Computability of the étale Euler-Poincaré characteristic
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The goal of this chapter is to describe an algorithm for Algorithm 2.2. A key ingredient is the fact that, for a finite étale Galois morphism \( g : X' \to X \) of smooth curves with Galois group \( \Gamma \), we can make the equivalence between the category of finite étale \( \Gamma \)-schemes over \( X' \) and the category of finite étale schemes over \( X \) explicit. Note that under this equivalence, a finite étale group scheme on \( X \) corresponds to a \( \Gamma \)-equivariant group scheme on \( X' \). We will use this to reduce to the following situation.

Let \( f : X \to \text{Spec} \, k \) be a smooth connected curve, let \( \Gamma, G \) be finite groups such that \( \Gamma \) acts on \( X \) and on \( G \) by automorphisms. Consider the stack \( T = T_{\Gamma, X}^{G} \) over \( k \) of \( \Gamma \)-equivariant \( G \)-torsors on \( X \) over \( k \). We then construct a groupoid scheme \( R \Rightarrow U \), together with a morphism of the corresponding fppf prestack \( [U/R] \to T \) such that \( U(k) \to T(k) \) induces a \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant bijection \( \pi_0(U_{k^{\text{sep}}}) \to T(k^{\text{sep}}) / \cong \). To this end, we only require a few properties.

**Proposition 3.1.** Let \( T \) be a separated étale algebraic stack over a field \( k \). Let \( R \Rightarrow U \) be a groupoid scheme such that \( R \) and \( U \) are affine and of finite type over \( k \). Let \( [U/R] \to T \) be a morphism of fppf prestacks such that for all perfect field extensions \( l \) of \( k \), the functor \( [U/R](l) \to T(l) \) is an equivalence, and such that the isomorphism classes in \( U(k) \) are connected.

Then the map \( U(k) \to \text{Ob} \, T(k^{\text{sep}}) / \cong = \text{Gal}(k^{\text{sep}}/k) \)-equivariant map that is surjective, and factors through a \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant bijection \( \pi_0(U_{k^{\text{sep}}}) \to \text{Ob} \, T(k) / \cong \).

If in addition the isomorphism classes in \( U(k) \) are irreducible, then the connected components of \( U_{k^{\text{sep}}} \) are irreducible.

**Proof.** The surjectivity of the map \( U(k) \to \text{Ob} \, T(k^{\text{sep}}) / \cong \) and its compatibility with the Galois action is clear from the identity \( \text{Gal}(k^{\text{sep}}/k) = \text{Gal}(\bar{k}/k^{\text{perf}}) \) and the assumption that \( [U/R](l) \to T(l) \) is an equivalence for every perfect field extension \( l \) of \( k \), so let us show that the map factors through \( \pi_0(U_{\bar{k}}) \).

Let \( \bar{x} \in U(\bar{k}) \) and let \( j : U \to U_{\bar{k}} \) be the open immersion of the connected component \( U \) containing \( \bar{x} \) into \( U_{\bar{k}} \). Moreover, let \( f : U \to \text{Spec} \, \bar{k} \) denote the structure morphism, and let \( i : U_{\bar{k}} \to U \) be the projection morphism. Let \( Y \in U(U) \) denote the “universal object”; i.e. the object corresponding to the identity map on \( U \).

Define \( Y_1 = j^{-1}i^{-1}Y, Y_2 = f^{-1}\bar{x}^{-1}Y \in U(U) \), and consider their images in \( \text{Ob} \, T(U) \). Then the Hom-sheaf \( \text{Hom}_{T(U)}(Y_1, Y_2) \) is representable by a finite étale \( U \)-scheme as \( T \) is separated and étale over \( k \).
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Moreover, it is surjective as by construction $\text{Hom}_{T(U)}(Y_1, Y_2) (x)$ is non-empty and $U$ is connected. Hence for any point $x' \in U(k)$, the set $\text{Hom}_{T(U)}(Y_1, Y_2) (x')$ is non-empty as well. Therefore the morphism $U(k) \to \text{Ob } T(k) / \cong$ factors through a surjective $\text{Gal}(k_{\text{sep}} / k)$-equivariant map $\pi_0(U_{k}) \to \text{Ob } T(k) / \cong$. Note that the source of this map is equal to $\pi_0(U_{k_{\text{sep}}})$ as $\text{Spec } k \to \text{Spec } k_{\text{sep}}$ is a universal homeomorphism, and that the target of this map is $\text{Ob } T(k_{\text{sep}}) / \cong$ as $T$ is étale over $k$.

As the map $\pi_0(U_{k_{\text{sep}}}) \to \text{Ob } T(k_{\text{sep}}) / \cong$ is surjective and isomorphism classes in $U(k_{\text{sep}})$ are connected (and therefore are disjoint unions of connected components of $U(k_{\text{sep}})$), it follows that map $\pi_0(U_{k_{\text{sep}}}) \to \text{Ob } T(k_{\text{sep}}) / \cong$ is bijective. Finally, if the isomorphism classes in $U(k_{\text{sep}})$ are irreducible, then it follows from this that connected components of $U_{k_{\text{sep}}}$ must be irreducible as well.

In particular, since in our case $T$ is a separated étale algebraic stack over $k$, we see that as soon as we construct an explicit groupoid scheme $\mathcal{R} \rightrightarrows U$ satisfying the conditions of Proposition 3.1, we have a proof of the finiteness of $\text{Ob } T(k_{\text{sep}}) / \cong$ as $\pi_0(\text{Ob } X_{k_{\text{sep}}})$ is finite.

We give a rough description of the construction of such a groupoid scheme in the case in which $X$ is projective; the affine case is then reduced to this one. We fix a finite locally free morphism $\pi : X \to \mathbb{P}^1_k$ (such that $\Gamma$ acts on $X / \mathbb{P}^1_k$), and we note that for a $k$-scheme $S$, giving a $\Gamma$-equivariant $G$-torsor on $X_S$ is equivalent to giving

- a finite locally free $\mathcal{O}_{\mathbb{P}^1_S}$-algebra $\mathcal{O}_X$, together with compatible actions of the finite groups $\Gamma$ and $G$,
- a morphism $\varphi : \pi_* \mathcal{O}_X \to \mathcal{O}_Y$ that is $\Gamma$-equivariant and $G$-equivariant for the trivial action of $G$ on $\pi_* \mathcal{O}_X$,

such that $\varphi$ corresponds to a $G$-torsor on $X$; in particular, $\mathcal{O}_Y$ is finite, étale, and of constant rank $\# G$ as an $\pi_* \mathcal{O}_X$-algebra. The desired groupoid scheme $\mathcal{R} \rightrightarrows U$ then is one in which $U$ has a moduli interpretation in terms of the objects above.

This chapter is roughly divided into four parts. In the first part, we set up a language for “universal linear algebra over $\mathbb{P}^1$.” In the second part (starting from Section 3.7) and the third part (starting from Section 3.13), we use this in order to describe a groupoid scheme with the desired properties, in the projective and affine case, respectively. Finally, we then use this description in the last part (starting from Section 3.17) to compute $R^q f_*$ and $R^q f_!$ for $q = 0, 1, 2$.

3.1 Category schemes

In our construction of the desired groupoid scheme, categories that are not groupoids will arise naturally. Therefore we will need the notion of a category scheme, which is (for the cognoscenti) an internal category (see e.g. Johnstone [23, Sec. B.2.3]) in the category of schemes over a fixed base scheme. We work out what this means below.

Definition 3.2. Let $S$ be a scheme. A category scheme $\mathcal{C}$ over $S$ consists of:

- $S$-schemes $\mathcal{R}_C, U_C$ (the scheme of morphisms, resp. scheme of objects);
- $S$-morphisms $\alpha, \omega : \mathcal{R}_C \to U_C$ (the source, resp. target morphisms);
- an $S$-morphism $1 : U_C \to \mathcal{R}_C$ (the unit morphism);
- an $S$-morphism $\circ : \mathcal{R}_C \times_{\alpha, \omega} \mathcal{R}_C \to \mathcal{R}_C$ (the composition morphism),
such that the following diagrams commute.

- (source and target of unit)

\[
\begin{array}{ccc}
\mathcal{R}_C & \xrightarrow{\pi_1} & \mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C \\
\mathcal{U}_C & \xrightarrow{\pi_2} & \mathcal{R}_C \\
\omega & \xleftarrow{\omega} & \alpha \\
\mathcal{R}_C & \xleftarrow{\alpha} & \mathcal{U}_C
\end{array}
\]

- (source and target of composition)

\[
\begin{array}{ccc}
\mathcal{R}_C & \xrightarrow{\pi_1} & \mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C \\
\mathcal{U}_C & \xrightarrow{\omega} & \mathcal{R}_C \\
\mathcal{U}_C & \xrightarrow{\alpha} & \mathcal{R}_C \\
\mathcal{R}_C & \xrightarrow{\alpha} & \mathcal{U}_C
\end{array}
\]

- (unit)

\[
\begin{array}{ccc}
\mathcal{R}_C & \xrightarrow{(1\omega,\text{id})} & \mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C \\
\text{id} & \xleftarrow{(\text{id},1\alpha)} & \mathcal{R}_C \\
\mathcal{R}_C & \xleftarrow{\alpha} & \mathcal{U}_C
\end{array}
\]

- (associativity)

\[
\begin{array}{ccc}
\mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C & \xrightarrow{(\text{id},\text{id})} & \mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C \\
\mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C & \xleftarrow{\text{id},\text{id}} & \mathcal{R}_C \\
\mathcal{R}_C \times_{\mathcal{U}_C} \mathcal{R}_C & \xrightarrow{\omega} & \mathcal{R}_C
\end{array}
\]

We will use the notation $\text{Ob}\mathcal{C}$ instead of $\mathcal{U}_C$ if we view it as a scheme of objects. We also define the corresponding notion of a functor.

**Definition 3.3.** Let $S$ be a scheme, and let $\mathcal{C}$ and $\mathcal{D}$ be category schemes. A functor $F: \mathcal{C} \to \mathcal{D}$ consists of $S$-morphisms $\mathcal{R}_F: \mathcal{R}_C \to \mathcal{R}_D$ and $\mathcal{U}_F: \mathcal{U}_D \to \mathcal{U}_D$ such that the following diagrams (representing respectively the compatibility of $F$ with source and target, unit, and composition) commute.

\[
\begin{array}{ccc}
\mathcal{R}_C & \xrightarrow{\mathcal{R}_F} & \mathcal{R}_D \\
\mathcal{U}_C & \xrightarrow{\mathcal{U}_F} & \mathcal{U}_D
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{R}_C & \xrightarrow{\mathcal{R}_F} & \mathcal{R}_D \\
\mathcal{U}_C & \xrightarrow{\mathcal{U}_F} & \mathcal{U}_D \\
\mathcal{U}_C & \xrightarrow{1} & \mathcal{U}_D
\end{array}
\]
Definition 3.4. Let $S$ be a scheme. A single-sorted category scheme consists of:

- an $S$-scheme $C$;
- $S$-morphisms $\alpha, \omega: C \to C$ (the source, resp. target morphisms);
- an $S$-morphism $\circ: C \times_{\alpha, \omega} C \to C$ (the composition morphism),

such that the following diagrams (representing respectively the relations regarding the source and target of the unit, the source and target of the composition, the unit morphism, and associativity) commute.

The term single-sorted refers to the fact that this notion of a category uses only one “type”, namely that of morphisms. We now define functors for this notion of category scheme.
3.2 The category scheme of standard modules

**Definition 3.5.** Let $S$ be a scheme, and let $C$ and $D$ be single-sorted category schemes over $S$. A functor $F : C \to D$ is a morphism of schemes such that the following diagrams (representing the compatibility of $F$ with source and target, and with composition, respectively) commute.

\[
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\omega \downarrow & & \downarrow \omega \\
C & \xrightarrow{F} & D
\end{array}
\quad
\begin{array}{ccc}
C & \xrightarrow{F} & D \\
\alpha \downarrow & & \downarrow \alpha \\
C & \xrightarrow{F} & D
\end{array}
\quad
\begin{array}{ccc}
C \times_{\alpha, \omega} C & \xrightarrow{F \times F} & D \times_{\alpha, \omega} D \\
\circ \downarrow & & \downarrow \circ \\
C & \xrightarrow{F} & D
\end{array}
\]

Note that we obtain a category $\text{Cat}_S$ of category schemes over $S$, as well as a category $\text{Cat}'_S$ of single-sorted category schemes over $S$. In order to avoid confusion with the category $\text{Cat}$ of small categories, we will never drop the base scheme $S$ from the notation. By construction, we have the following.

**Proposition 3.6.** The functor $\text{Cat}_S \to \text{Cat}'_S$ sending an object $(U, \mathcal{R}, \alpha, \omega, 1, \circ)$ to the object $(\mathcal{R}, 1\alpha, 1\omega, \circ)$ is an equivalence and has as quasi-inverse the functor $\text{Cat}'_S \to \text{Cat}_S$ sending an object $(\mathcal{C}, \alpha, \omega, \circ)$ to $(E, \mathcal{C}, \alpha, \omega, i, \circ)$, where $i : E \to \mathcal{C}$ is the equaliser of $\alpha$ and $\omega$.

**Proof.** By construction. \(\square\)

Finally, using the Yoneda lemma, one also has descriptions of category schemes in terms of the functor(s) of points. We will use all of these descriptions interchangably from now on.

3.2 The category scheme of standard modules

We start to carry out the program outlined in the introduction of this chapter by constructing an affine category scheme of finite type over $k$ modeling the category of vector bundles on $\mathbb{P}_k^1$, presented as in Section 1.6.2. Its objects over a $k$-scheme $S$ will be of the following kind. Let $\text{Seq} = \text{Seq}(\mathbb{Z})$ denote the set of all finite sequences of integers.

**Definition 3.7.** Let $S$ be a scheme, and let $a = (a_i)_{i=1}^s \in \text{Seq}$ be a finite sequence of integers. The standard module of type $a$ over $S$ is the $\mathcal{O}_{\mathbb{P}_k^1}$-module

\[\mathcal{O}_{\mathbb{P}_k^1}(a) = \bigoplus_{i=1}^s \mathcal{O}_{\mathbb{P}_k^1}(a_i).\]

Let $S \subseteq \text{Seq}$. A locally standard module of type in $S$ is an $\mathcal{O}_{\mathbb{P}_k^1}$-module $\mathcal{E}$ such that there exists a locally constant map $q : S \to S$ such that $\mathcal{E}|_{q^{-1}(a)} = \mathcal{O}_{\mathbb{P}_k^1}(a)$ for all $a \in S$. 39
So, as mentioned before, Theorem 1.23 states that every vector bundle over \( O_{\mathbb{P}^1_k} \) for a field \( k \) is isomorphic to a locally standard module over \( k \). In the remainder of this section, we define the category scheme of locally standard modules, together with some additional (“linear algebraic”) structure we will be using in later constructions. Our choice of base scheme will be \( \text{Spec} \mathbb{Z} \); this doesn’t cause any loss of generality, and will be easier on the notation.

### 3.2.1 Construction

We construct the category scheme of locally standard modules over \( k \) as a disjoint union of affine schemes of finite type over \( k \), which we describe first. To this end, we identify, for a scheme \( S \), the scheme \( \mathbb{P}^1_S \) with \( \text{Proj}_S O_S[x,y] \), and for all integers \( n \), the \( O_S \)-module \( O_{\mathbb{P}^1_S}(n)(\mathbb{P}^1_S) \) with \( O_S[x,y]_n \), the degree \( n \) part of \( O_S[x,y] \).

**Definition 3.8.** Let \( S \subseteq \text{Seq} \). The category scheme \( \text{Mod}^\text{st}_S \) of locally standard modules of type in \( S \) is the functor \( \text{Sch}^{\text{op}} \rightarrow \text{Cat} \) that sends a scheme \( S \) to the category of locally standard modules of type in \( S \).

We show that this functor is representable, so that it indeed is a category scheme. To this end, we introduce an auxiliary functor.

**Definition 3.9.** Let \( a,b \in \text{Seq} \). The functor \( \text{Mod}^\text{st}_{b,a} \) is the functor \( \text{Sch}^{\text{op}} \rightarrow \text{Set} \) that sends a scheme \( S \) to the set of \( O_{\mathbb{P}^1_S} \)-linear maps \( O_{\mathbb{P}^1_S}(a) \rightarrow O_{\mathbb{P}^1_S}(b) \).

