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Anomalous Maxwell equations for inhomogeneous chiral plasma

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Using the chiral kinetic theory we derive the electric and chiral current densities in inhomogeneous relativistic plasma. We also derive equations for the electric and chiral chemical potentials that close the Maxwell equations in such a plasma. The analysis is done in the regimes with and without a drift of the plasma as a whole. In addition to the currents present in the homogeneous plasma (Hall current, chiral magnetic, chiral separation, and chiral electric separation effects, as well as Ohm's current) we derive several new terms associated with inhomogeneities of the plasma. Apart from various diffusionlike terms, we find also new dissipationless terms that are independent of relaxation time. Their origin can be traced to the Berry curvature modifications of the kinetic theory.

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I. INTRODUCTION

Nowadays there is a widespread interest in relativistic plasmas with a chiral asymmetry. This interest is driven by a recent progress in the understanding of basic properties of numerous relativistic or pseudorelativistic systems, ranging from Dirac and Weyl semimetals in condensed matter [1], whose low-energy quasiparticle excitations are described by the Dirac and Weyl equations, to strongly coupled quark-gluon plasma created experimentally in heavy-ion collisions [2–4], and to the primordial plasma in the early Universe [5,6]. One of the most unusual features of a chiral relativistic plasma, which is absent in a conventional nonrelativistic one, is the possibility of a macroscopic realization of the celebrated quantum anomalies [7]. For example, in a chirally asymmetric plasma in a magnetic field $\mathbf{B}$ with an imbalance between the number densities of right- and left-handed fermions (described semirigorously by the chiral chemical potential μ_5), the chiral anomaly induces a new type contribution to the electric current: $e\mathbf{j} = e^2\mu_5\mathbf{B}/(2\pi^2 c)$ [8–11]. The latter is known as the *chiral magnetic effect* [12,13]. The Maxwell equations, amended by such a contribution to the electric current, become *anomalous Maxwell equations* [14–23].

The inclusion of the anomalous currents drastically changes the self-consistent evolution of chiral charge densities and helical magnetic fields [14,18,19,21–23]. Moreover, the nonlinear interactions due to anomalous processes induce an effective mechanism for transferring the energy from magnetic modes with short wavelengths (strongly affected by dissipation) to modes with longer wavelengths (and longer lifetimes)—a phenomenon similar to the inverse cascade in ordinary magnetohydrodynamics [24], but driven not by turbulence. Depending on the chosen initial conditions, it was found that the helicity can be transferred from the fermions to the magnetic fields or vice versa. These results support the suggestion made in Ref. [25] that all degrees of freedom with nonvanishing axial charge are equally excited in the equilibrium.

The chiral anomaly in a relativistic matter exhibits itself not only via the chiral magnetic effect. Even if a chiral asymmetry is absent, a nonzero magnetic field can induce an axial current $\mathbf{j}_5 = e\mu\mathbf{B}/(2\pi^2 c)$ in a plasma with a nonzero electric chemical potential. This phenomenon is known as the *chiral separation effect* [26]. Another prediction of the anomalous Maxwell equations in a relativistic plasma, that utilizes an interplay of the chiral separation and chiral magnetic effects, is a new type of collective excitation known as the *chiral magnetic wave* [27]. Further development of these ideas is anomalous hydrodynamics, which contains new terms due to the chiral anomaly [22,28–35].

The present study investigates the role of inhomogeneities in the chiral plasma evolution. Since the chiral anomaly relation is local, it is clear that the electric and chiral chemical potentials should be inhomogeneous just like the helical electromagnetic fields. An important question is whether additional contributions to the electric current exist in the inhomogeneous case. The authors of Ref. [15] postulated the absence of such currents, while a different set of equations was proposed in a recent study [22]. The present paper derives such inhomogeneous currents in a systematic way. Our starting point in the analysis is the chiral kinetic theory [36–39].

This paper is organized as follows. We briefly review the chiral kinetic theory and kinetic equations in Sec. II. The expressions for the electric and axial currents and equations for the local equilibrium chemical potentials in inhomogeneous chiral plasma in the drifting state are derived in Sec. III. The electric and chiral currents to the second order in electromagnetic field and derivatives are calculated in Sec. IV in the case where neutral particles exert a substantial drag on the system. The summary and conclusions are given in Sec. V. Some table integrals and useful relations are collected in the Appendix.

II. CHIRAL KINETIC THEORY

The chiral kinetic theory describes a time evolution of the one-particle distribution functions $f_\lambda(t,\mathbf{x},\mathbf{p})$ for the right- ($\lambda=+$) and left-handed ($\lambda=-$) fermions. It was recently suggested [36,40,41] that chiral fermions in external electromagnetic fields are described by the *chiral kinetic theory* given by [36–39]

$$\frac{\partial f_\lambda}{\partial t}+\frac{1}{1+\frac{e}{c}\mathbf{B}\cdot\Omega_\lambda}\left[\left(e\mathbf{E}+\frac{e}{c}\mathbf{v}\times\mathbf{B}+\frac{e^2}{c}(\mathbf{E}\cdot\mathbf{B})\Omega_\lambda\right)\cdot\frac{\partial f_\lambda}{\partial\mathbf{p}}+\left(\mathbf{v}+e\mathbf{E}\times\Omega_\lambda+\frac{e}{c}(\mathbf{v}\cdot\Omega_\lambda)\mathbf{B}\right)\cdot\frac{\partial f_\lambda}{\partial\mathbf{x}}\right]=I_{\mathrm{coll}},\qquad(1)$$

where the factor $(1+e\mathbf{B}\cdot\Omega_\lambda/c)^{-1}$ accounts for the correct phase-space volume [40,41] and $\Omega_\lambda=\lambda\mathbf{p}/(2|\mathbf{p}|^3)$ is the *Berry curvature* [42]. The Berry curvature is a crucial ingredient in the chiral kinetic theory that allows one to capture the fermionic nature of particles [36–39]. The group velocity is defined from the quasiparticle energy as follows: $\mathbf{v}\equiv\partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$. By imposing the constraint of the Lorentz invariance in Ref. [36], it was suggested that the dispersion relation for chiral fermions in a magnetic field $\mathbf{B}$ should be taken in the form $\epsilon_{\mathbf{p}}=c|\mathbf{p}|-\lambda e\mathbf{p}\cdot\mathbf{B}/|\mathbf{p}|^2$, which is valid to linear order in the field when $|e\mathbf{B}|/(c\mathbf{p}^2)\ll1$. Interestingly, however, such a definition for $\epsilon_{\mathbf{p}}$ may be problematic because the absolute value of the resulting group velocity, $v=c\sqrt{1+2e(\mathbf{B}\cdot\Omega_\lambda)/c+O(\mathbf{B}^2)}$, appears to be larger than the speed of light when $(\mathbf{B}\cdot\Omega_\lambda)>0$. In the present study, we will use the dispersion relation $\epsilon_{\mathbf{p}}=c|\mathbf{p}|$ for which the group velocity equals c.

Equation (1) is the kinetic equation for the distribution function f^i_λ of particles ($i=p$). A separate equation can be written down for antiparticles ($i=a$). The corresponding equation can be obtained by simply replacing $e\to-e$ and $\lambda\to-\lambda$ in Eq. (1). Below, when there is no risk of confusion, we will omit index i and assume that the expressions are given for particles. In the end, the results for charge densities and current densities will have to contain both particle and antiparticle contributions.

The term on the right-hand side of the kinetic equation (1) is a collision integral. In the simplest approximation, one can take $I_{\mathrm{coll}}=0$. This corresponds to the so-called collisionless limit, which is useful when the collective particle dynamics is driven primarily by averaged electromagnetic fields. One of the simplest approximations beyond the collisionless limit is the relaxation-time approximation with $I_{\mathrm{coll}}=-(f_\lambda-f^{(\mathrm{eq})}_\lambda)/\tau$ [43,44], where $\tau\sim1/[e^4T\ln(1/|e|)]$ [45] is the relaxation time and $f^{(\mathrm{eq})}_\lambda$ is the local equilibrium distribution function. In the absence of electromagnetic fields, it is the standard Fermi-Dirac distribution function

$$f^{(\mathrm{eq})}_\lambda(t,\mathbf{x},\mathbf{p})=\frac{1}{e^{[\epsilon_{\mathbf{p}}-\mu_\lambda(t,\mathbf{x})]/T}+1},\qquad(2)$$

where $\epsilon_{\mathbf{p}}=c|\mathbf{p}|$. The corresponding equilibrium distribution function for antiparticles is obtained by replacing $\mu_\lambda\to-\mu_\lambda$. Here we introduced the notation for the chemical potentials of the right- and left-handed fermions, $\mu_\lambda(t,\mathbf{x})=\mu(t,\mathbf{x})+\lambda\mu_5(t,\mathbf{x})$. It should be noted that, in the case of local equilibrium, the temperature T in distribution functions could also depend on the space-time coordinates, $T(t,\mathbf{x})$. However, in order to simplify our analysis below, we will neglect such a dependence in what follows. In the presence of electromagnetic fields, the choice of $f^{(\mathrm{eq})}_\lambda(t,\mathbf{x},\mathbf{p})$ is a delicate issue and we will discuss it in more detail below.

In a general case, the local equilibrium chemical potentials μ_λ evolve with time. Therefore, one of the central and crucial points of our analysis is the evolution equations for μ_λ which we derive from the kinetic equation. Integrating the left-hand side of the kinetic equation (1) over momentum leads to the continuity equations for the electric and chiral currents, where the latter equation includes the chiral anomaly term. This means that in order that the particles densities be conserved it is necessary that the integral of the collision integral over momentum be equal to zero. This requirement will play a crucial role in our analysis below.

