bar{\beta}} k_\Lambda^\alpha k_\Sigma^{\bar{\beta}}$ of the scalar potential is not present for abelian gauging.

6.5.1 Moment map and Fayet-Iliopoulos terms in $\mathcal{N} = 2$ supergravity

The triplet of moment maps $\mathcal{P}_\Lambda^x$ appearing in the scalar potentials are defined as a solution of the following equation [148, 139, 149]:

$$\nabla_X \vec{\mathcal{P}}_\Lambda \equiv \partial_X \vec{\mathcal{P}}_\Lambda + 2\vec{\omega}_X \times \vec{\mathcal{P}}_\Lambda = \frac{1}{2} k_\Lambda^X \vec{J}_{XY}. \quad (6.5.5)$$

In contrast with $\mathcal{N} = 1$ supergravity, in $\mathcal{N} = 2$ due to the non-trivial SU(2) connection, the triplet of moment maps cannot be shifted by arbitrary constants. Actually the freedom to choose the magnitude of the Fayet-Iliopoulos terms is only present in the absence of hypermultiplets, $n_H = 0$. Indeed, thanks to the identity satisfied by any moment map $\mathcal{P}_\Lambda^x$ [150]:

$$\nabla^X \nabla_X \vec{\mathcal{P}}_\Lambda = 2n_H \vec{\mathcal{P}}_\Lambda, \quad (6.5.6)$$

they are uniquely defined by

$$4n_H \vec{\mathcal{P}}_\Lambda = -\vec{J}_Y{}^Z \nabla_Z k_\Lambda^Y. \quad (6.5.7)$$

The uniqueness of $\vec{\mathcal{P}}_\Lambda$ implies in particular that the following equation

$$\nabla_X \vec{\eta} = 0, \quad (6.5.8)$$

has no nontrivial solutions. Otherwise, $\vec{\mathcal{P}}_\Lambda + \vec{\eta}$ would be another solution of equation (6.5.5). Moreover, if there is a non-trivial solution, the integrability condition $[\nabla_X, \nabla_Y] \vec{\eta} = 0$ implies that the SU(2) curvature vanishes and therefore that the SU(2) curvature is trivial. This is clearly not the case for a quaternionic-Kähler manifold as we can see from (6.4.13).

The moment map can also be described in another way. A Killing vector preserves the connection $\vec{\omega}_X$ and Kähler two forms $\vec{J}$ only modulo an SU(2) rotation. Denoting by $\mathcal{L}_\Lambda$ the Lie derivative with respect to k_Λ , we have that the gauge transformations of the SU(2) connection and the Kähler two form are

$$\begin{aligned} \delta_\Lambda \vec{\omega}_X &= (\mathcal{L}_\Lambda \vec{\omega})_X = -\frac{1}{2} \nabla_X \vec{r}_\Lambda, \\ \delta_\Lambda \vec{J} &= (\mathcal{L}_\Lambda \vec{J})_X = \vec{r}_\Lambda \times \vec{J}, \end{aligned} \quad (6.5.9)$$

Here $\vec{r}_\Lambda$ is known as an SU(2) *compensator*. The SU(2) curvature of a quaternionic manifold is non-trivial and therefore it is impossible to get rid of the compensator $\vec{r}_\Lambda$ by a redefinition of the SU(2) connections. The moment map can be expressed in terms of the triplet of connections $\vec{\omega}$ and the compensator $\vec{r}_\Lambda$ in the following way [148]:

$$\vec{\mathcal{P}}_\Lambda = k_\Lambda^Y \vec{\omega}_Y + \frac{1}{2} \vec{r}_\Lambda, \quad (6.5.10)$$

which is analogous to the relation between the moment map and the compensator in $\mathcal{N} = 1$ supergravity (2.2.16).

6.6 SU(2) R-symmetry transformations

In section 6.4 we introduced the SU(2) R-symmetry, which rotates the triplet of complex structures of the quaternionic manifold. Actually, the $\mathcal{N} = 2$ supergravity action is invariant under the R-symmetry. The SU(2) vector quantities transform under these rotations as follows (see [151]):

$$\delta_l \vec{J} = \vec{l} \times \vec{J}, \quad \delta_l \vec{\mathcal{P}}_\Lambda = \vec{l} \times \vec{\mathcal{P}}_\Lambda, \quad (6.6.1)$$

where $\vec{l}(q)$ is an arbitrary triplet of functions on the hyperscalars. Notice that the the gaugini and the gravitini have an SU(2) index and therefore they are also affected by these reparametrizations. In particular the gravitini transform as

$$\delta_l \psi_{\mu i} = -\frac{i}{2} \vec{l} \cdot \vec{\sigma}_i^j \psi_{\mu j}, \quad (6.6.2)$$

$$\delta_l \psi_\mu^i = \frac{i}{2} \psi_\mu^j \vec{l} \cdot \vec{\sigma}_j^i. \quad (6.6.3)$$

The SU(2) connection $\vec{\omega}_X$ is the gauge field associated to the R-symmetry, and transforms as

$$\delta_l \vec{\omega}_X = -\frac{1}{2} \nabla_X \vec{l} = -\frac{1}{2} \partial_X \vec{l} - \vec{\omega}_X \times \vec{l} \quad (6.6.4)$$

In general a change in the U(1) gauge symmetry associated to a killing vector k_Λ induces an SU(2) rotation. This can be seen comparing the gauge transformations of the Kähler forms and the SU(2) connection (6.5.9) with (6.6.1) and (6.6.4). Indeed, the compensator $\vec{r}_\Lambda$ can be identified with the triplet of functions $\vec{l}$ characterizing the SU(2) rotations induced by the change of U(1) gauge.

As a consequence, in order to have a non zero contribution from the compensator in the moment map (6.5.10) the gauge symmetry must induce an SU(2) rotation, in other words we have to gauge a U(1) subgroup of the R-symmetry. Moreover, whenever the compensator is non-vanishing the gravitini must transform under the gauge symmetry

$$\delta_\Lambda \psi_{\mu i} = -\frac{i}{2} \vec{r}_\Lambda \cdot \vec{\sigma}_i^j \psi_{\mu j}, \quad (6.6.5)$$

$$\delta_\Lambda \psi_\mu^i = \frac{i}{2} \psi_\mu^j \vec{r}_\Lambda \cdot \vec{\sigma}_j^i. \quad (6.6.6)$$

In general the compensator is field dependent but it might contain a constant contribution which would act as a U(1) charge for the gravitini. Such a contribution would be the $\mathcal{N} = 2$ supergravity analog of the FI-term.

Notice the strong similarities between this discussion and the one in section 5.2 for $\mathcal{N} = 1$ supergravity, where showed that the $\mathcal{N} = 1$ compensator also characterizes the gauge transformations of the gravitino.

6.7 $\mathcal{N} = 2$ Supersymmetry transformations

The supersymmetry transformations involve the geometrical objects that we just discussed: the moment map, the Killing vectors and the metric of the scalar manifold. For a bosonic configuration, the $\mathcal{N} = 2$ supersymmetry transformations of the left-handed fermionic fields are:

$$\begin{aligned}\delta\psi_\mu^i &= D_\mu\epsilon^i + \frac{1}{4}\gamma^{\rho\sigma}T_{\rho\sigma}^-\varepsilon^{ij}\gamma_\mu\epsilon_j + \gamma_\mu S^{ij}\epsilon_j, \\ \delta\lambda_i^\alpha &= \not{D}z^\alpha\epsilon_i - \frac{1}{2}e^{\frac{\mathcal{K}}{2}}g^{\alpha\bar{\beta}}\mathcal{D}_{\bar{\beta}}\bar{Z}^\Lambda\text{Im}\mathcal{N}_{\Lambda\Sigma}F_{\mu\nu}^{-\Sigma}\gamma^{\mu\nu}\varepsilon_{ij}\epsilon^j - N_{ij}^\alpha\epsilon^j, \\ \delta\zeta^A &= \frac{1}{2}if_X^{Ai}\not{D}q^X\epsilon_i - \mathcal{N}^{iA}\varepsilon_{ij}\epsilon^j.\end{aligned}\tag{6.7.1}$$

Here the fermionic shifts S^{ij} , $\mathcal{N}^{iA}$ and N_{ij}^α are given by

$$S^{ij} \equiv -e^{\mathcal{K}}\mathcal{P}_\Lambda^{ij}Z^\Lambda, \quad \mathcal{N}^{iA} \equiv -ie^{\mathcal{K}}f_X^{iA}k_\Lambda^X\bar{Z}^\Lambda,\tag{6.7.2}$$

$$N_{ij}^\alpha \equiv \varepsilon_{ij}e^{\mathcal{K}}k_\Lambda^\alpha\bar{Z}^\Lambda - 2e^{\mathcal{K}}\mathcal{P}_{\Lambda|ij}\bar{\mathcal{D}}_{\bar{\beta}}\bar{Z}^\Lambda g^{\alpha\bar{\beta}},\tag{6.7.3}$$

and the covariant derivatives are

$$\begin{aligned}D_\mu\epsilon^i &\equiv (\partial_\mu + \frac{1}{4}\omega_\mu^{mn}\gamma_{mn})\epsilon^i + \frac{1}{2}iA_\mu^{S\mathcal{K}}\epsilon^i + V_{\mu j}^i\epsilon^j, \\ D_\mu z^\alpha &= \partial_\mu z^\alpha - A_\mu^\Lambda k_\Lambda^\alpha, \\ D_\mu q^X &= \partial_\mu q^X - A_\mu^\Lambda k_\Lambda^X.\end{aligned}\tag{6.7.4}$$

The $\text{SU}(2)$ connection $V_{\mu i}^j$ is related to the quaternionic-Kähler $\text{SU}(2)$ connection and gets a contribution from the moment map when isometries of the quaternionic-Kähler manifold have been gauged:

$$V_{\mu i}^j = \partial_\mu q^X\omega_{Xi}^j - A_\mu^\Lambda\mathcal{P}_{\Lambda|i}^j.\tag{6.7.5}$$

$A_\mu^{S\mathcal{K}}$ is the $\text{U}(1)$ connection of the Special Kähler manifold:

$$A_\mu^{S\mathcal{K}} = \frac{i}{2}(\partial_{\bar{\alpha}}K\partial_\mu z^{\bar{\alpha}} - \partial_\alpha K\partial_\mu z^\alpha).\tag{6.7.6}$$

In the case of gauging in the vector multiplet sector, this is modified by a scalar moment map similar to the $\text{SU}(2)$ connection. The dressed graviphoton is given by

$$T_{\mu\nu}^- = e^{\frac{\mathcal{K}}{2}}F_{\mu\nu}^{-\Lambda}\text{Im}\mathcal{N}_{\Lambda\Sigma}Z^\Sigma.\tag{6.7.7}$$

6.8 Consistent reduction of supersymmetry

For phenomenological reasons it is important to have mechanisms to reduce the number of supersymmetries of extended supergravity models down to $\mathcal{N} = 1$. There are many ways to realize a reduction of the number of supersymmetries

of a given theory. One can break supersymmetry explicitly by introducing supersymmetry breaking terms in the Lagrangian. Two other methods that appear naturally in the study of string inspired supergravity theories are *spontaneous supersymmetry breaking* and *consistent reduction of supersymmetry* [106, 107, 135].

The consistent reduction of supersymmetry is closely related to the consistent truncations of $\mathcal{N} = 1$ supergravity theories we discussed in chapter 3. Recall that these truncations of $\mathcal{N} = 1$ theories consist on a reduction of the number of fields of the model while preserving both the $\mathcal{N} = 1$ supersymmetry transformations and the equations of motion. That is, the solutions to the equations of motion derived from the reduced action $\mathcal{S}$ should also solve the equations derived from the original action $\hat{\mathcal{S}}$

$$\frac{\delta \mathcal{S}}{\delta L^X} = 0 \quad \implies \quad \frac{\delta \hat{\mathcal{S}}}{\delta L^X} = 0, \quad (6.8.1)$$

where L^X represent the fields surviving the truncation. In the present section we review consistent reductions of $\mathcal{N} = 2$ supergravity models down to $\mathcal{N} = 1$, which also preserve the equations of motion, but where the reduced action is only invariant under $\mathcal{N} = 1$ supergravity.

The method of spontaneous breaking of supersymmetry is less restrictive than consistent reductions. Indeed, the solutions of the reduced theory are only required to solve approximately the equations of motion of the original action at energies much lower than the supersymmetry breaking scale, $E/M_{SSB} \ll 1$:

$$\frac{\delta \mathcal{S}}{\delta L^X} = 0 \quad \implies \quad \frac{\delta \hat{\mathcal{S}}}{\delta L^X} = 0 + \mathcal{O}(E/M_{SSB}), \quad (6.8.2)$$

In supergravity theories coming from string theory, spontaneous supersymmetry breakings arrive naturally from compactification with fluxes and/or torsion. In such a case, one ends up with gauged supergravity theories (or with a superpotential in $\mathcal{N} = 1$ supergravity) in which some of the gravitini become massive. Consistent truncations, on the other hand do not require any fermionic masses. Although they might appear quite artificial at first look from a purely supergravity point of view, consistent reductions are naturally realized in string theory, for example by models containing orbifolds and/or orientifolds. Moreover some spontaneous supersymmetry breaking are also consistent reduction in the sense that all solutions of the equations of motion of the reduced theory are also solutions of the mother theory.

6.8.1 Consistency conditions.

Consistent truncation of supersymmetry in the context of supergravity is not a trivial task. Actually, $\mathcal{N} = 2$ supergravity theories cannot be seen in general as

6.8. Consistent reduction of supersymmetry

a special case of a $\mathcal{N} = 1$ supergravity. The conditions ensuring a consistent truncation in supergravity have been analyzed carefully in [106, 107, 135] following a procedure quite similar to the one presented in chapter 3, but here we will just summarize their results.

The most obvious incompatibility between $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity is the different number of supersymmetry generators and gravitini, $\mathcal{N}$. Without loss of generality, we shall consider consistent truncations in which the first supersymmetry generator, (ϵ_1, ϵ^1) , is the one that respects the truncation. Then, in the reduced theory we must truncate the second gravitino

$$\epsilon_2 = \epsilon^2 = \psi_{\mu 2} = \psi_\mu^2 = 0, \quad (6.8.3)$$

since $(\psi_{\mu 2}, \psi_\mu^2)$ is the gauge field associated the second generator (ϵ_2, ϵ^2) , and then we can make the identifications

$$\epsilon_L = \epsilon^1, \quad \psi_{\mu L} = \psi_\mu^1. \quad (6.8.4)$$

The $\mathcal{N} = 2$ supersymmetry transformations of the gravitini involve the graviphoton $T_{\mu\nu}$, which is inconsistent with the $\mathcal{N} = 1$ supersymmetry transformations, and therefore it has to be truncated

$$T_{\mu\nu} = 0. \quad (6.8.5)$$

The n_H hypermultiplets of $\mathcal{N} = 2$ supergravity models cannot be seen as $2n_H$ chiral multiplets. Indeed, the kinetic terms of the scalar fields of n_H hypermultiplets define a quaternionic geometry and in general a quaternionic manifold is not even a complex manifold. Therefore the scalar manifold does not qualify to describe the kinetic terms of $\mathcal{N} = 1$ chiral multiplets as it is supposed to be Kähler-Hodge. The consistency of the truncation requires that each hypermultiplet has to be fully truncated or reduce to a unique chiral multiplet, and the surviving fields should parametrize a Kähler-Hodge submanifold $\mathcal{M}_Q^{KH}$ of the original quaternionic-Kähler manifold $\mathcal{M}_Q$. The $U(1)$ connection of $\mathcal{M}_Q^{KH}$ is determined by ω_X^3 , the only one of the components of the quaternionic $SU(2)$ -connection that can survive the truncation:

$$\omega_X^1 = \omega_X^2 = 0. \quad (6.8.6)$$

Here the scalar fields of the $\mathcal{N} = 2$ vector multiplets define a special manifold. A special manifold is a Kähler-Hodge manifold and therefore one would expect that n_V $\mathcal{N} = 2$ vector multiplets can be seen as n_V gauge multiplets of $\mathcal{N} = 1$ supergravity together with n_V chiral multiplets. However, this is not in general the case. Indeed, the coupling matrix $\mathcal{N}_{\Lambda\Sigma}$ appearing in the kinetic terms of $\mathcal{N} = 2$ gauge fields is not restricted to be a holomorphic function whereas this is mandatory in $\mathcal{N} = 1$ supergravity coupled to gauge fields. For instance, in a $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ consistent truncation, a vector multiplet has to be completely

truncated or reduce to a gauge or a chiral multiplet. After the truncation, in the reduced $\mathcal{N} = 1$ supergravity we can identify the following quantities

$$D^a = 2 \operatorname{Im} \mathcal{N}^{-1|ab} (\mathcal{P}_b^0 + \mathcal{P}_b^3), \quad W = 2Z^b (P_b^1 - i\mathcal{P}_b^2), \quad (6.8.7a)$$

$$\lambda_L^a = -\lambda_2^b e^{\frac{\mathcal{K}}{2}} \mathcal{D}_b Z^a, \quad f_{ab} = i\mathcal{N}_{ab}, \quad (6.8.7b)$$

which, in particular, implies the surviving components of the matrix $\mathcal{N}$ should be holomorphic. To write these equations we have decomposed the $\mathcal{N} = 2$ vector indices Λ as $\Lambda \rightarrow (a, \tilde{a})$, with $a = 1, \dots, n_G$ running over the n_G gauge multiplets that survive the truncation to the $\mathcal{N} = 1$ theory, and $\tilde{a} = 0, n_G + 1, \dots, n_V$ over the complementary indices, which label the chiral multiplets of the truncated theory coming from $\mathcal{N} = 2$ vector multiplets.

The scalar manifold of the reduced $\mathcal{N} = 1$ theory is a direct product $\mathcal{M}_{S\mathcal{K}}^{KH} \times \mathcal{M}_Q^{KH}$ where $\mathcal{M}_{S\mathcal{K}}^{KH}$ is the reduced manifold coming from the scalar manifold M_V of vector multiplets:

$$\mathcal{M} = \mathcal{M}_{S\mathcal{K}} \times \mathcal{M}_Q \implies \mathcal{M}_{KH} = \mathcal{M}_{S\mathcal{K}}^{KH} \times \mathcal{M}_Q^{KH}. \quad (6.8.8)$$

6.9 $\mathcal{N} = 1$ FI terms from $\mathcal{N} = 2$ supergravity.

In the previous chapters we showed that in $\mathcal{N} = 1$ supergravity the FI-terms, which appear as a constant contribution to the moment map, are related to the charge of the gravitino under $U(1)$ gauge transformations. Actually the gravitino $U(1)$ charge is proportional to the value of the moment map at the fixed point of the killing vector. Moreover, given the gauge couplings of the scalar fields, i.e. the killing vector, the $\mathcal{N} = 1$ moment map is only determined up to an arbitrary real constant and thus the gravitino $U(1)$ charge is still a free parameter of the theory.

