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19th September 2017

Dedicated to Ya.G. Sinai on the occasion of his 70th birthday

Abstract

In this paper we study a two-dimensional directed self-avoiding walk model of a random copolymer in a random emulsion. The polymer is a random concatenation of monomers of two types, A and B, each occurring with density $\frac{1}{2}$. The emulsion is a random mixture of liquids of two types, A and B, organised in large square blocks occurring with density p and $1-p$, respectively, where $p \in (0,1)$. The polymer in the emulsion has an energy that is minus α times the number of AA-matches minus β times the number of BB-matches, where $\alpha, \beta \in \mathbb{R}$ are interaction parameters. Symmetry considerations show that without loss of generality we may restrict to the cone $\{(\alpha,\beta) \in \mathbb{R}^2 \colon\ \alpha \geq |\beta|\}$.

We derive a variational expression for the quenched free energy per monomer in the limit as the length n of the polymer tends to infinity and the blocks in the emulsion have size L_n such that $L_n \to \infty$ and $L_n/n \to 0$. To make the model mathematically tractable, we assume that the polymer can only enter and exit a pair of neighbouring blocks at diagonally opposite corners. Although this is an unphysical restriction, it turns out that the model exhibits rich and physically relevant behaviour.

Let $p_c \approx 0.64$ be the critical probability for directed bond percolation on the square lattice. We show that for $p \geq p_c$ the free energy has a phase transition along *one* curve in the cone, which turns out to be *independent* of p. At this curve, there is a transition from a phase where the polymer is fully A-delocalized (i.e., it spends almost all of its time deep inside the A-blocks) to a phase where the polymer is partially AB-localized (i.e., it spends a positive fraction of its time near those interfaces where it diagonally crosses the A-block rather than the B-block). We show that for $p < p_c$ the free energy has a phase transition along *two* curves in the cone, both of which turn out to *depend* on p. At the *first* curve there is a transition from a phase where the polymer is fully A,B-delocalized (i.e., it spends almost all of its time deep inside the A-blocks and the B-blocks) to a partially BA-localized phase, while at the *second* curve there is a transition from a partially BA-localized phase to a phase where both partial BA-localization and partial AB-localization occur simultaneously.

We derive a number of qualitative properties of the critical curves. The supercritical curve is non-decreasing and concave with a finite horizontal asymptote. Remarkably, the first subcritical curve does not share these properties and does not converge to the supercritical curve as $p \uparrow p_c$. Rather, the second subcritical curve converges to the supercritical curve as $p \downarrow 0$.

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AMS 2000 *subject classifications.* 60F10, 60K37, 82B27.
Key words and phrases. Random copolymer, random emulsion, localization, delocalization, phase transition, percolation, large deviations.

1 Introduction and main results

1.1 Background

(Linear) copolymers are polymer chains consisting of two or more types of monomer. Random copolymers are copolymers where the order of the monomers along the polymer chain is determined by a random process. In any particular chain, the sequence of monomers once determined is fixed, so a random copolymer is an example of a *quenched random system*. In this paper we will be concerned with copolymers consisting of two types of monomer, labelled A and B. We write $\omega_i \in \{A, B\}$ to denote the type of the i-th monomer, and $\omega = \{\omega_1, \omega_2, \cdots\}$ to denote the full order along the chain, which is truncated at ω_n when the polymer has length n. We will only consider the case where the random variables ω_i are independent and identically distributed (i.i.d.), assuming the values A and B with probability $\frac{1}{2}$ each. In principle, the properties of the polymer depend on ω, and we write P_n^ω for the value of a property P when the polymer has length n and order ω. If $\lim_{n\to\infty} P_n^\omega$ exists ω-a.s. and is non-random, then we say that the property P is *self-averaging*.

Several different physical situations are of interest. For instance, if the monomer-monomer interactions differ for pairs AA, BB and AB, then we may investigate the effect of the randomness on the *collapse transition*, where the polymer collapses from a random coil to a ball-like structure as the temperature decreases or the solvent quality varies. Alternatively, if the two types of monomer interact differently with an impenetrable surface, then we may investigate the effect of the randomness on the *adsorption transition*, where the polymer adsorbs onto the surface as the temperature decreases or the surface quality varies. There are interesting questions about how the location of the collapse transition or the adsorption transition, and the values of associated critical exponents, depend on the parameters controlling the randomness. Many of these questions remain unresolved. For background and references, the reader is referred to Orlandini *et al* [26], [27], Janse van Rensburg *et al* [19], Brazhnyi and Stepanow [5], Whittington [33], and Soteros and Whittington [30].

The problem that we will consider here is the *localization transition* of a random copolymer near an interface. Suppose that we have two immiscible liquids and that it is energetically favourable for monomers of one type to be in one liquid and for monomers of the other type to be in the other liquid. At high temperatures the polymer will delocalize into one of the liquids in order to maximise its entropy, while at low temperatures energetic effects will dominate and the polymer will localize close to the interface between the two liquids in order to be able to place more than half of its monomers in their preferred liquid. In the limit as $n \to \infty$, we may expect a phase transition. A typical example here would be an oil-water interface and a copolymer with hydrophobic and hydrophilic monomers.

Given such a physical situation, the polymer can be modelled in a variety of ways, e.g. as a random walk or as a self-avoiding walk, either directed or undirected. Such examples have been investigated for the situation where *the interface is flat and infinite*. In addition, there is some flexibility in the details of the Hamiltonian that is chosen to model the interactions.

A simple model with a single interface was proposed and analysed by Garel *et al* [9], with a Hamiltonian that depends on temperature and interaction bias. A first mathematical

treatment of this model was given by Sinai [29] and by Grosberg *et al* [15], in the absence of interaction bias, for a directed random walk version of the model. For this version, Bolthausen and den Hollander [4] proved that the quenched free energy is non-analytic along a critical curve in the plane of inverse temperature vs. interaction bias, and derived several qualitative properties of this curve, among which upper and lower bounds. Albeverio and Zhou [1], for the unbiased case, and Biskup and den Hollander [2], for the biased case, extended this work by deriving path properties of the model, in particular, ergodic limits along the interface and exponential tightness perpendicular to the interface in the localized phase, as well as zero limiting frequency of hits of the interface in the interior of the delocalized phase. The latter result was recently strengthened by Giacomin and Toninelli [11], who showed that in the interior of the delocalized phase the number of times the path intersects the interface grows at most logarithmically with its length. The conjecture is that the number of intersections is actually bounded. In Giacomin and Toninelli [13] it was proved that the free energy is infinitely differentiable inside the localized phase. Thus, there is no phase transition of finite order anywhere off the critical curve. Morover, in Giacomin and Toninelli [12] it was proved that the free energy is twice differentiable across the critical curve, i.e., the phase transition is at least of second order.

Maritan *et al* [22] considered both random walk and self-avoiding walk models and derived rigorous bounds on the free energy, under an assumption on the asymptotics of a certain class of self-avoiding walks. Martin *et al* [23] proved the existence of a localization transition for a self-avoiding walk model and obtained qualitative results about the shape of the phase transition curve. These results were extended and improved by Madras and Whittington [21], who also gave a rigorous version of the result of Maritan *et al* [22] for the self-avoiding walk model. Orlandini *et al* [25] derived rigorous bounds on the critical curve for the directed random walk model, while Causo and Whittington [7] and James, Soteros and Whittington [18] obtained sharp numerical estimates for the self-avoiding walk model.

An interesting recent development concerns the slope of the critical curve in the limit of small inverse temperature and interaction bias in the directed random walk version of the model. In Bolthausen and den Hollander [4] it was proved that this slope exists, is strictly positive and is at most 1, the latter being a corollary of an upper bound on the full critical curve. Garel *et al* [9] had earlier hinted at the possibility that this slope be 1, a viewpoint that was taken up by Trovato and Maritan [32]. However, Stepanow *et al* [31] conjectured the slope to be $\frac{2}{3}$, based on replica symmetry arguments. Monthus [24], using a general renormalization scheme, conjectured a simple explicit formula for the full critical curve, which indeed has slope $\frac{2}{3}$ in the limit of small inverse temperature and interaction bias. Based on this work, Bodineau and Giacomin [3] proved that this formula is a lower bound for the critical curve, so that we now know that the slope is at least $\frac{2}{3}$. Numerical work by Garel and Monthus [10] and Caravenna, Giacomin and Gubinelli [6] indicates that the upper and lower bounds on the critical curve are not sharp, nor are the bounds 1 and $\frac{2}{3}$ for the slope. So far all attempts to improve these bounds have failed. The slope seems to be close to 0.82.

The reason why the above issue is of interest is that, while the full shape of the critical curve is model-dependent, the slope in the limit of small inverse temperature and interaction bias is believed to be insensitive to the details of the model.

The goal of the present paper is to study a model where *the interface has a more complex geometry*. A first attempt in this direction was made by den Hollander and Wüthrich [17], where *an infinite array of parallel flat infinite interfaces* was considered and the average hopping time between interfaces was computed for a directed random walk model. In the

present paper we investigate the situation in which the lattice is divided into large blocks, and each block is independently labelled A or B with probability p and $1-p$, respectively, i.e., *the interface has a percolation type structure*. This is a primitive model of an emulsion (e.g. oil dispersed as droplets in water as the dispersing medium). As before, the copolymer consists of a random concatenation of monomers of type A and B. It is energetically favourable for monomers of type A to be in the A-blocks and for monomers of type B to be in the B-blocks of the emulsion. *Under the restriction that the polymer can only enter and exit a pair of neighbouring blocks at diagonally opposite corners*, we show that there is a phase transition between a phase where the polymer is *fully delocalized* away from the interfaces between the two types of blocks and a phase where the polymer is *partially localized* near the interfaces. It turns out that the critical curve *does not* depend on p in the supercritical percolation regime, but *does* depend on p in the subcritical percolation regime. In the latter regime, a *second* critical curve appears separating *two* partially localized phases.

Our paper is organised as follows. In the rest of Section 1 we define the model, formulate our main theorems, discuss these theorems, and formulate some open problems. Section 2 contains some preparatory results about path entropies and free energies per pair of neighbouring blocks. In Sections 3 and 4 we provide the proofs of the main theorems, focussing on the free energy, respectively, the critical curves.

1.2 The model

Each positive integer is randomly labelled A or B, with probability $\frac{1}{2}$ each, independently for different integers. The resulting labelling is denoted by

$$\omega = \{\omega_i\colon\ i \in \mathbb{N}\} \in \{A,B\}^{\mathbb{N}}. \tag{1.2.1}$$

Fix $p \in (0,1)$ and $L_n \in \mathbb{N}$. Partition $\mathbb{R}^2$ into square blocks of size L_n:

$$\mathbb{R}^2 = \bigcup_{x\in\mathbb{Z}^2} \Lambda_{L_n}(x), \qquad \Lambda_{L_n}(x) = xL_n + (0,L_n]^2. \tag{1.2.2}$$

(Note that the blocks contain their north and east side but not their south and west side.) Each block is randomly labelled A or B, with probability p, respectively, $1-p$, independently for different blocks. The resulting labelling is denoted by

$$\Omega = \{\Omega(x)\colon\ x \in \mathbb{Z}^2\} \in \{A,B\}^{\mathbb{Z}^2}. \tag{1.2.3}$$

Consider the set of n-step *directed self-avoiding paths* starting at the origin and being allowed to move *upwards, downwards and to the right*. Let $\mathcal{W}_{n,L_n}$ be the subset of those paths that enter blocks at a corner, exit blocks at one of the two corners *diagonally opposite* the one where it entered, and in between *stay confined* to the two blocks that are seen when entering. In other words, after the path reaches a site xL_n, it must make a step to the right, it must subsequently stay confined to the pair of blocks labelled x and $x+(0,-1)$, and it must exit this pair of blocks either at site $xL_n + (L_n, L_n)$ or at site $xL_n + (L_n, -L_n)$ (see Figure 1). This restriction is put in to make the model mathematically tractable.

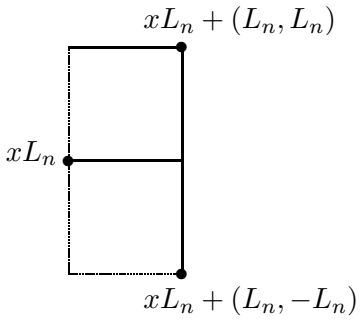

Fig. 1. Two neighbouring blocks. The dots are the sites of entrance and exit. The drawn lines are part of the blocks, the dashed lines are not.

Given ω, Ω and n, with each path $\pi \in \mathcal{W}_{n,L_n}$ we associate an energy given by the Hamiltonian

$$H^{\omega,\Omega}_{n,L_n}(\pi) = -\sum_{i=1}^{n} \Big(\alpha 1\{\omega_i = \Omega^{L_n}_{\pi_i} = A\} + \beta 1\{\omega_i = \Omega^{L_n}_{\pi_i} = B\}\Big), \tag{1.2.4}$$

where π_i denotes the i-th step of the path and $\Omega^{L_n}_{\pi_i}$ denotes the label of the block that step π_i lies in. What this Hamiltonian does is count the number of AA-matches and BB-matches and assign them energy $-\alpha$ and $-\beta$, respectively, where $\alpha, \beta \in \mathbb{R}$. Note that *the interaction is assigned to bonds rather than to sites*: we identify the monomers with the steps of the path.

For $\alpha, \beta > 0$, the above definitions are to be interpreted as follows: ω plays the role of the random monomer types, with A denoting hydrophobic and B denoting hydrophilic; Ω plays the role of the random emulsion, with A denoting oil and B denoting water; n is the number of monomers; the Hamiltonian assigns negative energy to matches of affinities between polymer and emulsion, with α and β the interaction strengths (it assigns zero energy to mismatches).

Given ω, Ω and n, we define the *quenched free energy per step* as

$$\begin{aligned} f^{\omega,\Omega}_{n,L_n} &= \frac{1}{n} \log Z^{\omega,\Omega}_{n,L_n}, \\ Z^{\omega,\Omega}_{n,L_n} &= \sum_{\pi \in \mathcal{W}_{n,L_n}} \exp\Big[-H^{\omega,\Omega}_{n,L_n}(\pi)\Big]. \end{aligned} \tag{1.2.5}$$

We are interested in the limit $n \to \infty$ subject to the restriction

$$L_n \to \infty \qquad \text{and} \qquad L_n/n \to 0. \tag{1.2.6}$$

This is a *coarse-graining* limit where the path spends a long time in each single block yet visits many blocks. Throughout the paper we assume that this restriction is in force, which is necessary to make the model mathematically tractable. It will turn out that the free energy does not depend on the choice of the sequence $(L_n)_{n\in\mathbb{N}}$.

1.3 Free energy

Theorem 1.3.1 below says that the quenched free energy per step is self-averaging and can be expressed as a variational problem involving the free energies of the polymer in each of the four possible pairs of adjacent blocks it may encounter and the frequencies at which the polymer visits each of these pairs of blocks on the coarse-grained block scale. To formulate this theorem we need some more definitions.

First, for $L \in \mathbb{N}$ and $a \geq 2$ (with aL integer), let $\mathcal{W}_{aL,L}$ denote the set of aL-step directed self-avoiding paths starting at $(0,0)$, ending at (L,L), and in between not leaving the two adjacent blocks of size L labelled $(0,0)$ and $(-1,0)$.

