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Chapter 1

Complete noetherian local rings

The aim of this chapter is to recall several standard facts from commutative algebra and category theory that will be crucial for the rest of the thesis. We introduce the basic definitions, set the notation and recall the main properties that can be found in the literature. Proofs are omitted for brevity of the exposition.

1.1 Categories $\hat{\mathcal{C}}$ and $\mathcal{C}$

1.1.1 Definitions

The rings that are of main interest in this thesis are the complete, noetherian and local ones. Moreover, we will require their residue fields to be finite.

Notation 1.1. *The following notation will be widely used throughout the thesis:*

- *we reserve the symbols k and p for a finite field and its characteristic,*
- *the symbol $\mathrm{W}(k)$ stands for the ring of Witt vectors over k.*
- *whenever an element of some ring is denoted by ε, it is assumed that $\varepsilon \neq 0$ and $\varepsilon^2 = 0$. In particular, $k[\varepsilon] \cong k[X]/(X^2)$.*

Definition 1.2. Let k be a finite field. We will denote by $\hat{\mathcal{C}}$ the category of all complete noetherian local commutative rings with residue field k. Morphisms of $\hat{\mathcal{C}}$ are the local ring homomorphisms inducing the identity on k.

Definition 1.3. By $\mathcal{C}$ we will denote the full subcategory of artinian rings in $\hat{\mathcal{C}}$.

In what follows we will refer to the objects and morphisms of the category $\hat{\mathcal{C}}$ shortly as "$\hat{\mathcal{C}}$-rings" and "$\hat{\mathcal{C}}$-morphisms" (and analogously for the objects and morphisms of $\mathcal{C}$).

Remark 1.4. It is easy to check that, due to the finiteness of k, the category $\mathcal{C}$ coincides with the category of all finite $\hat{\mathcal{C}}$-rings.

Example 1.5. The ring $\mathrm{W}(k)$ is an object of $\hat{\mathcal{C}}$, but not of $\mathcal{C}$. The rings k and $k[\varepsilon]$ are examples of objects of both $\mathcal{C}$ and $\hat{\mathcal{C}}$.

The ring $k[\varepsilon]$ can be seen as a particular case of the following construction.

Example 1.6. We can identify the category $\mathfrak{V}$ of finite dimensional k-vector spaces with a full subcategory of $\mathcal{C}$. If $V \in \mathfrak{V}$, then we introduce the ring structure on the k-vector space $k[V] := k \oplus V$ by requiring $V^2 = 0$ and obtain an object of $\mathrm{Ob}(\mathcal{C})$. Moreover, for every V, $W \in \mathfrak{V}$ there is a bijective correspondence $f \leftrightarrow \mathrm{id} \oplus f$ between k-linear maps $f : V \to W$ and morphism $k[V] \to k[W]$ of $\mathcal{C}$.

Notation 1.7. *Let R be a $\hat{\mathcal{C}}$-ring. We will use the following notation:*

- $\mathfrak{m}_R$ *denotes the maximal ideal of R,*
- $R^\times$ *denotes the multiplicative group of R and $R^\times_{\equiv 1}$ denotes its subgroup $1 + \mathfrak{m}_R$,*
- μ_R *denotes the set $\{x \in R \mid x^{\#k-1} = 1\}$ of multiplicative representatives of the non-zero residue classes modulo $\mathfrak{m}_R$,*
- $\tau_R : k^\times \to \mu_R$ *denotes the Teichmüller lift of $k^\times$ to R.*

Remark 1.8. Note that, using the introduced notation, we have $R^\times \cong \mu_R \times R^\times_{\equiv 1}$.

The existence of the Teichmüller lift is a consequence of the following general and very useful property of complete rings.

Theorem 1.9 (Hensel's lemma)**.** *Let R be a ring that is complete with respect to an ideal I and let $f \in R[X]$ be a polynomial. If $a \in R$ is such that $f'(a)$ is invertible and $f(a) \equiv 0 \pmod{I}$, then there exists a uniquely determined $b \in R$ such that $f(b) = 0$ and $b \equiv a \pmod{I}$.*

Proof. See [Ei, Theorem 7.3]. □

In some of our arguments we will also use the following easy and well-known result.

