# Filter-based reconstruction methods for tomography

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# 3 Approximating SIRT by filtered backprojection

## 3.1 Introduction

In computed tomography, two common approaches to reconstruct objects from their projections are *analytical* methods and *algebraic* methods. Analytical reconstruction is based on inverting a continuous model of the problem, and discretizing the result. The popular filtered backprojection method for parallel-beam projections is a result of this approach [KS01], as well as the FDK method for cone-beam projection data [FDK84]. Analytical methods assume that projection data is available for *all* angles, and that the data is free of noise. In many practical applications, it is either impossible or undesirable to acquire a sufficient number of noise-free projections to accurately reconstruct the scanned object with an analytical method. In such cases, algebraic methods are often able to yield more accurate reconstructions.

Algebraic methods are based on a discretized model of the problem, resulting in a linear system of equations. A reconstruction is then computed by solving this linear system using an iterative method. Since algebraic methods are based on a model of the data that is available instead of assuming perfect data, algebraic reconstructions are often of higher quality than analytical reconstructions when presented with a limited

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number of projections. Furthermore, the effect of noise on the reconstruction can be minimized by using certain forms of regularization in the iterative method.

Despite the advantages of algebraic methods for imperfect data, analytical methods remain very popular in practice. In [PSV09], several reasons for the popularity of analytical methods are discussed, one of which is the gap that exists between a mathematical definition of an algebraic method and its application in actual real-world problems. When implementing an algebraic method in real-world applications, many difficulties can arise, for example with computational requirements, which are typically much higher compared to analytical methods. In this chapter, we aim to bridge the gap by introducing a method that approximates the algebraic SIRT method by computing a special filter for the filtered backprojection (FBP) method. The resulting method can achieve a reconstruction quality similar to algebraic methods using existing, computationally efficient, FBP implementations.

Recently, a number of reconstruction methods have been proposed that aim to improve FBP by changing its convolution filter. In [Zen12], a window function for the standard ramp filter is derived that approximates an algebraic method. During the derivation, however, it is assumed that projection data is available for enough angles such that a certain approximation can be made, which may not be the case in practice. A different approach is taken in Chapter 2 of this thesis, where a data-dependent filter is computed that minimizes the projection error of the resulting FBP reconstruction. Since a different filter has to be computed for each scanned object, the computational requirements of this method are higher than for standard FBP. Finally, in [BP12], a way of computing angle-dependent filters that approximate algebraic methods is proposed. The method for computing the filters is very computationally demanding, however, which severely limits its applicability in practice. In this chapter, we propose a method to compute filters that are similar to those in [BP12], but can be computed much faster, using an approach that is, in part, similar to [Zen12].

## 3.2 Method

In this section, we propose a method for computing filters for the parallel-beam FBP method that approximate the algebraic SIRT method [KS01], which is a method from the class of Landweber iteration methods [Lan51]. We assume that projection data is acquired for $N_\theta$ angles, with $N_d$ detector elements per projection. In this case, we can write the acquired projection data as a vector $\boldsymbol{p} \in \mathbb{R}^{N_\theta N_d}$. Similarly, we can write the reconstructed image, which is defined on a $N \times N$ pixel grid, as a vector $\boldsymbol{x} \in \mathbb{R}^{N^2}$. An element $w_{ij}$ of the *projection matrix* $\boldsymbol{W}$ gives the contribution of pixel $j$ to detector element $i$. Using these definitions, we can write the FBP method as:

$$\mathit{FBP}(\boldsymbol{p}, \boldsymbol{h}) = \boldsymbol{W}^T \boldsymbol{C}_{\boldsymbol{h}} \boldsymbol{p} \tag{3.1}$$

Here, $\boldsymbol{C}_{\boldsymbol{h}}$ is a convolution of each detector with filter $\boldsymbol{h}$, which can be angle-dependent. Note that in parallel-beam tomography, an FBP reconstruction can also be calculated by first backprojecting the projections, and filtering the result:

$$\mathit{FBP}(\boldsymbol{p}, \boldsymbol{h}') = \boldsymbol{H}_{\boldsymbol{h}'} \boldsymbol{W}^T \boldsymbol{p} \tag{3.2}$$

Here, $\boldsymbol{H}_{\boldsymbol{h}'}$ is a two-dimensional convolution of an image with filter $\boldsymbol{h}'$, and $\boldsymbol{h} = \boldsymbol{W}\boldsymbol{h}'$.