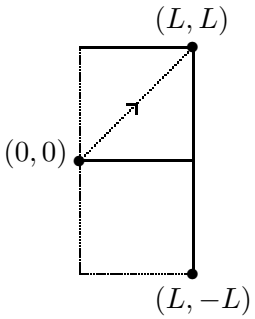

Fig. 2. Two neighbouring blocks. The dashed line with arrow indicates that the coarse-grained path makes a step diagonally upwards.

For $k, l \in \{A, B\}$, let

$$\begin{aligned} \psi^{\omega}_{kl}(aL, L) &= \frac{1}{aL} \log Z^{\omega}_{aL,L}, \\ Z^{\omega}_{aL,L} &= \sum_{\pi \in \mathcal{W}_{aL,L}} \exp\big[- H^{\omega,\Omega}_{aL,L}(\pi)\big] \text{ when } \Omega(0,0) = k \text{ and } \Omega(0,-1) = l, \end{aligned} \tag{1.3.1}$$

denote the free energy per step in a kl-block when the number of steps inside the block is a times the size of the block. Let

$$\lim_{L \to \infty} \psi^{\omega}_{kl}(aL, L) = \psi_{kl}(a) = \psi_{kl}(\alpha, \beta; a). \tag{1.3.2}$$

Note here that k labels the type of the block that is diagonally crossed, while l labels the type of the block that appears as its neighbour at the starting corner (see Fig. 2). We will prove in Section 2.2 that the limit exists ω-a.s. and is non-random. It will turn out that ψ_{AA} and ψ_{BB} take on a simple form, whereas ψ_{AB} and ψ_{BA} do not.

Second, let $\mathcal{W}$ denote the class of all *coarse-grained paths* $\Pi = \{\Pi_j \colon\ j \in \mathbb{N}\}$ that step diagonally from corner to corner (see Fig. 3, where each dashed line with arrow denotes a single step of Π). For $n \in \mathbb{N}$, $\Pi \in \mathcal{W}$ and $k, l \in \{A, B\}$, let

$$\rho^{\Omega}_{kl}(\Pi, n) = \tfrac{1}{n} \textstyle\sum_{j=1}^{n} 1\{ \quad \begin{array}{l} \Pi_j \text{ diagonally crosses a } k\text{-block in } \Omega \text{ that has an } l\text{-block} \\ \text{in } \Omega \text{ appearing as its neighbour at the starting corner } \}. \end{array} \tag{1.3.3}$$

Abbreviate

$$\rho^{\Omega}(\Pi, n) = \big(\rho^{\Omega}_{kl}(\Pi, n)\big)_{k,l \in \{A,B\}}, \tag{1.3.4}$$

which is a 2×2 matrix with nonnegative elements that sum up to 1. Let $\mathcal{R}^{\Omega}(\Pi)$ denote the set of all limits points of the sequence $\{\rho^{\Omega}(\Pi, n) \colon\ n \in \mathbb{N}\}$, and put

$$\mathcal{R}^{\Omega} = \text{the closure of the set } \bigcup_{\Pi \in \mathcal{W}} \mathcal{R}^{\Omega}(\Pi). \tag{1.3.5}$$

Clearly, $\mathcal{R}^{\Omega}$ exists for all Ω. Moreover, since Ω has a trivial sigma-field at infinity (i.e., all events not depending on finitely many coordinates of Ω have probability 0 or 1) and $\mathcal{R}^{\Omega}$ is measurable with respect to this sigma-field, we have

$$\mathcal{R}^{\Omega} = \mathcal{R}(p) \qquad \Omega - a.s. \tag{1.3.6}$$

for some *non-random closed* set $\mathcal{R}(p)$. This set, which depends on the parameter p controlling Ω, will be analysed in Section 3.2. It is the set of all possible limit points of the frequencies at which the four pairs of adjacent blocks can be seen along an infinite coarse-grained path.

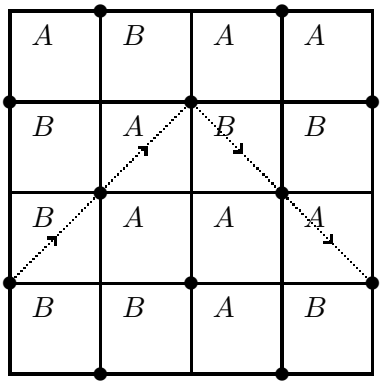

Fig. 3. Π sampling Ω. The dashed lines with arrows indicate the steps of Π.

Let $\mathcal{A}$ be the set of 2×2 matrices whose elements are ≥ 2. The starting point of our paper is the following representation of the free energy.

Theorem 1.3.1 *(i) For all* $(\alpha,\beta)\in\mathbb{R}^2$ *and* $p\in(0,1)$,

$$\lim_{n\to\infty} f_{n,L_n}^{\omega,\Omega} = f = f(\alpha,\beta;p) \tag{1.3.7}$$

exists ω,Ω*-a.s., is finite and non-random, and is given by*

$$f = \sup_{(a_{kl})\in\mathcal{A}} \sup_{(\rho_{kl})\in\mathcal{R}(p)} \frac{\sum_{k,l}\rho_{kl}a_{kl}\psi_{kl}(a_{kl})}{\sum_{k,l}\rho_{kl}a_{kl}}. \tag{1.3.8}$$

(ii) The function $(\alpha,\beta)\mapsto f(\alpha,\beta;p)$ *is convex on* $\mathbb{R}^2$ *for all* $p\in(0,1)$.
(iii) The function $p\mapsto f(\alpha,\beta;p)$ *is continuous on* $(0,1)$ *for all* $(\alpha,\beta)\in\mathbb{R}^2$.
(iv) For all $(\alpha,\beta)\in\mathbb{R}^2$ *and* $p\in(0,1)$,

$$\begin{aligned} f(\alpha,\beta;p) &= f(\beta,\alpha;1-p),\\ f(\alpha,\beta;p) &= \frac{1}{2}(\alpha+\beta)+f(-\beta,-\alpha;p). \end{aligned} \tag{1.3.9}$$

Theorem 1.3.1(i), which will be proved in Section 3.1, says that the limiting free energy per step is *self-averaging in both* ω *and* Ω, and equals the average of the limiting free energies per step associated with the four pairs of adjacent blocks, weighted and optimised according to the frequencies at which these four pairs are visited by the coarse-grained path and the fraction of time spent in each of them by the path. Assumption (1.2.6) is crucial, since it provides the separation of the path scale and the block scale, thereby separating the self-averaging in ω and Ω. Theorem 1.3.1(ii), which will be proved in Section 3.1 also, is standard. Theorem 1.3.1(iii) is a consequence of the fact that $p\mapsto\mathcal{R}(p)$ is continuous in the Hausdorff metric, which will be proved in Section 3.2. Theorem 1.3.1(iv) is immediate from (1.2.4) upon interchanging the two monomer types and/or the two block types.

In view of Theorem 1.3.1(iv), we may *without loss of generality restrict to the cone*

$$\text{CONE} = \{(\alpha,\beta)\in\mathbb{R}^2\colon\ \alpha\geq|\beta|\}. \tag{1.3.10}$$

The upper half of the cone is the physically most relevant part, but we will see that also the lower half of the cone is of interest. Note that *AA-matches are favored over BB-matches*. This will be crucial throughout the paper.

The behaviour of f as a function of (α,β) is different for $p\geq p_c$ and $p<p_c$, where $p_c\approx 0.64$ is the critical percolation density for directed bond percolation on the square lattice. The reason is that the coarse-grained paths Π, which determine the set $\mathcal{R}(p)$, sample Ω just like paths in *directed bond percolation* on the square lattice rotated by 45 degrees sample the percolation configuration (see Fig. 3).

1.4 Supercritical case $p \geq p_c$

The entropy per step of the walk in a single block, subject to (1.2.6), is

$$\kappa = \lim_{n\to\infty} \frac{1}{n} \log |\mathcal{W}_{n,L_n}|. \tag{1.4.1}$$

In Section 2.1 we will see that $\kappa = \frac{1}{2}\log 5$. This number is special to our model.

Our first theorem identifies the two phases, which turn out *not to depend on* p.

Theorem 1.4.1 *Let $p \geq p_c$. Then $f(\alpha,\beta;p) = f(\alpha,\beta)$, and $(\alpha,\beta) \mapsto f(\alpha,\beta)$ is non-analytic along the curve in* CONE *separating the two regions*

$$\begin{aligned} \mathcal{D} = \ \textit{delocalized regime} \ &= \left\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta) = \frac{1}{2}\alpha + \kappa\right\}, \\ \mathcal{L} = \ \textit{localized regime} \ &= \left\{(\alpha,\beta) \in \text{CONE}\colon\ f(\alpha,\beta) > \frac{1}{2}\alpha + \kappa\right\}. \end{aligned} \tag{1.4.2}$$

The intuition behind Theorem 1.4.1, which will be proved in Section 4.1.1, is as follows. The A-blocks (almost) percolate. Therefore the polymer has the option of moving to the (incipient) infinite cluster of A-blocks and staying in that infinite cluster forever, thus seeing only AA-blocks. In doing so, it loses an entropy of at most $o(n/L_n) = o(n)$, it gains an energy $\frac{1}{2}\alpha n + o(n)$ (because only half of its monomers are matched), and it gains an entropy $\kappa n + o(n)$. Alternatively, the path has the option of following the boundary of the infinite cluster, at least part of the time, during which it sees AB-blocks and (when $\beta \geq 0$) gains more energy by matching more than half of its monomers. Consequently,

$$f(\alpha,\beta) \geq \frac{1}{2}\alpha + \kappa. \tag{1.4.3}$$

The boundary between the two regimes in (1.4.2) corresponds to the crossover where one option takes over from the other.

Our second theorem gives an explicit classification of the two phases in terms of the free energies ψ_{AA} and ψ_{AB}.

Theorem 1.4.2 *Let $p \geq p_c$. Then*

$$\begin{aligned} \mathcal{D} &= \{(\alpha,\beta) \in \text{CONE}\colon\ S_{AB} = S_{AA}\}, \\ \mathcal{L} &= \{(\alpha,\beta) \in \text{CONE}\colon\ S_{AB} > S_{AA}\}, \end{aligned} \tag{1.4.4}$$

where

$$S_{kl} = S_{kl}(\alpha,\beta) = \sup_{a\geq 2} \psi_{kl}(\alpha,\beta;a). \tag{1.4.5}$$

We have $S_{AB} \geq S_{AA}$ for all (α,β), because in an AB-block the path may spend all of its time in the half that is A, in which case it is not aware of the presence of the half that is B (see Fig. 4). Theorem 1.4.2, which will be proved in Section 4.1.1 also, says that the critical curve marks those parameter values where $=$ changes to $>$.

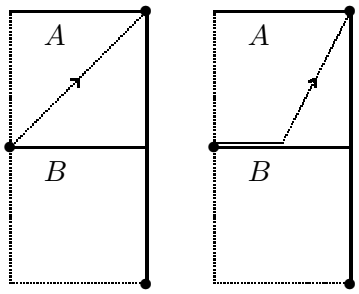

Fig. 4. Two possible strategies inside an AB-block: The path can either move straight across or move along the interface for awhile and then move across. Both strategies correspond to a coarse-grained step diagonally upwards.

Our third theorem gives the qualitative properties of the critical curve separating $\mathcal{D}$ and $\mathcal{L}$ (see Fig. 5).

Theorem 1.4.3 *Let $p \geq p_c$.*
(i) For every $\alpha \geq 0$ there exists a $\beta_c(\alpha) \in [0, \alpha]$ such that the copolymer is

$$\begin{array}{ll} \textit{delocalized} & \textit{if } -\alpha \leq \beta \leq \beta_c(\alpha), \\ \textit{localized} & \textit{if } \beta_c(\alpha) < \beta \leq \alpha. \end{array} \tag{1.4.6}$$

(ii) The function $\alpha \mapsto \beta_c(\alpha)$ is continuous, non-decreasing and concave on $[0, \infty)$.
(iii) There exists an $\alpha^* \in (0, \infty)$ such that

$$\begin{array}{ll} \beta_c(\alpha) = \alpha & \textit{if } \alpha \leq \alpha^*, \\ \beta_c(\alpha) < \alpha & \textit{if } \alpha > \alpha^*. \end{array} \tag{1.4.7}$$

Moreover,

$$\lim_{\alpha \downarrow \alpha^*} \frac{\alpha - \beta_c(\alpha)}{\alpha - \alpha^*} \in [0, 1). \tag{1.4.8}$$

(iv) There exists a $\beta^* \in [\alpha^*, \infty)$ such that

$$\lim_{\alpha \to \infty} \beta_c(\alpha) = \beta^*. \tag{1.4.9}$$

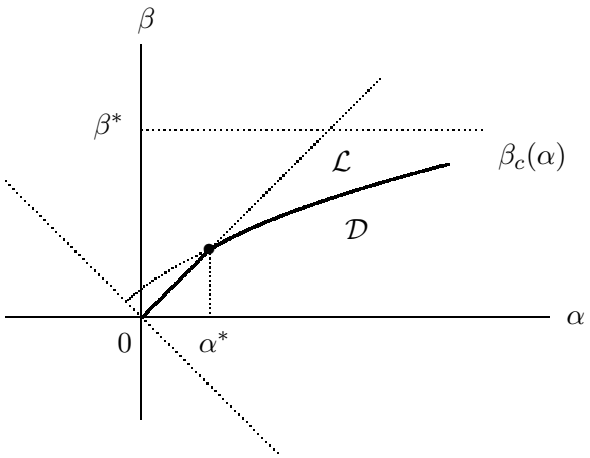

Fig. 5. Qualitative picture of $\alpha \mapsto \beta_c(\alpha)$ for $p \geq p_c$. The curved dotted line is the analytic continuation outside CONE of the part off the diagonal.

It is clear from (1.4.2) that the part off the diagonal is a critical line. We will see in Section 4.2.3 that also the part on the diagonal is a critical line. Theorem 1.4.3, which will be proved in Section 4.1.2, says that the critical curve follows the diagonal for $\alpha \in [0, \alpha^*]$, moves off the diagonal at $\alpha = \alpha^*$ with a slope discontinuity, and has a finite asymptote for large α. The concavity of the curve implies that it is strictly increasing as long as it is below the asymptote. We are not able to exclude that the curve hits the asymptote, nor that it follows the diagonal all the way up to the asymptote, but we expect this not to happen. We will see in Section 4.1.2 that the curved dotted line crosses the vertical axis at $(0, \alpha_0)$ with $\alpha_0 \approx 0.125$. We have no numerical values for α^* and β^*. We will show in Section 4.1.2 that $\beta^* \in [\log 2, 8 \log 3)$. Clearly, $\alpha^* \in [\alpha_0, \beta^*]$.

To prove Theorem 1.4.3, we will *reformulate* the criterion $S_{AB} > S_{AA}$ in terms of a criterion for the free energy of a model with a *single linear interface*. This reformulation, which will be given in Section 2.3, is crucial in allowing us to get a handle on the critical curve in Fig. 5.

We will see in Section 4.1.1 that $\mathcal{D}$ corresponds to the situation where the polymer is *fully A-delocalized* (i.e., it spends almost all of its time away from the interface deep inside the A-blocks), while $\mathcal{L}$ corresponds to the situation where the polymer is *partially AB-delocalized* (i.e., it spends a positive fraction of its time near those interfaces where it diagonally crosses the A-block rather than the B-block).

1.5 Subcritical case $p < p_c$

In the subcritical percolation regime, the analogue of the critical curve in Fig. 5 turns out *to depend on* p and to be much more difficult to characterise. We begin with some definitions.