A collision integral $I^{\mathrm{BGK}}_{\mathrm{coll}}=-(f_\lambda-\frac{n_\lambda}{n^{(0)}_\lambda}f^{(0)}_\lambda)/\tau$ of the Bhatnagar-Gross-Krook (BGK)-type [46] was used in Ref. [21]. Here n_λ is a local fermion number density, $f^{(0)}_\lambda$ is a given distribution function (for example, in the analysis of non-relativistic particle dynamics in Ref. [46], a Maxwell velocity distribution function was used), and $n^{(0)}_\lambda$ is determined by $f^{(0)}_\lambda$. Clearly, such a collision integral automatically conserves the particle number and agrees with the chiral anomaly relation. We found that when studying the evolution of electromagnetic fields and the chiral asymmetry in the presence of strong magnetic fields, where the chiral chemical potential evolves with time, it is crucial to use the local equilibrium function $f^{(\mathrm{eq})}_\lambda$ rather than a fixed $f_\lambda^{(0)}$, otherwise, the kinetic equation is not entirely consistent. We checked, however, that the results obtained for the BGK-type collision integral with $f_\lambda^{(\mathrm{eq})}$ instead $f_\lambda^{(0)}$ are not much different from those found in the relaxation-time approximation described above. Since the relaxation-time approximation with $I_{\mathrm{coll}} = -(f_\lambda - f_\lambda^{(\mathrm{eq})})/\tau$ is slightly simpler, we will use it in our analysis below.

The evolution of electric and magnetic fields is determined by the Maxwell equations

$$\nabla \cdot \mathbf{E} = 4\pi e n, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} e\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}. \tag{6}$$

By definition, the electric charge density is given by $en(\mathbf{x}) = \sum_i \sum_{\lambda=\pm} e_i n_\lambda^i(\mathbf{x})$, where the sum over i includes the contributions due to particles ($i = p$) and antiparticles ($p = a$). Note that the latter comes with the opposite sign. The number density of particles of a given chirality λ is given by

$$n_\lambda(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\left(1 + \frac{e}{c}\mathbf{B}\cdot\Omega_\lambda\right) f_\lambda(\mathbf{p}, \mathbf{x}). \tag{7}$$

Here, the factor $1 + e\mathbf{B}\cdot\Omega_\lambda/c$ in the integrand takes care of the correct phase-space volume. After multiplying the kinetic equation (1) by $1 + e\mathbf{B}\cdot\Omega_\lambda/c$, integrating over momentum $\mathbf{p}$, and using the Maxwell equations, we obtain the following relation:

$$\begin{aligned}\partial_t n_\lambda + \nabla\cdot\mathbf{j}_\lambda &= -\frac{e^2}{c}\int \frac{d^3p}{(2\pi)^3}(\Omega_\lambda\cdot\nabla_{\mathbf{p}} f_\lambda)\mathbf{E}\cdot\mathbf{B} \\ &= \frac{\lambda e^2 \mathbf{E}\cdot\mathbf{B} f_\lambda(\mathbf{p}=0)}{4\pi^2 c},\end{aligned} \tag{8}$$

where we integrated by parts in the last equality and used the following identity for the Berry curvature $\nabla_{\mathbf{p}}\cdot\Omega_\lambda = 2\pi\lambda\delta^3(\mathbf{p})$. The electric current density is given by $e\mathbf{j}(\mathbf{x}) = \sum_i \sum_{\lambda=\pm} e_i \mathbf{j}_\lambda^i(\mathbf{x})$, where the contribution due to particles of a given chirality is determined by [36,37,43]

$$\mathbf{j}_\lambda = \int \frac{d^3p}{(2\pi)^3}\left(\mathbf{v} + e\mathbf{E}\times\Omega_\lambda + \frac{e}{c}\mathbf{B}(\mathbf{v}\cdot\Omega_\lambda)\right) f_\lambda + \mathbf{j}_\lambda^{(\mathrm{curl})}. \tag{9}$$

We should note that the definition of the current in Ref. [43] differs from that in Ref. [36]. The latter has an extra term $\mathbf{j}_\lambda^{(\mathrm{curl})}$, the explicit form of which can be obtained by integrating the definition of current in Ref. [36] by parts,

$$\mathbf{j}_\lambda^{(\mathrm{curl})} = \nabla\times\int\frac{d^3p}{(2\pi)^3} f_\lambda \epsilon_{\mathbf{p}} \Omega_\lambda. \tag{10}$$

As we see, this contribution is a total curl and, thus, does not contribute to the continuity equation. However, such a current affects the Maxwell equation (6). From the structure of the corresponding equation and the structure of the current, we see that the integral on the right hand on Eq. (10) plays the role of a magnetization. Then, the current itself can be viewed as a "magnetization" current [47].

It is instructive to emphasize that, in the case of hot relativistic plasmas, it is essential that the complete expressions for the electric/chiral charge densities, as well as the corresponding currents contain the contributions of both particles and antiparticles. This is in contrast to the case of dense relativistic plasmas at low temperatures, $T \ll |\mu_\lambda|$, when the contributions of antiparticles are exponentially suppressed and, therefore, could be safely neglected. In the high-temperature regime, $T \gg |\mu_\lambda|$, which is of prime importance in cosmology, the antiparticle number densities are given by the same expressions as in Eq. (7), but in terms of the antiparticle distribution functions f_λ^{a}. Taking into account that antiparticles carry the opposite electric charge, we will find that, as expected, the corresponding high-temperature plasma will be almost neutral. Perhaps even more importantly, antiparticles will contribute approximately as much as particles to the electric current and, thus, will effectively double the result.

It is interesting to note that taking the contribution of antiparticles into account is critical also in order to obtain the correct chiral anomaly relation from Eq. (8). The right-hand side of the corresponding equation for particles is proportional to the local equilibrium distribution function $f_\lambda^{p(\mathrm{eq})}(\mathbf{p}=0)$, which depends on μ_λ and temperature. This seems to be at odds with the topological nature of the corresponding relation. The same is true for antiparticles. However, in view of the identity $\sum_i f_\lambda^i(\mathbf{p}=0) = 1$, the total chiral current $\mathbf{j}_5 = \sum_i \sum_{\lambda=\pm} \lambda \mathbf{j}_\lambda^i$ does satisfy the conventional continuity equation with the correct anomalous term $e^2\mathbf{E}\cdot\mathbf{B}/(2\pi^2 c)$. Note that, in the low-temperature regime, the correct result is saturated almost exclusively by the contribution of particles. (Of course, it is easy to check that the electric current $e\mathbf{j} = \sum_i \sum_{\lambda=\pm} e_i \mathbf{j}_\lambda^i$ satisfies the usual nonanomalous continuity equation.)

The complete set of the Maxwell equations (3) through (6), the kinetic equation (1) in the relaxation-time approximation supplemented by the definitions of the number densities (7) and currents (9) form a system of self-consistent equations for the one-particle distribution functions of the left- and right-handed fermions and electromagnetic fields. Therefore, to study, for example, the evolution of inhomogeneous magnetic fields and chiral asymmetry in magnetized plasma, one should solve the corresponding system of equations. The corresponding task is formidable. In order to simplify it, we will derive an approximate set of equations in the case where electromagnetic fields are weak.

Before proceeding to the derivation of a complete set of equations that describe the evolution of an inhomogeneous chiral plasma coupled to electromagnetic fields, it is instructive to identify generic classes of such plasmas, depending on their composition and underlying dynamics. In particular, as will become clear below, an important role in the analysis is played by a possible presence of additional neutral particles and their interactions with the charged carriers of the plasma.

In order to understand the role of neutral particles better, let us remind a very special property of a plasma that contains no such particles. When a configuration of perpendicular electric and magnetic fields, $\mathbf{E}\perp\mathbf{B}$ (assuming only that $E<B$), is applied, such a plasma drifts as a whole with the velocity perpendicular to both electric and magnetic fields [48]

$$\bar{\mathbf{v}} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{11}$$

see also Sec. III below. It is crucial that the drift velocity $\bar{\mathbf{v}}$ does *not* depend on the specific values of charges (and masses, if present) of particles. In fact, despite the drift, the plasma is in perfect equilibrium. It is described by a boosted, rather than the usual form of the Fermi-Dirac distribution function. (This can be also understood from a different angle: there is no electric field in the boosted reference frame, moving with the velocity $\bar{\mathbf{v}}$ with respect to the laboratory frame.)

It should be clear that the above-mentioned drifting state of the plasma should be profoundly affected whenever a background is present that exerts a drag on the system. In solid state materials, for example, the corresponding background could be due to the lattice of ions or impurities. In other plasmas, it could be due to neutral particles, which are not affected by electromagnetic fields directly. In the latter case, of course, it is assumed that the time scale for the neutral component to develop its own drift (via the interaction with the charged particles) is much longer than the characteristic time scales for the evolution of electromagnetic fields and inhomogeneities.

In this paper, we will discuss both cases with and without a drift of the plasma as a whole. Perhaps one of the best realistic examples of the plasma that is subject to the drift is a relativistic QED electron-positron plasma. An example of a plasma where the drift may not fully develop to involve the background of neutral particles is the quark-gluon plasma at sufficiently high temperatures. The (electromagnetically) neutral particles in the latter case are gluons. In the case of chiral plasmas in Dirac/Weyl semimetals, the background is due to the lattice ions or impurities that do not develop any drift at all.

III. CHIRAL PLASMA IN THE DRIFTING STATE: EXPANSION IN $\mathbf{E}_\parallel$, B, AND DERIVATIVES

In this section, we will derive the equations describing the evolution of a weakly inhomogeneous chiral plasma and electromagnetic fields without any additional components (e.g., neutral particles, pinned impurities, or ion lattices) present that could exert a substantial drag on the charged particles. If additional components are present, it is assumed that their interaction with charged particles is negligible and has no qualitative effect on the electromagnetic dynamics. We will use the kinetic equation (1) in the relaxation-time approximation. The distribution functions for charged particles ($i=p$) and their antiparticles ($i=a$) satisfy the following equations:

$$\begin{aligned}&\left(1+\frac{e_i}{c}\mathbf{B}\cdot\mathbf{\Omega}_\lambda^i\right)\frac{\partial f_\lambda^i}{\partial t}+\left(e_i\mathbf{E}+\frac{e_i}{c}\mathbf{v}\times\mathbf{B}+\frac{e_i^2}{c}(\mathbf{E}\cdot\mathbf{B})\mathbf{\Omega}_\lambda^i\right)\cdot\frac{\partial f_\lambda^i}{\partial\mathbf{p}}\\&+\left(\mathbf{v}+e_i\mathbf{E}\times\mathbf{\Omega}_\lambda^i+\frac{e_i}{c}(\mathbf{v}\cdot\mathbf{\Omega}_\lambda^i)\mathbf{B}\right)\cdot\frac{\partial f_\lambda^i}{\partial\mathbf{x}}\\&=-\frac{1}{\tau}\left(1+\frac{e_i}{c}\mathbf{B}\cdot\mathbf{\Omega}_\lambda^i\right)(f_\lambda^i-f_\lambda^{i\,(\mathrm{eq})}),\end{aligned} \tag{12}$$

where $e_i=e$, $\mathbf{\Omega}_\lambda^i=\mathbf{\Omega}_\lambda$ for particles and $e_i=-e$, $\mathbf{\Omega}_\lambda^i=-\mathbf{\Omega}_\lambda$ for antiparticles.

