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Phenomenology and dynamics of a Majorana Josephson junction

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We derive a generic phenomenological model of a Majorana Josephson junction that accounts for avoided crossing of Andreev states, and investigate its dynamics at constant bias voltage to reveal an unexpected pattern of an any-π Josephson effect in the limit of slow decoherence: sharp peaks in noise not related to any definite fraction of Josephson frequency.

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Recently, attention has been paid to proposals of solid-state realizations of Majorana fermions. While the first proposal[1] concerned non-Abelian excitations in a 5/2 fractional quantum Hall effect (FQHE) in semiconductor heterostructures, most proposals[2,3] exploited exotic superconductors where Majorana fermions correspond to zero-energy states of an effective Bogoliubov–de Gennes (BdG) Hamiltonian. The Majorana states are instrumental for the realization of topological quantum computation.[4] More recent contributions[5] utilize the proximity effect from a conventional superconductor, either in nanowires in a strong magnetic field and with a strong spin-orbit interaction,[6–8] or in topological insulators.[9,10] This brings the Majoranas close to experimental realization, and underlines the importance of reliable experimental signatures of their presence. Among the signatures are half-integer conductance quantization[11] and the 4π Josephson effect in superconductor-superconductor (SS) junctions.[7,12,13]

No 4π periodicity is to be seen in the stationary ground state of the junction. It can only be observed[14,15] in dynamics induced, for instance, by a dc voltage bias. The unambiguous signature of this 4π periodicity is the current noise peak at half of the Josephson frequency $\omega_j = 2\,eV/\hbar$.[14] The avoided crossing of Andreev states is intrinsic for finite systems and restores the 2π periodicity of the junction ground state.[2,12,16] This has been recently confirmed by detailed calculations of the Andreev spectrum of the nanowire-based SS junctions.[17]

In this Rapid Communication, we put forward a generic phenomenological model of a Majorana Josephson junction and demonstrate that the dynamics in the junction are substantially richer than thought. In particular, the sharp peaks in the noise spectrum of a voltage-biased junction are not generally confined to any definite fraction of ω_j: One can talk of an any-π Josephson effect in this context. Experimental observation of these singularities would give a robust proof of the existence of Majoranas and open up the possibilities for quantum manipulation of these states. Our treatment of dynamics encompasses the Landau-Zener tunneling at the avoided crossings, decoherence, relaxation, and quasiparticle poisoning.

We exemplify with a nanowire setup (Fig. 1), although the same phenomenology extends to topological insulators. A nanowire mounted on a single superconducting lead develops a topologically nontrivial state in a parameter range of magnetic fields and gate voltages.[7] Two Majorana states emerge at the wire ends. A Majorana Josephson junction is formed by mounting the wire on two leads biased with a superconducting phase difference ϕ. Two extra Majorana states $\gamma_{2,3}$ emerge at the junction, in addition to the end states $\gamma_{1,4}$. The overlap between γ_2 and γ_3 is strong but does depend on phase and vanishes at a certain ϕ_0. If one disregards the end states,[7,12,13] the resulting energies are 4π periodic in ϕ and the resulting states are of indefinite parity. We exemplify this dependence with $E(\phi) = \pm\tilde{\epsilon}\sin(\frac{\phi-\phi_0}{2})$, $\tilde{\epsilon}$ being a typical coupling energy of $\gamma_{2,3}$. To fix the parity, it is paramount to bring the end states to the picture. We developed[16] a scattering matrix theory where the 2π periodicity is proven from the topological properties of the scattering matrix. In a nutshell, the crossing of Andreev levels is avoided. We need a practical Hamiltonian to describe the details of the situation in the vicinity of ϕ_0. That can be rigorously derived from the scattering approach, yet we opt here for a simple heuristic deviation in terms of overlaps of Majorana states.

