RAPID COMMUNICATIONS

PHYSICAL REVIEW B **87**, 041105(R) (2013)

Holographic symmetries and generalized order parameters for topological matter

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(Received 2 November 2012; revised manuscript received 12 January 2013; published 28 January 2013)

We introduce a universally applicable method, based on the bond-algebraic theory of dualities, to search for generalized order parameters in disparate systems including non-Landau systems with topological order. A key notion that we advance is that of *holographic symmetry*. It reflects situations wherein global (or bulk) symmetries become, under a duality mapping, symmetries that act solely on the system's boundary. Holographic symmetries are naturally related to edge modes and localization. The utility of our approach is illustrated by systematically deriving generalized order parameters for pure and matter-coupled Abelian gauge theories, and for some models of topological matter.

DOI: 10.1103/PhysRevB.87.041105 PACS number(s): 05.30.Rt, 11.15.Ha, 75.10.Jm, 75.10.Kt

Introduction. Landau's concept of an order parameter (OP) and spontaneous symmetry breaking are central in physics.[1] In systems with long-range Landau orders, two-point correlation functions of an OP field $\mathcal{O}(\boldsymbol{r})$, in their large-distance limit, tend to a finite (i.e., nonzero) value, $\lim_{|\boldsymbol{r}-\boldsymbol{r}'|\to\infty}\lim_{N^d\to\infty}\langle\mathcal{O}(\boldsymbol{r})\mathcal{O}^\dagger(\boldsymbol{r}')\rangle\neq 0$, where N is the linear size of the d-dimensional system, and $\mathcal{O}(\boldsymbol{r})$ is *local* in the (spatial) variable $\boldsymbol{r}$. It is in Landau's spirit to use the OP as a macroscopic variable characterizing the ordered phase *and* as an indicator of a possible phase transition (classical or quantum) to a disordered state where the OP becomes zero.

There is much experience, including systematic methods,[2,3] for deriving Landau OPs and their effective-field theories.[1] Landau's ideas of a (*local*) OP cannot be extended to topological states of matter because, by definition,[4,5] these lie beyond Landau's paradigm. However, the notion of long-range order or the design of a *witness correlator* (i.e., a correlator discerning the existence of various phases and related transitions) can be extended to topological phases—phases that can only be meaningfully examined by *nonlocal probes*.[5] Topological orders appear in gauge theories, quantum Hall and spin liquid states (when defined as deconfined phases of emergent gauge theories[6]), including well-studied exactly solvable models.[7,8]

In this paper we demonstrate that generalized *nonlocal* OPs may diagnose topological phases of matter. Most importantly, we outline a method based on bond-algebraic duality mappings to search systematically for generalized OPs. Dualities have the striking capability of mapping Landau to topological orders and vice versa for essentially two reasons: First, dualities in general represent nonlocal transformations of elementary degrees of freedom[9] and may even perform transmutation of statistics.[10] Second, bond-algebra techniques[10–12] allow for the generation of dualities in finite- and infinite-size systems. As we will show, in systems with a boundary, dualities realize a form of *holography*[13] capable of transforming a *global* symmetry, that may be spontaneously broken, into a *boundary* symmetry. We term these distinguished boundary symmetries *holographic*. They are, under suitable further conditions, connected to *edge (boundary) states*. To illustrate the method, we derive explicitly a (nonlocal) witness correlator and a generalized OP, suited to diagnose the transition between deconfined and confined phases of matter-coupled gauge theories, undetectable by standard OPs or Wilson loops. Other examples are reported in Ref. 14.

