

Proofs as Acts and Proofs as Objects:
Some questions for Dag Prawitz.*

by

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SEVENTEEN YEARS AGO I carried out an exegetical investigation into Heyting's original meaning-explanations for the intuitionistic logical constants and the accompanying notion of a proposition.¹ Central to my analysis was a tripartite distinction with respect to the notion of *construction*, which has at least three different meanings, namely:

- (i) a construction-act (or process);
- (ii) the construction-object constructed in the construction-act;
- (iii) the process (i) considered as an object.

The proofs occurring in the meaning explanations are constructions (ii), whereas what is required for an assertion that a certain proposition is true is that a construction in the sense (i) has been carried out. Finally, I also suggested that Brouwer's notion of construction could be understood as an amalgamation of all three notions.

At the time, Dag Prawitz was not convinced that the notion of proofs as (construction-)objects was needed, or even legitimate. Ever since, in the course of numerous seminar-discussions and informal conversations, in and out of Stockholm, I have been trying to persuade Professor Prawitz of the legitimacy and utility of the proof-act/proof-object distinction, so far without much success.

For the past three decades, Professor Prawitz has been one of the foremost commentators on the relation between proof theory and semantics. Thus, in his work, he has pinpointed several crucial issues, as well as difficulties, concerning his favoured anti-realist stance. On the present occasion I wish to confront the rejected distinction

* I wish to thank Professor D. T. Runia, Dean of the Faculty of Philosophy at Leyden University, for putting extra computer facilities at my disposal.

¹ Sundholm (1983).

with a number of such topics within the anti-realist philosophy of logic and mathematics. The issues to be discussed have been raised by Professor Prawitz himself, or are highly germane to his work. The tool I use for resolving the quandaries in question will be precisely the proof-act/proof-object distinction, in the hope that this will make him concede, if not the legitimacy, then at least the utility of the distinction in question.

First I consider the historical issue:

- How does the distinction fare in the writings of Brouwer and Heyting?

Furthermore the following points will be dealt with:

- the semantic interpretation of natural deduction derivation-trees.
- the relation between Conservativeness and Harmony;
- the proper formulation of Dummett's Fundamental Assumption;
- assumptions and the constructive meaning of implication.

1. *The historical issue*

This I regard as settled: both notions – proofs as acts and proofs as objects – can be found in the seminal writings of Brouwer and Heyting. It is a commonplace that, for Brouwer, proofs are mental acts.² Already in the second of the theses that were defended together with his dissertation on February 19, 1907, we read:

The admissibility of complete induction cannot only not be proved, but does also not deserve a place as a separate axiom or separate intuitively evident truth. Complete induction is *an act of mathematical construction*, which has its justification already in the Ur-intuition of mathematics.³

² Not just for Brouwer, really. A proof (demonstration), according to the OED, is that whereby one acquires knowledge. What but the act (or process) of getting to know could serve in this office? The unease, or even hostility, with which Brouwer's formulations are met, stems, I suppose, from their strongly mentalistic slant that they exhibit, possibly under the influence of Schopenhauer. With a proper notion of mental act, taking into account the post-Wittgensteinian primacy of the exterior aspects of our acts, though, Brouwer's view ought to be unexceptional.

³ Brouwer (1981, p. 223), my translation and emphasis.

Proofs as construction-objects – *Beweisführungen* – are present already in the early (1924) versions of Brouwer's famous proof of the Bar Theorem. They are given mathematical notations and behave very much like functions or well-founded trees.⁴ In Heyting's original treatment from 1930, the construction demanded by the mathematical proposition

Euler's constant C is rational

is an ordered pair of positive integers such that $a/b = C$, that is, an object of a certain kind. Such examples could be multiplied; here I confine myself to the above and refer the reader to my articles (1983, 1993, 1994).

Professor Prawitz has granted these observations, as indeed one must, I think, as far as history is concerned. He has, however, countered with the remark that for Brouwer (and perhaps also for Heyting), given their mentalistic ontology and concomitant idiom, there is no difference to be made between the construction-object and the act of construction. Accordingly, it would be improper to say that both crucial notions occur in early intuitionistic writings in the sense intended by *me*: if they did so occur, they would clearly have separate roles to fill and could not be considered identical. Brouwer's notion of construction, on the other hand, seems to oscillate between construction-acts and construction-objects.

To what an extent can we accept the rejoinder made by Professor Prawitz? Matters are reasonably straightforward regarding Brouwer, but they are more complex as far as Heyting is concerned. A completed act is individuated in terms of its agent and time of execution. Thus, when mathematical construction-objects are identified with acts of construction, their timeless objectivity is lost: constructions become mental entities. Indeed, it is hard to see how such a psychologistic position is compatible with an objectivistic, meaning-theoretical stance on mathematical language. Communicable meaning appears to be ruled out. This, of course, is also the position to which Brouwer was driven: mathematics is an essentially *languageless* activity. So, on such a view, what purpose would a meaning theory serve? In mathematics there is then no such thing as meaning to be explained, and the Brouwerian intuitionist is able to carry on

very well without, thank you very much.⁵ Thus, for meaning-theoretical purposes, one must not draw substantial bills of exchange on *Brouwer's* writings: such a traffic they cannot sustain.

Heyting, however, lead a double life in matters intuitionistic: on the one hand, he was a straightforward Brouwerian mathematical intuitionist, who made important contributions to the development of intuitionistic mathematics, in particular, geometry, algebra and measure theory, and, on the other hand, he was the semanticist of intuitionism, who developed a constructivist notion of proposition (with concomitant explanations of logical constants). In the latter work his meaning explanations are formulated in terms of objectivist notions such as those of task, or problem. Heyting offered no meaning-theoretical analyses for specifically *intuitionistic* conceptions, such as choice sequences and quantification with respect to them. *Heyting's semantics is confined to basic constructivism*. The mathematical practice of A. Heyting, the Brouwerian intuitionist, is not covered by the meaning-theoretical reflections of A. Heyting, the semanticist of constructivism.

