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On regularly perturbed fundamental matrices [☆]

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Abstract

In this note we characterize regular perturbations of finite state Markov chains in terms of continuity properties of its fundamental matrix. A perturbation turns out to be regular if and only if the fundamental matrix can be approximated by the discounted deviation from stationarity for small perturbation parameters. We also give bounds to assess the quality of the approximation.

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1. Introduction

In the theory of Markov chains the fundamental matrix (denoted by Z) is a key notion that is often used to establish identities for mean hitting and occupation times (see, for instance, [11] and [1]). In recent years a lot of work has been done on the analysis of perturbed Markov chains, a line of research stemming from Schweitzer's seminal paper [14]. In these works, the probability transition matrix P is replaced by P_ε , where ε is called the *perturbation parameter* and the asymptotic properties (as $\varepsilon \rightarrow 0$) of the resulting Markov chain are the subject of investigations. A recent survey of results emerging from these studies—many of which concern the limit of Z_ε , the fundamental matrix of the perturbed process as $\varepsilon \rightarrow 0$ —can be found in Avrachenkov

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et al. [2]. However, it is well known that the fundamental matrix can also be obtained as an “Abel limit” of powers of deviations of the probability transition matrix and the stationary distribution matrix of the underlying Markov chain. Here the *discount factor* α is the parameter of interest and the limit is as $\alpha \rightarrow 1$, from below.

In view of the above it is evident that the asymptotic analysis of Z_ε as $\varepsilon \rightarrow 0$, actually constitutes a study of an iterated limit of the matrix $U(\alpha, \varepsilon) = \sum_{t \geq 0} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t$, first as $\alpha \rightarrow 1$ and then as $\varepsilon \rightarrow 0$, where P_ε^* is the stationary distribution matrix of the perturbed Markov chain. This immediately raises the question of whether the latter is consistent with the reverse iterated limit of $U(\alpha, \varepsilon)$. To the best of our knowledge, this question has not been studied up to now.

The fundamental matrix of a perturbed Markov chain is also used in applications of stochastic analysis to combinatorial optimisation and control theory. For instance, in [4,7] the [1, 1]-element of the fundamental matrix of Markov chains associated with the Hamiltonian cycle problem embedded in Markov decision processes provides the global minimum precisely for the chains corresponding to Hamiltonian cycles. These Markov decision processes, in fact, constitute a family of perturbed Markov chains. The problem is that they may be periodic and in general they allow a multichain structure. Thus the question of whether the corresponding fundamental matrices can be adequately approximated by matrices of the form $U(\alpha, \varepsilon)$ becomes quite relevant. Of course, the interchangeability of the limits (with respect to α and ε) is a key requirement in this context.

Consequently, in this note we derive necessary and sufficient conditions for the above interchangeability of limits and, in the process, we characterise certain continuity properties of the fundamental matrix in terms of the perturbation parameter, for perturbed Markov chains with a possibly periodic and/or multichain structure.

Consider a time-homogenous Markov chain $\xi_t, t = 0, 1, \dots$, on a finite state space S with transition matrix P , which has entries $p_{ij} = P\{\xi_{t+1} = j \mid \xi_t = i\}$. The *Cesaro sum limit* [18]

$$P^* = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T P^{(t-1)}$$

always exists and is also called the *stationary matrix*. Here $P^{(0)} = \mathbf{I}$ and $P^{(t)}$ is the t th iterate of P . The *fundamental matrix*

$$Z = (\mathbf{I} - P + P^*)^{-1}$$

exists as well and is given by

$$Z = \lim_{\alpha \uparrow 1} \sum_{t=0}^{\infty} \alpha^t (P - P^*)^t = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^T \sum_{n=1}^t (P - P^*)^{n-1}. \tag{1.1}$$

The stationary matrix P^* is a solution to

$$P^* P = P P^* = P^* \tag{1.2}$$

and the fundamental matrix Z is a solution to

$$\begin{aligned} Z P^* &= P^* Z = P^*, \\ P Z &= Z P = Z + P^* - \mathbf{I}. \end{aligned} \tag{1.3}$$

Note that the stationary matrix is, generally, not the unique solution to (1.2). In contrast, the fundamental matrix is the unique solution to (1.3). A less restricted system suffices: suppose that the matrix A solves

$$P^*X = P^*, \tag{1.4}$$

$$PX = X + P^* - \mathbf{I}, \tag{1.5}$$

then $A = Z$. Indeed, let A and Z be two solutions of (1.4)–(1.5). Then, by (1.5), $A - Z = P(A - Z)$. Iterating this, we obtain $A - Z = P^{(t)}(A - Z)$, $t = 0, 1, 2, \dots$. Hence

$$A - Z = \frac{1}{T} \sum_{t=0}^{T-1} P^{(t)}(A - Z).$$

Taking the limit $T \rightarrow \infty$ yields that $A - Z = P^*(A - Z) = 0$ by virtue of (1.4). We will need this result later on.