The search for generalized order parameters. A natural mathematical *language* to describe a physical system is that for which the system's degrees of freedom couple *locally*. This simple observation is key to understanding that topological order is a property of a state(s) relative to the algebra of observables (defining the language) used to probe the system experimentally.[5] In the language in which the system is topologically ordered, it is also robust (at zero temperature[15]) against perturbations local in that language. Spectral properties are invariant under unitary transformations of the local Hamiltonian H governing the system: $H\mapsto UHU^\dagger$. If $UHU^\dagger$ corresponds to a sensible local theory then the unitary transformation U establishes a *duality*.[10] A duality may map a system that displays topological order to one that does not.[5] Dualities for several of Kitaev's models[7,8,16] epitomize this idea .[5,12,15]

Since dualities are unitary transformations (or, more generally, partial isometries)[10] they cannot in general change a phase diagram, only its interpretation. This leads to a central point of our work: *A duality mapping a Landau to a topologically ordered system must map the Landau OP to a generalized OP characterizing the topological order.* Our method for searching for generalized OPs combines this observation with the advantages of the bond-algebraic theory of dualities.[10] In this framework, dualities in arbitrary size (finite or infinite) systems can be systematically searched for as alternative local representations of *bond algebras* of interactions associated to a Hamiltonian H. Hence it is possible for *any* system possessing topological order to systematically search for a duality mapping it to a Landau order. When a dual Landau theory is found, the dual system's OP can be mapped back to obtain a generalized OP for the topologically ordered system. In what follows and in Ref. 14, we study various quantum gauge and topologically ordered theories, and their duals, to illustrate our ideas.

Holographic symmetries and edge states: the gauged Kitaev wire. We next illustrate the concept of holographic symmetry and its relation to generalized OPs and edge modes. Consider the Kitaev wire Hamiltonian[16] with open boundary conditions, here generalized to include a $\mathbb{Z}_2$ gauge field (termed the *gauged*

Kitaev wire),

$$H_{\mathsf{GK}} = -ih\sum_{m=1}^{N} b_m a_m - \sum_{m=1}^{N-1}\left[iJb_m\sigma^z_{(m;1)}a_{m+1} + \kappa\sigma^x_{(m;1)}\right], \tag{1}$$

where $a_m = a_m^\dagger$, $b_m = b_m^\dagger$ denote two Majorana fermions ($\{a_m,a_n\} = 2\delta_{mn} = \{b_m,b_n\}$, $\{a_m,b_n\} = 0$) placed on each site of an open chain with N sites. The Pauli matrices $\sigma^\alpha_{(m;1)}, \alpha = x,z$, placed on the links $(m;1)$ connecting sites m and $m+1$ represent a $\mathbb{Z}_2$ gauge field. For the gauged Kitaev wire, fermionic parity is obtained as the product of the local (gauge) $\mathbb{Z}_2$ symmetries $ib_1a_1\sigma^x_{(1;1)}$, $\sigma^x_{(N-1;1)}ib_Na_N$, and $\sigma^x_{(m;1)}ib_{m+1}a_{m+1}\sigma^x_{(m+1;1)}$ $(m = 1,\ldots,N-2)$. Just like the standard Kitaev wire, $H_{\mathsf{GK}}[h=0]$ has two free edge modes a_1 and b_N.

The gauged Kitaev wire holds two important dualities. It is dual to the one-dimensional $\mathbb{Z}_2$ Higgs model[17]

$$H_{\mathsf{H}} = -h\sum_{i=1}^{N}\sigma^x_i - \sum_{i=1}^{N-1}\left[J\sigma^z_i\sigma^z_{(i;1)}\sigma^z_{i+1} + \kappa\sigma^x_{(i;1)}\right], \tag{2}$$

with Pauli matrices σ^α_i placed on sites i. Moreover, the gauge-reducing[10] duality mapping Φ_{d}

$$\begin{aligned} ib_m a_m &\xrightarrow{\Phi_{\mathsf{d}}} \sigma^z_m\sigma^z_{m+1}, \quad m = 1,\ldots,N,\\ ib_m\sigma^z_{(m;1)}a_{m+1} &\xrightarrow{\Phi_{\mathsf{d}}} \sigma^x_{m+1}, \quad m = 1,\ldots,N-1,\\ \sigma^x_{(m;1)} &\xrightarrow{\Phi_{\mathsf{d}}} \sigma^z_{m+1}, \quad m = 1,\ldots,N-1,\end{aligned} \tag{3}$$

