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(Received 18 October 2004; published 5 July 2005)

Plane Couette flow of viscoelastic fluids is shown to exhibit a purely elastic subcritical instability at a very small-Reynolds number in spite of being linearly stable. The mechanism of this instability is proposed and the nonlinear stability analysis of plane Couette flow of the Upper-Convected Maxwell fluid is presented. Above a critical Weissenberg number, a small finite-size perturbation is sufficient to create a secondary flow, and the threshold value for the amplitude of the perturbation decreases as the Weissenberg number increases. The results suggest a scenario for weakly turbulent viscoelastic flow which is similar to the one for Newtonian fluids as a function of Reynolds number.

DOI: 10.1103/PhysRevLett.95.024501 PACS numbers: 47.20.Ft, 47.50.+d, 83.60.Wc

Recently, it has been discovered that flows of viscoelastic fluids like polymer solutions and melts can lose their stability and become turbulent at very low Reynolds numbers [1–4]. In contrast to the Newtonian turbulence, where inertial forces destabilize the flow upon increasing the Reynolds number, this small-Reynolds number *elastic turbulence* or *turbulence without inertia* [5] arises when the flow-induced stretching of the polymers makes the elastic stresses in the fluid become large and anisotropic. It is a challenge to explain how the transition to this new type of turbulence can occur. In this Letter we show that viscoelastic plane Couette flow (PCF) exhibits a subcritical instability to finite-amplitude waves and argue for a scenario for weak elastic turbulence analogous to that for weak turbulence in PCF of Newtonian fluids.

The mechanism of linear elastic instability was identified for flows with curved stream lines [1,6]. One of the classical examples of such a flow is realized in Taylor-Couette cells where fluid fills the gap between two coaxial cylinders made to rotate with respect to each other. In the laminar state, the fluid moves around the cylinder axis, and the stretching of polymer molecules along the circular stream lines exerts extra pressure towards the inner cylinder. When these forces overcome viscous friction, the laminar state becomes linearly unstable—infinitesimal perturbations will push a polymer from the circular stream lines and create a secondary flow, reminiscent of the Newtonian Taylor vortices. Pakdel and McKinley generalized this mechanism to arbitrary flows [7] and proposed that there exists a universal relation between the properties of the fluid and the flow geometry which determines the conditions of the linear instability. One of the dimensionless parameters in their argument is the so-called *Weissenberg number* $\mathrm{Wi} = \lambda\dot{\gamma}$, where λ is the elastic relaxation time of the fluid, and the shear rate $\dot{\gamma} = \partial v/\partial y$ gives the relative velocity of two fluid layers moving with respect to each other. Pakdel and McKinley argued [7] that the critical Weissenberg number is related to the characteristic curvature of the flow stream lines and that the linear instability disappears when the curvature goes to zero. The known results on the viscoelastic instabilities in Taylor-Couette [1], cone-and-plate and parallel plate [2], Dean and Taylor-Dean [6] flows are in agreement with this *curved stream lines–linear instability* paradigm.

The linear stability of parallel viscoelastic shear flows has been investigated in detail. For essentially all studied viscoelastic models, laminar PCF is linearly stable [8,9] (note the exception [10]). In the case of pipe flow, the linear stability was demonstrated numerically by Ho and Denn [11] for any value of the Weissenberg and Reynolds numbers. Therefore, it has become common knowledge that the parallel shear flows of fluids obeying simple viscoelastic models (UCM, Oldroyd-B, etc.) are linearly stable, in agreement with the *curved stream lines–linear instability* paradigm. Clearly, if an instability does occur in practice, it has to be nonlinear.