These overlaps are exponentially small for long wires, $\propto\exp(-L/2\xi)$, L being the wire length, the localization length ξ being of the order of the spin-orbit length L_{so}. For InAs wires,[18] $L_{\mathrm{so}} = 0.2\ \mu$m, and L would not exceed 2 μm since inevitable disorder forbids a topological state for longer wires. This sets the biggest exponential suppression to $\simeq 10^{-2}$. Owing to the exponential suppression, the direct overlap t_{14} between the end states is much smaller than the overlaps between the end and the junction states, and can be disregarded. This brings us to the following Hamiltonian,

$$\hat{H} = i\tilde{\epsilon}(\phi-\phi_0)\hat{\gamma}_2\hat{\gamma}_3 + i(t_{12}\hat{\gamma}_2 + t_{13}\hat{\gamma}_3)\hat{\gamma}_1 + i(t_{42}\hat{\gamma}_2 + t_{43}\hat{\gamma}_3)\hat{\gamma}_4, \quad (1)$$

that is valid in the vicinity of the crossing point and provides a generic phenomenological model of a Majorana Josephson junction. Here, the overlaps t are real, and $\hat{\gamma}_{1\text{–}4}$ are self-conjugated anticommuting Majorana operators.[19] We present a detailed derivation of the Hamiltonian (1) in Ref. 20.

It is instructive to give the eigenenergies of the full many-body states of the Hamiltonian, rather than the associated Andreev levels. The energies of these states are sums over the energies of the Andreev levels with taking the filling of the levels into account. The Hamiltonian conserves the parity of the particle number and therefore gives rise to eigenstates with either an odd or even number of particles. There are two

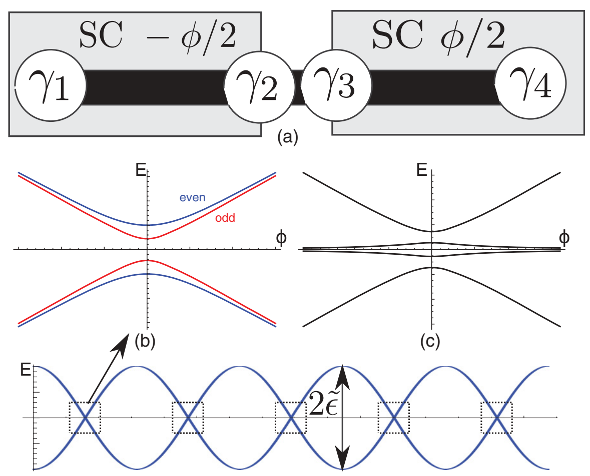

FIG. 1. (Color online) (a) A Majorana Josephson junction is formed by mounting a nanowire (black) on two superconducting leads (gray), resulting in four Majoranas γ_{1-4}. (b) The energies of the many-body junction states vs phase near an avoided crossing point. (c) Corresponding Andreev levels. (d) The energies of the junction states vs phase at a bigger scale. Far from the avoided crossing points, the energies of even and odd states are indistinguishably close. This corresponds to two zero-energy Andreev levels.

eigenvalues of opposite sign for each parity,

$$E_{\pm,o,e} = \pm\sqrt{G_{o,e}^2 + \frac{\tilde{\epsilon}^2}{4}(\phi - \phi_0)^2}, \tag{2}$$

where

$$G_{o,e} = \tfrac{1}{2}\sqrt{(t_{12} \pm t_{43})^2 + (t_{13} \mp t_{42})^2}, \tag{3}$$

and the $\pm$ sign is chosen such that $G_e > G_o$. Their phase dependence [Fig. 1(b)] gives a familiar glimpse of avoided level-crossing hyperbolas, with $G_{o,e}$ being the minimum energy splittings of odd or even states, respectively (when the difference between even and odd cases is insignificant, we will denote both values G). The two positive energies of the associated Andreev levels are given by $E_{1,2} = |E_e| \pm |E_o|$ [Fig. 1(c)]. This characteristic form is conformed by the numerical calculations based on microscopic models,[16,17] proving the validity of the Hamiltonian (1). The absence of the direct overlap t_{14} leads to a special property: The phase-dependent term describing the overlap of $\hat{\gamma}_2$,$\hat{\gamma}_3$ anticommutes with the rest of the terms. This guarantees the energies to be even in phase and to merge far from $\phi = \phi_0$; these properties would be absent for a most general four-Majorana Hamiltonian.