2. Two styles of natural deduction

It took a long time for the theoretical study of natural deduction to come off. From the pioneering works of Jaskowski and Gentzen onwards, natural deduction derivations had mainly been used as a *pedagogical device*, or *technique*, for making the often tedious task of actually finding derivations run smoother. *Natural Deduction*, Professor Prawitz's doctoral dissertation, began the theoretical study in 1965. It also inaugurated the renaissance for the use of proof-trees, rather than the lengthy annotated tabular arrangements preferred by previous writers, for instance Quine and Fitch, who had stayed with the Jaskowski format — sub-proofs and all.⁶ Professor Prawitz, who received his logical nursery-training between the firm covers of Church's *Introduction to Mathematical Logic* (in which work the use of natural deduction is scrupulously avoided), turned

⁴ See Sundholm (1994, p.121) for documentation.

⁵ In my (1984) I have dealt with these issues at some length.

⁶ Quine (1950), Fitch (1952).

back to the original format introduced by Gentzen in his dissertation.^{7,8}

In the original Gentzen format the derivable objects are well-formed formulae. As is by now very familiar, owing in particular to the writings of Dag Prawitz, the end-formula of a Gentzen proof-tree is not derived outright, but depends on certain assumptions. Thus, *prima facie*, when the formalism is cast in a natural deduction mould, the arithmetization of metamathematics, for instance, as needed in the proof of the Gödel incompleteness theorems, would become unduly complex.⁹ The Gödel-code for a natural deduction derivation would be a five-tuple comprising (i) (the Gödel-code of) the end-formula, (ii) (a code for) the rule used in the last step, (iii) (Gödel-codes for) the derivations of the premisses, (iv) (a code for) the (finite) class of assumptions that are still open after the final inference, and, finally, (v) (a code for) the (finite) class of assumption-formulae that were discharged by the last inference.¹⁰ In my opinion, no extra labour is involved, and, having taught the Incompleteness theorems on the basis of natural deduction, I can vouch for the elegance of the details in the resulting treatment.

Gentzen did not only design this familiar version of natural deduction. In the so called *First Consistency proof* from 1936, he provided a sequent-calculus formulation for natural deduction. Also in the earlier doctoral dissertation there occurs a sequent calculus. There, though, we find only introduction rules that operate on both sides of the (con)sequence arrow. Here, in the First Consistency proof, there are elimination rules as well, and rules of both kinds operate only to the right of the arrow. This formalism constitutes another formulation, in fact even an axiomatization, of natural deduction. The derivable objects at the nodes of derivation trees are sequents – not well-formed formulae – of the form

$$A_1, \dots, A_k \Rightarrow C,$$

⁷ Church (1956).

⁸ Gentzen 1934–35.

⁹ This alleged clumsiness of the ensuing arithmetization was the main reason for Church's aversion to natural deduction (personal communication from Prof. Church, Uppsala 1976).

¹⁰ The class at (v) of discharged assumptions will often be empty.

where the $A_1, \dots, A_k$, and C are well-formed formulae. According to Gentzen, such a sequent carries (what amounts to) the intended meaning

$$A_1 \supset (\dots \supset (A_k \supset C) \dots),$$

In particular, use of formula A as an assumption in standard natural deduction corresponds to the use of the axiom

$$A \Rightarrow A$$

in the sequential version. Note also that the sequent arrow and the sequents themselves are part of the *object*-language. From the point of view of an interpreted language, sequents are an object-language generalization of propositions. This sequential version has been seen as a mere notational variant of the original Gentzen formulation, among others by Professors Prawitz and Dummett, as well as by myself.¹¹

In the theory of meaning, we consider interpreted formalisms, that is, formal languages, with a natural deduction derivational apparatus, where the formulae have meaning, that is, express propositions. The theory of meaning must then issue a meaning theory for the language in question. However, one would expect such a meaning theory not only to provide meaning for the terms and formulae, but also for the derivations themselves. In short, the meaning theory ought to answer the question:

What is the semantic value of a natural deduction derivation?

The answer will depend on what type of natural deduction style formalism is chosen.

The proof-act/proof-object distinction that forms the topic of the present discussion must not be confused with the act/object distinction applied to proofs. The object of an act of proof, or better, demonstration, is the knowledge gained, the theorem proved, or made, in, or by, the act of proving. This object – or *theorem* – has the form

the construction(-object) c is a proof(-object) for the proposition A .

¹¹ Prawitz (1971. Remark 1.6. p. 243), Dummett (1977, p. 121–122) and (1991, p. 248), and Sundholm (1983a).

which form was hinted at already by Heyting:¹²

Every theorem has the form (if enunciated without abbreviations):

"A construction with such and such properties has been effected by a mathematician",

as well as by Freudenthal, according to whom

jeder Satz, wenn man ihn erst einmal intuitionistisch eindwandfrei formuliert, automatisch seinen ganzen Beweis enthält.¹³

The proof-object is the construction-object, that is, the object of the construction-act that forms part of the proof-act. At this point, great confusion would result if the relevant distinctions with respect to proofs and constructions were neglected. "The proof is the construction, that is, the object of the construction, which is part of the proof."¹⁴

However, the construction, that is, the proof-object, and the theorem proved, that is, the object of the proof-act, are not the only objective features of the act of proving; there is also what Per Martin-Löf has called the *trace*, or *track*, of the proof-act, and which corresponds to the third of my readings of 'construction'.¹⁵ This trace is what remains when the act has been completed, for instance, – Martin-Löf's example – literally the track resulting from a completed skiing-tour, possibly comprising several sub-tracks, in case there were several stations to be reached under way, in the race to the final goal, or object, of the skiing-tour in question. The traces of acts need not be material (physical); I have used the example of a written travelogue, which is a careful description of a journey, that is, an act of traveling, to a certain object, or goal.¹⁶ Such an objec-

¹² Heyting (1958, p. 278).