transforms H_{GK} into a spin-$\frac{1}{2}$ system

$$H^D_{\mathsf{GK}} = -h\sum_{m=1}^{N}\sigma^z_m\sigma^z_{m+1} - \sum_{m=2}^{N}\left[J\sigma^x_m + \kappa\sigma^z_m\right]. \tag{4}$$

defined on $(N+1)$ sites. The fermionic parity P maps to a holographic symmetry under this duality, since $P = \prod_{m=1}^N ib_m a_m \xrightarrow{\Phi_{\mathsf{d}}} \sigma^z_1\sigma^z_{N+1}$, i.e., the product of two (commuting) boundary symmetries. Holography is a relational phenomenon (see Ref. 14). A duality that uncovers a holographic symmetry links a global (higher-dimensional) symmetry of a system to a boundary (lower-dimensional) symmetry of its dual. Boundary symmetries need not in general be duals of global symmetries.

What is the physical consequence of having an holographic symmetry? Consider the not uncommon situation in which the holographic symmetry is supplemented by an additional (non-commuting) boundary symmetry *in some region of the phase diagram*. By definition, holographic symmetries are boundary symmetries which are dual to global symmetries. Thus, global symmetries linking degenerate states (and properties in the broken symmetry phase) in the dual system have imprints in their holographic counterparts. Then, the many-body level degeneracy of the ground state may be ascribed to boundary effects. If the couplings are now changed, the ground-state degeneracy may get removed, together with some boundary symmetries. However, so long as the system remains in a topological phase dual to the (broken-symmetry) ordered phase, the low-energy state splitting will be *exponentially small* in the system size, so that in the thermodynamic limit ground-state degeneracy is restored.

The language providing the most local operator description of the ground-state manifold is the one realizing the edge modes, which are expected to be exponentially localized to the boundary. Thus, as long as the thermodynamic-limit degeneracy remains, a suitable local probe will detect localization on the boundary for those states. Conversely, noncommuting edge-mode operators in a gapped phase reflect the existence of low-energy many-body states with energy splittings vanishing exponentially with system size. Many-body (*zero-energy*) edge states are thus simply a natural consequence of a degenerate ground-state manifold in a gapped system. They are witnesses of an ordered (degenerate) phase described in a most local language. Note that boundary operators that commute with the Hamiltonian at special values of the coupling(s) are a necessary but not sufficient condition to realize *exact* (zero-energy) edge modes.

The duality $H_{\mathsf{H}} \to H_{\mathsf{GK}}$ maps a global symmetry of $H_{\mathsf{H}}[\kappa,h=0]$ to a boundary symmetry of $H_{\mathsf{GK}}[\kappa,h=0]$, i.e., $\sigma^x_1\cdots\sigma^x_{N-1}\sigma^y_N \to b_N$, and one boundary symmetry to another, $\sigma^z_1 \to a_1$. If we now turn on $h < J$, keeping $\kappa = 0$, the edge-mode operators a_1,b_N evolve respectively into $\Gamma_1 = \sum_{m=1}^N(-h/J)^{m-1}a_m(\prod_{s=1}^{m-1}\sigma^z_{(s;1)})$ and $\Gamma_2 = \sum_{m=1}^N(h/J)^{N-m}b_m(\prod_{s=m}^{N-1}\sigma^z_{(s;1)})$. The modes (Γ_1,Γ_2) are exponentially localized as long as the system is in the ordered gapped phase within a gauge sector.[18] The Majorana language affords a *local* boundary description of these (partly nonlocal in the Higgs language) zero-energy modes. For $h > J$, and/or $\kappa > 0$, the ground state is unique, even in the thermodynamic limit, as we learn from the phase diagram of the one-dimensional Higgs model.[17] Hence the zero-energy modes disappear together with the ground-state degeneracy. For $\kappa > 0$, they disappear despite the fact that fermionic parity remains an exact symmetry and cannot be spontaneously broken.[19] Consider now H^D_{GK} of Eq. (4). At $h = 0$, it has zero-energy edge-mode operators $\sigma^z_1,\sigma^x_1,\sigma^z_{N+1},\sigma^x_{N+1}$. For $h > 0$ and $\kappa = 0$, two of these remain unchanged, and the other two evolve into $\Sigma_1 = \sigma^x_1 + \sum_{m=1}^{N-1}(h/J)^m\sigma^y_1(\prod_{s=2}^m\sigma^x_s)\sigma^y_{m+1}$ and $\Sigma_2 = \sigma^x_{N+1} + \sum_{m=2}^{N}(h/J)^m\sigma^y_m(\prod_{s=m+1}^N\sigma^x_s)\sigma^y_{N+1}$. These behave just as their Majorana relatives, yet they are recognized as nonlocal. The Majorana language distinguishes itself as the *most local one* for zero modes.