Let us notice that the junction in either an odd or even state is nothing but a qubit that is similar to other superconducting qubits that commonly exploit the avoided level crossing.[21–23] One can employ quantum manipulation of the resulting Majorana states by changing the superconducting phase in time. For instance, following Ref. 21, one can prepare the qubit in the ground state reasonably far from the crossing point $\phi = \phi_1$, and give a pulse that brings the junction to $\phi = \phi_0$. This will cause Rabi oscillations with frequency $2G/\hbar$ that can be detected by measuring the probabilities to find the qubit in the ground or excited state after the pulse as functions of pulse duration.

Here we restrict ourselves to the case of immediate experimental relevance where the junction is biased by a dc voltage V so that the phase ϕ is swept linearly with time, $\dot{\phi} = 2\,eV/\hbar$. In a usual Josephson junction where the energy levels are 2π periodic, such bias results in coherent oscillations of the supercurrent $I(\phi) = 2e/\hbar\partial E(\phi)/\partial\phi$ with Josephson frequency $\omega_j = 2\,eV/\hbar$.[24] The idea behind the 4π Josephson effect[25] is an apparent 4π periodicity of energy levels in the limit of vanishing G, and this suggests the oscillations at a half of Josephson frequency, $I(t) = \pm I_{\rm m}\cos(\omega_j t/2)$, $I_{\rm m} \equiv e\tilde{\epsilon}/\hbar$, although these oscillations cannot be coherent owing to random switching between the two branches $\pm$ of the energy spectrum. The signature of 4π periodicity is rather a sharp peak in the spectral density of the current *noise*,[14] with the width of the order of the switching rate, and integrated intensity being given by $I_m^2/2$, the average square of the current. For this simplified picture to hold, one should require sufficiently small voltages, $V \ll \tilde{\epsilon}$. Failure to satisfy this condition results in proliferation to higher energy levels and finally to a continuous spectrum, thus increasing the peak width to the values of the order of Δ and rendering noise peaks undetectable.[14,17]

In this Rapid Communication, we address the noise in Majorana Josephson junctions at smaller voltages. Evidently, the avoided level crossing results in the usual Josephson effect in the limit $V \to 0$. The complex and interesting crossover between 2π and 4π regimes involves Landau-Zener (LZ) tunneling upon crossing a point $\phi = \phi_0 + 2\pi n$ in the vicinity of the point. The parity obviously does not change, and for each parity we have a classic setup of LZ tunneling[27] between two levels. The values of the qubit wave function before and after LZ scattering are related by a 2×2 unitary matrix,

$$\hat{S}_{o,e} = \begin{pmatrix} \sqrt{1-P_{o,e}} & -e^{i\chi}\sqrt{P_{o,e}} \\ e^{-i\chi}\sqrt{P_{o,e}} & \sqrt{1-P_{o,e}} \end{pmatrix}, \tag{4}$$

where the probability of LZ tunneling is given by

$$P_{o,e} = \exp\left(-\frac{4\pi}{eV}\frac{G_{o,e}^2}{\tilde{\epsilon}}\right). \tag{5}$$

This suggests an importance of a voltage scale $eV_0 \equiv 4\pi G^2/\tilde{\epsilon} \ll \tilde{\epsilon}$ at which the probabilities are of the order of 1 and the crossover between 2π and 4π regimes is expected. We stress that the probabilities are generally different for odd and even states that permits the identification of these states that are hardly distinguishable otherwise.

The quantum dynamics are affected by the processes of relaxation, dephasing, and quasiparticle poisoning (Fig. 2) that occur throughout the timeline with no peculiarities near the crossing points. We assume low temperature $k_BT \ll \tilde{\epsilon}$, so that the relaxation is always from the higher to lower energy state with a rate $\Gamma_r(\phi)$. The decoherence suppresses the nondiagonal elements of the density matrix [with the rate $\Gamma_d(\phi)$] not affecting the diagonal ones. We assume the fluctuation of the phase ϕ to be the main source of the decoherence, in this case $\Gamma_d(\phi) \propto I^2(\phi)$. The quasiparticle transfer processes account for a parity change. They may be due to stray quasiparticles in the bulk superconductor that come to the junction with the energies of the order of the superconducting energy gap $\Delta > \tilde{\epsilon}$ and lose this energy either by adding or annihilating