Instead of the matrix P we consider a set of perturbed stochastic matrices P_ε , $\varepsilon \in (0, \varepsilon_0]$, such that P_ε is close to P for ε sufficiently small. In other words, we assume that

$$\lim_{\varepsilon \downarrow 0} P_\varepsilon = P.$$

Example 1. Take any multichain Markov chain with transition matrix P . Define $P_\varepsilon = (1 - \varepsilon)P + \varepsilon\mathbf{I}$.

Example 2. Take the same multichain Markov chain. Define \mathbf{E} by $\mathbf{E}_{ij} = 1$ for all $i, j \in S$. Define $P_\varepsilon = (1 - \varepsilon)P + \frac{\varepsilon}{\#S}\mathbf{E}$.

All quantities associated with an ε -perturbation have a subscript ε . For notational convenience, we indicate all operators associated with the original matrix by subscript 0, so that we write P_0 instead of P , and so on.

One might expect that Z_0 can be approached by $\sum_{t=0}^{\infty} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t$ for α close to 1 and ε sufficiently small. The latter expression may be computationally more tractable in some contexts. Preferably, one would also wish to have bounds on the difference between the limit and its approximation.

In order to investigate this problem, we recall the definition of a regular perturbation. A collection of ε -perturbations, $(0, \varepsilon_0]$, is called *regular*, if $\lim_{\varepsilon \downarrow 0} P_\varepsilon^* = P_0^*$. Schweitzer [14, Theorem 5] showed that $\lim_{\varepsilon \downarrow 0} P_\varepsilon^* = P_0^*$ if and only if $\lim_{\varepsilon \downarrow 0} \nu_\varepsilon = \nu_0$, where ν_ε is the number of closed classes in the ε -perturbed Markov chain. Since ν_ε has only integer values, the latter holds if and only if there exists ε_1 , $0 < \varepsilon_1 < \varepsilon_0$, such that $\nu_\varepsilon = \nu_0$ for $\varepsilon \leq \varepsilon_1$.

As a consequence, Example 1 is a regular perturbation, whereas Example 2 is not. The latter is called a *singular* perturbation.

2. Main results

As before, we define

$$U(\alpha, \varepsilon) = \sum_{t \geq 0} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t.$$

By virtue of (1.1), $\lim_{\alpha \uparrow 1} U(\alpha, \varepsilon) = Z_\varepsilon$. The posed problem now reduces to investigating the following two questions:

(i) Under what conditions

$$\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = \lim_{\varepsilon \downarrow 0} \lim_{\alpha \uparrow 1} U(\alpha, \varepsilon)? \tag{2.1}$$

(ii) Under what conditions is this limit equal to Z_0 ?

The following theorem supplies an essentially complete answer to the above questions.

Theorem 2.1. *The limit on either side of (2.1) is finite if and only if the ε -perturbation is regular. In that case we have*

$$\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = Z_0 = \lim_{\varepsilon \downarrow 0} \lim_{\alpha \uparrow 1} U(\alpha, \varepsilon).$$

While this result is not unexpected¹—to the best of our knowledge—it has not been formally stated and proved, hitherto.

The proof uses different techniques and hence it has been divided into two parts. The main part, Theorem 2.2, is based on the use of generating functions in a not entirely straightforward manner. The second part, Lemma 2.3, uses a coupling argument for showing the existence of uniform geometric bounds on the differences $\sum_j |p_{ij,\varepsilon}^{(t)} - p_{ij,\varepsilon}^*|$, for ε small and $i \in S$, in case of aperiodic perturbed Markov chains. This argument is analogous to [13, Theorem 16.2.4], but more involved since we allow a multichain structure and transient states. The periodic case finally follows from a data transformation argument due to Schweitzer [15].

Theorem 2.2.

(i) *Suppose that either limit*

$$\lim_{\varepsilon \downarrow 0} Z_\varepsilon = A$$

or

$$\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = A$$

exists for some finite $S \times S$ -matrix A . Then, $A = Z_0$ and the set of ε -perturbations is regular, that is, $\lim_{\varepsilon \downarrow 0} P_\varepsilon^ = P_0^*$.*

(ii) *If the set of ε -perturbations is regular, then $\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = Z_0$.*

Lemma 2.3. *If the set of ε -perturbations is regular, then $\lim_{\varepsilon \downarrow 0} Z_\varepsilon = \lim_{\varepsilon \downarrow 0} \lim_{\alpha \uparrow 1} U(\alpha, \varepsilon) = Z_0$.*

The tools used for proving the lemma imply an approximation result that is discussed next.

It is easy to see that (cf. Schweitzer [14, Theorem 5]) there is an enumeration of the recurrent classes R_0^n of P_0 (respectively R_ε^n of P_ε) for $0 < \varepsilon \leq \varepsilon_1$, such that $R_0^n \subset R_\varepsilon^n$, $n = 1, \dots, \nu_0$. Denote by $T_\varepsilon = S \setminus \bigcup_n R_\varepsilon^n$ the set of transient states.