**Lemma 3.10.** Let \( a,b \in \text{Seq} \). The functor \( \text{Mod}^\text{st}_{b,a} \) is representable by \( \mathbb{A}^N(b,a)_k \) where \( N(b,a) = \sum_{\sigma,\tau} \max(0, b_\tau - a_\sigma + 1) \).

The proof of Lemma 3.10 follows from the following. Note that for all schemes \( S \), we have

\[
\text{Hom}_{O_{\mathbb{P}^1_S}}(O_{\mathbb{P}^1_S}(a), O_{\mathbb{P}^1_S}(b)) = \bigoplus_{i,j} \text{Hom}_{O_{\mathbb{P}^1_S}}(O_{\mathbb{P}^1_S}(a_i), O_{\mathbb{P}^1_S}(b_j)),
\]

so therefore we obtain an identification

\[
\text{Hom}_{O_{\mathbb{P}^1_S}}(O_{\mathbb{P}^1_S}(a), O_{\mathbb{P}^1_S}(b)) = \left\{ M \in \text{Mat}_{t,s}(O_S(S)[x,y]) : M_{ji} \in O_S(S)[x,y]_{b_j-a_i} \right\},
\]

where \( s \) and \( t \) are the lengths of \( a \) and \( b \), respectively.

Note that this is an \( O(S) \)-module admitting an \( O(S) \)-basis given by, for all \( i,j \), and \( \lambda \in \{0,1,\ldots,b_j-a_i\} \), the \( t \times s \)-matrix \( E_{ji}^{\lambda} \) that has zeroes everywhere except for the \((j,i)\) entry, which is \( x^\lambda y^{b_j-a_i-\lambda} \) (taking the lexicographical order for triples \((j,i,\lambda)\)). The result follows.

We therefore get the representability of \( \text{Mod}^\text{st}_S \) as a consequence.

**Proposition 3.11.** Let \( S \subseteq \text{Seq} \). The functor \( \text{Mod}^\text{st}_S \) is representable by the category scheme \( (\text{Mod}^\text{st}_S, \alpha, \omega, \circ) \), where

- \( \text{Mod}^\text{st}_S = \bigsqcup_{a,b \in S} \text{Mod}^\text{st}_{b,a} \).
3.2 The category scheme of standard modules

- \( \alpha : \text{Mod}^S \to \text{Mod}^S \) is the morphism such that
  \[
  \alpha_{b,a} = \alpha\big|_{\text{Mod}^S_{b,a}} : \text{Mod}^S_{b,a} \to \text{Mod}^S_{a,a}
  \]
is the constant morphism with value the identity matrix;

- \( \omega : \text{Mod}^S \to \text{Mod}^S \) is the morphism such that
  \[
  \omega_{b,a} = \omega\big|_{\text{Mod}^S_{b,a}} : \text{Mod}^S_{b,a} \to \text{Mod}^S_{b,b}
  \]
is the constant morphism with value the identity matrix;

- \( \circ : \text{Mod}^S \times_{a,a} \text{Mod}^S \) is the morphism such that
  \[
  \circ_{c,b,a} = \circ\big|_{\text{Mod}^S_{c,b} \times \text{Mod}^S_{b,a}} : \text{Mod}^S_{c,b} \times \text{Mod}^S_{b,a} \to \text{Mod}^S_{c,a}
  \]
is the morphism given by matrix multiplication.

If \( S \) is finite, then \( \text{Mod}^S \) is affine and of finite type.

**Proof.** By definition, the functor \( \text{Mod}^S \) is the Zariski sheafification of the functor \( \coprod_{b,a \in S} \text{Mod}^S_{b,a} \). It follows that \( \text{Mod}^S \) is representable by \( \coprod_{b,a \in S} \text{Mod}^S_{b,a} \). Moreover, for all schemes \( S \) we have by construction that the maps \( \alpha(S), \omega(S), \) and \( \circ(S) \) coincide with the source, target, and composition maps on the category of locally standard modules of type in \( S \).

Finally, if \( S \) is finite, then \( \text{Mod}^S \) is a finite disjoint union of affine schemes of finite type, therefore affine and of finite type itself. \( \square \)

**Corollary 3.12.** For all \( a \in \text{Seq} \), the scheme \( \text{Ob Mod}^S_{\{a\}} \) is representable by \( \text{Spec} \mathbb{Z} \).

Let \( S \subseteq \text{Seq} \). The functor \( \text{Ob Mod}^S \) is representable by \( \coprod_{a \in S} \text{Ob Mod}^S_{\{a\}} \).

Note that by construction, for any field \( k \), the category of \( k \)-points of \( \text{Mod}^S \) is equal to the category \( \mathcal{P}(k) \) of Section 1.6.2.

Expanding everything in terms of the basis \( \{ E_{ji,\lambda} \} \) of \( \text{Mat}_{b,a} \) described above, we find that the following an explicit description of \( \text{Mod}^S_{1} \).

**Formulary 3.13.** Let \( a, b, c \in \text{Seq} \) be of lengths \( s, t, u \), respectively. Then
\[
\text{Mod}^S_{b,a} = \text{Spec} \mathbb{Z}[x_{ji,\lambda} : 1 \leq j \leq t, 1 \leq i \leq s, 0 \leq \lambda \leq b_j - a_i].
\]
The morphisms \( \alpha_{b,a} : \text{Mod}^S_{b,a} \to \text{Mod}^S_{a,a} \) and \( \omega_{b,a} : \text{Mod}^S_{b,a} \to \text{Mod}^S_{b,b} \) are given in terms of the coordinate rings by
\[
x_{ji,\lambda} \mapsto \begin{cases} 
1 & \text{if } i = j \text{ (so } \lambda = 0); \\
0 & \text{otherwise}.
\end{cases}
\]
The morphism \( \circ_{c,b,a} : \text{Mod}^S_{c,b} \times \text{Mod}^S_{b,a} \to \text{Mod}^S_{c,a} \) is given in terms of the coordinate rings by
\[
x_{ki,\lambda} \mapsto \sum_{j=1}^{t} \sum_{(\lambda_1, \lambda_2)} x_{kj,\lambda_1} \otimes x_{ji,\lambda_2},
\]
where \( (\lambda_1, \lambda_2) \) runs through all pairs of integers with the properties \( \lambda_1 + \lambda_2 = \lambda, 0 \leq \lambda_1 \leq c_k - b_j, \) and \( 0 \leq \lambda_2 \leq b_j - a_i \).

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3.2.2 Direct sums

We describe direct sums in \( \text{Mod}^{st} \). For a scheme \( S \), and \( a, a' \in \text{Seq} \) of lengths \( s, s' \), respectively, we identify the direct sum \( \mathcal{O}_{\mathbb{T}^1_S}(a) \oplus \mathcal{O}_{\mathbb{T}^1_S}(a') \) with \( \mathcal{O}_{\mathbb{T}^1_S}(aa') \), where \( aa' \) is the concatenation of \( a \) and \( a' \). Using this identification, we obtain a functor \( \oplus : \text{Mod}^{st} \times \text{Mod}^{st} \to \text{Mod}^{st} \) of categories, sending for all schemes \( S \) a pair \( (M, M') \in \text{Mod}^{st}(S) \times \text{Mod}^{st}(S) \) to \( M \oplus M' \).

Lemma 3.14. The functor \( \oplus : \text{Mod}^{st} \times \text{Mod}^{st} \to \text{Mod}^{st} \) is the morphism such that

\[
\oplus_{b,b',a,a'} = \bigoplus_{\text{Mod}^{st} \times \text{Mod}^{st}} : \text{Mod}^{st}_{b,a} \times \text{Mod}^{st}_{b',a'} \to \text{Mod}^{st}_{bb',aa'}
\]

sends, for any scheme \( S \), the pair \( (M, M') \in \text{Mod}^{st}_{b,a}(S) \times \text{Mod}^{st}_{b',a'}(S) \) to the direct sum \( M \oplus M' : \mathcal{O}_{\mathbb{T}^1_S}(b) \oplus \mathcal{O}_{\mathbb{T}^1_S}(b') \to \mathcal{O}_{\mathbb{T}^1_S}(a) \oplus \mathcal{O}_{\mathbb{T}^1_S}(a') \).

Expanding this in the same way as in \textbf{Formulary 3.13} we find the following.

\textbf{Formulary 3.15.} Let \( a, b, a', b' \in \text{Seq} \) of lengths \( s, t, s', t' \), respectively. Notation is as in \textbf{Formulary 3.13}. The morphism \( \oplus_{b,b',a,a'} : \text{Mod}^{st}_{b,a} \times \text{Mod}^{st}_{b',a'} \to \text{Mod}^{st}_{bb',aa'} \) is given in terms of the coordinate rings by

\[
x_{j,i,\lambda} \mapsto \begin{cases} x_{j,i,\lambda} \otimes 1 & \text{if } 1 \leq i \leq s \text{ and } 1 \leq j \leq t; \\ 1 \otimes x_{j-t,i-s,\lambda} & \text{if } s+1 \leq i \leq s+s' \text{ and } t+1 \leq j \leq t+t'; \\ 0 & \text{otherwise}. \end{cases}
\]

3.2.3 Tensor products and duals

We describe tensor products in \( \text{Mod}^{st} \). If \( S \) is a scheme, \( a, a' \in \text{Seq} \), then we identify the tensor product \( \mathcal{O}_{\mathbb{T}^1_S}(a) \otimes \mathcal{O}_{\mathbb{T}^1_S}(a') \) with \( \bigoplus_{i,i'} \mathcal{O}_{\mathbb{T}^1_S}(a_i + a'_i) \), together with the lexicographical ordering on the pairs \( (i, i') \).

Let us therefore introduce the notation \( a \otimes a' \) for the finite sequence \( (a_i + a'_i)_{i,i'} \) together with the lexicographical ordering on the pairs \( (i, i') \). The identification above now gives us a functor \( \otimes : \text{Mod}^{st} \times \text{Mod}^{st} \to \text{Mod}^{st} \) of category schemes, sending for all schemes \( S \) a pair \( (M, M') \in \text{Mod}^{st}(S) \times \text{Mod}^{st}(S) \) to \( M \otimes M' \).

Lemma 3.16. The functor \( \otimes : \text{Mod}^{st} \times \text{Mod}^{st} \to \text{Mod}^{st} \) is the morphism such that

\[
\otimes_{b,b',a,a'} = \otimes_{\text{Mod}^{st}_{b,a} \times \text{Mod}^{st}_{b',a'}} : \text{Mod}^{st}_{b,a} \times \text{Mod}^{st}_{b',a'} \to \text{Mod}^{st}_{bb',aa'}
\]

sends, for any scheme \( S \), the pair \( (M, M') \in \text{Mod}^{st}_{b,a}(S) \times \text{Mod}^{st}_{b',a'}(S) \) to the tensor product \( M \otimes M' : \mathcal{O}_{\mathbb{T}^1_S}(b) \otimes \mathcal{O}_{\mathbb{T}^1_S}(b') \to \mathcal{O}_{\mathbb{T}^1_S}(a) \otimes \mathcal{O}_{\mathbb{T}^1_S}(a') \).

Note that the tensor product is associative, and that the neutral element with respect to the tensor product is \( \mathcal{O}_{\mathbb{T}^1_S} = \mathcal{O}_{\mathbb{T}^1_S}(0_1) \), where \( 0_1 \) is the zero sequence of length 1.

Next, note that for any scheme \( S \) and any objects \( \mathcal{E}, \mathcal{E}' \in \text{Ob} \text{Mod}^{st}(S) \), we have an isomorphism \( \Sigma(\mathcal{E}, \mathcal{E}') : \mathcal{E} \otimes \mathcal{E}' \to \mathcal{E}' \otimes \mathcal{E} \). This defines a morphism \( \Sigma \) from the scheme \( \text{Ob} \text{Mod}^{st} \times \text{Ob} \text{Mod}^{st} \) to \( \text{Mod}^{st} \).
Lemma 3.17. The morphism $\Sigma: \text{Ob Mod}^\text{st} \times \text{Ob Mod}^\text{st} \to \text{Mod}^\text{st}$ is the morphism such that

$$\Sigma_{a,a'} = \Sigma|_{\text{Ob Mod}^\text{st}_{\{a\}} \times \text{Ob Mod}^\text{st}_{\{a'\}}} : \text{Spec} \mathbb{Z} \to \text{Mod}^\text{st}_{a \oplus a, a \oplus a'}$$

is given by the isomorphism $\mathcal{O}_{P^1}(a) \otimes \mathcal{O}_{P^1}(a') \to \mathcal{O}_{P^1}(a') \otimes \mathcal{O}_{P^1}(a)$ switching the two factors.

Finally, for a scheme $S$ and $a \in \text{Seq}$, we write $\mathcal{O}_{P^1}(a)^\lor = \text{Hom}_{\mathcal{O}_{P^1}(a)}(\mathcal{O}_{P^1}(a), \mathcal{O}_{P^1}(a))$, which we identify with $\mathcal{O}_{P^1}(-a)$, where $-a$ is the sequence $(-a_i)$.

Using this identification, we get a dualisation functor $\lor: (\text{Mod}^\text{st})^{\text{op}} \to \text{Mod}^\text{st}$ sending, for a scheme $S$, a morphism $M \in \text{Mod}^\text{st}(S)$ to $M^\lor$. This dualisation is compatible with tensor products.

Lemma 3.18. The functor $\lor: (\text{Mod}^\text{st})^{\text{op}} \to \text{Mod}^\text{st}$ is the morphism such that

$$\lor_{b,a} = \big|_{(\text{Mod}^\text{st})^{\text{op}}_{b,a}}: \text{Mod}^\text{st}_{b,a} \to \text{Mod}^\text{st}_{-a,-b}$$

sends, for any scheme $S$, the morphism $M \in \text{Mod}^\text{st}_{P, a}(S)$ to its dual $M^\lor$.

Remark 3.19. We have, for every scheme $S$ and for all $a, b, c \in \text{Seq}$, identifications

$$\text{Hom}_{\mathcal{O}_{P^1}(a)}(\mathcal{O}_{P^1}(b) \otimes \mathcal{O}_{P^1}(c), \mathcal{O}_{P^1}(a)) = \bigoplus_{i,j,k} \mathcal{O}_{P^1}(a_i - b_j - c_k)(P^1_S)$$

$$= \text{Hom}_{\mathcal{O}_{P^1}(a)}(\mathcal{O}_{P^1}(c), \mathcal{O}_{P^1}(a) \otimes \mathcal{O}_{P^1}(b)^\lor),$$

that preserve matrices (as tuples with the lexicographical order on the index set). This identification identifies the trace map $\mathcal{O}_{P^1}(a) \otimes \mathcal{O}_{P^1}(a)^\lor \to \mathcal{O}_{P^1}(a)$ with the identity map $\mathcal{O}_{P^1}(a)^\lor \to \mathcal{O}_{P^1}(a)^\lor$.

Consider the tuple $\mathcal{O}_{P^1}(\text{Mod}^\text{st}, \otimes, \lor, \Sigma, \mathcal{O}_{P^1})$. Expanding everything in the same way as in Formulary 3.13, we find the following.

Formulary 3.20. Let $a, b, a', b' \in \text{Seq}$ be of lengths $s, t, s', t'$, respectively. Notation is as in Formulary 3.13. Write for convenience

$$\text{Mod}^\text{st}_{b \oplus b', a \oplus a'} = \text{Spec} \mathbb{Z}[x_{(j,j')(i,i')\lambda}: 1 \leq j \leq t, 1 \leq j' \leq t', 1 \leq i \leq s, 1 \leq i' \leq s', 0 \leq \lambda \leq b_j + b'_j - a_i - a'_i].$$

The morphism $\otimes_{b, b', a, a'}: \text{Mod}^\text{st}_{b, a} \times \text{Mod}^\text{st}_{b', a'} \to \text{Mod}^\text{st}_{b \oplus b', a \oplus a'}$ is given in terms of the algebras by

$$x_{(j,j')(i,i')\lambda} \mapsto \sum_{(\lambda_1, \lambda_2)} x_{ji\lambda_1} \otimes x_{j'i'\lambda_2},$$

where $(\lambda_1, \lambda_2)$ runs through all integers such that $\lambda_1 + \lambda_2 = \lambda$, $0 \leq \lambda_1 \leq b_j - a_i$, and $0 \leq \lambda_2 \leq b'_j - a'_i$. The morphism $\lor_{b, a}: \text{Mod}^\text{st}_{b, a} \to \text{Mod}^\text{st}_{-a, -b}$ is given in terms of the algebras by

$$x_{ji\lambda} \mapsto x_{ij\lambda}.$$

$^1$This tuple, together with Remark 3.19, gives $\text{Mod}^\text{st}(S)$ the structure of a strict rigid symmetric monoidal category for all schemes $S$. 43
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The morphism $\Sigma_{a,a'} \in \text{Mod}^{\text{st}}_{a' \oplus a \oplus a'}(\text{Spec} \, \mathbb{Z})$ is given by

$$x_{i_2i_1i_1'} \mapsto \begin{cases} 1 & \text{if } i_1 = i_2 \text{ and } i_1' = i_1'; \\ 0 & \text{otherwise}. \end{cases}$$

The object $\mathcal{O}_{\mathbb{P}^1} \in \text{Mod}^{\text{st}}_{1,0,0}(\text{Spec} \, \mathbb{Z})$ (where $0_1$ is the zero sequence of length 1) is given by

$$x_{000} \mapsto 1.$$

### 3.2.4 Exterior powers

Fix a positive integer $n$. We describe the $n$-th exterior power on $\text{Mod}^{\text{st}}$, using the following.

