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Chapter 5

The special linear group and a solution to the inverse problem

This chapter contains the most important results of the thesis. Namely, we show that every \( \hat{\mathcal{C}} \)-ring \( R \) can be realized as the universal deformation ring of a continuous linear representation of a profinite group. The example we use for this goal is the special linear group \( G := \text{SL}_n(R) \) together with the natural representation (induced by the reduction \( R \twoheadrightarrow k \)) in \( \text{GL}_n(k) \), with the assumption \( n \geq 4 \). This is the main result of the chapter. We moreover discuss similar representations for \( n = 2, 3 \) and the results of our considerations may be summarized as follows:

**Theorem 5.1.** Let \( R \) be a complete noetherian local ring with a finite residue field \( k \), \( n \geq 2 \) and consider the natural representation \( \bar{\rho} \) of \( \text{SL}_n(R) \) in \( \text{GL}_n(k) \). Then \( R \) is the universal deformation ring of \( \bar{\rho} \) if and only if \( (n, k) \notin \{ (2, \mathbb{F}_2), (2, \mathbb{F}_3), (2, \mathbb{F}_5), (3, \mathbb{F}_2) \} \).

We conclude the chapter generalizing our considerations to the closed subgroups \( G \) of \( \text{GL}_n(R) \) that contain \( \text{SL}_n(R) \). We consider analogous representations \( \bar{\rho} \) of \( G \) (coming from the reduction \( R \twoheadrightarrow k \)) and discuss the problem whether, given \( G \), the corresponding \( \bar{\rho} \) has \( R \) as its universal deformation ring. Our results show, in particular, that this is not the case for \( G = \text{GL}_n(R) \), unless \( R = k \) and \( (n, k) \notin \{ (2, \mathbb{F}_2), (2, \mathbb{F}_3), (3, \mathbb{F}_2) \} \). On the other hand, for \( G = \{ A \in \text{GL}_n(R) \mid (\det A)^{#k-1} = 1 \} \), we obtain \( R \) as the universal deformation ring of the corresponding \( \bar{\rho} \) if and only if \( (n, k) \notin \{ (2, \mathbb{F}_2), (2, \mathbb{F}_3), (3, \mathbb{F}_2) \} \). We also show that, in contrast to the case
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\( G = \text{SL}_n(R) \), for some choices of \( G \) and \( R \) a universal deformation ring of the corresponding \( \overline{\rho} \) even does not exist.

This chapter is a modified version of author’s preprint [Dor]. Compared to the preprint, there are several changes in notation, order of exposition and even some of the proofs, but the mathematical content remains almost the same. The only improvement worth mentioning is presented in Remark 5.20, to which we would like to draw reader’s attention. It is a corollary of results discussed in the preceding chapter.

Remark. Similar results have been obtained independently by Eardley and Manoharmayum in their preprint [EM], published at almost the same time as the first version of [Dor]. However, the methods of both papers are different. The reader is encouraged to get familiar also with the approach of Eardley and Manoharmayum, which is based on cohomology computations. We present a more elementary and self-contained approach treating also some cases \( (n = 2; n = 4, k = \mathbb{F}_2; \text{the general linear group}) \) that [EM] does not cover.

5.1 Motivation

We begin describing and motivating a general framework in which we will be working in this chapter.

One often studies some naturally occurring group representations in order to understand better the structure of a given group. In our case, since we focus on the inverse problem, we are free to choose groups and representations the way it is convenient for us. Corollary 4.27 shows that it is sufficient to restrict to representations with an injective universal lift. Such a lift gives us a way of identifying the represented group with a subgroup of a general linear group and we naturally arrive at the following setup.

Let \( R \in \text{Ob}(\hat{\mathcal{C}}) \), \( n \in \mathbb{N} \) be given and suppose \( G \) is a closed subgroup of \( \text{GL}_n(R) \). Then it is a profinite group and the inclusion \( \iota_G : G \hookrightarrow \text{GL}_n(R) \) is a continuous representation of \( G \), lifting the residual representation \( \overline{\rho}_G := \pi_{mR} \iota_G \).
We are interested in finding a group $G$ such that $\iota_G$ is a universal lift of $\bar{\rho}_G$. The first and most obvious candidate to consider would be the group $\text{GL}_n(R)$ itself. Note that Rainone has studied the deformations of the identity map $\text{GL}_2(\mathbb{F}_p) \to \text{GL}_2(\mathbb{F}_p)$ in [Ra] and obtained $\mathbb{F}_p$ as the universal deformation ring for all $p > 3$. However, one quickly notices that in general the condition described in Proposition 2.33 may not be satisfied (see Example 5.23), so $\text{Def}_{\bar{\rho}_G}$ may even not be representable over $\tilde{\mathcal{C}}$. And even if it is, then not necessarily by $R$, as we will show at the end of this chapter.

The described problems with $G = \text{GL}_n(R)$ are some of the reasons why we turn our attention to the group $G = \text{SL}_n(R)$. A big advantage of this choice is that the special linear group has a nice set of generators satisfying many interesting properties (described in the next section), which will play a key role in our considerations.

## 5.2 Structure of the special linear group

In this section $R$ denotes a commutative ring and $n$ an integer. Moreover, we assume $n \geq 2$.

**Notation 5.2.** Let $a, b \in [n]$, $a \neq b$. We introduce the following notation for some of the elements of $\text{GL}_n(R)$.

- $t_{ab}^r := I_n + re_{ab}$, for $r \in R$.
- $d(r_1, \ldots, r_n)$, where $r_i \in R^\times$, is the diagonal matrix with consecutive diagonal entries $r_1, \ldots, r_n$.
- $d_{ab}^r := d(r_1, \ldots, r_n)$, where $r \in R^\times$ and $r_a := r$, $r_b := r^{-1}$, $r_i := 1$ for $i \neq a, b$.
- $\sigma_{ab}^r := I_n - e_{aa} - e_{bb} + re_{ab} - r^{-1}e_{ba}$, for $r \in R^\times$. 

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This notation suppresses \( n \) and \( R \), but that should not cause any problems as \( n \) will usually be fixed and \( R \) easily deducible from the element \( r \) used.

Lemma 5.3. The following relations hold for \( a, b, c, d \in [n] \), \( a \neq b \):

1. If \( r, s \in R \) then \( t^r_{ab} t^s_{ab} = t^{r+s}_{ab} \).
2. If \( r, s \in R \) and \( c \neq a, b \) then \( [t^r_{ab}, t^s_{bc}] = t^{rs}_{ac} \).
3. If \( r, s \in R \) and \( \{a, c\} \cap \{b, d\} = \emptyset \) then \( [t^r_{ab}, t^s_{cd}] = 1 \).
4. If \( r \in R \), \( D = d(\lambda_1, \ldots, \lambda_n), \lambda_i \in R^\times \) then \( D t^r_{ab} D^{-1} = t^{\lambda_i \lambda_r}_{ab} \).
5. If \( u \in R^\times \) then \( \sigma^u_{ab} = t^u_{ab} t^{-\frac{1}{u}}_{ba} t^u_{ab} \).
6. If \( u \in R^\times \) then \( d^u_{ab} = \sigma^u_{ab} \sigma^{-1}_{ab} \).
7. If \( u \in R^\times \), \( r, s \in R \) and \( u = 1 + rs \) then \( d^u_{ab} = t^r_{ab} t^s_{ba} t^{-s}_{ab} \).