To obtain a generalized OP for the gauged Kitaev wire, notice that $H^D_{\mathsf{GK}}[\kappa=0]$ reduces to the transverse-field Ising (TI) chain. Hence it exhibits a second-order phase transition at $J = h,\ \kappa = 0$. For $H^D_{\mathsf{GK}}[\kappa=0]$, this transition is witnessed by the Landau OP correlator $\lim_{|i-j|\to\infty}\langle\mathsf{TI}|\sigma^z_i\sigma^z_j|\mathsf{TI}\rangle$. (From now on $|\mathsf{label}\rangle$ represents the ground state of H_{label}). Our duality maps this correlator back to a generalized OP for the gauged Kitaev wire, the string correlator $\lim_{|i-j|\to\infty}\langle\mathsf{GK}|ib_ia_iib_{i+1}a_{i+1}\cdots ib_ja_j|\mathsf{GK}\rangle$.

Generalized OPs in higher-dimensional theories. We next show how to systematically *derive* generalized OPs in higher space dimensions. Our main goal is to illustrate the methodology in the challenging case of the Abelian $[U(1)]$ matter-coupled gauge (Higgs) theory. Previous works[20,21] *conjectured* generalized OPs for matter-coupled gauge theories and were numerically implemented, for instance, in Ref. 6. Unfortunately, a systematic mathematical *derivation* was missing and this is what our work is about. Our (nonlocal) witness correlator for the Higgs model turns out to be the one

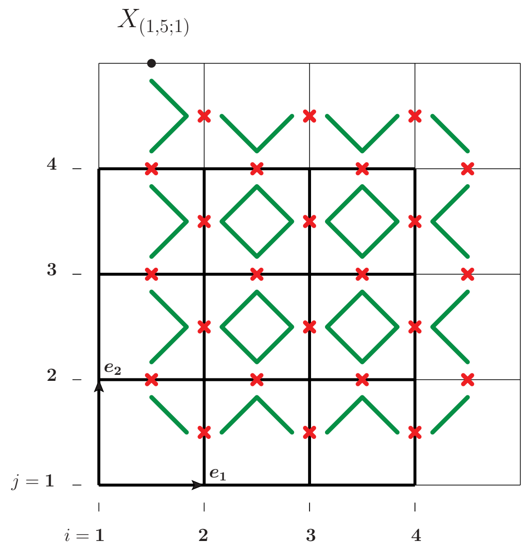

FIG. 1. (Color online) The $\mathbb{Z}$ gauge theory exactly dual to the quantum XY model must satisfy special boundary conditions and possesses a boundary symmetry. The lattice corresponding to the XY model is shown in thick lines, for $N=4$.

conjectured in Ref. 20. In Ref. 14, we study several other examples (displaying also holographic symmetries), including Ising and $\mathbb{Z}_p$ gauge and Higgs theories, the $\mathbb{Z}_p$ extended toric code[22,23] as an interesting example of topological order, and the XY model on the frustrated Kagome lattice. Non-Abelian extensions of our ideas based on Ref. 11 are currently under investigation.

