Enhanced Coinduction

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Chapter 5

Coinduction up-to

In the previous chapter, we have seen how up-to techniques enhance the proof method for bisimilarity. In the current chapter, we extend these results to a coalgebraic framework for up-to techniques that is applicable not only to bisimilarity but to a wide variety of coinductive predicates. For instance, this approach allows us to obtain sound up-to techniques for unary predicates such as divergence of processes and for binary predicates such as similarity, or language inclusion of weighted automata over an ordered semiring.

We build on the observation that coinductive predicates can be viewed as final coalgebras in a suitable category, so that the classical coinductive proof principle amounts to finality (explained in Section 3.2). We show that Pous's modular framework of compatible up-to techniques (Section 4.3) has a natural counterpart at this categorical level in terms of *compatible functors*, which are functors equipped with a suitable natural transformation. The modular aspect of this framework amounts to elementary manipulations and constructions on natural transformations. Moreover, the fact that every compatible functor yields a sound up-to technique turns out to be a basic result on distributive laws between functors.

In Section 3.3, we recalled how coinductive predicates can be studied in a structural and systematic way using *fibrations*, which provide an abstract notion of predicates. There, the coinductive predicate of interest is defined uniformly based on a lifting of the behaviour functor to a category of predicates. We instantiate the above mentioned framework of compatible functors within this fibrational setting, and consequently obtain a modular approach for defining and reasoning about up-to techniques for general coinductive predicates. In this setting, we introduce enhancements such as up-to-context, up-to-equivalence and up-to-behavioural equivalence. We prove their compatibility under conditions on the functor liftings under consideration.

By instantiating these abstract results we obtain concrete sound enhancements, with the results of Chapter 4 on bisimulation up-to as a special case. We treat divergence of processes as an example of a unary predicate, and inclusion of weighted automata as an example based on a non-standard version of up-to-context. Further, we apply the framework to prove the soundness of up-to techniques for *simulation* as introduced in [HJ04]. As a special case, we obtain that simulation up to context is compatible (sound) for any *monotone* GSOS specification (instantiated to GSOS for labelled transition systems, this means that there are no negative premises). This includes simulation up-to for languages as introduced in Chapter 2.

Outline. In the next section, we propose the notion of compatible functor. The (technical) heart of this chapter is Section 5.2, where we introduce the main up-to techniques and associated compatibility theorems. In Section 5.3, we show how to instantiate these theorems, and in Section 5.4 we derive the compatibility of simulation up-to for a mild restriction of abstract GSOS. In Section 5.5, we discuss related and future work, and provide a short summary of the soundness results.

5.1 Compatible functors

In Chapter 4, we have used Pous's lattice-theoretic framework of up-to techniques as a modular approach for proving the soundness of bisimulation up-to techniques. In the current section, we show how Pous's framework generalizes to a categorical setting, where complete lattices and monotone functions are replaced by categories and functors (Section 3.2.2).

In that categorical setting, proving a coinductive predicate determined by a given functor $F\colon \mathcal{C} \to \mathcal{C}$ amounts to the construction of a suitable F-invariant (F-coalgebra). In the current chapter, we introduce up-to techniques to construct F-invariants in an easier way; hence, these techniques can be seen as enhanced proof techniques for the coinductive predicate (final coalgebra) of F. However, we focus on proof techniques for constructing invariants and ignore the coinductive predicate, and therefore we do not depend on the existence of a final F-coalgebra.

In the definition below, the intuition is that F-invariants are the coinductive properties of interest, and $G\colon \mathcal{C} \to \mathcal{C}$ is a potential up-to technique.

- An *F-invariant up to G* is an FG-invariant, i.e., a coalgebra $R \to FGR$.
- G is *F-sound* if, for every FG-invariant, there exists a $\mathcal{C}$-arrow from its carrier into the carrier of an F-invariant.

It is easy to see that these definitions generalize the notions of invariants up-to and soundness from Section 4.3.

Recall that *compatibility* is the central notion of Pous's framework: given two monotone functions f, g on a complete lattice, g is said to be f-compatible if $g \circ f \subseteq f \circ g$. If g is f-compatible then it is sound, i.e., every f-simulation up to g is contained in an f-simulation (Theorem 4.3.2). This result is an instance of a more general fact from the theory of distributive laws between functors.

Theorem 5.1.1. *Suppose $\mathcal{C}$ is a category with countable coproducts, $F, G\colon \mathcal{C} \to \mathcal{C}$ are functors and $\gamma\colon GF \Rightarrow FG$ is a natural transformation. Then for any FG-coalgebra*

δ there is an F-coalgebra ϑ making the next diagram commute:

$$\begin{tikzcd} X \arrow[r, "\kappa_0"] \arrow[d, "\delta"'] & G^\omega X \arrow[d, "\vartheta"] \\ FGX \arrow[r, "F\kappa_1"'] & FG^\omega X \end{tikzcd}$$

Here $G^\omega X$ denotes the coproduct $\coprod_{i\in\mathbb{N}} G^i X$ of all finite iterations of G applied to X, with coproduct injections $\kappa_i \colon G^i X \to G^\omega X$.

This appears in the proof of [Bar03, Theorem 3.8], but for a complete presentation we include a proof.

Proof. Define $\vartheta_i \colon G^i X \to FG^{i+1}X$ inductively as $\vartheta_0 = \delta$ and

$$\vartheta_{i+1} = \; GG^i X \xrightarrow{G\vartheta_i} GFG^{i+1}X \xrightarrow{\gamma_{G^{i+1}X}} FGG^{i+1}X$$

Postcomposing these morphisms with the coproduct injections yields a cocone $(F\kappa_{i+1} \circ \vartheta_i \colon G^i X \to FG^\omega X)_{i\in\mathbb{N}}$ and by the universal property of $G^\omega X$ we obtain a coalgebra $\vartheta \colon G^\omega X \to FG^\omega X$. Commutativity of the diagram amounts to the base case ϑ_0. $\square$

(Alternatively, we can replace the countable coproduct G^ω by the free monad for G, assuming it exists. In this case, the result is an instance of the construction (3.14) in Section 3.5.1.)

If $\mathcal{C}$ is a preorder, then F and G are monotone functions, and the existence of a natural transformation amounts to compatibility as in Pous's framework. The fact that compatible functions are sound, is thus an instance of Theorem 5.1.1. Similarly, that f-compatible functions preserve the coinductive predicate defined by f (Lemma 4.3.6) is an instance of the fact that, if $\gamma \colon GF \Rightarrow FG$ is a distributive law, then a final F-coalgebra lifts to a final γ-bialgebra (Lemma 3.5.1). When $\mathcal{C}$ is a lattice, the fact that there is a G-algebra structure on the final coalgebra $Z = \mathsf{gfp}(F)$ simply means that $G(Z) \leq Z$ (cf. Lemma 4.3.6).

The main reason for studying compatible functions is their compositionality properties. To achieve a flexible approach to the construction of compatible functors, we define them as follows.

Definition 5.1.2. Let $F_1 \colon \mathcal{C}_1 \to \mathcal{C}_1$ and $F_2 \colon \mathcal{C}_2 \to \mathcal{C}_2$ be functors. We say a functor $G \colon \mathcal{C}_1 \to \mathcal{C}_2$ is (F_1, F_2)-*compatible* when there exists a natural transformation $\gamma \colon GF_1 \Rightarrow F_2 G$.

The pair (G, γ) is a morphism between endofunctors F_1 and F_2 in the sense of [LPW00]. In the remainder of this chapter, we often leave γ implicit, as the examples involve only categories that are preorders.

An important instance of the above definition is (F^n, F^m)-compatibility of a functor $G \colon \mathcal{C}^n \to \mathcal{C}^m$; in this case, we simply say that $G \colon \mathcal{C}^n \to \mathcal{C}^m$ is F-compatible. For example, coproduct then becomes a compatible functor by itself, rather than a way to compose compatible functors.

Proposition 5.1.3. *Compatible functors are closed under the following constructions:*

1. *composition: if G is (F_1, F_2)-compatible and G' is (F_2, F_3)-compatible, then $G' \circ G$ is (F_1, F_3)-compatible;*
2. *pairing: if $(G_i)_{i\in I}$ are (F_1, F_2)-compatible, then $\langle G_i\rangle_{i\in I}$ is (F_1, F_2^I)-compatible.*

Moreover, for any functor $F\colon \mathcal{C} \to \mathcal{C}$:

3. *the identity functor $\mathsf{Id}\colon \mathcal{C} \to \mathcal{C}$ is F-compatible;*
4. *the constant functor to the carrier of an F-coalgebra is F-compatible, in particular to the coinductive predicate defined by F (carrier of the final F-coalgebra), if it exists;*
5. *the coproduct functor $\coprod_I\colon \mathcal{C}^I \to \mathcal{C}$ is (F^I, F)-compatible.*

Proof. 1. By assumption we have natural transformations $\gamma\colon GF_1 \Rightarrow F_2G$ and $\gamma'\colon G'F_2 \Rightarrow F_3G'$, and composing them yields

$$G'GF_1 \xRightarrow{G'\gamma} G'F_2G \xRightarrow{\gamma' G} F_3G'G$$

which is a natural transformation of the desired type.

2. Given natural transformations $\gamma_i\colon G_iF_1 \Rightarrow F_2G_i$ for all $i \in I$, we have

$$\langle G_i\rangle_{i\in I}F_1 = \langle G_iF_1\rangle_{i\in I} \xRightarrow{\gamma} \langle F_2G_i\rangle_{i\in I} = F_2^I\langle G_i\rangle_{i\in I}$$

where $\gamma_X = ((\gamma_i)_X)_{i\in I}$ for any X.

Items 3 and 4 are trivial. For 5, we must find a natural transformation

$$\gamma\colon \coprod_I \circ F^I \Rightarrow F \circ \coprod_I .$$

On a component $(X_i)_{i\in I}$ it is defined using the universal property; applying F to the coproduct injections $\kappa_i\colon X_i \to \coprod_{i\in I} X_i$ yields a morphism $F\kappa_i\colon FX_i \to F\coprod_{i\in I} X_i$ for each $i \in I$. $\square$

In a lattice, the pointwise join of compatible functions is again compatible (Proposition 4.3.3). To retrieve this in the current setting, suppose $(G_i)_{i\in I}$ are (F_1, F_2)-compatible. Since the pairing of compatible functors is compatible, and the coproduct functor is compatible, composing them yields a compatible functor $\coprod_I \circ\langle G_i\rangle_{i\in I}$ (this is the coproduct of the functors G_i), which, in a lattice, is pointwise join of monotone functions. Further, in the next section we will see how to obtain the operator $\bullet$ defined in Equation (4.2) of Section 4.3, by combining a functor that composes relations with the pairing constructor.

Further compositionality could be obtained by defining a pair (G, G') of endofunctors to be F-compatible if there exists a natural transformation $\gamma\colon GF \Rightarrow FG'$. A suitable variant of Proposition 5.1.3 then allows to prove compatibility, modular in the shape of the functor F. A related approach is taken in [LLYL14]. In this chapter we do not consider such constructions, instead focusing on the combination of up-to techniques for a fixed functor F.

5.2 Compatibility results

In Section 3.3, we have seen how fibrations can be used to speak generally about coinductive predicates on coalgebras. In that approach, the invariants of interest are themselves coalgebras which live in the fibre above the carrier of a coalgebra in the base category.

In order to define both coinductive predicates and up-to techniques, we assume

- a bifibration $p\colon \mathcal{E} \to \mathcal{A}$ (see Section 3.3.1 for details);
- a coalgebra $\delta\colon X \to BX$ for a functor $B\colon \mathcal{A} \to \mathcal{A}$, and
- a lifting $\overline{B}\colon \mathcal{E} \to \mathcal{E}$ of B.

As explained in Section 3.3, the lifting $\overline{B}$ and the transition structure δ determine a functor on the fibre $\mathcal{E}_X$ above the carrier X of the coalgebra (X, δ), defined as follows:

$$\overline{B}_\delta = \delta^* \circ \overline{B}_X : \mathcal{E}_X \to \mathcal{E}_X\,.$$

We spell out the important definitions of invariants up-to, soundness and compatibility, for the functor $\overline{B}_\delta$. A *$\overline{B}_\delta$-invariant* is a coalgebra $R \to \overline{B}_\delta(R)$, where R is an object in $\mathcal{E}_X$. Given a functor $G\colon \mathcal{E}_X \to \mathcal{E}_X$, a *$\overline{B}_\delta$-invariant up to G* is a coalgebra $R \to \overline{B}_\delta(G(R))$.