The standard definition of the SIRT method is the following iterative equation:

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \alpha \boldsymbol{W}^T \left(\boldsymbol{p} - \boldsymbol{W}\boldsymbol{x}^k\right) \tag{3.3}$$

Here, $\boldsymbol{x}^k$ is the reconstructed image at iteration $k$, and $\alpha$ is a parameter that influences the stability and convergence of the method, for which the standard value of $\alpha = (N_\theta N_d)^{-1}$ is used throughout this chapter. By regrouping terms, we can write this equation in a matrix form:

$$\boldsymbol{x}^{k+1} = \left(\boldsymbol{I} - \alpha \boldsymbol{W}^T \boldsymbol{W}\right)\boldsymbol{x}^k + \alpha \boldsymbol{W}^T \boldsymbol{p} \tag{3.4}$$

Note that this is a recurrence relation of the form $\boldsymbol{x}^{k+1} = \boldsymbol{A}\boldsymbol{x}^k + \boldsymbol{b}$, which has as solution for the reconstruction at iteration $n$:

$$\boldsymbol{x}^n = \boldsymbol{A}^n \boldsymbol{x}^0 + \alpha \left[\sum_{i=0}^{n-1} \boldsymbol{A}^i\right] \boldsymbol{W}^T \boldsymbol{p} \tag{3.5}$$

Here, $\boldsymbol{A} = \boldsymbol{I} - \alpha \boldsymbol{W}^T \boldsymbol{W}$. In many cases, a zero image is used as the initial image $\boldsymbol{x}^0$, in which case Eq. (3.5) becomes:

$$\boldsymbol{x}^n = \alpha \left[\sum_{i=0}^{n-1} \boldsymbol{A}^i\right] \boldsymbol{W}^T \boldsymbol{p} \tag{3.6}$$

The similarities between Eq. (3.6) and Eq. (3.2) suggest that we can approximate the SIRT equation (Eq. (3.6)) by approximating $\sum_{i=0}^{n-1} \boldsymbol{A}^i$ by a two-dimensional convolution operation with filter $\boldsymbol{q}_n$:

$$\boldsymbol{x}^n \approx \alpha \boldsymbol{H}_{\boldsymbol{q}_n} \boldsymbol{W}^T \boldsymbol{p} \tag{3.7}$$

To find a good approximating filter, we can take the impulse response of $\sum_{i=0}^{n-1} \boldsymbol{A}^i$:

$$\boldsymbol{q}_n = \sum_{i=0}^{n-1} \boldsymbol{A}^i [0, \ldots, 0, 1, 0, \ldots, 0]^T \tag{3.8}$$

In other words, we start with an image with the central pixel set to one and the other pixels set to zero, and iteratively apply $\boldsymbol{A}$ to the image $n$ times, summing all images along the way. In parallel-beam tomography, we can write Eq. (3.7) in the standard FBP form by forward projecting $\boldsymbol{q}_n$:

$$\boldsymbol{x}^n \approx \boldsymbol{W}^T \boldsymbol{C}_{\boldsymbol{u}_n} \boldsymbol{p} = \mathit{FBP}(\boldsymbol{p}, \boldsymbol{u}_n) \tag{3.9}$$

$$\boldsymbol{u}_n = \alpha \boldsymbol{W} \boldsymbol{q}_n \tag{3.10}$$

We conclude that we can approximate the algebraic SIRT method by the FBP method with filter $\boldsymbol{u}_n$. The filter is computed by first computing $\boldsymbol{q}_n$ by applying $\boldsymbol{A}$ to a certain image $n$ times, summing the results. The resulting image is forward projected to obtain $\boldsymbol{u}_n$. Note that a single computed filter can be used to reconstruct many different objects,