Let

$$\rho^*(p) = \sup_{(\rho_{kl}) \in \mathcal{R}(p)} [\, \rho_{AA} + \rho_{AB} \,]. \tag{1.5.1}$$

This is the maximal frequency of A-blocks crossed by an infinite coarse-grained path (recall (1.3.3–1.3.6)). The graph of $p \mapsto \rho^*(p)$ looks like:

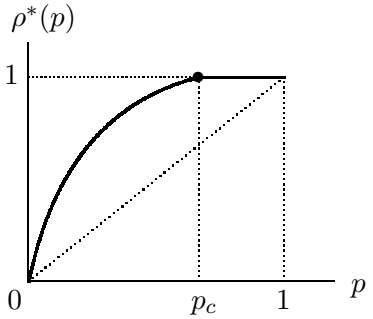

Fig. 6. Qualitative picture of $p \mapsto \rho^*(p)$.

Further details will be given in Section 3.2.

For $x, y \geq 2$, let $u(x) = u(\alpha; x)$ and $v(y) = v(\beta; y)$ be defined by

$$\begin{aligned} xu(x) &= \frac{1}{2}\alpha x + \log 2 + \frac{1}{2} x \log x - \frac{1}{2}(x-2)\log(x-2), \\ yv(y) &= \frac{1}{2}\beta y + \log 2 + \frac{1}{2} y \log y - \frac{1}{2}(y-2)\log(y-2). \end{aligned} \tag{1.5.2}$$

For $\rho \in (0,1)$, let

$$F(\rho) = F(\alpha, \beta; \rho) = \sup_{x,y \geq 2} \frac{\rho x u(x) + (1-\rho) y v(y)}{\rho x + (1-\rho) y}. \tag{1.5.3}$$

This variational formula will be analysed in Section 2.5. There we will see that $(\alpha, \beta) \mapsto F(\alpha, \beta; \rho)$ is analytic on $\mathbb{R}^2$ for all $\rho \in (0,1)$.

The following is the analogue of Theorem 1.4.1, and will be proved in Section 4.2.1.

Theorem 1.5.1 *Let $p < p_c$. Then $(\alpha, \beta) \mapsto f(\alpha, \beta; p)$ is non-analytic along the curve in* CONE *separating the two regions*

$$\begin{aligned} \mathcal{D} = \ & \textit{delocalized regime} \ = \{(\alpha,\beta) \in \text{CONE}\colon f(\alpha,\beta;p) = F(\alpha,\beta;\rho^*(p))\}, \\ \mathcal{L} = \ & \textit{localized regime} \ = \{(\alpha,\beta) \in \text{CONE}\colon f(\alpha,\beta;p) > F(\alpha,\beta;\rho^*(p))\}. \end{aligned} \tag{1.5.4}$$

The intuition behind Theorem 1.5.1 is as follows. We will see in Section 2.2.1 that $\psi_{AA}(a) = u(a)$ and $\psi_{BB}(a) = v(a)$. In the delocalized regime, the polymer stays away from the AB-interface. For the free energy this means that no difference is felt between ψ_{AB}, ψ_{AA} and between ψ_{BA}, ψ_{BB}. Therefore in this regime the variational formula in (1.3.8) effectively reduces to

$$f = \sup_{(a_{kl}) \in \mathcal{A}} \sup_{(\rho_{kl}) \in \mathcal{R}(p)} \frac{\rho_A a_{AA} \psi_{AA}(a_{AA}) + \rho_B a_{BB} \psi_{BB}(a_{BB})}{\rho_A a_{AA} + \rho_B a_{BB}}, \tag{1.5.5}$$

where $\rho_A = \rho_{AA} + \rho_{AB}$ and $\rho_B = \rho_{BA} + \rho_{BB}$ are the frequencies at which the polymer diagonally traverses A-blocks and B-blocks, while a_{AA} and a_{BB} are the respective times spent inside these blocks. The first supremum amounts to optimising over $a_{AA}, a_{BB} \geq 2$. Since AA-matches are preferred over BB-matches, implying $\psi_{AA} \geq \psi_{BB}$, the second supremum is taken at the largest possible value of $\rho_A = 1 - \rho_B$ in $\mathcal{R}(p)$, which is $\rho^*(p)$. Hence, putting $a_{AA} = x$ and $a_{BB} = y$, we get $f = F(\alpha, \beta; \rho^*(p))$. In the localized regime, on the other hand, the polymer spends part of its time near AB-interfaces or BA-interfaces, in which case a difference is felt between ψ_{AB}, ψ_{AA} and/or between ψ_{BA}, ψ_{BB}, and the free energy is larger. In Section 4.2.1 we will make the above intuition rigorous.

Comparing the first lines of (1.4.2) and (1.5.4), we see that the free energy in the supercritical delocalized regime is a function of α only and has a *simple linear form*, whereas the free energy in the subcritical delocalized regime is a function of $\alpha, \beta, \rho^*(p)$ and has a form that is *rather more complicated*. For $\rho = 1$, (1.5.2–1.5.3) yield $F(\alpha, \beta; 1) = \sup_{x \geq 2} u(x) = u(\frac{5}{2}) = \frac{1}{2}\alpha + \frac{1}{2}\log 5$. This explains the connection between (1.4.2) and (1.5.4).

The following is the analogue of Theorem 1.4.2, and will be proved in Section 4.2.1.

Theorem 1.5.2 *Let $p < p_c$. Then*

$$\begin{aligned} \mathcal{D} &= \{(\alpha,\beta) \in \text{CONE}\colon \ \psi_{AB}(\bar{x}) = \psi_{AA}(\bar{x}) \ \textit{and} \ \psi_{BA}(\bar{y}) = \psi_{BB}(\bar{y})\}, \\ \mathcal{L} &= \{(\alpha,\beta) \in \text{CONE}\colon \ \psi_{AB}(\bar{x}) > \psi_{AA}(\bar{x}) \ \textit{or} \ \psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})\}, \end{aligned} \tag{1.5.6}$$

where $\bar{x} = \bar{x}(\alpha, \beta; \rho^*(p))$ and $\bar{y} = \bar{y}(\alpha, \beta; \rho^*(p))$ are the unique maximisers of $F(\alpha, \beta; \rho^*(p))$, i.e., of the variational formula in (1.5.3) *for $\rho = \rho^*(p)$.*

Theorem 1.5.2 says that the crossover into the localized regime occurs when the difference between ψ_{AB}, ψ_{AA} or between ψ_{BA}, ψ_{BB} is felt at the minimisers of the variational formula for the delocalized regime.

Comparing (1.4.4) and (1.5.6), we see that the crossover into the supercritical localization regime occurs when the maxima of ψ_{AB}, ψ_{AA} separate, whereas the crossover into the subcritical localization regime occurs when ψ_{AB}, ψ_{AA} or ψ_{BA}, ψ_{BB} separate at specific locations, which themselves depend on $\alpha, \beta, \rho^*(p)$.

We will see in Section 4.2.2 that $\psi_{BA}(\bar{y}) = \psi_{BB}(\bar{y})$ implies $\psi_{AB}(\bar{x}) = \psi_{AA}(\bar{x})$. Hence, (1.5.6) in fact reduces to

$$\begin{aligned} \mathcal{D} &= \{(\alpha,\beta) \in \text{CONE}\colon\ \psi_{BA}(\bar{y}) = \psi_{BB}(\bar{y})\}, \\ \mathcal{L} &= \{(\alpha,\beta) \in \text{CONE}\colon\ \psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})\}. \end{aligned} \tag{1.5.7}$$

This is to be interpreted as saying that, when the critical curve is crossed from $\mathcal{D}$ to $\mathcal{L}$, localization occurs in the BA-blocks rather than in the AB-blocks. The intuitive explanation is as follows. In the delocalized phase the polymer spends positive fractions of its time in the A-blocks and in the B-blocks (the A-blocks do not percolate). Because AA-matches are preferred over BB-matches, there is a larger reward for the polymer to BA-localize (stay close to the interface when diagonally crossing a B-block) than to AB-localize (stay close to the interface when diagonally crossing an A-block).

The following is the analogue of Theorem 1.4.3, and will be proved in Section 4.2.2. Two constants $0 < \alpha_0 < \alpha_1 < \infty$ appear, which will be identified in Section 2.2.

Theorem 1.5.3 *Let $p < p_c$.*
(i) $\partial\mathcal{D}$ lies on or below the supercritical curve.
(ii) $\partial\mathcal{D}$ is continuous and intersects each line from the origin with slope in $[-1,1)$ at most once.
(iii) $\partial\mathcal{D}$ contains the diagonal segment $\{(\alpha,\alpha)\colon\ \alpha \in [0,\alpha^*]\}$, with α^* the same constant as in Theorem 1.4.3(iii)*, but lies below the diagonal elsewhere.*
(iv) There exists an $\alpha^*(p) \in (0,\infty)$ such that the intersection of $\partial\mathcal{D}$ with the lower half of CONE *is the linear segment $\{(\beta + \alpha^*(p), \beta)\colon\ \beta \in [-\frac{1}{2}\alpha^*(p), 0]\}$.*
(v) As $p \downarrow 0$, $\partial\mathcal{D}$ converges to the union of the diagonal segment in (iii) and the mirror image of the analytic continuation of the supercritical curve outside CONE *(i.e., the mirror image of the curved dotted line in Fig. 5). In particular, $\lim_{p\downarrow 0} \alpha^*(p) = \alpha_0$.*
(vi) As $p \uparrow p_c$, $\partial\mathcal{D}$ does not converge to the supercritical curve in Fig. 5. In particular, $\lim_{p\uparrow p_c} \alpha^*(p) = \alpha_1$.

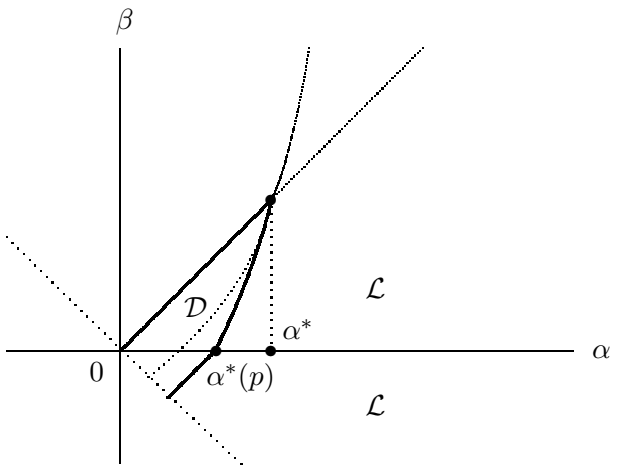

Fig. 7. Qualitative picture of $\partial\mathcal{D}$ for $p < p_c$. The curved dotted line is the mirror image of the union of the supercritical curve off the diagonal and its analytic continuation outside CONE (see Fig. 5).

It is clear from (1.5.4) that the part off the diagonal is a critical line. We will see in Section 4.2.3 that also the part on the diagonal is a critical line.

We will see in Section 2.2 that $\mathcal{D}$ corresponds to the situation where the polymer is *fully delocalized into the A-blocks and the B-blocks*, while $\mathcal{L}$ corresponds to the situation where the polymer is *partially BA-localized*. We expect that $\mathcal{D}$ is strictly increasing in p and that $\alpha^* > \alpha_1$, but we are unable to prove this. The curved dotted line crosses the horizontal axis at α_0.

We will see in Section 4.3 that $\mathcal{L}$ contains a second curve (see Fig. 8) at which a phase transition occurs from *partially BA-localized* to *partially BA-localized and partially AB-localized*. Qualitatively, this curve behaves like the supercritical curve (e.g. it also starts at the point (α^*, α^*)), but unfortunately we know little about it. We expect it to be strictly increasing in α. We expect it to move down as p increases. We do know that it converges to the supercritical curve as $p \uparrow p_c$.

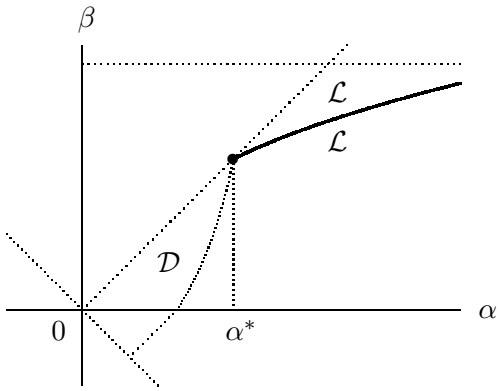

Fig. 8. Conjectured critical line inside $\mathcal{L}$ for $p < p_c$.

To prove Theorem 1.5.3, we will *reformulate* the criteria $\psi_{AB}(\bar{x}) > \psi_{AA}(\bar{x})$ and $\psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})$ in terms of criteria for the free energy of a model with a *single linear interface*. This reformulation, which will be given in Section 2.4, is again crucial in allowing us to get a handle on the critical curve in Fig. 7.

1.6 Heuristic explanation of the phase diagram

The physical background of the three critical curves in Figs. 5, 7 and 8 is as follows.

• $p \geq p_c$:

Consider the boundary $\partial\mathcal{D}$ sketched in Fig. 5. Pick a point $(\hat{\alpha}, \hat{\beta})$ inside $\mathcal{D}$. Then, since $p \geq p_c$ and $\alpha \geq \beta$, the polymer spends almost all of its time deep inside A-blocks. Now increase β but keep $\alpha = \hat{\alpha}$ fixed. Then there will be a larger energetic advantage for the polymer to move some of its monomers from the A-blocks to the B-blocks by crossing the interface inside the AB-block pairs. There is some entropy loss associated with doing so. The polymer has three options: (i) it may place all A-monomers in the A-blocks and no monomers in the B-blocks (resulting in all energy coming from AA-matches and some entropy); (ii) it may place all A-monomers in the A-blocks and a positive fraction of B-monomers in the B-blocks (resulting in a higher energy and a lower entropy); (iii) it may sacrifice some fraction

of AA-matches to get a larger fraction of BB-matches (resulting in an even higher energy and an even lower entropy). If β is large enough, then the energy advantage will dominate, so that AB-localization sets in. The value at which this happens depends on $\hat{\alpha}$ and is strictly positive. Since the entropy loss is finite, for $\hat{\alpha}$ large enough the energy-entropy competition plays out not only below the diagonal, but also below a horizontal asymptote. The larger the value of $\hat{\alpha}$, the larger the value of β where AB-localization sets in. This explains why the part of $\partial\mathcal{D}$ off the diagonal moves to the right and up.

• $p < p_c$:

First consider the boundary $\partial\mathcal{D}$ sketched in Fig. 7. Pick a point $(\hat{\alpha}, \hat{\beta})$ inside $\mathcal{D}$. Since $p < p_c$, the polymer spends almost all of its time deep inside A-blocks and B-blocks. Now increase α but keep $\beta = \hat{\beta}$ fixed. Then, while remaining delocalized, the polymer will spend more time in the A-blocks and less time in the B-blocks, trying to lower its energy with some attendant loss of entropy. As α increases further, there will be a larger energetic advantage for the polymer to move some of its monomers from the B-blocks to the A-blocks by crossing the interface inside the BA-block pairs. If α is large enough, then the energetic advantage will dominate, so that BA-localization sets in eventually. The value of α at which this happens depends on $\hat{\beta}$. A larger value of $\hat{\beta}$ means that the polymer spends more time in the B-blocks (at fixed α) with larger entropy. Consequently, more entropy will be lost on BA-localization and the value of α where BA-localization sets in will be larger. This explains why the part of $\partial\mathcal{D}$ off the diagonal moves to the right and up. Similarly, if p decreases, then the polymer hits more B-blocks, and to compensate for the loss of energy it will spend more time in an A-block when it hits one and less time in a B-block when it hits one (at fixed $\hat{\alpha}$ and $\hat{\beta}$). Consequently, less entropy will be lost on BA-localization and the value of α (at fixed $\hat{\beta}$) where BA-localization sets in will be smaller. This explains why $\mathcal{D}$ shrinks with p.