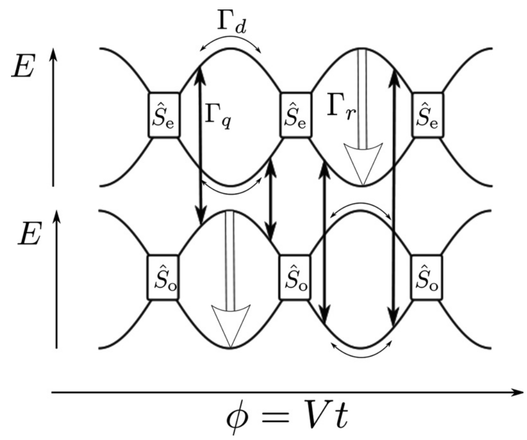

FIG. 2. Processes affecting the junction dynamics. In between the parity-dependent Landau-Zener scatterings (described by 2×2 matrices $\hat{S}_{o,e}$) the junction is subject to dephasing (with a rate Γ_d), relaxation (Γ_r), and quasiparticle poisoning (Γ_q).

a quasiparticle in Andreev levels under consideration. Due to a significant initial quasiparticle energy, the probabilities of finding the junction in either the upper or lower state after a quasiparticle transfer are the same. The quasiparticle rate Γ_q does not depend on the phase ϕ.

This results in the straightforward but lengthy equation for the density matrices $\hat{\rho}_{o,d}$ that is found in Ref. 20. We solve this equation with continuity conditions $\hat{\rho}_{o,d}(t_{ac}+0) = \hat{S}_{o,d}\hat{\rho}_{o,d}(t_{ac}-0)\hat{S}_{o,d}^{-1}$, t_{ac} corresponding to time moments of the crossings, and compute the correlator of current operators

$$S(\omega) = \int_{-\infty}^{\infty} dt_1 \int_0^{\frac{2\pi}{\omega_j}} dt_2 e^{i\omega(t_1-t_2)} \langle\langle I(t_1) I(t_2)\rangle\rangle \qquad (6)$$

that gives the spectral density of the current noise. We concentrate on two limiting cases of fast (Fig. 3) and slow

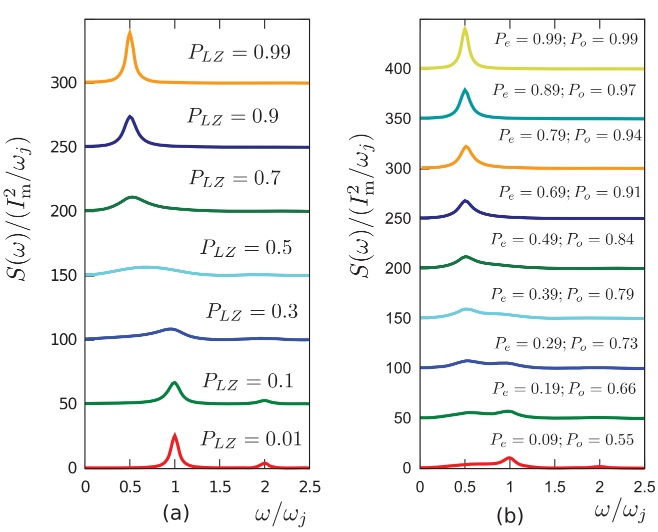

FIG. 3. (Color online) The spectral intensity $S(\omega)$ of the current noise for a set of V corresponding to LZ probabilities shown, in the limit of fast decoherence. (a) Indistinguishable parities $P_e = P_o = P_{\rm LZ}$; (b) $P_e \neq P_o$ ($G_e = 4G_o$). Distinct peaks at multiples of ω_j at $P \ll 1$ transmute into a single peak at $\omega_j/2$ at $P \approx 1$. ($\Gamma_r = \Gamma_q = 0.02\omega_j$ for all plots.)