¹³ Freudenthal (1937, p.112).

¹⁴ This is an extreme example, specifically designed for the purpose of exhibiting the implicit dangers, but formulations tending in this direction can be found in foundational writings on intuitionism – by writers belonging to both opposing camps.

¹⁵ In an – as yet unpublished – lecture *On the relation between mathematics, logic and the theory of knowledge*, Paris, April 1992. I am indebted to Professor Martin-Löf for allowing me to read his notes for the lecture in question.

¹⁶ In my (1993).

tive "blue-print" of travel-acts can be used by someone else for making the journey to the same goal, for instance, Venice. In cookery, or, more elaborately, culinary art, the traces can be material – sets of dirty pans, arranged in certain order, twigs of thyme, specks of butter, etc., – or descriptive in the form of written recipes. The *mathematical case resembles the culinary one: the material trace of the act of demonstration through which a certain theorem has been obtained, will consist, perhaps, of sheets of scrap-paper, empty coffee-mugs, chalk-marks on blackboards, etc.* This material part of the trace of an act of mathematical demonstration is often supplemented with a – no less objective – descriptive part in written form as a piece of mathematical text. This component is the important trace of a mathematical proof-act. The notion of a demonstration-trace plays a central role within the philosophy of mathematics. When a philosopher of mathematics speaks about a "proof", as often as not, he means a proof-trace. It is clearly what is at issue in such examples as:

"If you want to know what the theorem means, if you really want to understand it, don't look at the formulation, but look at the proof."

"Euclid's proof of the infinitude of primes is different from, and less interesting than, Euler's proof. The latter proof has an interesting topological significance."

Of the three notions of proof – act, trace, and object – the former two carry an epistemological import, whereas the last one does not: a proof-object is a mathematical object like any other, say, a function in a Banach space, or a complex contour-integral, whence, from an epistemological point of view, it is no more forcing than such objects.

Against the background of this tripartite distinction, there are two clear alternatives for the interpretation of natural deduction derivation-trees: proof-objects and proof-traces. We shall start our inquiry by considering the standard Gentzen-format, also favoured by Professor Prawitz. It operates with assumption-formulae, and the end-formulae of derivation-trees are ascribed the property of being derivable from (the remaining open) assumptions. In an interpreted Frege-calculus, *simple truth is the property ascribed to the propositions expressed by the (derivable) end-formulae of proof-*

trees. In a natural deduction calculus in Gentzen's original format, the proposition expressed by the end-formula of a derivation-tree is not ascribed simple truth, but the weaker property of truth under the assumed truth of certain propositions. On my preferred reading of Heyting, in order to have the right to ascribe truth to a proposition, one must have exhibited a proof-object for the proposition in question, that is, the enunciation

A is true

means the same as

Proof(A) is inhabited.¹⁷

This enunciation one is allowed to assert as soon as one has demonstrated that

c is a proof(-object) of A,

for some c. In order to have the right to assert the dependent, or *hypothetical, truth* of the proposition C, that is, that

C is true, *provided* that A_1 is true, ..., A_k is true,

one must exhibit a hypothetical, or dependent, proof

t is a proof of C, *provided* that x_1 is a proof of A_1 , ..., x_k is a proof of A_k .

The latter enunciation one is entitled to make when the inference-rule

$$\frac{c_1 \text{ is a proof of } A_1, \dots, c_k \text{ is a proof of } A_k}{t[c_1/x_1, \dots, c_k/x_k] \text{ is a proof of } C}$$

is valid.

¹⁷ The type Proof(A) is given as soon as we know that A is a proposition: to understand the meaning of a sentence is (partly) to know what would be a proof-object for the proposition expressed by the sentence in question, so the rule of inference

$$\frac{A \text{ is a proposition}}{\text{Proof}(A) \text{ is a type}}$$

is analytically valid. See my (1997), where these matters are spelled out a bit more.

In an interpreted mathematical calculus, the formulae express propositions and the terms stand for mathematical objects of certain kinds.¹⁸ The derivation-trees themselves also stand for certain objects, namely the (dependent) proof-objects that serve to make the proposition expressed by the end-formula of the tree a (dependently) true proposition.

This conclusion, we can say, gives philosophical content to the so called Curry-Howard *Propositions-as-Types* isomorphism between the logical formalism and a typed lambda-calculus.¹⁹ Professor Prawitz was one of many researchers who strove to formulate such a correspondence explicitly:

[N]ote a similarity between natural deduction and an extended λ -calculus for defining functionals. For instance, a derivation of B from the hypothesis A may be looked upon as an open term $t(x)$ in which x is a variable ranging over proofs of the hypothesis A and which yields a proof $t(p)$ of B whenever a proof p of A is substituted for the variable x . The proof of $A \supset B$ obtained by implication introduction applied to the derivation of B from A may then be identified with the function $\lambda x.t(x)$ which applied to any proof p of A yields the proof $t(p)$ of B.

...
[A] proof obtained by applying implication elimination to a proof q of A ... B and a proof p of A may be identified with the result of applying the function q to p . If q is written in the form $\lambda x.t(x)$, the proof may then be written $(\lambda x.t(x))(p)$, which is identical with $t(p)$.²⁰

Thus, the $\supset$ -reduction introduced by Professor Prawitz is seen to be essentially the same as the law of λ -conversion just stated.²¹ When a proof uses many intertwined maxima, be they of $\supset$ -form or not, that is, applications of introduction rules immediately followed by applications of the corresponding elimination rule, several reductions, or conversions, may be called for in order to bring it into normal, or *canonical*, form. The situation is exactly parallel to that

¹⁸ Note that in standard Natural Deduction the formulae play a double role; they also stand for assertions, that is, interpretations of derived end-formulae of proof-trees.

¹⁹ See Howard (1980). A partial anticipation of the Curry-Howard insight can be found in Ladrière (1951, Section 6, pp. 376–378) which bears the title 'Le théorème de Gentzen et le calcul de la conversion- λ '.