If $\beta \leq 0$, then there is a penalty for having B-monomers in B-blocks. Therefore, when the polymer BA-localizes, it will spend all the time it runs along the interface in the A-block and then shoot through the interface to spend its remaining time in the B-block (on its way to the diagonally opposite corner). Hence, the energy-entropy competition only depends on the difference $\alpha - \beta$. This explains why there is a degeneration of the critical curve into a linear segment.

In the limit as $p \downarrow 0$, the density of A-blocks tends to zero and so the polymer spends more and more of its time in B-blocks. Therefore the localization mechanism looks more and more like that for the supercritical curve with $\alpha \leftrightarrow \beta$ and $p \uparrow 1$.

Next consider the curve separating $\mathcal{L}$ sketched in Fig. 8. Pick a point $(\hat{\alpha}, \hat{\beta})$ inside $\mathcal{L}$. Now increase β but keep $\alpha = \hat{\alpha}$ fixed. Then, as before, an energy-entropy competition sets in. The polymer has the same three options inside AB-blocks as in the supercritical case, and therefore the curve has the same qualitative behaviour. In the limit as $p \uparrow p_c$, the polymer spends more and more of its time in A-blocks. Therefore the AB-localization mechanism looks more and more like that for the supercritical curve. If p decreases, then the polymer hits more B-blocks, and to compensate for the loss of energy it will spend more time in an A-block when it hits one and less time in a B-block when it hits one (at fixed $\hat{\alpha}$ and $\hat{\beta}$). Consequently, more entropy will be lost on AB-localization and the value of β (at fixed $\hat{\alpha}$) where AB-localization sets in will be larger. This explains why the curve moves up as p decreases.

Finally, with the help of the two symmetry properties stated in (1.3.9), the phase diagram can be extended from CONE to $\mathbb{R}^2$. When doing so, we obtain the following phase diagram.

Here, the label on $\mathcal{D}$ ($\mathcal{L}$) indicates the type of (de)localization.

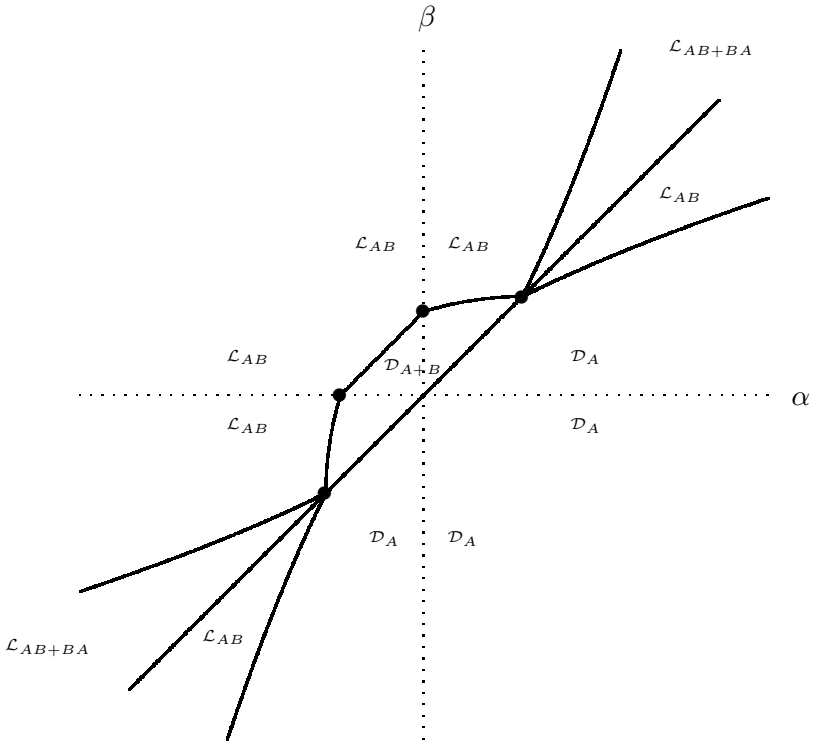

Fig. 9. Full phase diagram for $p \geq p_c$.

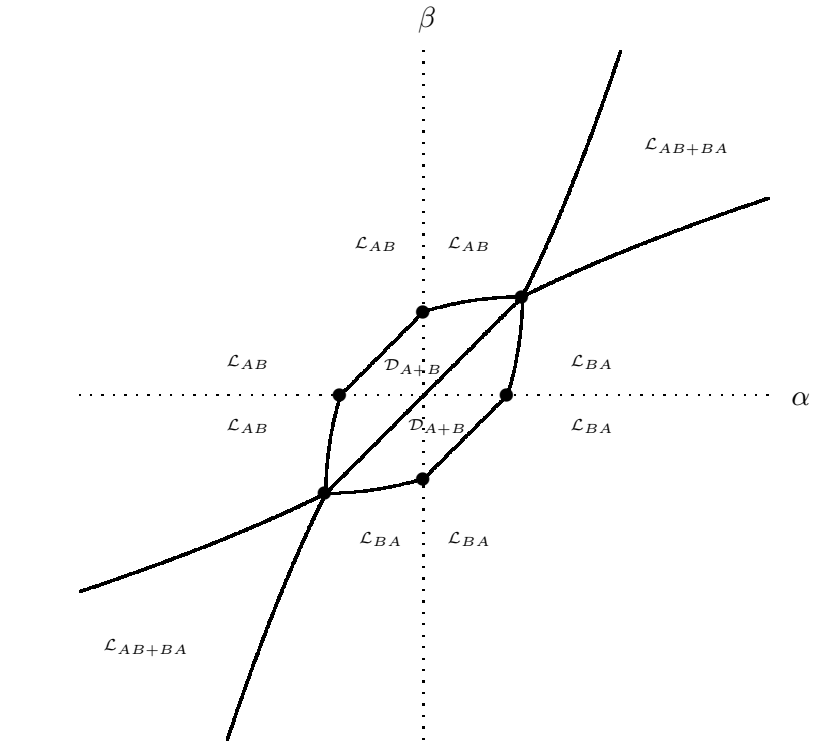

Fig. 10. Full phase diagram for $1 - p_c < p < p_c$.

The figure for $p \leq 1 - p_c$ is the same as for $p \geq p_c$, but with all the phases reflected in the first diagonal and with the labels A and B interchanged. Note that the phase diagram is discontinuous both at $p = p_c$ and $p = 1 - p_c$.

1.7 Open problems

The fine details of the two subcritical curves remain to be settled. Here are some further open problems:

1. Are the critical curves smooth off the diagonal and inside the first quadrant? Even for the model with a single linear interface this question has not been settled.

2. For $p \geq p_c$, is the free energy infinitely differentiable inside the localized phase? For the model with a single linear interface this was proved by Giacomin and Toninelli [13]. Is the same true for $p < p_c$ in the interior of the two subphases of the localized phase?

3. Our phase transitions are defined in terms of a non-analyticity in the free energy. Heuristically, they correspond to the path changing its behaviour from being fully delocalized away from the interfaces to being partially localized near the interfaces (in the subcritical case even in two possible ways). How can we prove that the path actually has this behaviour under the transformed path measure

$$P^{\omega,\Omega}_{n,L_n}(\pi) = \frac{1}{Z^{\omega,\Omega}_{n,L_n}} \exp\left[-H^{\omega,\Omega}_{n,L_n}(\pi)\right] \tag{1.7.1}$$

   for large n? For the model with a single linear interface this question was settled in Biskup and den Hollander [2] and in Giacomin and Toninelli [11].

4. How does the free energy behave near the critical curve? For the model with a single linear interface it was shown by Giacomin and Toninelli [12] that the phase transition is at least of second order. Numerical results in Causo and Whittington [7] suggest that the same is true for the self-avoiding walk model.

5. The coarse-graining expressed by (1.2.6) and the restriction that the polymer can enter and exit a pair of neighbouring blocks only at diagonally opposite corners are necessary to make the model mathematically tractable. Indeed, the corner restriction and $L_n \to \infty$ guarantee that the polymer "sees one pair of blocks at a time" and self-averages in ω in each block, which is why the free energy can be decomposed into contributions coming from single pairs of blocks, while $L_n/n \to 0$ guarantees that the polymer "sees many blocks" and self-averages in Ω, which is why percolation effects enter. What happens when we remove the corner restriction? What happens when the blocks have random sizes?

2 Preparations

In Section 2.1 we compute entropies for paths that cross a block and paths that run along an interface. In Section 2.2 we derive a formula for ψ_{kl} in (1.3.2) and S_{kl} in (1.4.5). In Section 2.3 we deduce a criterion for localization when $p \geq p_c$ *in terms of the free energy for the model with a single linear interface*. In Section 2.4 we do the same when $p < p_c$. In Section 2.5 we analyse the supremum $\rho^*(p)$ in (1.5.1) and the variational formula for $F(\rho)$ in (1.5.3).

2.1 Path entropies

The results in this section are based on straightforward computations, but are crucial for the rest of the paper.

2.1.1 Paths crossing a block

An important ingredient in the identification of $\psi_{kl}(a)$, $k,l \in \{A,B\}$, is the following combinatorial lemma. Let

$$\text{DOM} = \{(a,b)\colon\ a \geq 1+b, b \geq 0\}. \tag{2.1.1}$$

For $(a,b) \in$ DOM, let $N_L(a,b)$ denote the number of aL-step self-avoiding directed paths from $(0,0)$ to (bL,L) whose vertical displacement stays within $(-L,L]$ (aL and bL are integer). Let

$$\kappa(a,b) = \lim_{L\to\infty} \frac{1}{aL} \log N_L(a,b). \tag{2.1.2}$$

Lemma 2.1.1 *(i) $\kappa(a,b)$ exists and is finite for all $(a,b) \in$ DOM.*
(ii) $(a,b) \mapsto a\kappa(a,b)$ is continuous and strictly concave on DOM *and analytic on the interior of* DOM.
(iii) For all $a \geq 2$,

$$a\kappa(a,1) = \log 2 + \frac{1}{2}\left[a\log a - (a-2)\log(a-2)\right]. \tag{2.1.3}$$

(iv) $\sup_{a\geq 2} \kappa(a,1) = \kappa(a^*,1) = \frac{1}{2}\log 5$ with unique maximiser $a^* = \frac{5}{2}$.
(v) $(\frac{\partial}{\partial a}\kappa)(a^*,1) = 0$ and $a^*(\frac{\partial}{\partial b}\kappa)(a^*,1) = \frac{1}{2}\log\frac{9}{5}$.

Proof. (i) First we do the computation without the restriction on the vertical displacement. Later we show that putting in the restriction is harmless.

Let $N_L^0(a,b)$ denote the number of aL-step self-avoiding directed paths from $(0,0)$ to (bL,L). Since such paths make bL steps to the right, $\frac{a+1-b}{2}L$ steps upwards and $\frac{a-1-b}{2}L$ steps downwards, we have

$$N_L^0(a,b) = \sum_{k=1}^{bL} \binom{bL}{k} \binom{\frac{a+1-b}{2}L - 1}{k-1} \sum_{l=1}^{bL-k} \binom{bL-k}{l} \binom{\frac{a-1-b}{2}L - 1}{l-1}. \tag{2.1.4}$$

Here, k counts the number of columns where the path moves upward, l counts the number of colums where the path moves downward, the first and the third binomial coefficient count the number of choices for these columns, while the second and the fourth binomial coefficient count the number of ways in which the prescribed number of steps can be distributed over these columns. Since, by Stirling's formula,

$$\lim_{L\to\infty} \frac{1}{L} \log \binom{uL}{vL} = u\log u - v\log v - (u-v)\log(u-v), \qquad 0 \leq v \leq u, \tag{2.1.5}$$

we get, by putting $k = \delta L$, $l = \epsilon L$, that

$$a\kappa^0(a,b) = \lim_{L\to\infty} \frac{1}{L} \log N_L^0(a,b) = \sup_{\delta,\epsilon} f_{ab}(\delta,\epsilon) \tag{2.1.6}$$

with

$$f_{ab}(\delta,\epsilon) = b\log b - 2\delta\log\delta + \left(\frac{a+1-b}{2}\right)\log\left(\frac{a+1-b}{2}\right) - \left(\frac{a+1-b}{2}-\delta\right)\log\left(\frac{a+1-b}{2}-\delta\right) - 2\epsilon\log\epsilon - (b-\delta-\epsilon)\log(b-\delta-\epsilon) + \left(\frac{a-1-b}{2}\right)\log\left(\frac{a-1-b}{2}\right) - \left(\frac{a-1-b}{2}-\epsilon\right)\log\left(\frac{a-1-b}{2}-\epsilon\right). \tag{2.1.7}$$

Computing

$$\begin{aligned}\frac{\partial f_{ab}}{\partial\delta} &= \log\left[\frac{(\frac{a+1-b}{2}-\delta)(b-\delta-\epsilon)}{\delta^2}\right],\\ \frac{\partial f_{ab}}{\partial\epsilon} &= \log\left[\frac{(\frac{a-1-b}{2}-\epsilon)(b-\delta-\epsilon)}{\epsilon^2}\right],\end{aligned} \tag{2.1.8}$$

and setting these derivatives equal to zero, we find that the maximisers δ_{ab} and ϵ_{ab} of the right-hand side of (2.1.6) are solutions of quadratic equations, namely,

$$\begin{aligned}0 &= (1+b)\delta^2 - (a+1)b\delta + \frac{a+1-b}{2}\,b^2,\\ 0 &= (1-b)\epsilon^2 + (a-1)b\epsilon - \frac{a-1-b}{2}\,b^2,\end{aligned} \tag{2.1.9}$$

which leads to

$$\begin{aligned}\delta_{ab} &= \frac{b}{2(1+b)}\left[(a+1) - \sqrt{(a-b)^2+(b^2-1)}\right],\\ \epsilon_{ab} &= \frac{b}{2(1-b)}\left[-(a-1) + \sqrt{(a-b)^2+(b^2-1)}\right],\end{aligned} \tag{2.1.10}$$

for $b \neq 1$, and

$$\delta_{a1} = \frac{1}{2}, \qquad \epsilon_{a1} = \frac{a-2}{2(a-1)}, \tag{2.1.11}$$

for $b = 1$. Substitution of (2.1.10–2.1.11) into (2.1.6–2.1.7) yields a formula for $a\kappa^0(a,b)$ in closed form. From this formula it is obvious that $(a,b) \mapsto \kappa^0(a,b)$ is continuous on DOM and analytic on the interior of DOM.