Lemma 1.10. *Every surjective endomorphism of a noetherian ring is an automorphism.*

Finally, since we will very often be working with reductions modulo ideals and with quotient rings, we also introduce the following convention.

Notation 1.11. *For $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ and a proper ideal $I \lhd R$ the symbol π_I will denote the reduction homomorphism $R \to R/I$.*

Remark 1.12. Note that π_I, defined as above, is always a $\hat{\mathcal{C}}$-morphism. Indeed, it is clear that R/I is a local noetherian ring and that π_I induces an isomorphism on the residue fields. It is only less obvious that R/I is complete. Observe that its $\mathfrak{m}_{R/I}$-adic completion $\widehat{R/I}$ is isomorphic to $R/\widehat{I}$ ([Ei, Lemma 7.15]) and that $\widehat{I} = I$ follows from Krull's intersection theorem. Hence, $R/I \cong \widehat{R/I}$ is complete.

1.1.2 Structure theorems

The structure of complete noetherian local rings (with arbitrary residue fields) was studied by I. S. Cohen already in 1940's in his paper [Coh]. For the reader's convenience we quickly present here the most important implications of Cohen's results for $\hat{\mathcal{C}}$-rings. We refer to the original paper, but an interested reader can learn this topic also from popular books on commutative algebra, like [Mat] or [Ei].

Theorem 1.13. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a quotient of a power series ring in finitely many variables over $\mathrm{W}(k)$. Moreover, it contains precisely one ring that is a homomorphic image of $\mathrm{W}(k)$.*

Proof. See [Coh, Theorems 9 and 10.(b)] for the case $\operatorname{char} R = p$ and [Coh, Theorems 11, 12 and 13] for the case $\operatorname{char} R \neq p$. □

Note that Remark 1.12 implies a statement converse to the first claim: every quotient of a power series ring in finitely many variables over $\mathrm{W}(k)$ is in $\mathrm{Ob}(\hat{\mathcal{C}})$.

Corollary 1.14. *All $\hat{\mathcal{C}}$-rings have a natural $\mathrm{W}(k)$-algebra structure and $\hat{\mathcal{C}}$-morphisms coincide with local $\mathrm{W}(k)$-algebra homomorphisms.*

Remark 1.15. For a given $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, the structure map $\mathrm{W}(k) \to R$ takes $\mu_{\mathrm{W}(k)}$ to μ_R. In some applications we will find it useful to identify these two groups, cf. for example Definition 3.2.

We will also need the following result, which can be interpreted as an analog of E. Noether's normalization theorem.

Theorem 1.16. *Let $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ be such that either $\operatorname{char} R = 0$ and $\operatorname{ht} pR = 1$ or $\operatorname{char} R = p$. Then there exists a subring R_0 of R such that R_0 is isomorphic to a power series ring over $\mathrm{W}(k)/(\operatorname{char} R)$ and R is a finite R_0-module.*

Proof. See [Coh, Theorem 16]. □

Remark 1.17. The condition $\operatorname{ht} pR = 1$ is satisfied for example when p is not a zero-divisor in R (this is a consequence of Krull's "Hauptidealsatz").

Remark 1.18. Suppose R and R_0 are as in Theorem 1.16 and let $d := \dim R$ be the Krull dimension of R. By the properties of integral extensions, $\dim R = \dim R_0$, so $R_0 \cong k[[X_1, \ldots, X_d]]$ in the case $\operatorname{char} R = p$ and $R_0 \cong \mathrm{W}(k)[[X_1, \ldots, X_{d-1}]]$ in the case $\operatorname{char} R = 0$.

The structure of $\hat{\mathcal{C}}$-rings can also be better understood using the following observation connecting categories $\hat{\mathcal{C}}$ and $\mathcal{C}$.

Lemma 1.19. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is an inverse limit of $\mathcal{C}$-rings.*

Proof. For every $r \in \mathbb{N}$ the ring $R/\mathfrak{m}_R^r$ is artinian and $R \cong \varprojlim_{r \in \mathbb{N}} R/\mathfrak{m}_R^r$. □

Remark 1.20. Note that the converse statement is not true, i.e., not every limit of an inverse system of $\mathcal{C}$-rings is a $\hat{\mathcal{C}}$-ring. Indeed, such inverse limit need not be noetherian.

