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Gauge-covariant extensions of Killing tensors and conservation laws

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Abstract. In classical and quantum mechanical systems on manifolds with gauge-field fluxes, constants of motion are constructed from gauge-covariant extensions of Killing vectors and tensors. This construction can be carried out using a manifestly covariant procedure, in terms of covariant phase space with a covariant generalization of the Poisson brackets, c.q. quantum commutators. Some examples of this construction are presented.

1. Noether's theorem

This paper discusses symmetries and conservation laws in the context of hamiltonian dynamics. The discussion is framed predominantly in the language of classical dynamics, but the use of Poisson brackets and their correspondence with quantum commutators, guarantees that many results also apply to the operator formulation of quantum dynamics. The main difference is the operator ordering to be implemented in quantum theory, the technicalities of which are not relevant to the issues I focus on.

The connection between continuous symmetries and conservation laws is established by Noether's theorem [1]. I briefly review the theorem by considering infinitesimal transformations on phase-space variables (x, p) obtained from a generating function $G(x, p)$ through the Poisson brackets

$$\delta x = \{x, G\} = \frac{\partial G}{\partial p}, \quad \delta p = \{p, G\} = -\frac{\partial G}{\partial x}. \quad (1)$$

Observe, that these variations are defined such that G itself is an invariant:

$$\delta G = \delta x \frac{\partial G}{\partial x} + \delta p \frac{\partial G}{\partial p} = \{G, G\} = 0. \quad (2)$$

Under such transformations the hamiltonian of the system changes by

$$\delta H = \{H, G\} = -\frac{dG}{dt}, \quad (3)$$

the change of G along the phase-space trajectory $(x(t), p(t))$ generated by the hamiltonian H . It follows immediately, that G is a constant of motion if the hamiltonian is invariant under the transformations (1).



It is also of some interest to consider the variation of the action

$$S = \int_1^2 dt \left(p \frac{dx}{dt} - H(x, p) \right). \quad (4)$$

Applying the variations (1)

$$\delta S = \int_1^2 dt \left[\frac{d}{dt} \left(p \frac{\partial G}{\partial p} - G \right) - \{H, G\} \right] = \left[p \frac{\partial G}{\partial p} - G \right]_1^2. \quad (5)$$

Thus variations under which the hamiltonian is invariant, leave the action invariant modulo boundary terms. This is sufficient for G to be a constant of motion.

2. Isometries of manifolds

On a manifold with (local) co-ordinates x^μ and metric $g_{\mu\nu}(x)$ the geodesics can be obtained as the trajectories of test-particles with proper-time hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu. \quad (6)$$

Indeed, using the overdot notation for proper-time derivatives, the hamilton equations take the form

$$\dot{x}^\mu = g^{\mu\nu} p_\nu, \quad \dot{p}_\mu = -\frac{1}{2} \frac{\partial g^{\nu\lambda}}{\partial x^\mu} p_\nu p_\lambda = \frac{1}{2} \frac{\partial g_{\nu\lambda}}{\partial x^\mu} \dot{x}^\nu \dot{x}^\lambda. \quad (7)$$

The last expression is equivalent to the geodesic equation

$$\ddot{x}^\mu + \Gamma_{\lambda\nu}^{\mu} \dot{x}^\lambda \dot{x}^\nu = 0. \quad (8)$$

In this language, isometries of the manifold are found as constants of motion which are linear in the momentum:

$$J(x, p) = J^\mu(x) p_\mu, \quad \{J, H\} = (\nabla_\mu J_\nu) p^\mu p^\nu = 0, \quad (9)$$

where (in a somewhat hybrid notation) the contravariant components of the momentum are $p^\mu = \dot{x}^\mu$. Hence the covariant coefficient functions J^μ form a Killing vector, a solution of the Killing equation

$$\nabla_\mu J_\nu + \nabla_\nu J_\mu = 0 \quad \Leftrightarrow \quad J^\lambda \frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial J^\lambda}{\partial x^\mu} g_{\lambda\nu} + \frac{\partial J^\lambda}{\partial x^\nu} g_{\mu\lambda} = 0. \quad (10)$$

The second (contravariant) form of the equation states that the Lie-derivative of the metric w.r.t. the vector J^μ vanishes, which is the usual definition of an isometry. Also note, that the constants of motion defined by Killing vectors are precisely those, for which

$$p \frac{\partial G}{\partial p} = G, \quad (11)$$

and therefore generates transformations under which the action is strictly invariant. This is to be expected for an isometry which by construction leaves the line element $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ invariant.