It remains to show that the restriction on the vertical displacement has no effect in the limit as $L \to \infty$. This can be done by appealing to the reflection principle. Indeed, let $N_L^{\downarrow}(a,b)$ be the number of paths where the restriction of not moving above the line of height L is inserted. Then $N_L^{\downarrow}(a,b)$ is the difference of two terms of the type $N_L^0(a,b)$ in (2.1.4), one with the path ending at $(bL, L+2)$ and one with the path ending at (bL, L). A little computation shows that this difference equals $N_L^0(a,b)$ divided by a term that is growing at most polynomially fast in L. This polynomial factor does not affect the exponential asymptotics. A similar argument shows that the restriction of not moving below the line of height $-L+1$ is harmless as well. Hence

$$\kappa(a,b) = \kappa^0(a,b). \tag{2.1.12}$$

(ii) Pick any $b_1, b_2 \geq 0$ and $a_1 \geq 1+b_1, a_2 \geq 1+b_2$. Consider a block of height L and width $\frac{1}{2}(b_1+b_2)L$, and partition this block into four parts by cutting it at height $\frac{1}{2}L$ and width $\frac{1}{2}b_1L$. The number of paths that cross the large block in $\frac{1}{2}(a_1+a_2)L$ steps is larger than or

equal to the number of paths that cross the lower left block in $\frac{1}{2}a_1L$ steps times the number of paths that cross the upper right block in $\frac{1}{2}a_2L$ steps, i.e.,

$$N_L^0\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right) \geq N_{\frac{1}{2}L}^0(a_1,b_1)N_{\frac{1}{2}L}^0(a_2,b_2). \tag{2.1.13}$$

By (2.1.6) and (2.1.12), this proves that

$$\frac{a_1+a_2}{2}\kappa\left(\frac{a_1+a_2}{2},\frac{b_1+b_2}{2}\right) \geq \frac{1}{2}a_1\kappa(a_1,b_1)+\frac{1}{2}a_2\kappa(a_2,b_2), \tag{2.1.14}$$

which is the concavity desired. Strict concavity follows from analyticity on the interior of DOM, because $a\kappa(a,b)$ clearly is not linear in either a or b.

(iii) Substitute (2.1.11) into (2.1.6–2.1.7) to get the formula for $a\kappa(a,1)$ stated in (2.1.3).

(iv) Since $\frac{d}{da}\kappa(a,1) = -\frac{1}{a^2}\log[2(a-2)]$, the supremum is uniquely attained at $a^* = \frac{5}{2}$, giving the claim.

(v) Compute, from (2.1.7),

$$\begin{aligned}\left(\frac{\partial}{\partial a}\kappa\right)(a,b) &= \left(\frac{\partial}{\partial a}\left(\frac{1}{a}f_{ab}\right)\right)(\delta_{ab},\epsilon_{ab})\\ &= -\frac{1}{a^2}f_{ab}(\delta_{ab},\epsilon_{ab})+\frac{1}{a}\left(\frac{\partial}{\partial a}f_{ab}\right)(\delta_{ab},\epsilon_{ab})\\ &= -\frac{1}{a}\kappa(a,b)+\frac{1}{a}\,\frac{1}{2}\log\left[\frac{(\frac{a+1-b}{2})(\frac{a-1-b}{2})}{(\frac{a+1-b}{2}-\delta_{ab})(\frac{a-1-b}{2}-\epsilon_{ab})}\right]\end{aligned} \tag{2.1.15}$$

and

$$\begin{aligned}\left(\frac{\partial}{\partial b}\kappa\right)(a,b) &= \left(\frac{\partial}{\partial b}\left(\frac{1}{a}f_{ab}\right)\right)(\delta_{ab},\epsilon_{ab})\\ &= \frac{1}{a}\left(\frac{\partial}{\partial b}f_{ab}\right)(\delta_{ab},\epsilon_{ab})\\ &= \frac{1}{2}\,\frac{1}{a}\log\left[\frac{b^2(\frac{a+1-b}{2}-\delta_{ab})(\frac{a-1-b}{2}-\epsilon_{ab})}{(b-\delta_{ab}-\epsilon_{ab})^2(\frac{a+1-b}{2})(\frac{a-1-b}{2})}\right].\end{aligned} \tag{2.1.16}$$

Setting $a = a^* = \frac{5}{2}$, $b = 1$, $\delta_{ab} = \delta_{a^*1} = \frac{1}{2}$ and $\epsilon_{ab} = \epsilon_{a^*1} = \frac{1}{6}$, we get the claim. □

2.1.2 Paths running along an interface

We also need the following analogue of Lemma 2.1.1. For $\mu \geq 1$, let $\hat{N}_L(\mu)$ denote the number of μL-step self-avoiding paths from $(0,0)$ to $(L,0)$ with no restriction on the vertical displacement (μL is integer). Let

$$\hat{\kappa}(\mu) = \lim_{L\to\infty}\frac{1}{\mu L}\log\hat{N}_L(\mu). \tag{2.1.17}$$

Lemma 2.1.2 *(i) $\hat{\kappa}(\mu)$ exists and is finite for all $\mu \geq 1$.*
(ii) $\mu \mapsto \mu\hat{\kappa}(\mu)$ is continuous and strictly concave on $[1,\infty)$ and analytic on $(1,\infty)$.
(iii) $\hat{\kappa}(1) = 0$ and $\mu\hat{\kappa}(\mu) \sim \log\mu$ as $\mu\to\infty$.
(iv) $\sup_{\mu\geq 1}\mu[\hat{\kappa}(\mu) - \frac{1}{2}\log 5] < \frac{1}{2}\log\frac{9}{5}$.

Proof. (i) Similarly as in (2.1.4),

$$\hat{N}_L(\mu) = \sum_{k=1}^{L} \begin{pmatrix} L \\ k \end{pmatrix} \begin{pmatrix} \frac{\mu-1}{2}L - 1 \\ k-1 \end{pmatrix} \sum_{l=1}^{L-k} \begin{pmatrix} L-k \\ l \end{pmatrix} \begin{pmatrix} \frac{\mu-1}{2}L - 1 \\ l-1 \end{pmatrix}. \tag{2.1.18}$$

Again putting $k = \delta L$, $l = \epsilon L$, we get

$$\mu\hat{\kappa}(\mu) = \lim_{L\to\infty} \frac{1}{L} \log \hat{N}_L(\mu) = \sup_{\delta,\epsilon} f_\mu(\delta, \epsilon) \tag{2.1.19}$$

with

$$\begin{aligned} f_\mu(\delta,\epsilon) = &-2\delta\log\delta - 2\epsilon\log\epsilon - (1-\delta-\epsilon)\log(1-\delta-\epsilon) \\ &- \left(\frac{\mu-1}{2} - \delta\right)\log\left(\frac{\mu-1}{2} - \delta\right) - \left(\frac{\mu-1}{2} - \epsilon\right)\log\left(\frac{\mu-1}{2} - \epsilon\right) \\ &+ (\mu-1)\log\left(\frac{\mu-1}{2}\right). \end{aligned} \tag{2.1.20}$$

Computing

$$\begin{aligned} \frac{\partial f_\mu}{\partial\delta} &= \log\left[\frac{(\frac{\mu-1}{2} - \delta)(1-\delta-\epsilon)}{\delta^2}\right], \\ \frac{\partial f_\mu}{\partial\epsilon} &= \log\left[\frac{(\frac{\mu-1}{2} - \epsilon)(1-\delta-\epsilon)}{\epsilon^2}\right], \end{aligned} \tag{2.1.21}$$

and setting these derivatives equal to zero, we find that the maximisers δ_μ and ϵ_μ of the right-hand side of (2.1.19) are equal, $\delta_\mu = \epsilon_\mu$, with δ_μ the solution of the quadratic equation

$$0 = \delta^2 - \mu\delta + \frac{\mu-1}{2}, \tag{2.1.22}$$

which leads to

$$\delta_\mu = \frac{1}{2}\left[\mu - \sqrt{(\mu-1)^2+1}\right]. \tag{2.1.23}$$

Substitution of (2.1.23) into (2.1.20) yields a formula for $\mu\hat{\kappa}(\mu)$ in closed form. From this formula it is obvious that $\mu \mapsto \hat{\kappa}(\mu)$ is continuous on $[1,\infty)$ and analytic on $(1,\infty)$.

(ii) Pick any $\mu_1, \mu_2 \geq 1$. The number of $\frac{1}{2}(\mu_1+\mu_2)L$-step paths from $(0,0)$ to $(L,0)$ is larger than or equal to the number of $\frac{1}{2}\mu_1 L$-step paths from $(0,0)$ to $(\frac{1}{2}L, 0)$ times the number of $\frac{1}{2}\mu_2 L$-step paths from $(\frac{1}{2}L, 0)$ to $(L,0)$, i.e.,

$$\hat{N}_L\left(\frac{\mu_1+\mu_2}{2}\right) \geq \hat{N}_{\frac{1}{2}L}(\mu_1)\hat{N}_{\frac{1}{2}L}(\mu_2). \tag{2.1.24}$$

Via (2.1.17), this proves that

$$\frac{\mu_1+\mu_2}{2}\hat{\kappa}\left(\frac{\mu_1+\mu_2}{2}\right) \geq \frac{1}{2}\mu_1\hat{\kappa}(\mu_1) + \frac{1}{2}\mu_2\hat{\kappa}(\mu_2), \tag{2.1.25}$$

which is the concavity desired. Strict concavity follows from smoothness on $(1,\infty)$, because $\mu\kappa(\mu)$ clearly is not linear in μ.

(iii) From (2.1.23) we see that $\delta_1 (= \epsilon_1) = 0$. Hence (2.1.19–2.1.20) give $\hat{\kappa}(1) = 0$. Similarly, if $\mu \to \infty$, then $\delta_\mu = \frac{1}{2}[1 - \frac{1}{2\mu} + O(\frac{1}{\mu^2})]$ and hence $\mu\hat{\kappa}(\mu) \sim \log\mu$.

(iv) For any $a \geq 2$, $0 < b \leq 1$, $\mu \geq 1$ such that $(\mu - 1)b \leq a - 2$, we have

$$a\kappa(a,1) \geq b\mu\hat{\kappa}(\mu) + (a - b\mu)\kappa(a - b\mu, 1 - b). \tag{2.1.26}$$

Indeed, any aL-step self-avoiding path from $(0,0)$ to (L,L) may follow the interface over a distance bL during $b\mu L$ steps and then wander away from the interface to the diagonally opposite corner over a distance $(1-b)L$ during $(a - b\mu)L$ steps (see Fig. 4). Rewrite (2.1.26) as

$$\mu\hat{\kappa}(\mu) \leq \frac{1}{b}\left[a\kappa(a,1) - (a - b\mu)\kappa(a - b\mu, 1 - b)\right]. \tag{2.1.27}$$

Pick $a = a^*$ and let $b \downarrow 0$, to obtain

$$\mu\hat{\kappa}(\mu) \leq \mu\left(\frac{\partial}{\partial a}(a\kappa)\right)(a^*,1) + \left(\frac{\partial}{\partial b}(a\kappa)\right)(a^*,1). \tag{2.1.28}$$

By Lemma 2.1.1(iv,v), the right-hand side equals $\mu\frac{1}{2}\log 5 + \frac{1}{2}\log\frac{9}{5}$. Since $\mu \geq 1$ is arbitrary, this proves that $\sup_{\mu\geq 1}\mu[\hat{\kappa}(\mu) - \frac{1}{2}\log 5] \leq \frac{1}{2}\log\frac{9}{5}$, which is the claim with $\leq$ instead of $<$. A calculation with MAPLE gives that the supremum in the left-hand side is attained at $\mu \approx 2.12$ and equals ≈ 0.16. The right-hand side equals 0.29. □

In Section 4.1 we will need two special values of α, namely, α_0 and α_1 given by

$$\begin{aligned}\sup_{\mu\geq 1}\mu\left[\hat{\kappa}(\mu) + \frac{1}{2}\alpha_0 - \frac{1}{2}\log 5\right] &= \frac{1}{2}\log\frac{9}{5},\\ \sup_{\mu\geq 1}\mu\left[\hat{\kappa}(\mu) - \frac{1}{2}\log 5\right] &= \frac{1}{2}\log\left[\frac{4e^{-\alpha_1}(5+e^{-\alpha_1})^2}{5(5-e^{-\alpha_1})^2}\right].\end{aligned} \tag{2.1.29}$$

It follows from Lemma 2.1.2(iii-iv) that $\alpha_0, \alpha_1 > 0$. A calculation with MAPLE gives the values

$$\alpha_0 \approx 0.125, \qquad \alpha_1 \approx 0.154. \tag{2.1.30}$$

2.2 Free energies per pair of blocks

In this section we identify $S_{kl} = S_{kl}(\alpha,\beta)$.

2.2.1 Identification of S_{AA} and S_{BB}

Proposition 2.2.1 *For all* $(\alpha,\beta) \in \mathbb{R}^2$,

$$S_{AA} = \sup_{a\geq 2}\psi_{AA}(a) = \frac{1}{2}\alpha + \frac{1}{2}\log 5, \qquad S_{BB} = \sup_{a\geq 2}\psi_{BB}(a) = \frac{1}{2}\beta + \frac{1}{2}\log 5. \tag{2.2.1}$$

Proof. Recall (1.2.4) and (1.3.1–1.3.2). For any aL-step path in an AA-block, about half of the monomers contribute α to the energy, because $\sum_{i=1}^{aL} 1\{\omega_i = A\} = \frac{1}{2}aL[1 + o(1)]$ ω-a.s. as $L \to \infty$, while the remaining monomers contribute 0 to the energy. Hence

$$\psi_{AA}(a) = \frac{1}{2}\alpha + \kappa(a,1). \tag{2.2.2}$$

Now use Lemma 2.1.1(iv) to get the claim for S_{AA}. The proof for S_{BB} is the same. □

2.2.2 Identification of S_{AB} and S_{BA}

It is harder to obtain information on $S_{AB} = \sup_{a\geq 2} \psi_{AB}(a)$ and $S_{BA} = \sup_{a\geq 2} \psi_A(a)$, because these embody the effect of the presence of the AB-interface. We first consider the free energy per step when the path moves in the vicinity of a *single linear interface* $\mathcal{I}$ separating a liquid A in the upper halfplane from a liquid B in the lower halfplane including the interface itself. To that end, for $a \geq b > 0$, let $\mathcal{W}_{aL,bL}$ denote the set of aL-step directed self-avoiding paths starting at $(0,0)$ and ending at $(bL,0)$. Define

$$\psi_L^{\omega,\mathcal{I}}(a,b) = \frac{1}{aL} \log Z_{aL,bL}^{\omega,\mathcal{I}} \tag{2.2.3}$$

with

$$\begin{aligned} Z_{aL,bL}^{\omega,\mathcal{I}} &= \sum_{\pi\in\mathcal{W}_{aL,bL}} \exp\left[-H_{aL}^{\omega,\mathcal{I}}(\pi)\right], \\ H_{aL}^{\omega,\mathcal{I}}(\pi) &= -\sum_{i=1}^{aL} \Big(\alpha 1\{\omega_i = A, \pi_i > 0\} + \beta 1\{\omega_i = B, \pi_i \leq 0\}\Big), \end{aligned} \tag{2.2.4}$$

where $\pi_i > 0$ means that the i-th step lies in the upper halfplane and $\pi_i \leq 0$ means that the i-th step lies in the lower halfplane or in the interface.

Lemma 2.2.2 *For all $(\alpha,\beta) \in \mathbb{R}^2$ and $a \geq b > 0$,*

$$\lim_{L\to\infty} \psi_L^{\omega,\mathcal{I}}(a,b) = \psi^{\mathcal{I}}(a,b) = \psi^{\mathcal{I}}(\alpha,\beta;a,b) \tag{2.2.5}$$

exists ω-a.s. and is non-random.

