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By solving a master equation in the Sierpiński lattice and in a planar random-resistor network, we determine the scaling with size L of the shot noise power P due to elastic scattering in a fractal conductor. We find a power-law scaling $P \propto L^{d_f-2-\alpha}$, with an exponent depending on the fractal dimension d_f and the anomalous diffusion exponent α. This is the same scaling as the time-averaged current $\bar{I}$, which implies that the Fano factor $F = P/2e\bar{I}$ is scale-independent. We obtain a value of $F = 1/3$ for anomalous diffusion that is the same as for normal diffusion, even if there is no smallest length scale below which the normal diffusion equation holds. The fact that F remains fixed at $1/3$ as one crosses the percolation threshold in a random-resistor network may explain recent measurements of a doping-independent Fano factor in a graphene flake.

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Diffusion in a medium with a fractal dimension is characterized by an anomalous scaling with time t of the root-mean-squared displacement Δ. The usual scaling for integer dimensionality d is $\Delta \propto t^{1/2}$, independent of d. If the dimensionality d_f is noninteger, however, an anomalous scaling $\Delta \propto t^{1/(2+\alpha)}$, with $\alpha > 0$, may appear. This anomaly was discovered in the early 1980s [1–5] and has since been studied extensively (see Refs. [6,7] for reviews). Intuitively, the slowing down of the diffusion can be understood as arising from the presence of obstacles at all length scales—characteristic of a self-similar fractal geometry.

A celebrated application of the theory of fractal diffusion is to the scaling of electrical conduction in random-resistor networks (reviewed in Refs. [8,9]). According to Ohm's law, the conductance G should scale with the linear size L of a d-dimensional network as $G \propto L^{d-2}$. In a fractal dimension, the scaling is modified to $G \propto L^{d_f-2-\alpha}$, depending both on the fractal dimensionality d_f and on the anomalous diffusion exponent α. At the percolation threshold, the known [6] values for $d = 2$ are $d_f = 91/48$ and $\alpha = 0.87$, leading to a scaling $G \propto L^{-0.97}$. This almost inverse-linear scaling of the conductance of a planar random-resistor network contrasts with the L-independent conductance $G \propto L^0$ predicted by Ohm's law in two dimensions.

All of this body of knowledge applies to classical resistors, with applications to disordered semiconductors and granular metals [10,11]. The quantum Hall effect provides one quantum mechanical realization of a random-resistor network [12], in a rather special way because time-reversal symmetry is broken by the magnetic field. Very recently [13], Cheianov *et al.* announced an altogether different quantum realization in zero magnetic field. Following experimental [14] and theoretical [15] evidence for electron and hole puddles in undoped graphene [16], Cheianov *et al.* modeled this system by a degenerate electron gas [17] in a random-resistor network. They analyzed both the high-temperature classical resistance as well as the low-temperature quantum corrections by using the anomalous scaling laws in a fractal geometry.

These very recent experimental and theoretical developments open up new possibilities to study quantum mechanical aspects of fractal diffusion, both with respect to the Pauli exclusion principle and with respect to quantum interference (which are operative in distinct temperature regimes). To access the effect of the Pauli principle, one needs to go beyond the time-averaged current $\bar{I}$ (studied by Cheianov *et al.* [13]) and consider the time-dependent fluctuations $\delta I(t)$ of the current in response to a time-independent applied voltage V. These fluctuations exist because of the granularity of the electron charge, hence their name "shot noise" (for reviews, see Refs. [18,19]). Shot noise is quantified by the noise power

$$P = 2\int_{-\infty}^{\infty} dt \langle \delta I(0)\delta I(t)\rangle \tag{1}$$

and by the Fano factor $F = P/2e\bar{I}$. The Pauli principle enforces $F < 1$, meaning that the noise power is smaller than the Poisson value $2e\bar{I}$—which is the expected value for independent particles (Poisson statistics).