**Lemma 3.21.** Let $S$ be a scheme, let $a, b \in \text{Seq}$ be of lengths $s, t$, respectively, and let $n$ be a positive integer. Then the multilinear alternating map

$$\bigwedge: \mathcal{O}_{\mathbb{P}^1}(a)^n \to \sum_{I \subseteq \{1,2,\ldots,s\}, \#I=n} \mathcal{O}_{\mathbb{P}^1}(\sum_{i \in I} a_i)$$

mapping, locally on sections,

$$e \mapsto \left( \sum_{I} \text{sgn}(t^{-1} \pi) \prod_{k=1}^{n} e_{k\pi(k)} \right)_{I \subseteq \{1,2,\ldots,s\}, \#I=n}$$

is the $n$-th exterior power of $\mathcal{O}_{\mathbb{P}^1}(a)$. Here, $\pi$ runs through the set of bijections from $\{1,2,\ldots,n\}$ to $I$ and $t: \{1,2,\ldots,n\} \to I$ is the unique order-preserving bijection.

Moreover, the $n$-th exterior power $\bigwedge^n \varphi: \bigwedge^n \mathcal{O}_{\mathbb{P}^1}(a) \to \bigwedge^n \mathcal{O}_{\mathbb{P}^1}(b)$ of a $\mathcal{O}_{\mathbb{P}^1}$-linear map $\varphi: \mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{O}_{\mathbb{P}^1}(b)$, is given by the matrix indexed by subsets $I \subseteq \{1,2,\ldots,s\}$ and $J \subseteq \{1,2,\ldots,t\}$ of size $n$ of which the $(J,I)$-entry is the $n \times n$-minor of $\varphi$ (viewed as a $t \times s$-matrix with entries in $\mathcal{O}_S(S)[x,y]$) corresponding to $I$ and $J$.

Therefore, if we, for $a \in \text{Seq}$ of length $s$, write $\bigwedge^n a = (\sum_{i \in I} a_i)_{I \subseteq \{1,2,\ldots,s\}, \#I=n}$, where we take the lexicographical order on the set of subsets $I$ of $\{1,2,\ldots,s\}$, we can identify $\bigwedge^n \mathcal{O}_{\mathbb{P}^1}(a)$ with $\mathcal{O}_{\mathbb{P}^1}(\bigwedge^n a)$. This defines a morphism of schemes $\bigwedge^n: \text{Mod}^{\text{st}} \to \text{Mod}^{\text{st}}$ sending for any scheme $S$ the morphism $M \in \text{Mod}^{\text{st}}(S)$ to $\bigwedge^n M$.

**Lemma 3.22.** The functor $\bigwedge^n: \text{Mod}^{\text{st}} \to \text{Mod}^{\text{st}}$ is the morphism of schemes such that

$$\bigwedge^n_{b,a} = \bigwedge^n |_{\text{Mod}^{\text{st}}_{b,a}}: \text{Mod}^{\text{st}}_{b,a} \to \text{Mod}^{\text{st}}_{\bigwedge^n b,\bigwedge^n a}$$

sends, for any scheme $S$, the morphism $M \in \text{Mod}^{\text{st}}_{b,a}(S)$ to its $n$-th exterior power $\bigwedge^n M$.

Working this out as in [Formulary 3.13](#), we get the following.

**Formulary 3.23.** Let $n$ be a positive integer, and let $a, b \in \text{Seq}$ be of lengths $s, t$, respectively. Notation is as in [Formulary 3.13](#) for convenience

$$\text{Mod}^{\text{st}}_{\bigwedge^n b,\bigwedge^n a} = \text{Spec} \, \mathbb{Z}\left[ x_{IJ}: I \subseteq \{1,2,\ldots,s\}, J \subseteq \{1,2,\ldots,t\}, \#I = \#J = n, \right. \left. 0 \leq \lambda \leq \sum_{j \in J} b_j - \sum_{i \in I} a_i \right].$$
3.3 The category scheme of standard algebras

Then the morphism \( \Lambda^n_{b,a} : \text{Mod}^{st}_{b,a} \to \text{Mod}^{st}_{b,a,\Lambda^n} \) is given by

\[
x_{I\lambda} \mapsto \sum_{\pi} \text{sgn}(i^{-1} \pi) \sum_{\sum \lambda_i = \lambda} \prod_{i \in I} x_{\pi(i)\lambda_i}
\]

where \( \pi \) runs through the set of bijections \( I \to J \), and \( \iota : I \to J \) is the unique order-preserving bijection.

3.3 The category scheme of standard algebras

**Definition 3.24.** Let \( S \) be a scheme, let \( a \in \text{Seq} \), and let \( S \subseteq \text{Seq} \). An standard algebra of type \( a \) (resp. locally standard algebra of type in \( S \)) over \( S \) is an \( O_{P_S^1} \)-algebra of which the underlying \( O_{P_S^1} \)-module is standard of type \( a \) (resp. locally standard of type in \( S \)).

Equivalently, for \( a \in \text{Seq} \), a standard algebra of type \( a \) is given by a multiplication map \( \mu : O_{P_S^1}(a) \otimes O_{P_S^1}(a) \to O_{P_S^1}(a) \) and a unit map \( \iota : O_{P_S^1} \to O_{P_S^1}(a) \), such that the following diagrams commute.

- **(associativity)**

\[
\begin{array}{ccc}
O_{P_S^1}(a) \otimes O_{P_S^1}(a) \otimes O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a) \otimes O_{P_S^1}(a) \otimes O_{P_S^1}(a) \\
\mu \otimes \text{id} & & \mu \\
\text{id} \otimes \mu & \downarrow & \downarrow \\
O_{P_S^1}(a) \otimes O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a)
\end{array}
\]

- **(commutativity)**

\[
\begin{array}{ccc}
O_{P_S^1}(a) \otimes O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a) \otimes O_{P_S^1}(a) \\
\mu & & \mu \\
\Sigma & & \\
\mu & & \mu \\
O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a)
\end{array}
\]

- **(unit)**

\[
\begin{array}{ccc}
O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a) \otimes O_{P_S^1}(a) \\
\iota \otimes \text{id} & & \mu \\
\text{id} & \downarrow & \downarrow \\
O_{P_S^1}(a) & \longrightarrow & O_{P_S^1}(a)
\end{array}
\]

With this description, we see that a morphism \( \varphi : O_{P_S^1}(a) \to O_{P_S^1}(b) \) of standard algebras is a morphism of \( O_{P_S^1} \)-modules such that the following diagram commutes.
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\[ O_{P^1_S}(a) \otimes O_{P^1_S}(a) \xrightarrow{\varphi} O_{P^1_S}(b) \otimes O_{P^1_S}(b) \]

\[ \mu \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \mu \]

\[ O_{P^1_S}(a) \xrightarrow{\varphi} O_{P^1_S}(b) \]

\[ i \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad i' \]

\[ O_{P^1_S} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad O_{P^1_S} \]

**Definition 3.25.** Let \( S \subseteq \text{Seq} \). The category scheme \( \text{Alg}^{st}_S \) of locally standard algebras with type in \( S \) is the functor \( \text{Sch}^{op} \rightarrow \text{Cat} \) sending a scheme \( S \) to the category of locally standard algebras over \( S \) with type in \( S \).

We show that this functor is indeed representable, again using an auxiliary functor.

**Definition 3.26.** Let \( a, b \in \text{Seq} \). The functor \( \text{Alg}^{st}_{b,a} \) is the functor \( \text{Sch}^{op} \rightarrow \text{Set} \) sending a scheme \( S \) to the set of tuples \((\mu_b, l_b, \varphi, \mu_a, t_a)\) such that \((\mu_a, l_a)\) defines a structure of an \( O_{P^1_S} \)-algebra on \( O_{P^1_S}(a) \), \((\mu_b, l_b)\) defines a structure of an algebra on \( O_{P^1_S}(b) \), and \( \varphi: O_{P^1_S}(a) \rightarrow O_{P^1_S}(b) \) is a morphism of \( O_{P^1_S} \)-algebras.

**Lemma 3.27.** Let \( a, b \in \text{Seq} \). The functor \( \text{Alg}^{st}_{b,a} \) is representable by an affine scheme of finite type.

**Proof.** In fact, it is clear from the description of the category of standard algebras given above that \( \text{Alg}^{st}_{b,a} \) is a subscheme of \( \mathbb{A}^{N(b,b \otimes b) + N(b,0) + N(b,a) + N(a,a \otimes a) + N(a,0)}_k \).

**Corollary 3.28.** Let \( S \subseteq \text{Seq} \). Then \( \text{Alg}^{st}_S \) is representable by the tuple \((\text{Alg}^{st}_S, \alpha, \omega, \circ)\), where

- \( \text{Alg}^{st}_S = \bigsqcup_{a,b \in S} \text{Alg}^{st}_{b,a} \);
- \( \alpha: \text{Alg}^{st}_S \rightarrow \text{Alg}^{st}_S \) is the morphism such that
  \[ \alpha_{b,a} = \alpha|_{\text{Alg}^{st}_{b,a}}: \text{Alg}^{st}_{b,a} \rightarrow \text{Alg}^{st}_{a,a} \]
  is given by \((\mu_b, l_b, \varphi, \mu_a, t_a) \mapsto (\mu_a, l_a, \text{id}, \mu_a, t_a)\);
- \( \omega: \text{Alg}^{st}_S \rightarrow \text{Alg}^{st}_S \) is the morphism such that
  \[ \omega_{b,a} = \omega|_{\text{Alg}^{st}_{b,a}}: \text{Alg}^{st}_{b,a} \rightarrow \text{Alg}^{st}_{b,b} \]
  is given by \((\mu_b, l_b, \varphi, \mu_a, t_a) \mapsto (\mu_b, l_b, \text{id}, \mu_b, l_b)\);
- \( \circ: \text{Alg}^{st}_S \times_{\alpha, \omega} \text{Alg}^{st}_S \rightarrow \text{Alg}^{st}_S \) is the morphism such that
  \[ \circ_{c,b,a} = \circ|_{\text{Alg}^{st}_{c,b} \times_{\alpha, \omega} \text{Alg}^{st}_{b,a}}: \text{Alg}^{st}_{c,b} \times_{\alpha, \omega} \text{Alg}^{st}_{b,a} \rightarrow \text{Alg}^{st}_{c,a} \]
  is given by
  \[ (\mu_c, l_c, \psi, \mu_b, l_b) \circ (\mu_b, l_b, \varphi, \mu_a, t_a) = (\mu_c, l_c, \psi \circ \varphi, \mu_a, t_a) \].

If \( S \) is finite, then \( \text{Alg}^{st}_S \) is affine and of finite type.
3.4 Group actions

Proof. By definition, the functor \( \text{Alg}_{S}^{st} \) is the Zariski sheafification of the disjoint union of the representable functors \( \text{Alg}_{b,a}^{st} \) for \( a, b \in S \). Therefore it is representable by \( \coprod_{a,b \in S} \text{Alg}_{b,a}^{st} \). Moreover, for all schemes \( S \) we have by construction that the morphisms \( \alpha(S), \omega(S), \circ(S) \) coincide with the source, target, and composition on the category of locally standard algebras of type in \( S \) over \( S \).

If \( S \) is finite, then \( \text{Alg}_{S}^{st} \) is a finite disjoint union of affine schemes of finite type and therefore is itself affine and of finite type. \( \square \)

Note that the relative spectrum functor over \( \mathbb{P}^1 \) embeds, for all \( k \)-schemes \( S \), the category \( \text{Alg}^{st}(S) \) contravariantly and fully faithfully in the category of finite locally free \( \mathbb{P}^1_S \)-schemes. We will call the objects in the essential image standard schemes.

3.4 Group actions

We present finite groups by their multiplication tables.

Let \( \Gamma \) and \( G \) be multiplicatively written finite groups, with \( \Gamma \) acting by automorphisms on \( G \). Then recall that a \( \Gamma \)-equivariant \( G \)-action on a set \( X \) consists of an action of \( \Gamma \) and of \( G \) on \( X \), such that for all \( \gamma \in \Gamma \), \( g \in G \), and \( x \in X \), we have \( (\gamma g)x = \gamma(g(\gamma^{-1}x)) \). By the Yoneda lemma, this extends to arbitrary categories.

Example 3.29. The smallest example with non-trivial \( \Gamma \)-action on \( G \) and non-trivial action of \( G \) on \( X \) is the following; let \( G = \mathbb{Z}/3\mathbb{Z} \), and let \( \Gamma = \{ \pm 1 \} \) act on \( G \) via the unique non-trivial automorphism of \( G \). Let \( X = G \), together with the \( \Gamma \)-action. Then the regular action of \( G \) on \( X \) is \( \Gamma \)-equivariant.

One other way to describe this action is as \( G = X = \mu_3, \mathbb{R}(\mathbb{C}) \), and \( \Gamma = \text{Gal}(\mathbb{C}/\mathbb{R}) \) acting on \( G \) and \( X \) by complex conjugation.

Definition 3.30. Let \( S \) be a scheme, let \( a \in \text{Seq} \), and let \( S \subseteq \text{Seq} \). A standard \( \Gamma \)-equivariant \( G \)-algebra of type \( a \) (resp. locally standard \( \Gamma \)-equivariant \( G \)-algebra of type in \( S \)) over \( S \) is a \( \Gamma \)-equivariant \( G \)-algebra over \( \mathcal{O}_{\mathbb{P}^1_S} \) which is standard of type \( a \) (resp. locally standard of type in \( S \)) as an \( \mathcal{O}_{\mathbb{P}^1_S} \)-module.

An equivalent description is the following.

Note that an action of \( \Gamma \) on a standard algebra is given by a set \( \{ \rho_\gamma \} \) of endomorphisms such that

- \( \rho_1 = \text{id} \);
- \( \rho_\gamma \rho_\gamma' = \rho_{\gamma \gamma'} \) for all \( \gamma, \gamma' \in \Gamma \).

A morphism \( \varphi: \mathcal{O}_{\mathbb{P}^1_S}(a) \rightarrow \mathcal{O}_{\mathbb{P}^1_S}(b) \) between standard algebras that have a \( \Gamma \)-action is \( \Gamma \)-equivariant if and only if for all \( \gamma \in \Gamma \) the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^1_S}(a) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{P}^1_S}(b) \\
\rho_\gamma \downarrow & & \downarrow \rho_\gamma \\
\mathcal{O}_{\mathbb{P}^1_S}(a) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbb{P}^1_S}(b)
\end{array}
\]

Now a \( \Gamma \)-equivariant \( G \)-action on a standard \( \Gamma \)-algebra is given by a set \( \{ r_\gamma \} \) of endomorphisms such that
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- \( r_1 = \text{id} \);
- \( r_g r_{g'} = r_{gg'} \) for all \( g, g' \in G \);
- \( r_{g\gamma} = \rho_{\gamma} r_g (\rho_{\gamma})^{-1} \) for all \( \gamma \in \Gamma, g \in G \).

A \( \Gamma \)-equivariant morphism \( \varphi: \mathcal{O}_{\mathbf{P}_S^1}(a) \rightarrow \mathcal{O}_{\mathbf{P}_S^1}(b) \) between standard \( \Gamma \)-equivariant \( G \)-algebras is \( G \)-equivariant if and only if for all \( g \in G \) the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{O}_{\mathbf{P}_S^1}(a) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbf{P}_S^1}(b) \\
r_g & & r_g \\
\mathcal{O}_{\mathbf{P}_S^1}(a) & \xrightarrow{\varphi} & \mathcal{O}_{\mathbf{P}_S^1}(b)
\end{array}
\]

**Definition 3.31.** Let \( S \subseteq \text{Seq} \). The category scheme \((\Gamma, G)\)-\( \text{Alg}_{S}^{\text{st}} \) of locally standard \( \Gamma \)-equivariant \( G \)-algebras of type in \( S \) is the functor \( \text{Sch}^{\text{op}} \rightarrow \text{Cat} \) sending a scheme \( S \) to the category of locally standard \( \Gamma \)-equivariant \( G \)-algebras over \( S \) of type in \( S \).