To derive the generalized OP for the Abelian Higgs theory, our starting point is the XY model defined in terms of continuous $U(1)$ degrees of freedom $s_r \equiv e^{-i\theta_r}$, $\theta_r \in [0,2\pi)$, placed at sites $\boldsymbol{r} = i\boldsymbol{e_1} + j\boldsymbol{e_2} = (i,j)$ of a square lattice. The model's Hamiltonian reads (see Fig. 1)

$$H_{\mathsf{XY}} = h\sum_{i,j=1}^{N} L_{(i,j)}^2 + \frac{J}{2} \times \left[\sum_{i=1}^{N-1}\sum_{j=1}^{N} S_{(i,j;1)} + \sum_{i=1}^{N}\sum_{j=1}^{N-1} S_{(i,j;2)} + \text{H.c.}\right], \quad (5)$$

with $L_{\boldsymbol{r}} \equiv -i\partial/\partial\theta_{\boldsymbol{r}}$, and $S_{(\boldsymbol{r};\mu)} \equiv s_{\boldsymbol{r}} s^\dagger_{\boldsymbol{r}+\boldsymbol{e}_\mu}$. The XY model is dual to a $\mathbb{Z}$ (solid-on-solid–like) gauge theory also defined on a square lattice, but with degrees of freedom X and R associated to links $(\boldsymbol{r};\mu=1,2)$, see Fig. 1. (In matter-coupled gauge theories we will also have operators acting on sites $\boldsymbol{r}$.) These operators satisfy $X|m\rangle = m|m\rangle$, $R|m\rangle = |m-1\rangle$, $R^\dagger|m\rangle = |m+1\rangle$, with $m\in\mathbb{Z}$, and commute on different links (and/or sites). Then, the *exact* dual of H_{XY} for *open boundary conditions* reads

$$H_{\mathsf{ZG}} = h\sum_{i,j=1}^{N} b_{(i,j)}^2 + \frac{J}{2} \times \left[\sum_{i=2}^{N}\sum_{j=1}^{N} R_{(i,j;2)} + \sum_{i=1}^{N}\sum_{j=2}^{N} R_{(i,j;1)} + \text{H.c.}\right]. \quad (6)$$

We will call *system indices* the link indices $(i,j;\,\mu=1,2)$ labeling R operators that explicitly appear in H_{ZG}, and *extra indices* the remaining link indices. In the bulk, the plaquette operator $b_{(i,j)}$ reads

$$b_{(i,j)} \equiv X_{(i,j;1)} + X_{(i+1,j;2)} - X_{(i,j+1;1)} - X_{(i,j;2)}. \quad (7)$$

On the lattice boundary, the plaquette operators are set by two rules: (i) $b_{(1,N)} = X_{(1,N;1)} - X_{(2,N;2)} - X_{(1,N+1;1)}$. Thus, $b_{(1,N)}$ involves one degree of freedom $X_{(1,N+1;1)}$ labelled by an extra link index. (ii) The remaining boundary plaquettes are determined by Eq. (7) *provided operators labelled by extra link indices are omitted*. With these definitions in tow, the mapping of bonds

$$\begin{aligned} b_{(i,j)} &\xrightarrow{\Phi_{\mathsf{d}}} L_{(i,j)}, \quad 1\leqslant i,j\leqslant N,\\ R_{(i,j;1)} &\xrightarrow{\Phi_{\mathsf{d}}} S^\dagger_{(i,j-1;2)}, \quad 1\leqslant i\leqslant N, \quad 2\leqslant j\leqslant N, \\ R_{(i,j;2)} &\xrightarrow{\Phi_{\mathsf{d}}} S_{(i-1,j;1)}, \quad 2\leqslant i\leqslant N, \quad 1\leqslant j\leqslant N, \end{aligned} \quad (8)$$