Our interest is to find functors G that are *sound*, so that invariants up to G are a valid proof principle for the construction of $\overline{B}_\delta$-invariants. Instead of proving soundness, we focus on proving the stronger notion of *compatibility*. By definition, a functor $G\colon \mathcal{E}_X \to \mathcal{E}_X$ is $\overline{B}_\delta$-compatible if there exists a natural transformation

$$\gamma\colon G \circ \overline{B}_\delta \Rightarrow \overline{B}_\delta \circ G\,.$$

In the remainder of this section, we introduce three families of up-to techniques:

- behavioural equivalence (Section 5.2.1),
- equivalence closure (Section 5.2.2) and
- contextual closure (Section 5.2.3).

We prove their compatibility, based on conditions on the lifting $\overline{B}$ of B. As explained in the previous section, this suffices to show that they are sound, and that they can be combined in various ways to form new sound up-to techniques.

In Section 3.2.1 we associated to each coalgebra $\delta\colon X \to BX$ for a functor $B\colon \mathsf{Set} \to \mathsf{Set}$ a function b_δ, whose invariants are bisimulations. In the current setting, this can be obtained by choosing $\overline{B}$ to be the canonical relation lifting $\mathsf{Rel}(B)$ of B. Then:

$$\overline{B}_\delta(R) = \mathsf{Rel}(B)_\delta(R) = (\delta \times \delta)^{-1}(\mathsf{Rel}(B)(R)) = \mathsf{b}_\delta(R)$$

which means that $\overline{B}_\delta$-invariants are bisimulations on δ (Lemma 3.2.3). For all three types of up-to techniques, we study the canonical relation lifting as a special case, and retrieve all the b_δ-compatibility results from the previous chapter.

In Section 5.3 and Section 5.4, we consider examples and instances for liftings other than $\mathsf{Rel}(B)$, to obtain proof techniques for other coinductive predicates than bisimilarity.

5.2.1 Behavioural equivalence

The first technique that we introduce is up-to-behavioural equivalence. If $\delta\colon X \to BX$ is a coalgebra for a functor $B\colon \mathsf{Set} \to \mathsf{Set}$, then behavioural equivalence is the relation $\approx$ on its carrier given by $x \approx y$ iff $h(x) = h(y)$, where h is the coinductive extension of δ, i.e., the unique coalgebra morphism into the final coalgebra (assumed to exist), see Section 3.1. Now consider the function $\mathsf{bhv}_\delta\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ defined by

$$\mathsf{bhv}_\delta(R) = \approx \circ\, R \circ \approx .$$

To define bhv_δ more generally in the setting of a bifibration, observe that

$$\begin{aligned}\mathsf{bhv}_\delta(R) &= \{(x,y) \mid \exists u,v.\, h(x) = h(u),\, h(y) = h(v) \text{ and } (u,v) \in R\} \\ &= h^{-1}(\{(h(u),h(v)) \mid (u,v) \in R\}) \\ &= h^{-1}(h(R)) .\end{aligned}$$

But $h^{-1} \circ h$ is simply direct image followed by reindexing in the fibration $\mathsf{Rel} \to \mathsf{Set}$, i.e., $h^{-1}(h(R)) = h^* \circ \coprod_h(R)$ (see Section 3.3.1). Therefore, we can generalize the above function bhv_δ to an arbitrary bifibration $p\colon \mathcal{E} \to \mathcal{A}$, a functor $B\colon \mathcal{A} \to \mathcal{A}$ with a final coalgebra, and a coalgebra $\delta\colon X \to BX$ by defining the *behavioural equivalence closure* bhv_δ as

$$\mathsf{bhv}_\delta = h^* \circ \textstyle\coprod_h : \mathcal{E}_X \to \mathcal{E}_X$$

where h is the coinductive extension of δ. We sometimes write bhv instead of bhv_δ, if δ is clear from the context. In the predicate fibration $\mathsf{Pred} \to \mathsf{Set}$, we have

$$\mathsf{bhv}_\delta(P) = h^{-1}(h(P)) = h^{-1}(\{h(u) \mid u \in P\}) = \{x \mid \exists u \in P.\, h(x) = h(u)\} .$$

Our aim is to prove $\overline{B}_\delta$-compatibility of bhv_δ. This is an instance of the following result, which concerns a generalization of bhv_δ to arbitrary coalgebra morphisms (rather than the coinductive extension h).

Theorem 5.2.1. *Suppose that $(\overline{B}, B)$ is a fibration map. For any B-coalgebra morphism $h\colon (X,\delta) \to (Y,\vartheta)$, the functor $h^* \circ \coprod_h$ is $\overline{B}_\delta$-compatible.*

Proof. We exhibit a natural transformation

$$(h^* \circ \textstyle\coprod_h) \circ (\delta^* \circ \overline{B}_X) \Rightarrow (\delta^* \circ \overline{B}_X) \circ (h^* \circ \coprod_h)$$

obtained by pasting the 2-cells (natural transformations) $(a), (b), (c), (d)$ in the following diagram:

$$
\begin{tikzcd}
\mathcal{E}_X \arrow[r, "\overline{B}"] \arrow[dd, equal] \arrow[ddr, phantom, "\Downarrow(b)"] & \mathcal{E}_{BX} \arrow[r, "\delta^*"] \arrow[dr, "\coprod_{Bh}"'] & \mathcal{E}_X \arrow[r, "\coprod_h"] \arrow[d, phantom, "\Downarrow(d)"] & \mathcal{E}_Y \arrow[r, "h^*"] \arrow[ddr, phantom, "\Downarrow(c)"] & \mathcal{E}_X \arrow[dd, equal] \\
 & & \mathcal{E}_{BY} \arrow[ur, "\vartheta^*"'] \arrow[dr, "(Bh)^*"] \arrow[d, phantom, "\Downarrow(a)"] & & \\
\mathcal{E}_X \arrow[r, "\coprod_h"'] & \mathcal{E}_Y \arrow[r, "h^*"'] \arrow[ur, "\overline{B}"] & \mathcal{E}_X \arrow[r, "\overline{B}"'] & \mathcal{E}_{BX} \arrow[r, "\delta^*"'] & \mathcal{E}_X
\end{tikzcd}
$$

(a) $(\overline{B}, B)$ is a fibration map, so $\overline{B} \circ h^* \cong (Bh)^* \circ \overline{B}$.

(b) $\overline{B}$ is a lifting of B; this is an instance of Lemma 3.3.4.

(c) h is a coalgebra homomorphism, i.e., $\vartheta \circ h = Bh \circ \delta$, and consequently $(\vartheta \circ h)^* = (Bh \circ \delta)^*$. Combining this with the natural isomorphisms $h^* \circ \vartheta^* \cong (\vartheta \circ h)^*$ and $(Bh \circ \delta)^* \cong \delta^* \circ (Bh)^*$ shows that the required 2-cell is a natural isomorphism.

(d) follows from (c); see the proof of Proposition 3.3.7. For convenience we repeat the construction of the natural transformation:

$$\coprod_h \circ \delta^* \Longrightarrow \coprod_h \circ \delta^* \circ (Bh)^* \circ \coprod_{Bh} \Longrightarrow \coprod_h \circ h^* \circ \vartheta^* \circ \coprod_{Bh} \Longrightarrow \vartheta^* \circ \coprod_{Bh} .$$

The natural transformation on the left is the unit of the adjunction $\coprod_{Bh} \dashv (Bh)^*$, the middle is (c), and the one on the right is the counit of $\coprod_h \dashv h^*$. $\square$

We first instantiate this to the canonical relation lifting $\mathsf{Rel}(B)$ of a Set functor B. To this end, we use that $(\mathsf{Rel}(B), B)$ is a fibration map whenever B preserves weak pullbacks (Lemma 3.3.3). The functor $\mathsf{Rel}(B)_\delta$ coincides with b_δ, so from Theorem 5.2.1 we directly obtain:

Corollary 5.2.2. *If B is a* Set *functor preserving weak pullbacks then the behavioural equivalence closure functor* bhv_δ *is* b_δ*-compatible.*

If (X, δ) is a coalgebra for a functor B that preserves weak pullbacks, then behavioural equivalence $\approx$ coincides with bisimilarity $\sim$ (Lemma 3.1.6). Hence, in that case, the bisimilarity closure bis_δ defined in Section 4.2 coincides with the behavioural equivalence closure bhv_δ:

$$\mathsf{bis}_\delta(R) = (\sim \circ R \circ \sim) = (\approx \circ R \circ \approx) = \mathsf{bhv}_\delta(R) .$$

Thus, the fact that bis_δ is b_δ-compatible if B preserves weak pullbacks (Theorem 4.4.6) follows from Corollary 5.2.2, and hence is a special case of Theorem 5.2.1.

From Theorem 5.2.1 we also derive the soundness of up-to bhv for unary predicates that are defined by a *modality* $m \colon B2 \to 2$, where B is a functor on Set.

Modalities are in one-to-one correspondence to *predicate liftings*, which are natural transformations of the form $2^{\mathsf{Id}} \Rightarrow 2^B$ [Sch05, Proposition 20]. If such a predicate lifting is monotone, then it defines a lifting $\overline{B}\colon \mathsf{Pred} \to \mathsf{Pred}$ of B, which maps a predicate $X \to 2$ to $BX \to B2 \xrightarrow{m} 2$. Recall that with predicates viewed as functions $X \to 2$ reindexing is precomposition; then it is easy to show that the lifting induced by a modality is a fibration map. Consequently, we have:

Corollary 5.2.3. *If $\overline{B}\colon \mathsf{Pred} \to \mathsf{Pred}$ arises from a modality $m\colon B2 \to 2$ as explained above, then* bhv *is $\overline{B}_\delta$-compatible.*

5.2.2 Relational composition and equivalence

We propose a general approach for deriving the compatibility of the reflexive, symmetric and transitive closure. Composing these functors yields compatibility of the equivalence closure.

For transitive closure, it suffices to show that relational composition is compatible. Relational composition can be expressed in a fibrational setting by considering the category $\mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel}$ obtained as a pullback (in the category Cat of categories[1]) of the fibration $\mathsf{Rel} \to \mathsf{Set}$ along itself:

$$\begin{tikzcd}
\mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel} \arrow[r] \arrow[d] & \mathsf{Rel} \arrow[d] \\
\mathsf{Rel} \arrow[r] & \mathsf{Set}
\end{tikzcd}$$

The objects of $\mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel}$ are pairs of relations $R, S \subseteq X \times X$ on a common carrier X. An arrow from $R, S \subseteq X \times X$ to $R', S' \subseteq Y \times Y$ is a pair of morphisms in Rel above a common $f\colon X \to Y$; thus, it is a map $f\colon X \to Y$ such that $f(R) \subseteq R'$ and $f(S) \subseteq S'$. Then relational composition can be presented as a functor

$$\otimes\colon \mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel} \to \mathsf{Rel}$$

mapping relations $R, S \subseteq X \times X$ to their composition.

The pullback $\mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel}$ above is, in fact, a product in the category $\mathsf{Fib}(\mathsf{Set})$ of fibrations over Set. Indeed, $\mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel} \to \mathsf{Set}$ is again a fibration. In order to treat not only relational composition but also, e.g., symmetric and reflexive closure, we move to a more general setting of n-fold products. Consider for an arbitrary fibration $\mathcal{E} \to \mathcal{A}$ its n-fold product in $\mathsf{Fib}(\mathcal{A})$ (see [Jac99, Lemma 1.7.4]), denoted by $\mathcal{E}^{\times^n_{\mathcal{A}}} \to \mathcal{A}$ and defined by pullback in Cat. We have

$$(\mathcal{E}^{\times^n_{\mathcal{A}}})_X = (\mathcal{E}_X)^n \quad \text{and} \quad \mathcal{E}^0 = \mathcal{A}\,.$$

Concretely, the objects in $\mathcal{E}^{\times^n_{\mathcal{A}}}$ are n-tuples of objects in $\mathcal{E}$ belonging to the same fibre, and an arrow from $(R_1, \ldots, R_n)$ above X to $(S_1, \ldots, S_n)$ above Y consists

[1] We assume that Cat contains large categories such as Set and Rel; see [Lan98] for various ways to justify this at a foundational level.

of a tuple of arrows $(f_1 \colon R_1 \to S_1, \dots, f_n \colon R_n \to S_n)$ that sit above a common $f \colon X \to Y$.