In the case of a plasma made of only charged degrees of freedom, the standard Fermi–Dirac distribution (2) may not be the best choice for the zeroth order of the equilibrium distribution function. Ideally, one would want to capture the drift of the plasma as a whole by a modified distribution function. The line of arguments that allows one to obtain the corresponding function is well known.

Before considering a general configuration of electromagnetic fields, let us start by recalling that a field configuration with constant $\mathbf{E}\perp\mathbf{B}$ (and $E<B$) has no dissipative effects on the plasma. (Of course, this will change when the parallel component $\mathbf{E}_\parallel$ is added later as a perturbation.) The whole system simply drifts with the velocity $\bar{\mathbf{v}}$ [48], see Eq. (11). Of course, this is connected with the fact that, in the reference frame K', moving with the drift velocity $\bar{\mathbf{v}}$ with respect to the laboratory reference frame K, the electric field is absent [49]. In the absence of electric field in the K' frame, the equilibrium state is naturally described by the standard Fermi-Dirac distribution function in the relativistic notation [50] (for the sake of simplicity, we suppress the particle/antiparticle index i)

$$f_\lambda'^{\,(\mathrm{eq})} = \frac{1}{\exp\left(\frac{p'^\nu u'_\nu-\mu'_\lambda}{T'}\right)+1} = \frac{1}{\exp\left(\frac{\epsilon_{\mathbf{p}'}-\mu'_\lambda}{T'}\right)+1}, \tag{13}$$

where $p'^\nu=(\epsilon_{\mathbf{p}'}/c,\mathbf{p}')$ and $u'^\nu\equiv(u'^0,\mathbf{u}')=(c,0,0,0)$ is the proper velocity. We emphasize that this consideration assumes that either no neutral particles are present or that their interaction is too weak to change the evolution of electromagnetic fields in the inhomogeneous plasma.

By performing the inverse Lorentz transformation in Eq. (13), we easily find the equilibrium distribution function in the laboratory frame, i.e.,

$$f_\lambda^{(\mathrm{eq})} = \frac{1}{\exp(\frac{\epsilon_{\mathbf{p}} - \mathbf{p}\cdot\bar{\mathbf{v}} - \mu_\lambda}{T}) + 1}, \tag{14}$$

where $u^\nu \equiv (u^0, \mathbf{u}) = (c/\sqrt{1-(\bar{v}/c)^2}, \bar{\mathbf{v}}/\sqrt{1-(\bar{v}/c)^2})$, $T = T'\sqrt{1-(\bar{v}/c)^2}$, and $\mu_\lambda = \mu'_\lambda\sqrt{1-(\bar{v}/c)^2}$. Note that T' and μ'_λ are Lorentz scalars that have the meaning of the temperature and chemical potentials in the local rest frame of the plasma. From the form of the distribution function (14), the parameters T and μ_λ appear to play the role of the temperature and chemical potentials in the laboratory reference frame. Such an interpretation of T and μ_λ should be used with caution, however, because the plasma is not stationary with respect to the laboratory frame. It is not difficult to check that the distribution function (14) is indeed a stationary solution of the kinetic equation (12) for constant perpendicular electric and magnetic fields.

The velocity $\bar{\mathbf{v}} = c\mathbf{E}\times\mathbf{B}/B^2$ is known in the plasma physics as the drift velocity [48] because the motion of charged particles in constant perpendicular electric and magnetic fields is the superposition of circular motion around a point called the guiding center and a drift of this point with the velocity $\bar{\mathbf{v}}$. It is crucially important for us that the drift velocity does not depend on the charges (and masses) of particles. In fact, this remains true also in a nonrelativistic plasma.

It should be emphasized that the plasma drift is well defined only when $E_\perp < B$. In this case the drift speed $\bar{v} = cE_\perp/B$ is smaller than the speed of light c. In the opposite case, $E_\perp > B$, there is no reference frame K', in which the perpendicular component of the electric field vanishes.

By using function (14) as the zeroth order approximation for the distribution function, let us proceed to the analysis of the general case when the parallel component of the electric field $\mathbf{E}_\parallel$ is also present and drives the system out of equilibrium. We will seek the solution for f_λ in the form of an expansion, i.e.,

$$f_\lambda = f_\lambda^{(\mathrm{eq})} + \delta f_\lambda^{(1)} + \cdots, \tag{15}$$

where $\delta f_\lambda^{(1)}$ defines a deviation from the local equilibrium to the first order in the parallel electric field $\mathbf{E}_\parallel$, magnetic field $\mathbf{B}$, and the first derivatives of electromagnetic fields and chemical potentials.

By substituting expansion (15) into the kinetic equation (12) and keeping only the terms up to linear order, we obtain

$$\frac{\bar{D}_\lambda}{T}\frac{\partial(\mu_\lambda + \mathbf{p}\cdot\bar{\mathbf{v}})}{\partial t} - \frac{\bar{D}_\lambda}{T} e\frac{(\mathbf{E}\cdot\mathbf{B})(\mathbf{v}\cdot\mathbf{B})}{B^2} + \frac{\bar{D}_\lambda}{T}\mathbf{v}\cdot\frac{\partial(\mu_\lambda + \mathbf{p}\cdot\bar{\mathbf{v}})}{\partial\mathbf{x}} + \frac{\delta f_\lambda^{(1)}}{\tau} = 0, \tag{16}$$

where we used the notation $\bar{D}_\lambda(\mu_\lambda) \equiv f_\lambda^{(\mathrm{eq})}(1 - f_\lambda^{(\mathrm{eq})}) = e^{(cp - \mathbf{p}\cdot\bar{\mathbf{v}} - \mu_\lambda)/T}/(e^{(cp - \mathbf{p}\cdot\bar{\mathbf{v}} - \mu_\lambda)/T} + 1)^2$. Note that the drift velocity $\bar{\mathbf{v}}$ is defined in terms of electromagnetic fields and, thus, may depend on the spacetime coordinates. By solving the above equation, we obtain

$$\delta f_\lambda^{(1)} = \frac{\tau\bar{D}_\lambda}{T}\left(e\frac{(\mathbf{E}\cdot\mathbf{B})(\mathbf{v}\cdot\mathbf{B})}{B^2} - \mathbf{v}\cdot\frac{\partial(\mu_\lambda + \mathbf{p}\cdot\bar{\mathbf{v}})}{\partial\mathbf{x}} - \frac{\partial(\mu_\lambda + \mathbf{p}\cdot\bar{\mathbf{v}})}{\partial t}\right). \tag{17}$$

By making use of this first-order correction to the distribution function $f_\lambda^{i(\mathrm{eq})}$, we can now calculate the charge densities and current densities to the same order.

The first-order result for the chiral charge densities reads

$$\begin{aligned} n_\lambda &= \sum_i \mathrm{sign}(e_i)\int\frac{d^3p}{(2\pi)^3}\left(f_\lambda^{i(\mathrm{eq})} + \delta f_\lambda^{i,(1)} + \frac{e_i}{c}(\mathbf{B}\cdot\Omega_\lambda^i)f_\lambda^{i(\mathrm{eq})}\right) \\ &= n_\lambda^{(0)} - \tau\frac{\partial n_\lambda^{(0)}}{\partial\mu_\lambda}\left(\bar{\mathbf{v}}\cdot\frac{\partial\mu_\lambda}{\partial\mathbf{x}} + \frac{\partial\mu_\lambda}{\partial t}\right) - \tau n_\lambda^{(0)}\left[(\boldsymbol{\nabla}\cdot\bar{\mathbf{v}}) + \frac{4\bar{v}}{(c^2 - \bar{v}^2)}\left(\bar{\mathbf{v}}\cdot\frac{\partial\bar{v}}{\partial\mathbf{x}} + \frac{\partial\bar{v}}{\partial t}\right)\right], \end{aligned} \tag{18}$$

where, by definition,

$$n_\lambda^{(0)} = \frac{\mu_\lambda^3 + \pi^2T^2\mu_\lambda}{6\pi^2c^3[1 - (\bar{v}/c)^2]^2}. \tag{19}$$

In deriving expression (18), we took into account that $(\mathbf{B}\cdot\bar{\mathbf{v}}) = 0$.

To the same first order, the current densities are given by

$$\mathbf{j}_\lambda = \sum_i \mathrm{sign}(e_i)\int \frac{d^3p}{(2\pi)^3}\left[\left(\mathbf{v} + e_i\mathbf{E}\times\mathbf{\Omega}^i_\lambda + \frac{e_i}{c}\mathbf{B}(\mathbf{v}\cdot\mathbf{\Omega}^i_\lambda)\right)f^{i(\mathrm{eq})}_\lambda + \mathbf{\nabla}\times(\epsilon_{\mathbf{p}}\mathbf{\Omega}_\lambda f^{i(\mathrm{eq})}_\lambda) + \mathbf{v}\delta f^{i,(1)}_\lambda\right]$$
$$= \mathbf{j}^{(\mathrm{Hall})}_\lambda + \mathbf{j}^{(1)}_\lambda + \mathbf{j}^{(2)}_\lambda + \mathbf{j}^{(\mathrm{curl})}_\lambda + \mathbf{j}^{(3)}_\lambda, \tag{20}$$

where the first four contributions are independent of the relaxation time, while the last one, $\mathbf{j}^{(3)}_\lambda$, is linear in τ. The Hall current has the standard form, i.e.,

$$\mathbf{j}^{(\mathrm{Hall})}_\lambda = \sum_i \mathrm{sign}(e_i)\int \frac{d^3p}{(2\pi)^3}\mathbf{v}f^{i(\mathrm{eq})}_\lambda = cn^{(0)}_\lambda\frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{21}$$

