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Magnetohydrodynamics of chiral relativistic fluids

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We study the dynamics of a plasma of charged relativistic fermions at very high temperature $T \gg m$, where m is the fermion mass, coupled to the electromagnetic field. In particular, we derive a magnetohydrodynamical description of the evolution of such a plasma. We show that, compared to conventional magnetohydronamics (MHD) for a plasma of nonrelativistic particles, the hydrodynamical description of the relativistic plasma involves new degrees of freedom described by a pseudoscalar field originating in a local asymmetry in the densities of left-handed and right-handed fermions. This field can be interpreted as an effective axion field. Taking into account the chiral anomaly we present dynamical equations for the evolution of this field, as well as of other fields appearing in the MHD description of the plasma. Due to its nonlinear coupling to helical magnetic fields, the axion field significantly affects the dynamics of a magnetized plasma and can give rise to a novel type of inverse cascade.

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I. INTRODUCTION

It was suggested [1,2], more than a decade ago, that the chiral anomaly may be accompanied by the appearance of some new degrees of freedom described by an effective axion field that couples to the index density of the electromagnetic field. More recently, it has been understood [3] that this axion field plays an important role in the dynamics of the electromagnetic field in a relativistic plasma of charged fermions, (e.g., the plasma described by the Standard Model at high temperature and/or large matter density). In the spatially homogeneous case, the time derivative of the axion field is proportional to the difference between the chemical potentials (densities) of left-handed and right-handed charged particles [1,2]. As a consequence of the chiral anomaly, a left-right asymmetry, as described by a nonvanishing time-dependent axion field, affects the dynamics of the electromagnetic field (see also [4–7]) and gives rise to an instability of solutions of the coupled Maxwell-axion equations [1].

This kind of instability is encountered in different physical systems exhibiting a left-right asymmetry. For example, it has recently also appeared in the electrodynamics of topological insulators (see, e.g., [8]). Instabilities leading to the growth of helical magnetic fields in the plasma of relativistic charged particles, as discussed in this paper, are expected to have played an important role in the dynamics of the early Universe, assuming that a potent source of chiral asymmetry was present (see, e.g., [1,3,5, 9–11]). Magnetic fields pervading the primordial plasma at early times would have had significant effects on many processes, such as baryogenesis [12], primordial nucleosynthesis [13,14], physics of cosmic microwave background (see [15] for a review), the growth of various inhomogeneities (see [16], and refs. therein), etc. Thus, understanding the dynamics of primordial magnetic fields in the presence of sources of left-right asymmetries, as described by time-dependent effective axion fields, is essential in attempts to describe the evolution of the early Universe.

An important question in cosmology is whether traces of primordial magnetic fields could have served as seeds for galactic magnetic fields [17,18] or given rise to the large-scale magnetic fields observed in voids [19–21]. The astrophysical origin of the magnetic fields in voids is still under active investigation; (see e.g. [22,23]). If their origin turned out to be primordial then the difficulty that causally independent homogeneous magnetic fields survive [24–26] would have to be faced. To cope with this difficulty one might argue that the mechanism of generation of those fields is "super-horizon", presumably related to inflation [27,28]. However, the generation and survival of somewhat sizeable magnetic fields during inflation is difficult to reconcile with slow-roll conditions [29–32].

In an expanding Universe filled with Standard-Model particles, chirality flipping reactions, which erase chiral asymmetries, are in thermal equilibrium at temperatures below ~80 TeV [33]. It has therefore been expected that an initial growth of magnetic fields, as derived from the chiral anomaly, could only have taken place at such truly enormous temperatures (if they ever existed in the early Universe). In an expanding Universe, the wavelength of any perturbation grows like the scale factor $a(t) \propto t^{\alpha}$ (with $\alpha < 1$, for any noninflationary expansion), and the distance, R_H, to the horizon in the Universe increases linearly in time, $R_H \propto t$. As a result, even if a magnetic field were generated during some very early epoch, with correlations extending over a horizon-size scale (maximal scale possible for any causal generation mechanism), it would soon become stronger at subhorizon scales. But it is known that subhorizon-scale magnetic fields die out because of dissipation caused by resistivity and viscosity effects [24,34–38]. It has been argued that the only way for causally generated magnetic fields to survive is the turbulence-driven inverse cascade [24,25,39,40] of helical magnetic fields—a process that makes the characteristic scale of the magnetic fields grow faster than redshifting does. At present, the efficiency of this type of mechanism is actively discussed.

We have demonstrated previously [3] that there exists another (new) mechanism for the transfer of magnetic helicity from short wavelengths to long wavelengths. This mechanism is *not* related to magnetohydrodynamical turbulence, but has its root in the chiral anomaly. We have then argued that, because of their coupling to the effective axion field (which describes chiral asymmetries), helical magnetic fields may survive down to rather low temperatures of a few MeV, even though chirality flipping reactions (caused by mass terms for the charged particles) are in thermal equilibrium. It has also been argued in [41] that, as a consequence of the parity-violating nature of weak interactions and the chiral anomaly, quantum corrections in states of finite lepton- or baryon number density may give rise to an asymmetry between left-handed and right-handed particles. A time-dependent effective axion field is then generated, which, in turn, excites a magnetic field [1]. One concludes that a homogeneous, isotropic stationary state of the Standard-Model plasma might be unstable. Thus, one has to study *dynamical* features of this system, in order to identify its true equilibrium state.

Large-scale electromagnetic fields in the plasma excite macroscopic flows of matter consisting of charged particles. The dynamics of low-energy modes of the plasma can be described with the help of the equations of relativistic magnetohydrodynamics (RMHD). One might expect that the hydrodynamical description of a magnetized relativistic plasma does not differ in an essential way from the one of its nonrelativistic cousin, the only important difference being a different relation between energy density, ρ, and pressure, p (for reviews see, e.g., [24,42,43]). However, the chiral anomaly actually leads to an important modification of the Navier-Stokes equation; see e.g. [44–47] for the approach based on relativistic hydrodynamics with symmetries or [48,49] and Refs. therein for the anomalous fluid dynamics description based on a co-adjoint orbit description of relativistic point-particles. Furthermore, helical electromagnetic fields affect the evolution of the difference of chemical potentials of left- and right-handed particles, which becomes (space- and) time-dependent [3]. As a result, not only the Navier-Stokes equations, but also the connection between the electric current density and the magnetic field are modified in accordance with the chiral anomaly, and an equation of motion for an effective axion field must be added. For a spatially homogeneous state, the time derivative of this axion field is nothing but the chiral chemical potential, (i.e., the difference of chemical potentials for left- and right-handed particles).

A magnetohydrodynamical description of a chiral relativistic plasma of charged *massive* particles, as studied in this paper, provides an approximate description of the Standard-Model plasma at high temperatures and positive matter density; (additional effects not related to the chiral anomaly must, however, be taken into account, too). We expect that our magnetohydrodynamical description of the plasma is a useful and reliable tool to determine properties of (local) thermal equilibrium of the plasma at nonvanishing lepton- and baryon densities.

Apart from applications in studies of the early Universe, our magnetohydrodynamical approach may lead to a reasonably accurate description of astrophysical systems, such as relativistic jets, and of heavy ion collisions [45]. Moreover, it may change our current views of the origin and evolution of primordial magnetic fields and their relationship to the presently observed large-scale cosmic magnetic fields [19–21]. Our results and the effect reported in [41] could affect our current understanding of cold but very dense systems, such as neutron stars. The effects of anomalous currents in neutron stars has been previously explored in [50–52].

Last but not least, our magnetohydrodynamical approach may be useful in the analysis of models of beyond-the-Standard-Model particle physics, such as the νMSM [53–55].

We stress that the effects studied in this paper do *not* involve any "new physics"; they can be described within the Standard Model. The axion field introduced below might not be a fundamental field but, rather, an emergent one that appears in the description of certain states of the Standard-Model plasma. If instead a fundamental axion *does exist*, then the high-temperature states of the Standard Model plasma, augmented by this new field (as in [1,2]) can also be described by the chiral RMHD and the effects, discussed in this paper could be encountered in a wider class of states.

A. Main goals of the paper

The physical system of primary interest in this paper is a relativistic plasma of very light charged particles at a temperature T large enough that the masses of the lightest charged particles, the electrons and positrons, can be neglected, (i.e., set to 0). For simplicity, one may consider a plasma consisting only of electrons and positrons, with a charge-neutralizing background of protons (and neutrons). But our analysis readily extends to more general systems.

Let $J_5^\mu(x)$ denote the usual axial vector current density, where $x = (t, \vec{x})$ are coordinates of a space-time point; $\vec{E}(x)$ and $\vec{B}(x)$ denote the electric field and magnetic induction, respectively. (In our notations, we will not distinguish the quantized electromagnetic field from its expectation value in physically interesting states.) Furthemore, $A(x) = (A_0(x) \equiv \phi(x), \vec{A}(x))$ is the electromagnetic gauge field, with ϕ the electrostatic potential and $\vec{A}$ the vector potential. The chiral anomaly says that J_5^μ is *not* a conserved current; more precisely that

$$\partial_\mu J_5^\mu(x) = \frac{2\alpha}{\pi}\vec{E}\cdot\vec{B}(x), \tag{1}$$

where α is the fine structure constant. Introducing the current density

$$\tilde{J}_5^\mu \equiv J_5^\mu - \frac{\alpha}{2\pi}\epsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}, \tag{2}$$

where $F_{\rho\sigma}$ are the components of the electromagnetic field tensor, we find that

$$\partial_\mu \tilde{J}_5^\mu = 0, \tag{3}$$

i.e. that $\tilde{J}_5^\mu$ is a conserved current density. Unfortunately, it is *not* invariant under electromagnetic gauge transformations, $A_\mu \to A_\mu - \partial_\mu \eta$, where η is an arbitrary function on space-time. However, the charge

$$\tilde{Q}_5 \equiv \int_{t=\text{const}} d^3\vec{x}\tilde{J}_5^0(\vec{x},t) \tag{4}$$

is not only conserved but also gauge invariant. In order to describe thermal equilibrium states of the plasma, we are advised to introduce an "axial chemical potential," μ_5, conjugate to $\tilde{Q}_5$. The chemical potential conjugate to the conserved total electric charge is tuned in such a way that the system is neutral (i.e., the charge density of electrons and positrons in the system is canceled, in average, by the one of the heavy charged hadrons, in particular the protons).