As usual, we define the corresponding auxiliary functor in order to show that \((\Gamma, G)\)-\( \text{Alg}_{S}^{\text{st}} \) is representable.

**Definition 3.32.** Let \( a, b \in \text{Seq} \). The functor \((\Gamma, G)\)-\( \text{Alg}_{b,a}^{\text{st}} \) is the functor from \( \text{Sch}^{\text{op}} \) to Set sending a scheme \( S \) to the set of \( ((r'_g), (\rho'_g), (r_g), (\rho_g)) \) such that \( \varphi \in \text{Alg}_{b,a}^{\text{st}}(S) \), the tuples \( (r_g) \) and \( (\rho_g) \) define a \( \Gamma \)-equivariant \( G \)-action on the source of \( \varphi \), and the tuples \( (r'_g) \) and \( (\rho'_g) \) define a \( \Gamma \)-equivariant \( G \)-action on the target of \( \varphi \).

**Lemma 3.33.** Let \( a, b \in \text{Seq} \). The functor \((\Gamma, G)\)-\( \text{Alg}_{b,a}^{\text{st}} \) is representable by an affine \( k \)-scheme of finite type.

**Proof.** By construction, it is representable by a closed subscheme of \( (\text{Alg}_{b,a}^{\text{st}})^{1+2e+2n} \).

**Corollary 3.34.** Let \( S \subseteq \text{Seq} \). Then \((\Gamma, G)\)-\( \text{Alg}_{S}^{\text{st}} \) is representable. If \( S \) is finite, then \((\Gamma, G)\)-\( \text{Alg}_{S}^{\text{st}} \) is affine and of finite type.

Note that for any finite group \( G \) we have \((G,1)\)-\( \text{Alg}_{S}^{\text{st}} = (1,G)\)-\( \text{Alg}_{S}^{\text{st}} \); in this case, we will use the notation \( G\)-\( \text{Alg}_{S}^{\text{st}} \) instead. Moreover, we have \( 1\)-\( \text{Alg}_{S}^{\text{st}} = \text{Alg}_{S}^{\text{st}} \).

As seen in the previous section, we have for every scheme \( S \) a fully faithful contravariant functor from \((\Gamma, G)\)-\( \text{Alg}_{S}^{\text{st}}(S) \) into the category of \( \Gamma \)-equivariant \( G \)-schemes finite locally over \( \mathbf{P}_S^1 \); let us call objects in its essential image standard \( \Gamma \)-equivariant \( G \)-schemes.

### 3.5 Category schemes of free modules and algebras

We first consider the free modules over \( \mathcal{O}_S \).

**Definition 3.35.** Let \( S \) be a scheme, let \( \mathcal{R} \subseteq \mathbb{Z}_{\geq 0} \). A component-locally free \( \mathcal{O}_S \)-module of rank in \( \mathcal{R} \) is an \( \mathcal{O}_S \)-module \( \mathcal{E} \) such that there exists a locally constant map \( q: S \rightarrow \mathcal{R} \) such that \( \mathcal{E}|_{q^{-1} r} = \mathcal{O}_S^{r} \) for all \( r \in \mathcal{R} \). A component-locally free \( \mathcal{O}_S \)-module is a component-locally free \( \mathcal{O}_S \)-module with rank in \( \mathbb{Z}_{\geq 0} \).
We define the corresponding category scheme.

**Definition 3.36.** Let $\mathcal{R} \subseteq \mathbb{Z}_{\geq 0}$. The category scheme $\text{Mod}_{\text{free}}^{s_r}$ of free modules of rank in $\mathcal{R}$ is the functor $\text{Sch}^{\text{op}} \to \text{Cat}$ sending a scheme $S$ to the category of component-locally free $\mathcal{O}_S$-modules with rank in $\mathcal{R}$.

In order to show that $\text{Mod}_{\text{free}}^{s_r}$ is indeed a category scheme, we could repeat the same strategy as in Section 3.2. Alternatively, we note that if we have a free module $\mathcal{O}_S^r$ on $S$, it pulls back to a standard module $\mathcal{O}_S^{r_{\mathcal{R}^1}}$ over $S$, giving a natural transformation $\text{Mod}_{\text{free}}^{s_r} \to \text{Mod}_{\text{st}}^{s_r}$ of functors $\text{Sch}^{\text{op}} \to \text{Cat}$ such that for all schemes $S$, the functor $\text{Mod}_{\text{free}}^{s_r}(S) \to \text{Mod}_{\text{st}}^{s_r}(S)$ is fully faithful, and has as essential image those objects of $\text{Mod}_{\text{st}}^{s_r}(S)$ that have as type a finite sequence of zeroes. Hence we have the following.

**Proposition 3.37.** Let $\mathcal{R} \subseteq \mathbb{Z}_{\geq 0}$. Then $\text{Mod}_{\mathcal{R}}^{\text{free}} = \text{Mod}_{\mathcal{R}}^{\text{st}}$, where $\mathcal{S}$ is the set of finite sequences of zeroes with length in $\mathcal{R}$.

Recall that $\text{Mod}_{\text{st}}^{s_r}$, and therefore $\text{Mod}_{\text{free}}^{s_r}$ as well, is a disjoint union of affine schemes. Let us introduce some notation for these affine schemes.

**Definition 3.38.** Let $r, s \in \mathbb{Z}_{\geq 0}$. Then the scheme $\text{Mod}_{s_r}^{s_r}$ is $\text{Mod}_{0r,0s}^{s}$, where $0r,0s$ are the zero sequences of lengths $r$ and $s$, respectively.

**Proposition 3.39.** Let $\mathcal{R} \subseteq \mathbb{Z}_{\geq 0}$. Then $\text{Mod}_{\mathcal{R}}^{\text{free}} = \bigsqcup_{r,s \in \mathcal{R}} \text{Mod}_{s_r}^{s_s}$.

We do the same for the category scheme $(\Gamma, G)$-Alg$_{\text{st}}^{s_r}$.

**Definition 3.40.** Let $\Gamma, G$ be finite groups, together with an action of $\Gamma$ on $G$ by automorphisms. A component-locally free $\Gamma$-equivariant $G$-algebra over $\mathcal{O}_S$ of degree in $\mathcal{D}$ is a $\Gamma$-equivariant $G$-algebra over $\mathcal{O}_S$ that is component-locally free as a $\mathcal{O}_S$-module.

**Definition 3.41.** Let $\mathcal{D} \subseteq \mathbb{Z}_{\geq 0}$. The category scheme $(\Gamma, G)$-Alg$_{\text{st}}^{\text{free}}_{\mathcal{D}}$ of component-locally free $\Gamma$-equivariant $G$-algebras of degree in $\mathcal{D}$ is the functor $\text{Sch}^{\text{op}} \to \text{Cat}$ sending a scheme $S$ to the category of component-locally free $\Gamma$-equivariant $G$-algebras over $\mathcal{O}_S$ of degree in $\mathcal{D}$.

In the same way as before, we show the following.

**Proposition 3.42.** Let $\Gamma, G$ be finite groups, together with an action of $\Gamma$ on $G$ by automorphisms. The functor $(\Gamma, G)$-Alg$_{\text{free}}^{\text{st}}_{\mathcal{D}}$ is representable by $(\Gamma, G)$-Alg$_{\mathcal{S}}^{\text{st}}$, where $\mathcal{S}$ is the set of sequences of zeroes of length in $\mathcal{D}$.

Finally, we describe $(\Gamma, G)$-Alg$_{\text{free}}^{\text{st}}$ as a disjoint union of affine schemes.

**Definition 3.43.** Let $d,e \in \mathbb{Z}_{\geq 0}$. Then the scheme $(\Gamma, G)$-Alg$_{\text{free}}^{\text{st}}_{d}$ is $(\Gamma, G)$-Alg$_{0d,0e}^{\text{st}}$, where $0d,0e$ are the zero sequences of lengths $d$ and $e$, respectively.

**Proposition 3.44.** Let $\mathcal{D} \subseteq \mathbb{Z}_{\geq 0}$. Then $(\Gamma, G)$-Alg$_{\mathcal{D}}^{\text{free}} = \bigsqcup_{e,d \in \mathcal{D}} (\Gamma, G)$-Alg$_{0d,0e}^{\text{free}}$.

As in the previous two sections, we have for all schemes $S$ a fully faithful contravariant functor from $(\Gamma, G)$-Alg$_{\text{free}}^{\text{st}}(S)$ to the category of $\Gamma$-equivariant $G$-schemes finite locally free over $S$. We call the objects of its essential image the component-locally free $\Gamma$-equivariant $G$-schemes.
3.6 The slice category scheme

We treat this construction in general for clarity, although we only need it for the category schemes \((\Gamma, G)\)-\text{Alg}_{S}^{st} and \((\Gamma, G)\)-\text{Alg}_{D}^{free}\) from the previous section.

**Definition 3.45.** Let \(C\) be a category scheme over a scheme \(S\), and let \(X \in \text{Ob} C(S)\). Then the slice category scheme \(C_X\) of \(C\) over \(X\) is the functor \(\text{Sch}_{S}^\text{op} \to \text{Cat}\) sending an \(S\)-scheme \(T\) to the category \(C(T)_X\) of objects in \(C(T)\) with a morphism to \(X\) (or rather the image of \(X\) under \(C(S) \to C(T)\)).

**Lemma 3.46.** Let \(C\) be a category scheme over a scheme \(S\), and let \(X \in \text{Ob} C(S)\). Then \(C_X\) is representable.

**Proof.** Note that for any \(S\)-scheme \(T\), a morphism in \(C_X(T)\) is given by a commutative triangle

\[
\begin{array}{ccc}
  Z & \longrightarrow & Y \\
  \downarrow & & \downarrow \\
  X & \longrightarrow & \ast
\end{array}
\]

of morphisms in \(C(T)\). Therefore \(C_X\) is representable by a closed subscheme of \(C^3\). \(\square\)

Note that if \(X\) is the terminal object of \(C(S)\), then \(C_X = C\).

3.7 Torsors over smooth projective curves

Let \(k\) be a factorial field. Let \(\Gamma, G\) be finite groups, together with an action of \(\Gamma\) on \(G\) by automorphisms. Let \(f: X \to \text{Spec} k\) be a smooth projective curve, together with a given finite locally free morphism \(X \to \mathbb{P}^1_k\), and an action of \(\Gamma\) on \(X\) over \(\mathbb{P}^1_k\). Then note that \(X\) is (isomorphic to) an object of \((\Gamma, G)\)-\text{Alg}_{S}^{st}(k), say of type \(a\). In the next few sections, we will construct a category scheme over \(k\) of \(\Gamma\)-equivariant \(G\)-torsors on \(X\).

3.8 Fibre functors

Let \(p = (0 : 1) \in \mathbb{P}^1_k(k)\), and let \(N\) be a positive integer. What we describe below will work for any \(p \in \mathbb{P}^1_k(k)\) with some fixed pre-computation per finite sequence of integers – namely the expressions of a number of powers of \(x\) as linear combinations of powers of \(x - a\) in \(k[x]\). Since we do not need a lot of distinct points in the end (in fact we will only need two), we will restrict to the case \(p = (0 : 1)\) (and by symmetry the case \(p = (1 : 0)\) as well).

Denote for \(N \geq 2\) by \(p^{(N)}\) the \((N - 1)\)-th infinitesimal neighbourhood of \(p\) in \(\mathbb{P}^1_k\), we set \(p^{(1)} = p\), although we will usually omit the superscript \((1)\) from the notation. Note that for any \(k\)-scheme \(S\) the pullback of the standard module of type \(a\) (with \(a\) a sequence of length \(s\)) along the closed immersion \(p^{(N)}_S \to \mathbb{P}^1_S\) is as \(O_S\)-module isomorphic to \((O_S^{N})^s\). Therefore, for any \(k\)-scheme \(S\), the pullback of a locally standard module along \(p^{(N)}_S \to \mathbb{P}^1_S\) is component-locally free.
Definition 3.47. Let $N$ be a positive integer. The fibre functors

\[ \Phi_{p(N)} : \text{Mod}^{\text{st}} \to \text{Mod}^{\text{free}}, \quad \Phi_{p(N)} : (\Gamma, G)-\text{Alg}^{\text{st}} \to (\Gamma, G)-\text{Alg}^{\text{free}} \]

are the functors sending, for any scheme $S$, an object to its pullback along $p_S^{(N)} \to \mathbb{P}_S^1$.

We have the following description of $\Phi_{p(N)}$.

Proposition 3.48. Let $N$ be a positive integer, and let $a, b \in \text{Seq}$ be of lengths $s$ and $t$, respectively. Then $\Phi_{p(N),b,a} : \text{Mod}_{b,a}^{\text{st}} \to \text{Mod}_{Nt,Ns}^{\text{free}}$ sends, for any $k$-scheme $S$, the matrix $M \in \text{Mat}_{b,a}(S)$ to the $Nt \times Ns$-matrix of which the $(j', i')$-entry is the coefficient of $x^{j'-i'}$ in $M_{ji}(x, 1)$.

3.9 Finite flat covers

We now use the previous section to construct a category scheme of standard schemes that are finite locally over $X$ of constant rank. The following criterion guarantees that any morphism $Y \to X$ of standard schemes is automatically finite locally free.

Lemma 3.49. Let $S$ be a scheme, let $X$ be a finite locally free $\mathbb{P}_S^1$-scheme that is smooth over $S$. Let $Y$ be an $X$-scheme that is finite locally free over $\mathbb{P}_S^1$. Then $Y$ is a finite locally free $X$-scheme.

Proof. As $X$ and $Y$ are finite and of finite presentation over $\mathbb{P}_S^1$, $Y$ is also finite and of finite presentation over $X$. Hence it suffices to show that $Y$ is flat over $X$. By the fibre-wise criterion for flatness (see e.g. Görtz and Wedhorn [16, Cor. 14.25]), it suffices to prove that for all points $s \in S$, we have $Y_s$ flat over $X_s$. Hence we assume without loss of generality that $S$ is the spectrum of a field.

Let $y \in Y$ be a point, and let $x \in X$ and $p \in \mathbb{P}_S^1$ be their images. As $X$ is smooth over $S$, the ring $\mathcal{O}_{X,x}$ is a torsion-free $\mathcal{O}_{\mathbb{P}_S^1,p}$-algebra, which is either a discrete valuation ring, or a field. As $\mathcal{O}_{Y,y}$ is finite free, hence torsion-free, over $\mathcal{O}_{\mathbb{P}_S^1,p}$, it follows that $\mathcal{O}_{Y,y}$ is torsion-free over $\mathcal{O}_{X,x}$ as well, hence flat. Hence $Y$ is flat over $X$, as desired. \qed

We want to be able to check when (or otherwise enforce that) a morphism $Y \to X$ of standard schemes has constant rank. To this end, we use the following criterion.

Lemma 3.50. Let $S$ be a scheme, let $X$ be a finite locally free $\mathbb{P}_S^1$-scheme that is smooth over $S$. Let $Y$ be an $X$-scheme that is finite locally free over $\mathbb{P}_S^1$, such that $Y_{(0:1)}$ is finite locally free over $X_{(0:1)}$ of constant rank $n$. Then $Y$ is a finite locally free $X$-scheme of constant rank $n$.

Proof. We can check fibrewise on $S$ that $X_{(0:1)}$ intersects all components of $X$, from which our claim follows. \qed

Finally, note that $X_{(0:1)}$ is finite over $k$, so any finite locally free $\mathcal{O}_{X_{(0:1)}}$-module is in fact free. This motivates the following.
Definition 3.51. Let $S \subseteq \text{Seq}$. The category scheme $\text{Flat}_{X,S}^{\text{proj}}$ of standard finite flat covers of $X$ of type in $S$ is the functor $\text{Sch}_k^{\text{op}} \to \text{Cat}$ that sends a $k$-scheme $S$ to the category in which the objects are triples $(\mathcal{O}_Y, n, B)$, where $\mathcal{O}_Y$ is an object of $(\Gamma, G)\text{-Alg}_{X,S}^{\text{st}}(S)$, $n \geq 0$ is an integer, and $B : \mathcal{O}_X^{n}_{X(0:1)} \to \mathcal{O}_Y^{(0:1)}$ is an isomorphism of $\mathcal{O}_{X(0:1)}$-modules.