²⁰ (1981, p. 248, where I have changed the implication-symbol), but see also the earlier (1970) and (1971, Ch. IV) for more discussion.

²¹ (1965, p. 37)

of arithmetical computation, where a number in non-primitive form, that is, a number whose expression contains defined rather than primitive function-symbols, may need evaluation through several computation-steps before it reaches standard, or canonical, form as an (Arabic) numeral. For instance, $(6!/8) + 3 = (720/8) + 3 = 90 + 3 = 93$.²²

From the passage just quoted it is clear that Professor Prawitz grants that proofs are (isomorphic to) terms ("functionals") in an extended λ -calculus. Given this, it is not clear to me why he does not accept proofs as ordinary mathematical objects and that the semantic value of an interpreted natural deduction derivation is a (possibly dependent) proof-object for the proposition expressed by the end-formula of the derivation in question.

It should be stressed that it is for derivations in the standard Gentzen-format that the Curry-Howard isomorphism runs smoothly and it is such derivation-trees that provide notations for proof-objects. For the derivations in the sequential formulation from the First Consistency proof, matters are different. Here the derivable objects are not formulae, but sequents. A sequent expresses that the consequent proposition is a consequence of the propositions that serve as antecedents. The sequent S :

$$A_1, \dots, A_k \Rightarrow C,$$

expresses the relationship

that if A_1 is true, then if A_2 is true..., then if A_k , then C is true,

irrespective of whether the antecedent propositions $A_1, \dots, A_k$ really are true. When matters are as the sequent states we say that S *holds*.

The Curry-Howard isomorphism can be extended to sequents so that, just as propositions are true when suitable proof-objects exist, a sequent holds when there exists a suitable verification, namely, in the case of S , a function f of the type

$$(\text{Proof}(A_1)) \dots (\text{Proof}(A_k)) \text{Proof}(C),$$

that is, a function taking proof(-objects) for the propositions $A_1, \dots, A_k$

²² See Martin-Löf (1994).

into a proof-object for the proposition C, irrespective of whether there are any proofs of the propositions in question.

The nodes in a sequential natural deduction derivation-tree are occupied, not by formulae, but by sequents. The claim made in the theorem that is demonstrated by such a derivation is that the end-sequent holds, whereas the individual steps, out of which the derivation is composed, indicate separate acts of inference from claims that certain sequents hold to other claims of this form. A sequential derivation-tree provides a notation for an act-trace. The execution of the corresponding act of demonstration results in the claim that the consequence-relationship expressed by the end-sequent holds. From an epistemological point of view, the sequential form is the proper one in which to display natural deduction derivations: they are the representations of the demonstrations through which theorems become known. Possession of such a derivation allows one to carry out a demonstration for oneself, just in the same way that a proof given in a mathematical text allows us to do so.

A contentual difference between the two formulations of natural deduction is now clear: standard derivations stand for dependent proof-objects, whereas sequential derivations are act-traces. However, as is well-known, we can readily step from one to the other. In particular, from a derivation in the Gentzen standard format, we easily find a corresponding sequential derivation. What significance does this metamathematical observation carry on the level of interpreted, meaningful formalisms?

In order to spell this out, consider an objectively *correct* "statement" (enunciation) in the fully explicit form

(*) c is a proof-object for the proposition A.

We have no right to assume, of course, that the enunciation (*) has been demonstrated. The correctness of (*) only ensures us that it *can be demonstrated*, but it does not, as such, show us how to set about in order to obtain the desired demonstration in question. This is where the proof-object c comes in handy. Such proof-objects are meaningful counterparts to Gödel-codes of the sort that was described above. Thus, the proof-object c comprises information as to (i) the rule according to which it was formed, (ii) the inferentially relevant components out of which the proposition A was built, and

(iii) proof-objects for the premisses of the rule in (i) according to which the object c was formed. For instance, in the very simplest case, that of a proof formed by $\&$ -introduction (*) has the following form:

$\&I(A, B, a, b)$ is a proof of $A \& B$,

where A and B are the propositions out of which the conjunction $A \& B$ is formed and where a and b are proof-objects of the propositions A and B , respectively. Accordingly, the information contained in such a proof-object allows us to span – from below – a formal structure, which can be read with content as a proof-*trace* P with (*) at its conclusion. In the simple case of $\&I$, the first step will be:

a is a proof of A	b is a proof of B
$\&I(A, B, a, b)$ is a proof of $A \& B$	

This structure can be extended upwards by means of suitable similar steps, that depend on the particular forms of the objects and propositions involved. It should be noted that a certain stages of this process, for instance, those corresponding to an $\supset$ -I, the required steps will produce dependent proof-objects. Also, at some steps what might be required is not a proof-object of a proposition, but a verification-object for a sequent. All steps, however, will be completely deterministic, owing to the careful design of the proof-object machinery. Eventually, because of the presupposed correctness of (*), the various branches of the edifice that has been generated will break off in suitable axioms, that is, claims that are immediately – analytically – evident from the concepts they contain, for instance, of the form

0 is an element of N ,

or of the form

x is proof of B , provided that x is a proof of B .

In this fashion, then, a formal structure is spanned. When considered in the direction opposite to that of its generation, that is, from top to bottom, it can be read as a blue-print for acts of demonstration with the claim (*) as its object.

Michael Dummett has drawn

a distinction between a *proof proper* – a *canonical proof* – and the sort of argument which will normally appear in a mathematical article or textbook, an argument which we may call a '*demonstration*'. A demonstration is just as cogent a ground for the assertion of its conclusion as is a canonical proof, and is related to it in this way: that a *demonstration of a proposition provides an effective means* for finding a canonical proof. But it is in terms of a canonical proof that the meanings of the logical constants are given.²³

Armed with the distinction between proof-acts, proof-objects and proof-traces, Dummett's observation calls for emendation. On my view, one does not assert a proposition, but an enunciation that a proposition is true, which in fully explicit form has the shape

(**) c is a proof of A,

where A is the proposition in question.²⁴ I must have demonstrated such a claim in order to have the right to make it. The proof-object c need not be canonical: it can certainly contain defined, non-primitive, parts. It must however, be determinately evaluable to canonical form, just as a number(-expression) must be evaluable to the canonical form of a numeral; to this extent it is, indeed, a program, or 'effective means', for obtaining a *canonical* proof. It is not, however, on its own a demonstration of the claim (**). The written demonstrations occurring in mathematical texts, on the other hand, are traces of proof-acts, that is, demonstration-traces. The canonical/non-canonical dichotomy applies to proof-objects for propositions, whereas the Dummettian demonstrations are the *traces* of proof-acts.