The explicit expressions of the other two nondissipative terms are given by

$$\mathbf{j}^{(1)}_\lambda = \sum_i \mathrm{sign}(e_i)\int \frac{d^3p}{(2\pi)^3}e_i(\mathbf{E}\times\mathbf{\Omega}^i_\lambda)f^{i(\mathrm{eq})}_\lambda = \frac{\lambda e\mu_\lambda}{4\pi^2\bar{v}^3}(\mathbf{E}\times\bar{\mathbf{v}})\left(\frac{c}{2}\ln\frac{c+\bar{v}}{c-\bar{v}} - \bar{v}\right), \tag{22}$$

$$\mathbf{j}^{(2)}_\lambda = \sum_i \mathrm{sign}(e_i)\int \frac{d^3p}{(2\pi)^3}\frac{e_i}{c}\mathbf{B}(\mathbf{v}\cdot\mathbf{\Omega}^i_\lambda)f^{i(\mathrm{eq})}_\lambda = \frac{\lambda e\mu_\lambda}{8\pi^2\bar{v}}\mathbf{B}\ln\frac{c+\bar{v}}{c-\bar{v}}. \tag{23}$$

By making use of the definition for the drift velocity, it is easy to check that, in addition to a perpendicular component, the current $\mathbf{j}^{(1)}_\lambda$ also contains a contribution parallel to the magnetic field. By combining the corresponding parallel component with the other current, $\mathbf{j}^{(2)}_\lambda$, we obtain the usual currents of the chiral magnetic and chiral separation effects

$$\mathbf{j}^{(\mathrm{CME})}_\lambda = \frac{\lambda e\mu_\lambda}{4\pi^2 c}\mathbf{B}. \tag{24}$$

The remaining contribution is perpendicular to the magnetic field. Its explicit form reads

$$\mathbf{j}^{(\perp)}_\lambda = \mathbf{j}^{(1)}_\lambda + \mathbf{j}^{(2)}_\lambda - \mathbf{j}^{(\mathrm{CME})}_\lambda = \frac{\lambda e\mu_\lambda}{4\pi^2 c}\mathbf{E}_\perp\frac{(\mathbf{E}\cdot\mathbf{B})}{E^2_\perp}\left(\frac{c}{2\bar{v}}\ln\frac{c+\bar{v}}{c-\bar{v}} - 1\right). \tag{25}$$

This is a new topological contribution, associated with the drift of plasma. It is induced when there are both parallel and perpendicular components of the electric field. While it is intimately connected with the chiral magnetic effect, it is not just a Lorentz boosted form of it in the laboratory frame.

To the first order in gradients and fields, the magnetization current is given by

$$\mathbf{j}^{(\mathrm{curl})}_\lambda = \frac{\lambda c}{24\pi^2}\mathbf{\nabla}\times\left[\frac{\bar{\mathbf{v}}}{\bar{v}^3}(3\mu^2_\lambda + \pi^2T^2)\left(\frac{c\bar{v}}{c^2-\bar{v}^2} - \frac{1}{2}\ln\frac{c+\bar{v}}{c-\bar{v}}\right)\right]. \tag{26}$$

An interesting byproduct of this result is that the drift may induce a nonzero magnetization in a chirally asymmetric plasma.

Finally, the last (dissipative) term in the complete expression for the current (20) reads

$$\mathbf{j}^{(3)}_\lambda = \sum_i \mathrm{sign}(e_i)\int \frac{d^3p}{(2\pi)^3}\mathbf{v}\delta f^{i,(1)}_\lambda = \frac{c^2\tau(3\mu^2_\lambda + \pi^2T^2)}{12\pi^2}\left(e\frac{(\mathbf{E}\cdot\mathbf{B})\mathbf{B}}{B^2} - \frac{\partial\mu_\lambda}{\partial\mathbf{x}}\right)g_0$$
$$- \frac{c\tau(\mu^3_\lambda + \pi^2T^2\mu_\lambda)}{4\pi^2}\left[\frac{\bar{\mathbf{v}}(\mathbf{\nabla}\cdot\bar{\mathbf{v}}) + (\bar{\mathbf{v}}\cdot\mathbf{\nabla})\bar{\mathbf{v}} + \bar{v}\mathbf{\nabla}\bar{v}}{\bar{v}}g_1 + \frac{\bar{\mathbf{v}}(\bar{\mathbf{v}}\cdot\mathbf{\nabla}\bar{v})}{\bar{v}^2}g_2 + \frac{2}{3(c^2-\bar{v}^2)^2}\frac{\partial\bar{\mathbf{v}}}{\partial t} + \frac{8\bar{v}\,\bar{\mathbf{v}}}{3(c^2-\bar{v}^2)^3}\frac{\partial\bar{v}}{\partial t}\right], \tag{27}$$

where we used the shorthand for the following functions of $\bar{v}/c$:

$$g_0 \equiv \frac{c^3}{2}\int_{-1}^{1}\frac{(1-\xi^2)d\xi}{(c-\bar{v}\xi)^3} = \frac{c^3}{\bar{v}^3}\left(\frac{c\bar{v}}{c^2-\bar{v}^2} - \frac{1}{2}\ln\frac{c+\bar{v}}{c-\bar{v}}\right), \tag{28}$$

$$g_1 \equiv \frac{c^4}{2}\int_{-1}^{1}\frac{\xi(1-\xi^2)d\xi}{(c-\bar{v}\xi)^4} = \frac{c^4}{3\bar{v}^4}\left(\frac{c\bar{v}(5\bar{v}^2-3c^2)}{(c^2-\bar{v}^2)^2}+\frac{3}{2}\ln\frac{c+\bar{v}}{c-\bar{v}}\right), \tag{29}$$

$$g_2 \equiv \frac{c^4}{2}\int_{-1}^{1}\frac{\xi(5\xi^2-3)d\xi}{(c-\bar{v}\xi)^4} = \frac{c^4}{3\bar{v}^4}\left(\frac{c\bar{v}(15c^4-40c^2\bar{v}^2+33\bar{v}^4)}{(c^2-\bar{v}^2)^3}+\frac{15}{2}\ln\frac{c+\bar{v}}{c-\bar{v}}\right). \tag{30}$$

Note that, in the limit of small drift velocities $\bar{v}/c \to 0$, these functions are nonsingular: $g_0 \simeq 2/3 + O(\bar{v}^2/c^2)$, $g_1 \simeq O(\bar{v}/c)$, and $g_2 \simeq O(\bar{v}^3/c^3)$.

The result in Eq. (24) renders the standard chiral separation and chiral magnetic effect currents. In addition, Eq. (21) gives the following currents due to the Hall effect as well as its generalization to the case of axial current:

$$\mathbf{j} = cn^{(0)}\frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{\mu(\mu^2+3\mu_5^2+\pi^2T^2)}{3\pi^2c^2[1-(\bar{v}/c)^2]^2}\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{31}$$

$$\mathbf{j}_5 = cn_5^{(0)}\frac{\mathbf{E}\times\mathbf{B}}{B^2} = \frac{\mu_5(\mu_5^2+3\mu^2+\pi^2T^2)}{3\pi^2c^2[1-(\bar{v}/c)^2]^2}\frac{\mathbf{E}\times\mathbf{B}}{B^2}. \tag{32}$$

These have the expected structure and are not very surprising. Much more surprising are the new contributions to current densities in Eq. (25). The corresponding currents appear to be of topological origin. Indeed, they appear due to the presence of the Berry connection in the definition of current (9) and do not depend on temperature or the relaxation time τ. They give rise to the following electric current and axial current densities:

$$\begin{aligned}\mathbf{j} &= \frac{e\mu_5}{2\pi^2c}\mathbf{E}_\perp\frac{(\mathbf{E}\cdot\mathbf{B})}{E_\perp^2}\left(\frac{c}{2\bar{v}}\ln\frac{c+\bar{v}}{c-\bar{v}}-1\right)\\ &\xrightarrow{E_\perp\to 0}\frac{e\mu_5}{6\pi^2c}\mathbf{E}_\perp\frac{(\mathbf{E}\cdot\mathbf{B})}{B^2},\end{aligned} \tag{33}$$

$$\begin{aligned}\mathbf{j}_5 &= \frac{e\mu}{2\pi^2c}\mathbf{E}_\perp\frac{(\mathbf{E}\cdot\mathbf{B})}{E_\perp^2}\left(\frac{c}{2\bar{v}}\ln\frac{c+\bar{v}}{c-\bar{v}}-1\right)\\ &\xrightarrow{E_\perp\to 0}\frac{e\mu}{6\pi^2c}\mathbf{E}_\perp\frac{(\mathbf{E}\cdot\mathbf{B})}{B^2}.\end{aligned} \tag{34}$$

These currents resemble the chiral magnetic/separation currents, but flow perpendicularly to the magnetic field. They are of the first order in electromagnetic fields and appear to be nondissipative. The fact that these currents are proportional to $\mathbf{E}\cdot\mathbf{B}$ may hint at their possible connection with the chiral anomaly. We also observe that currents (33) and (34) are deeply connected with the existence of the plasma drift. They exist only when both the perpendicular and parallel components of the electric field are nonvanishing. This latter suggests that the new topological currents cannot be eliminated, or reduced to the chiral magnetic/separation currents, by a simple boost transformation.

The Ohm's current in Eq. (27) has only the longitudinal projection with respect to the magnetic field. This is due to the fact that the perpendicular component of the electric field is exactly accounted in the drift velocity. In addition to the standard Ohm's and diffusion currents, given by the first term in Eq. (27), there are also other dissipative contributions in $\mathbf{j}_\lambda^{(3)}$, which are associated with the inhomogeneity of the drift flow. These appear to be connected with a nonzero viscosity of chiral plasma.