Hereafter, we are interested in functors $G \colon \mathcal{E}^{\times^n_\mathcal{A}} \to \mathcal{E}$ that are liftings of the identity functor on $\mathcal{A}$, meaning that the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}^{\times^n_\mathcal{A}} \arrow[rr, "G"] \arrow[dr] & & \mathcal{E} \arrow[dl] \\ & \mathcal{A} & \end{tikzcd}$$

Given such a functor G, for each X in $\mathcal{A}$ we have functors $G_X \colon (\mathcal{E}_X)^n \to \mathcal{E}_X$.

For the relation fibration $\mathsf{Rel} \to \mathsf{Set}$, we have three interesting instances of such functors G:

- $(n = 0)$: $\mathsf{diag} \colon \mathsf{Set} \to \mathsf{Rel}$ mapping a set X to the diagonal relation Δ_X;
- $(n = 1)$: $\mathsf{inv} \colon \mathcal{E} \to \mathcal{E}$ mapping a relation R to its converse R^{op};
- $(n = 2)$: $\otimes \colon \mathcal{E} \times_{\mathsf{Set}} \mathcal{E} \to \mathcal{E}$ mapping relations R, S to their composition $R \circ S$.

Next, we provide a general condition on functors $G \colon \mathcal{E}^{\times^n_\mathcal{A}} \to \mathcal{E}$ as above and on the lifting $\overline{B}$ that guarantees G_X to be $\overline{B}_\delta$-compatible.

Theorem 5.2.4. *Let $\delta \colon X \to BX$ be a coalgebra. Let $G \colon \mathcal{E}^{\times^n_\mathcal{A}} \to \mathcal{E}$ be a lifting of the identity functor on $\mathcal{A}$ such that there exists a natural transformation*

$$\gamma \colon G_{BX} \circ (\overline{B}_X)^n \Rightarrow \overline{B}_X \circ G_X : (\mathcal{E}_X)^n \to \mathcal{E}_{BX} \, .$$

Then G_X is $\overline{B}_\delta$-compatible.

Proof. The goal is to construct a natural transformation of the form

$$G_X \circ (\delta^* \circ \overline{B}_X)^n \Rightarrow (\delta^* \circ \overline{B}_X) \circ G_X \, .$$

First, observe that there is a natural transformation

$$\theta \colon G_X \circ (\delta^*)^n \Rightarrow \delta^* \circ G_{BX} : (\mathcal{E}_{BX})^n \to \mathcal{E}_X \, .$$

by Lemma 3.3.4 (instantiated to $B = \mathsf{Id}$ and $\overline{B} = G$), using that reindexing along an $\mathcal{A}$-morphism f in $\mathcal{E}^{\times^n_\mathcal{A}}$ is $(f^*)^n$, where f^* is the reindexing functor in $\mathcal{E}$. (To see this, one can use the characterization of Cartesian morphisms in fibrations obtained by change-of-base and composition, which are the basic operations used to construct the fibration $\mathcal{E}^{\times^n_\mathcal{A}} \to \mathcal{A}$ [Jac99, Lemma 1.7.4].)

The desired natural transformation is now obtained as follows:

$$\begin{tikzcd} G_X \circ (\delta^* \circ \overline{B}_X)^n \arrow[r, equal] & G_X \circ (\delta^*)^n \circ (\overline{B}_X)^n \arrow[d, Rightarrow, "\theta(\overline{B}_X)^n"] & \\ & \delta^* \circ G_{BX} \circ (\overline{B}_X)^n \arrow[r, Rightarrow, "\delta^*\gamma"] & \delta^* \circ \overline{B}_X \circ G_X \end{tikzcd}$$

The first equality follows from the definition of $(-)^n$ as the mediating arrow into the product $(\mathcal{E}_X)^n$. $\square$

The use of the above theorem is that compatibility is reduced to checking the existence of a natural transformation that does not mention the coalgebra under consideration. We list several applications of the theorem for the fibration $\mathsf{Rel} \to \mathsf{Set}$. In this case, a natural transformation $G_{BX} \circ (\overline{B}_X)^n \Rightarrow \overline{B}_X \circ G_X$ exists precisely if for all relations $R_1, \ldots, R_n$ on the carrier X:

$$G(\overline{B}(R_1), \ldots, \overline{B}(R_n)) \subseteq \overline{B}G(R_1, \ldots, R_n) .$$

Instantiating this, we obtain concrete compatibility results for functors $\mathsf{Rel}^{\times^n_{\mathsf{Set}}} \to \mathsf{Rel}$, including relational composition.

Corollary 5.2.5. *Suppose $\overline{B}\colon \mathsf{Rel} \to \mathsf{Rel}$ is a lifting of B, and $\delta\colon X \to BX$ a B-coalgebra.*

1. *Let $\mathsf{diag}\colon \mathsf{Set} \to \mathsf{Rel}$ be the functor mapping each set to the associated diagonal relation. The functor $\mathsf{diag}_X : 1 \to \mathsf{Rel}_X$ is $\overline{B}_\delta$-compatible if:*

$$\Delta_{BX} \subseteq \overline{B}(\Delta_X) . \tag{5.1}$$

2. *Let $\mathsf{inv}\colon \mathsf{Rel} \to \mathsf{Rel}$ be the functor mapping each relation to its converse. The functor $\mathsf{inv}_X : \mathsf{Rel}_X \to \mathsf{Rel}_X$ is $\overline{B}_\delta$-compatible if for all relations $R \subseteq X^2$:*

$$(\overline{B}R)^{op} \subseteq \overline{B}(R^{op}) . \tag{5.2}$$

3. *Let $\otimes\colon \mathsf{Rel} \times_{\mathsf{Set}} \mathsf{Rel} \to \mathsf{Rel}$ be the relational composition functor. The functor $\otimes_X : \mathsf{Rel}_X \times \mathsf{Rel}_X \to \mathsf{Rel}_X$ is $\overline{B}_\delta$-compatible if for all $R, S \subseteq X^2$:*

$$\overline{B}(R) \otimes \overline{B}(S) \subseteq \overline{B}(R \otimes S) . \tag{5.3}$$

Note that $\overline{B}_\delta$-compatibility of diag_X simply means that $\Delta_X \subseteq \overline{B}_\delta(\Delta_X)$, i.e., the diagonal is a $\overline{B}_\delta$-invariant.

If relational composition is $\overline{B}_\delta$-compatible, and $F_1, F_2\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ are two $\overline{B}_\delta$-compatible functors, then their pointwise composition

$$F_1 \bullet F_2 = \otimes_X \circ \langle F_1, F_2 \rangle$$

is $\overline{B}_\delta$-compatible. This way of combining compatible functors corresponds to the operator $\bullet$ in Section 4.3 (4.2).

This operator $\bullet$ was used to prove the compatibility of transitive closure in the more concrete setting of the previous chapter (Theorem 4.4.6). We follow the same reasoning and define the transitive closure functor as follows:

$$\mathsf{tra} = \coprod \circ \langle (-)^{\bullet i} \rangle_{i \geq 1} : \mathsf{Rel}_X \to \mathsf{Rel}_X$$

where $(-)^{\bullet i}\colon \mathsf{Rel} \to \mathsf{Rel}$ is defined inductively: $(-)^{\bullet 1} = \mathsf{Id}$ and $(-)^{\bullet n+1} = \mathsf{Id} \bullet (-)^{\bullet n}$. By Proposition 5.1.3, compatibility of $\bullet$ implies compatibility of tra.

The above conditions (5.1) and (5.2) always hold for the canonical lifting $\overline{B} = \mathsf{Rel}(B)$; (5.3) holds for $\mathsf{Rel}(B)$ when B preserves weak pullbacks (Theorem 3.2.5). Thus, we retrieve the b_δ-compatibility of reflexive, symmetric and transitive closure (and hence also the equivalence closure eq), as proved in Theorem 4.4.6, as a special case of Corollary 5.2.5.

When $\overline{B}_\delta$ has a final coalgebra with carrier Z, one can define a *self closure* functor $\mathsf{slf}\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ by

$$\mathsf{slf}_\delta(R) = (\mathsf{cst}_Z \bullet \mathsf{Id} \bullet \mathsf{cst}_Z)(R) = Z \otimes R \otimes Z$$

where $\mathsf{cst}_Z\colon \mathsf{Rel}_X \to \mathsf{Rel}_X$ is the constant-to-Z functor. By Proposition 5.1.3 and the above, the functor slf is compatible whenever $\otimes$ is. If B is a Set functor and $\overline{B}$ is instantiated to the canonical relation lifting, then Z is the bisimilarity relation $\sim$, so

$$\mathsf{slf}_\delta(R) = \sim \circ R \circ \sim = \mathsf{bis}_\delta(R)$$

where bis_δ is the bisimilarity closure, defined in Section 4.2.

5.2.3 Contextual closure

In this section, we study the compatibility of the contextual closure. To this end, we assume an algebra $\alpha\colon TX \to X$ for some functor $T\colon \mathcal{A} \to \mathcal{A}$. Then contextual closure is defined using the bifibrational structure of p, parameterized by a lifting $\overline{T}$ of T:

$$\mathcal{E}_X \xrightarrow{\overline{T}_X} \mathcal{E}_{TX} \xrightarrow{\coprod_\alpha} \mathcal{E}_X$$

If T is a Set functor, then instantiating $\overline{T}$ to the canonical relation lifting $\mathsf{Rel}(T)$ yields the usual contextual closure, denoted ctx_α, as defined in Section 4.2.

However, taking different liftings of $\overline{T}$ yields different types of contextual closure, similar to the fact that taking different liftings of $\overline{B}$ to define $\overline{B}_\delta$ yields different coinductive predicates. Indeed, in the next section we consider the *left contextual closure* for reasoning about divergence, and the *monotone contextual closure* for weighted automata; both contextual closures differ from ctx_α.

Given liftings of T and B, compatibility of the associated contextual closure requires a λ-bialgebra, similar to the case of bisimulation up to context in Theorem 4.4.7. Additionally, it is required that λ lifts to a natural transformation between the lifted functors. All this is stated in Theorem 5.2.7 below; we require the following basic result for its proof.

Lemma 5.2.6. *Let $p\colon \mathcal{E} \to \mathcal{A}$ be a fibration, and F, G endofunctors on $\mathcal{A}$ with liftings $\overline{F}$ and $\overline{G}$ respectively. Given a natural transformation $\overline{\lambda}\colon \overline{F} \Rightarrow \overline{G}$ above some $\lambda\colon F \Rightarrow G$, there exists for every object X in $\mathcal{A}$ a natural transformation*

$$\theta\colon \overline{F}_X \Rightarrow (\lambda_X)^* \circ \overline{G}_X : \mathcal{E}_X \to \mathcal{E}_{FX}$$

Proof. For any R in $\mathcal{E}_X$ we use the universal property of the Cartesian lifting $\widetilde{(\lambda_X)}_{\overline{G}R}$ to define θ_R:

$$\begin{tikzcd}
\overline{F}(R) \arrow[d, dashed, "\theta_R"'] \arrow[drr, "\overline{\lambda}_R"] & & \\
\lambda_X^*(\overline{G}(R)) \arrow[rr, "\widetilde{(\lambda_X)}_{\overline{G}R}"'] & & \overline{G}(R) \\
FX \arrow[rr, "\lambda_X"] & & GX
\end{tikzcd}$$

Naturality is straightforward using the uniqueness of the factorisation and the definition of the reindexing functor on morphisms. $\square$

Theorem 5.2.7. *Suppose (X, α, δ) is a λ-bialgebra for some natural transformation $\lambda\colon TB \Rightarrow BT$, and suppose there exists a natural transformation $\overline{\lambda}\colon \overline{T}\,\overline{B} \Rightarrow \overline{B}\,\overline{T}$ sitting above λ. Then $\coprod_\alpha \circ \overline{T}$ is $\overline{B}_\delta$-compatible.*

Proof. The desired natural transformation is formed by composing basic pieces:

$$\begin{tikzcd}
\mathcal{E}_X \arrow[r, "\overline{B}"] \arrow[ddd, equal] \arrow[dddr, phantom, "\Downarrow(b)"] & \mathcal{E}_{BX} \arrow[rr, "\delta^*"] \arrow[drr, "\overline{T}"'] & & \mathcal{E}_X \arrow[r, "\overline{T}"] \arrow[d, phantom, "\Downarrow(d)"] & & \mathcal{E}_{TX} \arrow[r, "\coprod_\alpha"] \arrow[dddr, phantom, "\Downarrow(c)"] & \mathcal{E}_X \arrow[ddd, equal] \\
& & & \mathcal{E}_{TBX} \arrow[urr, "(T\delta)^*"'] \arrow[dr, "\coprod_{\lambda_X}"] \arrow[d, phantom, "\Downarrow(a)"] & & & \\
& & \mathcal{E}_{BTX} \arrow[ur, "\lambda_X^*"] \arrow[rr, equal] & {} & \mathcal{E}_{BTX} \arrow[dr, "\coprod_{B\alpha}"] & & \\
\mathcal{E}_X \arrow[r, "\overline{T}"'] & \mathcal{E}_{TX} \arrow[ur, "\overline{B}"] \arrow[rr, "\coprod_\alpha"'] & & \mathcal{E}_X \arrow[r, "\overline{B}"'] \arrow[u, phantom, "\Downarrow(e)"] & & \mathcal{E}_{BX} \arrow[r, "\delta^*"'] & \mathcal{E}_X
\end{tikzcd}$$

The pieces (natural transformations) are obtained as follows:

(a) This is the counit of the adjunction $\coprod_{\lambda_X} \dashv \lambda_X^*$.