(Fig. 4) decoherence. In both cases, we assume slow relaxation and poisoning, $\omega_j \gg \Gamma_r, \Gamma_q$.

"Fast" implies the quantum coherence is lost during a period of the Josephson oscillations, $\Gamma_d \gg \omega_j$, and the equation for the density matrix reduces to a master equation. Figure 3(a) shows the spectral density for equal LZ probability for even and odd states, $G_e \approx G_o$. The voltage growth from the lowermost to upper curve results in increased $P_{\rm LZ}$. At low voltage ($P_{\rm LZ} \ll 0$), the noise peaks at ω_j as well as at its multiples, the latter manifesting nonsinusoidal $I(\phi)$. This proves a usual periodicity. At higher voltage where $P_{\rm LZ} \approx 1$, we see a single peak at $\omega_j/2$ manifesting 4π periodicity. In both limiting cases, the peak widths $\simeq \Gamma_{r,qp}$. The important feature is the absence of any distinguishable peaks at intermediate $P_{\rm LZ}$. The reason is the LZ tunneling causing incoherent switching at almost any crossing point. The peaks acquire a width $\simeq \omega_j \gg \Gamma_{r,qp}$ and correspondingly reduce their height to the background level.

In Fig. 3(b) the LZ probabilities are very different at the crossover corresponding to $G_e/G_o = 4$. Now one can distinguish the peaks at both $\omega_j/2$ and ω_j in the crossover region, though they are reduced in height in comparison with the limiting cases. The explanation is the parity separation in the time domain. Since $\Gamma_q \ll \omega_j$, the parity persists over many periods between the random switches. While the junction is in an even parity state, $P_{\rm LZ} \approx 1$, and during this time interval the noise at $\omega_j/2$ is generated. While the junction is in an odd parity state, almost no LZ tunneling takes place, and the noise is generated at Josephson frequency. The experimental observation of two peaks would thus prove the parity effect. One can also think of a more challenging observation where the noise can be resolved quickly, that is, at a time scale $<\Gamma_q^{-1}$. Such a noise measurement will monitor the parity of the junction in real time.

The results in the opposite limit of slow decoherence $\Gamma_d \ll \omega_j$ are decisively more complex and intriguing (Fig. 4). In this limit, the dynamics are truly quantum over many periods. An analytical analysis gives the positions of the noise peaks as well as the integrated noise intensities around each peak.[20] The most striking feature is an oscillatory dependence of the peak intensities and positions on voltage. This is a manifestation of quantum interference between the subsequent LZ tunneling events not suppressed by decoherence. Similar interference patterns have been predicted and observed for superconducting qubits in Refs. 23 and 27. We have found that a voltage-biased Majorana Josephson junction presents the simplest and most striking framework for this interference effect.

The quantum phase θ accumulated between the subsequent crossing points is estimated as

$$\theta = \int_{\rm period} dt \frac{\Delta E(\phi(t))}{\hbar} = \frac{8\tilde{\epsilon}}{\hbar\omega_j}, \qquad (7)$$

where $\Delta E(\phi)$ is the energy difference between levels of the same parity. The phase is big on the scale $eV/\tilde{\epsilon}$, and its increment by 2π gives an estimate of the oscillation period in voltage $\Delta V = (\pi/8)V(eV/\tilde{\epsilon}) \ll V$.

Importantly, the frequency positions of the additional noise peaks [Fig. 4(a)], which are the main Josephson peaks at multiples of ω_j, are not at any integer fractions of ω_j. In the context, we can dub this the any-π Josephson effect. It stems from a quasienergy splitting in a periodically driven qubit. At