The functor $\text{Flat}_{X,S}^{\text{proj}}$ is indeed a category scheme.

Proposition 3.52. Let $S \subseteq \text{Seq}$. Then $\text{Flat}_{X,S}^{\text{proj}}$ is representable. If $S$ is finite, then $\text{Flat}_{X,S}^{\text{proj}}$ is affine and of finite type over $k$.

Proof. Let $S$ be a $k$-scheme, and let $(\mathcal{O}_Y, e, B)$ be an object of $\text{Flat}_{X,S}^{\text{proj}}(S)$. Note that for $B$, being an isomorphism of $\mathcal{O}_X^{(0:1)}$-modules is equivalent to being an $\mathcal{O}_S$-linear isomorphism such that the following diagram commutes.

\[
\begin{array}{ccc}
\mathcal{O}_X^{(0:1)} \otimes \mathcal{O}_X^{n}_{X(0:1)} & \xrightarrow{id \otimes B} & \mathcal{O}_X^{(0:1)} \otimes \mathcal{O}_Y \\
\downarrow & & \downarrow \\
\mathcal{O}_X^{n}_{X(0:1)} \otimes \mathcal{O}_X^{n}_{X(0:1)} & \xrightarrow{\mu} & \mathcal{O}_Y \otimes \mathcal{O}_Y \\
\downarrow & & \downarrow \mu \\
\mathcal{O}_X^{n}_{X(0:1)} & \xrightarrow{B} & \mathcal{O}_Y
\end{array}
\]

This establishes $\text{Flat}_{X,S}^{\text{proj}}$ as a closed subscheme of $(\Gamma, G)\text{-Alg}_{X,S}^{\text{st}} \times (\text{Mod}_{\text{free}}^+)^4$ (as we add the data of the isomorphism and its inverse of both the source and the target).

3.10 Finite étale covers

We now construct a category subscheme of $\text{Flat}_{X,S}^{\text{proj}}$ of those objects that are étale over $\mathcal{O}_X$. The next criterion allows us to restrict our attention to non-positive sequences of integers, as finite étale covers of a smooth projective curve are again smooth and projective.

Lemma 3.53. Let $S$ be a scheme, let $a \in \text{Seq}$, and let $X$ be a standard scheme of type $a$, and such that $X$ has geometrically reduced fibres over $S$. Then $a$ is non-positive (i.e. all of its elements are non-positive).

Proof. By taking a geometric fibre if necessary, we assume without loss of generality that $S$ is the spectrum of an algebraically closed field $k$. Let $X_1, \ldots, X_t$ be the components of $X$. Then there exist finite sequences $a_1, \ldots, a_t$ such that for all $i$, the algebra $\mathcal{O}_{X_i}$ is of type $a_i$. These have the property that their concatenation is equal to $a$ up to a permutation. Hence we assume without loss of generality that $X$ is connected. In this case $X$ is a reduced curve over $S$, so $\mathcal{O}_X(\mathbb{P}_S^1) = \mathcal{O}_X(X) = \mathcal{O}_S(S) = k$, where $\pi$ is the structure morphism of $X$, so we deduce that $a$ is non-positive.

Remark 3.54. Of course, the converse is not true; a counterexample is the $\mathcal{O}_{\mathbb{P}_k^1}$-module $\mathcal{O}_{\mathbb{P}_k^1} \oplus \mathcal{O}_{\mathbb{P}_k^1}(-1) \epsilon$ with multiplication given by $\epsilon^2 = 0$. 


Now we need a criterion for a morphism $Y \to X$ of constant rank of standard schemes to be étale. To this end, we will use the transitivity of the discriminant.

First, we recall the definitions of the discriminant and the norm of a finite locally free morphism $Y \to X$. Recall that, for a finite locally free morphism $Y \to X$ of schemes, we view $\mathcal{O}_Y$ as a (finite locally free) $\mathcal{O}_X$-algebra.

**Definition 3.55.** Let $f: Y \to X$ be a finite locally free morphism of schemes of constant rank, and let $\mu$ be the multiplication map $\mathcal{O}_Y \otimes \mathcal{O}_X \to \mathcal{O}_Y$. The trace form $\tau_f$ of $f$ is the morphism $\mathcal{O}_Y \to \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_X)$ corresponding to the composition $\text{Tr} \mu: \mathcal{O}_Y \otimes \mathcal{O}_X \mathcal{O}_Y \to \mathcal{O}_X$. The discriminant $\Delta_f$ of $f$ is the determinant (over $\mathcal{O}_X$) of the trace form $\tau_f$.

**Definition 3.56** (cf. Ferrand [11]). Let $f: Y \to X$ be a finite locally free morphism of schemes of constant rank, and let $\mathcal{L}$ be a line bundle on $Y$. The norm $N_f \mathcal{L}$ of $\mathcal{L}$ is the line bundle $\text{Hom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} f^* \mathcal{O}_Y, \text{det}_{\mathcal{O}_X} f^* \mathcal{L})$.

Let $f: Y \to X$ be a finite locally free morphism of schemes of constant rank, and let $\mathcal{E}$ and $\mathcal{F}$ be finite locally free $\mathcal{O}_Y$-modules of the same constant rank. By Deligne [8, Eq. 7.1.1] and the fact that norms (of line bundles) commute with tensor products and duals (see EGA2 [17, Sec. 6.5] and Ferrand [11, Prop. 3.3]), we see that there is a unique isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E}, \text{det}_{\mathcal{O}_X} \mathcal{F}) = \text{Hom}_{\mathcal{O}_X}(N_f \text{det}_{\mathcal{O}_Y} \mathcal{E}, N_f \text{det}_{\mathcal{O}_Y} \mathcal{F})$$

satisfying the following properties.

- It is compatible with base change by open immersions.
- For any isomorphism $\alpha: \mathcal{F} \to \mathcal{E}$, we have induced isomorphisms

  $$\text{Hom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E}, \text{det}_{\mathcal{O}_X} \mathcal{F}) \to \text{End}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E})$$

  and

  $$\text{Hom}_{\mathcal{O}_X}(N_f \text{det}_{\mathcal{O}_Y} \mathcal{E}, N_f \text{det}_{\mathcal{O}_Y} \mathcal{F}) \to \text{End}_{\mathcal{O}_X}(N_f \text{det}_{\mathcal{O}_Y} \mathcal{E}).$$

Therefore they induce isomorphisms

$$\text{Isom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E}, \text{det}_{\mathcal{O}_X} \mathcal{F}) \to \text{Aut}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E}) = \mathbb{G}_{m,X}$$

and

$$\text{Isom}_{\mathcal{O}_X}(N_f \text{det}_{\mathcal{O}_Y} \mathcal{E}, N_f \text{det}_{\mathcal{O}_Y} \mathcal{F}) \to \text{Aut}_{\mathcal{O}_X}(N_f \text{det}_{\mathcal{O}_Y} \mathcal{E}) = \mathbb{G}_{m,X}.$$ 

These isomorphisms are equal.

Therefore, we have the following.

**Corollary 3.57.** Let $f: Y \to X$ be a finite locally free morphism of schemes of constant rank, and let $\mathcal{E}$ be a finite locally free $\mathcal{O}_Y$-module of constant rank $r$. Then

$$\text{det}_{\mathcal{O}_X} \mathcal{E} = N_f \text{det}_{\mathcal{O}_Y} \mathcal{E} \otimes_{\mathcal{O}_X} (\text{det}_{\mathcal{O}_X} \mathcal{O}_Y)^{\otimes r}$$

$$\text{Hom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{E}, \mathcal{O}_X) = N_f \text{det}_{\mathcal{O}_Y} \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y) \otimes_{\mathcal{O}_X} (\text{Hom}_{\mathcal{O}_X}(\text{det}_{\mathcal{O}_X} \mathcal{O}_Y, \mathcal{O}_X))^{\otimes r}$$

Using the two identifications above, we may now state the transitivity of the discriminant. A proof can be found in e.g. Lieblich [28, Sec. 4.1].
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**Theorem 3.58** (Transitivity of the discriminant). Let \( f : Y \to X \) and \( g : Z \to Y \) be finite locally free morphisms of schemes of constant rank, and suppose that \( g \) has rank \( r \). Then
\[
\Delta_{f,g} = N_{f} \Delta_{g} \otimes \Delta_{f}^{\otimes r}.
\]

**Corollary 3.59.** Let \( f : Y \to X \) and \( g : Z \to Y \) be finite locally free morphisms of schemes of constant rank, and suppose that \( g \) has rank \( r \). Then \( g \) is étale if and only if we have
\[
\det \mathcal{O}_X \mathcal{O}_Z \cong (\det \mathcal{O}_X \mathcal{O}_Y)^{\otimes r} \text{ and } \Delta_{f,g} \text{ and } \Delta_{f}^{\otimes r} \text{ differ by a unit.}
\]

**Definition 3.60.** Let \( R \) be a set of non-negative integers. The category scheme \( \text{Et}^{\text{proj}}_{X,D} \) of standard finite étale \( \Gamma \)-equivariant \( G \)-schemes over \( X \) with degree in \( D \) is the functor \( \text{Sch}_{op}^k \to \text{Cat} \) sending a \( k \)-scheme \( S \) to the subcategory of \( \text{Flat}^{\text{proj}}_{X,S}(S) \) of finite étale morphisms \( Y \to X \) of standard schemes over \( S \), with degree in \( D \).

We show that \( \text{Et}^{\text{proj}}_{X,D} \) is indeed a category scheme.

**Proposition 3.61.** Let \( D \) be a set of non-negative integers. Then \( \text{Et}^{\text{proj}}_{X,D} \) is representable. If \( D \) is finite, then \( \text{Et}^{\text{proj}}_{X,D} \) is affine and of finite type over \( k \).

**Proof.** Let \( S \) be the set of non-positive sequences \( b \) of integers (of length \( t \)) such that
\[
t = \sum_{\sigma} a_{\sigma} \in D.
\]
Note that \( S \) is finite if \( D \) is finite. Then by [Lemma 3.49](#), [Lemma 3.53](#), and [Corollary 3.59](#), the desired category scheme is the category scheme in which the objects are the objects \( \mathcal{O}_Y \) of \( \text{Flat}^{\text{proj}}_{X,S}(S) \) such that \( \Delta_{\mathcal{O}_Y/\mathcal{O}_{P^1}} \) and \( \Delta_{\mathcal{O}_X/\mathcal{O}_{P^1}}^{\otimes \frac{1}{r}} \) differ by a unit \( \varepsilon \). This establishes \( \text{Et}^{\text{proj}}_{X,D} \) (explicitly if \( D \) is finite) as a closed subscheme of \( \text{Flat}^{\text{proj}}_{X,S} \times \mathbb{A}_k^4 \) (as we add the data of the units and their inverses corresponding to the source and target).

### 3.11 Torsors

We now identify the closed subscheme of \( \text{Et}^{\text{proj}}_{X,n} \) consisting of (morphisms between) \( G \)-torsors. To this end, we first prove the following lemmas, in order to show that checking whether an object of \( \text{Et}^{\text{proj}}_{X,n} \) is a \( G \)-torsor can essentially be done in the category scheme \( (\Gamma, G) \)-\( \text{Alg}^{st}_{S_X} \).

**Lemma 3.62.** Let \( f : Y \to X \) be a morphism of schemes, and let \( G \) be a finite group acting on \( Y / X \). Then \( Y \) is a \( G \)-torsor on \( X \) if and only if \( f \) is flat, surjective, locally of finite presentation, and \( G \) acts freely and transitively on geometric fibres.

**Proof.** The necessity of the condition is clear. Hence suppose that \( f \) is flat, surjective, locally of finite presentation, and \( G \) acts freely and transitively on geometric fibres. Then for any geometric point \( \overline{x} \) of \( S \), \( Y_{\overline{x}} \) is the trivial \( G \)-torsor, hence étale. As the property of being étale is fpqc local on the base, it follows that all fibres of \( f \) are étale, and since \( f \) is flat and locally of finite presentation, it follows that \( f \) is finite étale.

Now consider the morphism \( \varphi : G_X \times_X Y \to Y \times_X Y \) of finite étale \( Y \)-schemes given on the functor of points by \( (g, y) \mapsto (gy, y) \), where the occurring schemes are
viewed as $Y$-schemes via the projection on the second coordinate. Then $\varphi$ is itself finite étale surjective, and as $G_X \times_X Y$ and $Y \times_X Y$ have the same rank over $Y$, it follows that $\varphi$ is an isomorphism. After base change with itself, it admits a section, so as $f$ is finite étale, it also follows that $Y$ is a $G$-torsor, as desired. □

**Lemma 3.63.** Let $f : Y \to X$ be a finite étale morphism of schemes of constant rank $n$, and let $G$ be a finite group of order $n$ acting on $Y / X$. Then the locus in $X$ where $f$ is a $G$-torsor is open and closed in $X$.

**Proof.** Consider the locus $U$ in $Y \times_X Y$ on which the morphism $G_X \times_X Y \to Y \times_X Y$ given on the functor of points by $(g, y) \mapsto (gy, y)$ is an isomorphism (i.e. where the rank is equal to 1). It is an open and closed subset of $Y \times_X Y$ as this morphism is finite étale. As the rank of $f$ is equal to $n$, the $X$-locus where the same morphism is an isomorphism is the image of $U$ in $X$, and hence is open and closed as well. This locus equals the $X$-locus where $f$ is a $G$-torsor, as desired. □

**Corollary 3.64.** Let $S$ be a scheme, let $f : Y \to X$ be a finite étale morphism of $\mathbb{P}^1_S$-schemes, and let $G$ be a finite group of order $n$ acting on $Y / X$. Then $Y$ is a $G$-torsor on $X$ if and only if $Y_{(0:1)}$ is a $G$-torsor on $X_{(0:1)}$.

**Proof.** Note that by Lemma 3.62 we may assume that $S$ is the spectrum of a field, in which case it follows from Lemma 3.63 and the fact that $X_{(0:1)}$ intersects all components of $X$. □

Recall that for a $k$-scheme $S$, objects of $\text{Et}^{\text{proj}}_{\text{X}, n}(S)$ are “locally” of the form $(\mathcal{O}_Y, n, B)$, where $\mathcal{O}_Y$ is an object of $(\Gamma, G)$-$\text{Alg}^{\text{st}}_X(S)$, and $B$ is an isomorphism $\mathcal{O}_{X_{(0:1)}}^n \to \mathcal{O}_{Y_{(0:1)}}$ of $\mathcal{O}_{X_{(0:1)}}$-modules.

**Definition 3.65.** Let $X$ be a smooth projective curve together with an action of a finite group $\Gamma$ that is a standard scheme over $k$, say of type $a$. Let $G$ be a finite group of order $n$ on which $\Gamma$ acts by automorphisms. The category scheme $\text{Tors}^{\text{proj}}_{\text{X}}$ of $G$-torsors on $X$ is the functor $\text{Sch}^{\text{op}}_k \to \text{Cat}$ sending a $k$-scheme $S$ to the full subcategory of $\text{Et}^{\text{proj}}_{\text{X}, n}(S)$ of $G$-torsors on $X$.

**Proposition 3.66.** The functor $\text{Tors}^{\text{proj}}_{\text{X}}$ is representable by an affine category scheme of finite type over $k$.

**Proof.** It suffices to express explicitly the condition that an object $(\mathcal{O}_Y, n, B)$ of $\text{Et}^{\text{proj}}_{\text{X}, n}$ is a $G$-torsor. By Corollary 3.64, this is equivalent to

$$\mathcal{O}_{Y_{(0:1)}} \otimes \mathcal{O}_{X_{(0:1)}} \otimes \mathcal{O}_{Y_{(0:1)}} \to \mathcal{O}_{Y_{(0:1)}}, \quad y_1 \otimes y_2 \mapsto (gy_1y_2)_{g \in G}$$

being an isomorphism of $\mathcal{O}_{\mathbb{P}^1}$-modules. Note that we can compute the morphism from $\text{Ob} \text{Et}^{\text{proj}}_{\text{X}, n}$ to $\text{Alg}^{\text{free}}_{X_{(0:1)}}$ sending an object $(\mathcal{O}_Y, n, B)$ to the following composition in $\text{Alg}^{\text{free}}_{X_{(0:1)}}$. 