(b) $\overline{\lambda}$ is a lifting of λ, see Lemma 5.2.6.

(c) (X, α, δ) is a bialgebra, which implies that $(B\alpha \circ \lambda_X \circ T\delta)^* = (\delta \circ \alpha)^*$ and thus there is a natural isomorphism

$$(T\delta)^* \circ \lambda_X^* \circ (B\alpha)^* \cong \alpha^* \circ \delta^* \,. \tag{5.4}$$

The desired natural transformation (b) is defined from (5.4):

$$\begin{tikzcd}
\coprod_\alpha \circ (T\delta)^* \arrow[r, Rightarrow] & \coprod_\alpha \circ (T\delta)^* \circ \lambda_X^* \circ (B\alpha)^* \circ \coprod_{B\alpha} \circ \coprod_{\lambda_X} \arrow[d, Rightarrow, "(5.4)"] & \\
& \coprod_\alpha \circ \alpha^* \circ \delta^* \circ \coprod_{B\alpha} \circ \coprod_{\lambda_X} \arrow[r, Rightarrow] & \delta^* \circ \coprod_{B\alpha} \circ \coprod_{\lambda_X}
\end{tikzcd}$$

using the unit of the composite adjunction $\coprod_{B\alpha} \circ \coprod_{\lambda_X} \dashv \lambda_X^* \circ (B\alpha)^*$ and the counit of $\coprod_\alpha \dashv \alpha^*$.

(d) This is an instance of Lemma 3.3.4, using that $\overline{T}$ is a lifting of T.

(e) This is an instance of Lemma 3.3.4, using that $\overline{B}$ is a lifting of B. □

The canonical relation lifting $\mathsf{Rel}(-)$ of a Set functor preserves natural transformations [Jac12, Exercise 4.4.6]. Therefore, if $\overline{T}$ and $\overline{B}$ are instantiated to $\mathsf{Rel}(T)$ and $\mathsf{Rel}(B)$ respectively, then the condition that there exists a $\overline{\lambda}$ above λ is satisfied. Thus we obtain the b_δ-compatibility of the contextual closure (Theorem 4.4.7) as a special case of Theorem 5.2.7.

In order to apply Theorem 5.2.7 for situations when either $\overline{T}$ or $\overline{B}$ is not the canonical relation lifting, one has to exhibit a $\overline{\lambda}$ sitting above λ. In Rel, such a $\overline{\lambda}$ exists if and only if for all relations $R \subseteq X^2$, the restriction of $\lambda_X \times \lambda_X$ to $\overline{T}\,\overline{B}R$ corestricts to $\overline{B}\,\overline{T}R$:

$$(\lambda_X \times \lambda_X)(\overline{T}\,\overline{B}(R)) \subseteq \overline{B}\,\overline{T}(R)\,.$$

A similar condition has to be checked for $\mathsf{Pred} \to \mathsf{Set}$. In Section 5.3 we consider several examples for which we check the above condition.

Abstract GSOS

Recall from Section 3.5.2 that an *abstract GSOS specification* is a natural transformation of the form $\rho\colon \Sigma(B\times \mathsf{Id}) \Rightarrow B\Sigma^*$, where Σ^* is the free monad for $\Sigma\colon \mathcal{A} \to \mathcal{A}$ (the $(-)^*$ notation is used both to denote reindexing functors of morphisms in $\mathcal{A}$ and to denote free monads of endofunctors, but the distinction should be clear). Any such specification induces a distributive law $\rho^\dagger\colon \Sigma^*(B \times \mathsf{Id}) \Rightarrow (B \times \mathsf{Id})\Sigma^*$.

To prove compatibility of the contextual closure for bialgebras for a distributive law $\rho^\dagger$ generated from an abstract GSOS specification, one could exhibit a natural transformation $\overline{\rho^\dagger}\colon \overline{\Sigma^*}(\overline{B} \times \overline{\mathsf{Id}}) \Rightarrow (\overline{B} \times \overline{\mathsf{Id}})\overline{\Sigma^*}$ above $\rho^\dagger$ directly, and then apply Theorem 5.2.7. We next show how to simplify such a task by proving that, under mild additional conditions, it suffices to show that there exists $\overline{\rho}\colon \overline{\Sigma}(\overline{B}\times\overline{\mathsf{Id}}) \Rightarrow \overline{B}\,\overline{\Sigma^*}$ above ρ. The lifting of $\overline{\Sigma^*}$ here is induced by the given lifting of $\overline{\Sigma}$; the functor $\overline{\mathsf{Id}}$ lifts the identity (it does not need to be the identity itself), and will be subject to a condition involving $\overline{\Sigma}$.

The construction of $\overline{\rho^\dagger}$ from $\overline{\rho}$ is similar to the construction of $\rho^\dagger$ from ρ. In order to show that it is a lifting, we need some properties relating algebras in the total category $\mathcal{E}$ to those in the base category $\mathcal{A}$.

Lemma 5.2.8. *Consider a lifting $\overline{\Sigma}$ of an $\mathcal{A}$-endofunctor Σ and assume $\overline{\Sigma}$ has free algebras.*

1. *The functor $p\colon \mathcal{E} \to \mathcal{A}$ has a right adjoint $\mathbf{1}\colon \mathcal{A} \to \mathcal{E}$, and this adjunction lifts*

as follows:

$$\begin{tikzcd} \overline{\Sigma}\text{-alg} \arrow[r, bend left, "\overline{p}"] \arrow[r, phantom, "\perp"] \arrow[d] & \Sigma\text{-alg} \arrow[l, bend left, "\overline{\mathbf{1}}"] \arrow[d] \\ \mathcal{E} \arrow[r, bend left, "p"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, bend left, "\mathbf{1}"] \end{tikzcd}$$

2. *The functor $\overline{p}$ preserves initial algebras.*

3. *When $P \in \mathcal{E}_X$ for some X in $\mathcal{A}$, the functor $\overline{p}$ maps the free $\overline{\Sigma}$-algebra for P to the free Σ-algebra for X.*

4. *The free monad $\overline{\Sigma}^*$ over $\overline{\Sigma}$ exists and is a lifting of the free monad Σ^* over Σ.*

Proof. 1. By assumption, the fibration considered here has fibred finite products, so one can define $\mathbf{1}(X)$ as the terminal object 1_X in $\mathcal{E}_X$, and $\mathbf{1}(f\colon X \to Y)$ as the Cartesian lifting $\overline{f}_{1_Y}\colon (1_Y)^* \to 1_Y$ which is well-defined since the p preserves terminal objects by assumption; thus $(1_Y)^* = 1_X$.

The functor $\overline{p}$ maps an algebra $\alpha\colon \overline{\Sigma}P \to P$ to $p(\alpha)\colon p(\overline{\Sigma}(P)) \to p(P)$ which is indeed a Σ-algebra since $\overline{\Sigma}$ lifts Σ, i.e., $\Sigma p(P) = p(\overline{\Sigma}(P))$. The existence of a right adjoint $\overline{\mathbf{1}}$ to $\overline{p}$ is a consequence of [HJ98, Theorem 2.14].

2. Since $\overline{p}$ is a left adjoint, it preserves initial objects.

3. This follows from item 2 applied to the lifting $\overline{\Sigma} + P$ of $\Sigma + X$.

4. This is a consequence of item 3. □

Lemma 5.2.8 allows us to prove the desired result on lifting distributive laws induced by GSOS specifications. Rather than assuming that $\overline{\mathsf{Id}}$ is itself the identity (so that the lifted natural transformation is itself an abstract GSOS specification), we assume that $\overline{\mathsf{Id}}$ is a lifting that comes together with a natural transformation $\gamma\colon \overline{\Sigma}\,\overline{\mathsf{Id}} \Rightarrow \overline{\mathsf{Id}}\,\overline{\Sigma}$. We shall apply this result in Section 5.4.1, involving the relation fibration, where $\overline{\mathsf{Id}}$ maps any relation $R \subseteq X \times X$ to the diagonal Δ_X.

Theorem 5.2.9. *Suppose:*

- *$\overline{\Sigma}$ is a lifting of an $\mathcal{A}$-endofunctor Σ;*
- *$\overline{\Sigma}$ has free algebras;*
- *$\overline{\mathsf{Id}}$ is a lifting of the identity functor;*
- *there is a natural transformation $\gamma\colon \overline{\Sigma}\,\overline{\mathsf{Id}} \Rightarrow \overline{\mathsf{Id}}\,\overline{\Sigma}$ that sits above the identity;*
- *there is a natural transformation $\overline{\rho}\colon \overline{\Sigma}(\overline{B} \times \overline{\mathsf{Id}}) \Rightarrow \overline{B\Sigma}^*$ above $\rho\colon \Sigma(B \times \mathsf{Id}) \Rightarrow B\Sigma^*$, where $\overline{\Sigma}^*$ is the lifting of Σ^* induced by $\overline{\Sigma}$ as in Lemma 5.2.8.*

Then there is a natural transformation $\overline{\rho^\dagger}\colon \overline{\Sigma}^*(\overline{B}\times\overline{\mathsf{Id}}) \Rightarrow (\overline{B}\times\overline{\mathsf{Id}})\overline{\Sigma}^*$ *that sits above* $\rho^\dagger$.

Proof. The idea of the proof is to construct $\overline{\rho^\dagger}$ from the given natural transformation $\overline{\rho}$, by initiality, similar to the construction of a distributive law from a GSOS law (in this case, $\overline{\rho}$ is not a GSOS law in general since $\overline{\mathsf{Id}}$ does not need to be the identity functor in $\mathcal{E}$). Using Lemma 5.2.8 we can then show that this resulting distributive law (between functors) sits above $\rho^\dagger$.

For an object X in $\mathcal{A}$, we know that Σ^*X is the free Σ-algebra on X. Let

$$[\kappa_X, \eta_X]\colon \Sigma\Sigma^*X + X \to \Sigma^*X$$

denote the initial $\Sigma + X$-algebra. Similarly, let

$$[\overline{\kappa}_P, \overline{\eta}_P]\colon \overline{\Sigma}\,\overline{\Sigma}^*P + P \to \overline{\Sigma}^*P$$

denote the initial $\overline{\Sigma} + P$-algebra, where P is in $\mathcal{E}_X$. By Lemma 5.2.8 we know that $[\overline{\kappa}_P, \overline{\eta}_P]$ is above $[\kappa_X, \eta_X]$.