Proof. Since the polymer starts and ends at the interface, the proof can be done via a standard subadditivity argument in which two pieces of the polymer are concatenated (see e.g. Bolthausen and den Hollander [4] or Orlandini *et al* [27]). Indeed, fix a and b. Then, for any L_1 and L_2,

$$Z_{a(L_1+L_2),b(L_1+L_2)}^{\omega,\mathcal{I}} \geq Z_{aL_1,bL_1}^{\omega,\mathcal{I}} Z_{aL_2,bL_2}^{\sigma^{aL_1}\omega,\mathcal{I}}, \tag{2.2.6}$$

where σ is the left-shift acting on ω. Define

$$\Psi_K^{\omega,\mathcal{I}}(a,b) = \log Z_{K,(b/a)K}^{\omega,\mathcal{I}}. \tag{2.2.7}$$

Then, for any $K_1 (= aL_1)$ and $K_2 (= aL_2)$,

$$\Psi_{K_1+K_2}^{\omega,\mathcal{I}}(a,b) \geq \Psi_{K_1}^{\omega,\mathcal{I}}(a,b) + \Psi_{K_2}^{\sigma^{K_1}\omega,\mathcal{I}}(a,b). \tag{2.2.8}$$

We can now apply Kingman's superadditive ergodic theorem, noting that $\frac{1}{K}\Psi_K^{\omega,\mathcal{I}}(a,b)$ is bounded from above, to conclude that

$$\lim_{K\to\infty} \frac{1}{K} \Psi_K^{\omega,\mathcal{I}}(a,b) = \psi^{\mathcal{I}}(a,b) \tag{2.2.9}$$

exists ω-a.s. and is non-random. □

The relation linking $\psi_{AB}(a)$ to $\psi^{\mathcal{I}}(a,b)$ is the following.

Lemma 2.2.3 *For all* $(\alpha,\beta)\in\mathbb{R}^2$ *and* $a\geq 2$,

$$\begin{aligned}\psi_{AB}(a) &= \psi_{AB}(\alpha,\beta;a)\\ &= \sup_{0\leq b\leq 1,\, a_1\geq b,\, a_2\geq 2-b,\, a_1+a_2=a} \frac{a_1\psi^{\mathcal{I}}(a_1,b)+a_2[\frac{1}{2}\alpha+\kappa(a_2,1-b)]}{a_1+a_2}.\end{aligned} \tag{2.2.10}$$

Proof. The idea behind this relation is that the polymer follows the AB-interface over a distance bL during a_1L steps and then wanders away from the AB-interface to the diagonally opposite corner over a distance $(1-b)L$ during a_2L steps. The optimal strategy is obtained by maximising over b, a_1 and a_2 (recall Figure 4).

A formal proof goes as follows. Look at the last time u and the last site $(v,0)$ on the AB-interface before the polymer wanders off. This allows us to write the associated partition sum as

$$Z^{\omega}_{AB}(aL,L)=\sum_{v=0}^{L}\sum_{u=v}^{aL-(2L-v)} Z^{\omega,\mathcal{I}}_{u,v}\,Z^{\sigma^v\omega}_{aL-u,L-v}, \tag{2.2.11}$$

where $Z^{\omega,\mathcal{I}}_{u,v}$ is the partition sum for the single interface model to go in u steps from $(0,0)$ to $(v,0)$, and $Z^{\sigma^u\omega}_{aL-u,L-v}$ is the partition sum to go in $aL-u$ steps from $(v,0)$ to (L,L) without returning to the interface. Rewrite (2.2.11) as

$$Z^{\omega}_{AB}(aL,L)=[1+o(1)]\,L^2\int_0^1 db\int_b^{a-(2-b)} da_1\, Z^{\omega,\mathcal{I}}_{a_1L,bL}Z^{\sigma^{bL}\omega}_{(a-a_1)L,(1-b)L}. \tag{2.2.12}$$

From Lemmas 2.1.1 and 2.2.2 we know that, as $L\to\infty$,

$$\begin{aligned}\frac{1}{L}\log Z^{\omega,\mathcal{I}}_{a_1L,bL} &= [1+o(1)]\,a_1\psi^{\mathcal{I}}(a_1,b)\quad \omega-a.s.,\\ \frac{1}{L}\log Z^{\sigma^{bL}\omega}_{(a-a_1)L,(1-b)L} &= [1+o(1)]\,(a-a_1)\left(\frac{1}{2}\alpha+\kappa(a-a_1,1-b)\right)\quad \omega-a.s.\end{aligned} \tag{2.2.13}$$

For the latter, note that $\sigma^{bL}\omega$ changes with L. However, this causes no problem, because the distribution of ω is invariant under shifts and the shift length bL is independent of ω. Substitution of (2.2.13) into (2.2.12), and of the resulting expression into (1.3.1), yields the claim after we put $a_2=a-a_1$. Indeed, the right-hand sides of (2.2.13) are continuous in b and a_1. □

By obvious scaling, there exists a function $\phi^{\mathcal{I}}$ such that

$$\psi^{\mathcal{I}}(a,b)=\phi^{\mathcal{I}}(a/b). \tag{2.2.14}$$

Therefore Lemma 2.2.3 yields the following.

Proposition 2.2.4 *For all* $(\alpha,\beta)\in\mathbb{R}^2$,

$$S_{AB}=\sup_{a\geq 2}\psi_{AB}(a)=\sup_{0\leq b\leq 1,\, a_1\geq b,\, a_2\geq 2-b}\frac{a_1\phi^{\mathcal{I}}(a_1/b)+a_2[\frac{1}{2}\alpha+\kappa(a_2,1-b)]}{a_1+a_2}, \tag{2.2.15}$$

Proof. Insert (2.2.14) into (2.2.10) and take the supremum over a. □

This completes the identification of S_{AB}. The same formula applies for S_{BA} but with α and β interchanged, i.e.,

$$S_{BA}(\alpha,\beta) = S_{AB}(\beta,\alpha). \tag{2.2.16}$$

Recall from the remark made below (1.3.2) that the first index labels the type of the block that is diagonally crossed, while the second index labels the type of the block that appears as its neighbour.

Note that $\phi^{\mathcal{I}}$ is symmetric in α and β. The asymmetry in (2.2.4), coming from the fact that the interface is labelled B while the polymer starts at the interface, is not felt in the limit as $L \to \infty$. Further note that

$$\begin{aligned} &\phi^{\mathcal{I}}(\alpha,\beta;\mu) \in \left[\frac{1}{2}\alpha + \hat\kappa(\mu), \alpha + \hat\kappa(\mu)\right] && \forall\, \alpha \geq \beta \geq 0, \\ &\phi^{\mathcal{I}}(\alpha,\beta;\mu) = \frac{1}{2}\alpha + \hat\kappa(\mu) && \forall\, \alpha \geq 0 \geq \beta. \end{aligned} \tag{2.2.17}$$

We close with the following facts.

Lemma 2.2.5 *Let $k, l \in \{A, B\}$.*
(i) For all $(\alpha,\beta) \in \mathbb{R}^2$, $a \mapsto a\psi_{kl}(\alpha,\beta;a)$ is continuous and concave on $[2,\infty)$.
(ii) For all $a \in [2,\infty)$, $\alpha \mapsto \psi_{kl}(\alpha,\beta;a)$ and $\beta \mapsto \psi_{kl}(\alpha,\beta;a)$ are continuous and non-decreasing on $\mathbb{R}$.

Proof. (i) The claim is trivial for $k = l$, because of the simple form of ψ_{AA} and ψ_{BB} (recall (2.1.3) and (2.2.2)). The proof for $k \neq l$ runs as follows. Rewrite (2.2.10) as

$$a\psi_{AB}(a) = \sup_{0\leq b\leq 1, a_1\geq b, a_2\geq 2-b, a_1+a_2=a} \left\{ a_1\psi^{\mathcal{I}}(a_1,b) + a_2\left[\frac{1}{2}\alpha + \kappa(a_2, 1-b)\right]\right\}. \tag{2.2.18}$$

From this it follows that

$$\begin{aligned} &\frac{1}{2}a^1\psi_{AB}(a^1) + \frac{1}{2}a^2\psi_{AB}(a^2) = \sup_{0\leq b^1\leq 1, a_1^1\geq b^1, a_2^1\geq 2-b^1, a_1^1+a_2^1=a^1} \ \sup_{0\leq b^2\leq 1, a_1^2\geq b^2, a_2^2\geq 2-b^2, a_1^2+a_2^2=a^2} \\ &\left\{\frac{1}{2}a_1^1\psi^{\mathcal{I}}(a_1^1,b^1) + \frac{1}{2}a_1^2\psi^{\mathcal{I}}(a_1^2,b^2) + \frac{1}{2}(a_2^1+a_2^2)\frac{1}{2}\alpha + \frac{1}{2}a_2^1\kappa(a_2^1,1-b^1) + \frac{1}{2}a_2^2\kappa(a_2^2,1-b^2)\right\}. \end{aligned} \tag{2.2.19}$$

A standard concatenation argument gives

$$\begin{aligned} \frac{1}{2}a_1^1\psi^{\mathcal{I}}(a_1^1,b^1) + \frac{1}{2}a_1^2\psi^{\mathcal{I}}(a_1^2,b^2) &= \bar a_1\psi^{\mathcal{I}}(\bar a_1,\bar b), \\ \frac{1}{2}a_2^1\kappa(a_2^1,1-b^1) + \frac{1}{2}a_2^2\kappa(a_2^2,1-b^2) &= \bar a_2\kappa(\bar a_2,1-\bar b), \end{aligned} \tag{2.2.20}$$

where we abbreviate

$$\bar a_1 = \frac{a_1^1+a_1^2}{2}, \quad \bar a_2 = \frac{a_2^1+a_2^2}{2}, \quad \bar b = \frac{b^1+b^2}{2}. \tag{2.2.21}$$

Since the double supremum in (2.2.19) is more restrictive than the single supremum over $0 \leq \bar b \leq 1$, $\bar a_1 \geq \bar b$, $\bar a_2 \geq 2-\bar b$, $\bar a_1 + \bar a_2 = \bar a$, with $\bar a = (a^1+a^2)/2$, it follows from (2.2.10) and (2.2.19–2.2.20) that

$$\frac{1}{2}a^1\psi_{AB}(a^1) + \frac{1}{2}a^2\psi_{AB}(a^2) \leq \bar a\psi_{AB}(\bar a). \tag{2.2.22}$$

A similar argument applies to ψ_{BA}, after replacing $\frac{1}{2}\alpha$ by $\frac{1}{2}\beta$ and noting that $\psi^{\mathcal{I}}$ is symmetric in α and β.

(ii) The claim is again trivial for $k = l$. For $k \neq l$, note that $\psi^{\mathcal{I}}$ has the same property, as is evident from (2.2.3–2.2.5). Hence the claim follows from Lemma 2.2.3. □

2.3 Criterion for $S_{AB} > S_{AA}$

For all $(\alpha, \beta) \in \mathbb{R}^2$, we have

$$S_{AB} \geq S_{AA}. \tag{2.3.1}$$

The following gives us a criterion for when strict inequality occurs. In Section 4.1.1 this will be proved to be the criterion for localization when $p \geq p_c$.

Proposition 2.3.1 *$S_{AB} > S_{AA}$ if and only if*

$$\sup_{\mu \geq 1} \mu[\phi^{\mathcal{I}}(\mu) - S_{AA}] > \frac{1}{2} \log \frac{9}{5}. \tag{2.3.2}$$

Proof. From Propositions 2.2.1 and 2.2.4, together with the reparametrisation $\mu = a_1/b$ and $\nu = a_2/b$, it follows that

$$S_{AB} - S_{AA} = \sup_{\mu \geq 1, \nu \geq 1} \frac{\mu[\phi^{\mathcal{I}}(\mu) - S_{AA}] - \nu[\frac{1}{2}\log 5 - f(\nu)]}{\mu + \nu} \tag{2.3.3}$$

with

$$f(\nu) = \sup_{\frac{2}{\nu+1} \leq b \leq 1} \kappa(b\nu, 1-b), \qquad \nu \geq 1. \tag{2.3.4}$$

Abbreviate $g(\nu) = \nu[\frac{1}{2}\log 5 - f(\nu)]$. Below we will show that

$$\begin{aligned} &\text{(i)} \;\; g(\nu) > \frac{1}{2}\log\frac{9}{5} \;\; \text{ for all } \nu \geq 1, \\ &\text{(ii)} \;\; \lim_{\nu \to \infty} g(\nu) = \frac{1}{2}\log\frac{9}{5}. \end{aligned} \tag{2.3.5}$$

This will imply the claim as follows. If $\mu[\phi^{\mathcal{I}}(\mu) - S_{AA}] \leq \frac{1}{2}\log\frac{9}{5}$ for all μ, then by (i) the numerator in (2.3.3) is strictly negative for all μ and ν, and so by (ii) the supremum is taken at $\nu = \infty$, resulting in $S_{AB} - S_{AA} = 0$. On the other hand, if $\mu[\phi^{\mathcal{I}}(\mu) - S_{AA}] > \frac{1}{2}\log\frac{9}{5}$ for some μ, then, for that μ, by (i) and (ii) the numerator is strictly positive for ν large enough, resulting in $S_{AB} - S_{AA} > 0$.

To prove (2.3.5), we will need the following inequality. Abbreviate $\chi(a,b) = a\kappa(a,b)$. Then by Lemma 2.1.1(ii) we have, for all $(s,t) \neq (u,v)$ in DOM,

$$\begin{aligned} \chi(s,t) - \chi(u,v) &= \int_0^1 dw \, \frac{\partial}{\partial w} \chi(u + w(s-u), v + w(t-v)) \\ &> \left[\frac{\partial}{\partial w} \chi(u + w(s-u), v + w(t-v)) \right]_{w=1} \\ &= (s-u) \left(\frac{\partial}{\partial a} \chi \right)(s,t) + (t-v) \left(\frac{\partial}{\partial b} \chi \right)(s,t). \end{aligned} \tag{2.3.6}$$

To prove (2.3.5)(i), put $b = a/\nu$ in (2.3.4) and use Lemma 2.1.1(iv,v) to rewrite the statement in (2.3.5)(i) as

$$\kappa\left(a, 1-\frac{a}{\nu}\right) < \kappa(a^*,1) - \frac{a^*}{\nu}\left(\frac{\partial}{\partial b}\kappa\right)(a^*,1) \quad \text{for all } \nu \geq 1 \text{ and } \frac{2\nu}{\nu+1} \leq a \leq \nu. \tag{2.3.7}$$

But this inequality follows from (2.3.6) by picking $s = a^*$, $t = 1$, $u = a$, $v = 1 - \frac{a}{\nu}$, cancelling a term $a^*\kappa(a^*,1)$ on both sides, using that $(\frac{\partial}{\partial a}\kappa)(a^*,1) = 0$, and afterwards cancelling a common factor a on both sides.

To prove (2.3.5)(ii), we argue as follows. Picking $b = \frac{a^*}{\nu}$ in (2.3.4), we get from Lemma 2.1.1(iv) that

$$g(\nu) \leq \nu\left[\kappa(a^*,1) - \kappa\left(a^*, 1-\frac{a^*}{\nu}\right)\right]. \tag{2.3.8}$$

Letting $\nu \to \infty$, we get from Lemma 2.1.1(v) that

$$\limsup_{\nu\to\infty} g(\nu) \leq a^*\left(\frac{\partial}{\partial b}\kappa\right)(a^*,1) = \frac{1}{2}\log\frac{9}{5}. \tag{2.3.9}$$

Combine this with (2.3.5)(i) to get (2.3.5)(ii). □

Proposition 2.3.1 says that the free energy per step for an AB-block exceeds that for an AA-block if and only the free energy per step for the single linear interface exceeds the free energy per step for an AA-block by a certain positive amount. This excess is needed to *compensate* for the loss of entropy that occurs when the path runs along the interface for awhile before moving upwards from the interface to end at the diagonally opposite corner (recall Fig. 4). The constant $\frac{1}{2}\log\frac{9}{5}$ is special to our model.