Choose $b_n \in R_0^n$, $n = 1, \dots, \nu_0$, and denote $\mathfrak{B} = \{b_1, \dots, b_{\nu_0}\}$. We will call b_n a *reference state* and \mathfrak{B} a *set of reference states* (for the chain $\xi_{0,t}$ associated with P_0). Note that \mathfrak{B} is a set of reference states for all ε -perturbations, with $\varepsilon \leq \varepsilon_1$. Let d denote the period of the Markov chain associated with P_0 . One can choose ε_1 small enough, so that the period of all ε -perturbations, $\varepsilon \leq \varepsilon_1$, is at most equal to d .

Choose $0 < \delta < 1$, t_0 and ε_2 , $0 < \varepsilon_2 \leq \varepsilon_1$, such that for all $i \in S$ there exists $k \in \{0, \dots, d - 1\}$ for which

$$\sum_{b \in \mathfrak{B}} p_{ib,\varepsilon}^{(t_0-k)} \geq \delta \quad \text{for } \varepsilon \leq \varepsilon_2. \tag{2.2}$$

¹ It may, indeed, be known to some experts.

This is possible for δ small enough, that is, for $\delta < \min_{i \in S, b \in \mathfrak{B}} \pi_{ib}$.

Furthermore, let

$$\eta = \left(1 - \frac{d-1}{t_0}\right)^{t_0}, \quad c = 2 \cdot \frac{1}{(1-\eta\delta)^{1-t_0^{-1}}}, \quad \beta = (1-\eta\delta)^{1/t_0}. \tag{2.3}$$

The approximation result of this note can now be summarized in the statement of the following theorem.

Theorem 2.4. *Let a set of regular ε -perturbations, be given, such that $v_\varepsilon = v_0$, $\varepsilon \leq \varepsilon_1$. Let $0 < \delta < 1$ and t_0 be such that (2.2) holds. Choose constants c and β as in (2.3). Fix time $N > 1$ and let positive constants γ and $\varepsilon_3 \leq \varepsilon_2$ be such that*

$$\sup_i \sum_j |p_{ij,\varepsilon}^{(t)} - p_{ij,0}^{(t)}| \leq \gamma \quad \text{for } \varepsilon \leq \varepsilon_3, \quad t < N.$$

Then

$$\begin{aligned} \sum_j |U_{ij}(\alpha, \varepsilon) - Z_{ij,0}| &\leq 2cN \cdot \frac{\beta^{N-1}}{1-\beta} \left(1 + \mathbf{1}_{\{i \in T_\varepsilon\}} \frac{\delta N}{t_0}\right) + 2N\gamma \\ &\quad + c \cdot \frac{(1-\alpha)}{(1-\beta)^2} \left(1 + \mathbf{1}_{\{i \in T_\varepsilon\}} \frac{\delta}{t_0(1-\beta)^2}\right), \quad \varepsilon \leq \varepsilon_3. \end{aligned}$$

For the sake of brevity, many proofs have been omitted. A reader interested in technical derivations of the claimed results is referred to an extended version [8] containing the omitted arguments.

3. Characterizing regularity

In this section we prove the central Theorem 2.2 of this note.

Proof of Theorem 2.2. Assume that the limit $\lim_{\varepsilon \downarrow 0} Z_\varepsilon = A$ exists for some finite matrix A . We adapt an argument from Spieksma [16, p. 72]. Consider the set of matrix limit points of P_ε^* as $\varepsilon \downarrow 0$: these are also stochastic matrices. Take any limit point B and a sequence $\{\varepsilon_n\}_n \downarrow 0$ such that $P_{\varepsilon_n}^* \rightarrow B, n \rightarrow \infty$.

First note that $P_{\varepsilon_n} P_{\varepsilon_n}^* = P_{\varepsilon_n}^* P_{\varepsilon_n} = P_{\varepsilon_n}^*$, (1.2) implies

$$P_0 B = B P_0 = B, \tag{3.1}$$

by taking the limit $n \rightarrow \infty$ and using that all matrices are finite matrices. Similarly, (1.4) for ε_n implies that

$$B A = B. \tag{3.2}$$

By (1.5),

$$A = \lim_{n \rightarrow \infty} Z_{\varepsilon_n} = \lim_{n \rightarrow \infty} (P_{\varepsilon_n} Z_{\varepsilon_n} + \mathbf{I} - P_{\varepsilon_n}^*) = P_0 A + \mathbf{I} - B.$$