The investigation of shot noise in a fractal conductor is particularly timely in view of two different experimental results [20,21] that have been reported recently. Both experiments measure the shot noise power in a graphene flake and find $F < 1$. A calculation [22] of the effect of the Pauli principle on the shot noise of undoped graphene predicted $F = 1/3$ in the absence of disorder, with a rapid suppression upon either p-type or n-type doping. This prediction is consistent with the experiment of Danneau *et al.* [21], but the experiment of DiCarlo *et al.* [20] gives instead an approximately *doping-independent* F near $1/3$. Computer simulations [23,24] suggest that disorder in the samples of DiCarlo *et al.* might cause the difference.

Motivated by this specific example, we study here the fundamental problem of shot noise due to anomalous diffusion in a fractal conductor. While *equilibrium* thermal noise in a fractal has been studied previously [25–27], it remains unknown how anomalous diffusion might affect the *nonequilibrium* shot noise. Existing studies [28–30] of shot noise in a percolating network were in the high-voltage regime where *inelastic* scattering dominates, leading to hopping conduction [31], while in the low-voltage regime of diffusive conduction we have predominantly *elastic* scattering.

We demonstrate that anomalous diffusion affects P and $\bar{I}$ in such a way that the Fano factor (their ratio) becomes scale-independent as well as independent of d_f and α. Anomalous diffusion, therefore, produces the same Fano factor $F = 1/3$ as is known [32,33] for normal diffusion. This is a remarkable property of diffusive conduction, given that hopping conduction does not produce a scale-independent Fano factor [28–30]. Our general findings are consistent with the doping independence of the Fano factor in disordered graphene observed by DiCarlo *et al.* [20].

To arrive at these conclusions, we work in the experimentally relevant regime where the temperature T is sufficiently high that the phase coherence length is $\ll L$ and sufficiently low that the inelastic length is $\gg L$. Quantum interference effects can then be neglected, as well as inelastic scattering events. The Pauli principle remains operative if the thermal energy kT remains well below the Fermi energy, so that the electron gas remains degenerate. We neglect Coulomb interactions between the electrons. In the application to graphene, this requires that the electron and hole puddles have a conductance large compared to e^2/h (no Coulomb blockade).

We first briefly consider the case that the anomalous diffusion on long length scales is preceded by normal diffusion on short length scales. This would apply, for example, to a percolating cluster of electron and hole puddles with a mean free path l which is short compared to the typical size a of a puddle. We can then rely on the fact that $F = 1/3$ for a conductor of any shape, provided that the normal diffusion equation holds locally [34,35], to conclude that the transition to anomalous diffusion on long length scales must preserve the one-third Fano factor.

This simple argument cannot be applied to the more typical class of fractal conductors in which the normal diffusion equation does not hold on short length scales. As representative for this class, we consider fractal lattices of sites connected by tunnel barriers. The local tunneling dynamics then crosses over into global anomalous diffusion, without an intermediate regime of normal diffusion.

A classic example is the Sierpiński lattice [36] shown in Fig. 1 (inset). Each site is connected to four neighbors by bonds that represent the tunnel barriers, with equal tunnel rate Γ through each barrier. (The tunnel rate is voltage-independent in the low-voltage limit considered here.) The