However, this is not the case for $\mathcal{N} = 2$ supergravity theories with hypermultiplets. As we discussed in section 6.5.1 in that case the triplet of moment maps $\vec{\mathcal{P}}$ is completely determined, and thus we are not allowed to shift it with an arbitrary constant vector $\vec{\eta}$. This is due to the nontrivial $SU(2)$ curvature of the quaternionic-Kähler manifold. Nevertheless it still makes sense to ask whether if the moment map $\vec{\mathcal{P}}$, contains a constant contribution which could act as an FI-term. One way to identify such a constant contribution is using a consistent reduction of supersymmetry from $\mathcal{N} = 2$ down to a $\mathcal{N} = 1$ theory, where we can identify the magnitude of the FI-term using the arguments presented in chapter 5. In particular in this section we will show how to construct $\mathcal{N} = 2$ theories which can be truncated down to an $\mathcal{N} = 1$ theory containing a non-vanishing FI-term.

Consider a killing vector k_1 associated to a compact isometry of a simply connected quaternionic manifold $\mathcal{M}_Q$. Suppose that the killing vector induces a

6.9. $\mathcal{N} = 1$ FI terms from $\mathcal{N} = 2$ supergravity.

rotation of the complex structures such that the corresponding compensator is constant

$$\delta_{k_1} \vec{J} = \vec{r}_{k_1} \times \vec{J}, \quad r_{k_1}^x = \delta_3^x. \quad (6.9.1)$$

Since in simply connected manifolds any compact isometry always admits fixed points, it is possible to find a field configuration where $k_1 = 0$, and thus the moment map reads

$$\vec{\mathcal{P}}_{k_1}|_{k_1=0} = [k_1^X \vec{\omega}_X + \tfrac{1}{2} \vec{r}_{k_1}]_{k_1=0} = \tfrac{1}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (6.9.2)$$

Then, we can consider a reduced theory defined by the condition $k_1 = 0$, which in components is given by

$$k_1^X (q^Y) = 0, \quad X = 1, \dots, 4n_H. \quad (6.9.3)$$

Since the truncation implies that the moment map $\vec{\mathcal{P}}_{k_1}$ is a constant, the reduced theory can not be invariant under $\mathcal{N} = 2$ supersymmetry. Indeed, in order to obtain a non-trivial compensator we need to gauge the $SU(2)$ R-symmetry group. Since the R-symmetry acts non-trivially in all the hypermultiplets, the condition $k_1 = 0$ leads to constraints that affect all of them. In particular, within each hypermultiplet at least half of the fields have to be truncated, and the reduced scalar manifold is a Kähler-Hodge submanifold of $\mathcal{M}_{\mathcal{Q}}$ [90]. Therefore, the reduced theory defined by $k_1 = 0$ cannot be described by $\mathcal{N} = 2$ supergravity, but it is consistent with a reduction of supersymmetry $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$.

If we gauge the isometry associated to the killing vector $k = \eta k_1$, according to equations (6.8.7b), the corresponding reduced $\mathcal{N} = 1$ theory has a zero superpotential and a constant moment map, that is an FI-term, given by the third component of $\vec{\mathcal{P}}_{k_1}$

$$\vec{\mathcal{P}}_{k_1}|_{k_1=0} = \begin{pmatrix} 0 \\ 0 \\ \frac{\eta}{2} \end{pmatrix} \implies D^{k_1} \propto \eta. \quad (6.9.4)$$

The mechanism that we have just discussed was first presented in [73] in a model where $\mathcal{N} = 2$ supergravity is coupled to one vector multiplet and one chiral multiplet. After truncation it yields a $\mathcal{N} = 1$ supergravity theory coupled to a gauge and a chiral multiplet and admitting a D -term potential endowed with a constant FI term. The FI term was used to construct the first example of a half-BPS cosmic string solutions of $\mathcal{N} = 2$ supergravity. As we shall see in the examples below, it is possible to consider more general scenarios, where a theory with a field dependent compensator also leads to a constant FI-term in the reduced theory.

Using the property of consistent truncations, the construction of [73] can be described as the embedding of a $\mathcal{N} = 1$ half-BPS D -term cosmic string solution in a $\mathcal{N} = 2$ supergravity theory. Indeed, any solution of the reduced $\mathcal{N} = 1$ theory is also a solution of the mother $\mathcal{N} = 2$ theory. One can alternatively consider the use of consistent truncation in [73] as a trick to simplify the $\mathcal{N} = 2$ BPS equations so that they look similar to those of a $\mathcal{N} = 1$ supergravity model.

6.9.1 Example: the quaternionic space $\frac{\text{SO}(4,1)}{\text{SO}(4)}$.

We will now present two explicit realizations of the mechanism we just described. In order to study how FI-terms arise in the reduced $\mathcal{N} = 1$ theory we only need to consider the quaternionic manifold and its isometries. We will consider a model with a single hypermultiplet with the kinetic terms characterized by the quaternionic manifold

$$\mathcal{M}_Q = \frac{\text{SO}(4,1)}{\text{SO}(4)} \cong \frac{\text{Sp}(1,1)}{\text{Sp}(1)\text{Sp}(1)}. \quad (6.9.5)$$

We define the scalars in the hypermultiplets $\{h, b_1, b_2, b_3\}$ so that the corresponding quaternionic vielbein and $\text{SU}(2)$ connection are given by

$$f^{iA} = \frac{1}{\sqrt{2}}(dh\mathbb{1}_2 + ie^{-h}\sigma^x db_x), \quad \omega^x = -\frac{1}{2}e^{-h}db_x. \quad (6.9.6)$$

Therefore, the kinetic terms of the hyperscalars read:

$$\mathcal{T} = -\partial_\mu h \partial^\mu h - e^{-2h} \partial^\mu \vec{b} \partial_\mu \vec{b}, \quad \vec{b} = (b_1, b_2, b_3). \quad (6.9.7)$$

We will study the effect of gauging the isometries of the quaternionic manifold characterized by the following killing vectors:

$$\begin{aligned} k_1 &= 2b_1 \frac{\partial}{\partial b_2} - 2b_2 \frac{\partial}{\partial b_1}, \\ k_2 &= 4b_3 \frac{\partial}{\partial h} + 4b_1 b_3 \frac{\partial}{\partial b_1} + 4b_2 b_3 \frac{\partial}{\partial b_2} + 2[b_3^2 - b_1^2 - b_2^2 + 1 - e^{2h}] \frac{\partial}{\partial b_3} \end{aligned} \quad (6.9.8)$$

These two killing vectors induce a rotation of the complex structures, and therefore the corresponding compensator is non-trivial (6.5.9):

$$\vec{r}_{k_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{r}_{k_2} = 2 \begin{pmatrix} -b_2 \\ b_1 \\ -e^h \end{pmatrix}. \quad (6.9.9)$$

The moment maps can be computed using the equations (6.5.10),

$$\vec{\mathcal{P}}_{k_1} = \begin{pmatrix} e^{-h}b_2 \\ -e^{-h}b_1 \\ 1 \end{pmatrix}, \quad \vec{\mathcal{P}}_{k_2} = \begin{pmatrix} -2b_2 - 2b_1 b_3 e^{-h} \\ 2b_1 - 2b_2 b_3 e^{-h} \\ -e^{-h}[b_3^2 - b_1^2 - b_2^2 + 1] - e^h \end{pmatrix}. \quad (6.9.10)$$

6.9. $\mathcal{N} = 1$ FI terms from $\mathcal{N} = 2$ supergravity.

Field independent compensator.

From (6.9.9), we can see that the gauging of the killing vector $\eta_1 k_1$ leads to the situation described in the previous paragraph. We can define a reduction of supersymmetry consistent with the condition $k_1 = 0$, which implies $b_1 = b_2 = 0$. The kinetic terms of the reduced theory read:

$$\mathcal{T} = -\partial_\mu h \partial^\mu h - e^{-2h} \partial^\mu b_3 \partial_\mu b_3, \quad (6.9.11)$$

and defining $S = e^h + i b_3$ they can be derived from the Kähler potential

$$K(S, \bar{S}) = -\log(S + \bar{S}). \quad (6.9.12)$$

Actually, the scalar manifold of the reduced $\mathcal{N} = 1$ supergravity theory is $\frac{\text{SU}(1,1)}{\text{U}(1)}$, which describes the axio-dilaton system that we studied in previous chapter. According to the identifications (6.8.7b) the reduced $\mathcal{N} = 1$ theory has a vanishing superpotential, and the compensator $\vec{r}_{k_1}$ leads to a constant contribution in the moment map of the reduced theory, a FI-term:

$$\mathcal{P}|_{\mathcal{N}=1} = 2\mathcal{P}_{k_1}^3|_{k_1=0} = 2\eta_1. \quad (6.9.13)$$

Note that, as the killing vector k_1 vanishes identically after the truncation, there is no gauged symmetry in the reduced action. However, if we gauge a symmetry $k = \eta_1 k_1 + k_3$, where k_3 acts non-trivially on h and b_3 , we can end in a $\mathcal{N} = 1$ supergravity action with an abelian gauge symmetry and a constant FI-term.

Field dependent compensator.

Consider now the gauging of $\eta_2 k_2$. In the reduced theory defined by $b_1 = b_2 = 0$, k_2 still acts non trivially on h and b_3 :

$$k_2|_{\mathcal{N}=1} = 4\eta_2 b_3 \frac{\partial}{\partial h} + 2\eta_2 [b_3^2 + 1 - e^{2h}] \frac{\partial}{\partial b_3}, \quad (6.9.14)$$

which can be written in terms of S as

$$k_2|_{\mathcal{N}=1} = i2\eta_2(1 - S^2) \frac{\partial}{\partial S}. \quad (6.9.15)$$

Thus the killing vector k_2 represents the same isometry we used in sections 5.4 and 5.4.2 to obtain constant FI-terms from field dependent moment maps (5.4.6) after the consistent truncation of the axio-dilaton. Actually, after the truncation $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ the moment map of this killing vector reduces to

$$\mathcal{P}|_{\mathcal{N}=1} = 2\mathcal{P}_{k_2}^3|_{b_1=b_2=0} = -2\eta_2 [e^{-h}(b_3^2 + 1) + e^h] = -4\eta_2 \frac{1 + |S|^2}{(S + \bar{S})}. \quad (6.9.16)$$

As we discussed in the previous chapter, when the field S is truncated at $S = 1$, the fixed point of $k_2|_{\mathcal{N}=1}$, this moment map also becomes an effective FI-term:

$$\mathcal{P}(S = 0)|_{\mathcal{N}=1} = -4\eta_2. \quad (6.9.17)$$

Note that, although the compensator associated to k_2 was not a constant in the original $\mathcal{N} = 2$ theory, it has led to a constant FI-term in the reduced theory. Thus, the mechanisms to generate FI-terms from consistent reductions of $\mathcal{N} = 2$ theories are more general than the one discussed in [73].

Supersymmetric cosmic strings in $\mathcal{N} = 2$ supergravity.

7.1 Introduction.

The present chapter is dedicated to the embedding of local supersymmetric cosmic string solutions in $\mathcal{N} = 2$ supergravity. The first known example of such string solutions was constructed in [73]. The authors of this work considered a model with the minimal matter content needed to obtain a half-BPS cosmic string solution in $\mathcal{N} = 2$ supergravity action: one hypermultiplet and one vector multiplet.

As in $\mathcal{N} = 1$ supergravity, (see section 2.5), the supersymmetric cosmic strings are solutions of the BPS equations, which are obtained imposing the field configuration to preserve half of the supersymmetries of the system. In [73], only field configurations compatible with a consistent truncation to $\mathcal{N} = 1$ were considered, i.e. the cosmic strings are also valid solutions for a $\mathcal{N} = 1$ supergravity model. Although these string configurations involve only the fields surviving the truncation, supersymmetry is not spontaneously broken to $\mathcal{N} = 1$ supergravity, and the full $\mathcal{N} = 2$ supersymmetry is preserved in the vacuum far away from the string. From a $\mathcal{N} = 2$ point of view, the use of an

ansatz related to a consistent truncation to $\mathcal{N} = 1$ supergravity in [73] is not required but is useful to simplify the calculation, as the $\mathcal{N} = 2$ BPS equations are in general much more difficult to solve than those of $\mathcal{N} = 1$ theories. The cosmic string solution found in [73] can be seen as an embedding of a $\mathcal{N} = 1$ D-term string in $\mathcal{N} = 2$ supergravity. This is consistent with the discussion in [61] (reviewed in chapter 2), where it is argued that the only $\mathcal{N} = 1$ models admitting supersymmetric vortex solutions involve a D -term potential endowed with a constant Fayet-Iliopoulos term.

The main purpose of the work presented in this chapter is to enlarge the type of $\mathcal{N} = 2$ supergravity theories that can generate constant FI terms in $\mathcal{N} = 1$ supergravity. The special geometry used in [73] is a very particular case, it corresponds to the so-called *minimal special geometry*, which is based on a quadratic prepotential. Here we realize the construction of [73] with a special geometry based on a *cubic prepotential*.

For simplicity we will study a model with abelian gauging of isometries, and thus we will not be able to avoid the presence of hypermultiplets. Indeed, the hypermultiplet is required in order to provide the scalar acting as a Higgs field since, for Abelian gauging, supersymmetry forbids the scalars of vector multiplets to be charged under gauge transformations. As we discussed in the previous chapter, in the presence of hypermultiplets it is not possible to have constant FI-term in a $\mathcal{N} = 2$ supergravity model. Therefore, the FI-terms of the corresponding reduced $\mathcal{N} = 1$ theories are generated from field dependent moment maps in the mother $\mathcal{N} = 2$ theory, using mechanisms described in the previous chapter.

The field content of the model that we are going to study consists of one hypermultiplet and two vector multiplets, and the couplings are characterized by the following choice of quaternionic $\mathcal{M}_{\mathcal{Q}}$ and special Kähler manifolds $\mathcal{M}_{SK}$:

$$\mathcal{M}_{\mathcal{Q}} = \frac{\mathrm{SO}(4,1)}{\mathrm{SO}(4)} \quad \mathcal{M}_{SK} = ST[2,n] = \frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2,n)}{\mathrm{SO}(2) \times \mathrm{SO}(n)}. \quad (7.1.1)$$

In order to define the gauge couplings we will use the so called *Calabi-Vesentini* symplectic section [139, 152], well-known from different compactifications of string theory [145, 146]. An interesting feature of this section is that it can be used to construct models that exhibit partial breaking of supersymmetry $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ [153]. In this model we gauge a $\mathrm{U}(1)$ subgroup of the R -symmetry that rotates the complex structures of the quaternionic manifold. The corresponding compensator reduces to a constant after the truncation, which acts as an effective FI -term (see section 6.9). The truncated theory of this model contains an axiodilaton field $S = a - ie^\rho$, which appears in the gauge kinetic function that defines the kinetic term of the vector field is $f = iS$. We shall see that the $\mathcal{N} = 2$ BPS equations imply that, in the background of the string, the axiodilaton must be an arbitrary constant. Once the winding number

7.2. The model.

is fixed, the value of S in the string configuration parametrizes a one dimensional family of cosmic strings solutions degenerate in energy but with different radii.

7.2 The model.

7.2.1 Couplings in the hypermultiplets.

In the present model the couplings of the hypermultiplets are described by the quaternionic geometry we studied in the previous chapter, the quaternionic manifold of quaternionic dimension one $\frac{\text{SO}(4,1)}{\text{SO}(4)}$. This has a very simple quaternionic structure which can be derived from the vielbein and $\text{SU}(2)$ -connection are respectively

$$f = \frac{1}{\sqrt{2}} (dh\mathbb{1} + ie^{-h}db^x\sigma^x), \quad \omega^x = -\frac{1}{2}e^{-h}db^x. \quad (7.2.1)$$

where $x = 1, 2, 3$ and h, b^x are real fields. The kinetic terms of the hyperscalars read

$$\mathcal{T} = -\partial_\mu h \partial^\mu h - e^{-2h} (\partial_\mu b_1 \partial^\mu b_1 + \partial_\mu b_2 \partial^\mu b_2 + \partial_\mu b_3 \partial^\mu b_3). \quad (7.2.2)$$

We will consider the same Abelian gauging as in [73]. The $\text{U}(1)$ symmetry that we gauge is $k = k_2 + (\eta + 2)k_1$, where k_1 and k_2 where defined in (6.9.8):

$$k = \begin{pmatrix} 4b_3 \\ 4b_1b_3 \\ 4b_2b_3 \\ 2[b_3^2 - e^{2h} + 1 - b_1^2 - b_2^2] \end{pmatrix} + (\eta + 2) \begin{pmatrix} 0 \\ -2b_2 \\ 2b_1 \\ 0 \end{pmatrix}. \quad (7.2.3)$$

Here we have arranged the tangent vector in the order $(\frac{\partial}{\partial h}, \frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3})$. The parameter in front of the second term of the left hand side, $(\eta + 2)$, has been written in this way for later convenience. Although the previous Killing vector seems complicated at first sight it is defined in a precise and simple way on any symmetric normal quaternionic manifold using a solvable parametrization of the quaternionic manifold. The moment map corresponding to k reads

$$\vec{\mathcal{P}} = \begin{pmatrix} -2b_2 - 2b_1b_3e^{-h} \\ 2b_1 - 2b_2b_3e^{-h} \\ -e^{-h} [(b_3)^2 + 1 - (b_1)^2 - (b_2)^2] - e^h \end{pmatrix} + (\eta + 2) \begin{pmatrix} e^{-h}b_2 \\ -e^{-h}b_1 \\ 1 \end{pmatrix}. \quad (7.2.4)$$

The killing vector k has a unique fixed point, $k = 0$, at the origin of the quaternionic manifold :

$$k = 0 \implies b_3 = b_2 = b_1 = h = 0. \quad (7.2.5)$$

It is easy to check that the only non-vanishing component the moment map at the fixed point of the killing vector is $\mathcal{P}^3$:

$$\mathcal{P}^1|_{k=0} = \mathcal{P}^2|_{k=0} = 0, \quad \mathcal{P}^3|_{k=0} = \eta. \quad (7.2.6)$$

We will see that η is proportional to the FI-term of the reduced $\mathcal{N} = 1$ theory, and thus it will also be proportional to the tension of the cosmic string solution.