²³ Dummett (1975) at (1978, p. 240).

²⁴ For some reasons that the logical form of an assertion is not propositional, but takes the form

proposition A is true

see my (1998, section 3).

3. *Conservativeness and harmony*

Dummett has also noted that

there are always two aspects of the use of a given form of sentence: the conditions under which an utterance of the sentence is appropriate, which includes, in the case of an assertoric sentence, what counts as an acceptable ground for asserting it; and the consequences of an utterance of it, which comprise both what the speaker commits himself to by the utterance and the appropriate response on the part of the hearer, including in the case of assertion, what he is entitled to infer from it if he accepts it.

It is plain, ..., that we may legitimately demand a certain consonance between the two aspects of the use of a given form of expression, In the simplest cases, however, it is plain that the requirement of consonance may be expressed as the demand that the addition of the given expression to the language yields a conservative extension of it.²⁵

Later Dummett came to replace the term 'consonance' with that of *harmony*.²⁶

For those familiar with *Natural Deduction*, the resemblance of the Dummett harmony-condition to the inversion principle of Prawitz should be manifest.²⁷

Now, Dummett's point is, taking his cue from Belnap's response to Prior on Tonk, that harmony should impose conservativeness of the expanded language over the old one.²⁸

Professor Prawitz was, around 1985, the first to observe that harmony – in the form of introduction- and elimination-rules that permit inversion – does not suffice for conservativeness.²⁹ A clear statement can be found in his (1994) review of Dummett's *The Logical Basis of Metaphysics*:

Dummett ... suggests ... that the requirement of harmony between the two aspects of the use of an expression can be made more precise by saying that it is equivalent to requiring that the addition of the expression to a language should not license a use of the old vocabulary which was not already licensed in the original language. This can hardly be correct, however, because from Gödel's

²⁵ Dummett (1973, pp. 396–397).

²⁶ For instance, at (1973, p. 454) and (1991, *passim*).

²⁷ Prawitz (1965, p. 33)

²⁸ Belnap (1962).

²⁹ Prawitz (1987, p. 147), but see also (1985, p.166) and (1987a, 476–477).

incompleteness theorem we know that the addition to arithmetic of higher order concepts may lead to an enriched system that is not a conservative extension of the original one in spite of the fact that some of these concepts are governed by rules that must be said to satisfy the requirement of harmony.³⁰

Surely Professor Prawitz is right here, and the point is a substantial one that is well worth spelling out. Gödel's proof of his Incompleteness theorem is perfectly constructive. It is also applicable to constructive systems such as Heyting arithmetic. The Gödel proof provides a specific closed formula G , in the language of the chosen formalism, with the following properties:

- (i) if the formalism is consistent, the truth of the (proposition expressed by the) formula G cannot be demonstrated therein,

and

- (ii) the proposition (expressed by) G is true.

We are contentually – *inhaltlich* – convinced of the consistency of the chosen formalism. Consequently, the truth of the proposition G has not been demonstrated solely according to the rules and axioms of the original formalism. In the case of arithmetic, for instance, what is called for, over and above the arithmetical means of proof, is the use of the concept of truth for sentences in the arithmetical language. This can be obtained, as Professor Prawitz rightly remarks, through the use of certain higher order concepts, for instance, quantification with respect to propositional functions. Constructivistic purists might here object that these higher-order concepts are *impredicative*, and, therefore, non-constructive, whence the whole reasoning is null and void. I concur in their rejection of impredicativity, but, in order to obtain a truth-predicate for the sentences of arithmetic, there is no need to opt for full second-order quantification. Much weaker principles of inductive definition, the constructive content of which is not open to doubt, suffice for the required demonstration. Be that as it may, in either case the rules for the novel concepts are in harmony: both higher-order arithmetic and the theories of inductive definitions satisfy the Prawitz inversion-principle.

³⁰ Prawitz (1994, p. 375).

The original arithmetical formalism could also be extended with a truth-predicate for sentences of the original language, plus the so-called Tarski T-schema, which apparatus would then be used to show that the Gödel-number $G^\#$ of the Gödel-sentence G falls under the T-predicate, that is, that the proposition expressed by $T(G^\#)$ is true. The use of this theorem

$T(G^\#)$ is true,

together with the relevant instance of the T-schema

$(T(G^\#) \leftrightarrow G)$ is true,

yields our desired conclusion

G is true.

This, in fact, is the route that the demonstration would take also in the other cases; rather than using the T-predicate as primitive and the T-schema as a postulate, however, these would then be defined or derived, respectively, using either second-order quantification or inductive definitions, depending on the means chosen for extending the original formalism.

The Gödel incompleteness phenomena presuppose that the theorems of the interpreted formalism have the form:

proposition A is true.

From an intuitionistic standpoint, such a theorem means the same as

$\text{Proof}(A)$ is inhabited,

and the latter enunciation is construed as an ellipsis – or “incomplete communication” – for a completed assertion of the form

c is a proof of the proposition A .