Although the coordinate transformations δx^μ generated by Killing vectors thus have an elegant interpretation, this does not hold for the corresponding transformations of the canonical momenta p_μ :

$$\delta p_\mu = A_\mu^\nu(x) p_\nu, \quad A_\mu^\nu = -\frac{\partial J^\nu}{\partial x^\mu}. \quad (12)$$

This transformation rule is not general covariant; as $A_\mu^\nu(x)$ is point-dependent, covariance requires δp to be corrected for the parallel displacement generated by the translation δx^μ . Thus a set of covariant transformations in phase space is defined by

$$\Delta x^\mu = \delta x^\mu, \quad \Delta p_\mu = \delta p_\mu - \delta x^\lambda \Gamma_{\lambda\mu}^\nu p_\nu. \quad (13)$$

In order for these transformations to respect the Poisson brackets (2) and (3), it is then necessary to introduce a covariant derivative

$$\mathcal{D}_\mu G = \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}^\lambda p_\lambda \frac{\partial G}{\partial p_\nu} = -\Delta p_\mu, \quad (14)$$

such that

$$\Delta G = \Delta x^\mu \mathcal{D}_\mu G + \Delta p_\mu \frac{\partial G}{\partial p_\mu} = \{G, G\} = 0, \quad (15)$$

and

$$\Delta H = \Delta x^\mu \mathcal{D}_\mu H + \Delta p_\mu \frac{\partial H}{\partial p_\mu} = \{H, G\} = -\frac{dG}{d\tau}. \quad (16)$$

Thus we have constructed a covariant expression for the Poisson bracket of two arbitrary scalar phase-space functions:

$$\{G, K\} = \mathcal{D}_\mu G \frac{\partial K}{\partial p_\mu} - \frac{\partial G}{\partial p_\mu} \mathcal{D}_\mu K. \quad (17)$$

3. Killing tensors

The metric postulate guarantees that the geodesic hamiltonian (6) is covariantly constant:

$$\nabla_\lambda g_{\mu\nu} = 0 \quad \Leftrightarrow \quad \mathcal{D}H = 0. \quad (18)$$

The condition for a constant of geodesic motion then takes the simple form

$$\{G, H\} = p^\mu \mathcal{D}_\mu G = 0. \quad (19)$$

Applying this to a general expression of homogeneous rank n in the momenta

$$G(x, p) = G^{\mu_1 \dots \mu_n}(x) p_{\mu_1} \dots p_{\mu_n} \quad \Rightarrow \quad p^\mu \mathcal{D}_\mu G = (\nabla_{\mu_{n+1}} G_{\mu_1 \dots \mu_n}) p^{\mu_1} \dots p^{\mu_{n+1}}, \quad (20)$$

the condition for a constant of motion becomes a generalization of the Killing equation (10):

$$\nabla_{(\mu_{n+1}} G_{\mu_1 \dots \mu_n)} = 0. \quad (21)$$

The solutions of these equations are therefore known as *Killing tensors* [3, 4].

The geometrical interpretation of the transformations generated by constants of motion constructed from Killing tensors of rank 2 or higher, is more complicated than for Killing vectors. They do not generate transformations in the manifold, but in the tangent bundle, the physical phase-space:

$$\Delta x^\mu = \frac{\partial G}{\partial p_\mu} = n G^{\mu\mu_1 \dots \mu_{n-1}} p_{\mu_1} \dots p_{\mu_{n-1}}, \quad (22)$$

$$\Delta p_\mu = -\mathcal{D}_\mu G = -(\nabla_\mu G^{\mu_1 \dots \mu_n}) p_{\mu_1} \dots p_{\mu_n}.$$

Note, that for transformations with $n \geq 2$ the action actually transforms by boundary terms:

$$p_\mu \frac{\partial G}{\partial p_\mu} = nG \quad \Rightarrow \quad \Delta S = (n-1) [G(x_2, p_2) - G(x_1, p_1)]. \quad (23)$$

Obviously, as G is a constant of motion the action is still invariant along geodesics.