Sketch of the proof. These identities follow directly from definitions and straightforward computations in which one uses the fact that \( e_{ab} e_{cd} = \delta_{bc} e_{ad} \) (\( \delta_{bc} \) being the Kronecker delta symbol).

Lemma 5.4. Given an ideal \( \mathfrak{a} \leq R \) let \( U_\mathfrak{a} := \text{SL}_n(R) \cap (I_n + M_\mathfrak{a}(a)) \) and \( V_\mathfrak{a} := \langle t^r_{ab} \in \text{GL}_n(R) \mid r \in \mathfrak{a} \rangle \). If \( R \) is local then \( U_{\mathfrak{a}^2} \leq V_{\mathfrak{a}} \leq U_{\mathfrak{a}} \). In particular, \( U_R = \text{SL}_n(R) \) is generated by all the elements of the form \( t^r_{ab} \).

Sketch of the proof. The inclusion \( V_{\mathfrak{a}} \leq U_{\mathfrak{a}} \) is obvious. For the other inclusion, let \( M \in U_{\mathfrak{a}^2} \). Observe that multiplying \( M \) by \( t^r_{ab} \) amounts to adding a multiple of one of its rows or columns to some other. We claim that performing such operations on \( M \) we may obtain a diagonal matrix lying in \( U_{\mathfrak{a}^2} \). If \( \mathfrak{a} \) is contained in the maximal ideal \( \mathfrak{m} \) of the ring \( R \) then all the diagonal entries of \( M \) are invertible and we may simply cancel all other entries proceeding row by row. In case \( \mathfrak{a} = R \) every row contains an invertible element (since \( \text{det} M \not\in \mathfrak{m} \)), so each diagonal entry either is invertible or becomes such after one of the described operations. We proceed as follows: make \( M(n, n) \) invertible, cancel all other entries in the \( n \)-th row and column, repeat the procedure recursively on the leading \( (n-1) \times (n-1) \) submatrix.

Every diagonal matrix in \( U_{\mathfrak{a}^2} \) may be decomposed as a finite product of matrices of the form \( d^r_{ab}, r \in (1 + \mathfrak{a}^2) \cap R^\times \). To finish the proof, we show that each of them is generated by some elements of the form \( t^r_{ab}, r \in \mathfrak{a} \). If \( \mathfrak{a} = R \),
the claim follows from relations (R5) and (R6) described in Lemma 5.3. If \( a \subseteq m \) we use relation (R7) together with the observation that every element of \( 1 + a^2 \) is a finite product of elements of the form \( 1 + rs \), where \( r, s \in a \).

**Lemma 5.5.** Assume \( R \) is local with residue field \( k \).

(i) Using the notation of Lemma 5.4 we have that for every proper finitely generated ideal \( a \triangleleft R \) there exists \( r \in \mathbb{N} \) such that the commutator subgroup \( U_a' \) contains \( U_a^r \).

(ii) If either \( n \geq 3 \) or \( k \not= \mathbb{F}_2, \mathbb{F}_3 \) and \( n = 2 \) then \( SL_n(R)' = SL_n(R) \).

**Sketch of the proof.** (i) Suppose first that \( n \geq 3 \). Then relation (R2) from Lemma 5.3 implies that \( V_{a^2} \subseteq V_a' \) and hence, using Lemma 5.4, we obtain \( U_{a^2} \subseteq U_a' \).

If \( n = 2 \) and a proper ideal \( a \triangleleft R \) is given, define \( b := \langle x^2 - 2x \mid x \in a \rangle \triangleleft R \). Due to relation (R4) from Lemma 5.3, for every \( x, y \in a \) we have \( \begin{pmatrix} 1 & (x^2 - 2x)y \\ 0 & 1 \end{pmatrix} = [d_{12}^{(1-x)}, t_{12}^y] \in U_a' \) and, analogously, \( \begin{pmatrix} 1 & (x^2 - 2x)y \\ 0 & 1 \end{pmatrix} \in U_a' \). Hence, \( V_{ba} \subseteq U_a' \). Observe that \( \{ x^3 \mid x \in a \} \subseteq b \). Indeed, for every \( x \in a \) we have \( x^3 \equiv 2x^2 \equiv 4x \) (mod \( b \)) and \( 4x = (x^2 + 2x) - (x^2 - 2x) \in b \). If \( a \) is finitely generated and \( l \in \mathbb{N} \) is the cardinality of some finite set of its generators then it is easy to observe that \( a^{2l+1} \subseteq \langle x^3 \mid x \in a \rangle \). Hence, \( a^{2l+2} \subseteq ba \) and we conclude (using also Lemma 5.4) that \( U_{a^{2l+4}} \subseteq V_{a^{2l+2}} \subseteq V_{ba} \subseteq U_a' \).

(ii) It is sufficient to show that generators of \( SL_n(R) \) lie in \( SL_n(R)' \). To this end use Lemma 5.4 and suitable relations from Lemma 5.3: (R2) in case \( n \geq 3 \) or (R4) in case \( n = 2, k \not= \mathbb{F}_2, \mathbb{F}_3 \).

**Lemma 5.6.** If \( M \in M_n(R) \) commutes with all \( t_{ab}^1 \in GL_n(R) \) then \( M \) is a scalar matrix.

**Proof.** The claim follows from the observation that \( t_{ab}^1 M = M t_{ab}^1 \) is equivalent to \( e_{ab} M = M e_{ab} \), which holds if and only if \( M(a, a) = M(b, b) \) and \( \forall x \neq a, y \neq b : M(x, a) = M(y, b) = 0 \).

## 5.3 The special linear group and deformations

Let us fix a finite field \( k \) and work in the resulting category \( \hat{C} \). The following assumption will be made for the whole of this section.
Assumption 5.7. Let $R \in \text{Ob}(\mathcal{C})$ and $n \geq 2$ be given, define $G := \text{SL}_n(R)$ and consider the representation $\bar{\rho} : G \to \text{GL}_n(k)$ induced by the reduction $R \to k$. We will denote by $\iota$ the inclusion $G = \text{SL}_n(R) \hookrightarrow \text{GL}_n(R)$ and by $\mathcal{J}$ the set $\{(a, b) \in [n] \times [n] \mid a \neq b\}$.

As a closed subgroup of $\text{GL}_n(R)$, the group $G$ is profinite and $\bar{\rho}$ is continuous. We are interested in the following question: is $\text{Def}_{\bar{\rho}}$ represented by $R$?

### 5.3.1 General observations

Recall that, according to Notation 4.4, we will denote by $[\iota]^* : h_R \to \text{Def}_{\bar{\rho}}$ the natural transformation that, given $S \in \text{Ob}(\mathcal{C})$, associates with $f \in \text{Hom}_R(R, S)$ the deformation $[f \circ \iota] \in \text{Def}_{\bar{\rho}}(S)$.

**Lemma 5.8.** (i) There exists a universal deformation ring of $\bar{\rho}$.

(ii) $G$ satisfies the $p$-finiteness condition ($\Phi_p$).