implements the duality transformation $H_{\mathsf{ZG}} \xrightarrow{\Phi_{\mathsf{d}}} H_{\mathsf{XY}}$. Because the operators $R_{(1,N+1;1)}, R^\dagger_{(1,N+1;1)}$ *do not appear* in H_{ZG}, the operator $X_{(1,N+1;1)}$ constitutes a *boundary symmetry* of H_{ZG}. Similar to the duality between the one-dimensional theories of Eqs. (1) and (4), this is a *gauge-reducing* duality. The gauge symmetries of H_{ZG}, given by $A_{(i,j)} = R_{(i,j;1)}R_{(i,j;2)}R^\dagger_{(i-1,j;1)}R^\dagger_{(i,j-1;2)}$, $2\leqslant i,j\leqslant N$, are removed by the mapping since $A_{(i,j)} \xrightarrow{\Phi_{\mathsf{d}}} \mathbb{1}$.

In the thermodynamic ($N\to\infty$) limit, the strongly-coupled ($J\gg h$) phase of the XY model displays spontaneous symmetry breakdown of its global $U(1)$ symmetry with generator $L_{\mathsf{XY}} = \sum_{i,j=1}^{N} L_{(i,j)}$, as evinced by a nonvanishing $\langle \mathsf{XY}|s_{\boldsymbol{r}} s^\dagger_{\boldsymbol{r}'}|\mathsf{XY}\rangle$ in the limit $|\boldsymbol{r}-\boldsymbol{r}'|\to\infty$.[23] By virtue of being dual to the XY system, the gauge theory displays a nonanalyticity in its ground-state energy as h is varied and its symmetry is broken. However, the phase transition in the gauge theory cannot be characterized by a local OP. So, how can the duality connecting the two models bridge the drastic gap separating the physical interpretation of their common phase diagram? The answer lies in our notion of holography, since

$$-X_{(1,N+1;2)} = \sum_{i,j=1}^{N} b_{(i,j)} \xrightarrow{\Phi_{\mathsf{d}}} L_{\mathsf{XY}}. \quad (9)$$

Thus, the global symmetry of the XY model is holographically dual to the (local) boundary symmetry $X_{(1,N+1;2)}$ of its dual gauge theory and cannot be spontaneously broken in this dual theory.[19] This is how holographic symmetries explain the non-Landau nature of critical transitions in the $\mathbb{Z}$ gauge theory. There are no edge modes nor localization associated with this holographic symmetry as the ordered phase of the XY model is gapless.

We now derive a generalized OP for the $\mathbb{Z}$ gauge theory. Let Γ be an *oriented* path from $\boldsymbol{r}$ to $\boldsymbol{r}'$ made of directed links $\boldsymbol{l}\in\Gamma$, and we adopt the convention that $S_{\boldsymbol{l}} \equiv S_{(\boldsymbol{r};\mu)}$ if $\boldsymbol{l}$ points

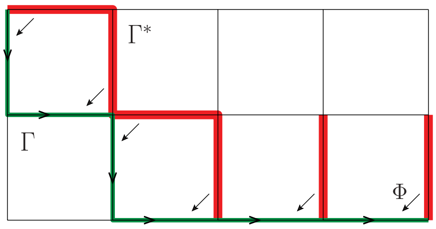

FIG. 2. (Color online) Dual sets of links Γ^* and Γ.