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This establishes $\text{Tors}_X^{\text{proj}}$ explicitly as a closed subscheme of $\text{Et}^{\text{proj}}_X \times (\text{Alg}^{\text{free}}_{X_{(0:1)}})^2$, by adding the data of the inverse of the morphism above for both the source and the target. □

3.12 The stack of $G$-torsors

We show that the category scheme $\text{Tors}_X^{\text{proj}}$ over $k$ defines a presentation of the stack $\mathcal{T} = \mathcal{T}^G_{\Gamma,X}$ over $k$ of $\Gamma$-equivariant $G$-torsors on $X$. More precisely, we show the following.

**Theorem 3.67.** Let $[\mathcal{U}_{\text{Tors}_X^{\text{proj}}}/\mathcal{R}_{\text{Tors}_X^{\text{proj}}}] \to \mathcal{T}$ be the functor sending for any $k$-scheme $S$, the object $T \in \text{Ob} \text{Tors}_X^{\text{proj}}(S)$ to the underlying $\Gamma$-equivariant $G$-torsor on $X$. For all field extensions $l$ of $k$, the functor $[\mathcal{U}_{\text{Tors}_X^{\text{proj}}}/\mathcal{R}_{\text{Tors}_X^{\text{proj}}}]_l \to \mathcal{T}(l)$ is an equivalence. The isomorphism classes in $[\mathcal{U}_{\text{Tors}_X^{\text{proj}}}/\mathcal{R}_{\text{Tors}_X^{\text{proj}}}]$ are irreducible. The stack $\mathcal{T}$ is the fppf stackification of $[\mathcal{U}_{\text{Tors}_X^{\text{proj}}}/\mathcal{R}_{\text{Tors}_X^{\text{proj}}}]$.

Therefore by Proposition 3.1 we have the following.

**Corollary 3.68.** The map $\text{Ob} \text{Tors}_X^{\text{proj}}(k) \to \text{Ob} \mathcal{T}(k) / \cong$ factors through a bijection $\pi_0(\text{Ob} \text{Tors}_X^{\text{proj},\text{sep}}) \to \text{Ob} \mathcal{T}(k_{\text{sep}}) / \cong$ that is $\text{Gal}(k_{\text{sep}}/k)$-equivariant, and the connected components of $\text{Ob} \text{Tors}_X^{\text{proj},\text{sep}}$ are irreducible.

We will first show the following.

**Theorem 3.69.** Let $k$ be a field, let $G$ be a finite group, let $X$ be a smooth projective curve together with a finite locally free morphism $X \to \mathbb{P}^1$. Then for all $k$-schemes $S$ and all étale $G$-torsors $Y$ over $X_S$, the $\mathcal{O}_{\mathbb{P}^1_S}$-algebra $\mathcal{O}_Y$ is as an $\mathcal{O}_{\mathbb{P}^1_S}$-module fppf locally isomorphic to a standard module.

**Proof.** Let $S$ be a $k$-scheme, and let $Y$ be an étale $G$-torsor over $X_S$. First note that we can reduce to the case that $S$ is affine, and since $Y$ is smooth over $S$, we can further reduce to the case that $S$ is of finite type over $k$ by Grothendieck [18 Prop. 17.7.8].

Let $s: \text{Spec} \kappa(s) \to S$ be a closed point of $S$, so that $\kappa(s)$ is finite over $k$. Let $Y_s / X_s$ denote the pullback of $Y / X_S$ along $s: \text{Spec} \kappa(s) \to S$. As $Y_s$ is finite locally free over $\mathbb{P}^1_{\kappa(s)}$, we see that $\mathcal{O}_{Y_s}$ is a standard module. Consider the base change $S_{\kappa(s)}$ of $S$ along $\text{Spec} \kappa(s) \to \text{Spec} k$. 

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Let $Y_1/X_{S(s)}$ be the pullback of $Y/X_S$ along $S_X(s) \to S$, and let $Y_2/X_{S(s)}$ be the pullback of $Y_2/X_S$ along $S_X(s) \to \text{Spec} \kappa(s)$. Note that $\mathcal{O}_{Y_2}$ is isomorphic to a standard module by Theorem 1.23. Let $I_{S(s)}(Y_1,Y_2)$ denote the functor sending $T$ to $\text{Isom}_{X,T}(Y_1,T,Y_2,T)$. By descent, it is representable by a finite étale $S_{k(s)}$-scheme and $I_{S(s)}(Y_1,Y_2)(s) \neq \emptyset$. Therefore (finite) étale locally around $s$ in $S_X(s)$, we have $Y_1 \cong Y_2$. As $S_X(s) \to S$ is an fppf cover, we deduce that the $\mathcal{O}_{\mathbb{P}_S^1}$-module $\mathcal{O}_Y$ is fppf locally isomorphic to a standard module over $S$.

**Corollary 3.70.** The stack $T$ is the fppf stackification of $[\mathcal{U}_{\text{Tors}_X}^{\text{proj}}/\mathcal{R}_{\text{Tors}_X}^{\text{proj}}]$.

**Proof.** We show that for any $k$-scheme $S$ and any $\Gamma$-equivariant $G$-torsor $Y$ on $X_S$, the torsor $Y$ is fppf locally on $S$ an object of $\text{Tors}_X^{\text{proj}}$.

Let $S$ be a $k$-scheme, and let $Y$ be a $\Gamma$-equivariant $G$-torsor on $X_S$. Then by Theorem 3.69, fppf locally on $S$ the $\mathcal{O}_{\mathbb{P}_S^1}$-module $\mathcal{O}_Y$ is a standard module, and $Y_{(0:1)}$ is a $G$-torsor on $X_S((0:1))$, so étale locally on $S$ the $G$-torsor $Y_{(0:1)}$ is isomorphic to $\bigsqcup_{g \in G} X_S((0:1))$. Therefore by construction of $\text{Tors}_X^{\text{proj}}$, it follows that $T$ is the stackification of $[\mathcal{U}_{\text{Tors}_X}^{\text{proj}}/\mathcal{R}_{\text{Tors}_X}^{\text{proj}}]$, as desired.

So in particular, the category scheme $\text{Tors}_X^{\text{proj}}$ defines a presentation of $T$. We now show the following.

**Proposition 3.71.** The stack $T$ is algebraic, and is presented by $\text{Tors}_X^{\text{proj}}$.

**Proof.** We show that the maps $\alpha, \omega: \mathcal{R}_{\text{Tors}_X}^{\text{proj}} \to \mathcal{U}_{\text{Tors}_X}^{\text{proj}}$ are smooth, so that $T$ is algebraic. Recall to this end that $\text{Tors}_X^{\text{proj}}$ is a disjoint union of category schemes $\text{Tors}^{\text{proj}}_{X,b,b}$, where $b$ runs through some finite subset of Seq. These category schemes are the category subschemes of $\text{Tors}_X^{\text{proj}}$ of those objects that are of type $b$. It therefore suffices to show that the morphisms $\alpha, \omega: \text{Tors}^{\text{proj}}_{X,b,b} \to \text{Ob} \, \text{Tors}^{\text{proj}}_{X,b,b}$ are smooth for all $b$. To this end, we construct for all $b$ a group scheme $A$ over $k$, as follows.

Let $A_1$ be the $\mathcal{O}_{\mathbb{P}_k}^1$-automorphism scheme over $k$ of $\mathcal{O}_{\mathbb{P}_k}^1(b)$, and let $A_2$ be the $\mathcal{O}_{X_{(0:1)}}$-automorphism scheme over $k$ of $\mathcal{O}_{X_{(0:1)}}^{\#G}$. Let $A = A_1 \times_k A_2$. We show that $A$ is smooth and geometrically irreducible. Of course, it suffices to show that both $A_1$ and $A_2$ are smooth and geometrically irreducible.

First consider $A_1$. Using the description of $\text{Mod}^{\text{st}}$, we easily see that $A_1$ is a product of factors of the form $\text{GL}_{i,k,\text{sep}}$ or $G_{a,k,\text{sep}}$, and therefore smooth and geometrically irreducible.

For $A_2$, we first note that $A_2$ is isomorphic to $\text{Hom}_k(X_{(0:1)},\text{GL}_{n,k})$. So we have an open immersion $A_2 \to \text{Hom}_k(X_{(0:1)},\mathbb{A}^{n^2})$, and its target is as a scheme isomorphic to $\mathbb{A}^{d(\#G)^2}$, where $d$ is the degree of $X_{(0:1)}$ over $k$. Hence $A_2$ is smooth and geometrically irreducible. We deduce that $A$ is smooth and geometrically irreducible.

Note that $\text{Ob} \, \text{Tors}^{\text{proj}}_{X,b,b}$ admits an obvious action of $A$, in other words, we have two maps suggestively denoted $\alpha, \omega: A \times_k \text{Ob} \, \text{Tors}^{\text{proj}}_{X,b,b} \to \text{Ob} \, \text{Tors}^{\text{proj}}_{X,b,b}$ given on functors.
of points by $\alpha(g, t) = t$ and $\omega(g, t) = gt$. In fact, the morphism from $A \times \text{Ob Tors}_{X,p,b}^{\text{proj}}$ to $\text{Tors}_{X,p,b}^{\text{proj}}$ sending $(g, t)$ to the morphism $t \to gt$ induced by $g$ is an isomorphism of schemes compatible with $\alpha$ and $\omega$. This shows that both $\alpha$ and $\omega$ are smooth, as $A$ is a smooth group scheme. □

We can now finish the proof of Theorem 3.67.

**Proof of Theorem 3.67.** By construction of $\text{Tors}_{X}^{\text{proj}}$ we see that it is fibred in groupoids and that there is an obvious fully faithful functor $\text{Tors}_{X}^{\text{proj}} \to T$ of fibred categories.

We show that for any field extension $l$ of $k$, any $\Gamma$-equivariant étale $G$-torsor over $X$ is isomorphic to one arising from $\text{Tors}_{X}^{\text{proj}}(l)$. Recall that by construction, the category $\text{Tors}_{X}^{\text{proj}}(l)$ is equivalent to the category of $\Gamma$-equivariant, $G$-invariant finite étale morphisms $f: Y \to X_l$ of constant rank $n$, together with an $A$-linear isomorphism $\phi: B \to \bigoplus_{g \in G} A$, and such that the $G$-equivariant $A$-algebra morphism $\psi: B \otimes_A B \to \bigoplus_{g \in G} B, b \otimes b' \mapsto (gbb')_{g \in G}$ is an isomorphism. In the above, $B$ and $A$ are the coordinate rings of the schemes $Y_{(0:1)}$ and $X_{l,(0:1)}$, respectively. Note that $A$ and $B$ are spectra of Artinian $l$-algebras. Hence it suffices to show that any such $f$ admits such $\phi$ if and only if it is an étale $G$-torsor over $X_l$. But this follows easily from Corollary 3.64.

It remains to show that isomorphism classes in $\text{Ob Tors}_{X}^{\text{proj}}(k_{\text{sep}})$ are irreducible, but as the isomorphism classes are precisely the $A(k_{\text{sep}})$-orbits by construction of $\text{Tors}_{X}^{\text{proj}}$, the result follows. □

### 3.13 Torsors over smooth affine curves

Let $k$ be a factorial field. Let $\Gamma, G$ be finite groups, together with an action of $\Gamma$ on $G$ by automorphisms. Let $f: X \to \text{Spec } k$ be a smooth affine curve, together with a finite locally free morphism $X \to \mathbb{A}^1_k$ and an action of $\Gamma$ on $X$ over $\mathbb{A}^1_k$. Let $\tilde{f}: \overline{X} \to \text{Spec } k$ be the normal completion of $X$, and let $\overline{X} \to \mathbb{P}^1_k$ be the finite locally free morphism induced by $X \to \mathbb{A}^1_k$. Then $\overline{X}$ is (isomorphic to) an object of $(\Gamma, G)$-$\text{Alg}_{\text{st}}(k)$. In the next few sections, we construct a category scheme over $k$ of torsors over $X$, or rather their completions over $\overline{X}$.

### 3.14 The differential morphism

First we find a criterion for a finite locally free $\mathbb{P}^1_k$-scheme to be smooth. We do this by constructing for each component-locally free algebra over a fixed one, its module of differentials, or rather a presentation thereof, functorially over the base. More precisely, we construct a morphism from $\text{Ob Alg}_{\text{st}}^k$ to a category scheme we define below. This category scheme is a fibred version of Definition 3.35.

We first define its objects.

**Definition 3.72.** Let $S$ be a $k$-scheme, and let $\mathcal{O}_X$ be an object of $\text{Alg}_{\text{st}}^k(S)$. Let $\mathcal{R} \subseteq \mathbb{Z}_{\geq 0}$. A component-locally free $\mathcal{O}_X$-module with rank in $\mathcal{R}$ is an $\mathcal{O}_X$-module $\mathcal{E}$ such
that there exists a locally constant map \( q: S \to \mathcal{R} \) such that \( E_{q^{-1}r} = \mathcal{O}_{q^{-1}r \times S}^r \) for all \( r \in \mathcal{R} \).

We define the corresponding category scheme.

**Definition 3.73.** Let \( \mathcal{D}, \mathcal{R} \subseteq \mathbb{Z}_{\geq 0} \). The category scheme \( \text{Mod Alg}_{k, \mathcal{R}, \mathcal{D}}^{\text{free}} \) of finite free modules of rank in \( \mathcal{R} \) over finite free algebras of degree in \( \mathcal{D} \) is the functor \( \text{Sch}_k^{\text{op}} \to \text{Cat} \) sending a \( k \)-scheme \( S \) to the fibred category of component-locally free modules over component-locally free algebras over \( S \).

Equivalently, for any \( k \)-scheme \( S \), the category \( \text{Mod Alg}_{k, \mathcal{R}, \mathcal{D}}^{\text{free}}(S) \) is the category in which the objects are pairs \( (\mathcal{E}, \mathcal{O}_Y) \) of a component-locally free algebra \( \mathcal{O}_Y \) and a component-locally free \( \mathcal{O}_Y \)-module \( \mathcal{E} \), and in which the morphisms from \( (\mathcal{E}, \mathcal{O}_Y) \) to \( (\mathcal{F}, \mathcal{O}_Z) \) are pairs \( (\psi, \varphi) \) of a morphism \( \varphi: \mathcal{O}_Y \to \mathcal{O}_Z \) and a \( \varphi \)-linear morphism \( \psi: \mathcal{E} \to \mathcal{F} \).

We show that \( \text{Mod Alg}_{k, \mathcal{R}, \mathcal{D}}^{\text{free}} \) is representable, using an auxiliary functor.

**Definition 3.74.** Let \( d, e, r, s \in \mathbb{Z}_{\geq 0} \). The functor \( \text{Mod Alg}_{k, s, r, e, d}^{\text{free}}: \text{Sch}_k^{\text{op}} \to \text{Set} \) is the functor sending a \( k \)-scheme \( S \) to the set of pairs \( (\psi, \varphi) \) with \( \varphi: \mathcal{O}_Y \to \mathcal{O}_Z \) in \( \text{Alg}_{k,r,e,d}^{\text{free}} \) and \( \psi \) a \( \varphi \)-linear map \( \mathcal{O}_Y^r \to \mathcal{O}_Z^s \).

**Lemma 3.75.** Let \( d, e, r, s \in \mathbb{Z}_{\geq 0} \). Then \( \text{Mod Alg}_{k, s, r, e, d}^{\text{free}} \) is representable by an affine scheme of finite type over \( k \).

**Proof.** Let \( S \) be a \( k \)-scheme. Then note that given an element \( \varphi: \mathcal{O}_Y^r \to \mathcal{O}_Z^s \) of \( \text{Mod Alg}_{k, s, r, e, d}^{\text{free}} \) is the same as giving \( r \) elements of \( (\mathcal{O}_Z(S))^s \), which is also the same as giving \( r \) elements of \( (\mathcal{O}_S(S))^{es} \). Hence \( \text{Mod Alg}_{k, s, r, e, d}^{\text{free}} \) is representable by the affine scheme \( \text{Alg}_{k,e,d}^{\text{free}} \times_k \mathbb{A}^e_k \).

**Corollary 3.76.** Let \( \mathcal{D}, \mathcal{R} \subseteq \mathbb{Z}_{\geq 0} \). Then \( \text{Mod Alg}_{k, \mathcal{R}, \mathcal{D}}^{\text{free}} \) is representable. If \( \mathcal{D} \) and \( \mathcal{R} \) are finite, then it is affine and of finite type over \( k \).

Now we are in a position to construct, a morphism \( \Omega: \text{Ob Alg}_{k}^{\text{free}} \to \text{Mod Alg}_{k}^{\text{free}} \) that sends a free algebra to a presentation of its module of differentials.