(iii) The ring $R$ is the universal deformation ring of $\bar{\rho}$ if and only if $\iota$ is a universal lift of $\bar{\rho}$.

(iv) The map $[\iota]^* : h_R \to \text{Def}_{\bar{\rho}}$ is injective.

**Proof.** (i) By Proposition 2.23, it is sufficient to show that $\text{Def}_{\bar{\rho}}$ satisfies properties (H3) and (H4). The latter follows from Lemma 5.6 and part (iv) of Proposition 2.23. For the former, we use part (iii) of the same proposition and Lemma 5.5. More precisely, we have $\ker \bar{\rho} = U_{m_R}$ and want to check that $\text{CHom}(U_{m_R}, \mathbb{Z}/p\mathbb{Z})$ is finite. This holds true since $U_{m_R}$ is open in $G$ (and hence the abelianization $U_{m_R}^{ab}$ is finite) by Lemma 5.5.

(ii) Similarly as above, for every $r \in \mathbb{N}_+$ the group $\text{CHom}(U_{m_R}^r, \mathbb{Z}/p\mathbb{Z})$ is finite, due to the finiteness of $U_{m_R}^{ab}$. Since $\{U_{m_R}^r \mid r \in \mathbb{N}_+\}$ forms a basis of open neighbourhoods of $G$, the claim follows.

(iii), (iv) Follow from Proposition 4.1.

We would like to point out that the first claim of Lemma 5.8 has only a motivating character. According to the last two claims of the lemma, in order to conclude that $R$ is a universal deformation ring of $\bar{\rho}$, it is enough to prove that $[\iota]^*$ is surjective. As the reader will observe, our arguments in
the next sections will not use the existence of a universal deformation ring of \( \hat{\rho} \), but will rather reprove it.

Preparing for the main argument, we present the following auxiliary result.

**Lemma 5.9.** Let \( S \in \text{Ob}(\mathcal{C}) \). If \( \xi \in \text{Def}_S(S) \) then \( \xi \in \text{im} [i]_S^k \) holds if and only if there exists a lift \( \rho \in \xi \) satisfying the following condition:

\[
\forall (a, b) \in \mathcal{J}, r \in R ~ \exists c_{ab}^r \in S : \rho(t_{ab}^r) = t_{ab}^{c_{ab}^r}.
\]

**Proof.** Every lift of \( \hat{\rho} \) of the form \( \text{GL}_n(f) \circ \iota \), \( f \in \text{Hom}_C(R, S) \) obviously satisfies (\( \diamond \)). Conversely, consider \( \rho \) satisfying (\( \diamond \)) and suppose first that \( n \geq 3 \). Conjugating with the diagonal matrix \( d(1, c_{12}^1, \ldots, c_{1n}^1) \in I_n + M_n(m_S) \) we obtain a lift \( \hat{\rho} \) strictly equivalent to \( \rho \). It satisfies (\( \diamond \)) as well and in addition \( \forall j \in [n] \setminus \{1\} : \hat{\rho}(t_{1j}^1) = t_{1j}^1 \), due to Lemma 5.3, (R4). We may thus suppose without loss of generality that \( c_{1j}^1 = 1 \) for all \( j \in [n] \setminus \{1\} \). Lemma 5.3, (R2) implies then that \( \forall (j, k) \in \mathcal{J} \), \( j, k \neq 1 \): \( c_{jk}^1 = c_{1k}^1/c_{1j}^1 = 1 \). Furthermore, for every \( j \in [n] \setminus \{1\} \) there exists \( k \in [n] \setminus \{1, j\} \), so \( c_{jk}^1 = c_{1k}^1/c_{1j}^1 = 1 \) as well. We conclude that \( \forall (a, b) \in \mathcal{J} : c_{ab}^1 = 1 \).

Due to Lemma 5.3, (R1) and (R2), the following relations are satisfied for all \( r, s \in R \) and pairwise distinct \( a, b, c \in [n] \):

\[
\begin{align*}
c_{ab}^{r+s} &= c_{ab}^r + c_{ab}^s \\
c_{ac}^{r+s} &= c_{ac}^r c_{bc}^s
\end{align*}
\]

Substituting in the second relation firstly \( r = 1 \), then \( s = 1 \), we see that the value \( c_{ab}^r \) with a fixed \( r \in R \) does not depend neither on \( a \), nor on \( b \). Denote this common value by \( \varphi(r) \). We have obtained a function \( \varphi : R \to S \), which is additive by the first relation and multiplicative by the second one, satisfies \( \varphi(1) = 1 \) and for which \( \varphi(r) \) and \( r \) have the same image in \( k \), i.e., \( \varphi \in h_R(S) \). Since \( G \) is generated by the elements of the form \( t_{ab}^r \) (Lemma 5.4) we conclude that \( \rho = \text{GL}_n(\varphi) \circ \iota \) and so \( [\rho] = [i]_S^k(\varphi) \in \text{im} [i]_S^k \).

Suppose now \( n = 2 \). We may similarly assume that \( c_{12}^1 = 1 \). Define \( \varphi, g : R \to S \) by \( \varphi(r) \) := \( c_{12}^r \) and \( g(r) := c_{21}^r \). We claim that \( g = \varphi \) and \( \varphi \in \text{Hom}_C(R, S) \), which clearly implies that \( [\rho] = [i]_S^k(\varphi) \in \text{im} [i]_S^k \). Since \( \varphi \) is additive by the relation (R1) of Lemma 5.3, \( \varphi(1) = 1 \) and for all \( r \in R \) the images of \( \varphi(r) \) and \( r \) in \( k \) coincide, we only need to check that \( \varphi \) is multiplicative. Furthermore, it is sufficient to check multiplicativity only
on $R^x$, because of additivity of $\varphi$ and the fact that every non-invertible $r \in R$ is a sum of two invertible elements (e.g. $r = (r - 1) + 1$). Similarly, it is sufficient to check that $g(r) = \varphi(r)$, for $r \in R^x$.

Let $r \in R^x$, $a := \varphi(r)$, $b := g(-r^{-1})$ and $\sigma_r := \begin{pmatrix} 0 & r \\ -r^{-1} & 0 \end{pmatrix}$. Applying relation (R5) of Lemma 5.3 twice, we get $t_{12}^{-1/r} t_{12}^{t_{12}^{-1/r} t_{12}^{-1/r} = \sigma_r = t_{21}^{-1/r} t_{21}^{t_{21}^{-1/r} t_{21}^{-1/r}}$, hence:

$$
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho(\sigma_r) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\begin{pmatrix} 1 + ab & 2a + a^2b \\ b & 1 + ab \end{pmatrix} = \rho(\sigma_r) = \begin{pmatrix} 1 + ab & a \\ 2b + ab^2 & 1 + ab \end{pmatrix}
$$

It follows that $a + a^2b = 0$, so $ab = -1$, since $a$ is invertible. This means $\varphi(r)g(r^{-1}) = 1$ and $\rho(\sigma_r) = \begin{pmatrix} 0 & \varphi(r) \\ -\varphi(r)^{-1} & 0 \end{pmatrix}$. Relation (R6) implies now $\rho(( R^{-1} )) = \rho(\sigma_r)\rho(\sigma_1)^{-1} = \begin{pmatrix} \varphi(r) \\ \varphi(r)^{-1} \end{pmatrix}$. As $R^x \ni r \mapsto ( R^{-1} )$ is a group homomorphism, we conclude that $\varphi$ is multiplicative on $R^x$. In particular, $\varphi(r)g(r^{-1}) = 1$ implies that $g(r^{-1}) = \varphi(r^{-1})$, so $g = \varphi$ on $R^x$. This finishes the proof. $\square$

5.3.2 Main result

**Theorem 5.10.** If $n \geq 4$ then $\iota$ is universal.

**Proof.** Let $S \in \text{Ob}(\mathcal{C})$. By Lemma 5.8, we only need to show that $[\iota]_S^*$ is surjective, i.e., that every lift of $\bar{\rho}$ to $S$ is strictly equivalent to $\rho_f := \text{GL}_n(f)|_C$ for some $f \in \text{Hom}_C(R, S)$. Moreover, we may restrict to the case $S \in \text{Ob}(\mathcal{C})$, since all rings in $\mathcal{C}$ are inverse limits of artinian rings.