Corollary 1.21. *Every $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a profinite ring.*

Proof. Combine the above lemma with Remark 1.4. □

Corollary 1.22. *For every R, $S \in \mathrm{Ob}(\hat{\mathcal{C}})$ we have*

$$\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S) = \varprojlim_{r \in \mathbb{N}} \mathrm{Hom}_{\mathcal{C}}(R/\mathfrak{m}_R^r, S/\mathfrak{m}_S^r).$$

Proof. It is sufficient to combine the following two facts: $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S) = \varprojlim_{r \in \mathbb{N}} \mathrm{Hom}_{\hat{\mathcal{C}}}(R, S/\mathfrak{m}_S^r)$ and $\mathrm{Hom}_{\hat{\mathcal{C}}}(R, S/\mathfrak{m}_S^r) \cong \mathrm{Hom}_{\mathcal{C}}(R/\mathfrak{m}_R^r, S/\mathfrak{m}_S^r)$ for every $r \in \mathbb{N}$. □

1.1.3 Some categorical constructions

Fiber products

Definition 1.23. Given two $\hat{\mathcal{C}}$-morphisms $\pi_1 : R_1 \to S$ and $\pi_2 : R_2 \to S$ let us define

$$R_1 \times_S R_2 := \{(r_1, r_2) \in R_1 \times R_2 \mid \pi_1(r_1) = \pi_2(r_2)\}.$$

We will consider this set with the subring structure inherited from the ring $R_1 \times R_2$. For $i = 1, 2$, the canonical projections $R_1 \times_S R_2 \to R_i$ will be denoted by p_i.

$$\begin{array}{ccccc} & & R_1 \times_S R_2 & & \\ & {\scriptstyle p_1}\swarrow & & \searrow{\scriptstyle p_2} & \\ R_1 & & & & R_2 \\ & {\scriptstyle \pi_1}\searrow & & \swarrow{\scriptstyle \pi_2} & \\ & & S & & \end{array}$$

Example 1.24. If V, $W \in \mathfrak{V}$ then $k[V] \times_k k[W] \cong k[V \oplus W]$.

Lemma 1.25. *Consider the setup of Definition 1.23 and set $\tilde{R} := R_1 \times_S R_2$. Then:*

(i) If R_1, $R_2 \in \mathrm{Ob}(\mathcal{C})$ then $\tilde{R} \in \mathrm{Ob}(\mathcal{C})$.

(ii) If π_1, π_2 are surjective then $\tilde{R} \in \mathrm{Ob}(\hat{\mathcal{C}})$.

(iii) If $\tilde{R} \in \mathrm{Ob}(\hat{\mathcal{C}})$ then it is the fiber product (in the category $\hat{\mathcal{C}}$) of π_1 and π_2. If R_1, $R_2 \in \mathrm{Ob}(\mathcal{C})$ then it is the fiber product of π_1 and π_2 also in $\mathcal{C}$.

Sketch of the proof. Let $\mathfrak{m} := \mathfrak{m}_{R_1} \times \mathfrak{m}_{R_2}$ and $\tilde{\mathfrak{m}} := \mathfrak{m} \cap \tilde{R}$. We see that $\tilde{R}/\tilde{\mathfrak{m}} \cong k$, so $\tilde{\mathfrak{m}}$ is a maximal ideal of $\tilde{R}$. It is actually its only such ideal, since $\tilde{R}\backslash\tilde{\mathfrak{m}} \subseteq (R_1\backslash\mathfrak{m}_{R_1} \times R_2\backslash\mathfrak{m}_{R_2}) \cap \tilde{R} = \tilde{R}^{\times}$. Moreover, as a closed subring of the $\mathfrak{m}$-adically complete ring $R_1 \times R_2$, the ring $\tilde{R}$ is $\tilde{\mathfrak{m}}$-adically complete. We conclude that $\tilde{R}$ is in $\mathrm{Ob}(\hat{\mathcal{C}})$ if and only if it is noetherian.