There is no general agreement, however, on whether parallel shear flows like PCF or pipe flow do in fact become unstable. At the moment, there has been no experiment that would clearly establish the presence or absence of a bulk hydrodynamic instability in parallel viscoelastic shear flows. One of the few indirect indications that a bulk instability might occur in pipe flow comes from the famous melt-fracture problem [12], which arises in extrusion of a dense polymer solution or melt through a thin capillary. There, when the extrusion rate exceeds some critical value, the surface of the extrudate becomes distorted and the extrudate might even break, giving the name to the phenomenon. It is possible that this is a manifestation of an instability taking place inside the capillary, though other mechanisms (such as stick-slip, influence of the inlet, etc.) have been proposed [12]. Recently, we presented arguments for the bulk instability being related to the melt-fracture phenomenon [13], but the issue stays highly controversial. There is also some evidence for nonlinear parallel shear flow instabilities from numerical simulations of viscoelastic hydrodynamic equations [14]. Partly because the numerical schemes used to solve these equations are known to break down when elastic stresses become large ($\mathrm{Wi} \gtrsim 1$)—the so-called *large Weissenberg number prob-*

lem [15]—it is open to debate whether an observed phenomenon is due to a numerical or a true physical instability.

In this Letter we show explicitly that a nonlinear instability mechanism does exist in agreement with the following argument. The laminar velocity profiles of the parallel shear flows have straight stream lines, and, therefore, their linear stability is in agreement with the *curved stream lines–linear instability* paradigm. The linear theory predicts that a small perturbation superimposed on top of the laminar flow will decay in time with the decay rate depending on the Weissenberg number. When Wi becomes larger than 1, the decay time becomes comparable with the elastic relaxation time λ, and the perturbation becomes long-lived. Thus, on short time scales, the superposition of the laminar flow and the slowly-decaying perturbation can be viewed as a new basis profile *with* curved stream lines. Applying the same *curved stream lines–linear instability* paradigm to the perturbed stream lines, we conclude that this new flow can become linearly unstable. The instability requires a subsequent creation of two perturbations, and thus is nonlinear. Since the initial perturbation has to be strong enough to become unstable, there exists a finite-amplitude threshold for the transition, which becomes smaller as the Weissenberg number increases. Note that this argument relies on the existence of the normal-stress effect only, and is independent of the precise constitutive equation. Moreover, this scenario resembles the transition to turbulence in parallel shear flows of Newtonian fluids. There, as well, one encounters the absence of the linear instability, and a subcritical transition with the threshold going down with the Reynolds number [16,17].

Our explicit results are for the nonlinear stability analysis of PCF. We consider the so-called UCM fluid [18] confined in the y direction between two plates ($y = \pm d$), which move with constant velocity v_0 in the opposite directions along the x axis. The hydrodynamic equations, consisting of the equations for momentum balance, the UCM model, and incompressibility, read [18]

$$\mathrm{Re}\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p - \nabla\cdot\boldsymbol{\tau}, \quad (1)$$

$$\boldsymbol{\tau} + \mathrm{Wi}\left[\frac{\partial \boldsymbol{\tau}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\tau} - (\nabla\mathbf{v})^{\dagger}\cdot\boldsymbol{\tau} - \boldsymbol{\tau}\cdot(\nabla\mathbf{v})\right] = -[(\nabla\mathbf{v}) + (\nabla\mathbf{v})^{\dagger}], \quad (2)$$

$$\nabla\cdot\mathbf{v} = 0, \quad (3)$$

where the Weissenberg number $\mathrm{Wi} = \lambda v_0/d$, the Reynolds number $\mathrm{Re} = \rho v_0 d/\eta$, ρ is the density and η is the viscosity of the fluid, and $\boldsymbol{\tau}$ and $\mathbf{v}$ are the stress tensor and the velocity, respectively; d is used as the unit of length, d/v_0 as the unit of time, and the stress tensor is scaled with $\eta v_0/d$; $(\ldots)^{\dagger}$ denotes the transposed matrix. As usual, we split these variables in two parts: the laminar values $\{\boldsymbol{\tau}_{\mathrm{lam}}, y\mathbf{e}_x\}$ and the deviation describing the disturbance $\{\boldsymbol{\tau}', \mathbf{v}'\}$. It is useful to organize all hydrodynamic fields of the disturbance in one vector $V = \{\tau'_{ij}, v'_i, p\}^{\dagger}$. Then, the Eqs. (1)–(3) can formally be rewritten as

$$\hat{\mathcal{L}}V + \frac{\partial}{\partial t}\{\mathrm{Re}\mathbf{v}', \mathrm{Wi}\boldsymbol{\tau}', 0\}^{\dagger} = N(V, V), \quad (4)$$

where the left hand side represents the linear terms in Eqs. (1)–(3), and the right hand side the quadratic nonlinearity.