For $S \in \text{Ob}(\mathcal{C})$ let $n(S)$ be the smallest $j \in \mathbb{N}$ such that $m_S^l = 0$. We proceed by induction on $n(S)$. For $n(S) = 1$, i.e., $S = k$, the statement is obvious. For the inductive step consider $S$ with $n(S) \geq 2$, a lift $\rho : G \to \text{GL}_n(S)$ of $\bar{\rho}$ and set $l := n(S) - 1$. By the inductive hypothesis, we may suppose (considering a strictly equivalent lift if necessary) that $\rho$ reduced to $S/m_S^l$ is induced by a morphism $g : R \to S/m_S^l$. For every $r \in R$ choose $p_r \in S$ such that $p_r \equiv g(r) \mod m_S^l$; for $r = 1$ we choose $p_1 = 1$. This way

$$
\forall (a,b) \in \mathfrak{J}, r \in R : \quad \rho(t_{ab}) = t_{ab}^p + M_{ab}^r, \quad \text{for some } M_{ab}^r \in M_{n \times n}(m_S^l).
$$

We will analyze the structure of the matrices $M_{ab}^r$ proving a series of claims. In the calculations we use the fact that $J := M_{n \times n}(m_S^l)$ is a two-sided ideal of $M_n(S)$ such that $m_S J = J^2 = 0$. 

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Claim 1. If \( a, b, c, d \in [n] \) are such that \( \{a, c\} \cap \{b, d\} = \emptyset \) then for all \( r, s \in R \):
\[
p_s(M_{ab}^{r}e_{cd} - e_{cd}M_{ab}^{r}) = p_r(M_{cd}^{s}e_{ab} - e_{ab}M_{cd}^{s})
\]

Proof. Since \( t_{ab}^{r} \) and \( t_{ab}^{s} \) commute (Lemma 5.3, (R3)), so do their lifts. Denoting \( t := t_{ab}^{pr} \), \( M := M_{ab}^{r} \), \( z := t_{cd}^{ps} \), \( N := M_{cd}^{s} \) we obtain:
\[
(t + M) (z + N) = (z + N) (t + M)
\]
\[
zt + tN + Mz = zt + zM + Nt
\]
\[
M(I_z + p_s e_{cd}) - (I_z + p_s e_{cd})M = N(I_z + p_r e_{ab}) - (I_z + p_r e_{ab})N
\]
\[
p_s(M_{ab}^{r}e_{cd} - e_{cd}M_{ab}^{r}) = p_r(M_{cd}^{s}e_{ab} - e_{ab}M_{cd}^{s})
\]

Claim 2. \( \forall(a, b) \in \mathfrak{A}, r \in R : \ M_{ab}^{r}(i, j) = 0 \) when \( i \neq a, j \neq b \) and \( i \neq j \).

Proof. If we fix \( r \in R \) and \( (a, b) \in \mathfrak{A} \) then given \( j \in [n] \backslash \{b\} \) we may choose \( d \in [n] \backslash \{a, b, j\} \) (here we use the assumption \( n \geq 4 \)). Such \( a, b, j, d \) satisfy the assumptions of Claim 1, so we obtain \( M_{ab}^{r}e_{jd} - e_{jd}M_{ab}^{r} = p_r(M_{jd}^{1}e_{ab} - e_{ab}M_{jd}^{1}) \). If \( i \in [n] \backslash \{a, j\} \) then a comparison of the \((i, d)\)-entries of both sides of the relation shows that \( M_{ab}^{r}(i, j) = 0 \).

Claim 3. \( \forall(a, b) \in \mathfrak{A}, r \in R : \ \tr M_{ab}^{r} = 0 \).

Proof. Lemma 5.5 implies that \( \det \rho(t_{ab}^{r}) = 1 \), while \( \det \rho(t_{ab}^{r}) = \prod_{i=1}^{n}(1 + M_{ab}^{r}(i, i)) = 1 + \tr M_{ab}^{r} \) by Claim 2.

Claim 4. \( \forall(a, b) \in \mathfrak{A}, r \in R : \ M_{ab}^{r}(i, i) = 0 \) for \( i \in [n] \backslash \{a, b\} \).

Proof. Consider \( r \in R \) and \( (a, b) \in \mathfrak{A} \). If \( c \in [n] \backslash \{a, b\} \) then since \( n \geq 4 \) we may choose \( d \in [n] \backslash \{a, b, c\} \). Let \( U_c := \{A \in \GL_n(S) \mid \forall x \in [n] \backslash \{c\} : A(x, c), A(c, x) \in m_{S}\} \). It is easy to see that \( U_c \) is a group and \( \chi : U_c \to S^\times \), \( A \to A(c, c) \) a group homomorphism (due to the fact that \( (m_{S})^2 = 0 \)). Moreover, \( p(t_{ab}^{r}), p(t_{cd}^{1}), p(t_{db}^{r}) \in U_c \) and since \([t_{ad}^{1}, t_{db}^{r}] = t_{ab}^{r} \) by Lemma 5.3, (R2), we have that \( \chi(p(t_{ab}^{r})) = [\chi(p(t_{ad}^{1})), \chi(p(t_{db}^{r}))] = 1 \). We conclude that \( M_{ab}^{r}(c, c) = 0 \).

Claim 5. \( \forall(a, b) \in \mathfrak{A}, r \in R : \ M_{ab}^{r}(a, a) = -M_{ab}^{r}(b, b) \).