Hence,

$$A = \mathbf{I} - B + P_0 A, \tag{3.3}$$

whence by subtracting zP_0A from both sides, $z \in \mathbf{C}$,

$$(\mathbf{I} - zP_0)A = \mathbf{I} - B + (1 - z)P_0A. \tag{3.4}$$

For z , with $|z| < 1$, the inverse matrix $(\mathbf{I} - zP_0)^{-1}$ exists and equals

$$(\mathbf{I} - zP_0)^{-1} = \sum_{t \geq 0} z^t P_0^{(t)}.$$

Multiplying both sides of (3.4) by this inverse matrix yields

$$\begin{aligned} A &= \left[\sum_{t \geq 0} z^t P_0^{(t)} \right] (\mathbf{I} - B) + (1 - z) \left[\sum_{t \geq 0} z^t P_0^{(t)} \right] P_0A \\ &= \sum_{t \geq 0} z^t (P_0^{(t)} - B) + (1 - z) \left[\sum_{t \geq 0} z^t P_0^{(t)} \right] P_0A, \end{aligned} \tag{3.5}$$

since $P_0^{(t)}B = B$, for each $t = 0, 1, \dots$, by virtue of (3.1). Now, putting $z = x \in \mathbf{R}_+$ and taking the limit as $x \uparrow 1$ in (3.5) implies

$$A = \lim_{x \uparrow 1} \sum_{t \geq 1} x^t (P_0^{(t)} - B) + P_0^*A,$$

because

$$\lim_{x \uparrow 1} (1 - x) \left[\sum_{t \geq 0} x^t P_0^{(t)} \right] P_0A = P_0^*P_0A = P_0^*A,$$

where the first equality above follows from finiteness of the state space and the fact that $\lim_{x \uparrow 1} (1 - x) \sum_{t \geq 0} x^t P_0^{(t)} = P_0^*$ by a now classical result due to Blackwell [5]. Next, for $x < 1$,

$$\sum_{t \geq 0} x^t (P_0^{(t)} - B) = \sum_{t \geq 0} x^t (P_0^{(t)} - P_0^*) + \sum_{t \geq 0} x^t (P_0^* - B), \tag{3.6}$$

because $\sum_t x^t$ converges. Since $\lim_{x \uparrow 1} \sum_{t \geq 0} x^t (P_0^{(t)} - B)$ and $\lim_{x \uparrow 1} \sum_{t \geq 0} x^t (P_0^{(t)} - P_0^*)$ both exist as finite limits, necessarily the limit $\lim_{x \uparrow 1} \sum_{t \geq 0} x^t (P_0^* - B)$ exists. However, $P_0^* - B$ is independent of t , and so this limit can only exist if $P_0^* = B$.

Any limit point of the sequence $P_{\varepsilon_n}^*$ must henceforth be equal to P_0^* and so we have $\lim_{\varepsilon \downarrow 0} P_\varepsilon^* = P_0^*$.

As a consequence, the limit matrix A is a solution to (3.3) and (3.2) for $B = P_0^*$. This is the determining set of Eqs. (1.4), (1.5) for the fundamental matrix Z_0 , so that $A = Z_0$.

Next, we assume that

$$\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = A$$

for some finite matrix A .

Again we consider the set of matrix limit points of the sequence $\{P_\varepsilon^*\}_{\varepsilon \downarrow 0}$. Take any limit point B and a sequence ε_n such that $\lim_{n \rightarrow \infty} P_{\varepsilon_n}^* = B$. As before, $BP_0 = P_0B = B$.

By dominated convergence

$$\lim_{n \rightarrow \infty} U(\alpha, \varepsilon_n) = \lim_{n \rightarrow \infty} \sum_{t \geq 0} \alpha^t (P_{\varepsilon_n} - P_{\varepsilon_n}^*)^t$$

$$\begin{aligned}
 &= \sum_{t \geq 0} \alpha^t (P_0 - B)^t \\
 &= \sum_{t \geq 0} \alpha^t (P_0^{(t)} - B) + B.
 \end{aligned} \tag{3.7}$$

One can now follow the argument from (3.6) onwards to obtain that $B = P_0^* = \lim_{\varepsilon \downarrow 0} P_\varepsilon^*$. Taking the limit $\alpha \uparrow 1$, yields $\lim_{\alpha \uparrow 1} \lim_{n \rightarrow \infty} U(\alpha, \varepsilon_n) = Z_0$.

In order to prove (ii), we assume that the set of ε -perturbations is regular. Then

$$\begin{aligned}
 \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) &= \lim_{\varepsilon \downarrow 0} \sum_{t \geq 0} \alpha^t (P_\varepsilon - P_\varepsilon^*)^t \\
 &= \sum_{t \geq 0} \alpha^t (P_0 - P_0^*)^t,
 \end{aligned}$$

by dominated convergence, regularity and the fact that $|(P_0 - P_0^*)^t_{ij}| \leq 2$ for $i, j \in S$ and $t \geq 0$. Taking the limit $\alpha \uparrow 1$ yields the required result. \square

4. Analysis of aperiodic perturbations

For completing the proof of Theorem 2.1, we have to show the validity of Lemma 2.3.

This follows largely from results by Dekker and Hordijk [6] on denumerable state Markov decision chains with compact action spaces. One can view the perturbation parameter ε as a control parameter. Their conditions translated to the finite state space case, reduce to: (i) continuity of the one-step transition probabilities and the one-step rewards as a function of the control parameter, and (ii) continuity of the number of closed classes as a function of the stationary, deterministic policies. A consequence of their theory is, that the fundamental matrix is a continuous matrix function of the stationary, deterministic policies.