If R_1 and R_2 are artinian, hence finite, then so is $\tilde{R}$ (see also Remark 1.4). Suppose now that π_1 and π_2 are surjective. Then so are p_1 and p_2. Let $K_i := \ker p_i$ ($i = 1, 2$) and observe that $K_1 \cap K_2 = \{0\}$. Since $\tilde{R}/K_2 \cong R_2$ is a noetherian $\tilde{R}$-module, so is its submodule $(K_1 + K_2)/K_2 \cong K_1/K_1 \cap K_2 = K_1$. We conclude that both K_1 and $\tilde{R}/K_1 \cong R_1$ are noetherian $\tilde{R}$-modules, so $\tilde{R}$ is noetherian as well.

The above arguments prove the first two claims. The last one can be easily deduced from the following facts. Firstly, $\tilde{R}$ is the fiber product of R_1 and R_2 in the category of rings. Secondly, p_1 and p_2 are $\hat{\mathcal{C}}$-morphisms ($\mathcal{C}$-morphisms in case R_1 and R_2 are artinian). □

Remark 1.26. In general $R_1 \times_S R_2$ need not be an object of $\hat{\mathcal{C}}$. For example, Mazur presents in [Maz1, p. 270] the following example, accredited to Brian Conrad:

$$\pi_1 : k[[X, Y]] \xrightarrow{\text{mod } Y} k[[X]], \qquad \pi_2 : k \hookrightarrow k[[X]].$$

The resulting ring $k[[X, Y]] \times_{k[[X]]} k \cong k + Yk[[X, Y]]$ is not noetherian. Indeed, its ideal $(Y, YX, YX^2, \ldots)$ is not finitely generated.

Coproducts

Let R_1, $R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ be given. It is known that, given a ring R, the coproduct in the category of commutative R-algebras is described by the tensor product. One can therefore expect the coproduct of R_1 and R_2 in $\hat{\mathcal{C}}$ to be related to $R_1 \otimes_{\mathrm{W}(k)} R_2$. Since this last ring does not necessarily belong to $\mathrm{Ob}(\hat{\mathcal{C}})$ (for example: it need not be complete), we make the following definition.

Definition 1.27. We define the completed tensor product $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ of $R_1,\ R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ as the completion of $R_1 \otimes_{\mathrm{W}(k)} R_2$ with respect to the maximal ideal $\mathfrak{m}_{R_1} \otimes R_2 + R_1 \otimes \mathfrak{m}_{R_2}$.

Lemma 1.28. *For every $R_1,\ R_2 \in \mathrm{Ob}(\hat{\mathcal{C}})$ the completed tensor product $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ is an object of $\hat{\mathcal{C}}$ and the coproduct (in category $\hat{\mathcal{C}}$) of R_1 and R_2.*

Sketch of the proof. One can check that $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ has also the following alternative descriptions (cf. [Maz1, §12]):

- $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2 := \varprojlim_{k \in \mathbb{N}} \left(R_1/\mathfrak{m}_{R_1}^k \otimes_{\mathrm{W}(k)} R_2/\mathfrak{m}_{R_2}^k \right)$,
- If $R_1 \cong \mathrm{W}(k)[[X_1, \ldots, X_n]]/(f_1, \ldots, f_s)$, $R_2 \cong \mathrm{W}(k)[[Y_1, \ldots, Y_m]]/(g_1, \ldots, g_r)$ then:

  $$R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2 \cong \mathrm{W}(k)[[X_1, \ldots, X_n, Y_1, \ldots, Y_m]]/(f_1, \ldots, f_s, g_1, \ldots, g_r).$$

It is clear from the definition that $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ is complete and local with residue field $k \otimes_{\mathrm{W}(k)} k \cong k$. The second of the above alternative descriptions shows that $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ is noetherian, while the first one, combined with Corollary 1.22, can be used for proving that $R_1 \hat{\otimes}_{\mathrm{W}(k)} R_2$ is the coproduct in category $\hat{\mathcal{C}}$. □

1.1.4 Tangent space

Definition 1.29. We define the cotangent space to $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ as the k-vector space $t_R^* := \mathfrak{m}_R/(\mathfrak{m}_R^2, p)$ and the tangent space as $t_R := \mathrm{Hom}_k(t_R^*, k)$. Given a $\hat{\mathcal{C}}$-morphism $R \to S$ we denote by $t_f^* : t_R^* \to t_S^*$ the k-linear map induced by f.