Proof. This is an immediate consequence of Claim 3 and Claim 4.
Claim 6. If \( a, b, c, d \in [n] \) are such that \( \{a, c\} \cap \{b, d\} = \emptyset \) and \((a, b) \neq (c, d)\) then for all \( r, s \in R\):

\[
\begin{align*}
\left\{ \begin{array}{l}
p_s M_{ab}^r(a, c) = -p_r M_{cd}^s(b, d), \\
p_s M_{ab}^r(d, b) = -p_r M_{cd}^s(c, a).
\end{array} \right.
\]

Proof. Thanks to Claim 2 and Claim 4 the formula of Claim 1 reduces to

\[
p_s \left( M_{ab}^r(a, c)e_{ad} - e_{cb} M_{ab}^r(d, b) \right) = p_r \left( M_{cd}^s(c, a)e_{cb} - e_{ad} M_{cd}^s(b, d) \right),
\]

If \((a, b) \neq (c, d)\) then the coefficients at \( e_{ad} \) (resp. \( e_{cb} \)) on both sides must be equal. \(\Box\)

Claim 7. There exists \( X \in M_n(m_1^r) \) such that \( \forall (a, b) \in \mathfrak{J}, r \in R \exists c_{ab}^r \in m_1^r: \)

\[
M_{ab}^r = p_r (e_{ab} X - X e_{ab}) + c_{ab}^r e_{ab}.
\]

Proof. Let \((a, b) \in \mathfrak{J}\) and \( c, d \in [n] \setminus \{a, b\} \). The quadruple \((a, b, a, d)\) satisfies the assumptions of Claim 6, so \( M_{ab}^1(a, a) = -M_{ab}^1(b, d) \) (the first relation). Combining with Claim 5 we obtain \( M_{ab}^1(b, b) = M_{ad}^1(b, d) \), so \( M_{ay}^1(b, y) \) is independent of the choice of \( y \in [n] \setminus \{a\} \). We will denote this common value by \( Y(b, a) \). Analogously, using the quadruple \((a, b, c, d)\) and the second relation of Claim 6 we prove that the value of \( M_{ab}^1(x, a) \), with \( x \) ranging over \([n] \setminus \{b\}\), is constant. We will denote it \( X(b, a) \).

Setting \( X(a, a) := Y(a, a) := 0 \) for all \( a \in [n] \) we obtain well defined matrices \( X, Y \in M_n(m_1^r) \). Since \( M_{ab}^1(a, a) = -M_{ab}^1(b, d) \) by Claim 5, we have \( X(b, a) = -Y(b, a) \) for all \((a, b) \in \mathfrak{J}\), hence \( X = -Y \).

Consider \((a, b) \in \mathfrak{J}\) and \( c \in [n] \setminus \{b\} \). Then it is possible to find \( d \in [n] \) such that \( a, b, c, d \) satisfy the assumptions of Claim 6 (if \( a = c \) choose any \( d \in [n] \setminus \{a, b\} \), if \( a \neq c \) let \( d := b \); note that this argument relies only on the fact that \( n \geq 3 \)). The first relation gives \( \forall r \in R : M_{ab}^r(a, c) = -p_r M_{cd}^1(b, d) = p_r X(b, c) \). Similarly, \( \forall d \in [n] \setminus \{a\}, r \in R : M_{ab}^r(d, b) = -p_r X(d, a) \). We conclude that \( M_{ab}^r = p_r (e_{ab} X - X e_{ab}) + c_{ab}^r e_{ab} \). \(\Box\)

Let \( X \) be as in the last claim and consider the representation \( \tilde{\rho} := (I_n + X) \rho (I_n + X)^{-1} \). It follows that \( \tilde{\rho}(t_{ab}^r) = t_{ab}^{pr} + M_{ab}^r + X t_{ab}^{pr} - t_{ab}^{pr} X = t_{ab}^{\phi_{ab}(r)} \), where \( \phi_{ab}(r) := p_r + c_{ab}^r \). The lift \( \tilde{\rho} \), strictly equivalent to \( \rho \), satisfies thus (\( \Diamond \)) and Lemma 5.9 finishes the proof of the theorem. \(\Box\)
Corollary 5.11. Every $R \in \text{Ob}(\hat{C})$ can be obtained as a universal deformation ring of a continuous representation of a profinite group satisfying the condition $(\Phi_p)$.

5.4 Lower dimensions

In this section we continue working under Assumption 5.7 and discuss the possibility of extending Theorem 5.10 to the cases $n = 2$ and $n = 3$.

5.4.1 Case $n = 3$

Theorem 5.12. Suppose $n = 3$, $k \neq \mathbb{F}_2$. Then $\iota$ is universal.

Proof. A closer look at the proof of Theorem 5.10 shows that assuming Claim 2 and Claim 4 the rest of the argument would hold also for $n \geq 4$. We provide thus a different argument for both of the claims in case $n = 3$ and $k \neq \mathbb{F}_2$ (this second assumption is actually needed only for proving Claim 4). In what follows, we assume $[n] = \{a, b, c\}$.

A proof of Claim 2: Considering $(a, b) \in \mathfrak{F}$, $r \in R$ we need to show that $M^r_{ab}(i, j) = 0$ for $(i, j) \in \{(b, a), (c, a), (b, c)\}$. We see that the fact that $t^r_{ab}$ and $t^1_{ac}$ commute implies $M^r_{ab}(i, a) = 0$ for $i \neq a$, just as in the case $n \geq 4$. Similarly, the fact that $t^r_{ab}$ and $t^1_{cb}$ commute implies $M^r_{ab}(b, j) = 0$ for $j \neq b$.

A proof of Claim 4: Let $(a, b) \in \mathfrak{F}$, $r \in R$ and define $U_c$, $\chi$ just as in the case $n \geq 4$. We need to show $M^r_{ab}(c, c) = 0$. Making use of the assumption $k \neq \mathbb{F}_2$ we choose $\lambda \in R$ such that $\lambda \neq 0, 1 \mod m_0$ and consider the elements $d := d^\lambda_{ac}$, $t := t^{\lambda c}_{ab}$. According to Lemma 5.3, relation (R4), we have $[d, t] = t^\lambda_{ab} t^{\lambda c}_{ab} = t^r_{ab}$. Since $\rho(d), \rho(t), \rho(t^r_{ab}) \in U_c$, evaluating $\chi$ at $\rho(t^r_{ab})$ we conclude that $M^r_{ab}(c, c) = 0$, just as in the case $n \geq 4$. \hfill $\square$

As the following lemma shows, the case $k = \mathbb{F}_2$ must really be excluded in Theorem 5.12.

Proposition 5.13. Assume $n = 3$ and $k = \mathbb{F}_2$.

(i) There exists a lift $\rho_0$ of $\bar{\rho}$ to $\mathbb{Z}_2$.

(ii) There is no $R \in \text{Ob}(\hat{C})$ for which $\iota$ is universal.
Proof. (i) Since $\text{im} \, p \subseteq \text{SL}_3(\mathbb{F}_2)$, it is enough to prove the claim for $R = k$. There exists an irreducible 3-dimensional representation of $\text{SL}_3(\mathbb{F}_2)$ over the ring $\mathbb{Z}[\omega]$, $\omega = \frac{-1+i\sqrt{7}}{2}$, defined in [ATL] by

$$A := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \omega & -1-\omega \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. $$

In order to check that a representation of $\text{SL}_3(\mathbb{F}_2) = \langle A, B \rangle$ may be defined this way recall that $\text{SL}_3(\mathbb{F}_2)$ is known to be isomorphic to $\text{PSL}_2(\mathbb{F}_7)$, which has an abstract presentation $\langle S, T \mid S^7 = T^2 = (ST)^3 = (S^4T^4)^4 = 1 \rangle$ due to Sunday ([Su]). One checks directly that the defining relations are satisfied both by $T := A$, $S := BA$ and their proposed images. We obtain $\rho_0$ by sending $\omega$ to the root of $X^2 + X + 2$ that lies in $1 + 2\mathbb{Z}_2$.