For $P \in \mathcal{E}_X$ the map $\overline{\rho}^\dagger_P$ is defined similarly to the construction of $\rho^\dagger_X$ from ρ_X (see (3.15) in Section 3.5.2); the difference is that it involves the natural transformation $\gamma\colon \overline{\Sigma}\,\overline{\mathsf{Id}} \Rightarrow \overline{\mathsf{Id}}\,\overline{\Sigma}$. Indeed, $\overline{\rho}^\dagger_P$ is the unique map arising from initiality:

$$\begin{tikzcd}
\overline{\Sigma}\,\overline{\Sigma}^*(\overline{B}\times\overline{\mathsf{Id}})P \arrow[r, "\overline{\Sigma}(\overline{\rho}^\dagger_P)"] \arrow[ddd, "\overline{\kappa}_{(\overline{B}\times\overline{\mathsf{Id}})P}"'] & \overline{\Sigma}(\overline{B}\times\overline{\mathsf{Id}})\overline{\Sigma}^*P \arrow[d, "\langle\overline{\rho}_{\overline{\Sigma}^*P}, \overline{\Sigma}\pi_2\rangle"] \\
 & \overline{B}\,\overline{\Sigma}^*\,\overline{\Sigma}^*P \times \overline{\Sigma}\,\overline{\mathsf{Id}}\,\overline{\Sigma}^*P \arrow[d, "\mathsf{id}\times\gamma_{\overline{\Sigma}^*P}"] \\
 & \overline{B}\,\overline{\Sigma}^*\,\overline{\Sigma}^*P \times \overline{\mathsf{Id}}\,\overline{\Sigma}\,\overline{\Sigma}^*P \arrow[d, "\overline{B}\overline{\mu}_P\times\overline{\mathsf{Id}}\overline{\kappa}_P"] \\
\overline{\Sigma}^*(\overline{B}\times\overline{\mathsf{Id}})P \arrow[r, dashed, "\overline{\rho}^\dagger_P"] & (\overline{B}\times\overline{\mathsf{Id}})\overline{\Sigma}^*P \\
(\overline{B}\times\overline{\mathsf{Id}})P \arrow[u, "\overline{\eta}_{(\overline{B}\times\overline{\mathsf{Id}})P}"] \arrow[ur, "(\overline{B}\times\overline{\mathsf{Id}})\overline{\eta}_P"']
\end{tikzcd} \tag{5.5}$$

By Lemma 5.2.8, and using that γ sits above the identity, we have that the $\overline{\Sigma} + (\overline{B}\times\overline{\mathsf{Id}})P$-algebras in the above diagram (5.5) sit above the $\Sigma + (B\times\mathsf{Id})X$-algebras defining $\rho^\dagger_X$ from ρ_X. By uniqueness of $\rho^\dagger_X$ it follows that $\overline{\rho}^\dagger_P$ sits above $\rho^\dagger_X$. $\square$

For a ρ-model (X, α, δ), the existence of $\overline{\rho^\dagger}$ above ρ ensures, via the above result and Theorem 5.2.7, compatibility of the contextual closure on the bialgebra $(X, \widehat{\alpha}, \langle\delta, \mathsf{id}\rangle)$ corresponding to the ρ-model. More precisely, it shows that $\coprod_{\widehat{\alpha}} \circ \overline{\Sigma^*}_X$

is $(\overline{B} \times \overline{\mathsf{Id}})_{\langle \delta, \mathsf{id} \rangle}$-compatible. In the remainder of this section, we address two technical issues regarding this result, which arise due to the fact that we present distributive laws by abstract GSOS specifications.

First, the above results provide compatibility for a contextual closure defined based on the free monad $\overline{\Sigma}^*$ rather than the lifted functor $\overline{\Sigma}$ itself, which is the one supplied in concrete examples. However, it turns out that the contextual closure defined by $\overline{\Sigma}$ is, in fibrations whose fibres are preorders, below the one defined by $\overline{\Sigma}^*$ (shown below in Lemma 5.2.10), so if the latter is compatible, the former is sound. Moreover, if the lifting $\overline{\Sigma}$ is given by a modality, then the lifting $\overline{\Sigma}^*$ is given in terms of the inductive extension of this modality (Lemma 5.2.11).

Second, $\overline{B} \times \overline{\mathsf{Id}}_{\langle \delta, \mathsf{id} \rangle}$-compatibility is not exactly $\overline{B}_\delta$-compatibility (the same phenomenon was discussed at a more concrete level at the end of Section 4.4.2). However, under some assumptions, any $\overline{B}_\delta$-invariant is also a $(\overline{B} \times \overline{\mathsf{Id}})_{\langle \delta, \mathsf{id} \rangle}$-invariant (shown below in Lemma 5.2.12).

Lemma 5.2.10. *Let $\Sigma, \overline{\Sigma}$, $\overline{\Sigma}^*$ and Σ^* be as in Lemma 5.2.8. Given an algebra $\alpha \colon \Sigma A \to A$ with induced algebra $\widehat{\alpha} \colon \Sigma^* A \to A$ for the monad Σ^*, there exists a natural transformation of type $\coprod_\alpha \circ \overline{\Sigma} \Rightarrow \coprod_{\widehat{\alpha}} \circ \overline{\Sigma}^*$.*

Proof. Let $\eta \colon \mathsf{Id} \Rightarrow \Sigma^*$ and $\kappa \colon \Sigma\Sigma^* \Rightarrow \Sigma^*$ be the canonical natural transformations defined by initiality; composing them yields a natural transformation $\iota \colon \Sigma \Rightarrow \Sigma^*$. Similarly, we can construct a natural transformation $\overline{\iota} \colon \overline{\Sigma} \Rightarrow \overline{\Sigma}^*$ sitting above ι.

The desired natural transformation consists of two pieces:

$$\begin{tikzcd}
\mathcal{E}_X \arrow[r, "\overline{\Sigma}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow(a)"] & \mathcal{E}_{\Sigma X} \arrow[r, "\coprod_\alpha"] \arrow[d, "\coprod_{\iota_X}"] \arrow[dr, phantom, "\Downarrow(b)"] & \mathcal{E}_X \arrow[d, equal] \\
\mathcal{E}_X \arrow[r, "\overline{\Sigma}^*"'] & \mathcal{E}_{\Sigma^* X} \arrow[r, "\coprod_{\widehat{\alpha}}"'] & \mathcal{E}_X
\end{tikzcd}$$

(a) Since $\overline{\iota}$ sits above ι, by Lemma 5.2.6 there is a natural transformation $\theta \colon \overline{\Sigma} \Rightarrow \iota_X^* \circ \overline{\Sigma}^*$. The natural transformation for (a) is its mate:

$$\coprod_{\iota_X} \circ \overline{\Sigma} \Longrightarrow \coprod_{\iota_X} \circ \iota_X^* \circ \overline{\Sigma}^* \Longrightarrow \overline{\Sigma}^*$$

using the counit of $\coprod_{\iota_X} \dashv \iota_X^*$.

(b) We have $\alpha = \widehat{\alpha} \circ \iota_X$, so $\coprod_\alpha = \coprod_{\widehat{\alpha} \circ \iota_X} \cong \coprod_{\widehat{\alpha}} \circ \coprod_{\iota_X}$. □

Lemma 5.2.11. *Suppose $\overline{\Sigma} \colon \mathsf{Pred} \to \mathsf{Pred}$ is a lifting of $\Sigma \colon \mathsf{Set} \to \mathsf{Set}$, given by a modality $n \colon \Sigma 2 \to 2$ (see the end of Section 5.2.1), and suppose $\overline{\Sigma}$ has free algebras. Then the lifting $\overline{\Sigma^*}$ of the free monad Σ^* (Lemma 5.2.8) is given by the modality $\widehat{n} \colon \Sigma^* 2 \to 2$.*

Proof. The lifting $\overline{\Sigma^*}$ of the free monad is itself a free monad $\overline{\Sigma}^*$, for $\overline{\Sigma}$ (see Lemma 5.2.8). We need to show that, for any $p \colon X \to 2$: $\overline{\Sigma}^* p = \widehat{n} \circ \Sigma^* p$.

First, observe that $\overline{\Sigma}^* p$ is the initial $\overline{\Sigma} + p$-algebra. By Lemma 5.2.8 it sits above the initial $\Sigma + X$-algebra $[\kappa_X, \eta_X] \colon \Sigma\Sigma X^* + X \to \Sigma^* X$. Let $q \colon \Sigma^* X \to 2$ be the carrier of the initial $\overline{\Sigma} + p$-algebra; then by definition of $\overline{\Sigma}$ and morphisms in Pred it makes the following diagram commute laxly:

$$\begin{tikzcd}
\Sigma\Sigma^* X + X \arrow[r, "{[\kappa_X, \eta_X]}"] \arrow[d, "\Sigma q + \mathsf{id}"'] & \Sigma^* X \arrow[dd, "q"] \\
\Sigma 2 + X \arrow[d, "{[n,p]}"'] \arrow[r, phantom, "\leq"] & \\
2 \arrow[r, equal] & 2
\end{tikzcd}$$

Since the initial algebra is an isomorphism, this is actually strict commutativity. Thus, we have a Σ-algebra morphism:

$$\begin{tikzcd}
\Sigma\Sigma^* X + X \arrow[r, "\Sigma q + \mathsf{id}"] \arrow[d, "{[\kappa_X, \eta_X]}"'] & \Sigma 2 + X \arrow[d, "{[n,p]}"] \\
\Sigma^* X \arrow[r, "q"'] & 2
\end{tikzcd}$$

But this is the unique Σ-algebra morphism from the initial algebra, so if we can prove that filling in $\widehat{n} \circ \Sigma^* p$ for q makes the above diagram commute, then we are done. Indeed, this follows from the commutativity of:

$$\begin{tikzcd}
\Sigma\Sigma^* X + X \arrow[r, "\Sigma\Sigma^* p + \mathsf{id}"] \arrow[d, "{[\kappa_X, \eta_X]}"'] & \Sigma\Sigma^* 2 + X \arrow[r, "\Sigma\widehat{n} + \mathsf{id}"] \arrow[d, "{[\kappa_2, \eta_2 \circ p]}"] & \Sigma 2 + X \arrow[d, "{[n,p]}"] \\
\Sigma^* X \arrow[r, "\Sigma^* p"'] & \Sigma^* 2 \arrow[r, "\widehat{n}"'] & 2
\end{tikzcd}$$

The left-hand square commutes by naturality of κ and the definition of Σ^* on morphisms, the right-hand square commutes by definition of $\widehat{n}$. □

Lemma 5.2.12. *Suppose G is an $\mathcal{E}$-endofunctor such that there exists a natural transformation $\eta \colon \mathsf{Id} \Rightarrow \overline{\mathsf{Id}} \circ G$ that sits above the identity. If R is a $\overline{B}_\delta$-invariant up to G then it is a $(\overline{B} \times \overline{\mathsf{Id}})_{\langle \delta, \mathsf{id} \rangle}$-invariant up to G.*

Proof. Given $R \to \delta^* \overline{B} G R$ and the natural transformation η we construct a morphism h using the universal property of the product $(\overline{B} \times \overline{\mathsf{Id}})(GR) = \overline{B} GR \times \overline{\mathsf{Id}} GR$:

$$\begin{tikzcd}
\delta^*(\overline{B} GR) \arrow[d, "\widetilde{\delta}_{\overline{B}GR}"'] & R \arrow[l] \arrow[d, dashed, "h"] \arrow[dr, "\eta_R"] & \\
\overline{B} GR & (\overline{B} \times \overline{\mathsf{Id}}) GR \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \overline{\mathsf{Id}} GR
\end{tikzcd}$$

The morphism h sits above $\langle \delta, \mathsf{id} \rangle$ (using that η sits above the identity). Thus we can use a Cartesian lifting of $\langle \delta, \mathsf{id} \rangle$ to get the desired invariant:

$$\begin{tikzcd}
R \arrow[d, dashed] \arrow[dr, "h"] & \\
\langle \delta, \mathsf{id} \rangle^*((\overline{B} \times \overline{\mathsf{Id}})(GR)) \arrow[r, "\widetilde{\langle \delta, \mathsf{id} \rangle}_{(\overline{B} \times \overline{\mathsf{Id}})(GR)}"'] & (\overline{B} \times \overline{\mathsf{Id}})(GR) \\
X \arrow[r, "\langle \delta, \mathsf{id} \rangle"] & BX \times X
\end{tikzcd}$$

□

If $\mathcal{A} = \mathsf{Rel}$ or $\mathcal{A} = \mathsf{Pred}$, then the existence of η means that $R \subseteq \overline{\mathsf{Id}} \circ G(R)$. A special case is when $\overline{\mathsf{Id}}$ is itself the identity and G is pointed; this holds, for instance, if G is the (canonical) contextual closure of a monad with respect to an algebra for that monad, see the end of Section 4.4.2.

5.3 Examples

We give examples of up-to techniques for several coinductive predicates, and prove their soundness by instantiating the results of Section 5.2.

5.3.1 Weighted language inclusion

Consider the Set functor $BX = \mathbb{S} \times X^A$, where $\mathbb{S}$ is a semiring. Recall that a B-coalgebra is a Moore automaton with output in $\mathbb{S}$, and that the final semantics assigns to every state a weighted language, i.e., a function in $\mathbb{S}^{A^*}$ (Example 3.1.1).