Here we would have the same situation, if we would allow to ‘mix’ the perturbation parameters across initial states. However, this is merely a formal matter. The proofs in Dekker and Hordijk for deriving continuity of the fundamental matrix, apply in the general setting of a compact parameter space, the parameter indexing policies or a perturbation parameter, etc.

However, these proofs do not yield any bounds. We prefer to give an alternative proof using coupling techniques, that does allow us to obtain the bounds as in Theorem 2.4. For the bounds we do not use any prior knowledge of the stationary distributions at play.

However, to achieve this we have to reduce the problem to the case of aperiodic ε -perturbations, that is, that all transition matrices P_ε correspond to aperiodic Markov chains, for ε sufficiently small. Assuming this, one has (cf. [1])

$$Z_\varepsilon = \sum_{t \geq 0} (P_\varepsilon - P_\varepsilon^*)^t = \sum_{t \geq 0} (P_\varepsilon^{(t)} - P_\varepsilon^*) + P_\varepsilon^*. \tag{4.1}$$

The proof for the general case (allowing periodicity) will follow from a transformation trick.

In order to obtain bounds, we will need a result on uniform exponentially quick convergence of the marginal distributions to the stationary matrix. To this end we will use a standard coupling argument (cf. Thorisson [17]), that we will describe first.

Let $\{X_t, X'_t\}_{t=0,1,\dots}$ be a bivariate stochastic process on $S \times S$ and let τ be a stopping time, such that $P\{X_t = X'_t, t \geq \tau\} = 1$, that is, from stage τ on the chains coincide. Then τ is called a coupling time and we have the following bound:

$$\sum_j |\mathbb{P}\{X_t = j\} - \mathbb{P}\{X'_t = j\}| \leq 2\mathbb{P}\{\tau > t\}. \tag{4.2}$$

We aim to apply this by constructing a bivariate stochastic process in such a way that $\mathbb{P}\{X_t = j\} = p_{ij,\varepsilon}^{(t)}$ and $\mathbb{P}\{X'_t = j\} = p_{ij,\varepsilon}^*$. Then the distance of marginal and stationary probabilities will be bounded by tail distribution of the coupling time.

Let us now be given a Markov chain with transition matrix P . Let R_1, \dots, R_ν be the recurrent classes of this chain, $R = \bigcup_{n=1}^\nu R_n$ and $T = S \setminus R$ is the set of transient states. Choose a set of reference states $B = \{b_1, \dots, b_\nu\}$, $b_n \in R_n$.

The construction of the bivariate process for recurrent initial states i is given in [13, Theorem 16.2.4]. We will describe the mechanism for our particular situation. Assume that $p_{iB} \geq \delta$ for all $i \in S$ (i.e. condition (2.2) for $t_0 = 1$ and period $d = 1$). Then $p_{iB}^* \geq \delta$ by iteration and taking limits.

Let $i \in R_n$ and let $\{X_t\}_t$ and $\{\tilde{X}_t\}_t$ be two Markov chains, with independent initial distributions given by $\mathbb{P}\{X_0 = j\} = \delta_{ij}$ and $\mathbb{P}\{\tilde{X}_0 = j\} = p_{b_n j}^*$, $j \in S$. At each time instant $t \geq 0$, given that $X_t = i$, $X_{t+1} = \tilde{X}_{t+1} = b_n$ with probability δ . With probability $(1 - \delta)$ choose for each chain independently the next position from the distribution $(\mathbb{P}\{X_{t+1} = k \mid X_t = \cdot\} - \delta \mathbf{1}_{\{k=b_n\}})/(1 - \delta)$.

Let $\tau = \min\{t \geq 1 \mid X_t = \tilde{X}_t = b_n\}$. Define a third Markov chain $\{X'_t\}_t$ by

$$X'_t = \begin{cases} \tilde{X}_t, & t \leq \tau, \\ X_t, & t > \tau. \end{cases} \tag{4.3}$$

Since $\mathbb{P}\{X_t = j\} = p_{ij}^{(t)}$ and $\mathbb{P}\{X'_t = j\} = p_{b_n j}^* = p_{ij}^*$, (4.2) implies

$$\sum_j |p_{ij}^{(t)} - p_{ij}^*| \leq 2\mathbb{P}\{\tau > t\}. \tag{4.4}$$

The construction for transient initial states is slightly more involved. Let $i \in T$. Let a_i^n denote the absorption probability in the class R^n :

$$a_i^n = \mathbb{P}\left\{ \bigcup_{t \geq 1} \{X_t \in R_n\} \mid X_0 = i \right\}.$$

Then $p_{ij}^* = a_i^n p_{b_n j}^*$, $j \in S$, $\sum_{j \in R_n} p_{ij}^* = a_i^n$ and $\sum_{n=1}^\nu a_i^n = 1$.

Divide the unit interval $(0, 1]$ up into ν left open, right closed intervals $I_i(n)$ of length a_i^n . Each of these intervals we further partition into denumerably many left open, right closed intervals $I(t, j)$ of length $|I(t, j)| = \mathbb{P}\{X_t = j, X_k \notin R_n, k < t \mid X_0 = i\}$, $t = 1, \dots, j \in R_n$.