2.4 Criterion for $\psi_{AB}(\bar{x}) > \psi_{AA}(\bar{x})$ and $\psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})$

For all $(\alpha,\beta) \in \mathbb{R}^2$ and $a \geq 2$, we have

$$\psi_{AB}(a) \geq \psi_{AA}(a), \qquad \psi_{BA}(a) \geq \psi_{BB}(a). \tag{2.4.1}$$

The following gives a criterion for when strict inequality occurs and is the analogue of Proposition 2.3.1.

Proposition 2.4.1 *For all $a \geq 2$, $\psi_{AB}(a) > \psi_{AA}(a)$ if and only if*

$$\sup_{\mu\geq 1}\mu\left[\phi^{\mathcal{I}}(\mu) - \frac{1}{2}\alpha - \frac{1}{2}\log\left(\frac{a}{a-2}\right)\right] > \frac{1}{2}\log\left[\frac{4(a-2)(a-1)^2}{a}\right]. \tag{2.4.2}$$

Proof. Return to Lemma 2.2.3. Fix $a \geq 2$. By (2.2.2), (2.2.10) and (2.2.14), we have

$$\psi_{AB}(a) - \psi_{AA}(a) = \sup_{0\leq b\leq 1,\, a_1\geq b,\, a_2\geq 2-b,\, a_1+a_2=a} \frac{a_1\phi^{\mathcal{I}}(a_1/b) + a_2[\frac{1}{2}\alpha + \kappa(a_2,1-b)] - (a_1+a_2)\frac{1}{2}\alpha - (a_1+a_2)\kappa(a_1+a_2,1)}{a_1+a_2}. \tag{2.4.3}$$

The denominator is fixed. Put $\mu = a_1/b$ and rewrite the numerator as

$$\begin{aligned}&\mu b\phi^{\mathcal{I}}(\mu) + (a-\mu b)\left[\frac{1}{2}\alpha + \kappa(a-\mu b, 1-b)\right] - a\left[\frac{1}{2}\alpha + \kappa(a,1)\right]\\ &\qquad = \mu b\phi^{\mathcal{I}}(\mu) - \mu b\frac{1}{2}\alpha - [a\kappa(a,1) - (a-\mu b)\kappa(a-\mu b, 1-b)].\end{aligned} \tag{2.4.4}$$

By picking $s = a$, $t = 1$, $u = a - \mu b$, $v = 1 - b$ in (2.3.6), we obtain that for all $\mu \geq 1$ and $0 < b \leq 1$ with $(\mu - 1)b \leq a - 2$,

$$a\kappa(a,1) - (a-\mu b)\kappa(a-\mu b, 1-b) > \mu b\left(\frac{\partial}{\partial a}(a\kappa)\right)(a,1) + b\left(\frac{\partial}{\partial b}(a\kappa)\right)(a,1). \tag{2.4.5}$$

Since the right-hand side of (2.4.5) is b times the derivative at $b = 0$ of the left-hand side, it follows that the difference in (2.4.4) is ≤ 0 for all $0 \leq b \leq 1$ if and only if its derivative at $b = 0$ is ≤ 0. This derivative equals

$$\mu\phi^{\mathcal{I}}(\mu) - \mu\frac{1}{2}\alpha - \mu\left(\frac{\partial}{\partial a}(a\kappa)\right)(a,1) - \left(\frac{\partial}{\partial b}(a\kappa)\right)(a,1). \tag{2.4.6}$$

After substituting the expressions for $\kappa(a,1)$, $(\frac{\partial}{\partial a}\kappa)(a,1)$ and $(\frac{\partial}{\partial b}\kappa)(a,1)$ that we computed in Section 2.1.1 (recall (2.1.3), (2.1.10–2.1.11), (2.1.15–2.1.16)), we find that (2.4.6) equals

$$\mu\left[\phi^{\mathcal{I}}(\mu) - \frac{1}{2}\alpha - \frac{1}{2}\log\left(\frac{a}{a-2}\right)\right] - \frac{1}{2}\log\left[\frac{4(a-2)(a-1)^2}{a}\right]. \tag{2.4.7}$$

Hence we get the claim. □

For $\psi_{BA}(a) > \psi_{BB}(a)$ the same criterion applies as in (2.4.2) with $\frac{1}{2}\alpha$ replaced by $\frac{1}{2}\beta$. (Recall that $\phi^{\mathcal{I}}$ is symmetric in α and β by the remark made below (2.2.16).)

2.5 Analysis of $F(\rho)$

In this section we analyse the variational problem in (1.5.3).

Proposition 2.5.1 *Let $(\alpha, \beta) \in$ CONE and $\rho \in (0,1)$. Abbreviate $C = \alpha - \beta \geq 0$. The variational formula in (1.5.3) has unique maximisers $\bar{x} = \bar{x}(C,\rho)$ and $\bar{y} = \bar{y}(C,\rho)$ satisfying:*
(i) $2 < \bar{y} < a^* < \bar{x} < \infty$ when $C > 0$ and $\bar{x} = \bar{y} = a^*$ when $C = 0$.
(ii) $u(\bar{x}) > v(\bar{y})$ when $C > 0$ and $u(\bar{x}) = v(\bar{y})$ when $C = 0$.
(iii) $\rho \mapsto \bar{x}(C,\rho)$ and $\rho \mapsto \bar{y}(C,\rho)$ are analytic and strictly decreasing on $(0,1)$ for all $C > 0$.
(iv) $C \mapsto \bar{x}(C,\rho)$ and $C \mapsto \bar{y}(C,\rho)$ are analytic and strictly increasing, respectively, strictly decreasing on $(0,\infty)$ for all $\rho \in (0,1)$.
(v) As $\rho \uparrow 1$, $\bar{x}(C,\rho) \downarrow a^*$ and $\bar{y}(C,\rho) \downarrow 10/(5 - e^{-C})$ for all $C \geq 0$.
(vi) As $\rho \downarrow 0$, $\bar{x}(C,\rho) \uparrow 10e^{-C}/(5e^{-C} - 1)$ and $\bar{y}(C,\rho) \uparrow a^*$ when $0 \leq C < \log 5$, while $\bar{x}(C,\rho) \uparrow \infty$ and $\bar{y}(C,\rho) \uparrow 2/(1 - e^{-C})$ when $C \geq \log 5$.
(vii) As $C \uparrow \infty$, $\bar{x}(C,\rho) \uparrow \infty$ and $\bar{y}(C,\rho) \downarrow 2$ for all $\rho \in (0,1)$.

Proof. Fix $(\alpha, \beta) \in$ CONE and $\rho \in (0,1)$. The supremum in (1.5.3) is attained at those x, y that solve the equations

$$\begin{aligned}0 &= -\log 2 + \frac{(1-\rho)y}{2}(\alpha - \beta) + \frac{(1-\rho)y}{2}\log\left(\frac{x(y-2)}{y(x-2)}\right) - \rho\log(x-2) - (1-\rho)\log(y-2),\\ 0 &= -\log 2 + \frac{\rho x}{2}(\beta - \alpha) + \frac{\rho x}{2}\log\left(\frac{y(x-2)}{x(y-2)}\right) - \rho\log(x-2) - (1-\rho)\log(y-2).\end{aligned} \tag{2.5.1}$$

Multiplying the first relation by ρx, the second relation by $(1-\rho)y$, and adding them up, we get

$$0 = -[\rho x + (1-\rho)y]\left\{\log 2 + \rho\log(x-2) + (1-\rho)\log(y-2)\right\}. \tag{2.5.2}$$

Alternatively, subtracting the second relation from the first, we get

$$0 = \frac{1}{2}[\rho x + (1-\rho)y]\left\{(\alpha-\beta) + \log\left(\frac{x(y-2)}{y(x-2)}\right)\right\}. \tag{2.5.3}$$

Hence, x, y solve the equations

$$\begin{aligned} 0 &= \log 2 + \rho\log(x-2) + (1-\rho)\log(y-2), \\ 0 &= (\alpha-\beta) + \log\left(\frac{x(y-2)}{y(x-2)}\right). \end{aligned} \tag{2.5.4}$$

These are two coupled equations depending on ρ, respectively, $C = \alpha - \beta$. Since the equations are linearly independent, their solution is unique.

(i) Let $\bar{x}$ and $\bar{y}$ denote the unique solution of (2.5.4). Clearly, $\bar{x} = \bar{y} = a^* = \frac{5}{2}$ when $C = 0$. Suppose that $C > 0$. Then it follows from the second line of (2.5.4) that $\bar{x}/(\bar{x}-2) < \bar{y}/(\bar{y}-2)$, or $\bar{x} > \bar{y}$. Moreover, it follows from the first line of (2.5.4) that it is not possible to have $\bar{x} > \bar{y} \geq a^*$ or $\bar{y} < \bar{x} \leq a^*$. Consequently,

$$\bar{y} < a^* < \bar{x}. \tag{2.5.5}$$

The fact that $(\bar{x}, \bar{y}) \neq (\infty, 2)$ follows from (2.5.4) as well.

(ii) By (1.5.2),

$$u(\bar{x}) - v(\bar{y}) = \frac{\alpha-\beta}{2} + \left(\frac{1}{\bar{x}} - \frac{1}{\bar{y}}\right)\log 2 + \frac{1}{2}\log\left(\frac{\bar{x}}{\bar{y}}\right) - \frac{\bar{x}-2}{2\bar{x}}\log(\bar{x}-2) + \frac{\bar{y}-2}{2\bar{y}}\log(\bar{y}-2). \tag{2.5.6}$$

Using (2.5.4), we may rewrite $\alpha - \beta$ and $\log 2$ in terms of $\bar{x}, \bar{y}, \rho$. This gives, after some cancellations,

$$u(\bar{x}) - v(\bar{y}) = \left(\frac{\rho}{\bar{x}} + \frac{1-\rho}{\bar{y}}\right)\log\left(\frac{\bar{x}-2}{\bar{y}-2}\right). \tag{2.5.7}$$

This is > 0 when $C > 0$, because then $\bar{x} > \bar{y}$, and is $= 0$ when $C = 0$, because then $\bar{x} = \bar{y}$.

(iii) The analyticity follows from the uniqueness of the solution of (2.5.4) and the implicit function theorem. From the second line of (2.5.4) it follows that $\rho \mapsto \bar{x}(C,\rho)$ and $\rho \mapsto \bar{y}(C,\rho)$ are either both non-increasing or both non-decreasing. Differentiating the first line of (2.5.4) w.r.t. ρ, we get

$$0 = \log\left(\frac{\bar{x}-2}{\bar{y}-2}\right) + \frac{\rho}{\bar{x}-2}\frac{\partial}{\partial\rho}\bar{x} + \frac{1-\rho}{\bar{y}-2}\frac{\partial}{\partial\rho}\bar{y}. \tag{2.5.8}$$

Since $\bar{x} > \bar{y}$ when $C > 0$, the sum of the last two terms is < 0. Therefore it is not possible that $\frac{\partial}{\partial\rho}\bar{x}, \frac{\partial}{\partial\rho}\bar{y} \geq 0$. Hence $\frac{\partial}{\partial\rho}\bar{x}, \frac{\partial}{\partial\rho}\bar{y} < 0$.

(iv) The analyticity again follows from the uniqueness of the solution of (2.5.4) and the implicit function theorem. From the first line of (2.5.4) it follows that $C \mapsto \bar{x}(C,\rho)$ and $C \mapsto \bar{y}(C,\rho)$ are either non-decreasing, respectively, non-increasing or vice versa. Differentiating the second line of (2.5.4) w.r.t. $C = \alpha - \beta$, we get

$$0 = 1 - \frac{2}{\bar{x}(\bar{x}-2)}\frac{\partial}{\partial C}\bar{x} + \frac{2}{\bar{y}(\bar{y}-2)}\frac{\partial}{\partial C}\bar{y}. \tag{2.5.9}$$

Since $\bar{x}, \bar{y} > 2$, it is not possible that $\frac{\partial}{\partial C}\bar{x} \leq 0 \leq \frac{\partial}{\partial C}\bar{y}$. Hence $\frac{\partial}{\partial C}\bar{y} < 0 < \frac{\partial}{\partial C}\bar{x}$.

(v) Abbreviate $\Delta = e^{-C} \in (0,1]$. Since $\bar{y} \leq a^*$, it follows from the first line of (2.5.4) that $\bar{x} \downarrow a^*$ as $\rho \uparrow 1$. The second line of (2.5.4) therefore gives $\bar{y}/(\bar{y}-2) \uparrow 5\Delta^{-1}$, i.e., $\bar{y} \downarrow 10/(5-\Delta)$.

(vi) It follows from (2.5.4) that, as $\rho \downarrow 0$, either $\bar{x} \uparrow A \in (a^*, \infty)$, $\bar{y} \uparrow a^*$ or $\bar{x} \uparrow \infty$, $\bar{y} \uparrow 2/(1-\Delta)$. Since $\bar{y} \leq a^* = \frac{5}{2}$, the latter is possible only when $\Delta \leq \frac{1}{5}$. The former applies when $\Delta > \frac{1}{5}$, in which case $A = 10/(5-\Delta^{-1}) = 10\Delta/(5\Delta-1)$.

(vii) This is immediate from (2.5.4). □

Lemma 2.5.2 *$(\alpha, \beta) \mapsto F(\alpha, \beta; \rho)$ is analytic on $\mathbb{R}^2$ for all $\rho \in (0,1)$.*

Proof. This is immediate from (1.5.2–1.5.3) and Proposition 2.5.1(iv). □

3 Free energy of the polymer

In Section 3.1 we prove Theorem 1.3.1. In Section 3.2 we analyse the set $\mathcal{R}(p)$ in (1.3.6) and the supremum $\rho^*(p)$ in (1.5.1).