from $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{e}_\mu$, or $S_l \equiv S^\dagger_{(\boldsymbol{r};\mu)}$ if $\boldsymbol{l}$ points oppositely from $\boldsymbol{r}+\boldsymbol{e}_\mu$ to $\boldsymbol{r}$. Then $s_{\boldsymbol{r}}s^\dagger_{\boldsymbol{r}'} = \prod_{l\in\Gamma} S_l$. Also let Γ^* denote the set of links $\boldsymbol{l}^*$ such that $\Phi_{\mathrm{d}}(R_{l^*}) = S_l$ (Γ^* need not be continuous, as for instance in Fig. 2). Then

$$\langle \mathsf{ZG}|\prod_{l^*\in\Gamma^*} R_{l^*}|\mathsf{ZG}\rangle \xrightarrow{\Phi_{\mathrm{d}}} \langle \mathsf{XY}|s_{\boldsymbol{r}}s^\dagger_{\boldsymbol{r}'}|\mathsf{XY}\rangle, \tag{10}$$

and so *the string correlator on the left-hand side is a generalized OP for the* $\mathbb{Z}$ *gauge theory, displaying long-range order in the ordered phase*. On a *closed* path, $\prod_{l^*\in\Gamma^*} R_{l^*}$ reduces to a product of gauge symmetries.

Finally, we couple the $\mathbb{Z}$ gauge theory to a $\mathbb{Z}$ matter field (defined on sites $\boldsymbol{r}$), $H_{\mathsf{ZH}} = H_{\mathsf{ZG}} + H_{\mathsf{M}}$, with

$$H_{\mathsf{M}} = \sum_{\boldsymbol{r}}\left[\lambda(R_{\boldsymbol{r}} + R^\dagger_{\boldsymbol{r}}) + \kappa \sum_{\mu=1,2} l^2_{(\boldsymbol{r};\mu)}\right], \tag{11}$$

and $l_{(\boldsymbol{r};\mu)} \equiv X_{\boldsymbol{r}+\boldsymbol{e}_\mu} - qX_{(\boldsymbol{r};\mu)} - X_{\boldsymbol{r}}$. The resulting matter-coupled theory H_{ZH} is dual to the Abelian Higgs model[17] Hamiltonian

$$H_{\mathsf{AH}} = \sum_{\boldsymbol{r}}\left\{\lambda(B_{\boldsymbol{r}} + B^\dagger_{\boldsymbol{r}}) + hL^2_{\boldsymbol{r}} + \sum_{\mu=1,2}\left[\kappa L^2_{(\boldsymbol{r};\mu)} + \frac{J}{2}\left(S^{(q)}_{(\boldsymbol{r},\mu)} + S^{(q)\dagger}_{(\boldsymbol{r},\mu)}\right)\right]\right\}. \tag{12}$$

Here $S^{(q)}_{(\boldsymbol{r},\mu)} \equiv s_{\boldsymbol{r}} s^q_{(\boldsymbol{r};\mu)} s^\dagger_{\boldsymbol{r}+\boldsymbol{e}_\mu}$ includes a coupling with integer charge q to the $U(1)$ gauge field $s_{(\boldsymbol{r};\mu)} \equiv e^{-i\theta_{(\boldsymbol{r};\mu)}}$, $s^q_{(\boldsymbol{r};\mu)} \equiv e^{-iq\theta_{(\boldsymbol{r};\mu)}}$, and $B_{\boldsymbol{r}} \equiv s_{(\boldsymbol{r};1)} s_{(\boldsymbol{r}+\boldsymbol{e}_1;2)} s^\dagger_{(\boldsymbol{r}+\boldsymbol{e}_2;1)} s^\dagger_{(\boldsymbol{r};2)}$. The correspondence between the two models, established by the mapping of bonds

$$\begin{aligned}
&R_{\boldsymbol{r}} \xrightarrow{\Phi_{\mathrm{d}}} B^\dagger_{\boldsymbol{r}-\boldsymbol{e}_1-\boldsymbol{e}_2}, \quad b_{\boldsymbol{r}} \xrightarrow{\Phi_{\mathrm{d}}} L_{\boldsymbol{r}},\\
&R_{(\boldsymbol{r};1)} \xrightarrow{\Phi_{\mathrm{d}}} S^{(q)\dagger}_{(\boldsymbol{r}-\boldsymbol{e}_2;2)}, \quad R_{(\boldsymbol{r};2)} \xrightarrow{\Phi_{\mathrm{d}}} S^{(q)}_{(\boldsymbol{r}-\boldsymbol{e}_1,1)},\\
&l_{(\boldsymbol{r};1)} \xrightarrow{\Phi_{\mathrm{d}}} L_{(\boldsymbol{r}-\boldsymbol{e}_2;2)}, \quad l_{(\boldsymbol{r};2)} \xrightarrow{\Phi_{\mathrm{d}}} -L_{(\boldsymbol{r}-\boldsymbol{e}_1;1)},
\end{aligned} \tag{13}$$