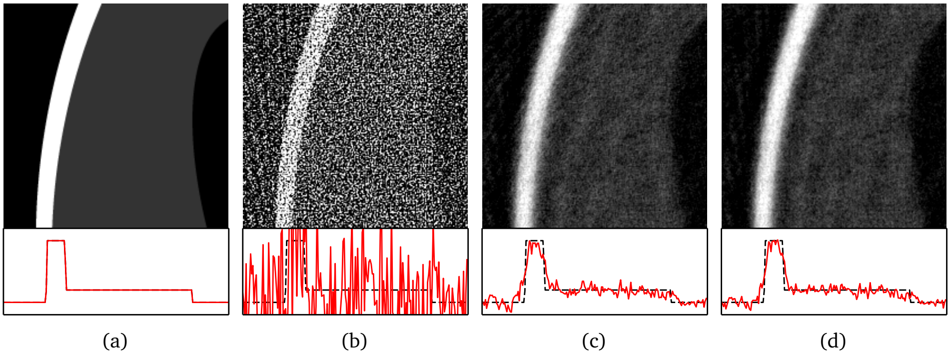


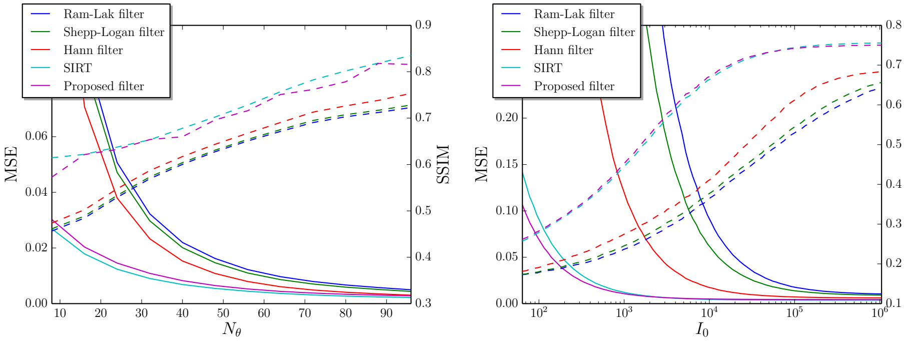

Figure 3.1: Cropped reconstructions of the Shepp-Logan head phantom (a) obtained by different reconstruction methods: FBP with the standard Ram-Lak filter (b), SIRT (c), and FBP with the proposed filter (d). The reconstructions are computed using simulated projection data of 64 angles, with 1024 samples per projection (rebinned from 4096 samples) and Poisson noise applied. A line profile of the central line of each cropped reconstruction is shown under each image, with the phantom shown by a dashed line.

as long as they are scanned with the same projection geometry. For computing the filter, $2n+1$ projection operations are needed, which is similar in computation time to a single run of the SIRT method. To compare, $\mathcal{O}(N_\theta N_d)$ runs of an algebraic method are needed to compute a similar filter in [BP12].

By approximating $\boldsymbol{A}$ by a convolution operation, we assume that the $\boldsymbol{W}^T\boldsymbol{W}$ operation is approximately shift-invariant. Whether this assumption is correct can depend on the actual implementation of the projection operator $\boldsymbol{W}$. If a ray is defined as a strip with the same width as a detector pixel, and the weight of $w_{ij}$ is given by the area of the intersection of the pixel $j$ and the ray $i$, $\boldsymbol{W}^T\boldsymbol{W}$ can be well approximated by a shift-invariant operation. If a ray is defined as a line of zero thickness, the approximation is not as accurate. In this case, however, the approximation can be improved by using supersampling, where multiple lines of zero thickness are cast through the volume per detector pixel. Note that supersampling is only needed during computation of the filter, and not during reconstruction using Eq. (3.9). In the rest of this chapter, we cast eight lines per detector pixel during computation of the filters.

## 3.3 Experiments

We implemented the proposed filter calculation method in Python 3.3.2 using the ASTRA toolbox [PBS11], which includes projection operations that use graphic processing units (GPUs) to improve performance. All reconstructions presented in this chapter are calculated by the ASTRA toolbox as well. To investigate the reconstruction quality of the proposed method compared to other reconstruction methods, we use reconstructions of the Shepp-Logan head phantom. For each experiment the phantom is generated on a $4096 \times 4096$ pixel grid, for which projections are simulated with 4096 detector elements per projection. The resulting projection data is rebinned to 1024 detector




(a) (b)

Figure 3.2: MSE (solid) and SSIM (dashed) of reconstructions of the Shepp-Logan head phantom computed by various methods. In (a), the metrics are shown as a function of the number of projections $N_\theta$, for noise-free projection data. In (b), the metrics are shown for reconstructions of 64 projections, as a function of the amount of applied Poisson noise ($I_0$). Higher values of $I_0$ correspond to lower amounts of applied noise.

elements per projection, which are used to reconstruct on a $1024 \times 1024$ pixel grid. The reconstructions are compared to the phantom, rebinned to $1024 \times 1024$ pixels. We give the mean squared error (MSE) and the structural similarity index (SSIM) [Wan+04] of each reconstruction compared to the rebinned phantom. For the error measures, we use all pixels that are within a disc with a radius of 512 pixels, centered in the pixel grid. For all experiments, we compute a filter that approximates 200 iterations of the SIRT method, unless stated otherwise. Each SIRT reconstruction is computed using 200 iterations as well.