(ii) The first part of the proposition implies the claim in case $R = k$. In the general case, the claim follows from this observation and Proposition 4.3. □

5.4.2 Case $n = 2$

Theorem 5.14. Suppose $n = 2$ and $k \neq \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$. Then $\iota$ is universal.

Proof. Let $S \in \text{Ob}(\hat{\mathcal{C}})$ and $\xi \in \text{Def}_p(S)$ be given. According to Lemma 5.8 we only need to show that $\xi \in \text{im} [i]^*_S$.

Define $H := \{ \bigl( \begin{smallmatrix} u & \alpha^{-1} \\ \alpha & 1 \end{smallmatrix} \bigr) \bigm| u \in \mu_R \bigr\}$, $M := R$ and let $\alpha : M \rightarrow k$ be the reduction modulo $m_0$. Due to the assumption $k \neq \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5$ there exists $\alpha \in \mu_R$ such that $\alpha^4 \neq 1$. Using the notation of Remark 3.6 we obtain $(\alpha \alpha^{-1}) \in H \cap \chi^{-1}(X_H)$. Therefore, restricting $\rho$ to the subgroup $G_{M,H}$ of $\text{SL}_2(R)$, we conclude from Remark 3.6 that there exist $\rho \in \xi$ and $f : R \rightarrow S$ such that $\rho|_H = \text{id}_H$ and $\rho(t_{12}^r) = t_{12}^r$ for every $r \in R$. Considering the representation $(\chi_1 \chi_0) \rho(\chi_0 \chi_1)$ we similarly obtain from Remark 3.6 that there exists $g : R \rightarrow S$ such that $\rho(t_{21}^r) = t_{21}^g(r)$ for every $r \in R$. We see that lift $\rho$ satisfies condition $(\diamond)$ of Lemma 5.9, hence $\xi = [\rho] \in \text{im} [i]^*_S$. □

Proposition 5.15. Assume $n = 2$ and $k \in \{ \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5 \}$.

(i) There exists a lift $\rho_0$ of $\bar{\rho}$ to, respectively, $\mathbb{Z}_2, \mathbb{Z}_3$ or $\mathbb{Z}_5[\sqrt{5}]$.

(ii) There is no $R \in \text{Ob}(\hat{\mathcal{C}})$ for which $\iota$ is universal.
Proof. (i) It is enough to prove the claim for \( R = k \). One easily checks that \( \text{SL}_2(\mathbb{F}_2) = \langle \tau \rangle \times \langle \varepsilon \rangle \), where \( \tau := \left( \begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix} \right) \), \( \varepsilon := \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \), and that \( \tau \mapsto \left( \begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix} \right) \), \( \varepsilon \mapsto \left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right) \) defines a lift of \( \text{SL}_2(\mathbb{F}_2) \) to \( \mathbb{Z}_2 \). In case \( p \in \{3, 5\} \) it is known ([Cox, §7.6]) that \( \text{SL}_2(\mathbb{F}_p) \) has presentation

\[ \langle A, B, C \mid A^p = B^3 = C^2 = ABC \rangle = \langle A, C \mid A^p = (A^{-1}C)^3 = C^2 \rangle, \]

realized for example by the following choice of generators: \( A := \left( \begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix} \right) \), \( C := \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \). Using this fact and defining \( t \in \mathbb{Z}_3 \) by \( t^2 = -2, t \equiv 2 \pmod{3} \) it is easy to check that

\[ \left( \begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix} \right) \mapsto \frac{1}{2} \left( \begin{smallmatrix} 1 & t+1 \\ t^{-1} & 1 \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \mapsto \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \]

extends to a lift of \( \text{SL}_2(\mathbb{F}_3) \) to \( \mathbb{Z}_3 \). Similarly, defining \( i, \varphi \in \mathbb{Z}_5[\sqrt{5}] \) by \( i^2 = -1, i \equiv 2 \pmod{5} \) and \( \varphi := \frac{1+i+\sqrt{5}}{2} \) we have that

\[ \left( \begin{smallmatrix} -1 & 0 \\ -1 & -1 \end{smallmatrix} \right) \mapsto \frac{1}{2} \left( \begin{smallmatrix} \varphi & i(\varphi^{-1}+1) \\ i(\varphi^{-1}-1) & \varphi \end{smallmatrix} \right), \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \mapsto \left( \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \]

extends to a lift of \( \text{SL}_2(\mathbb{F}_5) \) to \( \text{GL}_2(\mathbb{Z}_5[\sqrt{5}]) \).

(ii) The claim follows from (i) and Proposition 4.3, similarly as in Proposition 5.13. \( \square \)

Corollary 5.16. Combining Theorems 5.10, 5.12 and 5.14 with Propositions 5.13 and 5.15 we obtain Theorem 5.1 stated in the introduction to this chapter.

5.5 Special cases

It would be interesting to know what are the universal deformation rings of \( \bar{\rho} \) in the cases not treated by Theorem 5.12 and Theorem 5.14. We present a complete answer in case \( R = k \).

Proposition 5.17. The universal deformation ring of \( \bar{\rho} \) for \( n = 3 \) and \( R = \mathbb{F}_2 \) is \( \mathbb{Z}_2 \).

Proof. It is sufficient to check that the tangent space to \( \text{Def}_\bar{\rho} \) is zero dimensional. Indeed, this implies that a versal deformation ring \( R_\bar{\rho} \) of \( \text{Def}_\bar{\rho} \) exists (Proposition 2.23) and is a quotient of \( \mathbb{Z}_2 \) (Proposition 1.46). By
Proposition 5.13, it can not be a proper quotient of $\mathbb{Z}_2$, so it will follow that $R_\rho = \mathbb{Z}_2$. This last condition implies also that $R_\rho$ is a universal deformation ring of $\rho$.

We have to check that every deformation of $\rho$ to $S = \mathbb{F}_2[\varepsilon]$ is induced by a $\hat{\mathcal{C}}$-morphism $R \to S$. This can be done modifying the argument used in the inductive step of the proof of Theorem 5.10, in the special case $S = \mathbb{F}_2[\varepsilon]$, $R = \mathbb{F}_2$. In Theorem 5.10 we have assumed that $n \geq 4$, but, as mentioned in the proof of Theorem 5.12, this condition is crucial only for proving Claims 2 and 4; the rest of the argument uses only a weaker assumption $n \geq 3$. Moreover, in Theorem 5.12 we have presented an alternative argument for Claim 2 that holds in case $n = 3$. Therefore, the only difficulty lies in finding a different argument for Claim 4, which asserts that given $(a, b) \in \mathcal{J}$ and $r \in R$, we have $M_{ab}^r(i, i) = 0$ for all $i \in [n]\{a, b\}$. In our case, since $n = 3$, there is only one such $i$ for given $a$ and $b$. Moreover, since $R = \mathbb{F}_2$, the only non-trivial case is $r = 1$.