which holds only on physical gauge-invariant states. The reason is that Φ_{d} preserves all commutation relations while "trivializing" all gauge symmetries. More precisely, H_{ZH}'s gauge symmetries $G_{\boldsymbol{r}} = R_{\boldsymbol{r}}A_{\boldsymbol{r}}$ map to $\Phi_{\mathrm{d}}(G_{\boldsymbol{r}}) = \mathbb{1}$, while H_{AH}'s gauge generators $g_{\boldsymbol{r}} = L_{(\boldsymbol{r};1)} + L_{(\boldsymbol{r};2)} - L_{(\boldsymbol{r}-\boldsymbol{e}_1;1)} - L_{(\boldsymbol{r}-\boldsymbol{e}_2;2)} - qL_{\boldsymbol{r}}$ map to $\Phi^{-1}_{\mathrm{d}}(g_{\boldsymbol{r}}) = 0$ as follows from Eqs. (13) [Φ^{-1}_{d} is the mapping obtained from Eqs. (13) by reversing all the arrows].

If the $\mathbb{Z}$ matter field is weakly coupled to the $\mathbb{Z}$ gauge field, the string correlator of Eq. (10) will still change analytic behavior across transitions. Then, from Eqs. (13),

$$\langle \mathsf{ZH}|\prod_{l^*\in\Gamma^*} R_{l^*}|\mathsf{ZH}\rangle \xrightarrow{\Phi_{\mathrm{d}}} \langle \mathsf{AH}|s_{\boldsymbol{r}}s^\dagger_{\boldsymbol{r}'}\prod_{l\in\Gamma} s^q_l|\mathsf{AH}\rangle, \tag{14}$$

we obtain a witness correlator for the Abelian Higgs model that reduces to a Wilson loop on closed contours ($\boldsymbol{r} = \boldsymbol{r}'$) (here $s^q_l = s^q_{(\boldsymbol{r};\mu)}$ if a link $\boldsymbol{l}$ points from $\boldsymbol{r}$ to $\boldsymbol{r}+\boldsymbol{e}_\mu$ and $s^q_l = s^{q\dagger}_{(\boldsymbol{r};\mu)}$ otherwise). This nonlocal correlator is directly related to intuitively motivated generalized OPs like $\langle \mathsf{AH}|s_{\boldsymbol{r}}s^\dagger_{\boldsymbol{r}'}\prod_{l\in\Gamma} s^q_l|\mathsf{AH}\rangle/\langle \mathsf{AH}|\prod_{l\in\Gamma_C} s^q_l|\mathsf{AH}\rangle$ conjectured in earlier work[6,20,21] (Γ_C denotes a closed loop roughly twice as long as Γ and containing Γ as a proper segment).

Outlook. As demonstrated, holographic symmetries and generalized OPs appear in numerous systems once boundary conditions are properly accounted for in the framework of bond-algebraic dualities. By providing a systematic methodology and many examples, our results might bring the theory of generalized OPs and topological orders to a new level of development closer to that of Landau's theory. More key problems need to be tackled. First, the sufficient conditions under which a given topological order may be mapped to a Landau order and *vice versa* should be understood. Second, the problem of associating effective field theories to generalized OPs should be studied systematically.

Acknowledgment. This work was supported by the Dutch Science Foundation NWO/FOM and an ERC Advanced Investigator grant and, in part, under Grants No. NSF PHY11-25915 and CMMT 1106293.

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