7.2.2 Couplings in the vector multiplets.

In order to characterize the couplings of the vector multiplets we consider the Kähler-Hodge manifold

$$\mathcal{M}_{SK} = ST[2, 2 + n] = \frac{SU(1, 1)}{U(1)} \times \frac{SO(2, 2 + n)}{SO(2) \times SO(2 + n)}. \quad (7.2.7)$$

We will work in the Calabi-Visentini basis defined by the holomorphic section:

$$v = \begin{pmatrix} Z^\Lambda \\ F_\Lambda \end{pmatrix}, \quad \text{with} \quad Z^\Lambda = \begin{pmatrix} \frac{1}{2}(1 + y^2) \\ i\frac{1}{2}(1 - y^2) \\ y^a \end{pmatrix}, \quad \text{and} \quad F_\Lambda = S\pi_{\Lambda\Sigma}Z^\Sigma, \quad (7.2.8)$$

where $a = 1, \dots, n$ and $y^2 = y^a y^a$ and

$$\pi_{\Lambda\Sigma} = \begin{pmatrix} \mathbb{1}_2 & \\ & -\mathbb{1}_n \end{pmatrix}. \quad (7.2.9)$$

The fields $S = a - ie\rho$ and y^a parametrize the manifolds $\frac{SU(1,1)}{U(1)}$ and $\frac{SO(2,n)}{SO(2) \times SO(n)}$ respectively. The Calabi-Visentini basis does not admit a prepotential, but can be rotated to a symplectic section which can be obtained from the cubic prepotential

$$F(S, y) = -\frac{1}{2}S y^a y^a. \quad (7.2.10)$$

The kinetic terms of the bosons from the vector multiplet can be calculated using formulae (6.3.14) and (6.3.15) given in the previous chapter. The Kähler potential for the Calabi-Visentini section is given by

$$\mathcal{K} = -\log [i(S - \bar{S})] - \log \left[\frac{1}{2} (1 - 2\bar{y}^a y^a + |y^a y^a|^2) \right], \quad (7.2.11)$$

and the coupling matrix of the vector field takes the form

$$\mathcal{N}_{\Lambda\Sigma} = (S - \bar{S}) \frac{Z_\Lambda \bar{Z}_\Sigma + \bar{Z}_\Lambda Z_\Sigma}{\bar{Z}^T \pi Z} + \bar{S} \pi_{\Lambda\Sigma}. \quad (7.2.12)$$

The metric is given as usual by the second derivative of the Kähler potential

$$g_{S\bar{S}} = \frac{1}{(2\text{Im } S)^2}, \quad g_{b\bar{c}} = 2 \frac{(\delta^{b\bar{c}} - 2y^b \bar{y}^{\bar{c}})}{1 - 2\bar{y}^a y^a + |y^a y^a|^2} + 4 \frac{[\bar{y}^b - y^b (\bar{y}^{\bar{a}} \bar{y}^{\bar{a}})] [y^{\bar{c}} - \bar{y}^{\bar{c}} (y^a y^a)]}{(1 - 2\bar{y}^a y^a - |y^a y^a|^2)^2} \quad (7.2.13)$$

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In order to calculate the scalar potential (6.5.3) we will need the following quantity, which in the Calabi Visentini symplectic section is given by:

$$U^{\Lambda\Sigma} - 3e^{\mathcal{K}} \bar{Z}^{\Lambda} Z^{\Sigma} = -\frac{1}{i(S - \bar{S})} \pi^{\Lambda\Sigma}, \quad (7.2.14)$$

for any n . Since $\text{Im } S < 0$, it follows that the scalar potential is always positive and bounded from below in the Calabi-Visentini basis, provided that we gauge the vector A_{μ}^{Λ} with $\pi_{\Lambda\Delta}$ negative. This corresponds to a gauge field associated with any coordinate y^a .

In the example we shall consider in this paper, we will restrict ourselves to the case $n = 1$, as it requires the minimum amount of fields: two complex scalar fields S and y and three vector fields $A_{\mu}^0, A_{\mu}^1, A_{\mu}^2$. This specific case is immediately generalized to any n . Then for $n = 1$ the metric for the scalar manifold $\mathcal{M}_{S\mathcal{K}}$ and the coupling matrix $\mathcal{N}$ simplify to:

$$g_{S\bar{S}} = \frac{1}{(2\text{Im } S)^2}, \quad g_{y\bar{y}} = \frac{2}{(1 - y\bar{y})^2}, \quad \mathcal{N} = \begin{pmatrix} S\mathbb{1}_2 & \\ & -\bar{S} \end{pmatrix}. \quad (7.2.15)$$

7.2.3 The $\mathcal{N} = 2$ supergravity lagrangian

With our choice of special geometry, on the submanifold $y = 0$ the graviphoton (6.7.7) depends only on A_{μ}^0 and A_{μ}^1 . As we would like to put the graviphoton to zero on the string configuration, so that we can perform the reduction to $\mathcal{N} = 1$, we shall gauge the killing vector k with the gauge field A_{μ}^2 . Then the scalar potential (6.5.3) is given by

$$\mathcal{V} = 4e^{-\rho} k^2 \frac{y\bar{y}}{(1 - y\bar{y})^2} + 2e^{-\rho} \mathcal{P}^x \mathcal{P}^x, \quad (7.2.16)$$

where $k^2 = g_{XY} k^X k^Y$. Which is non-negative as we anticipated in the previous section.

The bosonic sector of the $\mathcal{N} = 2$ supergravity action is:

$$\begin{aligned} e^{-1} \mathcal{L}_{bos} = & -\frac{1}{2} R - \frac{1}{2} g_{XY} D_{\mu} q^X D^{\mu} q^Y - \frac{1}{(2\text{Im } S)^2} \partial_{\mu} S \partial^{\mu} \bar{S} - \frac{2}{(1 - y\bar{y})^2} \partial_{\mu} y \partial^{\mu} \bar{y} \\ & + \frac{1}{4} (\text{Im } \mathcal{N})_{\Lambda\Sigma} F^{\Lambda|\mu\nu} F_{\mu\nu}^{\Sigma} + \frac{e^{-1}}{8} (\text{Re } \mathcal{N})_{\Lambda\Sigma} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^{\Lambda} F_{\rho\sigma}^{\Sigma} \\ & + \frac{2g^2}{\text{Im } S} \left[2k^2 \frac{y\bar{y}}{(1 - y\bar{y})^2} + \mathcal{P}^x \mathcal{P}^x \right]^2, \end{aligned} \quad (7.2.17)$$

where:

$$(\text{Im } \mathcal{N})_{\Lambda\Sigma} = \text{Im } S \mathbb{1}_3 \quad (\text{Re } \mathcal{N})_{\Lambda\Sigma} = \begin{pmatrix} \text{Re } S \mathbb{1}_2 & \\ & -\text{Re } S \end{pmatrix}, \quad (7.2.18)$$

and the metric g_{XY} is given by (7.2.2), and the hyperscalars are organized as $q^X = (h, \vec{b})$. The covariant derivatives are defined by (6.5.2) and the killing vector (7.2.3) is gauged by A_μ^2 . The square of the moment map, $\mathcal{P}^x \mathcal{P}^x$ is given in (7.3.7).

7.3 Analysis of the scalar potential

The scalar potential derived in the previous section has the following properties:

1. *The scalar potential $\mathcal{V}$ is bounded from below*

$$\mathcal{V} \geq 0, \quad (7.3.1)$$

this is in sharp contrast to the case of the minimal special geometry, used in [73], where the scalar potential was not bounded from below and could be positive, negative or vanish depending on the value of the scalar fields.

2. *$y = 0$ is a critical point of the scalar potential $\mathcal{V}$:*

$$\frac{\partial \mathcal{V}}{\partial y} \Big|_{y=0} = 0. \quad (7.3.2)$$

3. *The scalar potential $\mathcal{V}$ has a runaway behaviour in the dilaton field ρ :*

$$\mathcal{V} \propto e^{-\rho}. \quad (7.3.3)$$

7.3.1 Minkowski vacua

Since the scalar potential is a sum of squares it is easy to compute all its Minkowski vacua by looking at the zeroes of the different terms:

$$\mathcal{V} = 0 \implies (k = 0 \text{ or } y = 0) \quad \text{and} \quad \mathcal{P}^x \mathcal{P}^x = 0. \quad (7.3.4)$$

To study the scalar potential, it is useful to introduce the following definitions:

$$\Phi = -b_3 + i e^h, \quad \tilde{\Phi} = b_1 + i b_2. \quad (7.3.5)$$

Using these coordinates the fixed point of the killing vector k is given by

$$\Phi = i \quad \text{and} \quad \tilde{\Phi} = 0. \quad (7.3.6)$$

On the other hand, the square of the moment map reads

$$\mathcal{P}^x \mathcal{P}^x = \frac{4}{(\text{Im } \Phi)^2} \left| \Phi - \frac{i}{2}(\eta + 2) \right|^2 |\tilde{\Phi}|^2 + \left(\frac{|\tilde{\Phi}|^2}{\text{Im } \Phi} - \frac{|\Phi - i|^2}{\text{Im } \Phi} + \eta \right)^2, \quad (7.3.7)$$

7.3. Analysis of the scalar potential

then, we can see that there is a Minkowski vacuum for each value of $\eta > -2$:

$$\mathcal{P}^x \mathcal{P}^x = 0 \implies \begin{cases} \text{Case I:} & \Phi = \frac{i}{2}(\eta + 2), \quad |\tilde{\Phi}|^2 = -\eta(1 + \frac{1}{4}\eta), \quad (-2 < \eta < 0), \\ \text{Case II:} & \frac{|\Phi - i|^2}{\text{Im } \Phi} = \eta, \quad \tilde{\Phi} = 0, \quad (0 \leq \eta). \end{cases} \quad (7.3.8)$$

Minkowski vacua	
No Minkowski vacuum	$\eta \leq -2$
$y = 0, \quad \Phi = \frac{i}{2}(\eta + 2), \quad \tilde{\Phi} ^2 = -\eta(1 + \frac{1}{4}\eta)$	$-2 < \eta < 0$
$\Phi = i, \quad \tilde{\Phi} = 0$	$\eta = 0$
$y = 0, \quad \frac{ \Phi - i ^2}{\text{Im } \Phi} = \eta, \quad \tilde{\Phi} = 0$	$\eta > 0$

Table 7.1 – Type of vacua of the scalar potential for different values of the parameter η . Non-singular cosmic string solutions are only possible for $\eta > 0$.

When $\eta \leq -2$ there are no Minkowski vacua. This implies in particular that for a gauging with $\eta \leq -2$, all the extrema of the potential are de Sitter vacua. However, the potential will not have an absolute minimum (for finite values of the fields) because of its runaway behaviour in the dilaton¹ (7.3.3).

We shall use table 7.1 to explain our choice for the cosmic string configuration. In order to have a cosmic string solution we need to have a circle in the vacuum manifold. If we want the string configuration to be compatible with a consistent reduction of supersymmetry, we shall have to truncate some of the scalar fields of the quaternionic manifold to end up with a Kähler-Hodge submanifold which is completely geodesic.

The appropriate choice of gauging to construct a cosmic string of the Nielsen-Olesen type is $\eta > 0$. Indeed, in that case, the vacuum is a circle

¹In order to obtain a stable de Sitter vacuum we would have to resort to the techniques presented in [154], which involve using a rotated version of the Calabi-Visentini basis together with non-abelian gauge couplings.

defined by $\frac{|\Phi - i|^2}{\text{Im } \Phi} = \eta$. The Higgs field of the cosmic string is Φ . We shall keep $y = \tilde{\Phi} = 0$ not only in the vacuum but for all the string solutions in order to have a consistent truncation.

In the case $-2 < \eta < 0$, we also have a circle in the vacuum. However, $\Phi = \frac{1}{2}(\eta + 2)$ does not define a consistent truncation of the quaternionic manifold. To see this note that a gauge transformation 6.5.1 with the killing vector k given by (7.2.3) does not respect this condition for every value of $\tilde{\Phi}$.

In the case where $\eta = 0$, the vacuum is just a point and therefore there is no room for a cosmic string solution of the Nielsen-Olesen type.

7.4 Consistent reduction of supersymmetry

The set of conditions that we impose on the bosonic fields defining the consistent reduction are:

$$\text{Consistent reduction ansatz : } \begin{cases} y &= 0, \\ \tilde{\Phi} &= 0, \\ A_\mu^0 = A_\mu^1 &= 0. \end{cases} \quad (7.4.1)$$

The condition $y = \tilde{\Phi} = 0$ was explained in the previous section. The conditions $A_\mu^0 = A_\mu^1 = 0$ ensure that the graviphoton (see equation (6.7.7)) vanishes as it should be in a consistent truncation to $\mathcal{N} = 1$ supergravity. Indeed, the graviphoton appears in the supersymmetry transformations of the gravitini in $\mathcal{N} = 2$ supergravity (6.7.1) but is absent in those of the gravitino of $\mathcal{N} = 1$ supergravity (2.2.22).

In this field configuration the quaternionic and special Kähler manifold reduce as follow:

$$\mathcal{M}_{SK} \times \mathcal{M}_Q \xrightarrow{y=\tilde{\Phi}=0} \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)_S \times \left(\frac{\text{SU}(1,1)}{\text{U}(1)} \right)_\Phi \simeq \left(\frac{\text{SO}(2,2)}{\text{SO}(2) \times \text{SO}(2)} \right)_{S,\Phi}, \quad (7.4.2)$$

with the corresponding Kähler potential given by

$$K = -\log [i(S - \bar{S})] - 2 \log [-i(\Phi - \bar{\Phi})]. \quad (7.4.3)$$

Here S is an axion-dilaton field and Φ is the scalar field whose Higgs mechanism generates the cosmic string. Once we impose the condition $\tilde{\Phi} = 0$, the Killing vector of the quaternionic manifold that we have gauged only acts on Φ as:

$$\delta\Phi = 2g(\Phi^2 + 1), \quad (7.4.4)$$

7.4. Consistent reduction of supersymmetry

which corresponds to the compact $U(1)$ symmetry of $\left(\frac{SU(1,1)}{U(1)}\right)_\Phi$.

In order to calculate the supersymmetry transformations in the reduced $\mathcal{N} = 1$ theory we will need the following relations:

$$\begin{aligned}
S^{ij} &= T_{\mu\nu}^- = N_{ij}^S = \mathcal{N}^{iA} = 0, \\
e^\mathcal{K} &= \frac{1}{-\text{Im } S}, \\
\mathcal{D}_S Z^\Lambda &= \frac{e^\mathcal{K}}{4} \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \quad \mathcal{D}_{y^a} Z^\Lambda = \delta_a^\Lambda, \\
V_{\mu i}^j &= i(\omega_\mu^3 + g A_\mu^2 \mathcal{P}^3)_i^j, \\
N_{ij}^y &= -e^{\mathcal{K}/2} \mathcal{P}_{ij}^3,
\end{aligned} \tag{7.4.5}$$

7.4.1 Truncated $\mathcal{N} = 1$ Lagrangian and supersymmetry transformations.

After setting to zero the truncated fields (7.4.1) the bosonic sector of the $\mathcal{N} = 1$ reduced action reads:

$$\begin{aligned}
e^{-1} \mathcal{L} &= -\frac{1}{2} R + \frac{\text{Im } S}{4} F^{\mu\nu} F_{\mu\nu} - \frac{e^{-1}}{8} \text{Re } S \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\
&\quad - \frac{1}{(2 \text{Im } S)^2} \partial_\mu S \partial^\mu \bar{S} - \frac{1}{2(\text{Im } \Phi)^2} D_\mu \Phi D^\mu \bar{\Phi} \\
&\quad - 2 \frac{g^2}{\text{Im } S} \left[\frac{|\Phi - i|^2}{\text{Im } \Phi} - \eta \right]^2.
\end{aligned} \tag{7.4.6}$$

For convenience, we denote by $A_\mu \equiv A_\mu^2$ the vector gauging the $U(1)$ isometry of the quaternionic manifold, and the corresponding field strength by $F_{\mu\nu} = F_{\mu\nu}^2$. Using this notation the covariant derivative is given by

$$D_\mu \Phi = \partial_\mu \Phi - 2g A_\mu (\Phi^2 + 1). \tag{7.4.7}$$

The supersymmetry transformations of the fermions in the reduced $\mathcal{N} = 1$ theory can be found from (6.7.1) using (7.4.5). The transformations corresponding to the supersymmetry parameter ϵ^1 are

$$\begin{aligned}
\delta\psi_\mu^1 &= (\partial_\mu + \tfrac{1}{4}\omega_{\mu|mn}\gamma^{mn} + \tfrac{1}{2}iA_\mu^{S\mathcal{K}} + \tfrac{1}{2}iA_\mu^{\mathcal{Q}}) \epsilon^1, \\
\delta\lambda_2^y &= -e^{\frac{\mathcal{K}}{2}} (\text{Im } S) F_{12} \gamma^{12} \epsilon^1 - i e^{\frac{\mathcal{K}}{2}} \mathcal{P}^3 \epsilon^1, \\
\delta\lambda_1^S &= \not{D} S \epsilon_1, \\
\delta\zeta^1 &= \frac{\sqrt{2}}{4 \text{Im } \Phi} \not{D} \Phi \epsilon_1,
\end{aligned} \tag{7.4.8}$$

and for the second supersymmetry parameter ϵ^2 we have

$$\begin{aligned}\psi_\mu^2 &= (\partial_\mu + \tfrac{1}{4}\omega_{\mu|mn}\gamma^{mn} + \tfrac{1}{2}iA_\mu^{SK} - \tfrac{1}{2}iA_\mu^Q)\epsilon^2, \\ \delta\lambda_1^y &= e^{\frac{\kappa}{2}}(\text{Im } S)F_{12}\gamma^{12}\epsilon^2 - ie^{\frac{\kappa}{2}}\mathcal{P}^3\epsilon^2, \\ \delta\lambda_2^S &= \not{D}S\epsilon_2, \\ \delta\zeta^2 &= -\frac{\sqrt{2}}{4\text{Im } \Phi}\not{D}\bar{\Phi}\epsilon_2.\end{aligned}\tag{7.4.9}$$

In these equations A_μ^Q is the quaternionic matter connection of the gravitini:

$$A_\mu^Q = 2\omega_\mu^3 + 2gA_\mu\mathcal{P}^3 = \frac{(\partial_\mu\Phi + \partial_\mu\bar{\Phi})}{\text{Im } \Phi} + 2gA_\mu\mathcal{P}^3,\tag{7.4.10}$$

and A_μ^{SK} is the U(1) connection of the Special Kähler manifold

$$A_\mu^{SK} = -\tfrac{1}{4}\frac{\partial_\mu S + \partial_\mu\bar{S}}{\text{Im } S}.\tag{7.4.11}$$

The main difference with the supersymmetry transformations obtained in [73] on the string configuration is the presence of the axion-dilaton field S coming from the special geometry and parametrizing the manifold $\frac{\text{SU}(1,1)}{\text{U}(1)}$. The gaugini λ_i^S and the U(1) connection A_μ^{SK} of the axion-dilaton scalar manifold do not distinguish between the two supersymmetry transformations :

- In the supersymmetry transformations of the gravitini fields, the U(1) connection A_μ^{SK} appears with the same charge for both transformations whereas the matter connection A_μ^Q (coming from the SU(2) of the hypermultiplet) comes with opposite charge for the supersymmetries.
- The axion-dilaton field S enters in the same way in the supersymmetric transformations of the gaugini λ_i^S in contrast to the way Φ appears in the supersymmetric transformation of the hyperini.

This difference of behaviour will be more clear in the next section where we analyze the different BPS projectors obtained from the BPS equations.