Additionally, we partition each interval $I(t, j)$, $t \geq 2$, into $(\#T)^{t-1}$ left open, right closed intervals $I(t, j, i_1, \dots, i_{t-1})$, with $i_1, \dots, i_{t-1} \in T$, of length

$$\begin{aligned} |I(t, j, i_1, \dots, i_{t-1})| &= \mathbb{P}\{X_1 = i_1, \dots, X_{t-1} = i_{t-1} \mid X_0 = i, X_t = j, X_k \notin R_n, k < t\} \\ &\quad \times |I(t, j)|. \end{aligned}$$

Finally, we partition each interval $I(t, j, i_1, \dots, i_{t-1})$ into left open, right closed intervals $I(t, j, i_1, \dots, i_{t-1}, j', i'_0, \dots, i'_{t-1})$ of length $p_{b_n i'_0}^* p_{i'_0 i'_1} p_{i'_1 i'_2} \cdots p_{i'_{t-1} j'}$ $|I(t, j, i_1, \dots, i_{t-1})|$, $j', i'_0, \dots, i'_{t-1} \in R_n$.

At time 0, we select a random number u from the unit interval $(0, 1]$. The interpretation is that—if $u \in I_i(n)$ —this selects the recurrent class that the two chains to be constructed, will be absorbed in. Furthermore, if u is also in $I(t, j)$, then this selects the time t and the particular

entrance state j into that recurrent class, for the chain starting at $i \in T$. If, additionally, $u \in I(t, j, i_1, \dots, i_{t-1})$, then this selects the sequence of transient states that leads to hitting R_n at time t in state j . Finally, if $u \in I(t, j, i_1, \dots, i_{t-1}, j', i'_0, \dots, i'_{t-1})$, then i'_0 is the initial state for the chain starting with initial distribution p_i^* , with j, i'_0 belonging to the same recurrent class. The states $i'_1, \dots, i'_{t-1}, j'$ are the successive states of the latter chain. At time t the chains are then in states j and j' , respectively.

Suppose now that the randomly sampled number from $(0, 1]$ is

$$u \in I(t_0, j_0, i_1, \dots, i_{t_0-1}, j'_0, i'_0, \dots, i'_{t_0-1}),$$

with $j_0, j'_0 \in R_n$. Run two Markov chains $\{X_{t_0+t}\}_{t \geq 0}$ and $\{\tilde{X}_{t_0+t}\}_{t \geq 0}$, with the same (coupled) transition mechanism as described for the case of recurrent initial states j_0 and j'_0 , respectively.

This whole construction forces both chains $\{X_t\}_{t \geq 0}$ and $\{\tilde{X}_t\}_{t \geq 0}$ to absorb in the same recurrent class with probability 1, while preserving the marginal probabilities of the states. And last, but not least, conditioning on $(X_t = i, \tilde{X}_t = i')$ yields a bivariate stochastic process, the conditional distribution of which at time $t + s$ is the same as the distribution at time s of the bivariate process starting with initial state $(X_0 = i, \tilde{X}_0 = i')$.

Define $\tau = \min\{t \geq 1 \mid X_t \in \tilde{X}_t, X_t \in \{b_1, \dots, b_\nu\}\}$. For the processes to couple, it is necessary for the first process X_t to have entered the set of recurrent states. Again, define a third chain $\{X'_t\}_{t \geq 0}$ by (2.2). Then (4.4) holds. In the aperiodic case $d = 1$, so that (2.2) reduces to

$$\inf_i \sum_{b \in B} p_{ib, \varepsilon}^{(t_0)} \geq \delta, \quad \varepsilon \leq \varepsilon_2,$$

and in (2.3) we have $\eta = 1$.

Lemma 4.1. *Assume that the collection of ε -perturbations is regular. Let ε_1 be such that the transition matrices P_ε are aperiodic and $\nu_\varepsilon = \nu_0$ for $\varepsilon \leq \varepsilon_1$. Then for the constants δ, ε_2, c and β given by (2.2) and (2.3) with constant $\eta = 1$, we have*

$$\sup_i \sum_j |p_{ij, \varepsilon}^{(t)} - p_{ij, \varepsilon}^*| \leq c \cdot \left(1 + \mathbf{1}_{\{i \in T_\varepsilon\}} \cdot t \frac{\delta}{t_0}\right) \cdot \beta^t, \quad t = 1, 2, \dots, \varepsilon \leq \varepsilon_2. \tag{4.5}$$

Proof. Fix $\varepsilon \leq \varepsilon_2$. We consider the ‘embedded’ transition matrices $P_\varepsilon^{(tt_0)}$, at times $0, t_0, 2t_0, \dots$. The embedded chain $\{X_t\}_t$ has the same stationary matrix as the original one. Define a coupling as described above, for the *embedded* chains. Note that given that $X_0 \in R_\varepsilon^n$, the stopping time τ is a coupling time only since $\nu_\varepsilon = \nu_0$!