To linear order in electromagnetic fields and derivatives, the continuity equation reads

$$\begin{aligned}&\frac{\partial n_\lambda^{(0)}}{\partial\mu_\lambda}\left(\frac{\partial\mu_\lambda}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\mu_\lambda}{\partial\mathbf{x}}\right)+\frac{\partial n_\lambda^{(0)}}{\partial\bar{v}}\left(\frac{\partial\bar{v}}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\bar{v}}{\partial\mathbf{x}}\right)\\ &\quad + n_\lambda^{(0)}\boldsymbol{\nabla}\cdot\bar{\mathbf{v}} = 0\end{aligned} \tag{35}$$

or equivalently,

$$\begin{aligned}&\left(\frac{3(\mu^2+\mu_5^2)}{\pi^2T^2}+1\right)\left(\frac{\partial\mu}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)+\frac{6\mu\mu_5}{\pi^2T^2}\left(\frac{\partial\mu_5}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}\right)\\ &\quad+\mu\left(\frac{\mu^2+3\mu_5^2}{\pi^2T^2}+1\right)\left[\frac{4\bar{v}}{c^2-\bar{v}^2}\left(\frac{\partial\bar{v}}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\bar{v}}{\partial\mathbf{x}}\right)+\boldsymbol{\nabla}\cdot\bar{\mathbf{v}}\right]\\ &= 0,\end{aligned} \tag{36}$$

$$\begin{aligned}&\left(\frac{3(\mu^2+\mu_5^2)}{\pi^2T^2}+1\right)\left(\frac{\partial\mu_5}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}\right)+\frac{6\mu\mu_5}{\pi^2T^2}\left(\frac{\partial\mu}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)\\ &\quad+\mu_5\left(\frac{\mu_5^2+3\mu^2}{\pi^2T^2}+1\right)\left[\frac{4\bar{v}}{c^2-\bar{v}^2}\left(\frac{\partial\bar{v}}{\partial t}+\bar{\mathbf{v}}\cdot\frac{\partial\bar{v}}{\partial\mathbf{x}}\right)+\boldsymbol{\nabla}\cdot\bar{\mathbf{v}}\right]\\ &= 0.\end{aligned} \tag{37}$$

In the complete set of the anomalous Maxwell equations, these two relations determine how local equilibrium electric and chiral chemical potentials evolve in a self-consistent way in a chiral plasma.

IV. EXPANSION IN POWERS OF ELECTROMAGNETIC FIELDS AND DERIVATIVES

In the previous section, we studied the chiral plasma in the drifting state. This allowed us to account for the plasma drift exactly in the zeroth approximation of the distribution function. In this section, we consider the case where neutral particles exert a substantial drag on the system so that one can use the Fermi–Dirac distribution function (2) as the equilibrium distribution function. We will treat both electric and magnetic fields as perturbations. From a physics viewpoint, this is the regime of a large collision rate [48,51]. In this case, an expansion in powers of both electric and magnetic fields is justified. Since the calculations in this case become much simpler than those in the previous section, we will derive the expressions for the densities and currents to the second order in electromagnetic fields and derivatives. For some earlier studies using the effective action formalism, see also Ref. [52].

A. Distribution function to quadratic order in electromagnetic fields and derivatives

We seek the solution to Eq. (12) in the form of a series in powers of electromagnetic fields, i.e.,

$$f_\lambda = f_\lambda^{(\mathrm{eq})} + \delta f_\lambda^{(1)} + \delta f_\lambda^{(2)} + \cdots \tag{38}$$

and treat each space/time derivative as an extra power of electromagnetic field. By substituting the above ansatz in Eq. (12) and keeping the terms up to linear order in electromagnetic fields, we obtain

$$\frac{D_\lambda}{T}\frac{\partial \mu_\lambda}{\partial t} - \frac{D_\lambda}{T} e(\mathbf{E}\cdot\mathbf{v}) + \frac{D_\lambda}{T}\mathbf{v}\cdot\frac{\partial \mu_\lambda}{\partial \mathbf{x}} + \frac{\delta f_\lambda^{(1)}}{\tau} = 0, \tag{39}$$

where we used the notation $D_\lambda(\mu_\lambda) = e^{(cp-\mu_\lambda)/T}/(e^{(cp-\mu_\lambda)/T}+1)^2$. The above equation is satisfied when

$$\delta f_\lambda^{(1)} = \frac{\tau D_\lambda}{T}\left(e(\mathbf{E}_\lambda\cdot\mathbf{v}) - \frac{\partial \mu_\lambda}{\partial t}\right). \tag{40}$$

Here, by definition, $\mathbf{E}_\lambda \equiv \mathbf{E} - e^{-1}\partial\mu_\lambda/\partial\mathbf{x}$. By making use of this result, we derive the following formal results for the densities and currents:

$$n_\lambda = \sum_i \mathrm{sign}(e_i)\int\frac{d^3p}{(2\pi)^3}\left(f_\lambda^{i(\mathrm{eq})} + \delta f_\lambda^{i,(1)} + \frac{e_i}{c}(\mathbf{B}\cdot\mathbf{\Omega}_\lambda^i) f_\lambda^{i(\mathrm{eq})}\right) = \frac{\mu_\lambda(\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3} - \frac{\tau(3\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}\frac{\partial\mu_\lambda}{\partial t}, \tag{41}$$

$$\mathbf{j}_\lambda = \sum_i \mathrm{sign}(e_i)\int\frac{d^3p}{(2\pi)^3}\left[\left(\mathbf{v} + e_i\mathbf{E}\times\mathbf{\Omega}_\lambda^i + \frac{e_i}{c}\mathbf{B}(\mathbf{v}\cdot\mathbf{\Omega}_\lambda^i)\right)f_\lambda^{i(\mathrm{eq})} + \mathbf{v}\delta f_\lambda^{i,(1)}\right] = \frac{\lambda e\mathbf{B}\mu_\lambda}{4\pi^2c} + \frac{\tau e\mathbf{E}_\lambda(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}, \tag{42}$$

where $\mathrm{sign}(e_i)$ was included in order to correctly account for the contribution of antiparticles. As is clear, to this order, the magnetization current in Eq. (10) does not contribute. Then, by making use of the continuity equation, to linear order in the fields and derivatives we obtain

$$\frac{3\mu_\lambda^2+\pi^2T^2}{6\pi^2c^3}\frac{\partial\mu_\lambda}{\partial t} = 0. \tag{43}$$

This result implies that the time derivative of μ_λ unlike the case of the drifting state considered in the previous section vanishes to linear order. In fact, as we will see below, it is of second order in fields. Taking this into account, the leading order correction to the distribution function takes the following form:

$$\delta f_\lambda^{(1)} = \frac{ec\tau D_\lambda}{T}(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}}), \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{p}, \tag{44}$$

where we also took into account that $\mathbf{v} = c\hat{\mathbf{p}}$.

Before proceeding to the derivation of the quadratic correction to the distribution function, let us note the following results:

$$\frac{\partial\delta f_\lambda^{(1)}}{\partial t} = \frac{ec\tau D_\lambda}{T}\frac{\partial}{\partial t}(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}}), \tag{45}$$

$$\frac{\partial\delta f_\lambda^{(1)}}{\partial\mathbf{p}} = -\frac{ec^2\tau D_\lambda}{T^2}(1-2f_\lambda^{(\mathrm{eq})})\hat{\mathbf{p}}(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}}) + \frac{\tau D_\lambda}{T}\frac{ec}{p}[\mathbf{E}_\lambda - \hat{\mathbf{p}}(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}})], \tag{46}$$

$$\frac{\partial \delta f_\lambda^{(1)}}{\partial \mathbf{x}} = \frac{ec\tau D_\lambda}{T^2}(1-2f_\lambda^{(\mathrm{eq})})(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}})\frac{\partial \mu_\lambda}{\partial \mathbf{x}} + \frac{ec\tau D_\lambda}{T}\frac{\partial}{\partial \mathbf{x}}(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}}). \tag{47}$$

To quadratic order in electromagnetic fields, the solution $\delta f_\lambda^{(2)}$ to the kinetic equation takes the form:

$$\begin{aligned}\delta f_\lambda^{(2)} = &-\frac{\tau D_\lambda}{T}\frac{\partial \mu_\lambda}{\partial t} + \frac{e^2c^2\tau^2 D_\lambda}{T^2}(1-2f_\lambda^{(\mathrm{eq})})(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}})^2 - \frac{ec\tau^2 D_\lambda}{T}\left(\frac{\partial}{\partial t} + c\hat{\mathbf{p}}\cdot\frac{\partial}{\partial \mathbf{x}}\right)(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}})\\ &+\frac{e^2\tau D_\lambda}{T}(\mathbf{E}_\lambda\cdot\mathbf{B})(\Omega_\lambda\cdot\hat{\mathbf{p}}) - \frac{e^2c\tau^2 D_\lambda}{pT}\hat{\mathbf{p}}\cdot(\mathbf{B}\times\mathbf{E}_\lambda) - \frac{e^2c\tau^2 D_\lambda}{pT}[(\mathbf{E}\cdot\mathbf{E}_\lambda) - (\mathbf{E}\cdot\hat{\mathbf{p}})(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}})]\\ &-\frac{e^2\tau D_\lambda}{T}(\mathbf{B}\cdot\Omega_\lambda)(\mathbf{E}_\lambda\cdot\hat{\mathbf{p}}) - \frac{e\tau D_\lambda}{T}(\mathbf{E}\times\Omega_\lambda)\cdot\frac{\partial \mu_\lambda}{\partial \mathbf{x}}.\end{aligned} \tag{48}$$

Having determined the corrections to the local equilibrium distribution function in the first and second order in electromagnetic field and derivatives, it is not difficult to find the corresponding charge and current densities.