7.5 Half-BPS cosmic string solution.

The coupling of the Higgs field to the gauge field is non standard (6.5.2), and thus it is difficult to see what would be the field configuration that corresponds to a cosmic string. To simplify the analysis we define, as in [73], the following field:

$$u = \frac{i - \Phi}{i + \Phi}.\tag{7.5.1}$$

7.5. Half-BPS cosmic string solution.

This parametrization of the scalar manifold corresponds to the Poincaré disc model the hyperbolic space H^2 . An appropriate Kähler potential to describe the geometry of the scalar manifold in terms of u and S is

$$K = -2 \log[1 - u\bar{u}] - \log[i(S - \bar{S})], \quad \text{with } |u|^2 < 1, \text{ and } \text{Im } S < 0. \quad (7.5.2)$$

This Kähler potential is related to the one given above in terms of Φ (7.4.3) by a Kähler tranformation

$$K^\Phi = K^u + h + \bar{h} \quad \text{where } h(u) = 2 \log[1 + u]. \quad (7.5.3)$$

The advantage of this parametrization is that the gauge transformations and the covariant derivatives of the field u have a simple form:

$$\delta u = 4giu, \quad D_\mu u = \partial_\mu u - 4giu A_\mu. \quad (7.5.4)$$

Thus gauge transformations correspond to a change of phase of u , and the winding of this phase will be the one inducing the magnetic flux of the string. The corresponding moment map and the D-term potential are given by

$$\mathcal{P} = -\frac{D}{\text{Im } S} = -2g\left(\frac{4u\bar{u}}{1 - u\bar{u}} - \eta\right), \quad V_D = -\frac{1}{2} \text{Im}(S) D^2 = -\frac{2g^2}{\text{Im } S} \left[\frac{4u\bar{u}}{1 - u\bar{u}} - \eta \right]^2. \quad (7.5.5)$$

Note that η is proportional to the value of the moment map at the fixed point of the gauged isometry $u = 0$, i.e. the FI-term, $\mathcal{P}|_{u=0} = -2g\eta$. In terms of the field u the bosonic sector of the reduced lagrangian (7.4.6) reads

$$\begin{aligned} e^{-1} \mathcal{L} = & -\frac{1}{2} R - \frac{1}{(2 \text{Im } S)^2} \partial_\mu S \partial^\mu \bar{S} - \frac{2}{(1 - u\bar{u})^2} D_\mu u D^\mu \bar{u} \\ & + \frac{1}{4} \text{Im } S F^{\mu\nu} F_{\mu\nu} - \frac{\epsilon^{-1}}{8} \text{Re } S \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \\ & - \frac{2g^2}{\text{Im } S} \left[\frac{4u\bar{u}}{1 - u\bar{u}} - \eta \right]^2. \end{aligned} \quad (7.5.6)$$

7.5.1 The BPS equations

The BPS equations for the cosmic string configuration are obtained by setting to zero the supersymmetry transformations (7.4.8) and (7.4.9). Note that the $\mathcal{N} = 2$ transformations in the reduced theory look like two sets of $\mathcal{N} = 1$ supersymmetry transformations, one for each parameter ϵ^i . Let us first consider the transformations of the gaugini and the chiralino corresponding to ϵ^1 written in terms of the field u

$$\begin{aligned} \delta \lambda_2^y &= -e^{\frac{\kappa}{2}} (\text{Im } S) F_{12} \gamma^{12} \epsilon^1 - i e^{\frac{\kappa}{2}} \mathcal{P}^3 \epsilon^1, \\ \delta \lambda_1^S &= \not{\partial} S \epsilon_1, \\ \delta \zeta^1 &= -\frac{i}{\sqrt{2}} (1 - u\bar{u})^{-1} \not{D} u \epsilon_1. \end{aligned} \quad (7.5.7)$$

The corresponding BPS equations can be found following the same steps we described in chapter 2 for the $\mathcal{N} = 1$ supersymmetric cosmic string

$$(D_1 \pm iD_2)u = 0, \quad (\partial_1 \pm i\partial_2)S = 0, \quad F_{12} \mp D = 0, \quad (7.5.8)$$

Furthermore, the supersymmetry parameter ϵ^1 must satisfy the projector condition

$$\gamma^{12}\epsilon^1 = \mp i\epsilon^1. \quad (7.5.9)$$

If we impose the cosmic string to preserve half of the supersymmetries of the full $\mathcal{N} = 2$ theory, we also have to set to zero the second set of supersymmetry transformations. The supersymmetry transformations of the gaugini and the chiralino for the supersymmetry parameter ϵ^2 read

$$\begin{aligned} \delta\lambda_1^y &= e^{\frac{\kappa}{2}}(\text{Im } S)F_{12}\gamma^{12}\epsilon^2 - ie^{\frac{\kappa}{2}}\mathcal{P}^3\epsilon^2, \\ \delta\lambda_2^S &= \not{D}S\epsilon_2, \\ \delta\zeta^2 &= \frac{i}{\sqrt{2}}(1 - u\bar{u})^{-1}\not{D}\bar{u}\epsilon_2. \end{aligned} \quad (7.5.10)$$

This extra requirement leads to the following BPS equations

$$(D_1 \pm iD_2)u = 0, \quad (\partial_1 \mp i\partial_2)S = 0, \quad F_{12} \mp D = 0, \quad (7.5.11)$$

and to a projector condition for ϵ^2

$$\gamma^{12}\epsilon^2 = \pm i\epsilon^2. \quad (7.5.12)$$

Thus, we see that in order to solve simultaneously the equations (7.5.8) and (7.5.11), the supersymmetry parameters ϵ^1 and ϵ^2 must have opposite chiralities on the cosmic string world sheet. Moreover, the two equations for the axio-dilaton S require it to be constant everywhere

$$S = \text{Constant}, \quad A_\mu^{S\kappa} = -\frac{1}{2}\frac{\partial_\mu(\text{Re } S)}{\text{Im } S} = 0. \quad (7.5.13)$$

The BPS equations we have obtained are the same as those obtained in [73] modulo the factor of $e^{-\rho}$ in the definition of the D -term.

Gravitini equations

We still have to take into account the supersymmetry transformations of the gravitini, which will lead to the equations that characterize the space-time metric in the background of the cosmic string

$$\begin{aligned} \delta\psi_\mu^1 &= (\partial_\mu \mp \frac{i}{2}\omega_{\mu|12} + \frac{i}{2}A_\mu^{\mathcal{Q}})\epsilon^1 = 0, \\ \delta\psi_\mu^2 &= (\partial_\mu \pm \frac{i}{2}\omega_{\mu|12} - \frac{i}{2}A_\mu^{\mathcal{Q}})\epsilon^2 = 0. \end{aligned} \quad (7.5.14)$$

7.5. Half-BPS cosmic string solution.

Here we have already used the projector conditions (7.5.9) and (7.5.12), and $A_\mu^\mathcal{Q}$ is given by

$$A_\mu^\mathcal{Q} = \frac{i}{1 - u\bar{u}}(u\partial_\mu\bar{u} - \bar{u}\partial_\mu u) - 2gA_\mu \left(\frac{4u\bar{u}}{1 - u\bar{u}} - \eta \right). \quad (7.5.15)$$

As we explained in section (2.5), in order to find the corresponding BPS equations we restrict ourselves to cylindrically symmetric cosmic string configurations. In that case we can take the space-time metric to be of the form

$$ds^2 = -dt^2 + dz^2 + dr^2 + C^2(r)d\theta^2, \quad (7.5.16)$$

where we have used cylindrical coordinates $\{t, z, r, \theta\}$. With the choice of space-time vielbein

$$e^1 = dr \quad \text{and} \quad e^2 = C(r)d\theta, \quad (7.5.17)$$

the only non vanishing component of the spin connection is

$$\omega_\theta^{12} = -C'(r). \quad (7.5.18)$$

Then, using the following ansatz for the supersymmetry parameters

$$\epsilon^i(\theta) = e^{\mp \frac{1}{2}i\theta} \epsilon_0^i, \quad (7.5.19)$$

we obtain that the equation for the profile function $C(r)$ is

$$1 - C'(r) = \pm A_\theta^\mathcal{Q}. \quad (7.5.20)$$

This equation guarantees that the supersymmetry transformations of the two gravitini (7.5.14) are vanishing.

7.5.2 Cosmic string profile functions

We will use the following time independent ansatz to solve the BPS equations

$$u = f(r)e^{in\theta}, \quad A_\theta = \frac{m}{4g}v(r), \quad A_0 = A_r = A_z = 0. \quad (7.5.21)$$

It represents a straight cylindrically symmetric cosmic string of winding m along the z -axis. The BPS equations, (7.5.8) and (7.5.20), for the profile functions of the string, $f(r)$, $v(r)$ and $C(r)$, are:

$$\begin{aligned} f'(r) &= \pm m C^{-1} f (1 - v), \\ v'(r) &= \mp e^{-\rho} \frac{8g^2 C}{m} \left(\frac{4f^2}{1 - f^2} - \eta \right), \\ C'(r) &= 1 \mp m \frac{2f^2}{1 - f^2} (1 - v) \mp \frac{1}{2} m \eta v. \end{aligned} \quad (7.5.22)$$

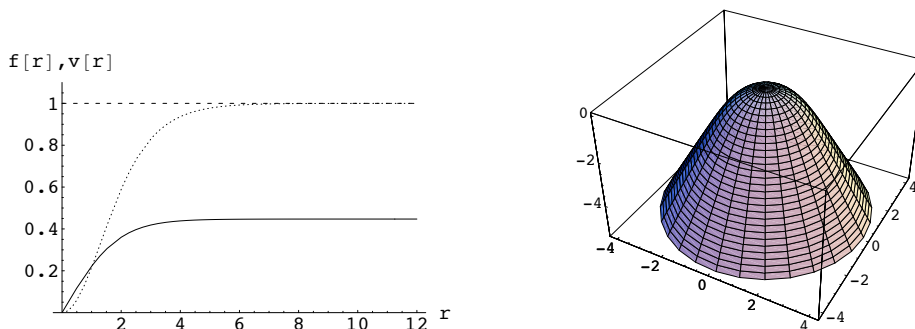


Figure 7.1 – *LEFT: Profile functions (7.5.21) $f(r)$ (solid line) and $v(r)$ (dotted line) which characterize the field configuration the $\mathcal{N} = 2$ supersymmetric cosmic string. We have chosen $m = 4g = \eta = 1$, and the dilaton is set to zero $\rho = 0$. RIGHT: Embedding of the metric on the plane orthogonal to the string in three dimensions. Far away from the string, which is located at the tip of surface, the metric approaches that of a cone with deficit angle $\Delta = \pi|m|\eta$.*

In order to have a regular solution at the origin $r = 0$ we have to impose the boundary conditions $f(0) = v(0) = 0$. We also require that far away from the string the field u is in the vacuum and its kinetic terms vanish, so that the string configuration has a finite action, thus

$$f(r) \rightarrow \sqrt{\frac{\eta}{\eta + 4}} \quad \text{and} \quad v(r) \rightarrow 1 \quad \text{for } r \rightarrow \infty. \quad (7.5.23)$$

A particular solution to these equations is shown in figure 7.1. The plots represents a unit winding cosmic string with coupling constant $g = 1/4$ and $\eta = 1$, and the dilaton has been set to zero $\rho = 0$.

From the BPS equations we can find the asymptotic behavior of the profile functions, which is similar to the cases of [61, 73]. In the case $r \rightarrow 0$ we have:

$$f(r) \sim r^{\pm m}, \quad C \sim r, \quad v(r) \sim \pm \frac{4g^2\eta}{m} r^2. \quad (7.5.24)$$

It can be seen that the equations only admit regular solutions provided we choose the upper sign for positive winding $m > 0$, and the lower sign for negative winding $m < 0$. In the opposite limit, $r \rightarrow \infty$ the metric is given by

$$ds^2 \approx -dt^2 + dz^2 + dr^2 + r^2 \left[1 - \frac{1}{2}|m|\eta\right]^2 d\theta^2. \quad (7.5.25)$$

At $r \rightarrow \infty$, the string creates a locally-flat conical metric with a deficit angle $\Delta = \pi|m|\eta$. The energy of the string per unit length can be computed as we discussed in section 2.5. One finds that the only non-vanishing contribution

7.5. Half-BPS cosmic string solution.

comes from the Gibbons-Hawking surface term:

$$\mu_{\text{string}} = - \int d\theta C' \Big|_{r=\infty} + \int d\theta C' \Big|_{r=0} = \pi |m| \eta > 0. \quad (7.5.26)$$

Note also that asymptotically, $r \rightarrow \infty$, since all the fields are in the vacuum the supersymmetry transformations (7.5.10), (7.5.10) and (7.5.14) become zero, and thus the full $\mathcal{N} = 2$ supersymmetry is restored.

7.5.3 Metastability of the string configuration.

An important issue is the study of the stability of these strings. Although supersymmetry ensures that these configurations are solutions to the equations of motion, a priori, there is nothing to prevent them from decaying into a different configuration. As we already mentioned in chapter 1, the stability of BPS cosmic string solutions has been investigated recently in the context of three and four dimensional $\mathcal{N} = 1$ supergravity in [99] [100] [101]. In these papers it was proven that these solutions are stable against all sorts of perturbations. However the present situation is more subtle since a complete analysis requires the study to be done in the full $\mathcal{N} = 2$ supergravity theory. In particular, it is not clear if the cosmic strings would survive a perturbation of the truncated fields. Moreover, these analyses do not prevent the presence of zero modes which, if they are excited, can also lead to the disappearance of the strings. The cosmic string solution we have presented in this chapter has one of such zero modes, the value of the axion-dilaton field.

The constant value of the axion-dilaton field is not fixed by the BPS equations nor by the scalar potential. The mass per unit length of the string is also independent of the value of the axion-dilaton field (7.5.26). The dilaton fixes the overall length scale of the configuration in the following sense. There are two natural lengths in the solution given by the inverse of the masses of the Higgs and the gauge field, and they are both functions of the dilaton field:

$$m_W^2 \propto -\frac{1}{\text{Im } S}, \quad m_\Phi^2 \propto -\frac{1}{\text{Im } S}, \quad (7.5.27)$$

so that the corresponding length scales are

$$l_W^2 \propto -\text{Im } S, \quad l_\Phi^2 \propto -\text{Im } S. \quad (7.5.28)$$

Suppose we have a solution to the BPS equations given by the profile functions $f(r)$, $W_\theta(r)$, $C(r)$ and ρ . Then it is easy to check that the functions $f(\lambda r)$, $W_\theta(\lambda r)$, $C(\lambda r)/\lambda$ and $\rho - 2 \log(\lambda)$ also satisfy the BPS equations for any real $\lambda > 0$. From here it is obvious that the value of the dilaton determines the length scales in the transverse direction to the string, in particular the core radius.

This situation looks similar to the case of semilocal strings [155], where there is also a one parameter family of solutions with equal energy and different core radii. In that case finite energy perturbations can excite the zero mode connecting solutions within the same family, leading to the spread of the magnetic flux and eventually to the disappearance of the strings. An instability of this type was also found in a class of BPS cosmic strings solutions which appear in a $\mathcal{N} = 1$ supersymmetric model proposed by Blanco-Pillado *et al.* [59] to describe the last stage of a brane-antibrane inflation. We review this analysis, published in [72], in appendix B.

This is not going to occur in our model. In order to go from one solution to a different one, the dilaton has to change its value everywhere in the plane transverse to the string. The kinetic energy needed in order to excite the value of the dilaton globally diverges, and this implies that, once the system has chosen a given value for the dilaton, finite energy perturbations cannot drive the system to a solution with a different value of S . The radius of the string will remain unchanged.

7.6 Discussion

As a generalization of the work done in [73], in this chapter we have enlarged the family of $\mathcal{N} = 2$ supergravity actions which allow the embedding of $\mathcal{N} = 1$ supergravity actions containing a D -term potential and a constant FI term. We have extended the result of [73] to a class of special geometries more familiar in compactifications of string theory. We are using here a “very special Kähler geometry” characterized by a cubic prepotential, instead of the minimal special geometry used in [73]. To be specific we take the special manifold to be:

$$ST[2, n] \equiv \frac{\mathrm{SU}(1, 1)}{\mathrm{U}(1)} \times \frac{\mathrm{SO}(2, n)}{\mathrm{SO}(2) \times \mathrm{SO}(n)},$$

in the Calabi-Visentini basis (7.2.8), which is related to the cubic prepotential by a symplectic rotation [139, 152].

This choice of special geometry has two important consequences. An axion-dilaton field, $S = a - i\mathrm{e}^\rho$, is present in the reduced $\mathcal{N} = 1$ theory after truncation from $\mathcal{N} = 2$. Moreover, it is possible to define a gauging for which the scalar potential is bounded from below. However, it has a runaway dependence on the dilaton:

$$\mathcal{V} \propto \mathrm{e}^{-\rho}.$$

As an application, we have shown how to construct a half-BPS cosmic string solution from a $\mathcal{N} = 2$ supergravity action in $D = 4$. Following [73] we have used a string ansatz compatible with a consistent truncation from $\mathcal{N} = 2$ to $\mathcal{N} = 1$.

7.6. Discussion

In order to obtain the scalar potential we have gauged the same isometry used in [73]. We have found that the BPS equations imply that the axion-dilaton has to be simultaneously holomorphic and anti-holomorphic, which can only be satisfied if it is a constant:

$$S = \text{Constant}, \quad \text{Im} S < 0.$$

Despite the runaway behavior of the potential, we have proved that all the string solutions have the same energy per unit length, regardless of the value of the dilaton, and it is given by the Gibbons-Hawking surface term [96, 73]. The value of the dilaton fixes the masses of the Higgs and the gauge field and, hence, also the radius of the string. We have argued that the system cannot evolve between two solutions with different values of the dilaton, since this would require an infinite amount of energy. Thus, once the strings are formed their radii remain fixed.

Observations of the timing of millisecond pulsars give the constraint $\mu_{\text{string}} \lesssim 2 \times 10^{-7}$ [51]. However this constraint depends on the specific model used to calculate it, what leads to a considerable uncertainty. This implies for our model that the FI term has to satisfy:

$$0 < \pi|m|\eta \lesssim 2 \times 10^{-7},$$

where the lower bound is coming from the study of Minkowski vacua in section 7.3.

Conventions.

In this thesis we use the units $c = \hbar = 1$. Thus, the Plank mass m_{pl} is simply given by

$$m_{pl}^{-2} \equiv \frac{G}{\hbar c} = G, \quad (\text{A.0.1})$$

where G is the Newtons constant. In general we will work in units of the reduced Plank mass M_p ,

$$M_p^{-2} = 8\pi G = 1, \quad (\text{A.0.2})$$

except in applications to cosmology where we might choose to keep M_p or G explicitly for clarity.

The use of indices is summarized in table A.1.

A.1 Space-time conventions

We choose the Minkowski space-time metric η_{mn} to have mostly positive Lorentz signature

$$\eta_{mn} = \text{Diag}(-1, 1, 1, 1). \quad (\text{A.1.1})$$

The Levi-Civita tensor is defined to be totally antisymmetric with $\varepsilon_{0123} = +1$.

Space-time indices		
μ	$0, \dots, 3$	space-time coordinates, with x^0 for the time
m	$0, \dots, 3$	local frame
Supersymmetric truncations in $\mathcal{N} = 1$ supergravity (Chapt. 2-5)		
I	$1, \dots, n_C$	chiral multiplets
α	$1, \dots, n_h$	truncated chiral multiplets
i	$1, \dots, n_l$	surviving chiral multiplets
a	$1, \dots, n_V$	vector multiplets
$\tilde{a}$	$1, \dots, \tilde{n}_V$	truncated vector multiplets
$\mathcal{N} = 2$ supergravity (Chapt. 6-7)		
s	$1, \dots, n_H$	hypermultiplets
X	$1, \dots, 4n_H$	scalar fields in hypermultiplets
A	$1, \dots, 2n_H$	spinors in hypermultiplets
Λ	$0, \dots, n_V$	vector multiplets
α	$1, \dots, n_V$	scalar fields and spinors in vector multiplets
i	$1, 2$	SU(2)
x	$1, 2, 3$	triplet of SU(2)

Table A.1 – Summary of indices.