For $i, i' \in R_\varepsilon^n, n = 1, \dots, \nu$, one can show by a standard induction argument (cf. [13, Theorem 16.2.4]), that

$$P\{\tau > t \mid X_0 = i\} \leq (1 - \delta)^t. \tag{4.6}$$

If i is transient for the ε -perturbed chain, one has

$$\begin{aligned} P\{\tau \leq 1 \mid X_0 = i\} &= \sum_{n=1}^\nu P\{I(1, b_n, b_n) \mid X_0 = i\} \\ &= \sum_{n=1}^\nu p_{ib_n, \varepsilon}^{(t_0)} p_{b_n b_n, \varepsilon}^* \\ &\geq \delta \sum_{n=1}^\nu p_{ib_n, \varepsilon}^{(t_0)} \geq \delta^2. \end{aligned}$$

By an induction argument and combination with (4.6) (see [8] for details) we obtain

$$P\{\tau > t \mid X_0 = i\} \leq (1 + t\delta) \cdot (1 - \delta)^t. \tag{4.7}$$

This yields the required result. \square

Finally, a dominated convergence argument and the regularity assumption directly implies the validity of the next lemma.

Lemma 4.2. *Assume that the set of ε -perturbations is regular. Assume that all transition matrices P_ε , $0 \leq \varepsilon \leq \varepsilon'$ are aperiodic for some $\varepsilon' \leq \varepsilon_0$. Then*

$$\lim_{\varepsilon \downarrow 0} Z_\varepsilon = Z_0.$$

5. The aperiodic transformation

This section discusses a data-transformation (cf. Schweitzer [15]), showing that is sufficient to restrict to the aperiodic case.

Following the construction of Example 1, let $\lambda \in (0, 1)$ and put $P_{\varepsilon,\lambda} = \lambda \mathbf{I} + (1 - \lambda) P_\varepsilon$. All previous notation with subscript λ added, denotes the corresponding quantity for the transformed chains. By inspection one has $P_{\varepsilon,\lambda}^* = P_\varepsilon^*$. First, we note that

$$\begin{aligned} U_\lambda(\alpha, \varepsilon) &= \sum_{t=0}^{\infty} \alpha^t (P_{\varepsilon,\lambda} - P_{\varepsilon,\lambda}^*)^t \\ &= \sum_{t=0}^{\infty} \alpha^t P_{\varepsilon,\lambda}^{(t)} - \frac{\alpha}{1 - \alpha} P_{\varepsilon,\lambda}^*. \end{aligned}$$

Rewriting yields the relation

$$U_\lambda(\alpha, \varepsilon) = \frac{1}{1 - \lambda\alpha} U(\alpha', \varepsilon) - \frac{\lambda\alpha}{1 - \lambda\alpha} P_\varepsilon^*, \tag{5.1}$$

where $\alpha' = (1 - \lambda)\alpha / (1 - \lambda\alpha)$. Note that $\alpha \rightarrow 1 \Leftrightarrow \alpha' \rightarrow 1$.

In view of (5.1), the fundamental matrices \tilde{Z}_ε and Z_ε are related by

$$Z_{\varepsilon,\lambda} = \frac{1}{1 - \lambda} Z_\varepsilon - \frac{\lambda}{1 - \lambda} P_\varepsilon^*. \tag{5.2}$$

As a consequence, for all $\lambda \in (0, 1)$,

$$U(\alpha, \varepsilon) - Z_0 = (1 - \lambda)(U_\lambda(\alpha', \varepsilon) - Z_{0,\lambda}) + \lambda(P_\varepsilon^* - P_0^*) + \lambda(1 - \alpha')(U_\lambda(\alpha', \varepsilon) - P_\varepsilon^*).$$

The aperiodic transformation together with Lemma 4.2 yields Lemma 2.3. Theorem 2.1 now follows immediately from Theorem 2.2 and Lemma 2.3. Theorem 2.4 finally can be derived by applying standard geometric bounding arguments together with Lemma 2.3.

6. Conclusion and open problems

In this paper we have characterised regular perturbations by means of the existence of finite limits in both sides of (2.1). For singular perturbations it is not clear whether either limit exists and if so, under which conditions they are equal.

Notice, however, that the limit

$$\begin{aligned} \lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) &= \lim_{\alpha \uparrow 1} \sum_{t \geq 0} \alpha^t (P_0 - P_0^l)^t \\ &= Z_0 + \lim_{\alpha \uparrow 1} \frac{\alpha}{1 - \alpha} (P_0^* - P_0^l) \end{aligned} \tag{6.1}$$

always exists componentwise in the extended reals, provided that the limit $P_0^l := \lim_{\varepsilon \downarrow 0} P_\varepsilon^*$ exists. Haviv and Ritov [12] (see also Avrachenkov and Lasserre [3]) studied the case of *analytic perturbations*, i.e.

$$P_\varepsilon = P_0 + \sum_{k=1}^{\infty} \varepsilon^k G_k,$$

where G_k are $S \times S$ matrices. Under the assumption that all perturbed Markov chains are unichain, they showed that the fundamental matrix Z_ε admits a Laurent expansion in a neighbourhood of $\varepsilon = 0$. This implies that also $\lim_{\varepsilon \downarrow 0} Z_\varepsilon$ exists in the extended reals.