Example: Kerr geometry [2, 3]

The Kerr geometry is a Ricci-flat geometry in 4-dimensional space-time: $R_{\mu\nu} = 0$, implying it is a solution of the Einstein equations of General Relativity in empty space-time. This solutions describes a rotating black hole with constant mass M and angular momentum J . The geodesic hamiltonian reads

$$H = \frac{1}{2\rho^2} \left[\Delta^2 p_r^2 + p_\theta^2 + \left(a \sin \theta p_t + \frac{p_\varphi}{\sin \theta} \right)^2 - \frac{1}{\Delta^2} \left((r^2 + a^2) p_t + a p_\varphi \right)^2 \right], \quad (24)$$

where $a = J/M$ is the angular momentum per unit of mass, and the notation follows standard conventions:

$$\Delta^2 = r^2 - 2Mr + a^2, \quad \rho^2 = r^2 + a^2 \cos^2 \theta.$$

This space-time geometry admits a rank-2 Killing tensor, coding the well-known Carter constant of motion:

$$K = \frac{1}{2\rho^2} \left[-\Delta^2 a^2 \cos^2 \theta p_r^2 + r^2 p_\theta^2 + r^2 \left(a p_t + \frac{p_\varphi}{\sin^2 \theta} \right)^2 + \frac{a^2 \cos^2 \theta}{\Delta^2} \left((r^2 + a^2) p_t + a p_\varphi \right)^2 \right]. \quad (25)$$

This constant of motion can be generalized to neutral and charged, spinless and spinning particles in Kerr-Newman space-time of charged black holes [3, 5, 6, 7].

4. Bracket algebra

The generators of hamiltonian symmetry transformations, defining constants of motion, define a Lie-algebra by their bracket relations (17); this follows from the Jacobi identity

$$\{\{G, K\}, J\} + \{\{K, J\}, G\} + \{\{J, G\}, K\} = 0. \quad (26)$$

Applying the identity to the special case where one of the functions is the hamiltonian: $J = H$, implies that constants of motion (G, K) satisfy

$$\{\{G, K\}, H\} = 0. \quad (27)$$

Hence the bracket of two constants of motion produces another constant of motion, and the set of such constants is closed under the bracket operation. Here we summarize some properties of this algebra [4].

The generators $J = J^\mu p_\mu$ linear in momentum define a Lie subalgebra of the full algebra:

$$\{J_1, J_2\} = J_3, \quad J_3^\mu = J_2^\nu \nabla_\nu J_1^\mu - J_1^\nu \nabla_\nu J_2^\mu. \quad (28)$$

The constants of motion of rank $n \geq 2$ then define representations of this subalgebra¹, characterized by the transformation rule

$$\{G_1, J\} = G_2, \quad G_2^{\mu_1 \dots \mu_n} = J^\lambda \nabla_\lambda G^{\mu_1 \dots \mu_n} - n G^{\lambda(\mu_1 \dots \mu_{n-1}} \nabla_\lambda J^{\mu_n)}. \quad (29)$$

The Jacobi identity

$$\{\{G, J_1\}, J_2\} - \{\{G, J_2\}, J_1\} = \{G, \{J_1, J_2\}\}, \quad (30)$$

¹ The parenthesis around the indices $(\mu_1 \dots \mu_n)$ denote complete symmetrization with unit weight.

then guarantees that the transformations generated by the linear $J(x, p)$ on the rank- n tensors $G(x, p)$ obey the composition rules of the subalgebra spanned by the linear $J(x, p)$ themselves.

As for the brackets of a generator $G^{(n)}$ of rank n , and a generator $G^{(m)}$ of rank m , with both $(n, m) \geq 2$, it is easily recognized that any non-vanishing brackets of such quantities must be a generator $G^{(n+m-1)}$, of rank $n + m - 1$ exceeding both n and m :

$$\{G^{(n)}, G^{(m)}\} \sim G^{(n+m-1)}. \quad (31)$$

Therefore non-vanishing brackets of higher-rank generators will potentially lead to an infinite set of generators of arbitrary high rank. Well-known examples of such algebras are the Virasoro and Kac-Moody algebras of 2-D conformal field theories.

5. Abelian gauge interactions

Geodesic motion on a manifold applies to the motion of a pure mass point subject only to geometrical forces. However, Noether's theorem is very general and can be applied also in presence of external force fields [8, 9]. In this section I consider abelian gauge interactions transmitted by a vector field $A_\mu(x)$, acting on a point mass m with charge q . For such a particle the canonical hamiltonian reads

$$H = \frac{1}{2m} g^{\mu\nu} (p_\mu - qA_\mu)(p_\nu - qA_\nu). \quad (32)$$

The hamiltonian equations of motion (7), (8) are generalized to

$$p_\mu = mg_{\mu\nu}\dot{x}^\nu + qA_\mu, \quad g_{\mu\nu} \left(\ddot{x}^\nu + \Gamma_{\kappa\lambda}^{\nu} \dot{x}^\kappa \dot{x}^\lambda \right) = \frac{q}{m} F_{\mu\nu} \dot{x}^\nu, \quad (33)$$

which is the Lorentz force law on curved manifolds.