Defining the space-time vielbein as

$$g_{\mu\nu} = e_\mu{}^m \eta_{mn} e_\nu{}^n, \quad (\text{A.1.2})$$

the spin connection and the Levi-Civita connection are given by

$$\begin{aligned} \omega_\mu{}^{mn}(e) &= 2e^{\nu[m} \partial_{[\mu} e_{\nu]}{}^{n]} - e^{\nu[m} e^{n]\sigma} e_{\mu\sigma} \partial_\nu e_\sigma{}^p, \\ \Gamma_{\mu\nu}^\rho &= \frac{1}{2} g^{\rho\lambda} (2\partial_{(\mu} g_{\nu)\lambda} - \partial_\lambda g_{\mu\nu}). \end{aligned} \quad (\text{A.1.3})$$

Both formulations are equivalent due to the constraint

$$\nabla_\mu e_\nu{}^m = \partial_\mu e_\nu{}^m + \omega_\mu{}^{mn}(e) e_{\nu n} - \Gamma_{\mu\nu}^\rho e_\rho{}^m = 0. \quad (\text{A.1.4})$$

The covariant derivative of a vector reads

$$\nabla_\mu k^m = \partial_\mu k^m + \omega_\mu{}^{mn} k_n. \quad (\text{A.1.5})$$

The Ricci tensor can be obtained using the following formulae

$$\begin{aligned} R_{\mu\nu}{}^{mn} &= 2\partial_{[\mu} \omega_{\nu]}{}^{mn}(e) + 2\omega_{[\mu}{}^{mp}(e) \omega_{\nu]p}{}^n(e), \\ R^\mu{}_{\nu\rho\sigma} &= R_{\rho\sigma}{}^{mn} e_m^\mu e_{\nu n} = 2\partial_{[\rho} \Gamma_{\sigma]\nu}^\mu + 2\Gamma_{\tau[\rho}^\mu \Gamma_{\sigma]\nu}^\tau, \\ R_{\mu\nu} &= R_{\mu\rho}{}^{nm} e_n^\rho e_{\nu m}, \quad R = g^{\mu\nu} R_{\mu\nu}. \end{aligned} \quad (\text{A.1.6})$$

$$(\text{A.1.7})$$

A.2 Spinor conventions

For the spinor conventions we follow [94]. The gamma matrices satisfy the relations

$$\gamma_m \gamma_n + \gamma_n \gamma_m = 2\eta_{mn}, \quad [\gamma_m, \gamma_n] \equiv \gamma_{mn}, \quad \gamma_5 \equiv i\gamma_0 \gamma_1 \gamma_2 \gamma_3, \quad \gamma_5^2 = 1. \quad (\text{A.2.1})$$

We define left and right projections by

$$P_L = \frac{1}{2}(1 + \gamma_5), \quad P_R = \frac{1}{2}(1 - \gamma_5), \quad (\text{A.2.2})$$

thus, denoting by ϵ_L and ϵ_R a left and a right handed chiral spinor respectively, they satisfy

$$P_L \epsilon_L = \epsilon_L, \quad P_R \epsilon_R = \epsilon_R. \quad (\text{A.2.3})$$

The covariant derivative of a fermion is given by

$$\nabla_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega_\mu^{mn} \gamma_{mn} \epsilon. \quad (\text{A.2.4})$$

A.3 SU(2) conventions

The SU(2) vector and matrix representations are related by

$$A_i^j \equiv iA^x (\sigma^x)_i^j, \quad A^x = -\frac{i}{2} \text{tr}(\sigma^x A), \quad (\text{A.3.1})$$

where the indices i, j run over 1, 2, and the index $x = 1, 2, 3$. We use the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.3.2})$$

SU(2) indices can be raised or lowered using the Levi-Civita tensor ε_{ij}

$$\varepsilon_{12} = 1, \quad \varepsilon^{12} = 1, \quad \varepsilon_{ij} \varepsilon^{jk} = -\delta_i^k, \quad (\text{A.3.3})$$

and we contract them using NorthWest-SouthEast convention (NW-SE)

$$A_i = A^j \varepsilon_{ji}, \quad A^i = \varepsilon^{ij} A_j. \quad (\text{A.3.4})$$

In $\mathcal{N} = 2$ supergravity the R-symmetry is SU(2). Since inside each multiplet, the fields are arranged into representations of the R-symmetry, in particular the fermions are labeled by an SU(2) index. We use the position of the index to denote left or right chirality:

$$P_L \epsilon^i = \epsilon^i, \quad P_R \epsilon_i = \epsilon_i. \quad (\text{A.3.5})$$

Stability of axionic D -term strings.

In this appendix we review the work presented in [72] which discusses the stability of a class of BPS solutions appearing in the globally supersymmetric model of Blanco-Pillado *et al.* [59]. The model describes a D-brane anti-D-brane unstable system after compactification to four dimensions, which is relevant for the study of the late stages of the brane antibrane inflationary model.

The cosmic strings appearing in the model were conjectured to be the low energy manifestation of D-strings that might form from tachyon condensation after D- anti-D-brane annihilation in type IIB superstring theory. The model describes the dynamics of two scalar fields, the tachyon and an axio-dilaton field, the later associated to the volume modulus and the Ramond-Ramond field which couples to the D-branes. The model admits three one-parameter families of cylindrically symmetric one-vortex solutions to the BPS equations, which are called tachyonic, axionic and hybrid strings, depending on which field sources the magnetic field in the core of the string. Different vortex solutions within a family have the same energy and varying core radius.

In particular we discuss the dynamics of the zero mode associated to the parameter connecting different string solutions within a family. We use a technique previously applied by R.A. Leese [156] in the case of semilocal strings [155]. We show that axionic strings behave similarly to semilocal strings, i.e.

if the string solution is perturbed, the evolution of the zero mode leads to the spread of the magnetic flux and the eventual disappearance of the string. In a cosmological setting these strings are expected to experience all sorts of perturbations, and therefore they would tend to become wider with time until they vanish.

This analysis is also useful to understand the stability of the cosmic strings found in the $\mathcal{N} = 2$ supergravity model presented in chapter 7. As in [59] the model admits a family of cosmic string solutions with equal energy and different radii. In section 7.5.3 we argued that the zero mode associated to the parameter connecting different solutions can not be excited by any finite energy perturbation. We will discuss again this statement in connection with the results we review here in the last section of this appendix.

B.1 The model

The lagrangian proposed in [59] describes a globally supersymmetric abelian Higgs model containing a vector multiplet and two chiral multiplets. One chiral multiplet contains a chiral field ϕ with canonical kinetic terms that represents the tachyon, and the other chiral multiplet involves an axion-dilaton field $S = s + ia$. The vector multiplet contains a gauge field denoted by A_μ .

As in the case of supergravity, globally supersymmetric models are defined in terms of a Kähler potential, a superpotential, the gauge kinetic functions and the gauge couplings (a review of globally supersymmetric models can be found in [18]). The Kähler potential is given by

$$K = \phi\bar{\phi} - \log(S + \bar{S}), \quad (\text{B.1.1})$$

the superpotential is chosen to be zero, $W(\phi, S) = 0$, and the gauge kinetic function is set to be constant, $f(S) = 1/g^2$, where g is the gauge coupling.

The gauge boson, A_μ , in the vector multiplet is coupled to the scalar fields gauging a combination of the $U(1)$ isometry of the tachyon and the shift symmetry of the axio-dilaton system:

$$\delta_{gauge}\phi = iq\phi\chi \quad \delta_{gauge}S = -i2\delta\chi. \quad (\text{B.1.2})$$

where q is the charge of the tachyon, δ is the coupling of the axion to the gauge field, and χ is the gauge parameter. Thus the corresponding covariant derivatives are given by:

$$D_\mu\phi = \partial_\mu\phi - iqA_\mu\phi, \quad D_\mu S = \partial_\mu S + i2\delta A_\mu. \quad (\text{B.1.3})$$

In globally supersymmetric models the D -term potential is obtained from the killing vectors and the Kähler potential using the same expressions as in su-

B.2. Cosmic string solutions

pergravity, (2.2.14) and (2.2.16). This particular model also includes a Fayet-Iliopoulos term ξ , and therefore the D -term potential reads:

$$V_D = \frac{1}{2}D^2, \quad D = g(\xi - \delta s^{-1} - |\phi|^2). \quad (\text{B.1.4})$$

The bosonic sector of the lagrangian, after eliminating the auxiliary field from the vector multiplet, is:

$$\mathcal{L} = -D_\mu \phi D^\mu \bar{\phi} - \frac{1}{4}s^{-2} D_\mu S D^\mu \bar{S} - \frac{1}{4}g^{-2} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}g^2 (\xi - \delta s^{-1} - q|\phi|^2)^2. \quad (\text{B.1.5})$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the abelian field strength associated to the gauge boson A_μ . It is convenient to write the bosonic part of the lagrangian using the rescalings:

$$\phi = \sqrt{\xi/q} \hat{\phi}, \quad s = \delta/\xi \hat{s}, \quad a = 2\delta/q \hat{a}, \quad A_\mu = g\sqrt{\xi/q} \hat{A}_\mu, \quad x = (g\sqrt{\xi q})^{-1} \hat{x}. \quad (\text{B.1.6})$$

With these definitions the axion $\hat{a}$ is defined modulo 2π , and δ is rescaled away. After dropping the hats, the bosonic sector of the lagrangian reads:

$$\begin{aligned} \mathcal{L}(\xi g)^{-2} = & -D_\mu \phi D^\mu \bar{\phi} - \frac{1}{4}(\alpha s)^{-2} \partial_\mu s \partial^\mu s - (\alpha/s)^2 (\partial_\mu a + A_\mu)(\partial^\mu a + A^\mu) \\ & - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2}(1 - s^{-1} - |\phi|^2)^2, \end{aligned} \quad (\text{B.1.7})$$

with $D_\mu \phi = \partial_\mu \phi - iA_\mu \phi$, and $\alpha^2 = \xi/q$. Note that α is the symmetry breaking scale in Plank units.

B.2 Cosmic string solutions

To study straight vortices along, say, the z -direction, we drop the z dependence and set $A_z = 0$. For time independent configurations and defining $\tilde{S} = s + 2i\alpha^2 a$, and $\tilde{D}_\mu \tilde{S} = \partial_\mu \tilde{S} + 2i\alpha^2 A_\mu$, the energy per unit length can be written in the Bogomolnyi form:

$$\begin{aligned} \mu_{string} = & (\xi g)^2 \int_M d^2x \left(|(D_x \pm iD_y)\phi|^2 + \frac{1}{4}(\alpha s)^{-2} |(\tilde{D}_x \pm i\tilde{D}_y)\tilde{S}|^2 \right. \\ & + \frac{1}{2}(F_{xy} \mp (1 - s^{-1} - |\phi|^2))^2 \pm F_{xy} \\ & \mp i[\partial_x(\phi^* D_y \phi) - \partial_y(\phi^* D_x \phi)] \\ & \left. \pm i\frac{1}{2}\alpha^{-2} [\partial_x(s^{-1} \tilde{D}_y \tilde{S}) - \partial_y(s^{-1} \tilde{D}_x \tilde{S})] \right). \end{aligned} \quad (\text{B.2.1})$$

Since the last two terms are boundary terms which do not contribute to the total energy of the string, this leads to a lower bound on the energy per unit length

$$\mu_{string} \geq \pm (\xi g)^2 \int d^2x F_{xy}. \quad (\text{B.2.2})$$

The bound is attained by the solutions of the Bogomolnyi equations

$$\begin{aligned} (D_x \pm iD_y)\phi &= 0 \\ (\tilde{D}_x \pm i\tilde{D}_y)\tilde{S} &= 0 \\ F_{xy} \mp (1 - s^{-1} - |\phi|^2) &= 0 \end{aligned} \quad (\text{B.2.3})$$

We will focus on cylindrically symmetric vortices, that can be described by the following ansatz:

$$\phi = f(r)e^{in\theta}, \quad s^{-1} = h^2, \quad a = m\theta, \quad A_\theta = v(r)/r, \quad (\text{B.2.4})$$

where $\{r, \theta\}$ are the polar coordinates in the plane transverse to the direction of the string. The ansatz for the dilaton is chosen for later convenience. It makes comparison to the semilocal case easier, and we will also see that h vanishes in various cases, thus we avoid having to deal with infinities.

With this ansatz the Bogomolnyi equations become

$$f' - f(|n| - v)/r = 0, \quad (\text{B.2.5})$$

$$h' - \alpha^2 h^3 (|m| - v)/r = 0, \quad (\text{B.2.6})$$

$$v'/r - (1 - f^2 - h^2) = 0, \quad (\text{B.2.7})$$

and the total energy per unit length saturates the bound giving $\mu_{string} = 2\pi v_\infty$, where v_∞ is the asymptotic value of the gauge field profile function for large values of r .

The conditions (B.2.5), (B.2.5) imply the following relation between the tachyon and the dilaton:

$$1/(\alpha h)^2 = 2(|n| - |m|) \log r - 2 \log f + \kappa. \quad (\text{B.2.8})$$

Depending on the asymptotic value of the profile functions for large values of r , f_∞ , h_∞ and v_∞ , we can classify the solutions to the Bogomolnyi equations in three different families. Each of them is parametrized by the integration constant κ .

- In the first case,

$$f_\infty = 1, \quad h_\infty = 0, \quad v_\infty = n, \quad (\text{B.2.9})$$

the tachyon acquires a non vanishing vacuum expectation value far from the center of the string. The magnetic flux of these vortices is induced by the winding of the tachyon, n , ($m < n$).

The profile function $h(r)$ tends very slowly, (logarithmically), to zero at large r . The plots of the profile functions and the details about the asymptotics of the fields can be found in [59]. Following Blanco-Pillado et al. we call these vortices ϕ -strings or tachyonic strings.

B.2. Cosmic string solutions

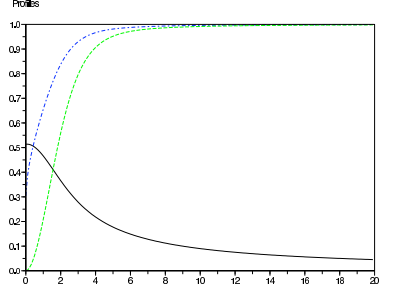


Figure B.1 – The axion string profile functions, for $m=1$, $n=0$, $a(r)$, (upper dashed line), $h(r)$, (middle dashed line) and $f(r)$, (lower solid line), with the size of the condensate, $f_0 = 0.51$.

- In the second case, Fig.(B.1), the dilaton alone is responsible for compensating the D-term. The function $h(r)$, approaches a non vanishing constant far from the centre of the string, while the tachyon expectation value tends to zero. In this family the magnetic flux is induced by the winding of the axion, m , ($n < m$).

$$f_\infty = 0 \quad h_\infty = 1 \quad v_\infty = m \quad (\text{B.2.10})$$

These vortices have been denominated *s-strings*, (or axionic). They are regular thanks to the constant Fayet-Iliopoulos term.

In the previous two families the constant κ has the same interpretation, each value of this parameter is associated to a particular width of the strings, as can be seen by setting:

$$k = 2(|n| - |m|) \log R \quad (\text{B.2.11})$$

In this way R gives the scale of the core radius. In the cases of ϕ -strings with $m = 0$, and s -strings with $n = 0$ the width of the strings, (or the core size), can be parametrized in a different way. For these particular cases, the ϕ -strings and s -strings develop a condensate at the core as they grow in width. For ϕ -strings the values of κ are in one to one correspondence with the value of the profile function $h(r)$ at $r = 0$. For s -strings with $n = 0$ each value of κ corresponds to a value of the profile function $f(r)$ at the origin. Therefore, the width of the static string solutions of the BPS equations can also be parametrized in terms of $h_0 = h(0)$ for ϕ -strings, and $f_0 = f(0)$ for s -strings.

As can be seen in (B.2.6) the derivatives of h scale as α^2 . In the case of the ϕ -strings, varying α for a fixed value of the integration constant κ does not change much the width of the string, but the condensate flattens. In the limit when α is very small the ϕ -strings are similar to a Nielsen-Olesen string. In the

case of s -strings the width of the string increases with decreasing α , while the condensate does not vary much. In fact, the main effect of decreasing α will be a slowing down of the dynamics.

- For strings of the third family both the tachyon and the dilaton contribute to compensate the D -term. This happens when the axion and dilaton have the same winding.

$$f_\infty^2 + h_\infty^2 = 1 \quad v_\infty = n = m \quad (\text{B.2.12})$$

In this case f_∞ and h_∞ can have any value as long as the previous relation is satisfied (B.2.12). Each particular f_∞ can be associated to a single κ , which means that the interpretation of this parameter is different to the previous two cases. Here the value of κ determines which of the fields, the dilaton or the tachyon, contributes more to compensate the Fayet-Iliopoulos term. Since it cannot be related any more to the width of the strings we will not discuss it any further.

B.3 Numerical simulation

In [72] we analyzed numerically the response of the BPS cosmic string solutions to cylindrically symmetric perturbations. Here we will just review the main results, the technical details of the simulation can be found in [72].

The initial field configurations correspond to static cosmic string solutions of the Bogomonlyi equations, and the perturbation is chosen in order to maximize the energy absorbed by the zero mode. Actually the perturbation gives the integration constant κ a dependence on the radius. Since both the initial configurations and the perturbation are cylindrically symmetric, the field configuration preserves the cylindrical symmetry at all times, and therefore it can be characterized completely giving the time dependence of the profile functions $f(r, t)$, $h(r, t)$ and $v(r, t)$. A cylindrically symmetric perturbation is not the most general one, it does not prove stability of the ϕ -strings, but it is sufficient to demonstrate the metastability of the s -strings.

B.4. Results

During the simulations we kept track of a set of observables:

$$E(t) = 2\pi \int_0^{r_{cal}} dr r \mathcal{E}(r, t), \quad (\text{B.3.1})$$

$$T(t) = 2\pi \int_0^{r_{cal}} dr r \mathcal{T}(r, t), \quad (\text{B.3.2})$$

$$E_T(t) = \frac{E(t) + T(t)}{E(0) + T(0)}, \quad (\text{B.3.3})$$

$$F(t) = 2\pi \int_0^{r_{cal}} dr r B_z(r, t), \quad (\text{B.3.4})$$

$$W(t) = \frac{\int_0^{r_{cal}} dr r^2 \mathcal{E}(r, t)}{\int_0^{r_{cal}} dr r \mathcal{E}(r, t)}, \quad (\text{B.3.5})$$

where $B_z = F_{xy}$ is the component of the magnetic field along the z direction and

$$\begin{aligned} \mathcal{E}(r, t) &= (\partial_r f)^2 + (\alpha h)^{-2} (\partial_r h)^2 + \frac{1}{2} \left(\frac{\partial_r v}{r} \right)^2 + \frac{1}{2} (1 - h^2 - f^2)^2 \\ \mathcal{T}(r, t) &= (\partial_t f)^2 + (\alpha h)^{-2} (\partial_t h)^2 + \frac{1}{2} \left(\frac{\partial_t v}{r} \right)^2. \end{aligned} \quad (\text{B.3.6})$$

The observable E is the static energy contained in a region defined by $r < r_{cal}$, and T the energy due to the time derivatives in the same area. r_{cal} is chosen to be much smaller than the size of the simulation box, so that the region $r < r_{cal}$ is causally disconnected from the boundaries during the whole simulation. E_T is the total energy normalized to the initial value and F gives the magnetic flux confined in the region $r < r_{cal}$.