In the words of Herman Weyl, that the proposition A is true is a *Urteilsabstrakt* that is, a partial judgement or incomplete communication.³¹ Such an incomplete communication is grounded in, or founded upon, a complete communication of the above form that a

³¹ Detailed references to Weyl, and others, concerning the provenance of *Partialurteile*, incomplete communications, and so on, are given in my (1994).

certain construction is a proof for A. Professor Prawitz is certainly right that, on the level of the *Partialurteile*, where proof-objects are suppressed, the addition of certain harmonizing principles of extension may yield a non-conservative extension with respect to the original language. His Gödel-based examples, though, do not work when theorems are not incomplete communications of the form

proposition A is true,

but complete communications to the effect that a certain construction(-object) is a proof of A. We can describe the situation arising from the Gödel theorem in the following way: since the proposition expressed by the Gödel formula G for the chosen formalism T is known by us to be true, we must have demonstrated its truth. That is, we must have demonstrated that

d is a proof of G,

for a certain proof-object d. In the course of this demonstration we have to have used various principles of definition (and perhaps also of demonstration) that are not available in T. Furthermore, the proof-object d cannot even be formulated in the theory T, that is, the claim that

d is a proof of G,

does not belong to the language of T.³² Accordingly, when this claim is established using suitable definition-principles for the stronger concepts that are needed for proving the truth of the (proposition expressed by the) Gödel-sentence G, we do not have a violation of, since the claim that is demonstrated using the stronger principles is not even formulated in the language of the old theory: the proof-object d cannot be expressed there. So when the proof-objects are suppressed, the Gödel-theorem provides a refutation of the entailment from Harmony to Conservativeness, whereas with their inclusion, it does not.

³² My analysis of the Gödel theorem in the context of proof-objects follows that of Martin-Löf (1994). I first heard him make the point in 1978 in discussions at Stockholm, in which Dag Prawitz also participated. Daniel Isaacson (1987) is also relevant.

4. Dummett's 'Fundamental Assumption'

Harmony is not the only topic where good use could have been made of the distinction with respect to proof: act, trace and object. An issue raised by Dag Prawitz serves to pinpoint another area where such use can be made, namely that of the **Fundamental Assumption** for Dummett's meaning-theoretical project:

The strategy of proof-theoretic justifications ... is ... to show that we can dispense with the rule up for justification: if we have a valid argument for the premisses of a proposed application of it, we already have a valid argument, not appealing to that rule, for the conclusion. But the justification depends heavily upon what we may call the 'fundamental assumption': that, if we have a valid argument for a complex statement, we can construct a valid argument for it which finishes with an application of one of the introduction rules governing its principal operator.³³

Dummett devotes a whole chapter of his work to an examination of the plausibility of this fundamental assumption, and at the end of his examination he has to concede that the assumption has been left 'very shaky'.³⁴ Similarly, with respect to the notion of 'canonical', the result reached at the end of a protracted discussion is of a disappointing complexity:

an argument is *canonical* if:

- (a) its final conclusion is a closed sentence;
- (b) all its initial premisses are closed atomic sentences;
- (c) every atomic sentence in the main stem is either an initial premiss or is derived by a boundary rule;
- (d) every closed complex sentence in the main stem is derived by means of one of the given set of introduction rules.³⁵

Professor Prawitz comments on the attempt to base a meaning theory on a verificationistic conception of the use of sentences in his review:

Dummett makes several interesting contributions to ...this project ... but, as he points out, it depends on the plausibility of what he dubs the fundamental as-

³³ (1991, p. 254).

³⁴ (1991, p. 277).

³⁵ (1991, p. 260).

sumption, viz. that a true complex statement always has a verification that ends with an application of an introduction rule. The viability of this ... thus depends on the extent to which it can be incorporated into a verificationistic meaning-theory of the language as a whole. The reviewer completely agrees with this and shares some (although not all) of Dummett's doubts about the fundamental assumption. However, a failure of this assumption is more serious than the book suggests: it probably means a failure of verificationism since we have hardly any other idea of the form of verifications of logically complex sentences.³⁶

These complexities can largely be avoided through a use of the proof-trace/proof-object distinction.

In his writings Dummett prefers to use the term 'statement'. This term, indeed, has had quite a good innings in recent British philosophy. Examples of all of the following uses of 'statement' can be found in the not too ancient Oxford literature:

- (a) meaningful sentence;
- (b) proposition;³⁷
- (c) state of affairs;
- (d) the act of stating/asserting;
- (e) what is asserted in an act of assertion,
- (ei) the assertion made;
- (eii) the assertion made, but stripped of its assertoric force.

The list is probably not complete.³⁸ With respect to the matter under discussion, we may ask: does an argument (for a statement) end with a proposition, or with an assertion? That is, should 'arguments' be understood as proofs for propositions or as demonstrations of assertions? A possible answer seems to be that Dummett's *statements* are his contentual counterparts of formulae, whence an *argument for a statement* would have to be the contentual counterpart of a derivation of a formula. The ambiguity of "statement" would then be accounted for, since, as has already been remarked, the

³⁶ (1994, p. 375).

³⁷ The most common use of 'statement' is to serve in the same roles as *proposition*, that is, Fregean *Gedanke*, but one which also takes phenomena of indexicality into account.

³⁸ There are authors – at least one! – where all five notions are conflated under the statement-umbrella, but I do not wish to imply that Michael Dummett is one of them. His *statements* seem to oscillate between (b), (ei) and (eii). Indeed, the point is moot whether Dummett would admit a distinction between (b) and (eii).

formulae of a standard Gentzen natural deduction formalism serve in two roles, namely those of being the formalistic simulacra for propositions, that is, arguments for the logical constants, and for asserted theorems, that is, the countentual counterparts of the derived end-formulae.³⁹

These, and other difficulties, as well as the complexities in the explanation of canonicity, can be obviated through the use of basic distinctions under consideration. In order to explain a proposition we must lay down what its canonical proof-, or verification-objects are and how they would be put together out of certain parts, such as proof-objects of other propositions.⁴⁰ Since the verifications are meant to form a type of objects, such an *application criterion* is not enough; we also have to provide an *identity criterion*, that is, in order to explain the proposition fully, we must explain what it is for two such canonical verifications to be equal verifications of the proposition in question. A non-canonical verification(-object) is a means, or *program*, which on execution yields a canonical proof. In order to have the right to make the assertion that a certain proposition A is true, I must exhibit a verification-object – not necessarily canonical – for A. When the claim that

(α) A is true

is demonstrable, but possibly not yet demonstrated, this means that a verification d of A *can* be found. When a non-canonical verification d has been found, so that

(β) d is a proof-object for A,

execution, or evaluation, of the program d will yield an object d' in canonical form, such that

³⁹ In this context it is also interesting to note that Professor Prawitz makes use of the term *verification* in the quote above. It also admits of a similar, tripartite distinction between (i) the act of verification, (ii) the verification-object resulting from that act, and (iii) the trace of the verification-act. There is also the object of the act of verification, namely what is verified, for instance, that a certain proposition is true.