A drawback of this hamiltonian formulation is, that the canonical momentum is not gauge invariant; under gauge transformations

$$A'_\mu = A_\mu + \nabla_\mu \Lambda, \quad p'_\mu = p_\mu + q\nabla_\mu \Lambda. \quad (34)$$

Therefore it is preferable to work with the covariant momentum [7, 10]

$$\pi_\mu = p_\mu - qA_\mu = mg_{\mu\nu}\dot{x}^\nu, \quad (35)$$

in terms of which the hamiltonian takes the simple form

$$H = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu. \quad (36)$$

If the covariant derivative \mathcal{D} now is generalized to

$$\mathcal{D}_\mu G = \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}^{\lambda} \pi_\lambda \frac{\partial G}{\partial \pi_\nu}, \quad (37)$$

then as before the metric postulate can be used to show that

$$\mathcal{D}_\mu H = 0. \quad (38)$$

Moreover, the correct dynamics is reproduced as usual by

$$\frac{dG}{d\tau} = \{G, H\}, \quad (39)$$

when supplemented by the covariant brackets

$$\{G, K\} = \mathcal{D}_\mu G \frac{\partial K}{\partial \pi_\mu} - \frac{\partial G}{\partial \pi_\mu} \mathcal{D}_\mu K + qF_{\mu\nu} \frac{\partial G}{\partial \pi_\mu} \frac{\partial K}{\partial \pi_\nu}. \quad (40)$$

In particular, this bracket reproduces the Ricci identity in the form

$$\{\pi_\mu, \pi_\nu\} = qF_{\mu\nu}. \quad (41)$$

In view of the identity (38) and the anti-symmetry of the field strength tensor $F_{\mu\nu}$, the condition for a scalar quantity $G(x, \pi)$ to be constant of motion becomes [10]

$$\{G, H\} = 0 \quad \Rightarrow \quad \pi^\mu \mathcal{D}_\mu G = q\pi^\mu F_{\mu\nu} \frac{\partial G}{\partial \pi_\nu}. \quad (42)$$

In particular, if G can be expressed as a power series

$$G(x, \pi) = \sum_n \frac{1}{n!} G^{(n)\mu_1 \dots \mu_n}(x) \pi_{\mu_1} \dots \pi_{\mu_n}, \quad (43)$$

then eq. (42) takes the form of a first-order p.d.e.

$$\nabla_{(\mu_1} G^{(n)}_{\mu_2 \dots \mu_{n+1})} = qF_{(\mu_1}{}^\nu G^{(n+1)}_{\mu_2 \dots \mu_{n+1})\nu}. \quad (44)$$

This is a generalization of equation (21) for Killing tensors, forming a hierarchy of equation connecting tensors of different rank. Nevertheless, the existence of a Killing tensor of rank n still provides a constant of motion: it allows the series expansion (43) to be truncated, with all higher-rank components vanishing:

$$\nabla_{(\mu_1} G^{(n)}_{\mu_2 \dots \mu_{n+1})} = 0 \quad \Rightarrow \quad G^{(n+k)}_{\mu_1 \dots \mu_{n+k}} = 0, \quad \forall k \geq 0, \quad (45)$$

while all lower-rank components are obtained by solving eq. (44) with the already known higher-rank ones as inhomogenous source terms on the right-hand side [10].

Finally note, that it is straightforward to include a scalar potential $\Phi(x)$ as well:

$$H = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu + \Phi, \quad \Rightarrow \quad \mathcal{D}_\mu H = \frac{\partial \Phi}{\partial x^\mu}. \quad (46)$$

This modifies the generalization of the Killing equation to include a gradient term [11]

$$\pi^\mu \mathcal{D}_\mu G = q\pi^\mu F_{\mu\nu} \frac{\partial G}{\partial \pi_\nu} + \frac{\partial \Phi}{\partial x^\mu} \frac{\partial G}{\partial \pi_\mu}. \quad (47)$$

Example: a quantum-dot model

A model of a quantum-dot, consisting of 2 electrons moving as a non-relativistic bound pair with Coulomb interaction in a confining harmonic potential and a magnetic field, was studied in ref. [12]. Ignoring the center-of-mass motion, the pair can be described in axial co-ordinates (ρ, z, φ) as a single particle with hamiltonian

$$H_{CM} = \frac{1}{2} g^{ij} \pi_i \pi_j + \Phi, \quad (48)$$

where

$$g_{ij} = \text{diag}(1, 1, \rho^2), \quad \Phi = \frac{1}{2} (\omega_0^2 \rho^2 + \omega_z^2 z^2) - \frac{\kappa}{\sqrt{\rho^2 + z^2}}. \quad (49)$$