W is a measure of the width of the string. In (B.3.5) the expression in the numerator is similar to the energy functional, but the extra factor r gives more weight to the energy far away from the core. Then, W increases as the energy spreads. Note that in all these expressions we have integrated over the polar angle θ assuming cylindrical symmetry over the whole evolution. The strength of the perturbations is characterized by the value of the time derivative of the width at the beginning of the simulation:

$$\dot{W}_0 \equiv \left. \frac{\dot{W}}{W} \right|_{t=0} \Delta t, \quad (\text{B.3.7})$$

where the constant Δt is the time step of the simulation.

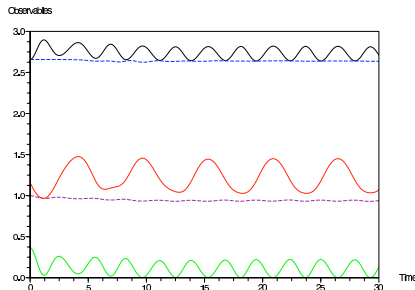


Figure B.2 – Response of a ϕ -string ($n = 1$, $m = 0$), with $\alpha = 1$ and condensate size $h_0 = 0.5$, to a perturbation with strength $\dot{W}_0 = -0.261$ (B.3.7). Time is measured in units proportional to the inverse of the tachyon mass, $g\sqrt{q\xi}/20$. From top to bottom, the plotted lines correspond to the quantities E , F , W , E_T and T defined in (B.3.1-B.3.5). Except for W and E_T , the rest of the plots have been rescaled by a factor of $1/2$ to fit in the window. The core width, W , oscillates but is constant on average showing that the zero mode is not excited.

B.4 Results

B.4.1 ϕ -strings

Fig.(B.2) shows the evolution of a ϕ -string, with $n = 1$, $m = 0$ and a core size $h_0 = 0.51$. We show the case $\alpha = 1$, which is also the choice made in [59]. The perturbation applied has a strength $\dot{W}_0 = -0.261$, which corresponds to a 0.4% perturbation in the energy. We have plotted the observables defined in the last section as a function of time. Upper solid line represents E , the dashed line just below is the magnetic flux in $r < r_{cal}$, which remains almost constant. The kinetic energy, T , is represented by the bottom solid line. The dashed line at the center of the figure is the total energy. Although it can not be clearly seen in the plot, the data tell us that during the period before $t = 12$, a fraction of the energy is lost. This is the initial burst of radiation emitted after the perturbation. After that the system reaches a stationary state where all quantities oscillate except the magnetic flux and the total energy.

The remaining solid line in the center is the string width. As the energy contained in the region $r \leq r_{cal}$ remains constant and the width of the vortex oscillates only around the initial value we conclude that this kind of string is stable under this perturbation. The experiment has been repeated for various types of perturbations, and for different initial widths (parametrized by h_0), and windings, but the results are similar to the ones presented here. Nielsen-Olesen vortices react in the same way to a perturbation, as was shown in [156].

B.4. Results

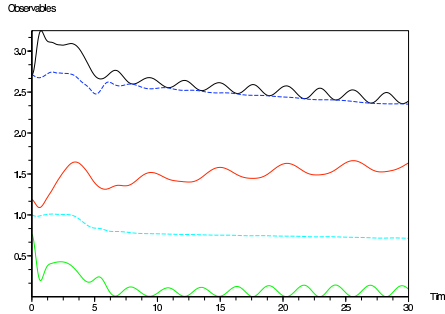


Figure B.3 – Response of an s -string ($n = 0$, $m = 1$), with $\alpha = 1$ and condensate size $f_0 = 0.49$, to a perturbation with strength $\dot{W}_0 = -0.095$. The plotted lines are, from top to bottom, E , F , W , E_T and T (except for W and E_T , the rest of the plots have been rescaled by a factor of $1/2$ to fit in the window). The core width, W , after a transient, oscillates and increases at a constant rate, the zero mode is excited in this case.

We have repeated the evolution for different values of α but no qualitative change has been observed. As was mentioned before, the smaller the value of α , the more similar the ϕ -string is to a regular Nielsen-Olesen string, which is known to be stable.

B.4.2 s -strings

The result of applying this perturbation with a strength of $\dot{W}_0 = -0.095$ to an s -string, with $\alpha = 1$, can be seen in Fig.(B.3). In this case the perturbation in energy is 2.4%. The string has windings $n = 0$ and $m = 1$, and core size $f_0 = 0.51$. The functions plotted are the same ones that appear in Fig.(B.2). One of the most relevant features of these plots is that the magnetic flux and the total energy are decreasing with time, which implies that the energy is flowing out from the region $r \leq r_{cal}$, and the magnetic flux is spreading. At the same time the width of the string, ignoring the oscillatory behavior, increases at a constant rate. Before $t = 5$ oscillations are noisy. In this period the shock wave produced by the perturbation is still inside $r \leq r_{cal}$.

Although the perturbation was chosen to reduce the core width, the time interval when the core is contracting cannot be seen clearly in the figures. The reason is that the contracting regime ends before the initial burst of radiation comes out from the observed region. As the system is not in a steady state yet, the data are difficult to interpret. We have chosen to show this case because the expanding regime is shown more clearly.

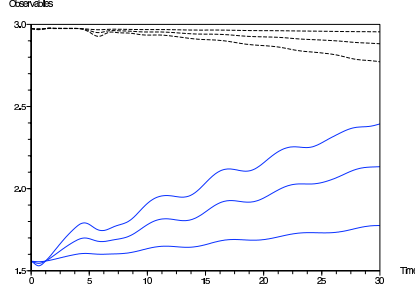


Figure B.4 – Response of an $n = 0$, $m = 2$, $\alpha = 1$ s -string with a core condensate of size $f_0 = 0.51$ to perturbations with different strengths. Dashed lines correspond to F , the strength is lowest for the top one. Solid lines represent W , the strength is highest for the top one. The strengths are: $\dot{W}_0 = -0.004$, -0.014 and -0.024 . F is rescaled by a factor of $1/4$. The figure shows how the growth rate of the core increases with the perturbation strength.

Fig.(B.4) shows the effect of applying perturbations of different strengths to a $n = 0$, $m = 2$ vortex. In this case the vortex has also a condensate size of $f_0 = 0.51$. The strengths applied are: $\dot{W}_0 = -0.004$, $\dot{W}_0 = -0.014$ and $\dot{W}_0 = -0.024$.

Notice that the bigger the strength of the perturbation, the larger the fraction of magnetic flux lost through the boundary $r = r_{cal}$. The rate of growth of the radius also increases with the strength of the perturbation.

In Fig.(B.5) it can be seen how the rate of expansion of an s -string, ($m = 1$, $n = 0$), is affected by varying the value of α . In this case the perturbation has been chosen to initially increase the core size $\dot{W}_0 = 0.233 > 0$. As we decrease α , keeping the perturbation strength fixed, the rate of expansion of the string decreases. This can be understood from equation (B.3.6). The energy associated to the field h scales as the inverse of α^2 . Deviations from the solution to the Bogomolnyi equations cost more energy for smaller values of α , thus for a fixed perturbation strength the evolution rates should decrease with α . The values of alpha are: $\alpha = 0.95$, 0.94 , 0.90 .

The precision of the technique used here does not allow to obtain reliable data for values of α lower than 0.7 , where already the evolution is so slow that it can hardly be appreciated during the time of the simulation.

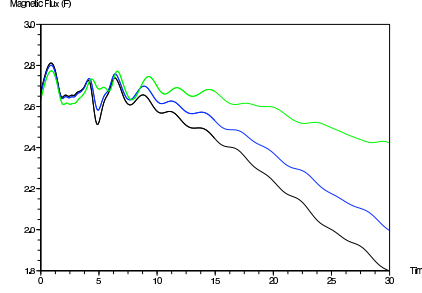


Figure B.5 – Response of an $n = 0$, $m = 1$ s -string with a core condensate of size $f_0 = 0.51$ to a perturbation with $\dot{W}_0 = 0.233$. The curves represent F , the magnetic flux rescaled by a factor of $1/4$. From bottom to top the values of α are: $\alpha = 0.95, 0.94$ and 0.90 . The figure illustrates the slowing down of the core expansion with decreasing α

B.5 Discussion

In order to analyze the results presented here we will use the moduli-space approximation, a semi-analytical method developed by N. Manton [33]. This technique was applied in [156] and [157] to study the dynamics of semilocal strings which, as we have mentioned previously, are closely related to the strings studied here.

According to the work by Manton, the low energy dynamics can be described restricting the Hamiltonian of the system to the solutions to the Bogomonly equations (B.2.3) with a time varying integration constant κ . In other words, if we substitute the static solutions to the Bogomonly equation (B.2.4) into the energy functional with κ promoted to be time dependent, we obtain an effective Hamiltonian for $\kappa(t)$. The evolution of $\kappa(t)$ obtained from this effective Hamiltonian approximately characterizes the evolution of the system at low energies.

For convenience we rewrite κ as

$$\kappa = 2(|n| - |m|) \log R \quad (\text{B.5.1})$$

and we study the dynamics of the parameter R which represents the width of the strings. The static cosmic string solutions of the Bogomonly equations can be written as functions of this parameter:

$$f = f(r, R), \quad h = h(r, R), \quad v = v(r, R), \quad (\text{B.5.2})$$

and promoting R to time dependent we have:

$$\dot{f} = \partial_R f \dot{R}, \quad \dot{h} = \partial_R h \dot{R}, \quad \dot{v} = \partial_R v \dot{R}. \quad (\text{B.5.3})$$

Substituting this ansatz into the energy functional associated to the lagrangian B.1.7 we find that the total energy of the string for time dependent (and homogeneous) parameter $R(t)$ reads:

$$(\xi g)^{-2} \mu_{string} = 2\pi \max\{|n|, |m|\} + \delta E_{pert}, \quad (\text{B.5.4})$$

where the energy of the perturbation is given by:

$$\delta E_{pert} = \int dr r \left((\partial_R f)^2 + \frac{1}{\alpha^2} \left(\frac{\partial_R h}{h} \right)^2 + \frac{(\partial_R v)^2}{2r} \right) \dot{R}^2. \quad (\text{B.5.5})$$

Note that, since the energy of the static string configuration stays the same for every value of R , the only new contribution to the energy due to the time dependence of $R(t)$ comes from the kinetic terms.

For ϕ -strings with $n = 1$ and $m = 0$ the asymptotic behavior of the profile functions for $r \rightarrow \infty$ is [59]:

$$f_\infty = 1, \quad h_\infty \sim \log(r/R)^{-1/2} \quad v_\infty = |n|. \quad (\text{B.5.6})$$

substituting this in the expression we just derived for the energy per unit length of the perturbation (B.5.5) we find that it has a divergent contribution of the form:

$$\delta E_{pert} \sim \int_{r_c}^{\Lambda} dr \frac{r}{\log(r)^2} \dot{R}^2 \sim \frac{\Lambda^2}{(\log \Lambda)^2} \dot{R}^2, \quad (\text{B.5.7})$$

where Λ is a cutoff, and r_c is the core radius of the string. Thus the amount of energy needed in order to excite the zero mode in a cylinder of radius Λ centered on the string scales almost as Λ^2 . In other words given an perturbation of energy δE_{pert} , the size of the region where the zero mode can be excited scales approximately as:

$$\Lambda \sim (\delta E_{pert})^{1/2}. \quad (\text{B.5.8})$$

Clearly any finite energy perturbation can not excite the zero mode everywhere in space, since this would require an infinite energy. If such a perturbation is localized around the core of the string it is not expected to propagate further away than a distance Λ from the string center. This is consistent with the observed results shown in figure B.2, which indicates that perturbing the static solution leads to oscillations in the string width and not to the spread of the magnetic flux.

It would seem that the infinity we have obtained for the kinetic energy of the perturbations is a result of promoting the parameter R to time dependent

B.5. Discussion

while keeping it homogeneous. However, a perturbation $R(t)$ independent of the coordinates does not necessarily imply that the profile functions are modified everywhere in space. Indeed, the profile functions $f(r)$, $h(r)$ and $v(r)$ are perturbed only for those values of r such that $\partial_R f(r) \neq 0$, $\partial_R h(r) \neq 0$ and $\partial_R v(r) \neq 0$ respectively. As we shall see in the case of s -strings, if these derivative decay sufficiently fast for $r \rightarrow \infty$, it is still possible to obtain finite results for the energy of the perturbations without introducing a space dependence on the parameter R .

The asymptotic behavior of s -strings is given by:

$$f_\infty \sim (r/R)^{|n|-|m|}, \quad h_\infty = 1, \quad v_\infty = |m|, \quad (\text{B.5.9})$$

and therefore, substituting into (B.5.5), we see that the perturbation has an energy per unit length of the form

$$\delta E_{\text{pert}} \sim \int_{r_c}^{\Lambda} dr r \sim r^{2(|n|-|m|)} \dot{R}^2. \quad (\text{B.5.10})$$

In particular, for the case studied here $m = 1$ and $n = 0$:

$$\delta E_{\text{pert}} \sim \int_{r_c}^{\Lambda} \frac{dr}{r} \dot{R}^2 \sim \log \Lambda \dot{R}^2. \quad (\text{B.5.11})$$

Therefore, a perturbation of energy δE_{pert} can only excite the zero mode in a cylinder centered on the string of radius Λ , which scales as

$$\Lambda \sim \exp(\delta E_{\text{pert}}). \quad (\text{B.5.12})$$

This analysis implies that an infinite energy is also needed in the case of s -strings in order to excite the zero mode everywhere in space. However the energy only scales logarithmically, in contrast with the almost quadratic divergence that appeared in the case of ϕ -strings. In a cosmological setting the string network has a natural length scale which acts as a cutoff, the typical inter-string distance. Due to this mild dependence on the cutoff, the perturbations inherent to any cosmological setting might be able to excite the zero mode, leading to the growth of the radius of the string and the eventual dilution of the magnetic flux. This is precisely confirmed in our simulations, where the cutoff is given by the size of the simulation box. As we have shown in figure B.3, perturbations with small amount of energy compared to the total energy of the string can excite the zero mode leading to the dilution of the magnetic flux. Note that for $|n| - |m| < -1$ the integral (B.5.10) becomes finite, and therefore any finite energy perturbation is enough to induce a growth of the parameter R everywhere in space.

This analysis is closely related to the discussion in section 7.5.3, where we consider the stability of cosmic string solutions in a $\mathcal{N} = 2$ supergravity model.

The model admits a family of string solutions parametrized by the value of one of the fields, the axio-dilaton S . In particular the radius of the string scales as the imaginary part of S . The imaginary part of the axio-dilaton $\text{Im } S = -e^\rho$ has a standard kinetic term of the form (7.4.6):

$$\int dr r \partial_\mu \rho \partial^\mu \rho, \quad (\text{B.5.13})$$

thus, in order to study the excitability of the zero mode associated to the value of ρ , we promote it to time dependent function. The time dependence of ρ introduces a contribution to the total energy of the configuration:

$$\delta E_{\text{pert}} \sim \int_{r_s}^{\Lambda} dr r \dot{\rho}^2 \sim \Lambda^2 \dot{\rho}^2, \quad (\text{B.5.14})$$

which scales with the cutoff in the same way as the perturbation of a ϕ -string. This suggests that this zero mode will not be excited by any finite energy perturbation, as was anticipated in section 7.5.3, and thus the radius of the cosmic string will not tend to grow in a cosmological context.

Bibliography

- [1] S. Weinberg, *Cosmology*, Oxford Univ. Press (2008).
- [2] G. Gamow, *Expanding universe and the origin of elements*, *Phys.Rev.* **70** (1946) 572–573.
- [3] R. A. Alpher and R. C. Herman, *Theory of the Origin and Relative Abundance Distribution of the Elements*, *Rev.Mod.Phys.* **22** (1950) 153–212.
- [4] A. A. Penzias and R. W. Wilson *Ap. J.* **142** (419) 1965.
- [5] D. Larson, J. Dunkley, G. Hinshaw, E. Komatsu, M. Nolte, *et. al.*, *Seven-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Power Spectra and WMAP-Derived Parameters*, *Astrophys.J.Suppl.* **192** (2011) 16, [[arXiv:1001.4635](#)].
- [6] G. Lemaitre, *A homogeneous Universe of constant mass and growing radius accounting for the radial velocity of extragalactic nebulae*, *Annales Soc.Sci.Brux.Ser.I Sci.Math.Astron.Phys.* **A47** (1927) 49–59.
- [7] A. Friedmann *Z. Phys.* **10** (1922) 377.
- [8] H. P. Robertson *Ap. J.* **82** (1935) 284.
- [9] A. G. Walker *Proc. Lond. Math. Soc.* **42** (1936) 90.
- [10] A. Wright *Nature Phys.* **7** (2011) 833.

-
- [11] E. J. Copeland, M. Sami, and S. Tsujikawa, *Dynamics of dark energy*, *Int.J.Mod.Phys.* **D15** (2006) 1753–1936, [[hep-th/0603057](#)].
 - [12] A. H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, *Phys.Rev.* **D23** (1981) 347–356.
 - [13] S. Weinberg, *The quantum theory of fields (vols. 1,2,3)*, Cambridge University Press (2000).
 - [14] S. Weinberg, *Gravitation and cosmology*, Wiley (1972).
 - [15] G. 't Hooft and M. Veltman, *Regularization and Renormalization of Gauge Fields*, *Nucl.Phys.* **B44** (1972) 189–213. *** Nobel Prize <http://www.nobel.se/announcement-99/physics99.html>; 1999; a; ***.
 - [16] J. R. Ellis, G. L. Fogli, and E. Lisi, *The Higgs boson mass from precision electroweak data*, *Z.Phys.* **C69** (1996) 627–634, [[hep-ph/9507424](#)].
 - [17] B. W. Lee, C. Quigg, and H. Thacker, *Weak Interactions at Very High-Energies: The Role of the Higgs Boson Mass*, *Phys.Rev.* **D16** (1977) 1519.
 - [18] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press.
 - [19] J. Polchinski, *String theory (vols. 1,2)*, Cambridge University Press (1998).
 - [20] S. B. Giddings, S. Kachru, and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys.Rev.* **D66** (2002) 106006, [[hep-th/0105097](#)].
 - [21] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, *De Sitter vacua in string theory*, *Phys.Rev.* **D68** (2003) 046005, [[hep-th/0301240](#)].
 - [22] S. C. Davis and M. Postma, *Successfully combining SUGRA hybrid inflation and moduli stabilisation*, *JCAP* **0804** (2008) 022, [[arXiv:0801.2116](#)].
 - [23] M. Dine, L. Randall, and S. D. Thomas, *Supersymmetry breaking in the early universe*, *Phys.Rev.Lett.* **75** (1995) 398–401, [[hep-ph/9503303](#)].
 - [24] C. Burgess, *Lectures on Cosmic Inflation and its Potential Stringy Realizations*, *Class.Quant.Grav.* **24** (2007) S795, [[arXiv:0708.2865](#)].
 - [25] S. Coleman, *Aspects of symmetry*, Cambridge University Press (1985).