Let \( S \) be a \( k \)-scheme, and let \( A = (\mu, \varepsilon) \) be an object of \( \text{Alg}_{k}^{\text{free}}(S) \). The idea is to view \( A \) as \( \mathcal{O}_S[t_1, t_2, \ldots, t_d]/I \), where \( I \) is the ideal generated by \( t_it'_j - \sum_j \mu_{jj'jj}t_j 

3.15 Finite flat covers

In this section, we construct from \( \text{Flat}_{k, \text{proj}} \) a category scheme such that for all field extensions \( l \) of \( k \), the category of \( l \)-points is the category of finite flat covers of \( X_l \) such
that its normal completion $\overline{X}$ is smooth over $l$ at points lying over $(1:0) \in \mathbb{P}^{1}(l)$. We do this by defining the following.

**Definition 3.77.** Let $S$ be a $k$-scheme, and let $Y$ be an object of $(\Gamma, G)$-$\text{Alg}_{X,b,b}^{st}$ $(S)$, with $b$ of length $t$. Smoothness data at $\infty$ on $Y$ consists of

- a morphism $i : \mathcal{O}_{Y(1:0)} \to \mathcal{O}_{Y(1:0)}^{2tn}$ of $\mathcal{O}_{Y(1:0)}$-modules;
- a morphism $j : \mathcal{O}_{Y(1:0)}^{2tn} \to \mathcal{O}_{Y(1:0)}^{2tn+2}$ of $\mathcal{O}_{Y(1:0)}$-modules;

such that

$$\left( (\Omega \Phi_{(1:0)}F)(Y) \oplus i \right) j = \text{id}_{\mathcal{O}_{Y(1:0)}^{2tn}}.$$

We check that this is the right notion for our purposes.

**Lemma 3.78.** Let $k$ be a field, let $X$ be a finite locally free $\mathbb{P}^{1}_{k}$-scheme that is smooth over $k$, and let $Y$ be a finite locally free $X$-scheme that is of degree $t$ over $\mathbb{P}^{1}_{k}$. Then smoothness data at $\infty$ on $Y$ exist if and only if $Y$ is smooth at all points lying over $(1:0) \in \mathbb{P}^{1}(k)$.

**Proof.** Write $B$ for the ring of global sections of $Y - Y_{(1:0)}$, and note that it is a finite locally free $k[y]$-algebra. Then $Y_{(1:0)} = \text{Spec} \ B/y^{2}B$. First suppose that smoothness data exist on $Y$ at $\infty$; i.e. there exist morphisms

$$i : (B/y^{2}B) \to (B/y^{2}B)^{2tn}, \quad j : (B/y^{2}B)^{2tn} \to (B/y^{2}B)^{2tn+2}$$

such that for the presentation $\varphi : (B/y^{2}B)^{2tn+1} \to (B/y^{2}B)^{2tn}$ of $\Omega_{(B/y^{2}B)/k}$ as a $(B/y^{2}B)$-module given in Section 3.14, we have $(\varphi \oplus i) j = \text{id}$. It then immediately follows that $\Omega_{(B/y^{2}B)/k}$ is generated by one element as a $B/y^{2}B$-module. Conversely, if $\Omega_{(B/y^{2}B)/k}$ is generated by one element, we let $i$ be a morphism from $(B/y^{2}B)$ to $(B/y^{2}B)^{2tn}$ sending 1 to (a lift of) a generator of $\Omega_{(B/y^{2}B)/k}$. Hence $(\varphi \oplus i)$ is a surjective morphism to a free $B/y^{2}B$-module, so it has a section $j$, as desired.

It remains to show that $\Omega_{(B/y^{2}B)/k}$ is generated as a $B/y^{2}B$-module by one element if and only if $Y$ is smooth over $k$ at all points lying over $(1:0) \in \mathbb{P}^{1}(k)$. Note that we have an isomorphism

$$\Omega_{B/k} \otimes_{B} (B/yB) \to \Omega_{(B/y^{2}B)/k} \otimes_{B} (B/y^{2}B),$$

and that by Nakayama’s lemma, the right hand side (and therefore the left hand side) is generated as a $B/yB$-module by one element if and only if $\Omega_{(B/y^{2}B)/k}$ is generated as a $B/y^{2}B$-module by one element. Therefore, again by Nakayama’s lemma, there exists some $f \in 1 + yB$ such that $\Omega_{B/k} \otimes_{B} Bf$ is generated as a $Bf$-module by one element. So the left hand side is a $B/yB$-module generated by one element if and only if there exists a neighbourhood of $Y_{(1:0)}$ that is smooth over $k$, which holds if and only if $Y$ is smooth over $k$ at all points lying over $(1:0) \in \mathbb{P}^{1}(k)$. 

**Definition 3.79.** Let $\mathcal{S} \subseteq \text{Seq}$. The category scheme $\text{Flat}_{X,\mathcal{S}}^{\text{aff}}$ of standard finite flat covers of $X$ of type in $\mathcal{S}$ is the functor $\text{Sch}_{k}^{\text{op}} \to \text{Cat}$ that sends a $k$-scheme $S$ to the category
in which the objects are triples \((O_Y, i, j)\) with \(O_Y\) an object of \(\text{Flat}^{\text{proj}}_{X, S}(S)\) and \((i, j)\) smoothness data on \(Y\) at \(\infty\).

As before, we have the following by construction.

**Proposition 3.80.** Let \(S \subseteq \text{Seq}\). Then \(\text{Flat}^{\text{aff}}_{X, S}\) is representable. If \(S\) is finite, then \(\text{Flat}^{\text{aff}}_{X, S}\) is affine and of finite type over \(k\).

### 3.16 Torsors

**Definition 3.81.** The category scheme \(\text{Tors}^{\text{aff}}_{X}\) of finite étale \(\Gamma\)-equivariant \(G\)-torsors over \(X\) is the functor \(\text{Sch}^\text{op}_k \to \text{Cat}\) sending a \(k\)-scheme \(S\) to the subcategory of \(\text{Flat}^{\text{aff}}_{X}(S)\) of objects \(Y\) (over \(\mathbb{P}^1_S\)) such that \(\Delta_{Y/\mathbb{P}^1_S}\) and \(\Delta^{\otimes \#G}_{X_S/\mathbb{P}^1_S}\) differ by a unit times a power of \(y\), and such that \(Y - Y_{(1:0)}\) is a \(G\)-torsor over \(X\).

We show that \(\text{Tors}^{\text{aff}}_{X}\) is a category scheme that is affine and of finite type over \(k\). We first note that the condition that \(\Delta_{Y/\mathbb{P}^1_S}\) and \(\Delta^{\otimes \#G}_{X_S/\mathbb{P}^1_S}\) is sufficient for an object of \(\text{Flat}^{\text{aff}}_{X}\) to be étale over \(X\), and that over fields, it is also necessary. This follows from the following lemma.

**Lemma 3.82.** Let \(S\) be a scheme, and let \(a\) and \(b\) be integers. Let \(A^1_S \subseteq \mathbb{P}^1_S\) be the complement of the section \((1 : 0)\), let \(\varphi: O_{\mathbb{P}^1_S}(b) \to O_{\mathbb{P}^1_S}(a)\) be a \(O_{\mathbb{P}^1_S}\)-linear map. If \(\varphi\) is multiplication by \(sy^{a-b}\) with \(s \in O_S(S)^\times\), then \(\varphi\) defines an isomorphism of \(O_{A^1_S}\)-modules. If \(S\) is the spectrum of a field, then \(\varphi\) defines an isomorphism of \(O_{A^1_S}\)-modules if and only if it is multiplication by \(sy^{a-b}\) with \(s \in O_S(S)^\times\).

**Proof.** Since \(y\) becomes invertible after restricting to \(A^1_S\), it follows that if \(\varphi\) is multiplication by \(sy^{a-b}\), then \(\varphi|_{A^1_S}\) is an isomorphism. Conversely, if \(S\) is the spectrum of a field \(l\), then \(\varphi\) is multiplication by some \(f \in l[x, y]_{a-b}\), which after restriction becomes the multiplication by \(f(x, 1)\) map \(l[x] \to l[x]\). Since this map is an isomorphism, \(f(x, 1)\) must be an invertible constant in \(l[x]\), i.e. \(f = sy^{a-b}\) for some \(s \in l^\times\).

Next, we want to bound the fibre-wise (classical) Euler characteristic of torsors, using the following lemmas.

**Lemma 3.83.** Let \(Y \to X\) be a finite locally free morphism of schemes, and let \(G\) be a finite group acting on \(Y\) over \(X\). Then the locus on \(X\) where \(G\) acts transitively on \(Y\) over \(X\) is closed.

**Proof.** Consider the morphism \(G_X \times_X Y \to Y \times_X Y\) given on the functor of points by \((g, y) \mapsto (gy, y)\). Let \(Z\) be the (topological) image of this morphism, and let \(V\) be its complement. As \(Y \times_X Y\) is finite locally free over \(X\), the image \(U\) of \(V\) is open, and the complement of \(U\) is the locus on \(X\) where \(G\) acts transitively on \(Y\) over \(X\).

**Corollary 3.84.** Let \(S\) be a scheme, let \(Y \to X\) be a finite locally free morphism of finite locally free \(\mathbb{P}^1_S\)-schemes that is étale over \(X = X_{(1:0)}\), and let \(G\) be a finite group acting on \(Y\) over \(X\). If \(G\) is a torsor on \(Y_{(0:1)}\) over \(X_{(0:1)}\), then \(G\) acts transitively on \(Y\) over \(X\).
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Proof. This follows from Lemma 3.63 and Lemma 3.83. □

Lemma 3.85. Let $S$ be a scheme, let $a \in \text{Seq}$, and let $X$ be a standard scheme over $S$ of type $a$, where $a$ has length $s$, and such that $X$ is smooth over $S$. Then $X$ is a family of curves over $S$ of Euler characteristic $s + \sum a_v$.

Proof. It suffices to check this on geometric fibres, so we may assume that $S$ is the spectrum of an algebraically closed field $k$. Then

$$\dim_k H^0(X, \mathcal{O}_X) - \dim_k H^1(X, \mathcal{O}_X) = \sum_i (1 + a_i) = s + \sum a_i.$$  

Proposition 3.86. Let $S$ be a scheme, and let $Y \to X$ be a morphism of standard schemes, with $X$ of type $a$ and with $Y$ of type $b$, where $a, b$ are of length $s, t$, respectively. Let $G$ be a finite group of order invertible in $S$ acting on $Y$ over $X$, such that $Y$ is a $G$-torsor over $X$. Then

$$\sum_j b_j \geq \frac{t}{s} \sum_i a_i - \frac{1}{2} t.$$  

Proof. It suffices to check this on geometric fibres, so we may assume that $S$ is the spectrum of an algebraically closed field $k$. As $G$ acts transitively on $Y$ over $X$, and the order of $G$ is invertible in $k$, it follows that $Y$ is tamely ramified over $X$. Therefore the ramification degree of $Y$ over $X$ is at most $t$, as $Y - Y(1:0)$ is étale over $X - X(1:0)$, and $Y$ has degree $t$ over $\mathbb{P}^1_k$. So by the Riemann-Hurwitz formula, we have

$$-2t - 2 \sum_j b_j \leq -2 \frac{1}{s} s - 2 \frac{1}{s} \sum_i a_i + t,$$

as desired. □

Hence we have the following.

Proposition 3.87. Then $\text{Tors}^\text{aff}_X$ is representable by an affine $k$-scheme of finite type.

Now write $\mathcal{T} = \mathcal{T}^G_{\Gamma, X}$.

Theorem 3.88. Let $[\mathcal{U}_{\text{Tors}^\text{aff}_X} / \mathcal{R}_{\text{Tors}^\text{aff}_X}] \to \mathcal{T}^G_{\Gamma, X}$ sending for any $k$-scheme $S$ the object $T$ of $\text{Tors}^\text{aff}_X(S)$ to the underlying $\Gamma$-equivariant $G$-torsor on $X$. For all perfect field extensions $l$ over $k$, the functor $[\mathcal{U}_{\text{Tors}^\text{aff}_X} / \mathcal{R}_{\text{Tors}^\text{aff}_X}](l) \to \mathcal{T}(l)$ is an equivalence. The isomorphism classes in $\mathcal{U}_{\text{Tors}^\text{aff}_X}$ are irreducible.

Proof. By construction, for all perfect field extensions $l$ over $k$, the functor from $[\mathcal{U}_{\text{Tors}^\text{aff}_X} / \mathcal{R}_{\text{Tors}^\text{aff}_X}](l) \to \mathcal{T}(l)$ is an equivalence. We show that the isomorphism classes in $\mathcal{U}_{\text{Tors}^\text{aff}_X}$ are irreducible.
3.17 Computation of cohomology

Note that \( \text{Tors}^{\text{aff}}_X \) admits an obvious forgetful functor to \( \text{Flat}^{\text{proj}}_X \), and we show that the image of an isomorphism class of \( \text{Ob} \text{Tors}^{\text{aff}}_X(\bar{k}) \) is irreducible in the same way as in \[\text{Theorem 3.67}\]. Therefore it suffices to show that the non-empty geometric fibres of the morphism \( \text{Ob} \text{Tors}^{\text{aff}}_X \to \text{Ob} \text{Flat}^{\text{proj}}_X \) are irreducible. So let \( Y \) be an object of \( \text{Flat}^{\text{proj}}_X(\bar{k}) \) in the image of this morphism, and write \( B \) for the ring of global sections of \( Y - Y_{(0:1)} \).

Then for smoothness data \((i, j)\) at \( \infty \) for \( s \) we have that

- \( i \) is unique up to a unique element of \( \text{im } \varphi \), which is free over \( \bar{k} \) and can therefore be parametrised by an affine space over \( \bar{k} \);
- \( j \) is unique up to a unique \( 2n\#G \)-tuple of \( \text{ker}(\varphi \oplus i) \), which again is free over \( \bar{k} \) and can therefore be parametrised by an affine space over \( \bar{k} \).

Hence the fibre over \( Y \) is irreducible, as desired. \( \Box \)

Therefore by \[\text{Proposition 3.1}\] we have the following.

**Corollary 3.89.** The induced map \( \text{Ob} \text{Tors}^{\text{aff}}_X(\bar{k}) \to \text{Ob } \mathcal{T}(\bar{k}) / \cong \) factors through a bijection \( \pi_0(\text{Ob} \text{Tors}^{\text{aff}}_{X,k^{\text{sep}}}) \to \text{Ob } \mathcal{T}(\bar{k}) / \cong \) that is \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant, and the connected components of \( \text{Ob} \text{Tors}^{\text{aff}}_{X,k^{\text{sep}}} \) are irreducible.

3.17 Computation of cohomology

We now use the previous sections to describe \[\text{Algorithm 2.2}\] which, as we recall, takes as input a factorial field \( k \), a finite ring \( \Lambda \) that is annihilated by an integer \( n \) that is invertible in \( k \) and that is injective as a \( \Lambda \)-module, a smooth connected curve \( f : X \to \text{Spec } k \) that is the composition of a finite étale morphism \( X \to U \), an open immersion \( U \to \mathbb{P}^1_k \) (given as the complement of the zero set of a single homogeneous polynomial), the structure morphism \( \mathbb{P}^1_k \to \text{Spec } k \), and a finite locally constant \( M \in \text{Ob } \Lambda - \text{Mod}_c(X_{\text{et}}) \), and outputs \( R^0f_!M, R^1f_!M, R^2f_!M \). Moreover, we do so functorially in \( M \).

Note that we can construct a finite locally free morphism \( X \to \mathbb{A}^1_k \) from these data if \( U \neq \mathbb{P}^1_k \).

**Algorithm 3.90.** Suppose that given as input is a factorial field \( k \) and a homogeneous polynomial \( h \in k[x, y] \) of degree \( d \geq 1 \).

Output: a finite locally free morphism \( U = \mathbb{P}^1_k - V_{\mathbb{P}^1_k}(h) \to \mathbb{A}^1_k \).

- If \( k = \mathbb{F}_q \), find the smallest integer \( D \leq d! \) such that \( x^{q^D} - x - 1 \) and \( h(x, 1) \) have no common zeroes, and output the morphism \( U \to \mathbb{A}^1_k \) given by

\[
(x : y) \mapsto \frac{(x^{q^D} - xy^{q^D-1} - y^{q^D})^d}{h(x, y)^{q^{D}}},
\]

and halt.
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- If \( k \) is infinite, compute an \( a \in k \) for which \( h(a) \neq 0 \), output the morphism \( U \to \mathbb{A}^1_k \) given by
  \[
  (x : y) \mapsto \frac{(x - ay)^d}{h(x, y)},
  \]
  and halt.