The definition of the dynamics is completed by the brackets

$$\{G, K\} = \mathcal{D}_i G \frac{\partial K}{\partial \pi_i} - \frac{\partial G}{\partial \pi_i} \mathcal{D}_i K - 2\omega_L \rho \left(\frac{\partial G}{\partial \pi_\rho} \frac{\partial K}{\partial \pi_\varphi} - \frac{\partial G}{\partial \pi_\varphi} \frac{\partial K}{\partial \pi_\rho} \right), \quad (50)$$

where $\omega_L = eB/2$ is the Larmor frequency.

To guarantee that a Killing tensor can be completed to a full constant of motion, solving eq. (47), it may be necessary to tune the parameters in the scalar potential. In the case of the quantum dot, there is a rank-4 Killing tensor which can be completed to a full constant of motion provided the magnetic field is tuned to a value such that the Larmor frequency is a fixed combination of the harmonic frequencies:

$$\omega_L^2 + \omega_0^2 = 4\omega_z^2. \quad (51)$$

Then the following expression is a solution of eq. (47) for this system:

$$\begin{aligned} G = & \rho^2 \pi_z^4 - 2\rho z \pi_\rho \pi_z^3 + z^2 \pi_\rho^2 \pi_z^2 + \frac{1}{\rho^2} \pi_\varphi^4 + \pi_\rho^2 \pi_\varphi^2 + \left(2 + \frac{z^2}{\rho^2} \right) \pi_z^2 \pi_\varphi^2 \\ & + 2\omega_L \pi_\varphi (\rho^2 \pi_\rho^2 + (2\rho^2 + z^2) \pi_z^2) + \left[2\omega_z^2 z^3 \rho + \frac{2\kappa z \rho}{\sqrt{\rho^2 + z^2}} \right] \pi_\rho \pi_z \\ & + \left[(2\omega_z^2 - \omega_0^2) z^2 \rho^2 + 2\omega_L^2 \rho^4 - \frac{2\kappa \rho^2}{\sqrt{\rho^2 + z^2}} \right] \pi_z^2 + \omega_L^2 \rho^4 \pi_\rho^2 \\ & + \left[2\omega_z^2 z^2 + (\omega_0^2 - 5\omega_L^2) \rho^2 - \frac{2\kappa}{\sqrt{\rho^2 + z^2}} \right] \pi_\varphi^2 \\ & - 2\omega_L \pi_\varphi \left[(3\omega_L^2 - \omega_0^2) \rho^4 - 2\omega_z^2 \rho^2 z^2 + \frac{2\kappa \rho^2}{\sqrt{\rho^2 + z^2}} \right] + \omega_z^4 \rho^2 z^4 + 2\omega_z^2 \omega_L^2 \rho^4 z^2 \\ & - \omega_L^2 (3\omega_L^2 - 4\omega_z^2) \rho^6 + \frac{2\kappa}{\sqrt{\rho^2 + z^2}} (\omega_z^2 \rho^2 z^2 - \omega_L^2 \rho^4) + \frac{\kappa^2}{2} \frac{\rho^2 - z^2}{\rho^2 + z^2}. \end{aligned} \quad (52)$$

6. Non-abelian gauge interactions

The dynamics of a particle with non-abelian gauge interactions is described by Wong's generalization of the Lorentz force law [13]

$$g_{\mu\nu} \left(\ddot{x}^\nu + \Gamma_{\kappa\lambda}^\nu \dot{x}^\kappa \dot{x}^\lambda \right) = \frac{g}{m} t_a F_{\mu\nu}^a \dot{x}^\nu, \quad \dot{t}_a + g f_{ab}{}^c t_c A_\mu^b \dot{x}^\mu. \quad (53)$$

Here g is the coupling constant, $f_{ab}{}^c$ are the structure constants of the Lie algebra of gauge charges t_a , and the non-abelian field-strength tensor is

$$F_{\mu\nu}^a = \nabla_\mu A_\nu^a - \nabla_\nu A_\mu^a + g f_{bc}{}^a A_\mu^b A_\nu^c. \quad (54)$$