Bibliography

- [26] A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects.*, Cambridge University Press (1994).
- [27] R. Jeannerot, J. Rocher, and M. Sakellariadou, *How generic is cosmic string formation in SUSY GUTs*, *Phys.Rev.* **D68** (2003) 103514, [[hep-ph/0308134](#)].
- [28] M. Goodband and M. Hindmarsh, *Bound states and instabilities of vortices*, *Phys.Rev.* **D52** (1995) 4621–4632, [[hep-ph/9503457](#)].
- [29] M. Hindmarsh and T. Kibble, *Cosmic strings*, *Rept.Prog.Phys.* **58** (1995) 477–562, [[hep-ph/9411342](#)].
- [30] E. Bogomolny, *Stability of Classical Solutions*, *Sov.J.Nucl.Phys.* **24** (1976) 449.
- [31] M. Prasad and C. M. Sommerfield, *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, *Phys.Rev.Lett.* **35** (1975) 760–762.
- [32] C. H. Taubes, *Arbitrary N: Vortex Solutions to the First Order Landau-Ginzburg Equations*, *Commun.Math.Phys.* **72** (1980) 277.
- [33] N. Manton, *A Remark on the Scattering of BPS Monopoles*, *Phys.Lett.* **B110** (1982) 54–56.
- [34] P. Ruback, *VORTEX STRING MOTION IN THE ABELIAN HIGGS MODEL*, *Nucl.Phys.* **B296** (1988) 669–678.
- [35] E. Shellard and P. Ruback, *VORTEX SCATTERING IN TWO-DIMENSIONS*, *Phys.Lett.* **B209** (1988) 262–270.
- [36] T. Kibble, *Topology of Cosmic Domains and Strings*, *J.Phys.A* **A9** (1976) 1387–1398.
- [37] M. Hindmarsh, S. Stuckey, and N. Bevis, *Abelian Higgs Cosmic Strings: Small Scale Structure and Loops*, *Phys.Rev.* **D79** (2009) 123504, [[arXiv:0812.1929](#)].
- [38] C. Martins and E. Shellard, *Fractal properties and small-scale structure of cosmic string networks*, *Phys.Rev.* **D73** (2006) 043515, [[astro-ph/0511792](#)].
- [39] K. D. Olum and V. Vanchurin, *Cosmic string loops in the expanding Universe*, *Phys.Rev.* **D75** (2007) 063521, [[astro-ph/0610419](#)].
- [40] J. J. Blanco-Pillado, K. D. Olum, and B. Shlaer, *Large parallel cosmic string simulations: New results on loop production*, *Phys.Rev.* **D83** (2011) 083514, [[arXiv:1101.5173](#)].

-
- [41] E. J. Copeland and T. Kibble, *Cosmic Strings and Superstrings*, *Proc.Roy.Soc.Lond.* **A466** (2010) 623–657, [[arXiv:0911.1345](#)].
- [42] M. Sakellariadou, *Cosmic Strings and Cosmic Superstrings*, *Nucl.Phys.Proc.Suppl.* **192-193** (2009) 68–90, [[arXiv:0902.0569](#)].
- [43] A. Achucarro and C. Martins, *Cosmic Strings*, [arXiv:0811.1277](#).
- [44] J. Urrestilla, N. Bevis, M. Hindmarsh, and M. Kunz, *Cosmic string parameter constraints and model analysis using small scale Cosmic Microwave Background data*, *JCAP* **1112** (2011) 021, [[arXiv:1108.2730](#)].
- [45] C. Dvorkin, M. Wyman, and W. Hu, *Cosmic String constraints from WMAP and the South Pole Telescope*, *Phys.Rev.* **D84** (2011) 123519, [[arXiv:1109.4947](#)].
- [46] P. Mukherjee, J. Urrestilla, M. Kunz, A. R. Liddle, N. Bevis, *et. al.*, *Detecting and distinguishing topological defects in future data from the CMBPol satellite*, *Phys.Rev.* **D83** (2011) 043003, [[arXiv:1010.5662](#)].
- [47] N. Bevis, M. Hindmarsh, M. Kunz, and J. Urrestilla, *CMB polarization power spectra contributions from a network of cosmic strings*, *Phys.Rev.* **D76** (2007) 043005, [[arXiv:0704.3800](#)].
- [48] M. Kamionkowski, A. Kosowsky, and A. Stebbins, *Statistics of cosmic microwave background polarization*, *Phys.Rev.* **D55** (1997) 7368–7388, [[astro-ph/9611125](#)].
- [49] A.-C. Davis and T. Kibble, *Fundamental cosmic strings*, *Contemp.Phys.* **46** (2005) 313–322, [[hep-th/0505050](#)].
- [50] N. Kaiser and A. Stebbins, *Microwave Anisotropy Due to Cosmic Strings*, *Nature* **310** (1984) 391–393.
- [51] T. Damour and A. Vilenkin, *Gravitational radiation from cosmic (super)strings: Bursts, stochastic background, and observational windows*, *Phys.Rev.* **D71** (2005) 063510, [[hep-th/0410222](#)].
- [52] M. Majumdar and A. Christine-Davis, *Cosmological creation of D-branes and anti-D-branes*, *JHEP* **0203** (2002) 056, [[hep-th/0202148](#)]. 30 pages, 5 eps figures: v2 more references added: v3 section 4 slightly improved Report-no: DAMTP 2002-12 Journal-ref: JHEP 0203 (2002) 056.
- [53] S. Sarangi and S. Tye, *Cosmic string production towards the end of brane inflation*, *Phys.Lett.* **B536** (2002) 185–192, [[hep-th/0204074](#)].
- [54] S. C. Davis, A.-C. Davis, and M. Trodden, *$N=1$ supersymmetric cosmic strings*, *Phys.Lett.* **B405** (1997) 257–264, [[hep-ph/9702360](#)].

Bibliography

- [55] S. C. Davis, P. Binetruy, and A.-C. Davis, *Local axion cosmic strings from superstrings*, *Phys.Lett.* **B611** (2005) 39–52, [[hep-th/0501200](#)].
- [56] R. H. Brandenberger, B. Carter, A.-C. Davis, and M. Trodden, *Cosmic vortons and particle physics constraints*, *Phys.Rev.* **D54** (1996) 6059–6071, [[hep-ph/9605382](#)].
- [57] K. Becker, M. Becker, and A. Strominger, *Three-dimensional supergravity and the cosmological constant*, *Phys.Rev.* **D51** (1995) 6603–6607, [[hep-th/9502107](#)].
- [58] R. Jeannerot and M. Postma, *Chiral cosmic strings in supergravity*, *JHEP* **0412** (2004) 043, [[hep-ph/0411260](#)].
- [59] J. J. Blanco-Pillado, G. Dvali, and M. Redi, *Cosmic D-strings as axionic D-term strings*, *Phys.Rev.* **D72** (2005) 105002, [[hep-th/0505172](#)].
- [60] P. Brax, C. van de Bruck, A. Davis, and S. C. Davis, *Fermionic zero modes of supergravity cosmic strings*, *JHEP* **0606** (2006) 030, [[hep-th/0604198](#)].
- [61] G. Dvali, R. Kallosh, and A. Van Proeyen, *D term strings*, *JHEP* **0401** (2004) 035, [[hep-th/0312005](#)].
- [62] J. Polchinski, *Introduction to cosmic F- and D-strings*, [hep-th/0412244](#).
- [63] M. Majumdar, *A Tutorial on links between cosmic string theory and superstring theory*, [hep-th/0512062](#).
- [64] G. Dvali and S. Tye, *Brane inflation*, *Phys.Lett.* **B450** (1999) 72–82, [[hep-ph/9812483](#)].
- [65] C. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh, *et. al.*, *The Inflationary brane anti-brane universe*, *JHEP* **0107** (2001) 047, [[hep-th/0105204](#)]. Dedicated to the memory of Detlef Nolte.
- [66] K. Dasgupta, J. P. Hsu, R. Kallosh, A. D. Linde, and M. Zagermann, *D3/D7 brane inflation and semilocal strings*, *JHEP* **0408** (2004) 030, [[hep-th/0405247](#)].
- [67] E. Halvø, *Cosmic D term strings as wrapped D3 branes*, *JHEP* **0403** (2004) 047, [[hep-th/0312268](#)].
- [68] S. S. Gubser, C. P. Herzog, and I. R. Klebanov, *Symmetry breaking and axionic strings in the warped deformed conifold*, *JHEP* **0409** (2004) 036, [[hep-th/0405282](#)].
- [69] A. Lawrence and J. McGreevy, *D-terms and D-strings in open string models*, *JHEP* **0410** (2004) 056, [[hep-th/0409284](#)].

-
- [70] L. Martucci, *D-branes on general $N=1$ backgrounds: Superpotentials and D-terms*, *JHEP* **0606** (2006) 033, [[hep-th/0602129](#)].
- [71] J. Urrestilla, A. Achucarro, and A. Davis, *D term inflation without cosmic strings*, *Phys.Rev.Lett.* **92** (2004) 251302, [[hep-th/0402032](#)].
- [72] A. Achucarro and K. Sousa, *A Note on the stability of axionic D-term strings*, *Phys.Rev.* **D74** (2006) 081701, [[hep-th/0601151](#)].
- [73] A. Achucarro, A. Celi, M. Esole, J. Van den Bergh, and A. Van Proeyen, *D-term cosmic strings from $N=2$ supergravity*, *JHEP* **0601** (2006) 102, [[hep-th/0511001](#)].
- [74] K. Dasgupta, C. Herdeiro, S. Hirano, and R. Kallosh, *$D3 / D7$ inflationary model and M theory*, *Phys.Rev.* **D65** (2002) 126002, [[hep-th/0203019](#)].
- [75] P. Binetruy, G. Dvali, R. Kallosh, and A. Van Proeyen, *Fayet-Iliopoulos terms in supergravity and cosmology*, *Class.Quant.Grav.* **21** (2004) 3137–3170, [[hep-th/0402046](#)].
- [76] A. Achucarro, S. Hardeman, and K. Sousa, *Consistent Decoupling of Heavy Scalars and Moduli in $N=1$ Supergravity*, *Phys.Rev.* **D78** (2008) 101901, [[arXiv:0806.4364](#)].
- [77] A. Achucarro and K. Sousa, *F-term uplifting and moduli stabilization consistent with Kahler invariance*, *JHEP* **0803** (2008) 002, [[arXiv:0712.3460](#)].
- [78] A. Achucarro, S. Hardeman, and K. Sousa, *F-term uplifting and the supersymmetric integration of heavy moduli*, *JHEP* **0811** (2008) 003, [[arXiv:0809.1441](#)].
- [79] L. Covi, M. Gomez-Reino, C. Gross, J. Louis, G. A. Palma, *et. al.*, *de Sitter vacua in no-scale supergravities and Calabi-Yau string models*, *JHEP* **0806** (2008) 057, [[arXiv:0804.1073](#)].
- [80] A. Achucarro, S. Mooij, P. Ortiz, and M. Postma, *Sgoldstino inflation*, [arXiv:1203.1907](#).
- [81] Z. Komargodski and N. Seiberg, *Comments on the Fayet-Iliopoulos Term in Field Theory and Supergravity*, *JHEP* **0906** (2009) 007, [[arXiv:0904.1159](#)].
- [82] F. Catino, G. Villadoro, and F. Zwirner, *On Fayet-Iliopoulos terms and de Sitter vacua in supergravity: Some easy pieces*, *JHEP* **1201** (2012) 002, [[arXiv:1110.2174](#)].

Bibliography

- [83] M. Esole and K. Sousa, *Half-BPS cosmic string in $N=2$ supergravity in the presence of a dilaton*, *JHEP* **0703** (2007) 079, [[hep-th/0610124](#)].
- [84] A. Van Proeyen, *Tools for supersymmetry*, [hep-th/9910030](#).
- [85] R. Kallosh, L. Kofman, A. D. Linde, and A. Van Proeyen, *Superconformal symmetry, supergravity and cosmology*, *Class.Quant.Grav.* **17** (2000) 4269–4338, [[hep-th/0006179](#)].
- [86] T. W. Grimm and J. Louis, *The Effective action of $N = 1$ Calabi-Yau orientifolds*, *Nucl.Phys.* **B699** (2004) 387–426, [[hep-th/0403067](#)].
- [87] S. Gates, M. T. Grisaru, M. Rocek, and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, *Front.Phys.* **58** (1983) 1–548, [[hep-th/0108200](#)].
- [88] M. Nakahara, *Geometry, topology and physics*, *IoP* (1990).
- [89] E. Cremmer, S. Ferrara, L. Girardello, and A. Van Proeyen, *Yang-Mills Theories with Local Supersymmetry: Lagrangian, Transformation Laws and SuperHiggs Effect*, *Nucl.Phys.* **B212** (1983) 413.
- [90] J. Bagger, *Supersymmetric sigma models*, .
- [91] K. Choi, A. Falkowski, H. P. Nilles, and M. Olechowski, *Soft supersymmetry breaking in KKLT flux compactification*, *Nucl.Phys.* **B718** (2005) 113–133, [[hep-th/0503216](#)].
- [92] P. Breitenlohner and D. Z. Freedman, *Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity*, *Phys.Lett.* **B115** (1982) 197.
- [93] M. Gomez-Reino and C. A. Scrucca, *Metastable supergravity vacua with F and D supersymmetry breaking*, *JHEP* **0708** (2007) 091, [[arXiv:0706.2785](#)].
- [94] A. Van Proeyen, *$N=2$ supergravity in $d=4,5,6$ and its matter couplings.*, In [preparation](#).
- [95] G. Gibbons and S. Hawking, *Action Integrals and Partition Functions in Quantum Gravity*, *Phys.Rev.* **D15** (1977) 2752–2756.
- [96] A. Comtet and G. Gibbons, *Bogomonly bounds for cosmic strings*, *Nucl.Phys.* **B299** (1988) 719.
- [97] J. A. Nester, *A New gravitational energy expression with a simple positivity proof*, *Phys.Lett.* **A83** (1981) 241.

-
- [98] E. Witten, *A Simple Proof of the Positive Energy Theorem*, *Commun.Math.Phys.* **80** (1981) 381.
- [99] J. D. Edelstein, C. Nunez, and F. A. Schaposnik, *Supergravity and a Bogomolny bound in three-dimensions*, *Nucl.Phys.* **B458** (1996) 165–188, [[hep-th/9506147](#)].
- [100] A. Collinucci, P. Smyth, and A. Van Proeyen, *The Energy and stability of D-term strings*, *JHEP* **0702** (2007) 060, [[hep-th/0611111](#)].
- [101] A.-C. Davis and S. Rajamanoharan, *Bogomol’nyi Bounds for Gravitational Cosmic Strings*, *JHEP* **0901** (2009) 027, [[arXiv:0809.4384](#)].
- [102] A. Achucarro and J. Urrestilla, *F-term strings in the Bogomol’nyi limit are also BPS states*, *JHEP* **0408** (2004) 050, [[hep-th/0407193](#)].
- [103] A. Achucarro, J.-O. Gong, S. Hardeman, G. A. Palma, and S. P. Patil, *Mass hierarchies and non-decoupling in multi-scalar field dynamics*, *Phys.Rev.* **D84** (2011) 043502, [[arXiv:1005.3848](#)].
- [104] A. Achucarro, J.-O. Gong, S. Hardeman, G. A. Palma, and S. P. Patil, *Features of heavy physics in the CMB power spectrum*, *JCAP* **1101** (2011) 030, [[arXiv:1010.3693](#)].
- [105] A. Achucarro, J.-O. Gong, S. Hardeman, G. A. Palma, and S. P. Patil, *Effective theories of single field inflation when heavy fields matter*, [arXiv:1201.6342](#).
- [106] L. Andrianopoli, R. D’Auria, and S. Ferrara, *Supersymmetry reduction of N extended supergravities in four-dimensions*, *JHEP* **0203** (2002) 025, [[hep-th/0110277](#)].
- [107] L. Andrianopoli, R. D’Auria, and S. Ferrara, *Consistent reduction of N=2 to N=1 four-dimensional supergravity coupled to matter*, *Nucl.Phys.* **B628** (2002) 387–403, [[hep-th/0112192](#)].
- [108] M. P. do Carmo, *Riemannian geometry*, Birkhauser (1992).
- [109] S. de Alwis, *On integrating out heavy fields in SUSY theories*, *Phys.Lett.* **B628** (2005) 183–187, [[hep-th/0506267](#)].
- [110] S. de Alwis, *Effective potentials for light moduli*, *Phys.Lett.* **B626** (2005) 223–229, [[hep-th/0506266](#)].
- [111] S. de Alwis, *Potentials for light moduli in supergravity and string theory*, [hep-th/0602182](#).

Bibliography

- [112] P. Griffiths and H. J., *Principles of algebraic geometry*, John Wiley & Sons (1978).
- [113] D. Bailin and A. Love, *Supersymmetric gauge field theory and string theory*, .
- [114] I. Ben-Dayan, R. Brustein, and S. P. de Alwis, *Models of Modular Inflation and Their Phenomenological Consequences*, *JCAP* **0807** (2008) 011, [[arXiv:0802.3160](#)].
- [115] E. Cremmer, S. Ferrara, C. Kounnas, and D. V. Nanopoulos, *Naturally Vanishing Cosmological Constant in $N=1$ Supergravity*, *Phys.Lett.* **B133** (1983) 61.
- [116] S. B. Giddings, S. Kachru, and J. Polchinski, *Hierarchies from fluxes in string compactifications*, *Phys.Rev.* **D66** (2002) 106006, [[hep-th/0105097](#)].
- [117] J. P. Conlon, F. Quevedo, and K. Suruliz, *Large-volume flux compactifications: Moduli spectrum and $D3/D7$ soft supersymmetry breaking*, *JHEP* **0508** (2005) 007, [[hep-th/0505076](#)].
- [118] J. P. Conlon, S. S. Abdussalam, F. Quevedo, and K. Suruliz, *Soft SUSY Breaking Terms for Chiral Matter in IIB String Compactifications*, *JHEP* **0701** (2007) 032, [[hep-th/0610129](#)].
- [119] J. P. Conlon, *Mirror Mediation*, *JHEP* **0803** (2008) 025, [[arXiv:0710.0873](#)].
- [120] M. Gomez-Reino and C. A. Scrucca, *Locally stable non-supersymmetric Minkowski vacua in supergravity*, *JHEP* **0605** (2006) 015, [[hep-th/0602246](#)].
- [121] M. Gomez-Reino and C. A. Scrucca, *Constraints for the existence of flat and stable non-supersymmetric vacua in supergravity*, *JHEP* **0609** (2006) 008, [[hep-th/0606273](#)].
- [122] C. Burgess, R. Kallosh, and F. Quevedo, *De Sitter string vacua from supersymmetric D terms*, *JHEP* **0310** (2003) 056, [[hep-th/0309187](#)].
- [123] G. Villadoro and F. Zwirner, *De-Sitter vacua via consistent D -terms*, *Phys.Rev.Lett.* **95** (2005) 231602, [[hep-th/0508167](#)].
- [124] A. Achucarro, B. de Carlos, J. Casas, and L. Doplicher, *De Sitter vacua from uplifting D -terms in effective supergravities from realistic strings*, *JHEP* **0606** (2006) 014, [[hep-th/0601190](#)].