The question is whether $\lim_{\varepsilon \downarrow 0} Z_\varepsilon = \lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon)$. We will give two examples, one of a linearly perturbed Markov chain and one of an analytically perturbed one, where this is not the case. In fact, the examples show that in general there does not seem to be much of a connection between the values of either limit for singularly perturbed Markov chains. For other interesting analyses carried out in a somewhat similar spirit, the reader is referred to Hunter [10].

Example 3. The first example is a three state, linearly perturbed Markov chain. More particularly, $S = \{1, 2, 3\}$ and

$$P_\varepsilon = \mathbf{I} + \varepsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then $P_0^* = \mathbf{I}$ and $Z_0 = \mathbf{I}$, but

$$P_\varepsilon^* = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad \varepsilon > 0.$$

Denote $P_0^l = \lim_{\varepsilon \downarrow 0} P_\varepsilon^*$. By virtue of (6.1) we have

$$\lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) = \mathbf{I} + \frac{\alpha}{1 - \alpha} \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix} \rightarrow \begin{pmatrix} \infty & -\infty & -\infty \\ -\infty & \infty & -\infty \\ -\infty & -\infty & \infty \end{pmatrix}, \quad \alpha \uparrow 1.$$

Note that $\pm\infty$ occurs precisely in those entries where P_0^l and P_0^* differ! This is consistent with the bounds in the previous section.

However,

$$Z_\varepsilon = P_\varepsilon^* + \begin{pmatrix} \frac{1}{3\varepsilon} & 0 & -\frac{1}{3\varepsilon} \\ -\frac{1}{3\varepsilon} & \frac{1}{3\varepsilon} & 0 \\ 0 & -\frac{1}{3\varepsilon} & \frac{1}{3\varepsilon} \end{pmatrix} \rightarrow \begin{pmatrix} \infty & \frac{1}{3} & -\infty \\ -\infty & \infty & \frac{1}{3} \\ \frac{1}{3} & -\infty & \infty \end{pmatrix}, \quad \varepsilon \downarrow 0.$$

Here we have the situation where $\lim_{\varepsilon \downarrow 0} Z_{12,\varepsilon} = 1/3$ is finite and positive, whereas $\lim_{\alpha \uparrow 1} \lim_{\varepsilon \downarrow 0} U_{12}(\alpha, \varepsilon) = -\infty$. The next example shows, that the reverse situation may happen as well.

Example 4. Again we consider a three state Markov chain, but now we have an *analytically* perturbed one of the following form:

$$P_\varepsilon = \mathbf{I} + \varepsilon \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \varepsilon^3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then, again, $P_0^* = \mathbf{I} = Z_0$. However,

$$P_\varepsilon^* = \frac{1}{1 + \varepsilon + \varepsilon^2} \begin{pmatrix} \varepsilon^2 & \varepsilon & 1 \\ \varepsilon^2 & \varepsilon & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix} \rightarrow P_0^l = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varepsilon \downarrow 0.$$

By (6.1), one has

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} U(\alpha, \varepsilon) &= \mathbf{I} + \frac{\alpha}{1 - \alpha} (P_0^* - P_0^l) \\ &= \mathbf{I} + \frac{\alpha}{1 - \alpha} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \infty & 0 & -\infty \\ 0 & \infty & -\infty \\ 0 & 0 & 1 \end{pmatrix}, \quad \alpha \uparrow 1. \end{aligned}$$

On the other hand,

$$\begin{aligned} Z_\varepsilon &= \frac{1}{\varepsilon^2(1 + \varepsilon + \varepsilon^2)} \begin{pmatrix} \varepsilon + \varepsilon^4 & 1 - \varepsilon + \varepsilon^3 & -1 + \varepsilon^2 \\ -\varepsilon^2 + \varepsilon^4 & 1 - \varepsilon + \varepsilon^2 + \varepsilon^3 & -1 + \varepsilon + \varepsilon^2 \\ \varepsilon^4 & -\varepsilon + \varepsilon^3 & \varepsilon + \varepsilon^2 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} \infty & \infty & -\infty \\ -1 & \infty & -\infty \\ 0 & -\infty & \infty \end{pmatrix}, \quad \varepsilon \downarrow 0. \end{aligned}$$

Structured cases, when the discount factor is of the form $\alpha = 1 - \varepsilon^k$, for a rational number k and $\varepsilon \downarrow 0$, may yield further insights as they did in a related analysis of the Cesaro sum limit reported in Filar et al. [9]. This is potentially important for the application to the Hamiltonian cycle problem considered in [4] and [7].

One can also consider to extend our results to the denumerable state space case. In this case the fundamental matrix need not always exist, even if the stationary matrix exists. However, even when it exists, it is not clear under what conditions it can be approximated by a discounted deviation from stationarity of a perturbation for a small perturbation parameter. It would be interesting to find sufficient regularity conditions for the validity of such an approximation.

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