We introduce the pseudoscalar density

$$\rho_5(\vec{x},t) \equiv \langle J_5^0(\vec{x},t)\rangle_{T,\mu_5}$$

and its spatial average

$$q_5 \equiv \overline{\rho_5(\vec{x},t)},$$

where we denote by $\langle(\cdot)\rangle_{T,\mu_5}$ the equilibrium state of the system at temperature T and axial chemical potential μ_5, and by $\overline{(\cdot)}$ we indicate spatial averaging. As one would expect, there is a response equation relating the average axial charge density, q_5, to the axial chemical potential μ_5 of the form

$$q_5(T,\mu_5) \approx \mu_5 \frac{\partial q_5}{\partial \mu_5}\bigg|_{\mu_5=0} \approx \frac{\mu_5}{6}T^2, \tag{5}$$

with $q_5(T,\mu_5=0)=0$. The last equality in (5) holds for relativistic particles in the weak magnetic field, $T^2 \gg eB$, and with $T \gg \mu_5$. We will comment on a more general case in Sec. III below. A "derivation" of this relation can be found in cosmology text books.[1]

Using the chiral anomaly, we see that Eq. (5) implies the equation

$$\dot{\mu}_5 = \frac{6}{T^2}\dot{q}_5(T,\mu_5) = \frac{12\alpha}{\pi T^2}\overline{\vec{E}\cdot\vec{B}}, \tag{6}$$

a relation derived and explained in [3]. (The dot indicates a derivative with respect to time, and we neglect terms $\propto \dot{T}$) It has been shown in [1,7] that, in the presence of a nonvanishing magnetic induction $\vec{B}(x)$, a nonzero axial chemical potential, $\mu_5 \neq 0$, induces an electric current density given by $\frac{\alpha}{\pi}\mu_5\vec{B}(x)$, so that the complete expression for the electric current density, $\vec{j}$, of the plasma is given by

$$\vec{j}(x) = \langle\vec{J}(x)\rangle_{T,\mu_5} = \frac{\alpha}{\pi}\mu_5\vec{B}(x) + \sigma\vec{E}(x), \tag{7}$$

where σ is the Ohmic conductivity.[2] In [1,7], the term $\frac{\alpha}{\pi}\mu_5\vec{B}$ on the right side of (7) has been derived from the chiral anomaly using current algebra (see [57]).

Inserting (7) in Ampère's law and applying Faradey's induction law, one readily finds the equation[3]

$$\frac{\partial\vec{B}}{\partial t} = \frac{1}{\sigma}\Box\vec{B} + \frac{\alpha}{\pi}\frac{\mu_5}{\sigma}\vec{\nabla}\wedge\vec{B}. \tag{8}$$

As shown in [1,5], the term proportional to μ_5 in Eq. (8) is responsible for an instability of the solutions of (8), namely an exponential growth of the magnetic induction $\vec{B}$, which leads to a nonvanishing $\vec{E}\cdot\vec{B}$, (i.e., to a nonvanishing helicity of the electromagnetic field). By Eq. (6), this, in turn, leads to a temporal variation of the axial chemical potential μ_5 and, hence, to a change in time of the pseudoscalar density q_5 that describes the asymmetry between the charge densities of left-handed and right-handed charged particles. (We note that the electric field is damped by Ohmic losses, which affects the time dependence of $\vec{E}\cdot\vec{B}$.)

Equation (6) implies that, in general, μ_5 depends on time. The anomaly equation (1) shows that the chiral density $\rho_5(\vec{x},t)$ usually not only depends on time, but also on *space*, $\vec{x}$. It is therefore necessary to generalize relation (5) to one where μ_5 not only depends *on* t, *but also on space* $\vec{x}$; to then derive the correct form of the Maxwell equations in the presence of a space- and time-dependent axial chemical potential, $\mu_5(\vec{x},t)$, and to derive the equations of motion for this field. To find such a generalization is one of the main

[1]It is assumed that in processes such as $e^+ + e^- \leftrightarrow \gamma+\gamma$ and $e^\pm + \gamma \leftrightarrow e^\pm + \gamma$ the rates of direct and inverse reactions are identical; see, e.g., [56], Sec. 3.4 and Chap. 5.

[2]For simplicity, the state of the plasma is assumed to be isotropic and homogeneous.

[3]The standard approximation made in MHD is to neglect Maxwell's displacement current and hence the second derivative, $\frac{\partial^2\vec{B}}{\partial t^2}$, on the right side of (8); see, e.g., [58].

goals of this paper. We will also study how to couple our system of equations to the equations of motion of a relativistic fluid, thereby deriving the correct equations of RMHD. In Sec. IV, we discuss some simple solutions of these equations and sketch possible applications to plasma physics, astrophysics and cosmology.

As was pointed out already in [1], electrodynamics coupled to an axion field, θ_5, can be used to describe the dynamics of a homogeneous system (with $\theta_5 = \theta_5(t)$ independent of $\vec{x}$) in the presence of an axial chemical potential $\mu_5(t) \equiv \dot{\theta}_5(t)$ satisfying Eq. (6). In this work, we show that, for an inhomogeneous chemical potential $\mu_5(\vec{x}, t)$ (defined below, Sec. II), equations analogous to Eqs. (6) and (8) can be written as local field equations for the electromagnetic field and for an effective axion field $\theta_5(x)$, with $\dot{\theta}_5(\vec{x}, t) = \mu_5(\vec{x}, t)$, [rather than in terms of the axial chemical potential $\mu_5(\vec{x}, t)$], as already pointed out in [1]. The axion field $\theta_5(\vec{x}, t)$ interacts with the electromagnetic field in the form of a term $\theta_5 \vec{E} \cdot \vec{B}$ in the Lagrangian density, but its dynamics is not necessarily described by a relativistic wave equation, but might be governed by an inhomogeneous diffusion equation.

So far, the effective axion field θ_5 has been treated as a classical field. The role played be the axial chemical potential $\mu_5(\vec{x}, t)$ in the description of inhomogeneous local thermal equilibrium (LTE) states suggests that μ_5 (and hence the axion field θ_5) does not describe Hamiltonian degrees of freedom that have to be quantized, but that they should be understood as c-number fields labeling a family of generally inhomogeneous nonstationary states of the quantum-mechanical plasma—as in the case of the local temperature (field) $T(x)$.

One could also consider a system with an axion field that does describe dynamical (Hamiltonian) degrees of freedom, which must then be quantized. In order to shed new light on this important issue, we show, in Appendix A, how an effective axion field emerges in theories with extra dimensions. In particular, we consider a model of (quantum) electrodynamics on a slab in five-dimensional Minkowski space in the presence of a five-dimensional Chern-Simons term. In this theory, the axion appears as a component of the electromagnetic gauge field (in the fifth direction transversal to the boundaries of the slab), and the state function $T(x)$ has an interpretation as the inverse width of the slab and is therefore related to the geometry of five-dimensional space-time (see also [59]). Comparing this five-dimensional formulation with the analysis presented in Secs. II and III, one may wonder whether, in the end, space-time geometry might merely encode properties of the state of quantum-mechanical degrees of freedom, rather than being described by *dynamical* quantum fields (in particular, a quantized metric field). These fundamental questions deserve further study with a view towards applications in cosmology.

II. EXPRESSION FOR THE ELECTRIC CURRENT DENSITY IN THE RESENCE OF AN INHOMOGENEOUS AXIAL CHEMICAL POTENTIAL

A. States describing local thermal equilibrium

In order to account for the local nature of the asymmetry between left-handed and right-handed particles, we propose to introduce a space- and time-dependent axial chemical potential, $\mu_5(x)$, and generalize Eq. (5) to a local relation between the pseudoscalar density ρ_5 and μ_5, valid in the regime where $\mu_5 \ll T$. We assume that LTE is reached at length scales *small* as compared to the scale of spatial variations of $\mu_5(x)$, i.e., $l_{\mathrm{LTE}} \ll \mu_5(x)/|\nabla\mu_5(x)|$), and that the scale of variation of electromagnetic fields and matter flows is much larger than l_{LTE}, too.

A quantum-mechanical system in LTE can be described similarly as one in perfect equilibrium (see, e.g., [60]). Its state is tentatively described by the density matrix

$$\varrho_{\mathrm{LTE}} \equiv \mathcal{Z}^{-1} \exp\left\{-\frac{\mathcal{H} - \int_y \mu_5(y)\tilde{J}_5^0(y)}{T}\right\}. \tag{9}$$

Here $\mathcal{H}$ is the Hamiltonian of the system, and $\mathcal{Z}$ is the partition function (chosen such that $\mathrm{tr}(\varrho_{\mathrm{LTE}}) \equiv 1$); $\tilde{J}_5^0$ is as in Eqs. (2) and (4), and T is the temperature (here assumed to be constant throughout space-time).

The integration in the exponent on the right side of (9) extends over a spatial hyperplane at fixed time. Expression (9) may be generalized as follows: Let $(u^\mu(x))$ be the hydrodynamical 4-velocity field of the plasma; (the condition that $u^0(x) \equiv 1$, $u^i(x) \equiv 0$, for $i = 1, 2, 3$ means that we are working in a comoving coordinate frame). Let $(T_{\mu\nu}(x))$ denote the components of the energy-momentum tensor of the system, and let $\beta(x) \equiv (T(x))^{-1}$ be a (possibly slowly space- and time-dependent) inverse-temperature field; (we set the Boltzmann constant k_B to 1). Let Σ be some spacelike hypersurface, and let $d\sigma^\mu(x)$ denote the (*dual* of the) surface element at a point $x \in \Sigma$. The covariant form of expression (9) is given by (see, e.g., [[60], Chap. 24]):

$$\rho_{\mathrm{LTE}} = \mathcal{Z}_\Sigma^{-1} \exp\left\{-\int_\Sigma [u^\mu(x) T_{\mu\nu}(x) - \mu_5(x)\tilde{J}_{5\nu}(x)]\beta(x) d\sigma^\nu(x)\right\}. \tag{10}$$

An equivalent expression is given in Sec. II B, Eq. (32). Expectation values of operators in the state introduced in (10) are denoted by $\langle(\cdot)\rangle_{\mathrm{LTE}}$. We recall from Eq. (2) that

$$\tilde{J}_5^\mu = J_5^\mu - \frac{\alpha}{2\pi}\varepsilon^{\mu\nu\rho\sigma} A_\nu F_{\rho\sigma},$$

which, by Eq. (1) (chiral anomaly), is conserved but *not* gauge invariant. Under a gauge transformation,

$A_\nu \to A_\nu - \partial_\nu \eta$, where η is an arbitrary (smooth) function on space-time, the term

$$\int_\Sigma \beta(x)\mu_5(x)\tilde{J}_{5\nu}(x)d\sigma^\nu(x) = \int_\Sigma \beta(x)\mu_5(x)\tilde{J}_5^\nu(x)d\sigma_\nu(x)$$

changes by

$$-\frac{\alpha}{2\pi}\int_\Sigma \varepsilon^{\nu\mu\rho\lambda}\partial_\mu(\beta(x)\mu_5(x))F_{\rho\lambda}(x)\eta(x)d\sigma_\nu(x),$$

as follows by integration by parts; using that $\partial_{[\mu}F_{\rho\lambda]} \equiv 0$. Requiring that expression (10) be gauge-invariant (i.e., independent of the gauge function η, for arbitrary η), we find the constraint

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu(\beta(x)\mu_5(x))F_{\rho\sigma}[d\sigma^\mu(x)]_\Sigma \equiv 0, \tag{11}$$

or, in the language of differential forms,

$$d(\beta\mu_5) \wedge F|_\Sigma \equiv 0. \tag{12}$$

If $\beta(x) \equiv \beta$ is constant and Σ is a hyperplane at fixed time then (11) implies that

$$\vec{B}(x)\cdot\vec{\nabla}\mu_5(x) \equiv 0, \tag{13}$$

i.e., gauge invariance implies that the gradient of the axial chemical potential is orthogonal to the magnetic induction. Condition (13) has a simple physical interpretation discussed in Sec. II C, below.