⁴⁰ The term *verification* is particularly apt as an alternative term here, since the ensuing notion of propositional truth, according to which proposition A is true = there exists a proof for A, is given a so called *truth-maker* analysis. The matter is dealt with at some length in my (1994).

(γ) d and d' are equal proof-objects of A ,

and thus we may rightly conclude

(δ) d' is a proof of A .

So we have found a canonical verification of A . That is, as soon as the truth of the proposition A is demonstrable, that is, as soon as the existence of any verification whatsoever for the proposition A is demonstrable, then a canonical verification can be found.

From the present perspective, the complexities in Dummett's formulation of canonicity can be put down to his using *argument* for both proof-objects and proof-traces. In the mathematical case, at least, a theory based on the distinction between proof-acts and demonstration-traces can be given a very smooth formulation, as is witnessed by the development of Martin-Löf's type theory, and its success as a so called logical framework in computer science. I cannot pretend that all problems that such a theory would have to overcome, say, concerning ordinary empirical discourse, have been resolved. In that area an act of demonstration will be an act of perception, and the verification-object will be the object of the act of perception, namely, what is perceived in the act in question. So my demonstration of the truth of the proposition that the cat sits on the mat will comprise a verification-act that issues in the perception-object, that is, a certain state of affairs, namely, the cat's sitting on the mat, which makes the proposition that the cat sits on the mat true. In this formulation *c is a proof of A* is replaced by (the state of affairs) *s is a truth-maker for A*, or *s makes A true*.

In the case of our cat we have the following situation:

the perception-object (verification-object)
 the cat's sitting on the mat
 makes the proposition
 that the cat sits on the mat
 true.

The verification-object serves as the truth-maker for the proposition.⁴¹

⁴¹ Note the difference between the object of the verification-act and the verification-object, that is the difference between the object of the proof-act, that is, the theorem proved, and the proof-object.

An elaboration of this idea, where states of affairs serve as truth-makers, that is, verification(-object)s would, if the parallel between the empirical and the mathematical cases is going to hold, demand that explanations be given of what a canonical truth-maker is, and how it may be put together out of certain constituents. For instance, a canonical proof-object would be what is seen in an act of watching the rain's falling, namely, the falling (of the) rain. So it is the falling rain, or perhaps the rain's falling, that serves as a truth-maker for the proposition that it is raining. Clearly, much remains to be done in order to provide a constructivist theory of states of affairs. Possibly, the *moments* – but then conceived of constructively – that are used by Mulligan, Simons and Smith in their seminal article will serve also in the constructive case.⁴² This is a matter for further investigation. It is equally the case for the alternatives advocated by Dummett, and by Prawitz, that much further work is needed in order to have a viable theory. In the present section I was mainly concerned to show that the neglect of the proof-act/proof-object distinction is a source of considerable complexities which are readily avoided through its observation.

5. Assumptions and the constructive meaning of implication

During the first half of 1994 I had the privilege, at the invitation of Dag Prawitz, to conduct a seminar on Truth in the Philosophical Institute at Stockholm University. Fierce discussion would often result from my presentations and, in particular, Bolzano's explanation of truth for his *Sätze an sich* proved controversial. According to Bolzano, a proposition (*Satz an sich*) has the form

A has b,

where A and b are suitable *Vorstellungen an sich*. For such propositions Bolzano explained truth in the following way:

The proposition A has b is true if and only if A really has b.⁴³

⁴² (1984).

⁴³ (1837, § 25).

The controversial point was whether *really* really has to occur in the definiens of the definition. It was first recast into the question whether there was a difference between:

- (i) the proposition A is true, and
- (ii) the proposition A is really true.

The realists that followed my seminar presentation, among whom Peter Pagin, joined their mentor Bolzano in finding the occurrence of *really* superfluous: according to them there is no relevant difference between (i) and (ii). I myself, together with Per Martin-Löf, argued that there was a difference. Dag Prawitz, as far as my recollection goes, was uneasy with both options, and would not commit himself, but, on balance, leaned towards the Bolzano position.

Using the battery of distinctions now at our disposal, at least one difference between (i) and (ii), can be registered regarding their respective behaviour in assumptions. Assumptions of the form (i) are, as we saw above, made in natural deduction derivations of the standard format. In the sequential form such an assumption corresponds to the axiom

$$A \text{ is true} \Rightarrow A \text{ is true,}$$

or, in fully explicit form, including proof-objects,

$$x : \text{Proof}(A) \Rightarrow x : \text{Proof}(A),$$

where the colon has been used to indicate the relation between object and type. In an interpreted formalism, an assumption in a Gentzen derivation, in fully explicit form, with proof-objects, will take the form

$$x \text{ is a proof of the proposition } A$$

or, using the notation just introduced

$$x : \text{Proof}(A).$$

Subsequently, under this assumption, the conclusion

$$b : \text{Proof}(B),$$

say, will be reached. This *b* is a dependent proof-object for the proposition B, under the assumption that the proposition A is true.