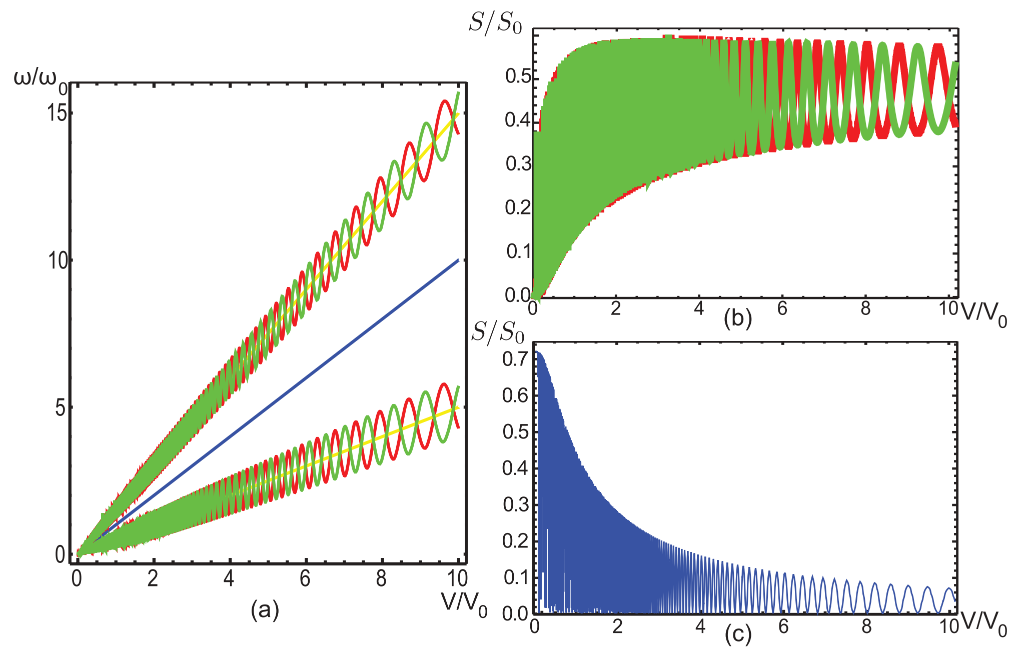

FIG. 4. (Color online) The slow decoherence limit. We chose $\tilde{\epsilon}/eV_0 = 30$ for all plots. (a) Frequency positions of the noise peaks vs V [$\omega_0 = (2e/\hbar)V_0$]. The main ones are at $n\omega_j$ while the positions of additional peaks oscillate, converging at $(n+1/2)\omega_j$. Only $n = 0,1$ are shown. (b) Integrated noise intensity (in units of $S_0 \equiv I_m^2/2$) of the first two additional peaks. (c) The same for the $n = 1$ main peak.

$V \gg V_0$, additional peaks converge at $(2n+1)\omega_j/2$ oscillation around this frequency. The spread of these oscillations $\Delta\omega$ does not vanish with increasing V: Rather, it increases following $\Delta\omega \simeq (2e/\hbar)\sqrt{VV_0}$. This proves that the any-π Josephson effect can be observed at voltages $V \gg V_0$ far beyond the crossover region. The width of the peaks is determined by Γ_d. From this, we estimate the minimum decoherence rate permitting the resolution of the peaks, $\Gamma_d \simeq (e/\hbar)\sqrt{VV_0}$. For the sake of simple drawing, we assumed indistinguishable parities such that $P_o = P_e$. If $P_o \neq P_e$, the additional peaks split once again, corresponding to the two parities.[20]

At $V \ll V_0$, the noise intensity is mainly concentrated at a main peak at ω_j. In the opposite limit, the intensity concentrates at the peaks converging to $\omega_j/2$ retaining oscillating features even at high voltage [Figs. 4(b) and 4(c)].

To summarize, we have derived a generic phenomenological Hamiltonian to describe a Majorana Josephson junction with an avoided Andreev level crossing, and investigated its quantum dynamics at a constant voltage bias at different regimes, putting emphasis on different periodicities of noise signatures. This is the only robust transport signature of the Majorana Josephson junction. While in the fast decoherence regime the signatures follow an expected pattern, the interference of the subsequent LZ tunneling events results in a complex any-π Josephson effect pattern in a slow decoherence regime. The experimental observation of the effects predicted will provide unambiguous signature of Majorana states in the Josephson junction and open up the perspectives of quantum manipulation and parity measurements in such junctions. One of the paths to observe the unusual periodicity is through measurement of Shapiro steps in voltage- or current-[28,29] biased setups.

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