Let $J^\mu(x)$ denote the electric (vector) current density, and let J_5^μ be as in Sec. I A [see Eq. (1)]. We set

$$j^\mu(x) = \langle J^\mu(x)\rangle_{\text{LTE}}, \qquad j_5^\mu(x) = \langle J_5^\mu(x)\rangle_{\text{LTE}}, \tag{14}$$

for $\mu = 0, 1, 2, 3$. In [1,2], the equal-time commutation relations between J^0 and J_5^0,

$$[J_5^0(\vec{y},t), J^0(\vec{x},t)] = \frac{\alpha}{\pi}\vec{B}(\vec{y},t)\cdot\vec{\nabla}_y\delta(\vec{x}-\vec{y}), \tag{15}$$

(see [57]) have been used to derive expression (7) for the $\vec{B}$ dependence of the electric current density $\vec{j}$:

$$\vec{j}(x) = \frac{\alpha}{\pi}\mu_5\vec{B}(x), \tag{16}$$

assuming that μ_5 does not depend on $\vec{x}$ and that $\sigma\vec{E} \equiv 0$. We sketch how (16) is derived from (15) and then propose a generalization of (16) where μ_5 may depend on space and time.

The conservation of the vector current density $J^\mu(x)$ implies that there exists a "current vector potential," $\vec{\Phi}(x)$, with

$$J^0(x) = -\vec{\nabla}\cdot\vec{\Phi}(x), \qquad \vec{J}(x) = \partial_t\vec{\Phi}(x). \tag{17}$$

These expressions are invariant under the transformations $\vec{\Phi} \to \vec{\Phi} + \vec{\nabla}\wedge\vec{X}$, for an arbitrary vector field $\vec{X}$ independent of time. The commutation relations of the field $\vec{\Phi}$ with the axial charge density J_5^0 are then given by

$$[J_5^0(\vec{y},t), \vec{\Phi}(\vec{x},t)] = \frac{\alpha}{\pi}\vec{B}(\vec{y},t)\delta(\vec{x}-\vec{y}) + \vec{\nabla}_x\wedge\vec{\Pi}(\vec{x},\vec{y},t), \tag{18}$$

where the second term on the right side of (18) is an "integration constant." Let $\vec{\pi}$ denote the expectation value of $\vec{\Pi}$ in the state $\langle(\cdot)\rangle_{\text{LTE}}$ of the plasma. If this state is homogeneous (in particular, $\langle\vec{B}\rangle_{\text{LTE}} = 0$) then $\vec{\pi}(\vec{x},\vec{y},t) \equiv \vec{\pi}(\vec{x}-\vec{y},t)$ must be space-translation invariant. It is plausible that $\vec{\pi}$ is independent of $\vec{B}$, so that $\vec{\pi}$ is space-translation invariant even if $\langle\vec{B}\rangle_{\text{LTE}} \neq 0$. In the following, we will write $\vec{B}$ for $\langle\vec{B}\rangle_{\text{LTE}}$, as above, in order to shorten our notation. Taking the expectation value of both sides of (18) in the state

$$\langle(\cdot)\rangle \equiv \langle(\cdot)\rangle_{\text{LTE}}$$

of the plasma and integrating over $\vec{y}$, we find that

$$\langle[Q_5, \vec{\Phi}(x)]\rangle = \frac{\alpha}{\pi}\vec{B}(x), \tag{19}$$

because the expectation of the second term on the right side of (18) integrates to 0 if $\vec{\pi} = \langle\vec{\Pi}\rangle$ is translation-invariant.

Since $\vec{\Phi}$ commutes with the electric charge operator Q, Eq. (19) implies that

$$\langle[Q_{L/R}, \vec{\Phi}(x)]\rangle = \pm\frac{\alpha}{2\pi}\vec{B}(x),$$

where $Q_{L/R} = \frac{1}{2}(Q \pm Q_5)$.

Next, we observe that

$$\vec{j}(x) = \langle\dot{\vec{\Phi}}(x)\rangle = \mathrm{i}\,\mathrm{tr}(\rho_{\text{LTE}}[\mathcal{H}, \vec{\Phi}(x)]), \tag{20}$$

where ρ_{LTE} is given by (9), with $\mu_5(x) \equiv \mu_5$ independent of x.

Formally, the right side of (20) appears to vanish for a constant μ_5, because $\tilde{j}_5^\mu$ is a conserved current. However, the field $\vec{\Phi}$ is so singular in the infrared that the formal calculation is deceptive. The right side of (20) must be regularized in the infrared by adding a mass term to the Hamiltonian, $\mathcal{H} \to \mathcal{H}_{\text{mass}}$. Then $\tilde{j}_5^\mu$ is *not* conserved, anymore, and $[\mathcal{H}_{\text{mass}}, Q_5] \neq 0$. However ρ_{LTE} continues to commute with $\mathcal{H}_{\text{mass}} - \mu_5\int\tilde{j}_5^0(\vec{y})d^3y$! Thus

$$\vec{j}(x) = \langle\dot{\vec{\Phi}}(x)\rangle = \mathrm{i}\,\mathrm{tr}\bigg(\rho_{\rm LTE}\bigg[\mathcal{H}_{\rm mass} - \mu_5\int \tilde{j}_5^0(\vec{y},t)d^3y + \mu_5\int \tilde{j}_5^0(\vec{y},t)d^3y, \vec{\Phi}(\vec{x},t)\bigg]\bigg) = \mu_5\langle[Q_5,\vec{\Phi}(x)]\rangle = \frac{\alpha}{\pi}\mu_5\vec{B}(x), \tag{21}$$

by Eq. (19).

Next, we consider the general situation where μ_5 may depend on time *and* space. Let $\vec{\pi}(\vec{x},\vec{y},t) \coloneqq \langle\vec{\Pi}(\vec{x},\vec{y},t)\rangle$, where $\vec{\Pi}$ is as in Eq. (18). Then Eq. (21) generalizes to

$$\vec{j}(x) = \frac{\alpha}{\pi}\mu_5(x)\vec{B}(x) + \vec{\nabla}_x \wedge \int \vec{\pi}(\vec{x},\vec{y},t)\mu_5(\vec{y},t)d^3y, \tag{22}$$

Note that the current $\vec{j}$ in (22) is conserved as a consequence of Eq. (13); (gauge invariance of $\rho_{\rm LTE}$). If μ_5 is independent of spatial coordinates and the distribution $\vec{\pi}$ is translation-invariant, the second term in Eq. (22) vanishes, and this equation reduces to (21).

B. Axial chemical potential and axion field

The second term in the expression (22) is nonlocal and the distribution $\vec{\pi}(\vec{x},\vec{y},t)$ has not been defined, yet. We attempt to find a local expression for the four-current density j^μ. This current density should be proportional to $(F_{\mu\nu})$ and linear in (μ_5); and it should be conserved, (i.e., satisfy the continuity equation). By introducing a pseudoscalar field, $\theta_5(x)$, indeed an axion field, related to the axial chemical potential $\mu_5(x)$, we arrive at the following ansatz for j^μ:

$$j^\mu_{\rm axion} \equiv \frac{\alpha}{2\pi}\epsilon^{\mu\nu\rho\sigma}(\partial_\nu\theta_5)(x)F_{\rho\sigma}(x), \tag{23}$$

with $\dot{\theta}_5$ proportional to μ_5 in comoving coordinates. The current $j^\mu_{\rm axion}$ defined by Eq. (23) is automatically conserved; ($j^\mu_{\rm axion}$ is dual to the 3-form $\frac{\alpha}{2\pi}d\theta_5 \wedge F$, which is closed, because $d^2 = 0 = dF$). For $j^\mu_{\rm axion}$ to transform as a vector under parity and time reversal, θ_5 must be a pseudoscalar field.

To define its relation to μ_5 let us separate the 0 components from the spatial components:

$$j^0_{\rm axion} = \frac{\alpha}{\pi}\vec{\nabla}\theta_5\cdot\vec{B}, \qquad \vec{j}_{\rm axion} = \frac{\alpha}{\pi}(\dot{\theta}_5\vec{B} + \nabla\theta_5\wedge\vec{E}). \tag{24}$$

Comparing these equations with Eq. (16) [see also Eq. (7)], we argue that $\mu_5(x) = \dot{\theta}_5(x)$. The identification of the axial chemical potential with an axion field ($\dot{\theta}_5$) has first been proposed in [1,2]; (later it also appeared in the context of the quark-gluon plasma in Refs. [47,61–67], with the pseudoscalar field being identified with a variable QCD theta angle).

We wish to explain the relation between an inhomogeneous $\mu_5(x)$ and $\theta_5(x)$. Equation (24) holds in any coordinate system, if we define $E_i = F_{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F^{jk}$. However, in plasma physics, it is convenient to express $F_{\mu\nu}$ in terms of electric and magnetic fields in comoving coordinates:

$$F_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}u^\lambda\mathcal{B}^\rho + (u_\mu\mathcal{E}_\nu - u_\nu\mathcal{E}_\mu), \tag{25}$$

where $\mathcal{E}_\mu, \mathcal{B}_\mu$ are four-vectors of electric and magnetic fields orthogonal to the 4-velocity field u^μ, (i.e., $\mathcal{E}_\mu u^\mu = \mathcal{B}_\mu u^\mu = 0$). For details see, e.g., the review [68]). Taking $F_{\mu\nu}$ in the form (25) and plugging it into Eq. (23), we find that

$$\vec{j}_{\rm axion} = \frac{\alpha}{\pi}[(\dot{\theta}_5 + \vec{v}\cdot\vec{\nabla}\theta_5)\vec{\mathcal{B}}(x) + (\vec{\nabla}\theta_5 + \vec{v}\dot{\theta}_5)\wedge\vec{\mathcal{E}}(x)] + \mathcal{O}(v^2) \tag{26}$$

[it should be stressed that in Eq. (26) the axion current is measured in the coordinate system where flow velocities are nonzero, while $\vec{\mathcal{E}}, \vec{\mathcal{B}}$ are the spatial components of the comoving four-vectors $\mathcal{E}_\mu, \mathcal{B}_\mu$, Eq. (25)]. Comparing Eq. (26) with (22) gives us the desired relation between μ_5 and θ_5,

$$\mu_5(\vec{x},t) \equiv u^\mu\partial_\mu\theta_5(\vec{x},t). \tag{27}$$

We note that if the relation $\mu_5(t) = \dot{\theta}_5(t)$ holds in comoving coordinates, then Eq. (27) follows by transformation to the laboratory frame.

Equations (27), (26), and (24) provide a local expression for the electric four-current density j^μ involving the axion field θ_5. Our expression for j^μ explains why the electric current density $\vec{j}$ in (22) is nonlocal in the axial chemical potential μ_5. The current density $\vec{j}$ is odd under time reversal, $\mathcal{T}$. The combination $\mu_5\vec{B}$ is $\mathcal{T}$-odd, too, because $\vec{B}$ is $\mathcal{T}$-odd and μ_5 is $\mathcal{T}$-even. The electric field ($\vec{E}$) is $\mathcal{T}$-even, hence a term proportional to (spatial derivatives of) μ_5 and $\vec{E}$ would have the wrong transformation properties under time reversal. However, when introducing the axion field θ_5, it is easy to write a local combination of (spatial derivatives of) θ_5 and $\vec{E}$ that transforms correctly under time reversal, namely $\vec{\nabla}\theta_5\cdot\vec{E}$, which is odd under $\mathcal{T}$, because the axion field is $\mathcal{T}$-odd, (as the expression in (27) shows). Clearly, $\vec{\nabla}\theta_5$ is a nonlocal functional of μ_5. (Thus, introducing an axion field enables one to find a local expression for the electric current, instead of the nonlocal one on the right hand side of Eq. (22), and the second equation in (24) is the correct generalization of Eq. (21) to a spatially inhomogeneous plasma.