Suppose $\mathbb{S}$ carries a partial order $\leq$. This can be extended pointwise to an order on weighted languages. For instance, if $\mathbb{S}$ is a two-element set of truth values then this order corresponds to plain language inclusion.

To obtain such a notion of inclusion as a coinductive predicate on any B-coalgebra, we define a lifting $\overline{B}\colon \mathsf{Rel} \to \mathsf{Rel}$ of B that maps a relation R on X to a relation on $\mathbb{S} \times X^A$:

$$\overline{B}(R) = \{((p, \varphi), (q, \psi)) \mid p \leq q \text{ and } \forall a \in A.\, (\varphi(a), \psi(a)) \in R\}\,. \qquad (5.6)$$

Given a coalgebra $\langle o, t \rangle \colon X \to \mathbb{S} \times X^A$, a relation $R \subseteq X \times X$ is a $\overline{B}_{\langle o,t \rangle}$-invariant iff for every pair $(x, y) \in R$: $o(x) \leq o(y)$ and for all $a \in A$: $(t(x), t(y)) \in R$. Notice that this generalizes simulation of deterministic automata (Definition 2.4.1, Example 3.1.2). The coinductive predicate defined by $\overline{B}_{\langle o,t \rangle}$, that is, the carrier of the final $\overline{B}_{\langle o,t \rangle}$-coalgebra, is the largest $\overline{B}_{\langle o,t \rangle}$-invariant. We call it *inclusion*, and denote it by $\precsim$. Thus, to prove that $x \precsim y$ it suffices to construct a $\overline{B}_{\langle o,t \rangle}$-invariant that contains (x, y).

Let $\langle o, t\rangle\colon X \to \mathbb{S} \times (\mathcal{M}X)^A$ be a weighted automaton (Example 3.1.1). Determinizing it yields a Moore automaton $\langle o^\sharp, t^\sharp\rangle\colon \mathcal{M}X \to \mathbb{S} \times (\mathcal{M}X)^A$, where the final semantics of a state x (viewed as a linear combination) is precisely the weighted language accepted by x on the original automaton (Example 3.5.2). Indeed, given states x and y, we have $x \precsim y$ if the weighted language accepted by x is (pointwise) less than the language accepted by y, and proving $x \precsim y$ amounts to exhibiting a $\overline{B}_{\langle o^\sharp, t^\sharp\rangle}$-invariant that contains (x, y).

As an example, consider the following weighted automaton, where $\mathbb{S} = \mathbb{R}^+$ is the semiring of non-negative real numbers and A is the singleton $\{a\}$:

$$\begin{tikzcd}
x \downarrow 0 \arrow[r, bend left, "{a,1}"] & y \downarrow 1 \arrow[l, bend left, "{a,1}"] \arrow[loop above, "{a,1}"]
\end{tikzcd}$$

Since the alphabet is a singleton, the language semantics assigns a sequence (of zeros and ones) to each state. To show that the semantics of x is pointwise less than that of y (i.e., the sequence generated by x is increasing) one can establish an invariant on the states of the determinized B-coalgebra associated to the above weighted automaton, as follows:

$$\begin{tikzcd}
x \downarrow 0 \arrow[r, "a"] \arrow[d, dash, dashed] & y \downarrow 1 \arrow[r, "a"] \arrow[d, dash, dashed] & x{+}y \downarrow 1 \arrow[r, "a"] \arrow[d, dash, dashed] & \cdots \\
y \downarrow 1 \arrow[r, "a"'] & x{+}y \downarrow 1 \arrow[r, "a"'] & x{+}2y \downarrow 2 \arrow[r, "a"'] & \cdots
\end{tikzcd} \tag{5.7}$$

where the solid arrows are transitions, and the dashed lines represent the relation. It is straightforward to see that this requires an infinite relation.

Now consider the finite relation $R = \{(x, y), (y, x{+}y)\}$. This is not an invariant, since $x + y$ is not related to $x + 2y$. However, $x + 2y$ is obtained from $x + y$ by substituting x for y and y for $x + y$, which means that $(x + y, x + 2y)$ is in the contextual closure $\mathsf{ctx}(R)$ as defined in Section 4.2, and thus R is an invariant up to ctx. Below, we define ctx properly and show that it is compatible. As a consequence, the relation R suffices to prove that $x \precsim y$.

Consider a determinized weighted automaton $(\mathcal{M}X, \langle o^\sharp, t^\sharp\rangle)$. The associated contextual closure ctx is formally defined by $\mathsf{ctx} = \coprod_{\mu_X} \circ \mathsf{Rel}(\mathcal{M})$, where μ_X is the multiplication of the monad $\mathcal{M}$ (Example 3.4.1). The canonical relation lifting $\mathsf{Rel}(\mathcal{M})$ is given on a relation $R \subseteq X \times X$ by

$$\mathsf{Rel}(\mathcal{M})(R) = \left\{ \left(\sum r_i x_i,\ \sum r_i y_i \right) \mid \forall i.\, (x_i, y_i) \in R \right\} .$$

To prove that ctx is $\overline{B}_\delta$-compatible, recall that $(\mathcal{M}X, \mu_X, \langle o^\sharp, t^\sharp\rangle)$ is a λ-bialgebra for some λ (Chapter 3). Compatibility follows from Theorem 5.2.7, if we show that there is a natural transformation $\overline{\lambda}\colon \mathsf{Rel}(\mathcal{M})\overline{B} \Rightarrow \overline{B}\mathsf{Rel}(\mathcal{M})$ sitting above λ. Concretely, this amounts to proving that

$$(\lambda_X \times \lambda_X)(\mathsf{Rel}(\mathcal{M})(\overline{B}(R))) \subseteq \overline{B}(\mathsf{Rel}(\mathcal{M})(R)) \tag{5.8}$$

for any relation $R \subseteq X \times X$ and any X. First, we compute $\mathsf{Rel}(\mathcal{M})(\overline{B}(R))$:

$$\left\{\left(\sum r_i(p_i,\varphi_i), \sum r_i(q_i,\psi_i)\right) \mid \forall i.\, p_i \leq q_i \text{ and } \forall a.\, (\varphi_i(a),\psi_i(a)) \in R\right\}$$

Applying $\lambda_X \times \lambda_X$ yields a relation on $B\mathcal{M}X$:

$$\left\{\left(\left(\sum r_i \cdot p_i, \lambda a. \sum r_i \cdot \varphi_i(a)\right), \left(\sum r_i \cdot q_i, \lambda a. \sum r_i \cdot \psi_i(a)\right)\right) \right.$$
$$\left. \mid \forall i.\, p_i \leq q_i \text{ and } \forall a.\, (\varphi_i(a),\psi_i(a)) \in R\right\}$$

Now we compute $\overline{B}(\mathsf{Rel}(\mathcal{M})(R))$:

$$\left\{\left(\left(p, \lambda a. \sum r_{a,i}x_{a,i}\right), \left(q, \lambda a. \sum r_{a,i}y_{a,i}\right)\right) \mid p \leq q \text{ and } \forall a.\, \forall i.\, (x_{a,i},y_{a,i}) \in R\right\}$$

It follows that the inclusion (5.8) holds whenever $\sum r_i \cdot p_i \leq \sum r_i \cdot q_i$ given that $p_i \leq q_i$ for all i. This is the case when for all $n_1, m_1, n_2, m_2 \in \mathbb{S}$ such that $n_1 \leq n_2$ and $m_1 \leq m_2$, we have

(a) $n_1 + m_1 \leq n_2 + m_2$, and

(b) $n_1 \cdot m_1 \leq n_1 \cdot m_2$.

These two conditions are satisfied, for instance, in the Boolean semiring or in $\mathbb{R}^+$. Thus, in these cases, the construction of invariants up to ctx is a sound proof technique for inclusion.

The above argument can possibly be reformulated by using the category Pos of posets and monotone functions as a base category rather than Set, since the conditions (a) and (b) assert that addition and multiplication are monotone. We leave this for future work.

Monotone contextual closure

Condition (b) fails for the semiring $\mathbb{R}$ of (all) real numbers. Nevertheless, our framework allows us to prove compatibility for a different up-to technique, based on a variant of contextual closure. The *monotone* contextual closure is obtained as the composition $\coprod_\mu \circ \overline{\mathcal{M}}$ involving the non-canonical lifting of the functor $\mathcal{M}$ (for the semiring $\mathbb{R}$) defined as follows:

$$\overline{\mathcal{M}}(R) = \left\{\left(\sum r_i x_i, \sum r_i y_i\right) \mid \forall i. \begin{array}{lll} r_i \geq 0 & \Rightarrow & (x_i,y_i) \in R \\ r_i < 0 & \Rightarrow & (y_i,x_i) \in R \end{array}\right\}$$

The rule-based inductive characterization of the monotone contextual closure differs from the standard contextual closure (presented in Example 4.2.5) in the rule for scalar multiplication, which now splits into two rules:

$$\frac{v \;\mathsf{ctx}(R)\; w \quad r \geq 0}{r \cdot v \;\mathsf{ctx}(R)\; r \cdot w} \qquad \frac{v \;\mathsf{ctx}(R)\; w \quad r < 0}{r \cdot w \;\mathsf{ctx}(R)\; r \cdot v}$$

To prove that this is compatible, we prove the inclusion

$$(\lambda_X \times \lambda_X)(\overline{\mathcal{M}}(\overline{B}(R))) \subseteq \overline{B}(\overline{\mathcal{M}}(R))\,. \tag{5.9}$$

We first compute $\overline{\mathcal{M}}(\overline{B}(R))$:

$$\left\{ \left(\sum r_i(p_i, \varphi_i), \sum r_i(q_i, \psi_i) \right) \mid \forall i. \begin{array}{lcl} r_i \geq 0 & \Rightarrow & p_i \leq q_i \text{ and } \forall a. \varphi_i(a)\ R\ \psi_i(a)) \\ r_i < 0 & \Rightarrow & q_i \leq p_i \text{ and } \forall a. \psi_i(a)\ R\ \varphi_i(a)) \end{array} \right\}$$

Then $(\lambda_X \times \lambda_X)(\overline{\mathcal{M}}(\overline{B}(R)))$ is:

$$\begin{gathered} \left\{ \left(\left(\sum r_i \cdot p_i, \lambda a. \sum r_i \cdot \varphi_i(a) \right), \left(\sum r_i \cdot q_i, \lambda a. \sum r_i \cdot \psi_i(a) \right) \right) \right. \\ \left. \mid \forall i. \begin{array}{l} r_i \geq 0 \Rightarrow p_i \leq q_i \text{ and } \forall a. (\varphi_i(a), \psi_i(a)) \in R \\ r_i < 0 \Rightarrow q_i \leq p_i \text{ and } \forall a. (\psi_i(a), \varphi_i(a)) \in R \end{array} \right\} \end{gathered}$$

Finally $\overline{B}(\overline{\mathcal{M}}(R))$ is

$$\begin{gathered} \left\{ \left(\left(p, \lambda a. \sum r_{a,i} x_{a,i} \right), \left(q, \lambda a. \sum r_{a,i} y_{a,i} \right) \right) \right. \\ \left. \mid p \leq q; \forall a. \forall i. \begin{array}{l} r_{a,i} \geq 0 \Rightarrow (x_{a,i}, y_{a,i}) \in R \\ r_{a,i} < 0 \Rightarrow (y_{a,i}, x_{a,i}) \in R \end{array} \right\} \end{gathered}$$

The desired inclusion (5.9) holds, since $r_i \cdot p_i \leq r_i \cdot q_i$ for all i. The reason is that $p_i \leq q_i$ when $r_i \geq 0$, whereas $q_i \leq p_i$ if $r_i < 0$.

Reflexive and transitive closure

Contextual closure can be combined with reflexive, transitive and symmetric closure to obtain the *congruence* closure (see Section 4.2), which is a useful technique for bisimulation up-to. For the lifting $\overline{B}$ of B (5.6) (with $BX = \mathbb{S} \times X^A$), we can not expect symmetric closure to be compatible, but we can nevertheless prove compatibility of reflexive and transitive closure.