Thus proposition B is *dependently* true, under the assumption in question. A use of $\supset$ -introduction then yields

$$\supset I(A, B, (x)b) : \text{Proof}(A \supset B),$$

where $(x)b$ is a function from $\text{Proof}(A)$ to $\text{Proof}(B)$, that is, an element of the function-type $(\text{Proof}(A))\text{Proof}(B)$, such that $((x)b)(a) = b[a/x]$, when $a : \text{Proof}(A)$.

In an assumption of this ordinary Gentzen type, *nothing* is assumed about what is actually the case: an assumption that A is true, that is, in fully explicit form, an assumption that

$$x \text{ is a proof-object for } A,$$

is not an assumption that it is possible to find a proof for A . The proposition A may turn out to be false – it may even be known to be false – in which case the type $\text{Proof}(A)$ will be empty, so that it is impossible to find a proof of A . The assumption that

$$x : \text{Proof}(A)$$

still makes good sense and can be used for establishing the truth of, say, the proposition

$$A \supset A \& A.$$

This situation we could not have when we assume that A is really true. Then we assume that reality is as is required for the truth of A , so that a proof-object can be found.⁴⁴ The former type of assumptions, that is, assumptions that propositions are true occur in the case of consequences: sequents have antecedents of this form. An assumption that the proposition A is *really* true is construed as an assumption that a proof-object for A can be found.

An alternative formulation can be made in terms of enunciations: if a proof-object for the proposition A can be found, then (the enunciation)

$$A \text{ is true}$$

⁴⁴ In metamathematical terms, the difference between the two types of assumptions comes out as: *assume the formula A* , and *assume that A has a closed derivation*, respectively.

can be known, that is, **A is true** is demonstrable. The *real* truth of the proposition A coincides with the demonstrability of the enunciation that

A is true.⁴⁵

This distinction between assuming that a proposition is true, and assuming that it is really true, that is, assuming that the truth of the proposition A can be known, seems to be of considerable importance in the history and philosophy of logic. On the historical side, Meinong's seminal work *Über Annahmen* is vitiated by its neglect, and it would be a worthwhile historical task to sort it out against the spectrum of distinctions drawn in the present essay.⁴⁶

Within the philosophy of logic, the distinction between the two sorts of assumptions seems to have considerable relevance for the theory of conditionals, including entailment theory and belief-revision theory. The theorems reached from assumptions that propositions are true serve to establish that certain consequence-relations – "sequents" – hold. Assumptions that the truth of propositions can be known do not serve to establish that sequents hold, but serve instead in attempts to validate *rules of inference*. Indeed, a sequent, when it holds, preserves propositional truth from its antecedents to the consequent. Valid inference, on the other hand, preserves knowability in the step from its premisses to the conclusion. One way in which to make sense of *intensional* propositional entailment, or relevant implication, is as an attempted propositionalization of the relation that obtains between the propositions that serve as contents of the premiss- and conclusion-judgements of a valid inference. That is, using "E" as an entailment-connective, the proposition AEB would have to be⁴⁷ true, when the inference

⁴⁵ Per Martin-Löf (1998) has convincingly argued that the knowability of truth pertains to *real* truth. I am indebted to his paper for my understanding of how *really* functions in combination with truth.

⁴⁶ Meinong's notion of an *Objektiv* is also notoriously slippery, and seems to oscillate between proposition, state of affairs and assertion/judgement made, but stripped of its assertoric force.

⁴⁷ Note that this explanation does not provide a constructive meaning for E. In order to do that one has to explain what a canonical proof of AEB is and when two such proofs are equal proofs of AEB.

A is true
B is true

is valid. Since the validity of inference is explained in terms of the preservation of knowability the required intensionality of the entailment connective E is ensured.

In Belief-revision theory, the so called Ramsey-test is pre-eminent:

If two people are arguing 'If p will q?' and both are in doubt as to p, they are *adding p hypothetically to their stock of knowledge and arguing on that basis about q*.⁴⁸

The hypothetical additions involved here are clearly not assumptions of the Gentzen standard form that a proposition is true; it is one's stock of *knowledge* that one must extend hypothetically. In the light of the foregoing discussion, assumptions of the form that a proposition is really true, that is, that its truth is demonstrable, or even known, seem worthwhile candidates here for explicating Ramsey. To explore this matter in full, however, would lead far outside the scope of the present essay.⁴⁹ I only make the suggestion here in order to exemplify the importance of the distinction between the two kinds of assumption.

Professor Prawitz himself has touched on such matters in an interesting early paper "On Constructive Logic and the Concept of Implication".⁵⁰ There he considers two notions of implication, namely

(i) implication as the existence of a chain of thought;

and

(ii) implication as the soundness of a rule.

Without entering into the details of his treatment, I note that the latter notion corresponds quite well with the notion of the holding

⁴⁸ F. P. Ramsey, 'General Propositions and Causality' (1929), quoted after (1978, p.143, f. n. 1), my emphasis.

⁴⁹ To judge from the avalanche of recent material, a sizable monograph, rather than a brief paragraph in an essay, seems to be called for in order to do justice to this topic.

⁵⁰ (1963). The English version of the Swedish title is mine.

of a sequent. A sequent holds when it preserves propositional truth. The existence of a chain of thoughts, on the other hand, under the present perspective corresponds most naturally to the validity of an inference. An inference from the premisses $J_1, \dots, J_k$ to the conclusion J – that is, an "implication" from premisses to conclusion – is valid when there exists a chain of thought (of *immediately* valid steps) from the premisses to the conclusion. The immediate validity of an inference step I has to be analytically grounded in the meaning of the terms – in the nature of the concepts – out of which I has been built.⁵¹

With this example my discussion comes to an end. It is with great pleasure, and with my warmest thanks for what I have learned from him, that I submit this *apologia pro obiecto verificationis* to the scrutiny of Dag Prawitz, in the hope that it might provoke him to continue our discussion.

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⁵¹ The notion of inferential validity has been dealt with at length along these lines in my (1997) and (1998). Lest anyone be convinced that I commit an *elementary* error at this point, permit me to observe that I am not unaware that several (Quinean) bullets have been bitten here.

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