Remark 1.30. Note that $R/(\mathfrak{m}_R^2, p) \cong k \oplus t_R^* = k[t_R^*]$.

One reason why this notion turns out to be very useful in the study of complete noetherian local rings is the following lemma.

Lemma 1.31. *A $\hat{\mathcal{C}}$-morphism $f : R \to S$ is surjective if and only if $t_f^* : t_R^* \to t_S^*$ is surjective.*

Proof. (cf. [Sch, Lemma 1.1]) Observe that f is surjective if and only if $\mathfrak{m}_S \subseteq \operatorname{im} f$, which by Nakayama's lemma holds true if and only if $\mathfrak{m}_S \subseteq (\operatorname{im} f, \mathfrak{m}_S^2)$. Using the fact that $p \in \mathfrak{m}_S$ and $p \in \operatorname{im} f$ we see that f is surjective if and only if the composition $R \xrightarrow{f} S \twoheadrightarrow S/(\mathfrak{m}_S^2, p)$ is surjective. It is sufficient to observe that this map factors via $R/(\mathfrak{m}_R^2, p)$ and apply Remark 1.30. □

As a consequence of Lemma 1.31, we can determine the minimal number of variables needed in the presentation described in Theorem 1.13.

Corollary 1.32. *A ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ can be presented as an epimorphic image of the ring $\mathrm{W}(k)[[X_1, \ldots, X_d]]$ if and only if $d \geqslant \dim_k t_R^*$.*

Proof. The tangent space to $\mathrm{W}(k)[[X_1, \ldots, X_d]]$ is d-dimensional, so $d \geqslant \dim_k t_R^*$ holds for every quotient ring R of $\mathrm{W}(k)[[X_1, \ldots, X_d]]$.

By Nakayama's lemma, $\dim_k \mathfrak{m}_R/\mathfrak{m}_R^2$ is equal to the minimal number of generators of the ideal $\mathfrak{m}_R$, so $\dim_k t_R^*$ is the minimal number of generators of its image in $R/(p)$. We conclude that for $d \geqslant \dim_k t_R^*$ there exist $x_1, \ldots, x_d \in \mathfrak{m}_R$ such that $\mathfrak{m}_R = (x_1, \ldots, x_d, p)$. Lemma 1.31 implies then that the map $\mathrm{W}(k)[[X_1, \ldots, X_d]] \xrightarrow{X_i \to x_i} R$ is a well-defined surjective $\hat{\mathcal{C}}$-morphism. □

1.2 Set valued functors on $\hat{\mathcal{C}}$

This thesis addresses several questions related to the problem of representability of some specific functors $\hat{\mathcal{C}} \to \mathrm{Sets}$, namely, the functors of deformations of group representations. Before introducing them (which will be done in the next chapter) we want to recall some standard results concerning the representability of (covariant) functors $\hat{\mathcal{C}} \to \mathrm{Sets}$ in general.

To learn more about this topic, we recommend the paper [Sch] or [Maz1, §14- §20]. The reader might also find useful the short introduction on this topic contained in [By2, Chapter 1].

1.2.1 Tangent space

Definition 1.33. If F is a functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ then we define its tangent space as $t_F := F(k[\varepsilon])$.

Remark 1.34. This definition and the definition of the tangent space to a $\hat{\mathcal{C}}$-ring are closely connected. Namely, for $R \in \mathrm{Ob}(\hat{\mathcal{C}})$, the tangent spaces t_R and $t_{\mathrm{Hom}_{\hat{\mathcal{C}}}(R,-)}$ may be identified. See [Maz1, Proposition on p. 271].

We note that under some additional assumptions on F a natural k-vector space structure can be introduced on t_F ([Maz1, §15]).