Suppose $[n] = \{a, b, c\}$ and set $t_{ab} := t_{ab}^1$, $M := M_{ab}^1$. We need to check that $M(c, c) = 0$. Note that $t_{ab}$ is of order 2 and so is its lift $\rho(t_{ab}) = t_{ab} + M$. Using the fact that $\text{char } S = 2$, we compute:

$$ I_n = (t_{ab} + M)^2 = I_n + t_{ab}M + Mt_{ab} = I_n + e_{ab}M + Me_{ab}. $$

In particular, comparison of $(a, b)$-entries yields: $M(a, a) + M(b, b) = 0$. Since $M(a, a) + M(b, b) + M(c, c) = \text{tr } M = 0$ by Claim 3, we conclude that $M(c, c) = 0$. \hfill $\Box$

**Lemma 5.18.** Let polynomials $f_n \in \mathbb{Z}[X]$ be defined recursively by $f_0 = 0$, $f_1 = 1$, $f_{n+1} = Xf_n - f_{n-1}$. Consider a commutative ring $R$ and a matrix $M \in M_2(R)$ such that $\det M = 1$ and at least one of its off-diagonal entries is not a zero divisor. If $n = 2k+1$ is an odd positive integer then $M^n = -I_n$ holds if and only if $t := \text{tr } M$ is a root of the polynomial $f_{k+1} - f_k$.

**Proof.** By Cayley-Hamilton, $M^2 = tM - 1$ and it is easy to check that $\forall n \geq 1: M^n = f_n(t)M - f_{n-1}(t)I_n$. It follows that $M^n = -I_n$ if and only if $f_n(t) = 0$ and $f_{n-1}(t) = 1$. If $I_n = (f_n, f_{n-1} - 1)$ is the ideal of $\mathbb{Z}[X]$ generated by $f_n$ and $f_{n-1} - 1$, then one easily proves by induction on $l$ that $\forall l \in \{0, \ldots, n-1\} : I_n = (f_{n-l} - f_l, f_{n-1-l} - f_{l+1})$. In particular, for $l = k$ we obtain $I_n = (f_{k+1} - f_k)$. \hfill $\Box$
Proposition 5.19. Assume \( n = 2 \) and \( k \in \{ \mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_5 \} \). The universal deformation rings of \( \bar{\rho} \) for \( R = k \) are, respectively: \( \mathbb{Z}_2, \mathbb{Z}_3[X]/(X^3 - 1) \) and \( \mathbb{Z}_5[\sqrt{5}] \).

Proof. For \( G = \text{SL}_2(\mathbb{F}_2) \) we observe that the \( \mathbb{F}_2G \)-module \( V_{\bar{\rho}} \) is projective. Indeed, for a field \( k \) of characteristic \( p \) and a finite group \( G \) with \( p \)-Sylow subgroup \( S \), a \( kG \)-module \( V \) is projective if and only if \( V \) is projective as a \( kS \)-module ([Alp, p. 66, Corollary 3]). In this case \( S \simeq \langle (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \rangle \cong C_2 \) (cyclic group of order 2) and \( V_{\bar{\rho}} \cong \mathbb{F}_2[C_2] \) is even a free \( \mathbb{F}_2S \)-module. The claim follows now from Proposition 2.34.

The case \( G = \text{SL}_2(\mathbb{F}_3) \) can be approached via Proposition 4.2. It follows from the discussion in [Cox, §7.2, §7.6] that \( G \cong G' \times C_3 \), where \( G' = \langle (\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \rangle \) is isomorphic to the quaternion group of order 8. Since \( G' \) has order coprime to \( p = 3 \), it follows from Proposition 2.34 that \( \mathbb{Z}_3 \) is the universal deformation ring for \( \bar{\rho}|_{G'} \). Proposition 5.15 shows that there exists a universal lift of \( \bar{\rho}|_{G'} \) that may be extended to \( G \). Moreover, it is easy to check that \( \text{Ad}(\bar{\rho})^{G'} = kI_n \). Thus, given that \( G/G' \cong C_3 \), the universal deformation ring of \( \bar{\rho} \) is \( \mathbb{Z}_3[C_3] \cong \mathbb{Z}_3[X]/(X^3 - 1) \).

In case \( R = \mathbb{F}_5 \) we will simply check that the lift described in Proposition 5.15 is universal. Consider \( S \in \text{Ob}(\hat{C}) \), \( \xi \in \text{Def}_{\bar{\rho}}(S) \) and let \( A := \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, C := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}_2(R) \); we moreover identify \( C \) with \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \text{SL}_2(S) \). Since \( H := \langle C \rangle \) is of order 4, it follows from Proposition 2.34 that there is precisely one deformation of \( \bar{\rho}|_H \) to \( S \). Hence, \( \xi \) has a representative \( \rho \in \xi \) satisfying \( \rho(C) = C \). We claim that there is precisely one \( \rho \in \xi \) satisfying this condition and such that the diagonal entries of \( \rho(A) \) are equal. Indeed, if \( \rho(A) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and \( X \in I_n + M_n(\mathfrak{m}_S) \) is a matrix commuting with \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) then there exist \( u, v \in S \) such that \( X = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \) and writing \( t := v/u \in \mathfrak{m}_S \) we obtain

\[
(\begin{pmatrix} u & v \\ -v & u \end{pmatrix} (\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\begin{pmatrix} u & v \\ -v & u \end{pmatrix})^{-1} = \frac{1}{1 + t^2} \left( \frac{a+ct+bt+dt^2}{c-at+dt-bt^2} \right).
\]

The equation \( a + ct + bt + dt^2 = d - bt - ct + at^2 \) is equivalent to \( t^2(d - a) + 2t(b + c) + (a - d) = 0 \) and has precisely one solution \( t \in \mathfrak{m}_S \), due to Hensel’s lemma.

Since \( A \) and \( C \) generate \( G \), a lift \( \rho \) is uniquely determined by \( \rho(A) \) and \( \rho(C) \). Note that \( \det \rho(A) = \det \rho(C) = 1 \) due to Lemma 5.5. Using all the above observations and the presentation of \( \text{SL}_2(\mathbb{F}_5) \) introduced in the proof
of Proposition 5.15, we see that deformations of $\tilde{\rho}$ to $S$ correspond bijectively with matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_n(S)$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \pmod{m_S}$, $\det M = 1$ and $M^5 = (M^{-1}C)^3 = -I_n$.

By Lemma 5.18, the last condition is equivalent to $\text{tr} M = 2a$ being a root of $f_3 - f_2 = X^2 - X - 1$ and $\text{tr}(M^{-1}C) = b - c$ being a root of $f_2 - f_1 = X - 1$. If $(2a)^2 = 2a + 1$ then solving the quadratic equation $b(b-1) + 1 - a^2 = 1 - \det M = 0$ we obtain that $b = \frac{1-i(1-2a)}{2}$ with $i^2 = -1$.