B. Equations for the chemical potentials

We have the following results for the densities and currents:

$$\begin{aligned}n_\lambda &= \sum_i \mathrm{sign}(e_i)\int\frac{d^3p}{(2\pi)^3}\left(f_\lambda^{i(\mathrm{eq})} + \delta f_\lambda^{i,(1)} + \frac{e_i}{c}(\mathbf{B}\cdot\Omega_\lambda^i)f_\lambda^{i(\mathrm{eq})} + \frac{e_i}{c}(\mathbf{B}\cdot\Omega_\lambda^i)\delta f_\lambda^{i,(1)} + \delta f_\lambda^{i,(2)}\right)\\ &= \frac{\mu_\lambda(\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3} - \frac{\tau(3\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}\frac{\partial \mu_\lambda}{\partial t} - \frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2 c}\nabla\cdot\mathbf{E}_\lambda + \frac{\lambda e^2\tau(\mathbf{E}_\lambda\cdot\mathbf{B})}{4\pi^2c} - \frac{e\tau^2\mu_\lambda}{3\pi^2c}\left(\mathbf{E}_\lambda\cdot\frac{\partial \mu_\lambda}{\partial \mathbf{x}}\right),\end{aligned} \tag{49}$$

$$\begin{aligned}\mathbf{j}_\lambda &= \sum_i \mathrm{sign}(e_i)\int\frac{d^3p}{(2\pi)^3}\left[\left(\mathbf{v} + e_i\mathbf{E}\times\Omega_\lambda^i + \frac{e_i}{c}\mathbf{B}(\mathbf{v}\cdot\Omega_\lambda^i)\right)f_\lambda^{i(\mathrm{eq})} + (\mathbf{v}+e_i\mathbf{E}\times\Omega_\lambda^i)\delta f_\lambda^{i,(1)} + \mathbf{v}\delta f_\lambda^{i,(2)}\right] + \mathbf{j}_\lambda^{(\mathrm{curl})}\\ &= \frac{\lambda e\mathbf{B}\mu_\lambda}{4\pi^2 c} + \frac{\tau e\mathbf{E}_\lambda(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c} - \frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\frac{\partial \mathbf{E}}{\partial t} - \frac{e^2\tau^2\mu_\lambda}{6\pi^2}(\mathbf{B}\times\mathbf{E}_\lambda) + \frac{\lambda e\tau}{12\pi^2}\boldsymbol{\nabla}\times(\mu_\lambda\mathbf{E}).\end{aligned} \tag{50}$$

To this quadratic order, we had to also take into account the magnetization current in Eq. (10). The corresponding additional contribution is the last term in Eq. (50). It can be rewritten in an equivalent form as follows:

$$\mathbf{j}_\lambda^{(\mathrm{curl})} = \frac{\lambda e\tau}{12\pi^2}\mu_\lambda\boldsymbol{\nabla}\times\mathbf{E} - \frac{\lambda e\tau}{12\pi^2}\mathbf{E}\times\frac{\partial \mu_\lambda}{\partial \mathbf{x}} = -\frac{\lambda e\tau\mu_\lambda}{12\pi^2 c}\frac{\partial \mathbf{B}}{\partial t} - \frac{\lambda e\tau}{12\pi^2}\mathbf{E}\times\frac{\partial \mu_\lambda}{\partial \mathbf{x}}, \tag{51}$$

where we used the Maxwell equation (4) in the last equality. In the continuity equation, of course, such a current plays no role. By substituting the results in Eqs. (49) and (50) into the continuity equation, we obtain the sought equations for the chemical potentials

$$\frac{3\mu_\lambda^2+\pi^2T^2}{6\pi^2c^3}\frac{\partial \mu_\lambda}{\partial t} + \frac{\tau e(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\nabla\cdot\mathbf{E}_\lambda + \frac{\tau e\mu_\lambda}{3\pi^2c}\left(\mathbf{E}_\lambda\cdot\frac{\partial \mu_\lambda}{\partial \mathbf{x}}\right) = \frac{\lambda e^2\mathbf{E}_\lambda\cdot\mathbf{B}}{4\pi^2c}. \tag{52}$$

These are equivalent to the following equations for the electric and axial charge chemical potentials:

$$\begin{aligned}&(3\mu^2+3\mu_5^2+\pi^2T^2)\left[\frac{\partial \mu}{\partial t} - \frac{\tau c^2}{3}\nabla^2\mu + \frac{\tau e c^2}{3}\nabla\cdot\mathbf{E}\right] + 6\mu\mu_5\left[\frac{\partial \mu_5}{\partial t} - \frac{\tau c^2}{3}\nabla^2\mu_5\right] + \frac{3}{2}ec^2\left(\mathbf{B}\cdot\frac{\partial \mu_5}{\partial \mathbf{x}}\right)\\ &+2\tau\mu c^2\left[e\mathbf{E}\cdot\frac{\partial \mu}{\partial \mathbf{x}} - \left(\frac{\partial \mu}{\partial \mathbf{x}}\right)^2 - \left(\frac{\partial \mu_5}{\partial \mathbf{x}}\right)^2\right] + 2\tau\mu_5 c^2\left[e\mathbf{E}\cdot\frac{\partial \mu_5}{\partial \mathbf{x}} - 2\frac{\partial \mu}{\partial \mathbf{x}}\cdot\frac{\partial \mu_5}{\partial \mathbf{x}}\right] = 0,\end{aligned} \tag{53}$$

$$(3\mu^2+3\mu_5^2+\pi^2T^2)\left[\frac{\partial\mu_5}{\partial t}-\frac{\tau c^2}{3}\nabla^2\mu_5\right]+6\mu\mu_5\left[\frac{\partial\mu}{\partial t}-\frac{\tau c^2}{3}\nabla^2\mu+\frac{\tau ec^2}{3}\nabla\cdot\mathbf{E}\right]+\frac{3}{2}ec^2\left(\mathbf{B}\cdot\frac{\partial\mu}{\partial\mathbf{x}}\right)$$
$$+2\tau\mu_5c^2\left[e\mathbf{E}\cdot\frac{\partial\mu}{\partial\mathbf{x}}-\left(\frac{\partial\mu}{\partial\mathbf{x}}\right)^2-\left(\frac{\partial\mu_5}{\partial\mathbf{x}}\right)^2\right]+2\tau\mu c^2\left[e\mathbf{E}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}-2\frac{\partial\mu}{\partial\mathbf{x}}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}\right]=0. \tag{54}$$

These equations determine how local equilibrium electric and chiral chemical potentials evolve in an inhomogeneous chirally asymmetric plasma. In order to complete our derivation of anomalous Maxwell equations for inhomogeneous chiral plasma, we should find the explicit expressions for the currents to the second order in electromagnetic field and derivatives.

To this order, the electric current and axial current densities are

$$\mathbf{j}=\frac{e\mathbf{B}\mu_5}{2\pi^2c}+\sum_\lambda\frac{\tau e\mathbf{E}_\lambda(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}-\sum_\lambda\frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\frac{\partial\mathbf{E}}{\partial t}-\sum_\lambda\frac{e^2\tau^2\mu_\lambda}{6\pi^2}(\mathbf{B}\times\mathbf{E}_\lambda)+\sum_\lambda\frac{\lambda e\tau}{12\pi^2}\boldsymbol{\nabla}\times(\mu_\lambda\mathbf{E}), \tag{55}$$

$$\mathbf{j}_5=\frac{e\mathbf{B}\mu}{2\pi^2c}+\sum_\lambda\lambda\frac{\tau e\mathbf{E}_\lambda(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}-\sum_\lambda\lambda\frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\frac{\partial\mathbf{E}}{\partial t}-\sum_\lambda\lambda\frac{e^2\tau^2\mu_\lambda}{6\pi^2}(\mathbf{B}\times\mathbf{E}_\lambda)+\sum_\lambda\frac{e\tau}{12\pi^2}\boldsymbol{\nabla}\times(\mu_\lambda\mathbf{E}), \tag{56}$$

while the electric and axial charge densities are

$$n=\sum_\lambda\frac{\mu_\lambda(\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}-\sum_\lambda\frac{\tau(3\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}\frac{\partial\mu_\lambda}{\partial t}-\sum_\lambda\frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\nabla\cdot\mathbf{E}_\lambda-\frac{e\tau}{2\pi^2c}\left(\mathbf{B}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}\right)-\sum_\lambda\frac{e\tau^2\mu_\lambda}{3\pi^2c}\left(\mathbf{E}_\lambda\cdot\frac{\partial\mu_\lambda}{\partial\mathbf{x}}\right), \tag{57}$$

$$n_5=\sum_\lambda\lambda\frac{\mu_\lambda(\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}-\sum_\lambda\lambda\frac{\tau(3\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}\frac{\partial\mu_\lambda}{\partial t}-\sum_\lambda\lambda\frac{e\tau^2(3\mu_\lambda^2+\pi^2T^2)}{18\pi^2c}\nabla\cdot\mathbf{E}_\lambda$$
$$+\frac{e\tau}{2\pi^2c}\left(e\mathbf{E}-\frac{\partial\mu}{\partial\mathbf{x}}\right)\cdot\mathbf{B}-\sum_\lambda\lambda\frac{e\tau^2\mu_\lambda}{3\pi^2c}\left(\mathbf{E}_\lambda\cdot\frac{\partial\mu_\lambda}{\partial\mathbf{x}}\right). \tag{58}$$

Since electric and chiral chemical potentials are much smaller than temperature in a primordial plasma, it is useful to determine the charge and current densities in the high-temperature limit.

C. Electric and chiral currents in the high-temperature limit

In the high-temperature limit, the electric current and axial current densities are

$$\mathbf{j}\simeq\frac{e\mathbf{B}\mu_5}{2\pi^2c}+\frac{\tau T^2}{9c}\left(e\mathbf{E}-\frac{\partial\mu}{\partial\mathbf{x}}\right)-\frac{e\tau^2T^2}{9c}\frac{\partial\mathbf{E}}{\partial t}, \tag{59}$$

$$\mathbf{j}_5\simeq\frac{e\mathbf{B}\mu}{2\pi^2c}-\frac{\tau T^2}{9c}\frac{\partial\mu_5}{\partial\mathbf{x}}, \tag{60}$$

while the electric and axial charge densities are

$$n\simeq\frac{T^2\mu}{3c^3}-\frac{\tau T^2}{3c^3}\frac{\partial\mu}{\partial t}+\frac{\tau^2T^2}{9c}(\nabla^2\mu-e\nabla\cdot\mathbf{E})-\frac{e\tau}{2\pi^2c}\left(\mathbf{B}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}\right), \tag{61}$$

$$n_5\simeq\frac{T^2\mu_5}{3c^3}-\frac{\tau T^2}{3c^3}\frac{\partial\mu_5}{\partial t}+\frac{\tau^2T^2}{9c}\nabla^2\mu_5+\frac{e\tau}{2\pi^2c}\left(e\mathbf{E}-\frac{\partial\mu}{\partial\mathbf{x}}\right)\cdot\mathbf{B}. \tag{62}$$