An alternative route towards understanding the origin of the axion field θ_5 is the following one (see [1,2]): We consider a system of massless fermions coupled to the electromagnetic field in a state corresponding to a non-vanishing axial chemical potential μ_5. The action of this system is given by

$$S[A,\psi] = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(\not{\partial} + e\not{A})\psi + \mu_5(\bar{\psi}_L^\dagger \psi_L - \bar{\psi}_R^\dagger \psi_R) \right] \tag{28}$$

One can remove the term proportional to μ_5 by performing a "*chiral redefinition*" of the fermion fields

$$\psi \to e^{i\theta_5(t)\gamma_5}\chi \tag{29}$$

where $\dot{\theta}_5 = \mu_5$ [see (27)]. This transformation yields a Fujikawa Jacobian [69]

$$\mathcal{J}[A_\mu, \theta_5] = \exp\left(i\frac{\alpha}{4\pi}\theta_5 \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} \right) \tag{30}$$

in the functional integral, yielding the modified action

$$S[A,\chi,\theta_5] = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}^2 + \frac{\alpha}{4\pi}\theta_5 \epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} + \bar{\chi}(\not{\partial} + \not{A})\chi \right). \tag{31}$$

We see that if the axial chemical potential is spatially homogeneous then it can be re-expressed in terms of a pseudoscalar field, θ_5, that has an axionlike interaction with the electromagnetic field.

Equation (27) enables us to present the general expression for the density matrix describing local thermal equilibrium of an inhomogeneous plasma in an arbitrary coordinate system.

$$\rho_{\mathrm{LTE}} = \mathcal{Z}_\Sigma^{-1} \exp\left\{ -\int_\Sigma [u^\mu(x)(T_{\mu\nu}(x) - \partial_\mu \theta_5(x) \tilde{J}_{5\nu}(x))] \beta(x) d\sigma^\nu(x) \right\}, \tag{32}$$

where u^μ is the 4-velocity field of the plasma and the expression $u^\mu \partial_\mu \theta_5$ plays the role of a local axial chemical potential $\mu_5(x)$. Expression (32) generalizes (10) to an arbitrary coordinate system.

In Sec. III we will search for the correct dynamics of the axion field. But, before addressing this problem, we propose to clarify the significance of the constraint (13).

C. Constraints on θ_5

In this subsection, we tentatively interpret θ_5 as a classical field, more precisely as a generalized thermodynamic parameter that labels states describing LTE with left-right (chiral) asymmetry. As discussed in Secs. II A and II B, the requirement that the density matrix ρ_{LTE}, supposed to describe LTE [see (10), (32)], be gauge invariant yields the constraint

$$d(\beta\mu_5) \wedge F|_\Sigma \equiv 0, \tag{33}$$

with

$$\mu_5 = u^\mu \partial_\mu \theta_5, \tag{34}$$

where $u^\mu(x)$ is the 4-velocity field of the plasma appearing in expressions (10) and (32) for ρ_{LTE}. Equation (33) is identical to (11)–(12), and Eq. (34) follows from comparing expressions (10) in Sec. II A and (32) in Sec. II B. In comoving coordinates, $u^0(x) \equiv 1$, $\vec{u}(x) \equiv 0$. Choosing Σ to be a hyperplane at fixed time (in comoving coordinates), and assuming that the temperature T is independent of $x = (\vec{x}, t)$, (33) becomes

$$\vec{B}(x) \cdot \vec{\nabla}\mu_5(x) = \vec{B}(x) \cdot \vec{\nabla}\dot{\theta}_5(x) \equiv 0, \tag{35}$$

which is Eq. (13).

It may be somewhat surprising that the constraint (35) must be imposed on the choice of μ_5. In order to illustrate the origin of this constraint, we consider a plasma in a homogeneous magnetic field $\vec{B} \neq 0$. We choose an initial state where all left-handed (spin parallel to momentum) charged particles are located in the half-space $x^1 < 0$, while the right-handed ones (spin antiparallel to momentum) are located at $x^1 > 0$. We assume that the particles are non-interacting. Then the one-particle states are labeled by the quantum number of a Landau level and the component of the momentum parallel to $\vec{B}$. If $\vec{\nabla}\mu_5$ is perpendicular to the field $\vec{B}$ then the densities of left-handed and right-handed particles are independent of time. But if $\vec{B} \cdot \vec{\nabla}\mu_5 \neq 0$ then left-handed and right-handed particles start to mix, as they move in the direction of $\vec{B}$. As a result, the chiral charge density changes in time, which implies that LTE is lost; see Fig. 1.

As a second example, we choose $\mu_5(x)$ to be nonzero and constant in the slab defined by $0 \leq x^1 \leq L$ and $\mu_5(x) \equiv 0$, elsewhere. We consider a $\vec{B}$-field parallel to the x^1 axis. An electric current then flows from $x^1 = 0$ to $x^1 = L$, thus creating charge densities proportional to $-B\delta(x^1)$ and $B\delta(x^1 - L)$, respectively, on the two boundary planes of the slab. These surface charges create a nonvanishing electric field $\vec{E}(x)$ inside the slab, which is parallel to the x^1 axis. Hence, $\vec{E}(x) \cdot \vec{B}(x) \neq 0$, in the region where

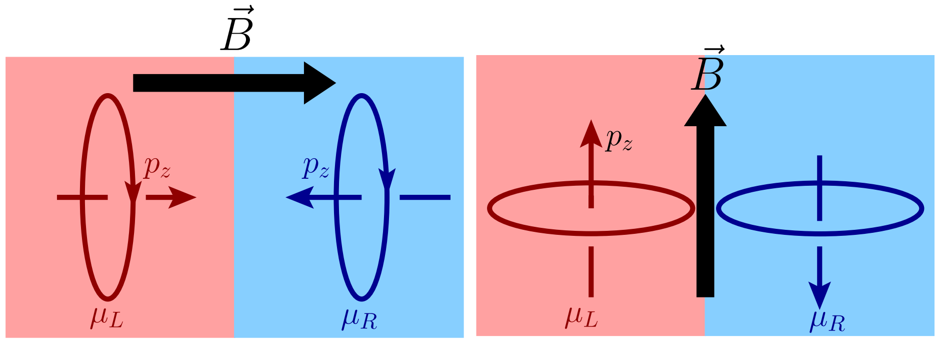

FIG. 1 (color online). The simplest configuration with a nonzero gradient of $\mu_5(x)$. The region $x^1 < 0$ has only left-handed particles and the region $x^1 > 0$ has only right-handed particles. If we consider a configuration where the magnetic field $\vec{B}$ has a nonvanishing one-component, B_1, the left-handed particles drift to the right, while the right-handed particles drift to the left, (because their state is parametrized by a nonzero p_1). Thus, the state is not on equilibrium state. But if $\vec{B}$ is perpendicular to the x axis the particles move parallel to the boundary plane $\{x = 0\}$, and the state is an equilibrium state.

$\mu_5(x) \neq 0$, i.e., inside the slab. By Eq. (6) (Sec. I A), $|\mu_5|$ then decreases in time, with $\mu_5(\vec{x}, t) \to 0$, as $t \to \infty$. Obviously, LTE is then only approached in the limit where $t \to \infty$.

D. Chiral asymmetry and electroneutrality

To conclude this section, we discuss consequences of electroneutrality in a high-temperature plasma and in the presence of an axion field. Electroneutrality means that the *total* electric charge density, $j^0_{\text{tot}}(x)$, vanishes. According to Eq. (24), the light charged particles (electrons and positrons, with $T \gg m_e$) make a contribution

$$j^0_{\text{axion}} = \frac{\alpha}{\pi} \vec{\nabla}\theta_5 \cdot \vec{B} \tag{36}$$

to the electric charge density. In a plasma containing *only* electrons and positrons, electroneutrality implies the absence of matter-antimatter asymmetry, and j^0_{axion} must vanish identically. If $j^0_{\text{axion}} \equiv 0$ then the 3-divergence of $\vec{j}_{\text{axion}}$ must vanish, i.e.,

$$0 = \vec{\nabla} \cdot \vec{j}_{\text{axion}} = \frac{\alpha}{\pi}(\vec{\nabla}\dot{\theta}_5 \cdot \vec{B} + \vec{\nabla}\theta_5 \cdot \dot{\vec{B}}) = \frac{\alpha}{\pi}\partial_t(\vec{\nabla}\theta_5 \cdot \vec{B}), \tag{37}$$

by (36). Expression (22) for the current density $\vec{j}$ implies that $\vec{\nabla} \cdot \vec{j} = 0$, as a consequence of Eq. (35). As a consequence, electroneutrality, i.e., $j^0_{\text{axion}} = 0$ follows.

If the plasma contains several types of charged particles (for example, $e^{\pm}$ and $p, \bar{p}$) then electroneutrality does *not* imply any symmetry between matter and antimatter, and there is no reason to expect that j^0_{axion}, as given in (36), vanishes identically. One must then introduce electric four-vector current densities for leptons and baryons and axial chemical potentials for all species of very light, charged particles. For simplicity, we consider a plasma consisting of electrons, positrons, protons and antiprotons only. Let J^μ_p denote the electric current density of protons and antiprotons, and let

$$j^\mu_p = \langle J^\mu_p \rangle_{\text{LTE}} \tag{38}$$

Electroneutrality then implies that

$$j^0_p = -j^0_{\text{axion}} = -\frac{\alpha}{\pi}\vec{\nabla}\theta_5 \cdot \vec{B} \tag{39}$$

Let J^μ_L be the total lepton vector current density, and let J^μ_B denote the total baryon vector current density. These vector currents are separately conserved; hence,

$$J^\mu_B - J^\mu_L \tag{40}$$

is conserved. In order to describe the matter-antimatter asymmetry observed in the Universe, we introduce a chemical potential $\mu_{\text{B-L}} = \dot{\theta}_{\text{B-L}}$ conjugate to $Q_L - Q_B$, where

$$Q_{L/B} = \int_{t=\text{const}} J^0_{L/B}(\vec{x}, t) d^3x \tag{41}$$

are the conserved lepton and baryon electric charges. If the distribution of asymmetry between matter and antimatter in the Universe turned out to be inhomogeneous the scalar quantity $\theta_{\text{B-L}}$ would have to be taken to depend on time *and* space, and a term

$$\int_\Sigma u^\mu(x)\partial_\mu\theta_{\text{B-L}}(x)(J^\nu_B(x) - J^\nu_L(x))\beta(x)d\sigma_\nu(x) \tag{42}$$

must be added in the exponent of the expression for the density matrix that describes LTE, on the right side of Eq. (32) (see Sec. II B). To complete the picture we will have to find an equation of motion for the scalar field $\theta_{\text{B-L}}$. We note that there are no constraints on $\theta_{\text{B-L}}$ similar to (33), (35), but there is an equation analogous to (5) discussed in the next section. Before we turn our attention to the dynamics of $\theta_{\text{B-L}}$, we study the dynamics of θ_5.

III. SEARCH FOR THE DYNAMICS OF THE AXION FIELD

Having introduced a time-dependent axial chemical potential μ_5, it is necessary to specify its dynamics. Equations (23) and (24) suggest that it may be convenient to derive the dynamics of μ_5 from the one of the axion field θ_5.