-
- [125] V. Balasubramanian and P. Berglund, *Stringy corrections to Kahler potentials, SUSY breaking, and the cosmological constant problem*, *JHEP* **0411** (2004) 085, [[hep-th/0408054](#)].
- [126] A. Westphal, *Eternal inflation with alpha-prime-corrections*, *JCAP* **0511** (2005) 003, [[hep-th/0507079](#)].
- [127] S. de Alwis, *Mediation of Supersymmetry Breaking in a Class of String Theory Models*, *JHEP* **0903** (2009) 023, [[arXiv:0806.2672](#)].
- [128] M. Dine, N. Seiberg, and E. Witten, *Fayet-Iliopoulos Terms in String Theory*, *Nucl.Phys.* **B289** (1987) 589.
- [129] M. B. Green and J. H. Schwarz, *Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory*, *Phys.Lett.* **B149** (1984) 117–122.
- [130] Z. Komargodski and N. Seiberg, *Comments on Supercurrent Multiplets, Supersymmetric Field Theories and Supergravity*, *JHEP* **1007** (2010) 017, [[arXiv:1002.2228](#)].
- [131] N. Seiberg, *Modifying the Sum Over Topological Sectors and Constraints on Supergravity*, *JHEP* **1007** (2010) 070, [[arXiv:1005.0002](#)].
- [132] J. Distler and E. Sharpe, *Quantization of Fayet-Iliopoulos Parameters in Supergravity*, *Phys.Rev.* **D83** (2011) 085010, [[arXiv:1008.0419](#)].
- [133] S. Groot Nibbelink, T. Nyawelo, and J. van Holten, *Construction and analysis of anomaly free supersymmetric $SO(2N) / U(N)$ sigma models*, *Nucl.Phys.* **B594** (2001) 441–476, [[hep-th/0008097](#)].
- [134] J. van Holten, *Kahler manifolds and supersymmetry*, *Acta Phys.Polon.* **B34** (2003) 5983–6004, [[hep-th/0309094](#)].
- [135] R. D’Auria, S. Ferrara, M. Trigiante, and S. Vaula, *$N=1$ reductions of $N=2$ supergravity in the presence of tensor multiplets*, *JHEP* **0503** (2005) 052, [[hep-th/0502219](#)].
- [136] B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged $N=2$ Supergravity: Yang-Mills Models*, *Nucl.Phys.* **B245** (1984) 89.
- [137] B. de Wit, P. Lauwers, and A. Van Proeyen, *Lagrangians of $N=2$ Supergravity - Matter Systems*, *Nucl.Phys.* **B255** (1985) 569.
- [138] A. Strominger, *SPECIAL GEOMETRY*, *Commun.Math.Phys.* **133** (1990) 163–180.

Bibliography

- [139] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, *et. al.*, *N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map*, *J.Geom.Phys.* **23** (1997) 111–189, [[hep-th/9605032](#)].
- [140] B. Craps, F. Roose, W. Troost, and A. Van Proeyen, *What is special Kahler geometry?*, *Nucl.Phys.* **B503** (1997) 565–613, [[hep-th/9703082](#)].
- [141] B. de Wit and A. Van Proeyen, *Special geometry and symplectic transformations*, *Nucl.Phys.Proc.Suppl.* **45BC** (1996) 196–206, [[hep-th/9510186](#)].
- [142] S. Ferrara, L. Girardello, and M. Porrati, *Minimal Higgs branch for the breaking of half of the supersymmetries in N=2 supergravity*, *Phys.Lett.* **B366** (1996) 155–159, [[hep-th/9510074](#)].
- [143] R. D'Auria, S. Ferrara, and S. Vaula, *N=4 gauged supergravity and a IIB orientifold with fluxes*, *New J.Phys.* **4** (2002) 71, [[hep-th/0206241](#)].
- [144] A. R. Frey and J. Polchinski, *N=3 warped compactifications*, *Phys.Rev.* **D65** (2002) 126009, [[hep-th/0201029](#)].
- [145] L. Andrianopoli, R. D'Auria, S. Ferrara, and M. A. Lledo, *4-D gauged supergravity analysis of type IIB vacua on $K3 \times T^{**2} / Z(2)$* , *JHEP* **0303** (2003) 044, [[hep-th/0302174](#)].
- [146] C. Angelantonj, R. D'Auria, S. Ferrara, and M. Trigiante, *$K3 \times T^{**2} / Z(2)$ orientifolds with fluxes, open string moduli and critical points*, *Phys.Lett.* **B583** (2004) 331–337, [[hep-th/0312019](#)].
- [147] J. Bagger and E. Witten, *Matter Couplings in N=2 Supergravity*, *Nucl.Phys.* **B222** (1983) 1.
- [148] K. Galicki, *a generalization of the momentum mapping construction for quaternionic Kahler manifolds*, *Commun.Math.Phys.* **108** (1987) 117.
- [149] E. Bergshoeff, S. Cucu, T. De Wit, J. Gheerardyn, R. Halbersma, *et. al.*, *Superconformal N=2, D = 5 matter with and without actions*, *JHEP* **0210** (2002) 045, [[hep-th/0205230](#)]. Erratum added online, 03/24/2004.
- [150] R. D'Auria and S. Ferrara, *On fermion masses, gradient flows and potential in supersymmetric theories*, *JHEP* **0105** (2001) 034, [[hep-th/0103153](#)].
- [151] A. Van Proeyen, *Supergravity with Fayet-Iliopoulos terms and R-symmetry*, *Fortsch.Phys.* **53** (2005) 997–1004, [[hep-th/0410053](#)].

- [152] A. Ceresole, R. D'Auria, S. Ferrara, and A. Van Proeyen, *Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity*, *Nucl.Phys.* **B444** (1995) 92–124, [[hep-th/9502072](#)].
- [153] P. Fre, L. Girardello, I. Pesando, and M. Trigiante, *Spontaneous $N=2$; $N=1$ local supersymmetry breaking with surviving compact gauge group*, *Nucl.Phys.* **B493** (1997) 231–248, [[hep-th/9607032](#)].
- [154] P. Fre, M. Trigiante, and A. Van Proeyen, *Stable de Sitter vacua from $N=2$ supergravity*, *Class.Quant.Grav.* **19** (2002) 4167–4194, [[hep-th/0205119](#)].
- [155] T. Vachaspati and A. Achucarro, *Semi-local cosmic strings*, .
- [156] R. Leese, *The Stability of semilocal vortices at critical coupling*, *Phys.Rev.* **D46** (1992) 4677–4684.
- [157] R. Leese and T. Samols, *Interaction of semilocal vortices*, *Nucl.Phys.* **B396** (1993) 639–669.

Samenvatting

De laatste 10–20 jaar is er veel interesse om kosmologische modellen tot stand te brengen binnen de context van de belangrijkste theorieën die de Natuur op zeer hoge energieschalen beschrijven, zoals Algehele Unificatie Theorieën (*Grand Unified Theories*), supersymmetrische theorieën en Snaartheorie.

De voornaamste reden hiervoor is dat de verschijnselen die door deze theorieën beschreven worden, zich afspelen op energieschalen ver boven het bereik van deeltjesversnellers op aarde, zowel de huidige deeltjesversnellers als die in de nabije toekomst. Kosmologie daarentegen biedt een uitstekende mogelijkheid om deze theorieën te testen, dankzij de extreem hoge energieën in het allereerste begin van de evolutie van het heelal.

Het standaard model binnen de kosmologie heet het Λ CDM-model. Dit is vernoemd naar de belangrijkste ingrediënten in het heelal: de kosmologische constante Λ , welke verantwoordelijk is voor de waargenomen versnelde uitdijning van het heelal, en de koude donkere materie (*Cold Dark Matter*), een exotische, nieuwe soort materie die alleen opgemerkt wordt door zijn zwaartekrachtinteractie met overige materie. Het Λ CDM-model geeft een zeer succesvolle verklaring voor onder andere de relatieve aanwezigheid van de lichtste elementen, de vorming van sterrenstelsels, en de verdeling van sterrenstelsels op grote lengteschaal. Veel vragen zijn echter nog onbeantwoord, zoals bijvoorbeeld de microscopische verklaring voor de kosmologische constante, en uit welk soort deeltjes donkere materie bestaat.

Een ander probleem van het Λ CDM-model is gerelateerd aan de beginvoorwaarden van het heelal, waaruit de nu waargenomen verdeling van sterrenstelsels is voortgekomen. In de tegenwoordig geaccepteerde ideeën over de vorming van sterrenstelsels komt de structuur van het heelal op de grootste schaal voort uit

kleine fluctuaties in de energiedichtheid in het zeer jonge heelal. Deze fluctuaties groeiden, met name door de gravitationele invloed van donkere materie, uit tot de huidige kosmische structuur, en kunnen direct waargenomen worden als temperatuurschommelingen in de kosmische microgolf achtergrondstraling (CMB, *Cosmic Microwave Background radiation*). De kleine fluctuaties vinden hun oorsprong in quantumfluctuaties, welke opgerekt werden tot kosmische proporties tijdens een vroege periode van extreem versnelde uitdijing, genaamd inflatie.

Hoewel het temperatuurspectrum van de CMB consistent is met de voorspellingen van inflatie, zijn de fysische processen die inflatie veroorzaken nog onduidelijk. In de simpelste modellen wordt inflatie veroorzaakt door de vacuümenergie van een quantumveld —net zoals in het Higgs-model— dat langzaam naar beneden rolt in een zeer vlakke potentiaal. Zo'n veld wordt het inflaton genoemd. Het ontwikkelen van modellen voor inflatie kan een zeer nuttige informatiebron zijn over het gedrag van de natuur bij zeer hoge energieën. Aan de ene kant verwachten we dat de eigenaardigheden van de theorieën waarbinnen inflatie wordt beschreven, hun sporen nalaten in de CMB, terwijl we aan de andere kant over uiterst precieze metingen beschikken van het spectrum van de CMB. Die metingen komen momenteel van de WMAP-missie, maar nauwkeuriger metingen worden binnenkort verwacht dankzij waarnemingen met de Atacama Cosmology Telescope en de PLANCK-satelliet.

De onderwerpen van dit proefschrift zijn relevant voor kosmologische modellen die gebaseerd zijn op theorieën welke invariant zijn onder supersymmetrie. Supersymmetrie, een symmetrie die bosonen in fermionen transformeert en vice versa, vormt een aantrekkelijk raamwerk dat bijvoorbeeld kan verklaren waarom de zwakke kernkracht zo veel sterker is dan zwaartekracht. In dit proefschrift richten we onszelf met name op superzwaartekracht, een uitbreiding van de Algemene Relativiteitstheorie die invariant is onder supersymmetrie. Superzwaartekracht werd oorspronkelijk voorgesteld om bepaalde divergenties die typisch verschijnen in elke theorie over quantumzwaartekracht, te repareren, maar dat idee bleek onsuccesvol. Tegenwoordig worden supersymmetrie en superzwaartekracht met name gebruikt om de belangrijkste quantumzwaartekrachttheorie, Snaartheorie, te beschrijven bij lage energieën.

De kosmologische modellen binnen Snaartheorie voorspellen typisch een groot aantal deeltjes, terwijl een model zijn voorspelbare kracht verliest naarmate het meer deeltjes bevat. Zodoende wordt in kosmologische modellen aangenomen dat het merendeel van de deeltjes een extreem grote massa heeft, zodat ze tijdens inflatie niet geproduceerd kunnen worden. Dit gebeurt deels om controle te krijgen over de analyse, maar ook omdat het spectrum van fluctuaties in de CMB accuraat voorspeld wordt door modellen van inflatie met slechts één deeltje. Aangezien de zware deeltjes niet voorkomen tijdens inflatie, kunnen we

een supersymmetrische theorie bij lage energie gebruiken, waarin inflatie wordt geïmplementeerd met een minimum aantal deeltjes. Bovendien weten we dat supersymmetrie gebroken wordt bij lagere energieën, zowel tijdens inflatie als nu (het spectrum van deeltjes in het Standaard Model is niet consistent met supersymmetrie), waardoor de sector die overblijft bij lage energieën automatisch het mechanisme moet bevatten dat supersymmetrie breekt.

De mogelijke interacties tussen deeltjes in de beschrijving van superzwaartekracht kennen sterke beperkingen, en doordat zwaartekracht door alles wordt gevoeld, is het niet-triviaal om een theorie consistent te beperken tot een enkele sector en tegelijkertijd supersymmetrie te behouden. Bovendien is er geen eenvoudige manier om te garanderen dat de deeltjes die men negeert zwaar blijven, aangezien supersymmetrie gebroken wordt in de overgebleven sector. Als de beperking tot één sector niet op een consistente manier gedaan wordt, kan het voorkomen dat het inflaton koppelt aan de genegeerde deeltjes, waardoor er karakteristieke verschijnselen zichtbaar zijn in het spectrum van de CMB. Door deze effecten te bestuderen, kunnen we belangrijke informatie verkrijgen over de natuur bij hoge energieën.

In dit proefschrift bestuderen we de noodzakelijke eisen om een theorie consistent te beperken tot één sector terwijl tegelijk supersymmetrie wordt behouden. Voor die modellen waarbij aan deze eisen is voldaan, beschouwen we hoe de massa's van de genegeerde deeltjes beïnvloed worden door het breken van supersymmetrie in de sector met lichte deeltjes.

Hoewel inflatie alom geaccepteerd wordt als de belangrijkste oorzaak voor het ontstaan van schommelingen van de energiedichtheid in het vroege heelal, zijn de waarnemingen nog steeds compatibel met kleine bijdragen afkomstig van andere oorzaken, zoals kosmische snaren. Kosmische snaren zijn zeer dunne concentraties van energie, zoals een hele dunne draad, die zich uitstrekken over heel het heelal en die met relativistische snelheden bewegen. Het bestaan van kosmische snaren is een algemene voorspelling van veel veelbelovende kosmologische modellen die gebaseerd zijn op supersymmetrische Algehele Unificatie Theorieën en Snaartheorie.

In het algemeen zijn oplossingen van kosmische snaren in supersymmetrische theorieën niet invariant onder supersymmetrie, oftewel de snaren breken supersymmetrie. Het is echter nog steeds mogelijk om oplossingen te vinden met kosmische snaren die een deel van de supersymmetrie ongebroken laten. Deze noemen we supersymmetrische kosmische snaren.

Supersymmetrische kosmische snaren zijn met name interessant omdat ze, dankzij hun specifieke quantumeigenschappen, informatie kunnen verschaffen over de natuurkundige beschrijving van het universum bij zeer hoge energieën.

Aangezien het heelal heter is en meer energie bevat naarmate de tijd wordt teruggedraaid, kan de studie van supersymmetrische kosmische snaren licht werpen op de kenmerken van het vroege, allereerste stadium in de evolutie van het heelal.

Supersymmetrische theorieën kunnen invariant zijn onder één of meerdere supersymmetrieën en worden als zodanig geklassificeerd middels het aantal supersymmetrieën $\mathcal{N}$ waaronder ze invariant zijn. Aangezien we geen supersymmetrie waarnemen in de natuur, is het lastiger om contact te maken met de experimentele data naarmate een theorie meer supersymmetrie bevat. Vandaar dat, wat betreft supersymmetrische theorieën, met name de theorieën die invariant zijn onder één enkele supersymmetrie, d.w.z. $\mathcal{N} = 1$ theorieën, gebruikt worden om de natuur te beschrijven.

Echter, bij het construeren van kosmologische modellen uit Snaartheorie, is het eindresultaat doorgaans een supersymmetrische theorie die invariant is onder meer dan één supersymmetrie, $\mathcal{N} > 1$. In dit proefschrift bestuderen we een nieuw type oplossingen van kosmische snaren in $\mathcal{N} = 2$ superzwaartekracht. Kosmologische modellen die gebaseerd zijn op $\mathcal{N} = 2$ theorieën zijn vooral interessant, omdat ze als brug gebruikt kunnen worden tussen de meer fenomenologische $\mathcal{N} = 1$ theorieën en Snaartheorie.

In dit proefschrift presenteren we één van de slechts twee bekende voorbeelden van oplossingen van kosmische snaren in een $\mathcal{N} = 2$ superzwaartekracht-model. We leiden af dat de breedte van de kosmische snaren kan veranderen zonder dat dat energie kost. Binnen een kosmologische context zijn kosmische snaren gevoelig voor verschillende verstoringen en, doordat het veranderen van de straal van de snaar geen energie kost, kunnen zulke verstoringen ertoe leiden dat de snaren breder worden tot op het punt waarop ze volledig oplossen. Ons werk verkent deze mogelijkheid.

Publications

- ‘*A Note on the stability of axionic D-term strings.*’ Ana Achúcarro and Kepa Sousa. *Phys.Rev.* **D74** (2006) 081701; [hep-th/0601151](#).
- ‘*Half-BPS cosmic string in $\mathcal{N} = 2$ supergravity in the presence of a dilaton.*’ Mboyo Esole and Kepa Sousa. *JHEP* **0703** (2007) 079; [hep-th/0610124](#).
- ‘*F-term uplifting and moduli stabilization consistent with Kahler invariance.*’ Ana Achúcarro and Kepa Sousa. *JHEP* **0803** (2008) 002; [arXiv:0712.3460](#).
- ‘*Consistent decoupling of heavy scalars and moduli in $\mathcal{N} = 1$ supergravity.*’ Ana Achúcarro Sjoerd Hardeman and Kepa Sousa. *Phys.Rev.* **D78** (2008) 101901; [arXiv:0806.4364](#).
- ‘*F-term uplifting and the supersymmetric integration of heavy moduli.*’ Ana Achúcarro, Sjoerd Hardeman and Kepa Sousa. *JHEP* **0811** (2008) 003; [arXiv:0809.1441](#).

Curriculum Vitæ

I was born in Berango, Spain, on the 7th of December of 1981. I attended high school in Getxo, where I graduated with honours in 1999. The same year I started my undergraduate studies in Physics at the University of the Basque Country (UPV/EHU). For my third year, that is in 2002, I was granted the SENECA scholarship which allowed me to continue my studies at the University of La Laguna (Spain). A year after, 2003-2004, I participated in the TASSEP exchange program between universities in Europe, Canada and U.S.A., finishing my fifth and last year of undergraduate studies at the University of Texas at Austin. During my stay in Austin I collaborated for four months in the Dr. Erskines Surface Physics Laboratory, working on a project about the response of films under magnetic fields. I graduated in 2004, receiving the award to the "Best Academic Record in Physics" granted by the University of the Basque Country for the students with the best grades. In 2004 I started my PhD under the supervision of Prof. dr. Ana Achúcarro at the University of Leiden, (The Netherlands), with the help of an FPI grant (Research Training Award) offered by the Basque Government. From 2007 to 2008 my position was funded by the NWO (Dutch Organization for Scientific Research) via the VICI grant 680-47-320. During my stay in Leiden I taught the exercise class of General Relativity, and also that of Elementary Particle Physics for two years. The last stages of my PhD have taken place at the Department of Theoretical Physics and History of Science of the University of the Basque Country, where I have spent 7 months, and I am currently collaborating with Dr. J. Urrestilla.

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