In the high-temperature limit, $|\mu_\lambda|\ll T$, the equations for the chiral μ_5 and fermion number μ chemical potentials read

$$\frac{\partial\mu}{\partial t}+\frac{3c^2}{2\pi^2T^2}e\mathbf{B}\cdot\frac{\partial\mu_5}{\partial\mathbf{x}}-\frac{\tau c^2}{3}(\nabla^2\mu-e\boldsymbol{\nabla}\cdot\mathbf{E})=0, \quad (63)$$

$$\frac{\partial\mu_5}{\partial t}+\frac{3c^2}{2\pi^2T^2}e\mathbf{B}\cdot\frac{\partial\mu}{\partial\mathbf{x}}-\frac{\tau c^2}{3}\nabla^2\mu_5=\frac{3e^2c^2\mathbf{E}\cdot\mathbf{B}}{2\pi^2T^2}. \quad (64)$$

It is interesting to point that the equations of motion for the chemical potentials contain the diffusion terms, proportional to $\nabla^2\mu_5/3$ and $\nabla^2\mu/3$, with the diffusion constant given by $\tau c^2/3$. This is in agreement with the general arguments in Ref. [27]. Our derivation in the present paper not only establishes such a term, but also leads to a formal expression for the diffusion constant in terms of the relaxation time. In general, in the presence of a nonzero magnetic field, one expects that there are two different diffusion terms, a longitudinal one proportional to $\partial_z^2\mu_\lambda$ and a transverse one proportional to $\Delta_T\mu_\lambda$, see, for example, Eq. (9) in Ref. [53]. By making use of the chiral kinetic theory, both diffusion terms can be rigorously derived. As we see from our analysis above, to the quadratic order in the fields, the longitudinal and transverse diffusion constants are the same. It can be shown, however, that the longitudinal diffusion constant will have a nonzero correction of order B^2 that comes from the term of the type $(\mathbf{B}\cdot\boldsymbol{\nabla})(\mathbf{B}\cdot\partial\mu_\lambda/\partial\mathbf{x})$.

In the case of a constant magnetic field and vanishing electric field, the above set of the equations has a solution in the form of a (diffusive) chiral magnetic wave. Indeed, by setting $\mathbf{E}=0$ and assuming that the magnetic field $\mathbf{B}$ is constant, we find that there is a solution that describes a diffusive chiral magnetic wave with the following dispersion relation:

$$\omega_{\rm CMW}=\pm\frac{3c^2}{2\pi^2T^2}(e\mathbf{B}\cdot\mathbf{k})-\frac{i}{3}\tau c^2|\mathbf{k}|^2. \quad (65)$$

Note that the speed of the chiral magnetic wave is given by

$$v_{\rm CMW}=\frac{3c^2|eB|}{2\pi^2T^2}\cos\theta_{Bk}, \quad (66)$$

where θ_{Bk} is the angle between the wave vector $\mathbf{k}$ and the magnetic field.

D. Explicit expressions for currents

It is instructive to discuss the physical meaning of separate contributions in the expressions for the electric current (55) and axial current (56) densities. Let us start from the electric current density. After performing the sum over λ, we derive

$$\begin{aligned}\mathbf{j}=&\frac{e\mathbf{B}\mu_5}{2\pi^2c}+\frac{\tau T^2}{9c}\left(1+\frac{3(\mu^2+\mu_5^2)}{\pi^2T^2}\right)\left(e\mathbf{E}-\frac{\partial\mu}{\partial\mathbf{x}}\right)\\&+\frac{e^2\tau^2\mu}{3\pi^2}\mathbf{E}\times\mathbf{B}-\frac{2e\tau\mu\mu_5}{3\pi^2c}\frac{\partial\mu_5}{\partial\mathbf{x}}+\frac{e\tau^2\mu}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu}{\partial\mathbf{x}}\right)\\&+\frac{e\tau^2\mu_5}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu_5}{\partial\mathbf{x}}\right)-\frac{e\tau^2T^2}{9c}\left(1+\frac{3(\mu^2+\mu_5^2)}{\pi^2T^2}\right)\frac{\partial\mathbf{E}}{\partial t}\\&-\frac{e\tau\mu_5}{6\pi^2c}\frac{\partial\mathbf{B}}{\partial t}-\frac{e\tau}{6\pi^2}\mathbf{E}\times\frac{\partial\mu_5}{\partial\mathbf{x}}. \end{aligned} \quad (67)$$

Obviously, the first term describes the current of the chiral magnetic effect, which is the current induced by a nonzero chiral chemical potential along the direction of the magnetic field. The second term combines the Ohm's and diffusion currents. Let us note that the conductivity equals

$$\sigma=\frac{e^2c^2\tau}{3}\chi^{(0)}, \quad (68)$$

where we introduced the shorthand notations

$$\chi^{(0)}=\frac{\partial n^{(0)}}{\partial\mu}=\frac{3\mu^2+3\mu_5^2+\pi^2T^2}{3\pi^2c^3}, \quad (69)$$

$$n^{(0)}=\sum_\lambda\frac{\mu_\lambda(\mu_\lambda^2+\pi^2T^2)}{6\pi^2c^3}=\frac{\mu(\mu^2+3\mu_5^2+\pi^2T^2)}{3\pi^2c^3}. \quad (70)$$

In order to apply our results for a deconfined quark-gluon plasma, one could fix the relaxation-time parameter by using the lattice results [54,55] for the quark-gluon plasma conductivity (obtained at $T=1.45T_c$) and Eq. (68):

$$\tau\simeq0.37\frac{9}{\alpha T}\simeq375\ \text{fm}/c\left(\frac{240\ \text{MeV}}{T}\right). \quad (71)$$

The last five terms in Eq. (67) are new types of terms that are produced by time-dependent electric and magnetic fields and gradients of the chemical potentials and, to the best of our knowledge, have not been discussed in the literature before. We will discuss the physical meaning of each of them, as well as their possible implications in the subsection below.

A few words are in order about the third term in Eq. (67), which is nothing else but the celebrated Hall current. At the first sight it appears to be strange that the corresponding current is proportional to the square of the relaxation time. Yet, we emphasize that this is a standard result in the limit of large collision rate (small τ), see for example, Sec. VI.10 in Ref. [48]. Moreover, the usual experimental setup for measuring the Hall effect, in which one enforces $j_y=0$, will lead to the well-known relation between the electric current in the x direction and the electric field in the y direction, i.e., $j_x\propto n^2E_y/(\mu B)$ up to small corrections suppressed by the second power of the magnetic field and the relaxation time [51]. Now, the leading order term in such a result is indeed standard and independent of the relaxation time.

Similarly, after performing the sum over λ in the expression for the axial current density in Eq. (56), we derive

$$\mathbf{j}_5 = \frac{e\mathbf{B}\mu}{2\pi^2 c} - \frac{e\tau T^2}{9c}\left(1+\frac{3(\mu^2+\mu_5^2)}{\pi^2 T^2}\right)\frac{\partial\mu_5}{\partial\mathbf{x}} + \frac{2e^2\tau\mu\mu_5}{3\pi^2 c}\mathbf{E} - \frac{e\tau^2\mu_5}{3\pi^2}\mathbf{B}\times\left(e\mathbf{E}-\frac{\partial\mu}{\partial\mathbf{x}}\right) + \frac{e\tau^2\mu}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu_5}{\partial\mathbf{x}}\right) - \frac{2e\tau\mu\mu_5}{3\pi^2 c}\frac{\partial\mu}{\partial\mathbf{x}} - \frac{2e\tau^2\mu\mu_5}{3\pi^2 c}\frac{\partial\mathbf{E}}{\partial t} - \frac{e\tau\mu}{6\pi^2 c}\frac{\partial\mathbf{B}}{\partial t} - \frac{e\tau}{6\pi^2}\mathbf{E}\times\frac{\partial\mu}{\partial\mathbf{x}}. \tag{72}$$

The first term in $\mathbf{j}_5$ is the celebrated chiral separation effect current. The second term is a diffusion current. The third term is the axial current associated with the chiral electric separation effect [56,57]. The rest are new terms.

E. New contributions to the electric current

Let us discuss the new terms in the electric current (67) connected with inhomogeneities of the electric and axial charge densities in chiral plasma. The first of the three new types of currents is associated in a simple way with a chiral diffusion,

$$\mathbf{j}_{\partial\chi} = -\frac{2e\tau\mu\mu_5}{3\pi^2 c}\frac{\partial\mu_5}{\partial\mathbf{x}}. \tag{73}$$

It is induced in a plasma in which both the fermion number and chiral chemical potentials are nonzero. The direction of the current coincides with the gradient $\partial\mu_5/\partial\mathbf{x}$. The current of the second type goes perpendicularly to the magnetic field, as well as to the gradient of the electric/chiral chemical potential, i.e.,

$$\mathbf{j}_{B\times\partial} = \frac{e\tau^2\mu}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu}{\partial\mathbf{x}}\right) + \frac{e\tau^2\mu_5}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu_5}{\partial\mathbf{x}}\right), \tag{74}$$

therefore, we will call this current the *Hall diffusion*. The currents in Eqs. (73) and (74) occur already in absence of electric fields. The current of the third type is driven by a time-dependent electric field,

$$\mathbf{j}_{\partial_t E} = -\frac{e\tau^2 T^2}{9c}\left(1+\frac{3(\mu^2+\mu_5^2)}{\pi^2 T^2}\right)\frac{\partial\mathbf{E}}{\partial t}. \tag{75}$$

This current is clearly a time-dependent electric field analogue of the Ohm's current, cf. the second term in Eq. (67). Finally, we also get the following contributions due to the magnetization current:

$$\mathbf{j}^{(\mathrm{curl})} = -\frac{e\tau\mu_5}{6\pi^2 c}\frac{\partial\mathbf{B}}{\partial t} - \frac{e\tau}{6\pi^2}\mathbf{E}\times\frac{\partial\mu_5}{\partial\mathbf{x}}. \tag{76}$$

The second term is very interesting. It describes a current perpendicular to the electric field and the gradient of the axial chemical potential. Such a current resembles the anomalous Hall effect current [58], which happens in the absence of magnetic field. In the case of the chiral plasma at hand, it describes the *anomalous chiral Hall effect*.

F. New contributions to the axial current

Let us now turn to the new terms in the axial current (72). The first of them, i.e.,

$$\mathbf{j}_{5,EB} = \frac{e^2\tau^2\mu_5}{3\pi^2}\mathbf{E}\times\mathbf{B}, \tag{77}$$

is a chiral analogue of the Hall effect with the axial current induced in a medium with $\mu_5 \neq 0$ [59]. The existence of this term is very interesting. In principle, it allows one to determine experimentally the sign of the chiral charge of dominant carriers in a chiral plasma. The corresponding sign could be extracted from the direction of $\mathbf{j}_5$ in orthogonal electric and magnetic fields.