In the homogeneous case, i.e., when $\mu_5(t) = \dot{\theta}_5(t)$, Eq. (1) can be rewritten as

$$\Lambda^2 \overline{\ddot{\theta}_5} = \frac{\alpha}{4\pi} \epsilon^{\mu\nu\lambda\rho} \overline{F_{\mu\nu} F_{\lambda\rho}}, \tag{43}$$

where Λ is a constant with the dimension of an "energy" (the field θ_5 is dimensionless, because $\mu_5 = \dot{\theta}_5$ must be an "energy"). By Eqs. (1)–(6), we have that

$$\Lambda^2 = \frac{T^2}{6}, \tag{44}$$

where T is the temperature of the plasma. (We have used that $\epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} = 8\vec{E} \cdot \vec{B}$, and $\overline{(\cdot)}$ indicates spatial averaging.) In (43), $(F_{\mu\nu})$ is the expectation value of the electromagnetic field tensor in the state $\langle(\cdot)\rangle_{\text{LTE}}$. The parameter Λ has the meaning of a susceptibility. Its exact relation to the thermodynamic variables depends on the density of states. The relation (44) holds in the weak magnetic field regime, $eB \ll T^2$ (we restrict ourselves to the massless fermions and so the Landau levels are given by $E_n^2 = p_z^2 + 2neB$, where $n = 0, 1, 2 \ldots$). In the regime of large magnetic field, the particles are confined to the lowest Landau level where the density of states is $eB \frac{dp_z}{(2\pi)^2}$ rather than $\frac{dp_x dp_y dp_z}{(2\pi)^3}$. In this regime Λ becomes B dependent. As the magnetic field continues to grow, the interaction of θ_5 with the electromagnetic field (proportional to Λ^{-1}) becomes weaker and weaker, effectively turning off the coupling.

We should ask what the correct equation of motion of θ_5 is that, upon taking spatial averages, yields Eq. (43). The axion field θ_5 has been introduced in order to describe plasmas exhibiting a chiral asymmetry; in other words, θ_5 may label states of a plasma—as do other thermodynamic parameters, in particular the temperature. For this reason, the equation of motion of θ_5 has no reason to be relativistic; (i.e., to preserve its form under arbitrary coordinate transformations). It may therefore be plausible to consider a diffusion equation for $\dot{\theta}_5$, which we discuss below.

The chiral anomaly tells us that

$$\partial_\mu J_5^\mu = \frac{2\alpha}{\pi} \vec{E} \cdot \vec{B},$$

see Eq. (1), Sec. I A. Taking the expectation in the state of the plasma (and viewing $\vec{E}$ and $\vec{B}$ as classical fields) yields

$$\partial_\mu j_5^\mu = \frac{2\alpha}{\pi} \vec{E} \cdot \vec{B}, \tag{45}$$

with $j_5^\mu = \langle J_5^\mu \rangle_{T,\mu_5}$. On the basis of Eqs. (43) and (45), one argues that

$$\partial_\mu j_5^\mu = \Lambda^2 \ddot{\theta}_5 + \varepsilon[\theta_5], \qquad \left(\text{with } \Lambda^2 = \frac{T^2}{6}\right), \tag{46}$$

where ε is a term whose spatial average vanishes, i.e., $\overline{\varepsilon[\theta_5]} = 0$.

If θ_5 is interpreted as a kind of thermodynamic parameter labeling an inhomogeneous state of the plasma in LTE then the following ansatz for j_5^μ as a function of θ_5 in comoving coordinates is reasonable:

$$j_5^0 = \Lambda^2 \dot{\theta}_5, \qquad \vec{j}_5 = \Lambda^2 D \vec{\nabla}(\dot{\theta}_5), \tag{47}$$

and

$$\Lambda^2 = \left[\frac{\partial j_5^0}{\partial \mu_5}\right]\Bigg|_{\mu_5=0}, \quad \text{with} \quad j_5^0|_{\mu_5=0} = 0. \tag{48}$$

Note that these equations imply that

$$D \vec{\nabla} j_5^0 = \vec{j}_5, \tag{49}$$

a relation that can also be confirmed by direct computations (see, e.g., [70]). In comoving coordinates, (45) and (47) then yield the equations of motion

$$\Lambda^2 (\ddot{\theta}_5 - D\Delta \dot{\theta}_5) = \partial_\mu j_5^\mu = \frac{2\alpha}{\pi} \vec{E} \cdot \vec{B}, \tag{50}$$

which imply (46). Together with Eq. (27) we find that

$$\Lambda^2 (\dot{\mu}_5 - D\Delta \mu_5) = \frac{2\alpha}{\pi} \vec{E} \cdot \vec{B}, \tag{51}$$

in accordance with Eq. (6), which is an inhomogeneous diffusion equation for μ_5. This diffusion equation can also be obtained from a more formal first-order expansion of the constitutive equation for the global current j_5^μ; (see, e.g., [71,72], or the book [73]). Taking into account chirality flips due to small masses of the charged particles, Eq. (51) should be generalized to the equation

$$\Lambda^2 (\dot{\mu}_5 - D\Delta \mu_5) + \Gamma_f \mu_5 = \frac{2\alpha}{\pi} \vec{E} \cdot \vec{B}. \tag{52}$$

Furthermore, LTE imposes the constraint

$$\vec{\nabla} \mu_5 \cdot \vec{B} = 0 \tag{53}$$

[see (13)]. The electric vector current density is given by

$$j^0_{\rm tot} = j^0_{\rm axion} + j^0_p = \frac{\alpha}{\pi}\vec{\nabla}\theta_5 \cdot \vec{B} + j^0_p = 0, \tag{54}$$

$$\vec{j}_{\rm tot} = \vec{j}_{\rm axion} + \vec{j}_p = \vec{j}_{\rm axion} + \vec{j}_{Ohm} = \frac{\alpha}{\pi}(\mu_5\vec{B} + \vec{\nabla}\theta_5 \wedge \vec{E}) + \sigma\vec{E}, \tag{55}$$

with $|\vec{j}_p| \ll |\vec{j}_l|$, because protons are much heavier than electrons and positrons.

Of course, we also require the Maxwell equations, with $j^\mu = j^\mu_{\rm tot}$ given by (54) and (55).

Assuming that the baryonic current $\vec{j}_p$ can be incorporated in $\vec{j}_{Ohm} = \sigma\vec{E}$, as we have done in (55), we have arrived at a complete system of field equations for $\theta_5, \vec{E}$ and $\vec{B}$.

The conductivity (tensor) σ must be calculated from transport equations for the plasma. Apparently, the equation of motion for the scalar field $\theta_{\text{B-L}}$ that determines the matter-antimatter asymmetry is not needed here; although it is of considerable interest to derive one, (as will be discussed elsewhere).

Equations (52)–(55) are supposed to be valid in comoving coordinates. In general coordinate systems equations (53) and (54) must be replaced by

$$\mu_5 = u^\mu\partial_\mu\theta_5 \approx \dot{\theta}_5 + \vec{u}\cdot\vec{\nabla}\theta_5 \tag{56}$$

and

$$j^0_{\rm tot} + \vec{u}\cdot\vec{j}_{\rm tot} = 0 \tag{57}$$

[see Eq. (27)].

The analogue of Eq (5) for the field $\theta_{\text{B-L}}$ is the relation

$$\overline{j^0_B - j^0_L} = \varepsilon^2\overline{\dot{\theta}_{\text{B-L}}}, \tag{58}$$

where $\overline{(\cdot)}$ denotes spatial averaging, and ε is a constant with the dimension of an energy. Since $j^\mu_B - j^\mu_L$ is a conserved current, it follows that

$$\overline{\ddot{\theta}_{\text{B-L}}} = 0. \tag{59}$$

A relativistic equation compatible with (59) is

$$\Box\theta_{\text{B-L}} = 0, \tag{60}$$

which describes a free, massless scalar field. As in our discussion of the equation of motion of θ_5, one could also envisage an equation of the form

$$\dot{\mu}_{\text{B-L}} - D\Delta\mu_{\text{B-L}} = 0, \tag{61}$$

with $\mu_{\text{B-L}} = \dot{\theta}_{\text{B-L}}$ in comoving coordinates. Choosing the hyperplane Σ appearing in Eq. (42) to correspond to a constant time t (in comoving coordinates, with $u^0 = 1, \vec{u} = 0$) and imposing the condition that expression (42) is independent of t (i.e., the matter-antimatter asymmetry does not change in time, anymore, in the old Universe), we find the constraint

$$\overline{(J^\alpha_B - J^\alpha_L)\partial_\alpha\mu_{\text{B-L}}} = 0. \tag{62}$$

A. Dynamics of a fundamental axion field

It is advisable to also consider the possibility that θ_5 is a fundamental field evolving in time according to some relativistic Hamiltonian dynamics. The equations of motion for θ_5 can then be derived from an action principle, the action of θ_5 being denoted by

$$S_{\rm eff}[A, \theta_5], \tag{63}$$

where A is the electromagnetic four-vector potential. In order for the action (63) to reproduce the desired expressions for $j^\mu_{\rm axion}$ given in (23)–(24) (Sec. II B), it *must* have the form

$$S_{\rm eff}[A, \theta_5] = S_0[\theta_5] + \frac{\alpha}{4\pi}\int d^4x\theta_5(x)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{64}$$

The equation of motion for θ_5 derived from (64) is given by

$$\frac{\delta S_{\rm eff}[A, \theta_5]}{\delta\theta_5} = 0, \tag{65}$$

or

$$\frac{\delta S_{\rm eff}[A, \theta_5]}{\delta\theta_5} - \frac{\delta S_0[\theta_5]}{\delta\theta_5} = \frac{\alpha}{2\pi}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} = \frac{2\alpha}{\pi}\vec{E}\cdot\vec{B} \tag{66}$$

By (45), this implies that

$$\frac{\delta S_{\rm eff}[A, \theta_5]}{\delta\theta_5} - \frac{\delta S_0[\theta_5]}{\delta\theta_5} = \partial_\mu j^\mu_5 \tag{67}$$

A reasonable ansatz for $S_{\rm eff}[A, \theta_5]$ is

$$S_{\rm eff}[A, \theta_5] = \int d^4x\Big\{\Lambda^2(\partial_\mu\theta_5)(x)(\partial^\mu\theta_5)(x) + U(\theta_5(x)) - \frac{\alpha}{4\pi}\theta_5(x)\varepsilon^{\mu\nu\lambda\rho}F_{\mu\nu}(x)F_{\lambda\rho}(x)\Big\} \tag{68}$$

This action yields the equations of motion

$$\Lambda^2\Box\theta_5 + U'(\theta_5) = \frac{\alpha}{4\pi}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}. \tag{69}$$

If θ_5 is spatially homogeneous and $U = 0$ this field equation reduces to (43). Equation (69) has the form (46), provided $\overline{U'(\theta_5)} = 0$ (e.g., if $U \equiv 0$). Equation (69), with a term $\Gamma_f\dot{\theta}_5$ added on the left side, represents an alternative to Eq. (52).

We conclude that, in a system with a fundamental axion field, the chiral MHD description of the plasma can be expected to be applicable even if the fermions are not ultrarelativistic. For example, if an axionlike particle exists and represents a contribution to dark matter, then the production of dark matter in such a system may be accompanied by the generation of magnetic fields (see for example [74]). The phenomenological viability of this scenario will be discussed elsewhere.