The first step of our analysis is to determine the eigenvalues λ and the eigenfunctions V_0 of the linear operator $\hat{\mathcal{L}}$. Gorodtsov and Leonov [8], and Reynardy *et al.* [9] have shown that the eigenfunctions in the form

$$V_0(x, y, z) = \tilde{V}_0(y)e^{i(kx+qz)} + \mathrm{c.c.} \quad (5)$$

have two types of physical eigenvalues: a pair of complex-conjugated ''elastic'' eigenvalues (Gorodtsov-Leonov modes), and an infinite discrete set of ''inertial'' ones. There also exists a continuous spectrum of eigenvalues which were shown to be unphysical [19] and which, therefore, will be discarded. Let us for the moment focus on the elastic mode, and suppose we choose the initial disturbance to be in the form of the elastic eigenmode

$$V(x, y, z, t) = \Phi(t)\tilde{V}_0^{(GL)}(y)e^{i(kx+qz)} + \mathrm{c.c.} \quad (6)$$

where $\Phi(t)$ is a complex amplitude (our normalization is such that when the amplitude $\Phi(t) = 1$, the strength of the shear rate created by the perturbation equals that of the laminar flow). In order to investigate how the amplitude $\Phi(t)$ will change in time depending on Wi, k, and q, we derive the equation governing the time evolution of $\Phi(t)$, or *the amplitude equation*. The standard technique used to derive the amplitude equations relies on the presence of a linear instability (occurring at, say, $\mathrm{Wi}_{\mathrm{lin}}$), and uses the distance to the instability $(\mathrm{Wi} - \mathrm{Wi}_{\mathrm{lin}})/\mathrm{Wi}_{\mathrm{lin}}$ as a small parameter. Then, the nonlinear evolution of $\Phi(t)$ near $\mathrm{Wi}_{\mathrm{lin}}$ can be deduced by the method of multiple scales [20]. The parallel shear flows, however, are linearly stable ($\mathrm{Wi}_{\mathrm{lin}} \to \infty$, effectively), and we have to use other methods. Instead, assuming $\Phi(t)$ to be small, we substitute Eq. (6) into Eq. (4), collect the terms proportional to $\exp[i(kx + qz)]$, and using the corresponding eigenmode of the adjoint operator $\hat{\mathcal{L}}^{\dagger}$, we project these terms on the original form Eq. (6). We then find the time derivative of $\Phi(t)$ as a series in $\Phi(t)$

$$\frac{d\Phi}{dt} = \lambda^{(\mathrm{GL})}\Phi + C_3\Phi|\Phi|^2 + C_5\Phi|\Phi|^4 + C_7\Phi|\Phi|^6 + C_9\Phi|\Phi|^8 + C_{11}\Phi|\Phi|^{10} + \cdots \quad (7)$$

where $\lambda^{(\mathrm{GL})}$ is the eigenvalue of the elastic (Gorodtsov-Leonov) mode, and the nonlinear coefficients C's are explicit functions of Wi, k, and q. Equation (7) has solution $\Phi(t) = |\Phi|e^{i\Omega t}$ which results in traveling waves in Eq. (6). For small amplitudes, it reproduces the linear decay