Notation 1.35. *Let $k[\varepsilon] \times_k k[\varepsilon]$ denote the fiber product of two copies of the reduction map $\pi : k[\varepsilon] \twoheadrightarrow k$. We introduce the operation $+ : k[\varepsilon] \times_k k[\varepsilon] \to k[\varepsilon]$ defined by $(x + y_1\varepsilon,\ x + y_2\varepsilon) \mapsto x + (y_1 + y_2)\varepsilon$. Moreover, given $\alpha \in k$ we will denote by a_α the $\hat{\mathcal{C}}$-morphism $k[\varepsilon] \to k[\varepsilon]$ sending $x + y\varepsilon$ to $x + \alpha y\varepsilon$.*

Lemma 1.36. *Let us use the above notation and conventions introduced in Definition 1.23. Suppose $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ is a covariant functor such that:*

(1) $F(k)$ is a one-element set.

(2) The map $\Phi := (F(p_1), F(p_2)) :\ F(k[\varepsilon] \times_k k[\varepsilon]) \to F(k[\varepsilon]) \times F(k[\varepsilon])$ is a bijection.

Then the following operations:

- *scalar multiplication $k \times t_F \to t_F$ defined as $(\alpha, \xi) \mapsto F(a_\alpha)(\xi)$,*
- *addition $t_F \times t_F \to t_F$ defined as $(\xi_1, \xi_2) \mapsto F(+)(\Phi^{-1}(\xi_1, \xi_2))$,*

define a structure of a k-vector space on $F(k[\varepsilon])$.

Proof. See [Maz1, §15] or [Sch, Lemma 2.10]. □

Remark 1.37. Note that this structure is natural, in the sense that for every natural transformation $\Phi : F \to G$ of functors F and G satisfying properties (1) and (2), the map $\Phi_{k[\varepsilon]} : F(k[\varepsilon]) \to G(k[\varepsilon])$ is k-linear with respect to the introduced structure.

Remark 1.38. Assuming that F satisfies the following slightly stronger assumption:

$$F(k[V] \times_k k[W]) \cong F(k[V]) \times F(k[W]) \text{ for every } V, W \in \mathfrak{V},$$

we obtain for every $V \in \mathfrak{V}$ a canonical k-vector space structure on $F(k[V])$, such that $F(k[V]) \cong t_F \otimes_k V$.

1.2.2 Continuous functors

The functors in which we will be interested are continuous in the following sense.

Definition 1.39. A functor $F : \hat{\mathcal{C}} \to \text{Sets}$ is called continuous if and only if for every $R \in \text{Ob}(\hat{\mathcal{C}})$ the canonical map $F(R) \to \varprojlim_{l \in \mathbb{N}} F(R/\mathfrak{m}^l)$ is an isomorphism.

Since continuous functors are completely determined by their restrictions to $\mathcal{C}$, we could see them simply as functors defined on $\mathcal{C}$. Note that this subcategory has, for example, the advantage of being closed under fiber products, while $\hat{\mathcal{C}}$ does not have this property (see Lemma 1.25 and Remark 1.26).

On the other hand, there is a good reason to work in the full category $\hat{\mathcal{C}}$. Namely, we are interested in representability problems (see the next section) and a continuous functor that is representable in $\hat{\mathcal{C}}$ may restrict to a functor that is not representable in $\mathcal{C}$.

1.2.3 Representable functors and versal hulls

Notation 1.40. *Given a $\hat{\mathcal{C}}$-ring R, we will denote the functor* $\text{Hom}_{\hat{\mathcal{C}}}(R, -) : \hat{\mathcal{C}} \to \text{Sets}$ *by* h_R.

Definition 1.41. A functor $F : \hat{\mathcal{C}} \to \text{Sets}$ is called representable if and only if there exists $R \in \text{Ob}(\hat{\mathcal{C}})$ representing it, i.e., $R \in \text{Ob}(\hat{\mathcal{C}})$ such that there exists a natural isomorphism $h_R \to F$.

Note that if a functor is representable and a natural isomorphism as in Definition 1.41 is fixed, then the object representing it is uniquely unique, i.e., unique up to a canonical isomorphism (this is a consequence of Yoneda's lemma). Observe also that h_R or, more generally, representable functors are continuous.

We introduce next a slightly weaker notion.

Definition 1.42. Let F and G be functors $\hat{\mathcal{C}} \to \text{Sets}$. A natural transformation $F \to G$ is called *smooth* if for every surjection $B \twoheadrightarrow A$ in $\hat{\mathcal{C}}$ the induced map

$$F(B) \to F(A) \times_{G(A)} G(B)$$

is surjective. It is called *étale* if it is smooth and bijective on $k[\varepsilon]$.