If Eq. (69) is the correct equation of motion it is tempting to argue that not only $F_{\mu\nu}$ corresponds to a quantized field, but the axion field θ_5 must be quantized, too. Furthermore, the action (68), with $U = 0$, suggests to unify the electromagnetic gauge potential A and θ_5 to a gauge potential, $A = (A, \Lambda\theta_5)$, on a slab of width $\propto \Lambda^{-1} = \mathrm{const}\beta$ in five-dimensional space-time, with β the inverse temperature [see (44)]. Then the term $\int d^4x\{\Lambda^2(\partial_\mu\theta_5)(x)(\partial^\mu\theta_5)(x)\}$ originates from a contribution to the five-dimensional Maxwell term, after dimensional reduction. These ideas are discussed in more detail in Appendix A.

IV. DYNAMICS OF CHIRAL MHD IN THE INHOMOGENEOUS CASE

It was demonstrated in [3] that the presence of a dynamical field $\mu_5(t)$ in the Maxwell-axion equations leads to the generation of an "inverse cascade"—i.e., to the transport of magnetic energy and helicity from shorter to longer scales. In what follows we present a preliminary analysis of the inhomogeneous equations derived in Sec. III and show that the behavior of solutions is qualitatively similar to that of spatially homogeneous solutions, as discussed in [3]. In our analysis, we neglect

(1) the diffusion term $\Delta\mu_5$ in Eq. (52), assuming that the timescales are much longer than typical diffusion times
(2) the coupling to the velocity field $\vec{u}$ of the plasma, as in Eqs. (56)–(57); i.e., we set $\vec{u} \approx 0$; we also neglect, as usual, Maxwell's displacement current proportional to $\dot{\vec{E}}$
(3) the chirality flipping term in Eq. (52); i.e., we set $\Gamma_f = 0$.

We thus analyze the following simplified system of equations:

$$\begin{aligned} \vec{\nabla} \wedge \vec{B} &= \sigma\vec{E} + \frac{\alpha}{\pi}(\dot{\theta}_5\vec{B} + \nabla\theta_5 \times \vec{E}), \\ \Lambda^2\ddot{\theta}_5 &= \frac{2\alpha}{\pi}\vec{E}\cdot\vec{B}, \\ \vec{B}\cdot\nabla\dot{\theta}_5 &= 0, \end{aligned} \tag{70}$$

with $\mu_5 = \dot{\theta}_5$. In the analysis below we will assume that $\Lambda \propto T$ and is independent of the magnetic field, B (see the discussion after Eq. (44) in Sec. III). That is, we ignore the backreaction of the magnetic field on the θ_5 coupling.

A. Exact single-mode solution

In this section, we set out to find the simplest (but nontrivial) exact solutions of Eqs. (70). These solutions can be considered to be direct generalizations of the "single-mode" solutions, considered in Ref. [3], to the inhomogeneous case.

We start with the following ansatz for the magnetic induction $\vec{B}$:

$$\vec{B}(x) = B(t)(\sin(kz), \cos(kz), 0) \tag{71}$$

For this choice of $\vec{B}$, we have that

$$\vec{\nabla} \wedge \vec{B} = k\vec{B}, \tag{72}$$

which is a special case of the force-free configuration first described in [75] (see also [76,77]). A general force-free configuration (72) is given by the following expression [77]:

$$\vec{B} = k^{-1}\vec{\nabla} \wedge^2 (\vec{e}\psi) + \vec{\nabla} \wedge (\vec{e}\psi) \tag{73}$$

where $\vec{e}$ is an arbitrary unit vector and ψ is a solution of the Helmholtz equation:

$$\Delta\psi + k^2\psi = 0. \tag{74}$$

The configuration (71) corresponds to the choice $\vec{e} = (1, 0, 0)$ and $\psi = \sin(kz)$. Therefore, a tracking solution, found in [3],

$$\begin{cases} \dot{\theta}_5 = \mu_5 = \frac{\alpha}{\pi}k \\ \vec{E} = 0 \\ \vec{B} = B_0(\sin(kz), \cos(kz), 0) \end{cases}, \tag{75}$$

is also a solution of the inhomogeneous Maxwell equations (70) for any value of the constant B_0. However, if one starts with nonzero $\vec{E}(0)$ *or* with $\mu_5 \neq \frac{\alpha}{\pi}k$ (or if the initial conditions are not monochromatic, see the next subsection) the value B_∞, to which the solution $B(t)$ arrives at late times, will depends nontrivially on the initial conditions as we demonstrate below.

By the Bianchi identity, the electric field corresponding to the configuration (71) is given by

$$\vec{E} = -\frac{1}{k}\dot{B}(t)(\sin(kz), \cos(kz), 0), \tag{76}$$

hence

$$\vec{E}(x)\cdot\vec{B}(x) = -\frac{1}{2k}\frac{\partial}{\partial t}B^2(t) \quad \text{is space-independent.} \tag{77}$$

Next, we plug this ansatz into the system of Eqs. (70). This yields

$$\Lambda^2 \frac{\partial \mu_5}{\partial t} = -\frac{\alpha}{2\pi k} \frac{\partial B^2(t)}{\partial t} \tag{78}$$

or

$$\mu_5(t) = \mu_5^0 - \frac{\alpha}{2\pi} \frac{B^2(t) - B_0^2}{k\Lambda^2} \tag{79}$$

where $B_0 = B(0)$ and $\mu_5^0 = \mu_5(0)$. This is the exact solution for $\mu_5(t)$, for all times t, derived from the one-mode ansatz (71). Such a solution has been first described in [3], and we see here that the local nature of the axial anomaly equation (70) does not invalidate the conclusion in [3].

Equations (70) allow us to determine $B(t)$ and its asymptotic behavior for large t. This is seen by plugging expression (79) for $\mu_5(t)$ into the Maxwell equations, which then yield a closed system of equations for $\vec{B}$:

$$\vec{\nabla} \wedge \vec{B} = \sigma \vec{E} + \frac{\alpha}{\pi} \left(\mu_5^0 - \frac{\alpha}{\pi} \frac{B^2(t) - B_0^2}{2k\Lambda^2} \right) \vec{B}. \tag{80}$$

Using Eq. (76), and recalling that $\vec{\nabla} \wedge \vec{B} = k\vec{B}$, for $\vec{B}$ as in (71), we arrive at a nonlinear ordinary differential equation (ODE) for one scalar function $B(t)$:

$$kB(t) = -\frac{\sigma}{k} \dot{B}(t) + \frac{\alpha}{\pi} \left(\mu_5^0 - \frac{\alpha}{\pi} \frac{B^2(t) - B_0^2}{2k\Lambda^2} \right) B(t) \tag{81}$$

This equation can be integrated explicitly and yields $B(t)$ as a function of B_0 and μ_5^0 and we can write down the explicit time-dependent solution of the system (70):

$$\begin{cases} B(t) = \dfrac{C_1}{\sqrt{1 + C_2 \exp(\frac{2(k^2-\gamma^2)}{\sigma} t)}}, \\ \mu_5(t) = \mu_5^0 - \frac{\alpha}{2\pi} \frac{B^2(t) - B_0^2}{k\Lambda^2} \end{cases} \tag{82}$$

where we have introduced a parameter γ given by

$$\gamma^2 \coloneqq \frac{\alpha}{\pi} k\mu_5^0 + \frac{\beta_0^2}{2}, \quad \text{with} \quad \beta_0^2 \coloneqq \frac{\alpha^2 B_0^2}{\pi^2 \Lambda^2}. \tag{83}$$

The constants C_1, C_2 can be expressed in terms of B_0, μ_5^0, and k, we will not provide their explicit form.

Next, we analyze the asymptotic behavior of $B(t)$ as $t \to \infty$. To this end we put $\dot{B}(t) = 0$ and find an algebraic equation for $B_\infty \equiv B(t \to \infty)$:

$$kB_\infty = \frac{\alpha}{\pi} \left(\mu_5^0 - \frac{\alpha}{\pi} \frac{B_\infty^2 - B_0^2}{2k\Lambda^2} \right) B_\infty. \tag{84}$$

There are two distinct solutions of Eq. (84). The trivial one, i.e., $B_\infty = 0$, is approached if

$$\text{trivial solution} \Leftrightarrow k^2 > \gamma^2 \quad \text{or}$$
$$k > \frac{\alpha\mu_5^0}{2\pi} + \sqrt{\frac{\beta_0^2}{2} + \left(\frac{\alpha\mu_5^0}{2\pi}\right)^2}. \tag{85}$$

Notice that although $B_\infty = 0$, the $\mu_5(t \to \infty)$ can still be nontrivial. In the case of zero magnetic field, the asymptotic form of the $\mu_5(t)$ is not bound to be equal to $\frac{\pi k}{\alpha}$ and its value is determined by Eq. (79).

The nontrivial solution of Eq. (86) exists if $k < \gamma$. It is given by

$$B_\infty^2 = B_0^2 \left[1 + \left(\frac{2k^2}{\beta_0^2} \right) \left(\frac{\mu_5^0}{\mu_5^\infty} - 1 \right) \right],$$
$$\text{where } \mu_5^\infty \equiv \mu_5(t \to \infty) = \frac{\pi k}{\alpha}, \tag{86}$$

The necessary condition for the nontrivial solution (86) to exist, i.e. $B_\infty^2 > 0$, is given by

$$\left[1 + \left(\frac{2k^2}{\beta_0^2} \right) \left(\frac{\mu_5^0}{\mu_5^\infty} - 1 \right) \right] > 0 \tag{87}$$

It is exactly opposite to the condition (85), as it should be.

In summary, we were able to find an exact solution of the nonlinear system of differential equations (70) and found condition (85) [or equivalently (87)] that relates the initial value of $\mu_5(t = 0)$, $B(t = 0)$ and the wave number k and determines the ultimate fate of the single-mode solution.

Notice that if, initially, $\mu_5^0 = \frac{\pi}{\alpha} k = \mu_5^\infty$ then the solution is stationary, i.e., $B_0 = B_\infty$. If, however, $\frac{\alpha}{\pi}\mu_5^0 \neq k$ then $b(t)$ will increase or decrease and hence cause $\mu_5(t)$ to change in time, as is seen from Eq. (79). Indeed, if $\frac{\alpha}{\pi}\mu_5^0 > k$ then $b(t) > 0$, and the amplitude of the $\vec{B}$ field grows, while, according to (79), $\mu_5(t)$ decreases—*as* it should. But if $\frac{\alpha}{\pi}\mu_5^0 < k$ then $b(t) < 0$, i.e., the amplitude of the $\vec{B}$ field decreases, and, according to (79), $\mu_5(t)$ increases towards μ_5^∞. To demonstrate this, we consider the behavior of the solution for small times when $B(t) \approx B_0$ and $\mu_5(t) \approx \mu_5^0$. To this end we linearize the system of Eq. (81) around some special solutions. We set $B(t) = B_0(1 + \epsilon b(t))$ where $b(0) = 0$. To first order in ϵ we obtain the following equation for $b(t)$:

$$\left(\frac{\alpha}{\pi}\mu_5^0 - k \right) = \left[\left(\frac{\alpha}{\pi}\mu_5^0 - k \right) b(t) - \frac{\beta_0^2}{k} b(t) - \frac{\sigma}{k} b'(t) \right] + \mathcal{O}(\epsilon). \tag{88}$$

The solution of Eq. (88) is given by

$$b(t) = \frac{1 - \exp(\frac{t(\Delta\mu k - \frac{\beta_0^2}{2})}{\sigma})}{1 - \frac{\beta_0^2}{2k\Delta\mu}},$$
$$\text{where } \Delta\mu = \frac{\alpha\mu_5^0}{\pi} - k. \tag{89}$$

and it demonstrates the conclusion.