By reflexivity of $\leq$ it follows that $\Delta_{BX} \subseteq \overline{B}(\Delta_X)$, and thus by Corollary 5.2.5 the functor $\mathsf{diag}_X \colon 1 \to \mathsf{Rel}$ is $\overline{B}_\delta$-compatible, i.e., Δ_X is a $\overline{B}_\delta$-invariant (this amounts to the elementary fact that the diagonal on any Moore automaton is a simulation). By Proposition 5.1.3 this implies that the endofunctor on Rel_X that maps any relation to Δ_X is $\overline{B}_\delta$-compatible. For the transitive closure, by transitivity of $\leq$ it follows that $\overline{B}(R) \otimes \overline{B}(S) \subseteq \overline{B}(R \otimes S)$, where $\otimes$ is relational composition. Again by Corollary 5.2.5 we obtain $\overline{B}_\delta$-compatibility of $\otimes_X$, and thus also of the transitive closure.

5.3.2 Divergence of processes

Consider the functor $BX = (\mathcal{P}_\omega X)^A$, where A is a set of labels that contains a distinguished $\tau \in A$. Let $\overline{B} \colon \mathsf{Pred} \to \mathsf{Pred}$ be the predicate lifting for divergence (Example 3.2.6), and recall that a process diverges if it has an infinite outgoing path labelled by τ-actions. In this section, we establish compatibility of the behavioural equivalence closure, and of a variant of the contextual closure.

As a motivating example, consider the processes p and q given by

$$p \xrightarrow{a} p|p \qquad q \xrightarrow{\overline{a}} q$$

where the parallel composition $|$ is defined as usual (Example 3.5.4). To prove that the process $p|q$ diverges, one can establish an invariant containing $p|q$. But this invariant should then contain all states occurring on the infinite path

$$p|q \xrightarrow{\tau} (p|p)|q \xrightarrow{\tau} \ldots$$

and thus it needs to contain infinitely many states.

Instead, an informal proof might go as follows: $p|q$ makes a τ-step to the process $(p|p)|q$. But $(p|p)|q$ is bisimilar to $(p|q)|p$, and now we would like to conclude that this suffices, since we have already inspected $p|q$. Formally, this argument corresponds to establishing an invariant up to the composition of the behavioural equivalence closure and a particular type of contextual closure.

More precisely, recall from Section 5.2.1 that the functor bhv closes a given predicate under behaviourally equivalent (i.e., bisimilar) states. Further, we define the *left contextual closure* as

$$\mathsf{ctx}^l(P \subseteq X) = \{(p|x) \mid p \in P,\ x \in X\}\,.$$

Then $P = \{p|q\}$ is a $\overline{B}_\delta$-invariant up to $\mathsf{bhv} \circ \mathsf{ctx}^l$ (where δ is the model). To conclude from this argument that $p|q$ diverges, we need to prove the soundness of $\mathsf{bhv} \circ \mathsf{ctx}^l$. We do this by proving the compatibility of bhv and ctx^l separately.

Observe that the lifting $\overline{B}$ is determined by a modality $m\colon (\mathcal{P}_\omega 2)^A \to 2$ (as in the end of Section 5.2.1). This modality is defined by: $m(f) = 1$ iff $1 \in f(\tau)$. It induces a monotone predicate lifting, so by Corollary 5.2.2, bhv is $\overline{B}_\delta$-compatible on any B-coalgebra δ.

For the contextual closure, we use a functor $\Sigma X = X \times X$ to syntactically represent the composition operator. Let $\rho\colon \Sigma(B \times \mathsf{Id}) \to B\Sigma^*$ be the GSOS specification giving its semantics, and ρ^* the induced distributive law (Example 3.5.4). We define the left contextual closure of a Σ-algebra α as the composite functor $\mathsf{ctx}^l = \coprod_\alpha \circ \overline{\Sigma}$. The lifting $\overline{\Sigma}$ is given by the modality $n\colon \Sigma 2 \to 2$, defined by $n(b,c) = b$.

Using Theorem 5.2.9, we prove the compatibility of the (left) contextual closure $\coprod_{\widehat{\alpha}} \circ \overline{\Sigma}^*$, involving the free monad for $\overline{\Sigma}$ (by Lemma 5.2.10, ctx^l is contained in this contextual closure). The main step is to show that there exists $\overline{\rho}\colon \overline{\Sigma}(\overline{B} \times \mathsf{Id}) \Rightarrow \overline{B}\,\overline{\Sigma^*}$ that sits above ρ (notice that we use the identity functor on the total category Pred as the lifting of the identity functor on Set).

The existence of $\overline{\rho}$ above ρ amounts to the inclusion

$$\rho(\overline{\Sigma}(\overline{B} \times \mathsf{Id})) \subseteq \overline{B}\,\overline{\Sigma^*} \tag{5.10}$$

which can be proved by hand, based on a careful analysis of ρ and the liftings. However, in the present situation, where both $\overline{B}$ and $\overline{\Sigma}$ are given by modalities (m

and n respectively), this condition can be proved in a neater way. Using the definition of the liftings $\overline{B}$ and $\overline{\Sigma}$ in terms of modalities, the inclusion (5.10) amounts to (lax) commutativity of the outside of the following diagram, for any predicate $p\colon X \to 2$:

$$\begin{tikzcd}
\Sigma(BX \times X) \arrow[r, "\rho_X"] \arrow[d, "\Sigma(Bp\times p)"'] & B\Sigma^* X \arrow[d, "B\Sigma^* p"] \\
\Sigma(B2 \times 2) \arrow[r, "\rho_2"] \arrow[d, "\Sigma m\circ\Sigma\pi_1"'] & B\Sigma^* 2 \arrow[d, "B\widehat{n}"] \\
\Sigma 2 \arrow[d, "n"'] \arrow[r, phantom, "\leq"] & B2 \arrow[d, "m"] \\
2 \arrow[r, equal] & 2
\end{tikzcd}$$

(The lifting $\overline{\Sigma^*}$ is given by $\widehat{n}$; this is Lemma 5.2.11.) The upper square commutes by naturality, which means that lax commutativity of the lower square suffices. To see that this requirement is satisfied, let $f, g \in B2 = (\mathcal{P}2)^A$. If $1 \in f(\tau)$ (which is the only situation where $n \circ \Sigma m \circ \Sigma\pi_1((f,x),(g,y)) = 1$) then $1|y \in \rho_2((f,x),(g,y))(\tau)$, which implies that $m \circ Bn \circ \rho_2((f,x),(g,y)) = 1$ holds, as required.

More interestingly, the property that ρ lifts reduces to checking an inclusion that only involves finite sets (given that the set of labels is finite). This suggest that in general, if B and Σ both preserve finite sets and the liftings are presented by modalities, then this property is decidable. We leave a general investigation for future work.

5.4 Compositional predicates

In this section, we describe a way of defining functor liftings by *composing* simpler liftings, using a generalization of relational composition. We show that proving compatibility of the contextual closure for such a composite lifting reduces to proving compatibility for its constituents. We instantiate this to relational composition in the fibration Rel $\to$ Set, and apply it to derive sound up-to techniques for the notion of *similarity*, studied in [HJ04].

Assume a fibration $p\colon \mathcal{E} \to \mathcal{A}$ and a functor $\otimes\colon \mathcal{E} \times_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ that lifts the identity functor (see Section 5.2.2). Suppose we have two liftings $\overline{B}_1, \overline{B}_2\colon \mathcal{E} \to \mathcal{E}$ of the same functor $B\colon \mathcal{A} \to \mathcal{A}$. One can then define a *composite* lifting

$$\overline{B}_1 \otimes \overline{B}_2 = \otimes \circ \langle \overline{B}_1, \overline{B}_2 \rangle \,. \tag{5.11}$$

Notice that $\overline{B}_1 \otimes \overline{B}_2$ is a lifting of B. This follows from the fact that $\langle \overline{B}_1, \overline{B}_2 \rangle$ lifts B and that $\otimes$ lifts the identity functor.

Let $\overline{T}\colon \mathcal{E} \to \mathcal{E}$ be a lifting of a functor $T\colon \mathcal{A} \to \mathcal{A}$. To obtain the compatibility of the contextual closure for a composite lifting $\overline{B}_1 \otimes \overline{B}_2$ using Theorem 5.2.7, one needs to prove that a distributive law $\lambda\colon TB \Rightarrow BT$ under consideration lifts to a distributive law of $\overline{T}$ over $\overline{B}_1 \otimes \overline{B}_2$. As a consequence of Theorem 5.4.1 below, it suffices to show that there are distributive laws for the two liftings $\overline{B}_1$ and $\overline{B}_2$ separately, both sitting above λ.

This additionally requires a natural transformation $\gamma\colon \overline{T}\otimes \Rightarrow \otimes\overline{T}^2$. Here

$$\overline{T}^2\colon \mathcal{E} \times_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \times_{\mathcal{A}} \mathcal{E}\,,$$

defined by the universal property of the pullback $\mathcal{E} \times_{\mathcal{A}} \mathcal{E}$, is simply the restriction of the functor $\overline{T}^2\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E} \times \mathcal{E}$. If $\overline{T}$ is the canonical relation lifting $\mathsf{Rel}(T)$ and $\otimes$ is relational composition, then the existence of γ in the theorem amounts to the inclusion $\mathsf{Rel}(T)(R \otimes S) \subseteq \mathsf{Rel}(T)(R) \otimes \mathsf{Rel}(T)(S)$, which holds for any Set functor T (Lemma 3.2.4).

Theorem 5.4.1. *Suppose we have*

1. *a lifting $\overline{T}$ of T;*
2. *a natural transformation $\gamma\colon \overline{T}\otimes \Rightarrow \otimes\overline{T}^2$ above $\mathsf{id}\colon T \Rightarrow T$;*
3. *two liftings $\overline{B}_1$ and $\overline{B}_2$ of B;*
4. *two natural transformations $\overline{\lambda}_1\colon \overline{T}\,\overline{B}_1 \Rightarrow \overline{B}_1\overline{T}$ and $\overline{\lambda}_2\colon \overline{T}\,\overline{B}_2 \Rightarrow \overline{B}_2\overline{T}$ sitting above the same $\lambda\colon TB \Rightarrow BT$.*

Then there exists $\overline{\lambda}\colon \overline{T}(\overline{B}_1 \otimes \overline{B}_2) \Rightarrow (\overline{B}_1 \otimes \overline{B}_2)\overline{T}$ above λ.

Proof. Define $\overline{\lambda}$ on a component P in $\mathcal{E}$ as follows:

$$\overline{T}(\overline{B}_1 P \otimes \overline{B}_2 P) \xrightarrow{\gamma_{\langle \overline{B}_1, \overline{B}_2\rangle P}} (\overline{T}\,\overline{B}_1 P) \otimes (\overline{T}\,\overline{B}_2 P) \xrightarrow{(\overline{\lambda}_1)_P \otimes (\overline{\lambda}_2)_P} \overline{B}_1\overline{T}P \otimes \overline{B}_2\,\overline{T}P$$

Notice that $((\overline{\lambda}_1)_P, (\overline{\lambda}_2)_P)$ is indeed a morphism in $\mathcal{E} \times_{\mathcal{A}} \mathcal{E}$ since $\overline{\lambda}_1$ and $\overline{\lambda}_2$ sit above a common λ. Naturality of $\overline{\lambda}$ follows from naturality of $\overline{\lambda}_1$, $\overline{\lambda}_2$ and γ. Finally, $\overline{\lambda}$ sits above λ since γ sits above $\mathsf{id}\colon T \Rightarrow T$ and $\otimes$ is a lifting of the identity functor. $\square$

5.4.1 Simulation up-to

We recall simulations for coalgebras as introduced in [HJ04]. An *ordered* functor is a pair $(B, \sqsubseteq)$ consisting of a functor $B\colon \mathsf{Set} \to \mathsf{Set}$ with a factorization through the category Pre of preorders and monotone maps:

$$\begin{tikzcd} & \mathsf{Pre} \arrow[d] \\ \mathsf{Set} \arrow[ur, "\sqsubseteq"] \arrow[r, "B"'] & \mathsf{Set} \end{tikzcd}$$

Such an ordered functor gives rise to a *constant* relation lifting $\overline{\sqsubseteq}$ of B defined by $\overline{\sqsubseteq}(R \subseteq X \times X) = \sqsubseteq_{BX}$. Then the *lax relation lifting* $\mathsf{Rel}(B)^{\sqsubseteq}$ is defined compositionally by

$$\mathsf{Rel}(B)^{\sqsubseteq} = \overline{\sqsubseteq} \otimes \mathsf{Rel}(B) \otimes \overline{\sqsubseteq}$$

where $\otimes\colon \mathcal{E} \times_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ is the relational composition functor (using the notation of (5.11) above).