In Fig. 3.1, cropped reconstructions of the Shepp-Logan phantom are shown for FBP with the standard Ram-Lak filter, SIRT, and FBP with the proposed filter, computed using data of 64 projections with a moderate amount of applied Poisson noise. The results for FBP (Fig. 3.1b) show that the noise present in the FBP reconstruction can be prohibitive for further analysis. The reconstructions of SIRT (Fig. 3.1c) and FBP with the proposed filter (Fig. 3.1d) are, at least visually, very similar.

To further investigate the reconstruction quality of the proposed method, we generated projection data for different numbers of projections $N_\theta$, and compared the MSE and SSIM of reconstructions of FBP with different standard filters, SIRT, and the proposed filter method. The results are shown in Fig. 3.2a. We also generated data of 64 projections with various amounts of applied Poisson noise, for which the results are given in Fig. 3.2b. Here, $I_0$ indicates the amount of applied Poisson noise, with higher values corresponding to lower amounts of applied noise. In both Fig. 3.2a and Fig. 3.2b, reconstructions using the proposed method have a significantly lower MSE and higher SSIM compared to FBP reconstructions with standard filters. Compared to SIRT, the proposed method has a similar MSE and SSIM. Note, however, that for 64 projections a SIRT reconstruction took 1.40 seconds to compute, while a reconstruction of the proposed method was computed in 9.67 milliseconds, which is roughly 144×

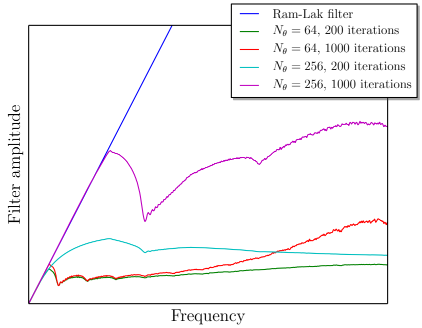



Figure 3.3: Comparison of the standard Ram-Lak filter and the computed filters for $N_d = 1024$ and various numbers of projections $N_\theta$ and numbers of iterations, averaged over all angles, shown in Fourier space.

faster.

Computed filters, averaged over all angles, are shown in Fourier space in Fig. 3.3, along with the standard Ram-Lak filter. The figure shows that the computed filters are identical to the Ram-Lak filter up to a certain frequency, which depends on the number of projections and iterations. Taking more angles and/or iterations results in a filter that is closer to the Ram-Lak filter. The figure also shows that by taking more iterations, the higher frequencies of the filters are amplified. A similar effect can also be observed in the SIRT method, where taking more iterations results in stronger high frequencies in the reconstructed image.

## 3.4 Conclusion

In this chapter, we presented a novel method to compute filters for the filtered backprojection method that approximate the algebraic SIRT method. The method is based on rewriting SIRT into a matrix form, and approximating the combined backprojection and forward projection operation ($\boldsymbol{W}^T\boldsymbol{W}$) by a 2D convolution operation. An approximating filter can be found by applying the combined projection operation repeatedly to a specific image. The result is an angle-dependent filter that can be used in the FBP method to produce reconstructions that are similar to those produced by SIRT. Computation of the filter is significantly faster than in similar approaches of previous work [BP12], enabling its use in large-scale real-world tomographic problems.

Several experiments on a phantom image show that the proposed method produces reconstructions of similar quality to the SIRT method, both in the case of a low number of projections and with noise present in the data. Compared to FBP with standard filters, the proposed method produces reconstructions with significantly lower MSE and higher SSIM. The computation time of reconstructing with the proposed method is identical to the FBP method, which is significantly lower than SIRT. These results show that by computing geometry-dependent convolution filters, it is possible to accurately approximate the SIRT method by filtered backprojection.