We conclude that the full set of conditions imposed on $a, b, c$ is as follows: $a = \frac{\varphi}{2}$, $b = \frac{1-i(1-\varphi)}{2}$, $c = b - 1$, where $\varphi^2 = \varphi + 1$, $\varphi \equiv -2 \pmod{m_S}$ and $i^2 = -1$, $i \equiv 2 \pmod{m_S}$. It follows that every deformation of $\tilde{\rho}$ to $S$ is induced by a morphism $\mathbb{Z}_5[\sqrt{5}] \to S$ applied to the universal lift defined in the proof of Proposition 5.15.

\begin{remark}
In view of Proposition 4.3, the results of this chapter provide a valuable information about the exceptional universal deformation rings in general. More precisely, we have that the universal deformation ring of $\tilde{\rho}$ has, depending on the case, $R \times \mathbb{F}_2 \mathbb{Z}_2$, $R \times \mathbb{F}_3 \mathbb{Z}_3[X]/(X^3-1)$ or $R \times \mathbb{F}_5 \mathbb{Z}_5[\sqrt{5}]$ as its quotient.
\end{remark}

\begin{remark}
The above results obtained for $n = 2$ seem to be not entirely new. For example, Rainone in [Ra] has considered the case $k = \mathbb{F}_2$ and Mazur mentions the case $k = \mathbb{F}_5$ in [Maz2, §1.9] though without giving a proof. Also Bleher and Chinburg obtained analogous results for an algebraically closed field in [BC3]. However, there does not seem to be an easy and complete treatment of all the cases in the literature.
\end{remark}

\begin{remark}
It is worth noting that even though in case $k = \mathbb{F}_2$ we have obtained the same universal deformation ring for $n = 2$ and $n = 3$, the $kG$-module $V_7$ is not projective when $n = 3$. Indeed, it is known ([Alp, p. 33, Corollary 7]) that if a $kG$-module $V$ is projective then the order of the $p$-Sylow subgroup $S$ of $G$ divides $\dim_k V$. Here $|S| = 8$ and $\dim_k V_p = 3$.
\end{remark}

## 5.6 The general linear group

Concluding this chapter, we turn back to the more general picture sketched in Section 5.1. Let $R \in \text{Ob}(\hat{\mathcal{C}})$ be given. We will consider the family

$$\mathfrak{S} := \{ G \leq \text{GL}_n(R) \mid G \text{ closed and } \text{SL}_n(R) \leq G \}$$
and the corresponding representations $\bar{\rho}_G$, defined as in Section 5.1. In particular, we want to analyze what results would be obtained in the preceding sections if we considered the general linear group instead of the special linear one.

All elements of $\mathfrak{F}$ are clearly profinite groups. Note that the determinant map gives a bijective correspondence between $\mathfrak{F}$ and closed subgroups of $R^\times$. In particular, every $G \in \mathfrak{F}$ is a normal subgroup of $GL_n(R)$. As mentioned in Section 5.1, not every $G \in \mathfrak{F}$ satisfies the necessary condition presented in Proposition 2.33.

Example 5.23. Let $R := \mathbb{F}_p[[X]]$. One may check that $R_1^\times \cong \mathbb{Z}_p^N$. Using the determinant map and isomorphism $R^\times \cong \mu_R \oplus R_1^\times$, we obtain that $\text{CHom}(GL_n(R), \mathbb{Z}/p\mathbb{Z})$ is infinite.

Consequently, $\text{Def}_{\bar{\rho}_G}$ need not be representable over $\hat{C}$. If it is, we will denote an object representing it by $R_u(G)$.

Proposition 5.24. Let $G, H \in \mathfrak{F}$ be such that $H \subseteq G$ and $\text{Def}_{\bar{\rho}_H}$ is represented by $R$. If $\text{CHom}(G/H, \mathbb{Z}/p\mathbb{Z})$ is finite then $\text{Def}_{\bar{\rho}_G}$ is represented by $R[(G/H)^P]]$. Otherwise it is not representable over $\hat{C}$.

Proof. It is an immediate consequence of Proposition 4.2 and Lemma 2.26.

We conclude that Rainone’s results about $GL_n(k)$, mentioned in Section 5.1, generalize much better and in a more natural way to the group $\mu L_n(R)$ defined below than to the group $GL_n(R)$:

Corollary 5.25. Suppose $(n, k) \notin \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (2, \mathbb{F}_5), (3, \mathbb{F}_2)\}$ and let $\bar{\rho} := \bar{\rho}_{GL_n(k)}$.

(i) Either $R_u(GL_n(R)) \cong R[[R_1^\times]]$ or $\text{Def}_{\bar{\rho}}$ is not representable over $\hat{C}$. In particular, $R$ represents $\text{Def}_{\bar{\rho}}$ if and only if $R = k$.

(ii) Let $\mu L_n(R) := \{A \in GL_n(R) \mid \det A \in \mu_R\}$. The set $\mathfrak{G}$ of all $G \in \mathfrak{F}$ for which $\text{Def}_{\bar{\rho}_G}$ is represented by $R$ coincides with the set $\{G \in \mathfrak{F} \mid G \subseteq \mu L_n(R)\}$.

Proof. (i) By Theorems 5.10, 5.12 and 5.14 we have $R_u(SL_n(R)) = R$, so we may apply Proposition 5.24 with $G = GL_n(R)$ and $H = SL_n(R)$. Since
$G/H \cong R^x \cong \mu_R \oplus R_1^x$, $\mu_R$ is of finite order coprime to $p$ and $R_1^x$ is a pro-$p$ group, we have that $(R^x)^p \cong R_1^x$. Hence, the first claim follows.

(ii) A similar reasoning as in part (i) shows that elements of $\mathcal{G}$ correspond (via the determinant map) with these closed subgroups of $R^x \cong \mu_R \oplus R_1^x$ that have a trivial pro-$p$ completion. Since $R_1^x$ is a pro-$p$ group and $\mu_R$ finite of order coprime to $p$, every closed subgroup of $\mu_R \oplus R_1^x$ is a product $A \oplus B$ of closed subgroups $A \leq \mu_R$ and $B \leq R_1^x$. Moreover, $(A \oplus B)^p \cong B$, so the elements of $\mathcal{G}$ correspond with subgroups of $\mu_R$. 

Remark 5.26. For $(n, k) \in \{(2, \mathbb{F}_2), (2, \mathbb{F}_3), (3, \mathbb{F}_2)\}$ there exists a lift of $\text{GL}_n(k)$ to $\mathbb{Z}_p$. Indeed, in case $k = \mathbb{F}_2$ we have $\text{GL}_n(k) = \text{SL}_n(k)$, so we already know it; for $n = 2$, $k = \mathbb{F}_3$ see [Ra] (it is also not difficult to check it directly, knowing that $\text{SL}_2(\mathbb{F}_3)$ lifts to $\mathbb{Z}_3$). This fact and a reasoning as in Proposition 5.13 show that in these cases $R$ does not represent $\text{Def}_{\rho_G}$ for any $G \in \mathfrak{F}$. 

If $n = 2$, $k = \mathbb{F}_5$ then $R_u(\text{SL}_n(R)) \not\cong R$, but $R_u(\mu L_n(R)) \cong R$. This can be proved using the proof of Theorem 5.14 with only a small modification. Namely, instead of $H = \{(u_{-1}) | u \in \mu_R\}$ we consider $H := \{(u_1) | u \in \mu_R\}$ and instead of $\alpha \in k^x$ satisfying the condition $\alpha^4 \neq 1$, we choose $\alpha$ such that $\alpha^2 \neq 1$. Then $(\alpha_1) \in H \cap \chi^{-1}(X_H)$ and a combination of Remark 3.6, Lemma 5.9 and Lemma 5.8 proves the claim. As a corollary, we conclude, using Proposition 5.24, that the first part of Corollary 5.25 holds also in the case $n = 2$, $k = \mathbb{F}_5$. 