The current of the second type is driven by gradients of the electric and chiral chemical potentials and a perpendicular magnetic field, i.e.,

$$\mathbf{j}_{5,B\times\partial} = \frac{e\tau^2\mu_5}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu}{\partial\mathbf{x}}\right) + \frac{e\tau^2\mu}{3\pi^2}\left(\mathbf{B}\times\frac{\partial\mu_5}{\partial\mathbf{x}}\right). \tag{78}$$

Obviously, this current is a chiral analogue to the Hall diffusion current in Eq. (74). The last contribution to the axial current is given by two terms

$$\mathbf{j}_{5,\partial_t E} = -\frac{2e\tau\mu\mu_5}{3\pi^2 c}\frac{\partial\mu}{\partial\mathbf{x}} - \frac{2e\tau^2\mu\mu_5}{3\pi^2 c}\frac{\partial\mathbf{E}}{\partial t}. \tag{79}$$

Since current (79) vanishes in a plasma where μ or μ_5 equals zero, this current defines diffusion and time-dependent electric field analogues of the current of the chiral electric separation effect given by the third term in Eq. (72) (note that the corresponding numerical coefficients of the currents match too). At last, the magnetization current gives the following contribution to the axial current density:

$$\mathbf{j}_5^{(\mathrm{curl})} = -\frac{e\tau\mu}{6\pi^2 c}\frac{\partial\mathbf{B}}{\partial t} - \frac{e\tau}{6\pi^2}\mathbf{E}\times\frac{\partial\mu}{\partial\mathbf{x}}. \tag{80}$$

The last term is analogous to the anomalous chiral Hall effect current, given by the last term in Eq. (76). However, the corresponding contribution to the axial current is perhaps even more interesting. It implies that, in an electric field, the perpendicular component of the gradient of the chemical potential should lead to a nonzero $\mathbf{j}_5$.

Before concluding this section, let us emphasize that the main results of this section are the explicit expressions for the currents in an inhomogeneous chiral plasma. These expressions provide a critical ingredient in the analysis of the anomalous Maxwell equations (3) through (6), together with the equations for time-dependent and spatially inhomogeneous chemical potentials. The corresponding complete set of equations is a starting point for the future studies of the inverse cascade scenarios with a realistic treatment of plasma inhomogeneities.

V. CONCLUSION

In this paper, by making use of the chiral kinetic equation, we derived a closed set of anomalous Maxwell equations relevant for the study of relativistic plasmas with chiral asymmetry and inhomogeneities. By utilizing an expansion in powers of electromagnetic fields and derivatives, we derived electric and axial currents as well as a closed set of the coordinate-space equations for the electric (or fermion number) and chiral chemical potentials. We studied the two regimes in which the zero order distribution function is given by the standard Fermi-Dirac distribution function and a boosted one. The latter realizes the drifting state where the plasma drifts as a whole with the drift velocity perpendicular to both electric and magnetic fields. In this case, the expansion proceeds only in powers of the component of electric field parallel to the magnetic field whereas the perpendicular component of electric field is taken into consideration exactly. In addition to the Hall current for the electric current, we found its analogue for the axial current generated by the axial density. The chiral magnetic effect current is reproduced exactly in the drifting state. What is surprising is that we also found two additional electric and axial currents of a possible topological origin. They resemble the chiral magnetic/separation currents, but flow perpendicular to the magnetic field and are driven by the perpendicular component of electric field, as well as the scalar product of the electric and magnetic fields.

While a relativistic QED electron-positron plasma provides perhaps one the best examples of a plasma in the drifting state, quark-gluon plasma at sufficiently large temperatures may give an example of plasma where the drift is not fully developed because electromagnetically neutral gluons provide an essential drag on the charged particles. For such a case, we treated both electric and magnetic fields as perturbations and derived the expressions for the densities and currents to the second order in electromagnetic fields and derivatives. In the special case of vanishing electric field, we found a solution in the form of a diffusive chiral magnetic wave with the propagation speed $v_{\mathrm{CMW}} \propto |eB|/T^2$, see Eq. (66). The diffusion constant is given by $c^2\tau/3$, where $\tau \sim 1/[e^4 T \ln(1/|e|)]$ is the relaxation time [45].

In the framework used, we also derived the explicit expressions for the fermion number and chiral densities, as well as the corresponding currents. The results are in agreement with the continuity equations, supplemented by the appropriate quantum anomaly term. The final results reproduce several previously known effects. In the case of the electric current, we reproduced the Ohm's and diffusion currents, as well as the chiral magnetic effect. In addition, we found several new types of contributions connected with inhomogeneities in relativistic plasma that have not been reported before. They are the chiral diffusion and two diffusion terms of the Hall type perpendicular to the magnetic field. There is also a term driven by a time-dependent electric field. One of the very interesting new contributions is the anomalous chiral Hall effect current, which describes an electric current perpendicular to the applied electric field and the gradient $\partial\mu_5/\partial\mathbf{x}$. The origin of this current is related to the Berry curvature. We also find that there is its counterpart in the axial current density, which is driven by the electric field and the gradient of the chemical potential. It is even more amazing because it can be induced even in a chirally symmetric plasma.

In the case of the chiral current, we reproduced the well-known results for the chiral separation and chiral electric separation effects. We also found an expected diffusion current and several new types of currents including a chiral analogue of the Hall effect, which may allow one to experimentally determine the sign of the chiral charge of dominant carriers in a chiral plasma. In addition, there are two new types of chiral diffusion currents of the Hall type and two diffusion and time-dependent electric field analogues of the chiral electric separation effect.

The theoretical framework of our study here is a starting point in the study of the inverse cascade scenario proposed in a number of recent papers. In the simplified analysis of the corresponding dynamics, it is often assumed that a space-average currents and density correctly capture the underlying dynamics of the inverse cascade. It is quite natural to suggest that such an assumption may not be justified. Indeed, the underlying mechanism relies on the clear separation of the short- and long-range modes at the scale set by the chiral chemical potential. If the latter is not uniform, a whole window of length scales opens, where the dynamics does not have a preferred direction of the cascade. If the underlying processes in this region would happen to enhance the degree of inhomogeneities, the corresponding window of length scales would widen and prematurely quench the cascade. Indirectly, this may have been suggested by a recent study in Ref. [60], although the conclusions of that study may not be conclusive. In fact, there are indications that the inverse cascade in the model of Ref. [60] should be realized when sufficiently large lattices are used [61].

The natural continuation of the present study is a critical reexamination of the cosmological inverse cascade scenario, in which plasma inhomogeneities are properly accounted for. The corresponding investigation seems possible only by making extensive use of numerical methods. That is beyond the scope of the present paper, but will be attempted in the future study and reported elsewhere.

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Noted added.—When finishing this paper, we became aware of a partially overlapping study by Jiunn-Wei Chen, Takeaki Ishii, Shi Pu, and Naoki Yamamoto [62].

APPENDIX: USEFUL TABLE INTEGRALS AND RELATIONS

In this appendix, for reader's convenience, we list the key table integrals and relations used in the main text.

By making use of the shorthand notation $D_\lambda(\mu_\lambda)\equiv f_\lambda^{(\mathrm{eq})}(1-f_\lambda^{(\mathrm{eq})}) = e^{(cp-\mu_\lambda)/T}/(e^{(cp-\mu_\lambda)/T}+1)^2$, is it straightforward to derive the following results of integrations over the momenta:

$$\int\frac{d^3p}{(2\pi)^3}\frac{D_\lambda(\mu_\lambda)}{p^2} = \frac{T}{2\pi^2 c}\frac{1}{1+e^{-\mu_\lambda/T}}, \qquad (A1)$$

$$\int\frac{d^3p}{(2\pi)^3}\frac{D_\lambda(\mu_\lambda)}{p} = \frac{T^2}{2\pi^2 c^2}\ln\left(1+e^{\mu_\lambda/T}\right), \qquad (A2)$$

$$\int\frac{d^3p}{(2\pi)^3}p^{n-2}f_\lambda^{(\mathrm{eq})}(\mu_\lambda) = -\frac{T^{n+1}\Gamma(n+1)}{2\pi^2 c^{n+1}}\mathrm{Li}_{n+1}(-e^{\mu_\lambda/T}), \qquad n\geq 0, \qquad (A3)$$

$$\int\frac{d^3p}{(2\pi)^3}p^{n-2}D_\lambda(\mu_\lambda) = -\frac{T^{n+1}\Gamma(n+1)}{2\pi^2 c^{n+1}}\mathrm{Li}_{n}(-e^{\mu_\lambda/T}), \qquad n\geq 0, \qquad (A4)$$

$$\int\frac{d^3p}{(2\pi)^3}D_\lambda(\mu_\lambda)(1-2f_\lambda^{(\mathrm{eq})}) = \frac{T^3}{\pi^2 c^3}\ln\left(1+e^{\mu_\lambda/T}\right). \qquad (A5)$$

When calculating the contributions of antiparticles, one also encounters similar integrals with $D_\lambda(\mu_\lambda)\to D_\lambda(-\mu_\lambda)$. In order to simplify the final expressions for currents and densities, then, it is often useful to take into account the following relations:

$$\frac{1}{1+e^x}+\frac{1}{1+e^{-x}} = 1, \qquad (A6)$$

$$\ln(1+e^x)-\ln(1+e^{-x}) = x, \qquad (A7)$$

$$\mathrm{Li}_2(-e^x)+\mathrm{Li}_2(-e^{-x}) = -\frac{x^2}{2}-\frac{\pi^2}{6}, \qquad (A8)$$

$$\mathrm{Li}_3(-e^x)-\mathrm{Li}_3(-e^{-x}) = -\frac{x^3}{6}-\frac{\pi^2 x}{6}, \qquad (A9)$$

$$\mathrm{Li}_4(-e^x)+\mathrm{Li}_4(-e^{-x}) = -\frac{x^4}{24}-\frac{\pi^2 x^2}{12}-\frac{7\pi^4}{360}. \qquad (A10)$$

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