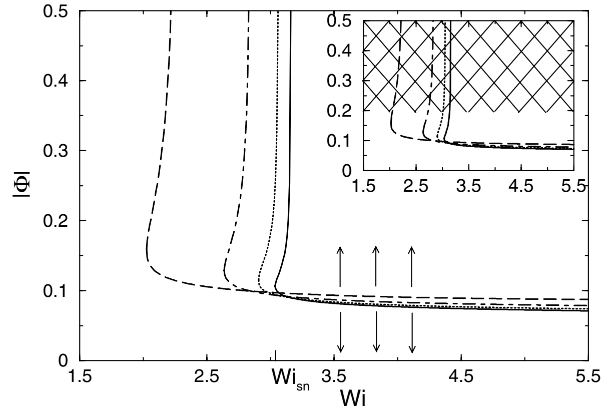

FIG. 1. Steady-state amplitude Φ for $k = 1$ and $q = 1$: dashed, dot-dashed, dotted, and solid lines show the solution to $\mathrm{Re}(\frac{d\Phi}{dt}) = 0$ up to the 5th, 7th, 9th, and 11th order. The inset illustrates qualitatively that the series converges only for small enough $|\Phi|$. The ratio $\mathrm{Re}/\mathrm{Wi} = 10^{-3}$ was kept constant.

$\Phi \sim \exp[\lambda^{(\mathrm{GL})} t]$, $\mathrm{Re}(\lambda^{(\mathrm{GL})}) < 0$. The instability threshold is determined by finding the steady-state solutions of Eq. (7). The main problem in dealing with the series like Eq. (7) is that it is not known *a priori* whether it converges. In order to check the convergence of the series upon inclusion of higher-order terms, we solve the equation $\mathrm{Re}(\frac{d\Phi}{dt}) = 0$ in successive order.

In Fig. 1 we show the solution to these equations for $k = 1$ and $q = 1$. The most important feature of these curves is that they show the existence of a subcritical instability for Weissenberg numbers larger than the saddle-node value $\mathrm{Wi}_{\mathrm{sn}}$ indicated in the figure and obtained by extrapolating the curves. As the arrows indicate, for $\mathrm{Wi} > \mathrm{Wi}_{\mathrm{sn}}$ the lower branch of the curves denotes the critical amplitude—amplitudes larger than this value will grow in time. Note that the instability threshold is small (consistent with our assumption $|\Phi| < 1$), and goes down as Wi increases. The inclusion of higher-order terms causes the whole curve to shift to the right, though the shift becomes roughly 2 times smaller with every coefficient included, indicating convergence.

While the lower branch of each curve gives the minimal amplitude of the disturbance sufficient to destabilize the laminar flow, the upper branch determines the saturated value of Φ after the transition. Surprisingly, it diverges in the vicinity of the saddle node where the highest coefficient in the expansion changes sign (see Table I). There could be several reasons for that. It may indicate that the nonlinear state in the form of Eq. (6) is unstable and will undergo a transition to another coherent state or to turbulence, or it may be a convergence problem. As indicated in the inset of Fig. 1, the upper branch may lie beyond the radius of convergence of (7). When we apply the method to the subcritical Swift-Hohenberg equation [21], we also find such behavior.

In Fig. 2 we plot the lowest Weissenberg number for which the nonlinear instability is possible, or the position of the saddle node $\mathrm{Wi}_{\mathrm{sn}}$, as a function of the wave vectors k and q. It clearly shows that the saddle-node position is only a weak function of the wave vectors, and a large number of modes with different k's and q's is nonlinearly unstable for given $\mathrm{Wi} > 2.1$. Even if each individual mode saturates at a given value of Φ, the superposition of a large number of such modes is likely to become chaotic, and we expect the elastic turbulence to set in close to or even at the instability.

Now we turn to the discussion of the inertial eigenmodes. Replacing $\lambda^{(\mathrm{GL})}$ and $\tilde{V}_0^{(\mathrm{GL})}$ in Eq. (6) with one of the inertial eigenvalues and the corresponding eigenmode, and repeating the same calculation for the coefficients in the amplitude Eq. (7), we find that this type of disturbances is nonlinearly stable for $\mathrm{Wi} \sim \mathrm{Wi}_{\mathrm{sn}}$. We have checked this result for the five eigenvalues with the smallest imaginary parts.