Remark 1.43. Suppose F and G are continuous functors. Then the above definition is equivalent to the one in which we require the surjectivity property only for every surjection $B \twoheadrightarrow A$ in $\mathcal{C}$.

Remark 1.44. Suppose functors F, $G : \hat{\mathcal{C}} \to \mathrm{Sets}$ are such that $F(k)$ and $G(k)$ are one-element sets. If $\Phi : F \to G$ is a smooth transformation, then Φ is surjective on every $\hat{\mathcal{C}}$-ring R. Indeed, it is sufficient to apply the surjectivity property of Definition 1.42 to the reduction morphism $\pi_{\mathfrak{m}_R} : R \twoheadrightarrow k$.

Definition 1.45. We say that a ring $R \in \mathrm{Ob}(\hat{\mathcal{C}})$ is a *versal hull* for a functor $F : \hat{\mathcal{C}} \to \mathrm{Sets}$ if there exists a natural transformation $h_R \to F$ that is étale.

Observe that if R represents some functor, then it is also its versal hull; the converse implication does not hold in general. The versal hull, if it exists, is uniquely determined up to isomorphism which, however, may be not canonical.

We finish this subsection showing how the notion of a tangent space can be useful in representability problems.

Proposition 1.46. *If R_v is a versal hull of a functor F, then R_v can be presented as a quotient of $\mathrm{W}(k)[[X_1, \dots X_d]]$ if and only if $d \geqslant \dim_k t_F$.*

Proof. Combine Corollary 1.32 with the definition of a versal deformation ring, by which t_F and t_{R_v} are isomorphic. □

1.2.4 Schlessinger criteria

The continuity assumption is very useful, since it allows us to use the criteria developed by Schlessinger in his paper [Sch].

Theorem 1.47 (Schlessinger Criteria)**.** *Let F be a continuous functor $\hat{\mathcal{C}} \to \mathrm{Sets}$ satisfying the following property (**H0**): $F(k)$ is a one-element set. Observe that for every $\mathcal{C}$-morphisms*[†] *$\pi_1 : R_1 \to S$, $\pi_2 : R_2 \to S$ we obtain an induced map*

$$\Psi : \quad F(R_1 \times_S R_2) \to F(R_1) \times_{F(S)} F(R_2)$$

and let us define the following conditions:

[†]Let us emphasize: morphisms of $\mathcal{C}$, not $\hat{\mathcal{C}}$. Recall that $\hat{\mathcal{C}}$ is not closed under fiber products.

(***H1***) *Ψ is surjective whenever π_2 is a surjection.*

(***H2***) *Ψ is bijective whenever π_2 is the reduction $k[\varepsilon] \twoheadrightarrow k$.*

(***H3***) *$\dim_k t_F$ is finite.*

(***H4***) *Ψ is bijective whenever $\pi_2 = \pi_1$ is a surjection.*

*Then F has a versal hull if and only if it satisfies properties (**H1**)–(**H3**) and is representable if and only if it satisfies properties (**H1**)–(**H4**).*

Proof. See [Sch, Theorem 2.11]. □

Remark 1.48. Compared to Schlessinger's original formulation, there are some minor changes in the statement of this theorem. Firstly, the theorem was originally stated for functors $\mathcal{C} \to$ Sets. Secondly, Schlessinger requires the properties described in conditions (**H1**) and (**H4**) only for the so called "small surjections" ([Sch, Definition 1.2]). However, it is easy to check that these formulations are equivalent. See also [Sch, Remark 2.14]. Finally, Schlessinger does not require the residue field to be finite.

Remark 1.49. Property (**H0**) coincides with condition (1) of Lemma 1.36 and property (**H2**) implies condition (2) of the same lemma (it even implies the stronger condition of Remark 1.38), which makes the symbol $\dim_k t_F$ appearing in property (**H3**) well-defined. Alternatively, to avoid recurring to the definition of the vector space structure on t_F, we could phrase (**H3**) simply as "t_F is finite", relying on the fact that k is finite.