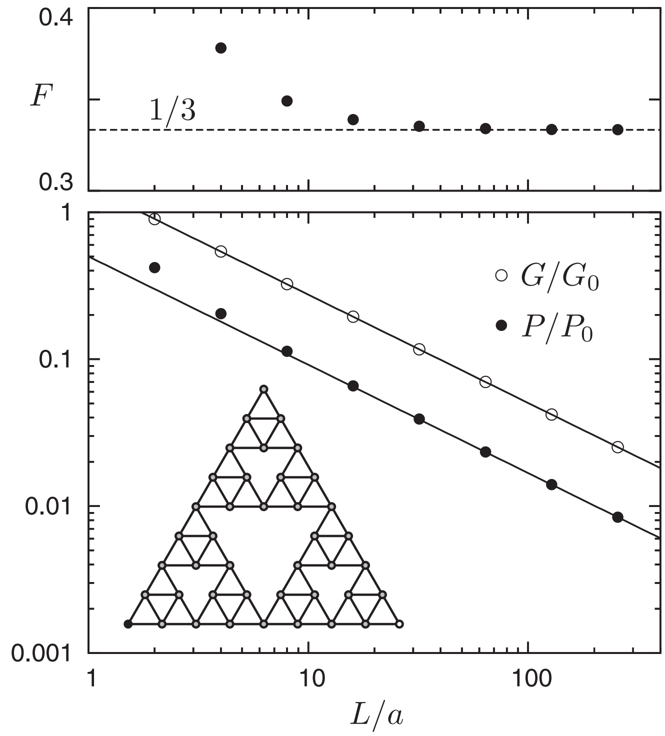

FIG. 1. Lower panel: Electrical conduction through a Sierpiński lattice. This is a deterministic fractal, constructed by recursively removing a central triangular region from an equilateral triangle. The recursion level r quantifies the size $L = 2^r a$ of the fractal in units of the elementary bond length a (the inset shows the third recursion). The conductance $G = \bar{I}/V$ (open dots, normalized by the tunneling conductance G_0 of a single bond) and shot noise power P (solid dots, normalized by $P_0 = 2eVG_0$) are calculated for a voltage difference V between the lower-left and lower-right corners of the lattice. Both quantities scale as $L^{d_f-2-\alpha} = L^{\log_2(3/5)}$ (solid lines on the double-logarithmic plot). The Fano factor $F = P/2e\bar{I} = (P/P_0)(G_0/G)$ rapidly approaches 1/3, as shown in the upper panel.

fractal dimension is $d_f = \log_2 3$, and the anomalous diffusion exponent is [6] $\alpha = \log_2(5/4)$. The Pauli exclusion principle can be incorporated as in Ref. [37], by demanding that each site is either empty or occupied by a single electron. Tunneling is therefore allowed only between an occupied site and an adjacent empty site. A current is passed through the lattice by connecting the lower-left corner to a source (injecting electrons so that the site remains occupied) and the lower-right corner to a drain (extracting electrons so that the site remains empty). The resulting stochastic sequence of current pulses is the "tunnel exclusion process" of Ref. [38].

The statistics of the current pulses can be obtained exactly (albeit not in closed form) by solving a master equation [39]. We have calculated the first two cumulants by extending to a two-dimensional lattice the one-dimensional calculation of Ref. [38]. To manage the added complexity of an extra dimension, we found it convenient to use the Hamiltonian formulation of Ref. [40]. The

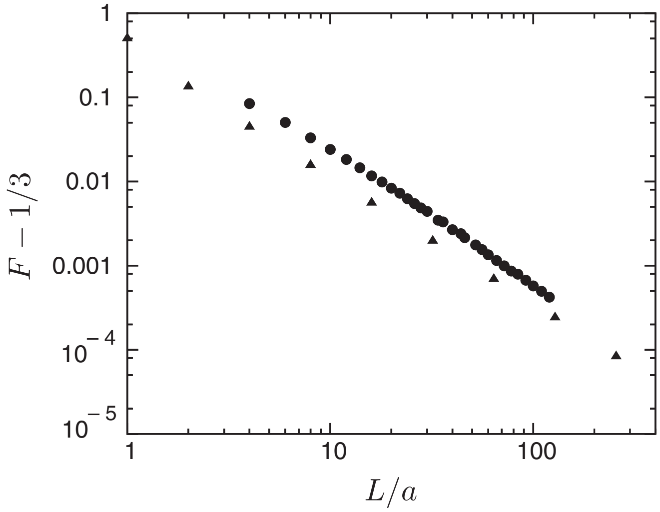

FIG. 2. The deviation of the Fano factor from 1/3 scales to zero as a power law for the Sierpiński lattice (triangles) and for the random-resistor network (circles).

hierarchy of linear equations that we need to solve in order to obtain $\bar{I}$ and P is derived in the supplement [41].