B. Stability analysis of the helical single-mode solution

In this section we sketch an analysis of stability of the solution found in the previous section. To this end, we assume that the initial configuration has reached its time-independent form with some k, B_∞ and μ_5^∞. We then perturb the vector potential $\vec{A}$ and the chiral chemical potential μ_5 by

$$\mu_5(\vec{x},t) = \frac{\pi k}{\alpha} + \delta\mu_5(\vec{x},t)$$
$$\vec{A}(\vec{x},t) = \frac{\vec{B}_0(z)}{k} + \vec{a}(\vec{x},t), \tag{90}$$

where $\vec{B}_0(z)$ has the form (71), with amplitude $B(t) \equiv B_\infty$ given by Eq. (86). In (90), $\mu_5(\vec{x},t) \equiv \mu_5^\infty$ if $\delta\mu_5(\vec{x},t) \equiv 0$; see (86).

The (equation expressing the) chiral anomaly [see Eq. (70)], linearized around the background $\mu_5(x) \equiv \alpha^{-1}\pi k$ and $\vec{A}(x) \equiv k^{-1}\vec{B}_0(z)$, then yields

$$\Lambda^2 \frac{\partial \delta\mu_5}{\partial t} = -\frac{2\alpha}{\pi}\vec{B}_0(z)\cdot\frac{\partial\vec{a}}{\partial t}, \tag{91}$$

which can be integrated to

$$\delta\mu_5(\vec{x},t) = -\frac{2\alpha}{\pi}\frac{1}{\Lambda^2}\vec{B}_0(z)\cdot(\vec{a}(\vec{x},t) - \vec{a}_0(\vec{x})) + \delta\mu_5(\vec{x},0) \tag{92}$$

where $\vec{a}_0(\vec{x}) \coloneqq \vec{a}(\vec{x},t=0)$. Below, we choose as the initial condition $\delta\mu_5(\vec{x},t=0) = 0$. This is obviously a special choice. However, physically, it is the most interesting one, because the spatial inhomogeneity of the chiral chemical potential is then caused by fluctuations of the magnetic field. The analysis of a more general situation will follow. Using (92), the equation of motion for $\vec{a}$ is seen to be given by

$$-\Delta\vec{a} = -\sigma\frac{\partial\vec{a}}{\partial t} + k\vec{\nabla}\wedge(\vec{a})$$
$$-\frac{2\alpha^2}{\pi^2}\frac{1}{\Lambda^2}(\vec{B}_0(z)\cdot\vec{a} - \vec{B}_0(z)\cdot\vec{a}_0)\vec{B}_0(z). \tag{93}$$

We analyze the special solution found by assuming that $\vec{a}$ depends only on the spatial coordinate z and on time t; i.e., $\vec{a}(\vec{x},t) = (a_x(z,t), a_y(z,t), 0)$. The equations then reduce to

$$a_x'' = \sigma\dot{a}_x + ka_y' + 2\beta^2(\sin^2(kz)(a_x - a_x^0) + \sin(kz)\cos(kz)(a_y - a_y^0)),$$
$$a_y'' = \sigma\dot{a}_y - ka_x' + 2\beta^2(\sin(kz)\cos(kz)(a_x - a_x^0) + \cos^2(kz)(a_y - a_y^0)), \tag{94}$$

where β^2 is given by

$$\beta^2 \equiv \frac{\alpha^2 B_\infty^2}{\pi^2\Lambda^2}. \tag{95}$$

(similar to the definition of β_0 in (83), but with B_0 replaced with B_∞).

We rewrite the system of equations (94) as a matrix equation,

$$\underbrace{\begin{pmatrix} \sigma\partial_t + \beta^2 - \partial_z^2 & k\partial_z \\ -k\partial_z & \sigma\partial_t + \beta^2 - \partial_z^2 \end{pmatrix}}_{\equiv\hat{L}}\begin{pmatrix} a_x \\ a_y \end{pmatrix} = -\beta^2\underbrace{\begin{pmatrix} -\cos(2kz) & \sin(2kz) \\ \sin(2kz) & \cos(2kz) \end{pmatrix}}_{\equiv O(z)}\begin{pmatrix} a_x - a_x^0 \\ a_y - a_y^0 \end{pmatrix}, \tag{96}$$

or, schematically, as $\hat{L}\,\vec{a} = -\beta^2 O(z)(\vec{a} - \vec{a}_0)$. Defining U by

$$U = \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \tag{97}$$

we find that

$$U^{-1}\hat{L}U = \begin{pmatrix} \sigma\partial_t + \beta^2 - \partial_z^2 - ik\partial_z & 0 \\ 0 & \sigma\partial_t + \beta^2 - \partial_z^2 + ik\partial_z \end{pmatrix} = \begin{pmatrix} \hat{L}_+ & 0 \\ 0 & \hat{L}_- \end{pmatrix}, \tag{98}$$

and

$$U^{-1}\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a_x + ia_y \\ a_x - ia_y \end{pmatrix} = \begin{pmatrix} a_+ \\ a_- \end{pmatrix}. \tag{99}$$

Furthermore,

$$U^{-1}O(z)U = \begin{pmatrix} 0 & -e^{-2ikz} \\ -e^{2ikz} & 0 \end{pmatrix}. \tag{100}$$

We then have that

$$\hat{L}_+ a_+ = \beta^2 e^{-2ikz}(a_- - a_-^0),$$
$$\hat{L}_- a_- = \beta^2 e^{+2ikz}(a_+ - a_+^0), \tag{101}$$

and we choose as an initial condition a plane wave: $a_-^0 = e^{iqz}$, $a_+^0 = e^{-iqz}$, hence $\vec{\nabla}\wedge\vec{a}_0 = q\vec{a}_0$. One may attempt to solve Eqs. (101) by iteration. In zeroth approximation, we set the right side of Eqs. (101) to zero and get

$$\hat{L}_+ a_+^{(0)} = 0; \qquad a_+^{(0)}(0,z) = e^{-iqz},$$
$$\hat{L}_- a_-^{(0)} = 0; \qquad a_-^{(0)}(0,z) = e^{iqz}. \tag{102}$$

These equations are solved by

$$a_+^{(0)} = e^{-iqz}e^{\lambda t}, \qquad a_-^{(0)}0 = e^{iqz}e^{\lambda t},$$
$$\lambda = \frac{kq - q^2 - \beta^2}{\sigma}. \tag{103}$$

The condition that perturbation grows in time, i.e. $\lambda > 0$, is translated into

$$\lambda > 0 \Leftrightarrow \frac{k - \sqrt{k^2 - 4\beta^2}}{2} < q < \frac{k + \sqrt{k^2 - 4\beta^2}}{2} \tag{104}$$

That is, for $\beta^2 < k^2/4$ and q obeying the inequalities in (104), a perturbation with wave number q grows in time; (i.e., the single-mode solution of Sec. IV is unstable). We stress that, compared to the results of Ref. [3], not every mode, longer than $2\pi/k$, can grow. The difference is given by the presence of β^2 term, which is a nonlinear contribution from the background field, B_∞. For $\beta^2 > k^2/4$, the solution of the previous Sec. IV A is stable with respect to the perturbation in the form.

To point out an important difference between the stability analysis of Eqs. (70) and the results of [3], we iterate Eqs. (101) again, yielding

$$\hat{L}_+ a_+^{(1)} = \beta^2 e^{-i(2k-q)z}(e^{\lambda t} - 1); \qquad a_+^{(1)}(0,z) = 0,$$
$$\hat{L}_- a_-^{(1)} = \beta^2 e^{i(2k-q)z}(e^{\lambda t} - 1); \qquad a_-^{(1)}(0,z) = 0, \tag{105}$$

which is solved by (we only write the explicit expression for $a_+^{(1)}$, a_- being its complex conjugate)

$$a_+^{(1)} = \frac{\beta^2 e^{-i(2k-q)z}}{2k(k-q)}\left[(e^{\lambda t} - 1) + \frac{\lambda}{\lambda_1}(e^{-\lambda_1 t} - 1)\right],$$
$$\lambda_1 = \frac{2k^2 - 3kq + q^2 + \beta^2}{\sigma} > 0, \tag{106}$$

for $q < k$. We see that $a_+^{(1)}$ is a wave with inverse wavelength $2k - q$. If $q < k$ (i.e., at $t = 0$, the perturbation has a longer wavelength than the original single-mode solution) then $2k - q > k$, i.e., a wave with shorter wavelength is excited. This result is a consequence of the nonlinear nature of Eqs. (70) and the coupling between the modes (i.e. the generation of mode $2k - q$ starting from two modes with the wave number k and q) is expected.

Below we show that the coefficient in front of the $e^{\lambda t}$ term in $a_+^{(1)}$ is always smaller than the term in front of the e^{-iqz} harmonic. The ratio of coefficients can be found to be (we take times large enough so that $\lambda_1 t \gg 1$ and $\lambda t \gg 1$)

$$\left.\frac{|a_+^{(1)}(t,z)|}{|a_+^{(0)}(t,z)|}\right|_{\lambda_1 t \gg 1; \lambda t \gg 1} = \frac{\beta^2}{2k(k-q)} \leq \frac{q}{2k} < \frac{1}{2} \tag{107}$$

(in view of the condition that $\beta^2 < kq - q^2$ that follows from $\lambda > 0$, Eq. (104) and considering only modes with $q < k$). As a result at most $\frac{1}{4}$ of the energy could be transformed into the short wavelength mode.

Summary: Starting from the simplified system of equations of chiral electrodynamics, see Eqs. (70), we have shown that the helical single-mode solution is stable for sufficiently strong magnetic fields. For weak magnetic fields, the long-wavelength perturbation grows in time. A short-wavelength mode also gets excited, but its amplitude is always parametrically smaller than the one of the long-wavelength mode. Thus, transfer of energy and helicity from a single helical field mode to a long-wavelength perturbation, as described in [3], is observed for solutions of the full system of equations with an inhomogeneous axial chemical potential.

V. CONCLUSION AND OUTLOOK

In a relativistic plasma of charged particles, the axial anomaly is accompanied by the appearance of new degrees of freedom described by a pseudoscalar field $\theta_5(x,t)$. This field has the property that its time derivative is equal to the axial chemical potential μ_5 parameterizing states of local equilibrium of the plasma. When coupled to the electromagnetic field and to the motion of the fluid the field θ_5 appears in a system of equations that we call the "equations of chiral magnetohydrodynamics" [see Eqs. (47) through (54)]. If θ_5 is spatially uniform and depends only on time $\dot{\theta}_5$ is nothing but the axial chemical potential μ_5 of the light charged fermions. In this paper, we have studied states of the plasma corresponding to *local* equilibrium, with θ_5 depending not only on time t, but also on $\vec{x}$. Our analysis is significant, because the presence of a nonzero (homogeneous) axial chemical potential μ_5 leads to an instability of some solutions of the Maxwell equations accompanied by the generation of helical magnetic fields [1,2,5]. This instability leads to the appearance of strong electromagnetic fields that will, in turn, backreact on μ_5, making it spatially inhomogeneous.