The above results give very strong indications for the existence of a branch of nonlinear finite-amplitude solutions which renders viscoelastic PCF nonlinearly unstable for $\mathrm{Wi} \gtrsim 2.1$. We believe these solutions do play an important role in organizing the dynamics of viscoelastic PCF for any constitutive equation that incorporates the normal-stress effect, although the critical value of Wi will be model-dependent. First of all, it is intriguing to note that in the direct numerical simulations of PCF of Atalik and Keunings [14], numerical instabilities were found for $\mathrm{Wi} \gtrsim 2$—could this be due to the occurrence of these modes? Secondly, we have found that the branch of solutions which we have established here have an analog in pipe flow where the transition is found to be at $\mathrm{Wi}_{\mathrm{sn}} \approx 5$ (results will be published elsewhere, see also [13]). Thirdly, and most importantly, the similarity of our results with what has been found for PCF of Newtonian fluids

TABLE I. Real parts of the coefficients C's for $k = 1$, $q = 1$, and $\mathrm{Re} = 10^{-3}\mathrm{Wi}$.

Wi	$\mathrm{Re}\lambda^{(\mathrm{GL})}$	$\mathrm{Im}\lambda^{(\mathrm{GL})}$	C_3	$C_5 \times 10^{-2}$	$C_7 \times 10^{-5}$	$C_9 \times 10^{-7}$	$C_{11} \times 10^{-9}$
1.50	−0.575 067	0.761 527	60.883 38	−48.012 18	−19.162 32	−39.640 80	−53.378 42
2.27	−0.340 756	0.812 308	26.405 54	0.077 370	−1.898 243	−3.942 051	−5.394 861
2.85	−0.254 964	0.842 388	19.479 66	6.524 338	0.014 115	−0.649 883	−1.345 933
3.07	−0.232 165	0.851 779	17.827 83	7.498 090	0.357 424	0.016 135	−0.297 758
3.15	−0.224 797	0.854 949	17.302 63	7.747 495	0.455 524	0.216 437	0.040 563
5.50	−0.113 814	0.911 950	9.295 180	7.426 946	1.204 062	2.424 150	5.223 760

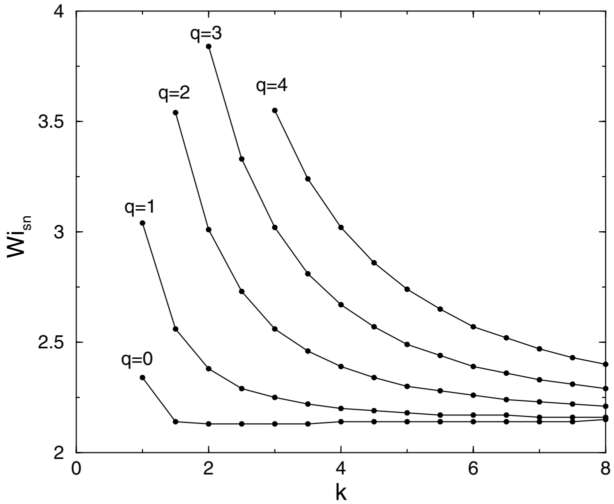

FIG. 2. The saddle-node Weissenber number as a function of the wave vectors. The dots are the calculated values; the lines serve as a guide to the eye.

leads us to suspect that these solutions could be the building blocks of weak viscoelastic turbulence: The viscoelastic periodic finite-amplitude solutions found here resemble the periodic finite-amplitude solutions in Newtonian fluids. There, they are known to be *exact* but *unstable* solutions of the Navier-Stokes equation [22] and take part in the self-sustaining cycle—a periodic sequence of instabilities in which streaks, streamwise vortices and rolls are continuously destroyed and regenerated [23,24]. It is tantalizing to speculate that a similar cycle can be proposed to sustain the weak elastic turbulence in PCF and Poiseuille flow. The first step in this direction was made in [25] where the influence of a minute amount of polymer on the weak Newtonian turbulence was studied. Together with our findings on the finite-amplitude solutions this gives hope that a viscoelastic version of the self-sustaining cycle can indeed be formulated.

In conclusion, we have presented evidence for nonlinear instability in viscoelastic PCF. Together with the vast experimental and numerical evidence in various flow geometries [2,4,13,26–28], this makes us believe that the nonlinear subcritical instability is an inherent feature of the viscoelastic parallel shear flows and that the finite-amplitude solutions may organize the dynamics of weak elastic turbulence.

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