The theory can be cast in hamiltonian form [10] by introducing covariant momenta π_μ and a canonical hamiltonian

$$H = \frac{1}{2m} g^{\mu\nu} \pi_\mu \pi_\nu, \quad (55)$$

supplemented with brackets

$$\begin{aligned} \{G, K\} &= \mathcal{D}_\mu G \frac{\partial K}{\partial \pi_\mu} - \frac{\partial G}{\partial \pi_\mu} \mathcal{D}_\mu K + g t_a F_{\mu\nu}^a \frac{\partial G}{\partial \pi_\mu} \frac{\partial K}{\partial \pi_\nu} + f_{ab}{}^c t_c \frac{\partial G}{\partial t_a} \frac{\partial K}{\partial t_b}, \\ \mathcal{D}_\mu G &\equiv \frac{\partial G}{\partial x^\mu} + \Gamma_{\mu\nu}{}^\lambda \pi_\lambda \frac{\partial G}{\partial \pi_\nu} + g f_{ab}{}^c t_c A_\mu^a \frac{\partial G}{\partial t_b}. \end{aligned} \quad (56)$$

In particular these brackets guarantee the Ricci identity and the Lie-algebra of gauge charges:

$$\{\pi_\mu, \pi_\nu\} = g t_a F_{\mu\nu}^a, \quad \{t_a, t_b\} = f_{ab}{}^c t_c. \quad (57)$$

It is now straightforward to derive the condition for the existence of constants of motion:

$$\{G, H\} = 0 \quad \Rightarrow \quad \pi^\mu \mathcal{D}_\mu G = g t_a F_{\mu\nu}^a \pi^\mu \frac{\partial G}{\partial \pi_\nu}. \quad (58)$$

In components this gives us a hierarchy of equations similar to (44) for the abelian case:

$$\begin{aligned} \mathcal{D}_\mu G^{(0)} &= g t_a F_\mu^{a\nu} G_\nu^{(1)}, \\ \mathcal{D}_\mu G_\nu^{(1)} + \mathcal{D}_\nu G_\mu^{(1)} &= g t_a \left(F_\mu^{a\lambda} G_{\lambda\nu}^{(2)} + F_\nu^{a\lambda} G_{\lambda\mu}^{(2)} \right), \\ \mathcal{D}_\mu G_{\nu\lambda}^{(2)} + \mathcal{D}_\nu G_{\lambda\mu}^{(2)} + \mathcal{D}_\lambda G_{\mu\nu}^{(2)} &= g t_a \left(F_\mu^{a\kappa} G_{\kappa\lambda\nu}^{(3)} + F_\nu^{a\kappa} G_{\kappa\lambda\mu}^{(3)} + F_\lambda^{a\kappa} G_{\kappa\mu\nu}^{(3)} \right), \\ &\dots \end{aligned} \quad (59)$$

Example: 2-D $SU(2)$ Yang-Mills point charge

As a simple example consider the dynamics of a non-abelian point charge in a static magnetic $SU(2)$ -field in the euclidean plane. In the 2-dimensional plane such a magnetic field can be factorised as

$$F_{ij}^a = \varepsilon_{ij} B^a, \quad (60)$$

where a represents an adjoint vector component in 3-dimensional internal $SU(2)$ -space. The free Yang-Mills equation then implies

$$\nabla_i B^a + g \varepsilon^{abc} A_i^b B^c = 0 \quad \Rightarrow \quad \nabla_i B^{a2} = 0, \quad (61)$$

and the modulus of B^a is constant. Now the direction of B^a can be rotated point-wise in the plane by a local gauge transformation, and this freedom can be used to make B^a constant. Such a constant B^a -field is derived from a vector potential

$$A_i^a = -\frac{1}{2} \varepsilon_{ij} x^j B^a. \quad (62)$$

Now the 2-dimensional euclidean plane is invariant under translations and rotations, guaranteeing conservation of momentum and angular momentum. Our construction implies, that the angular momentum will obtain a field dependent addition:

$$J = \varepsilon_{ij} x_i \pi_j + \frac{g}{2} t_a B^a x_i^2. \quad (63)$$

In addition, there is also a pair of rank-2 Killing tensors, generating a conserved Runge-Lenz vector:

$$K_i = x_i \pi_j^2 - \pi_i \pi_j x_j + g B^a t_a \left(\frac{1}{2} \varepsilon_{ij} \pi_j x_k^2 + x_i x_j \varepsilon_{jk} \pi_k \right) + \frac{1}{2} (g B^a t_a)^2 x_i x_j^2. \quad (64)$$

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