We have analyzed special solutions of the system of chiral MHD equations and have shown that the qualitative conclusions reached in [3] remain valid. In particular, we ave shown that there is a stationary solution (helical single-mode solution) of our system of nonlinear equations with the property that the growth caused by a nonvanishing axial chemical potential exactly compensates the Ohmic dissipation (subsection IV A). Another important property of solutions in the presence of a homogeneous axial chemical potential is the "inverse cascade" phenomenon, i.e., the transfer of energy and magnetic helicity from short to large scales, described in subsection IV B. This transfer is not caused by turbulence in the plasma, but is an effect derived from the chiral anomaly and caused by the chiral imbalance in the plasma. We have shown that, under suitable conditions, this phenomenon appears to persist for spatially inhomogeneous fields.

Our analysis can be expected to be relevant for the study of various physical systems exhibiting a chiral (left-right) imbalance, including the quark-gluon plasma and the plasma in the early Universe. In an analysis of the plasma in the early Universe, it might not be legitimate to neglect the coupling of the axion field θ_5 to the space-time curvature, (namely to the term $\frac{1}{2}\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu\sigma\tau}R^{\sigma\tau}_{\alpha\beta}$, where $R_{\mu\nu\sigma\tau}$ is the Riemann tensor). We plan to return to this topic elsewhere.

Many elements of our analysis remain unchanged when the axion is treated as a fundamental field with relativistic Hamiltonian dynamics. As discussed in Sec. IV, it can induce a left-right asymmetry in the distribution of electrically charged fermions and trigger the growth of magnetic fields.

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APPENDIX A: ELECTRODYNAMICS ON A SLAB IN FIVE-DIMENSIONAL MINKOWSKI SPACE

The connection between the chiral chemical potential μ_5 and the axion field θ_5 becomes manifest if one studies electrodynamics on a slab $\mathbb{R}^{3,1}\times\mathcal{I}$ of five-dimensional Minkowski space, where $\mathcal{I}$ is a finite interval extending into the fifth dimension (cf. [1,2,78]). Here we briefly review some important elements of this story. We consider very heavy charged four-component Dirac fermions populating the five-dimensional "bulk." After coupling these fermions to an external electromagnetic five-vector potential A and, subsequently, integrating them out, we obtain a low-energy bulk effective action for the gauge field A and a boundary effective action for boundary degrees of freedom, which turn out to be massless chiral fermions localized on the $(3+1)$-dimensional "top" and "bottom" boundary components of the slab (see Fig. 2).[4] The effective action is given by

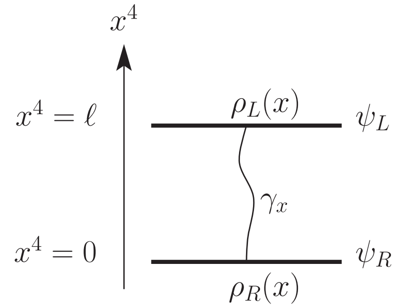

FIG. 2. Five-dimensional geometry. Light chiral fermions are localized on the boundaries of the five-dimensional slab ($x^4=0$ and $x^4=\ell$).

$$S^{(5)}=\int d^5x\left[-\frac{1}{4\ell}F^2_{ab}+\frac{\alpha}{32\pi}\epsilon^{abcde}A_aF_{bc}F_{de}\right]$$
$$+\text{ boundary action for chiral fermions}$$
$$+\text{ higher-order nonlocal terms,} \tag{A1}$$

where ℓ is the length of the interval $\mathcal{I}$, α is the four-dimensional fine-structure constant, and F_{ab} is the five-dimensional field strength ($a, b = 0, 1, 2, 3, 4$). Fermions of opposite chirality are located on opposite boundary components, i.e., at $x^4=0$ and $x^4=\ell$, respectively. We turn on an electric field pointing into the fifth dimension, i.e., $F_{0a}=0$, for $a=1, 2, 3$, and $F_{04}=\text{const}$. Then there are nonzero charge densities, $\rho_L(x)$ and $\rho_R(x)$, of chiral fermions located on opposite boundary components of the slab. These left- and right-chiral fermions couple to the electromagnetic gauge field restricted to the boundary hyperplanes. They give rise to anomalous currents on both boundary hyperplanes. The action $S^{(5)}$ in (A1) must, therefore, contain the Chern-Simons term that cancels the gauge anomalies of the chiral fermions on the two boundary components. The bulk electric current corresponding to the action $S^{(5)}$ is given by

$$j^a\coloneqq\frac{\delta S^{(5)}}{\delta A_a}.$$

It has a contribution corresponding to the Chern-Simons term,

$$j^a_{\text{CS}}=\frac{\alpha}{8\pi}\epsilon^{abcde}F_{bc}F_{de} \tag{A2}$$

Note that j^a_{CS} is divergence-free in the five-dimensional bulk. Its divergence does, however, *not* vanish on the boundaries, where it is proportional to $\pm\frac{\alpha}{8\pi}\epsilon^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho}$, respectively, and cancels the divergence of the anomalous currents of the massless chiral fermions located on the boundary components of the slab—a phenomenon known as anomaly inflow.

In order to reveal the connection between this model of five-dimensional electrodynamics and axion electrodynamics, we study the dimensional reduction of the

[4] These chiral fermions may acquire a mass through tunneling between the two boundary components, which shows that they may be used for purposes of reasonably realistic model building. We will ignore this possibility in the following.

five-dimensional theory. We define a scalar field depending only on the coordinates $x = (x^0 = t, x^1, x^2, x^3)$ of a space-time point $X \in \mathbb{R}^{3,1} \times \mathcal{I}$ by setting

$$\theta_5(x) = \int_\gamma dx^4 A_4(X), \tag{A3}$$

where γ is a straight path in the fifth direction connecting the two points $X_l = (x, x^4 = 0)$ and $X_u = (x, x^4 = \ell)$ located on opposite boundary components; see Fig. 2. We assume that the components $A_\mu, \mu = 0, 1, 2, 3$ of the gauge field A are independent of the x^4 coordinate (or are averaged over x^4). By Eq. (A3),

$$\dot\theta_5 = -\int_\gamma dx^4 \dot A_4 = \int_\gamma dx^4 E_4. \tag{A4}$$

The right side of this equation is the voltage drop between the two boundary components, which is nothing but the difference of the chemical potentials of left- and right-chiral fermions. The dimensional reduction of the action (A1) yields exactly the effective action given in Eq. (68) + a four-dimensional Maxwell term. This implies that the axion-photon coupling constant can be identified with the size of the interval $\mathcal{I}$ extending into the fifth dimension. Comparing the equations derived in this appendix with Eqs. (5) and (6) of Sec. I, we observe that the temperature T can be identified with the inverse of the width ℓ of the slab.

The theory outlined here can be viewed as a five-dimensional cousin of the quantum Hall effect.

APPENDIX B: LINEAR ANALYSIS

1. Growth of the long-wavelength modes

Let us now perturb the stationary solution for the one-mode (80) by a small long-wavelength mode $\vec B_1$:

$$\vec B_1(z,t) = \epsilon B_0 b_1(t)(\sin(k_1 z), \cos(k_1 z), 0). \tag{B1}$$

The linearized equation does not have the simple form Eq. (88) anymore. However the property (77) still holds for the mode $\vec B_1$ separately. What is not true, however, is that the overall $\mu_5(t)$ is now space independent. It is governed by the following equation (where again we introduced $B(t) = B_0(1 + \epsilon b(t))$

$$f_5^2 \dot\mu_5 = \frac{2\alpha}{\pi}\left[\frac{\partial}{\partial t}\frac{B^2(t)}{2k} + \epsilon^2 \frac{\partial}{\partial t}\frac{b_1^2(t)}{2k_1} + \epsilon\left(\frac{B(t)\dot b_1(t)}{2k_1} + \frac{b_1(t)\dot B(t)}{2k}\right)\cos((k-k_1)z)\right]. \tag{B2}$$

To the first order in ϵ Eq. (B2) has the following form

$$f_5^2 \frac{\partial \mu_5}{\partial t} = -\epsilon \frac{2\alpha}{\pi} B_0 \frac{\partial}{\partial t}\left[\frac{b(t)}{k} + \frac{b_1(t)}{k_1}\cos((k-k_1)z)\right]. \tag{B3}$$

We can integrate Eq. (B3) over time to get

$$\boxed{\mu_5(t) = \mu_5^0 - \epsilon \frac{2\alpha}{\pi}\frac{B_0}{f_5^2}\left[\frac{b(t)}{k} + \frac{b_1(t)}{k_1}\cos((k-k_1)z)\right]}. \tag{B4}$$

It is important to notice that in this order in ϵ,

$$\nabla\theta_5 \propto (0,0,1), \tag{B5}$$

points in the z direction, and therefore $\vec B \cdot \nabla\theta_5$ is indeed equal to zero.

If μ_5 is homogeneous in space [i.e. if we neglect the last term in Eq. (B4)] we end us with the usual two-mode equation:

$$\begin{aligned} kB(t) &= -\frac{\sigma}{k}\dot B(t) + \frac{\alpha}{\pi}\mu_5(t)B(t) \\ k_1 b_1(t) &= -\frac{\sigma}{k_1}\dot b_1(t) + \frac{\alpha}{\pi}\mu_5(t) b_1(t). \end{aligned} \tag{B6}$$

If instead of solving Eqs. (B6) directly we would expand them in ϵ (in view of the subsequent nonhomogeneous case) we would get the following.

We start with the $\boxed{\mu_5^0 = \frac{\pi k}{\alpha}}$ (i.e. the one that would make a single mode stationary). We then see that the correction to this mode obeys an ODE [compare the last two terms in Eq. (88)],

$$\sigma b'(t) = -\frac{\alpha^2}{\pi^2}\frac{B_0^2}{f_5^2} b(t). \tag{B7}$$

This equation has the solution $b(t) = 0$ if one starts from $b(0) = 0$. This probably means that $B(t)$ does not change in the linear order in ϵ. In the first order in ϵ the equation for $b_1(t)$ has the form expected from (B6):

$$\sigma b_1'(t) = (kk_1 - k_1^2) b_1(t), \tag{B8}$$

which gives exactly the exponential solution with $\mu_5 = \frac{\pi k}{\alpha}$.

Qualitatively one has constant amplitude for the short wavelength solution (the nontrivial evolution appears at the order ϵ^2). The growth of the mode $b_1(t)$ starts immediately and goes on until the neglected terms become important.

2. Two-mode inhomogeneous solution

Let us now turn to the case of the two-mode inhomogeneous solution. The expression for μ_5 is given by

Eq. (B4) and the following equations for the different modes appear:

The mode with the wave vector k (short wavelength):

$$\sigma b'(t) = -\frac{\alpha^2}{\pi^2}\frac{B_0^2}{f_5^2}b(t) \quad \text{[identical to (118)]}. \tag{B9}$$

The mode with the wave vector k_1 (long wavelength):

$$\sigma b_1'(t) = (kk_1 - k_1^2)b_1(t) + \frac{\alpha^2 B_0^2}{2\pi^2 f_5^2}(b_1(t) - b_1(0)). \tag{B10}$$

Due to the nonlinearity of the equations, we also have the very short wavelength mode $(2k - k_1)$ excited. If we assume that its initial amplitude is zero, we have

$$\sigma b_2'(t) = \frac{\alpha^2 B_0^2(2k - k_1)}{2\pi^2 k_1 f_5^2}(b_1(t) - b_1(0)), \tag{B11}$$

a term similar to the last term in Eq. (B10) but with an additional "enhancement" $k/k_1 \gg 1$. However, this term is sourced by $(b_1(t) - b_1(0))$ and will not get excited until the mode b_1 has grown significantly.

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