The results in Fig. 1 demonstrate, first, that the shot noise power P scales as a function of the size L of the lattice with the same exponent $d_f - 2 - \alpha = \log_2(3/5)$ as the conductance and, second, that the Fano factor F approaches $1/3$ for large L. More precisely (see Fig. 2), we find that $F - 1/3 \propto L^{-1.5}$ scales to zero as a power law, with $F - 1/3 < 10^{-4}$ for our largest L.

Turning now to the application to graphene mentioned in the introduction, we have repeated the calculation of shot noise and Fano factor for the random-resistor network of electron and hole puddles introduced by Cheianov *et al.* [13]. The construction of the network is explained in Fig. 3. While this model was inspired by a specific application [13], it is generic and representative for any model of two-dimensional percolation in the low-voltage limit. The results, shown in Fig. 3, demonstrate that the shot noise power P scales with the same exponent $L^{-0.97}$ as the conductance G (solid lines in the lower panel) and that the Fano factor F approaches $1/3$ for large networks (upper panel). This is a random, rather than a deterministic, fractal, so there remains some statistical scatter in the data, but the deviation of F from $1/3$ for the largest lattices is still $<10^{-3}$ (see the circular data points in Fig. 2).

In conclusion, we have found that the universality of the one-third Fano factor, previously established for normal diffusion [32–35], extends to anomalous diffusion as well. This universality might have been expected with respect to the fractal dimension d_f (since the Fano factor is dimension-independent), but we had not expected universality with respect to the anomalous diffusion exponent α. The experimental implication of the universality is that the Fano factor remains fixed at $1/3$ as one crosses the percolation threshold in a random-resistor network—thereby crossing over from anomalous diffusion to normal diffusion. This is consistent with the doping-independent Fano factor measured in a graphene flake by DiCarlo *et al.* [20].

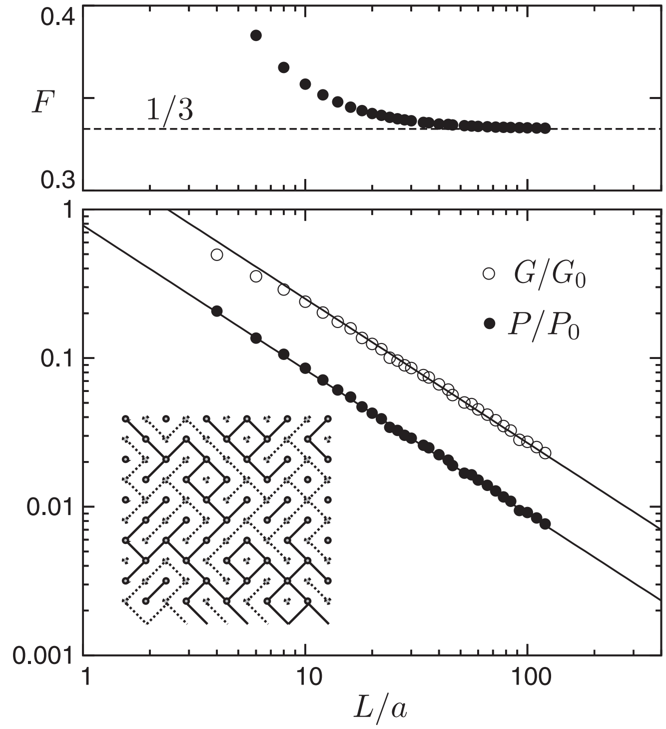

FIG. 3. The same as Fig. 1, but now for the random-resistor network of disordered graphene introduced by Cheianov *et al.* [13]. The inset shows one realization of the network for $L/a = 10$ (the data points are averaged over $\simeq 10^3$ such realizations). The alternating solid and dashed lattice sites represent, respectively, the electron (n) and hole (p) puddles. Horizontal bonds (not drawn) are p-n junctions, with a negligibly small conductance $G_{pn} \approx 0$. Diagonal bonds (solid and dashed lines) each have the same tunnel conductance G_0. Current flows from the left edge of the square network to the right edge, while the upper and lower edges are connected by periodic boundary conditions. This plot is for undoped graphene, corresponding to an equal fraction of solid (n-n) and dashed (p-p) bonds.

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