Let $\delta\colon X \to BX$ be a B-coalgebra. A $\mathsf{Rel}(B)^{\sqsubseteq}_{\delta}$-invariant, where $\mathsf{Rel}(B)^{\sqsubseteq}_{\delta}$ abbreviates $\delta^* \circ \mathsf{Rel}(B)^{\sqsubseteq}_{X}$, is called a *simulation*. The coinductive predicate defined by $\mathsf{Rel}(B)^{\sqsubseteq}_{\delta}$ is called *similarity*.

Example 5.4.2. We list a few examples of ordered functors and their associated notion of simulations, and refer to [HJ04] for many more.

1. Let $\mathbb{S}$ be a semiring equipped with a partial order $\leq$. The functor $BX = \mathbb{S} \times X^A$ is ordered, with $\sqsubseteq_{BX}$ defined as $(p, \varphi) \sqsubseteq_{BX} (q, \psi)$ iff $p \leq q$ and $\varphi = \psi$. Then $\mathsf{Rel}(B)^{\sqsubseteq}$ coincides with the lifting $\overline{B}$ defined in Section 5.3.1.

2. The functor $BX = (\mathcal{P}_{\omega}X)^A$ is ordered by pointwise subset inclusion. In this case, a simulation is the standard notion on transition systems: a relation $R \subseteq X \times X$ such that for all $(x, y) \in R$: if $x \xrightarrow{a} x'$ then there exists y' such that $y \xrightarrow{a} y'$ and $(x', y') \in R$. Given a transition system, similarity is the greatest simulation.

An ordered functor B is called *stable* if $(\mathsf{Rel}(B)^{\sqsubseteq}, B)$ is a fibration map [HJ04]. Since polynomial functors, as well as the one for LTSs, are stable [HJ04], the following results hold for the coalgebras in Example 5.4.2.

Proposition 5.4.3. *If B is a stable ordered functor, then the behavioural equivalence closure* bhv*, the self closure* slf *and the transitive closure* tra *(all defined in Section 5.2.2) are* $\mathsf{Rel}(B)^{\sqsubseteq}_{\delta}$*-compatible.*

Proof. Compatibility of bhv comes from Theorem 5.2.1, which only requires that $(\mathsf{Rel}(B)^{\sqsubseteq}, B)$ is a fibration map. Compatibility of slf and tra comes from Corollary 5.2.5: as shown in [HJ04, Lemma 5.3], stable functors satisfy condition (5.3), i.e., for all relations $R, S \subseteq X^2$: $\mathsf{Rel}(B)^{\sqsubseteq}(R) \otimes \mathsf{Rel}(B)^{\sqsubseteq}(S) \subseteq \mathsf{Rel}(B)^{\sqsubseteq}(R \otimes S)$. $\square$

If $BX = (\mathcal{P}_{\omega}X)^A$ then bhv maps a relation R to $\sim \circ R \circ \sim$ where $\sim$ is bisimilarity, whereas slf maps R to $\lesssim \circ R \circ \lesssim$, where $\lesssim$ is similarity.

We proceed to consider the compatibility of the contextual closure, for which we assume an abstract GSOS specification $\rho\colon \Sigma(B \times \mathsf{Id}) \Rightarrow B\Sigma^*$. Such a specification ρ is *monotone* if, for any X, the restriction of $\rho_X \times \rho_X$ to $\mathsf{Rel}(\Sigma)(\sqsubseteq_{BX} \times \Delta_X)$ corestricts to $\sqsubseteq_{B\Sigma^* X}$. If Σ is a polynomial functor representing a signature, then this means that for any operator σ (of arity n) we have

$$\frac{b_1 \sqsubseteq_{BX} c_1 \quad \ldots \quad b_n \sqsubseteq_{BX} c_n}{\rho_X(\sigma(\mathbf{b}, \mathbf{x})) \sqsubseteq_{B\Sigma^* X} \rho_X(\sigma(\mathbf{c}, \mathbf{x}))}$$

where $\mathbf{b}, \mathbf{x} = (b_1, x_1), \ldots, (b_n, x_n)$ with $x_i \in X$ and similarly for $\mathbf{c}, \mathbf{x}$. If $\sqsubseteq$ is the order on the functor for LTSs, then monotonicity corresponds to the *positive GSOS* format [FS10], which is GSOS without negative premises. Monotonicity turns out to be precisely the condition needed to apply Theorem 5.2.9.

Proposition 5.4.4. *Let* $\rho\colon \Sigma(B \times \mathsf{Id}) \Rightarrow B\Sigma^*$ *be a monotone abstract GSOS specification and* $(X, \alpha, \langle \delta, \mathsf{id} \rangle)$ *be a* $\rho^\dagger$*-bialgebra. Then* $\mathsf{ctx} = \coprod_\alpha \circ \mathsf{Rel}(\Sigma^*)$ *is* $(\mathsf{Rel}(B)^{\sqsubseteq} \times \mathsf{Id})_{\langle \delta, \mathsf{id} \rangle}$*-compatible.*

Proof. To obtain the desired compatibility from Theorem 5.2.7, we need to prove that there exists a distributive law $\overline{\rho^\dagger}$ of $\mathsf{Rel}(\Sigma^*)$ over $\mathsf{Rel}(B)^{\sqsubseteq} \times \mathsf{Id}$, sitting above $\rho^\dagger$.

First, observe that the lifting $\mathsf{Rel}(B)^{\sqsubseteq} \times \mathsf{Id}$ of $B \times \mathsf{Id}$ decomposes as

$$(\overline{\sqsubseteq} \times \overline{\mathsf{Id}}) \otimes (\mathsf{Rel}(B) \times \mathsf{Id}) \otimes (\overline{\sqsubseteq} \times \overline{\mathsf{Id}})$$

where $\overline{\mathsf{Id}}$ is the constant functor mapping $R \subseteq X \times X$ to Δ_X. Notice that $\overline{\mathsf{Id}}$ is a lifting of the identity functor (but it is not the identity functor itself).

By Theorem 5.4.1, proving the existence of $\overline{\rho^\dagger}$ above $\rho^\dagger$ reduces to proving that there exist two natural transformations

1. $\overline{\rho^\dagger}_1\colon \mathsf{Rel}(\Sigma^*)(\mathsf{Rel}(B) \times \mathsf{Id}) \Rightarrow (\mathsf{Rel}(B) \times \mathsf{Id})\mathsf{Rel}(\Sigma^*)$, and
2. $\overline{\rho^\dagger}_2\colon \mathsf{Rel}(\Sigma^*)(\overline{\sqsubseteq} \times \overline{\mathsf{Id}}) \Rightarrow (\overline{\sqsubseteq} \times \overline{\mathsf{Id}})\mathsf{Rel}(\Sigma^*)$,

both sitting above $\rho^\dagger$. (Notice that since the functor $\overline{T}$ of the theorem is a canonical relation lifting, the required γ exists.)

For item 1, observe that the required natural transformation exists since both functor liftings are canonical; see Section 5.2.3 (below Theorem 5.2.7).

For item 2, the task reduces by Theorem 5.2.9 to showing that there is

$$\overline{\rho}\colon \mathsf{Rel}(\Sigma)(\overline{\sqsubseteq} \times \overline{\mathsf{Id}}) \Rightarrow \overline{\sqsubseteq} \circ \mathsf{Rel}(\Sigma^*)$$

above ρ. But this is precisely monotonicity, as introduced above. Further, Theorem 5.2.9 requires that there exists a natural transformation $\gamma\colon \mathsf{Rel}(\Sigma) \circ \overline{\mathsf{Id}} \Rightarrow \overline{\mathsf{Id}} \circ \mathsf{Rel}(\Sigma)$. Since $\overline{\mathsf{Id}}$ is the functor mapping any relation to the diagonal over its carrier, γ exists if $\mathsf{Rel}(\Sigma)(\Delta_X) \subseteq \Delta_{\Sigma X}$, which holds for any Σ (Lemma 3.2.4). Thus, as a consequence of Theorem 5.2.9, we obtain the desired natural transformation.

The existence of $\overline{\rho^\dagger}_1$ and $\overline{\rho^\dagger}_2$ ensures, by Theorem 5.4.1 and Theorem 5.2.7, that ctx is $(\mathsf{Rel}(B)^{\sqsubseteq} \times \mathsf{Id})_{\langle \delta, \mathsf{id} \rangle}$-compatible. □

A direct consequence of this result is that simulation up-to is compatible on any model of a positive GSOS specification.

Further, Theorem 2.4.6 states that simulation up-to (precongruence) for languages is sound whenever the operations under consideration are given by monotone behavioural differential equations. But any such operation can also be expressed in monotone GSOS for the ordered functor $BX = 2 \times X^A$ (see Example 5.4.2). Thus, we obtain the compatibility of the contextual closure by Proposition 5.4.4, and since the reflexive and transitive closure are compatible as well (Section 5.3.1), composing them together yields an alternative proof of Theorem 2.4.6.

5.5 Discussion and related work

We showed how up-to techniques fit into the setting of coinduction in a fibration, yielding a general and modular theory of coinduction up-to. This goes beyond the previous chapter in several ways: first, it allows other predicates than bisimilarity, including other binary predicates but also, e.g., unary predicates. Second, it can be instantiated to different base categories (in [BPPR14] an example of this is given by up-to-congruence for nominal automata).

Bisimulation up-to at the level of coalgebras was studied by Lenisa [Len99, LPW00]. The up-to-context technique for coalgebraic bisimulation was later derived as a special case of so-called λ-coinduction [Bar04]. Combining up-to techniques remained an open problem. In [Luo06], Sangiorgi's framework of up-to techniques [San98] is adapted to prove soundness of several up-to techniques for bisimulation, based on relation lifting and thus strongly related to the development in Chapter 4, but combinations of enhancements are not considered there. Finally [ZLL$^+$10] introduces bisimulation up-to where the notion of bisimulation is based on a specification language for polynomial functors. All of the above works focus on bisimulation, rather than general coinductive predicates.

We conclude with a short, technical summary of the main soundness results of this chapter. The up-to techniques and soundness results are all formulated in terms of a bifibration $p\colon \mathcal{E} \to \mathcal{A}$, a coalgebra $\delta\colon X \to BX$ for a functor $B\colon \mathcal{A} \to \mathcal{A}$ (that models the system of interest) and a lifting $\overline{B}\colon \mathcal{E} \to \mathcal{E}$ of $\mathcal{A}$ (that determines the coinductive predicate of interest). By proving a functor G to be *$\overline{B}_\delta$-compatible*, the construction of invariants up to G is a sound proof technique for the coinductive predicate determined by the lifting $\overline{B}$ on the coalgebra δ. The table below lists the main compatibility results, based on conditions on the functors involved. For ctx_α, we assume an algebra $\alpha\colon TX \to X$ for a functor T with a lifting $\overline{T}$, and a distributive law of the functor T over the functor B.

Name	Notation	Condition $\overline{B}_\delta$-compatibility
Behavioural equivalence	bhv_δ	$(\overline{B}, B)$ is a fibration map
Contextual closure	ctx_α	(X, α, δ) is a λ-bialgebra, and there is a distributive law of $\overline{T}$ over $\overline{B}$ above λ

If $p\colon \mathsf{Rel} \to \mathsf{Set}$ is the relation fibration, then we have the following additional results.

Name	Notation	Condition $\overline{B}_\delta$-compatibility
Diagonal functor	diag	$\Delta_{BX} \subseteq \overline{B}(\Delta_X)$
Inverse functor	inv	$(\overline{B}R)^{op} \subseteq \overline{B}(R^{op})$ for all $R \subseteq X^2$
Relational comp.	$\otimes$	$\overline{B}(R) \otimes \overline{B}(S) \subseteq \overline{B}(R \otimes S)$ for all $R, S \subseteq X^2$
Self closure	slf_δ	$\otimes$ is $\overline{B}_\delta$-compatible
Transitive closure	tra	$\otimes$ is $\overline{B}_\delta$-compatible
Equivalence closure	eq	diag, inv and $\otimes$ are $\overline{B}_\delta$-compatible

While the techniques introduced in this chapter are very general, they are also quite technical and require significant background knowledge to be understood. It would be a worthwile effort to develop natural specification techniques for coinductive predicates, in which compatibility can be established easily, or even automatically. In this chapter we have suggested one approach in this direction: the use of *modalities* to specify coinductive predicates, so that, under suitable assumptions, the required condition for compatibility of the contextual closure is a decidable property. We leave a more extensive investigation for future work.