**Proposition 3.91.** [Algorithm 3.90] is correct and halts in an effectively bounded number of field operations.

**Proof.** In the case that \( k = \mathbb{F}_q \), we have by construction that \( ((x^q - x - 1)^d, h^q) \) is the unit ideal in \( k[x] \). In the case that \( k \) is infinite, there exists an \( a \in k \) with \( h(a) \neq 0 \), which can be found by enumerating at most \( d + 1 \) elements of \( k \), and we have that \( ((x - a)^d, h) \) is the unit ideal of \( k[x] \). The morphism constructed can therefore in both cases be extended to a morphism \( \mathbb{P}^1_k \to \mathbb{P}^1_k \) such that the inverse image of \( \{ \infty \} \) is \( \mathbb{Z}_{\mathbb{P}^1_k}(h) \); as this morphism is finite locally free, so is the morphism constructed. \( \Box \)

### 3.18 Computation of \( R^0 f_* \)

Let us first describe how to compute pushforwards, functorially; it suffices to consider sheaves of sets. We will insist on giving our output as a set of sections, since this makes it easier to compare elements.

The computation of pushforwards is a special case of the following computation.

**Algorithm 3.92.** Suppose that given as input is a factorial field \( k \), a finite locally free morphism \( X \to \mathbb{A}^1_k \) (or \( X \to \mathbb{P}^1_k \)) with \( X \) a smooth connected curve over \( k \), and finite étale \( X \)-schemes \( Y_1, Y_2 \).

Output: \( \text{Hom}_{X_{\text{sep}}} (Y_{1, \text{sep}}, Y_{2, \text{sep}}) \) as a finite Gal(\( k_{\text{sep}} / k \))-set.

- Compute a finite purely inseparable extension \( l \) of \( k \) such that the normal completions \( \overline{X}_l, \overline{Y}_{1,l}, \overline{Y}_{2,l} \) of \( X_l, Y_{1,l}, Y_{2,l} \), respectively, are smooth over \( l \), using [Algorithm 2.12].
- Compute the morphisms \( \overline{Y}_{1,l} \to \overline{X}_l \) and \( \overline{Y}_{2,l} \to \overline{X}_l \) in \( \text{Alg}_l^\text{st}(l) \), so that \( \overline{Y}_{1,l} \) and \( \overline{Y}_{2,l} \) are objects of \( \text{Alg}_l^\text{st}(l) \).
- Set \( H' = \omega^{-1}(\overline{Y}_{2,l}) \times_{\text{Alg}_l^\text{st}(l)} \alpha^{-1}(\overline{Y}_{1,l}), \) set \( H = \text{Res}_k^l H' \), and compute a finite Galois extension \( k' \) of \( k \) such that \( H_{k'} \) splits completely over \( k' \).
- Set \( l' = l \otimes k' \), and attach to every element of \( H(k') \) the corresponding morphism \( Y_{1,l'} \to Y_{2,l'} \).
- Attach to each such morphism \( \overline{Y}_{1,l'} \to \overline{Y}_{2,l'} \) its restriction to \( Y_{1,l'} \).
- Output the finite Gal(\( k_{\text{sep}} / k \))-set obtained in this way, and halt.

**Proposition 3.93.** [Algorithm 3.92] is correct and halts in an effectively bounded number of field operations.

**Proof.** We first show that \( H \) is finite étale over \( k \). First note that \( H' \) is a finite \( l \)-scheme, since it is of finite type over \( l \) by definition, and for all field extensions \( m \) of \( l \), we have \( H'(m) = \text{Hom}_{X_m}(\overline{Y}_{1,m}, \overline{Y}_{2,m}) = \text{Hom}_{X_m}(Y_{1,m}, Y_{2,m}) \), which is finite as \( Y_{1,m} \) and \( Y_{2,m} \) are finite étale over \( X_m \). Moreover, by the topological invariance of the small étale...
site, we see that \( H' \) is formally étale and therefore étale. Hence \( H = \text{Res}_k H' \) is finite étale over \( k \).

Now the proof follows from the fact that any morphism \( Y_{1,j'} \to Y_{2,j'} \) must already be defined over \( k' \). \( \square \)

As \( f_* \mathcal{F} \) corresponds to the \( \text{Gal}(k^{\text{sep}}/k) \)-set \( \text{Hom}_{X_{\text{ét}}} (X_{\text{ét}}, \mathcal{F}) \), we see that computing pushforwards is a special case of Algorithm 3.92. Next, we consider functoriality in \( \mathcal{F} \).

Algorithm 3.94. Suppose that given as input is a factorial field \( k \), a finite locally free morphism \( X \to \mathbb{A}_k^1 \) (or \( X \to \mathbb{P}_k^1 \)) with \( X \) a smooth connected curve over \( k \), a morphism \( \varphi: \mathcal{F} \to \mathcal{G} \) between finite locally constant sheaves \( \mathcal{F}, \mathcal{G} \) on \( X_{\text{ét}} \).

Output: the \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant map \( f_* \varphi: f_* \mathcal{F} \to f_* \mathcal{G} \).

1. Let \( l \) be a finite Galois extension of \( k \) such that \( f_* \mathcal{F}, f_* \mathcal{G} \) split completely over \( l \).
2. Output the map sending a section \( s: X_l \to \mathcal{F}_l \) to its composition with the morphism \( \mathcal{F}_l \to \mathcal{G}_l \), and halt.

Remark 3.95. It follows that we can also compute \( R^0 f_* \mathcal{F} = f_* \mathcal{F} \) for any finite locally constant sheaf \( \mathcal{F} \) of pointed sets and any smooth curve \( f: X \to \text{Spec} k \) given as a finite locally free scheme over \( \mathbb{A}_k^1 \) or \( \mathbb{P}_k^1 \); if \( X \) is affine, then \( f_* \mathcal{F} \) is simply the sheaf represented by \( \text{Spec} k \), and if \( X \) is projective, then \( f_* \mathcal{F} = f_* \mathcal{F} \).

3.19 Computation of \( R^1 f_* \)

We now describe the computation of \( R^1 f_* \); it suffices to do this functorially for sheaves of groups. As in the previous section, we insist on giving the elements of our output a geometric interpretation, namely as a set of representatives of isomorphism classes of torsors on \( X_{\text{ét}} \). In each of the algorithms in this section, the condition that \( \mathcal{G} \) be of degree coprime to the characteristic can be dropped in the case that the curve \( X \) is projective.

We first describe an algorithm deciding whether two torsors on \( X \) are isomorphic over \( k^{\text{sep}} \).

Algorithm 3.96. Suppose that given as input is a factorial field \( k \), a finite locally free morphism \( X \to \mathbb{A}_k^1 \) (or \( X \to \mathbb{P}_k^1 \)) with \( X \) a smooth connected curve over \( k \), a finite locally constant sheaf \( \mathcal{G} \) of groups on \( X_{\text{ét}} \) of degree coprime to the characteristic of \( k \), finite separable extensions \( k_1, k_2 \) of \( k \), and a \( \mathcal{G}_{k_i} \)-torsor \( T_i \) on \( X_{k_i} \) for \( i = 1, 2 \).

Output: “yes” if \( T_{1,k^{\text{sep}}} \cong T_{2,k^{\text{sep}}} \); otherwise nothing.

1. Compute a finite étale Galois morphism \( g: Y \to X \) with Galois group \( \Gamma \) and with \( Y \) connected, such that \( g^{-1} \mathcal{G} \) is constant, say with fibre \( G \) (with \( \Gamma \)-action).
2. Compute a finite purely inseparable extension \( l \) over \( k_1 k_2 \) such that the normal completions \( Y_l, T_{1,l}, T_{2,l} \) of \( Y_i, T_{1,i}, T_{2,i} \), respectively, are smooth over \( l \), using Algorithm 2.12.
3. Set \( \mathcal{T} = \text{Tors}^\text{aff}_{k_1, k_2} \) (or \( \mathcal{T} = \text{Tors}^\text{proj}_{k_1, k_2} \) in the projective case).
• Using an absolute primary decomposition algorithm, compute a finite separable extension $l'$ of $l$ over which the connected components of $\text{Ob} \ T_{\text{sep}}$ are defined.
• Compute $\overline{T_{1,l'}}$ and $\overline{T_{2,l'}}$ as objects of $T_\nu'(l')$.
• If they do not belong to the same connected component of $\text{Ob} \ T_\nu$, \textbf{halt}.
• Otherwise, \textbf{output} “yes” and \textbf{halt}.

**Proposition 3.97.** Algorithm 3.96 is correct and halts in an effectively bounded number of field operations.

**Proof.** By construction, the algorithm decides correctly whether $T_{1,k} \cong T_{2,k}$, but this is equivalent to $T_{1,k_{\text{sep}}} \cong T_{2,k_{\text{sep}}}$. □

We use this to compute $R^1f_*G$.

**Algorithm 3.98.** Suppose that given as input is a factorial field $k$, a finite locally free morphism $X \to \mathbb{A}_k^1$ (or $X \to \mathbb{P}_k^1$) with $X$ a smooth connected curve over $k$, $G$ a finite locally constant sheaf of groups on $X_{\text{et}}$ of degree coprime with the characteristic of $k$.

\textbf{Output:} $R^1f_*G$ as a finite $\text{Gal}(k_{\text{sep}}/k)$-set of representatives of isomorphism classes of $G_{k_{\text{sep}}}$-torsors on $X_{k_{\text{sep}}}$. 

• Compute a finite étale Galois morphism $g: Y \to X$ with Galois group $\Gamma$ and with $Y$ connected, such that $g^{-1}G$ is constant, say with fibre $G$ (with $\Gamma$-action).
• Compute a finite purely inseparable extension $l$ of $k$ such that the normal completion $\overline{Y}$ of $Y$ is smooth.
• Set $T = \text{Tors}_{l,Y}$ (or $T = \text{Tors}_{l,Y}$ in the projective case).
• Using an absolute primary decomposition algorithm, compute a finite Galois extension $l'$ of $l$ over which the connected components of $\text{Ob} \ T_{\text{sep}}$ are defined.
• Compute a finite extension $l''$ of $l'$, and for every connected component of $\text{Ob} \ T_\nu'$ an $l''$-rational point on it; i.e. a $\Gamma$-equivariant $G$-torsor on $Y_{l''}$.
• Attach to every such torsor its restriction to $Y_{l''}$, and then its quotient by $\Gamma$.
• Let $k''$ be the separable closure of $k$ in $l''$.
• Let $T$ denote the finite set of $G$-torsors on $X_{k''}$ obtained this way.
• For every $t \in T$ and $\gamma \in \text{Gal}(k''/k)$, find using Algorithm 3.96 $\gamma t \in T$ by enumeration.
• \textbf{Output} the finite $\text{Gal}(k_{\text{sep}}/k)$-set $T$, and \textbf{halt}.

**Proposition 3.99.** Algorithm 3.98 is correct and halts in an effectively bounded number of field operations.

**Proof.** This follows directly from Corollary 3.89 in the affine case, and from Corollary 3.68 in the projective case. □

Before considering functoriality in $G$, we first consider quotients of finite étale morphisms by finite locally constant sheaves of groups on $X_{\text{et}}$.

**Algorithm 3.100.** Suppose that given as input is a factorial field $k$, a finite locally free morphism $X \to \mathbb{A}_k^1$ (or $X \to \mathbb{P}_k^1$) with $X$ a smooth connected curve over $k$, a finite étale scheme $Y$ over $X$, a finite locally constant sheaf $G$ of groups on $X_{\text{et}}$ acting on $Y$.

\textbf{Output:} the quotient of $Y$ by the action of $G$. 

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• Compute a finite étale Galois morphism \( g : X' \to X \) with Galois group \( \Gamma \) and with \( Y \) connected, such that \( g^{-1} \mathcal{G} \) is constant, say with fibre \( G \) (with \( \Gamma \)-action).
• Set \( Y' = X' \times_X Y \).
• Output \( \Gamma \backslash (G \backslash Y') \) and halt.

Hence we can compute \( R^1 f_* \) functorially as follows.

**Algorithm 3.101.** Suppose that given as input is a factorial field \( k \), a finite locally free morphism \( X \to \mathbb{A}_k^1 \) (or \( X \to \mathbb{P}_k^1 \)) with \( X \) a smooth connected curve over \( k \), \( \varphi : \mathcal{G} \to \mathcal{H} \) a morphism of finite locally constant sheaves of abelian groups on \( X_{\text{ét}} \) of degree coprime with the characteristic of \( k \).

Output: the \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant map \( R^1 \varphi : R^1 f_* \mathcal{G} \to R^1 f_* \mathcal{H} \).

• Let \( l \) be a finite Galois extension of \( k \) such that \( R^1 f_* \mathcal{G} \) and \( R^1 f_* \mathcal{H} \) split completely over \( l \).
• Output the map sending a \( \mathcal{G}_l \)-torsor \( T \to X_l \) to a \( \mathcal{H}_l \)-torsor isomorphic to \( \mathcal{H}_l \otimes_{\mathcal{G}_l} T = \mathcal{G}_l \backslash (\mathcal{H}_l \times_{X_l} T) \to X_l \) (where \( \mathcal{G} \) acts by \( g(h,t) = (hg^{-1}, gt) \)), and halt.

Finally, if \( \mathcal{G} \) is commutative, then \( R^1 f_* \mathcal{G} \) is an abelian group, and we can compute its group structure.

**Algorithm 3.102.** Suppose that given as input is a factorial field \( k \), a finite locally free morphism \( X \to \mathbb{A}_k^1 \) (or \( X \to \mathbb{P}_k^1 \)) with \( X \) a smooth connected curve over \( k \), \( \varphi : \mathcal{G} \to \mathcal{H} \) a morphism of finite locally constant sheaves of abelian groups on \( X_{\text{ét}} \) of degree coprime with the characteristic of \( k \).

Output: the addition map \( R^1 f_* \mathcal{G} \times_k R^1 f_* \mathcal{G} \to R^1 f_* \mathcal{G} \).

• Let \( l \) be a finite Galois extension of \( k \) such that \( R^1 f_* \mathcal{G} \) splits completely over \( l \).
• Output the map sending a pair \((T_1, T_2)\) of \( \mathcal{G}_l \)-torsors to a \( \mathcal{G}_l \)-torsor isomorphic to \( T_1 \otimes_{\mathcal{G}_l} T_2 = \mathcal{G}_l \backslash (T_1 \times_{X_l} T_2) \) (where \( \mathcal{G} \) acts by \( g(t_1, t_2) = (t_1g^{-1}, gt_2) \)), and halt.

### 3.20 Poincaré duality

Note that we have now computed \( R^0 f_* \), \( R^0 f_+ \), and \( R^1 f_* \) of a smooth connected curve \( f : X \to \text{Spec} k \). We compute the rest using Poincaré duality; we recall its statement first.

Let \( \Lambda \) be a finite ring annihilated by \( n \in \mathbb{Z} \), let \( X \) be a scheme, and let \( \mathcal{M} \) be a finite locally constant sheaf of \( \Lambda \)-modules on \( X_{\text{ét}} \). Then we denote the \( d \)-th Tate twist \( \mathcal{M} \otimes_{\mathbb{Z}/n\mathbb{Z}} (\mu_n)'^d \) of \( \mathcal{M} \) by \( \mathcal{M}(d) \); note that this doesn’t depend the choice of the annihilator \( n \), and that we can compute this if \( X \) is a smooth curve or the spectrum of a field. Write moreover \( \mathcal{M}^\vee \) for \( \text{Hom}(\mathcal{M}, \Lambda) \), which we can compute by Algorithm 3.92.

**Theorem 3.103 (Poincaré duality, \([\text{SGA}4.3][\text{II}]\) Exp. XVIII, Sec. 3.2.6]).** Let \( \Lambda \) be a finite ring that is injective as a \( \Lambda \)-module, let \( f : X \to \text{Spec} k \) be a smooth curve over a field, and let \( \mathcal{M} \) be a finite locally constant sheaf of \( \Lambda \)-modules on \( X_{\text{ét}} \). Then for \( q = 0, 1, 2 \) we have \( R^{2-q} f_*(\mathcal{M}^\vee(1)) = (R^q f_*\mathcal{M})^\vee \).

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In other words, we have the identities
\[
R^1 f_! M = \left( R^1 f_*(M^\vee(1)) \right)^\vee
\]
\[
R^2 f_! M = \left( f_*(M^\vee(1)) \right)^\vee
\]
\[
R^2 f_* M = \left( f_!(M^\vee(1)) \right)^\vee.
\]
Therefore we indeed have an algorithm as in Algorithm 2.2 as desired.