3.1 Proof of Theorem 1.3.1

(i) Write out the partition sum in (1.2.5) in terms of partition sums in successive blocks:

$$Z_{n,L_n}^{\omega,\Omega} = \sum_{N=1}^{n/2L_n} \sum_{(\Pi_i)_{i=1}^N} \sum_{u_1=2L_n}^{\infty} \cdots \sum_{u_N=2L_n}^{\infty} \Big[\prod_{i=1}^{N-1} Z_{u_i}^{\sigma^{u_1+\cdots+u_{i-1}}\omega}(t^\Omega(\Pi_i)) \Big] \times Z_{n-(u_1+\cdots+u_{N-1})}^{\sigma^{u_1+\cdots+u_{N-1}}\omega}((t^\Omega(\Pi_i))\ 1\{u_1+\cdots+u_{N-1} \leq n < u_1+\cdots+u_{N-1}+u_N\}. \tag{3.1.1}$$

Here, N counts the number of blocks traversed, σ is the left-shift acting on ω, Π_i is the i-th step of the coarse-grained path, u_i counts the number of steps spent in the i-th block diagonally traversed by Π_i, $t^\Omega(\Pi_i)$ labels the type of the i-th block in Ω, and $Z_u^\omega(t)$ is the partition sum for spending u steps in a block of type t. We want to derive the asymptotics of this expression as $n \to \infty$. *For reasons of space the argument below is somewhat sketchy, but the technical details are easy to fill in.*

First, for the computation we pretend that after n steps the path has just completed traversing a block, i.e., we replace the indicator in (3.1.1) by

$$1\{u_1+\cdots+u_N = n\}. \tag{3.1.2}$$

The error made in doing so is at most a factor $O(n)$. Next, in (3.1.1) we insert a weight 2^{-N} under the sum over $(\Pi_i)_{i=1}^N$, scale u_i by putting $v_i = u_i/L_n$, change the sum over u_i to an

integral over v_i, and insert a weight $e^{-(v_i-2)}$ under the integral. This gives

$$\begin{aligned} Z^{\omega,\Omega}_{n,L_n} = O(n) \sum_{N=1}^{n/2L_n} \Big(2^N e^{(n/L_n)-2N} L_n^N\Big) \\ \times \Big[\sum_{(\Pi_i)_{i=1}^N} 2^{-N} \int_2^\infty dv_1\, e^{-(v_1-2)} \cdots \int_2^\infty dv_N\, e^{-(v_N-2)} \\ \times \prod_{i=1}^N Z^{\sigma^{(v_1+\cdots v_{i-1})L_n}\omega}_{v_i L_n}(t^\Omega(\Pi_i))\ 1\{v_1+\cdots+v_N = n/L_n\}\Big], \end{aligned} \tag{3.1.3}$$

where the term between round brackets compensates for the insertion of the weights (and can be computed because of (3.1.2)), while roundoff errors (coming from turning sums into integrals) disappear into the error term. The factor between round brackets is $e^{o(n)}$ and therefore is negligible.

The point of the rewrite in (3.1.3) is that the sum over $(\Pi_i)_{i=1}^N$ and the integrals over v_i are *normalised*. Therefore we can now introduce two independent sequences of *random variables*,

$$\hat\Pi = \{\hat\Pi_i\colon\ i\in\mathbb{N}\}, \qquad \hat v = \{\hat v_i\colon\ i\in\mathbb{N}\}, \tag{3.1.4}$$

which describe a random uniform coarse-grained path, repectively, a random sequence of scaled times that are i.i.d. and Exp(1) distributed on $[2,\infty)$. In terms of these random variables we can rewrite (3.1.3) as

$$Z^{\omega,\Omega}_{n,L_n} = e^{o(n)} \sum_{N=1}^{n/2L_n} \Big\langle \prod_{i=1}^N Z^{\sigma^{(\hat v_1+\cdots\hat v_{i-1})L_n}\omega}_{\hat v_i L_n}(t^\Omega(\hat\Pi_i))\ 1\{\hat v_1+\cdots+\hat v_N = n/L_n\}\Big]\Big\rangle, \tag{3.1.5}$$

where $\langle\cdot\rangle$ denotes expectation with respect to $(\hat\Pi,\hat v)$. As $L_n\to\infty$ we have, for every fixed realisation of $(\hat\Pi,\hat v)$,

$$Z^{\sigma^{(\hat v_1+\cdots+\hat v_{i-1})L_n}\omega}_{\hat v_i L_n}(t^\Omega(\hat\Pi_i)) = \exp\Big\{L_n[1+o(1)]\,\hat v_i\,\psi_{t^\Omega(\hat\Pi_i)}(\hat v_i)\Big\} \qquad \omega-a.s. \tag{3.1.6}$$

with $\psi_{kl}(\hat v_i)$ the free energy per step in a kl-block where the path spends $\hat v_i L_n$ steps, defined in (1.3.2). Here we use that the distribution of ω is invariant under shifts and that $(\hat v_1+\cdots+\hat v_{i-1})L_n$ is independent of ω.

Because of (3.1.5) and (3.1.6), we are in a position to use large deviation theory (for background see e.g. den Hollander [16], Chapters I and II). To that end, we introduce the *empirical distribution*

$$\mathcal{E}^\Omega_N = \mathcal{E}^\Omega_N(\hat\Pi,\hat v) = \frac{1}{N}\sum_{i=1}^N \delta_{(t^\Omega(\hat\Pi_i),\hat v_i)}, \tag{3.1.7}$$

where $\delta_{(t,v)}$ is the unit measure at (t,v). This $\mathcal{E}^\Omega_N$ counts the frequency at which the $(t^\Omega(\hat\Pi_i),\hat v_i)$ assume values in the space

$$\Theta = \{AA, AB, BA, BB\}\times[2,\infty) \tag{3.1.8}$$

and is an element of $\mathcal{P}(\Theta)$, the set of probability distributions Θ. With the help of (3.1.6), we may rewrite (3.1.5) as

$$Z^{\omega,\Omega}_{n,L_n} = e^{o(n)} \sum_{N=1}^{n/2L_n} \Big\langle \exp\Big\{NL_n[1+o(1)]\,(\mathcal{E}^\Omega_N,h_1)_*\Big\}\ 1\{(\mathcal{E}^\Omega_N,h_2)_* = n/NL_n\}\Big\rangle, \tag{3.1.9}$$

where we introduce two functions on Θ,

$$h_1(t,v) = v\psi_t(v) \quad \text{and} \quad h_2(t,v) = v \quad \text{for} \quad (t,v) \in \Theta, \tag{3.1.10}$$

and put $(\mu, h)_* = \int_\Theta h d\mu$ for $\mu \in \mathcal{P}(\Theta)$. At this point, our partition sum has been rewritten in terms of an expectation w.r.t. the empirical distribution $\mathcal{E}_N^\Omega$ (which depends on $(\hat{\Pi}, \hat{v})$).

Our next step is to scale N by putting $M = NL_n/n$ and to rewrite (3.1.9) as

$$Z_{n,L_n}^{\omega,\Omega} = e^{o(n)} \int_0^{\frac{1}{2}} dM \, \left\langle \exp\left[n \frac{(\mathcal{E}_N^\Omega, h_1)_*}{(\mathcal{E}_N^\Omega, h_2)_*}\right] \, 1\left\{(\mathcal{E}_N^\Omega, h_2)_* = \frac{1}{M}\right\} \right\rangle, \tag{3.1.11}$$

where again roundoff errors disappear into the error term. The idea that we now use is that $(\mathcal{E}_N^\Omega)_{N\in\mathbb{N}}$ satisfies the *large deviation principle* on $\mathcal{P}(\Theta)$ with rate N and with some rate function $\mu \mapsto I(\mu)$ that has compact level sets. This rate function will be Ω-a.s. constant. For techniques on how to prove this, we refer to Comets [8], Greven and den Hollander [14] and Seppäläinen [28]. Thus, the probability of $\mathcal{E}_N^\Omega$ being close to some $\mu \in \mathcal{P}(\Theta)$ is $\exp\{-N[1+o(1)]I(\mu)\}$. Since $N = Mn/L_n = o(n)$, we conclude that these large deviations have a *negligible cost*. Using (3.1.11), together with the relation

$$\int_0^{\frac{1}{2}} dM \, 1\left\{(\mathcal{E}_N^\Omega, h_2)_* = \frac{1}{M}\right\} = 1/O(n) \tag{3.1.12}$$

(which follows from reversing the calculations above), we arrive at

$$f = \lim_{n\to\infty} \frac{1}{n} \log Z_{n,L_n}^{\omega,\Omega} = \sup_{\mu\in\mathcal{P}(\Theta)} \frac{(\mu, h_1)_*}{(\mu, h_2)_*} \qquad \omega - a.s. \tag{3.1.13}$$

But the supremum in the right-hand side is precisely the formula for f stated in Theorem 1.3.1(i). The free energy is trivially finite (recall (1.2.4), (1.2.5) and (1.3.7)).

Remark: In the above, somewhat sketchy, computation the introduction of the rate function can be avoided by appealing to concentration of measure estimates, which are a crude yet flexible form of large deviations (see Madras and Whittington [20] for an application of this technique in the context of self-avoiding random copolymers near a single linear interface).

(ii) The proof is elementary. Fix ω and Ω, and rewrite the partition sum in (1.2.5) as

$$Z_{n,L_n}^{\omega,\Omega} = \sum_{v_A,v_B} c_{n,L_n}^{\omega,\Omega}(v_A, v_B) \, e^{\alpha v_A + \beta v_B} \tag{3.1.14}$$

with

$$\begin{aligned} &c_{n,L_n}^{\omega,\Omega}(v_A, v_B) \\ &\quad = \left| \left\{ \pi \in \mathcal{W}_{n,L_n} : \, \sum_{i=1}^n 1\{\omega_i = \Omega_{\pi_i} = A\} = v_A, \, \sum_{i=1}^n 1\{\omega_i = \Omega_{\pi_i} = B\} = v_B \right\} \right|. \end{aligned} \tag{3.1.15}$$

Pick any $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. Then, by Cauchy-Schwarz applied to (3.1.14),

$$Z_{n,L_n}^{\omega,\Omega}\left(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}\right) \le Z_{n,L_n}^{\omega,\Omega}(\alpha_1,\beta_1) Z_{n,L_n}^{\omega,\Omega}(\alpha_2,\beta_2), \tag{3.1.16}$$

from which the claim follows (recall (1.2.5) and (1.3.7)).

(iii) According to Proposition 3.2.1(i) below, $p \mapsto \mathcal{R}(p)$ is continuous in the Hausdorff metric. It is therefore immediate from the variational representation of f in (1.3.8), together with the continuity of ψ_{kl} stated in Lemma 2.2.5(i), that $p \mapsto f(\alpha, \beta; p)$ is continuous.

3.2 Analysis of $\mathcal{R}(p)$ and $\rho^*(p)$

The following proposition is crucial for the analysis of the phase transition curves. Recall (1.3.6), (1.5.1) and Fig. 6. The elements of $\mathcal{R}(p)$ are matrices

$$\begin{pmatrix} \rho_{AA} & \rho_{AB} \\ \rho_{BA} & \rho_{BB} \end{pmatrix} \tag{3.2.1}$$

whose elements are non-negative and sum up to 1.

Proposition 3.2.1 *(i) $p \mapsto \mathcal{R}(p)$ is continuous in the Hausdorff metric.*
(ii) If $p \geq p_c$, then

$$\begin{pmatrix} 1-\gamma & \gamma \\ 0 & 0 \end{pmatrix} \in \mathcal{R}(p) \quad \text{for some } \gamma \in (0,1). \tag{3.2.2}$$

(iii) If $p < p_c$, then

$$\begin{pmatrix} 1-\gamma & \gamma \\ 0 & 0 \end{pmatrix} \notin \mathcal{R}(p) \quad \text{for all } \gamma \in [0,1]. \tag{3.2.3}$$

Proof. (i) Return to (1.3.3–1.3.6). Pick $0 < p < p' < 1$. Let Ω and Ω' be two typical percolation configurations with parameter p and p', respectively, coupled such that the set of A's in Ω' contains the set of A's in Ω. Then

$$\limsup_{n\to\infty} |\rho_{kl}^{\Omega}(\Pi, n) - \rho_{kl}^{\Omega'}(\Pi, n)| \leq 2(p'-p) \quad \Omega, \Omega' - a.s. \qquad \forall\, k, l \in \{A, B\},\ \Pi \in \mathcal{W}. \tag{3.2.4}$$

Therefore, for all $k, l \in \{A, B\}$, we have $|(\mathcal{R}^{\Omega} - \mathcal{R}^{\Omega'})_{kl}| \leq 2(p'-p)$ Ω, Ω'-a.s. and hence $|[\mathcal{R}(p) - \mathcal{R}(p')]_{kl}| \leq 2(p'-p)$.

(ii) If $p > p_c$, then there is an infinite cluster of A-blocks with a strictly positive density. For the coarse-grained path Π that moves to the infinite cluster and afterwards stays inside this cluster, we have $\rho_A = \rho_{AA} + \rho_{AB} = 1$. For the coarse-grained path Π that moves to the infinite cluster, afterwards stays inside this cluster, but follows its boundary as much as possible, we have $0 < \rho_{AB} = 1 - \rho_{AA} < 1$, which proves the claim. Since $p \mapsto \mathcal{R}(p)$ is continuous in the Hausdorff metric, and since lowering p increases the density of the B blocks, the claim trivially extends to $p = p_c$.

(iii) If $p < p_c$, then there is no infinite cluster of A-blocks. In fact, any coarse-grained path Π visits B-blocks with a strictly positive density (as follows from a coupling argument similar as in (i)). Hence, $\rho_A = \rho_{AA} + \rho_{AB} < 1$ for all Π, and in fact $\sup_{\Pi \in \mathcal{W}} \rho_A(\Pi) < 1$. Since $\mathcal{R}(p)$ is a closed set, this proves the claim. □

It follows from Proposition 3.2.1 and the fact that $\mathcal{R}(p)$ is closed that $\rho^*(p)$ defined in (1.5.1) has the qualitative properties indicated in Fig. 6: $p \mapsto \rho^*(p)$ is continuous and non-decreasing on $(0,1)$, $\rho^*(p) = 1$ for $p \in [p_c, 1)$ and $\rho^*(p) \in (0,1)$ for $p \in (0, p_c)$.

4 Analysis of the critical curve

In Section 4.1 we prove Theorems 1.4.1, 1.4.2 and 1.4.3 for $p \geq p_c$. In Section 4.2 we prove Theorems 1.5.1, 1.5.2 and 1.5.3 for $p < p_c$. In Section 4.3 we make some observations about the separation of the localised phase into two subphases for $p < p_c$.

4.1 Supercritical case $p \geq p_c$

4.1.1 Proof of Theorems 1.4.1 and 1.4.2

The following proposition proves Theorems 1.4.1 and 1.4.2. Recall from (2.3.1) that $S_{AB} \geq S_{AA}$.

Proposition 4.1.1 *Fix $p \geq p_c$.*
(i) If $S_{AB} = S_{AA}$, then $f = S_{AA}$.
(ii) If $S_{AB} > S_{AA}$, then $f > S_{AA}$.

Proof. The proof uses Theorem 1.3.1(i) and Proposition 3.2.1(ii), in combination with the inequalities

$$S_{BB} \leq S_{AA}, \qquad S_{BA} \leq S_{AB}, \tag{4.1.1}$$

which hold because $\beta \leq \alpha$ (recall Propositions 2.2.1 and 2.2.4).

(i) Suppose that $S_{AB} = S_{AA}$. Then, because $\psi_{kl}(a) \leq S_{kl}$ for all $a \geq 2$ and $k, l \in \{A, B\}$ by (1.4.5), Theorem 1.3.1(i) and (4.1.1) yield

$$f \leq \sup_{(a_{kl}) \in \mathcal{A}} \sup_{(\rho_{kl}) \in \mathcal{R}(p)} \frac{\sum_{k,l} \rho_{kl} a_{kl} S_{kl}}{\sum_{k,l} \rho_{kl} a_{kl}} \leq \sup_{k,l} S_{kl} = S_{AB} = S_{AA}. \tag{4.1.2}$$

On the other hand, Theorem 1.3.1(i) and Proposition 3.2.1(ii) yield

$$f \geq \frac{(1-\gamma)\bar{a}_{AA} S_{AA} + \gamma \bar{a}_{AB} S_{AB}}{(1-\gamma)\bar{a}_{AA} + \gamma \bar{a}_{AB}} = S_{AA}, \tag{4.1.3}$$

where $\bar{a}_{AA}, \bar{a}_{AB}$ are the maximisers of S_{AA}, S_{AB} (and the value of γ is irrelevant). Combine (4.1.2) and (4.1.3) to get $f = S_{AA}$.

(ii) Suppose that $S_{AB} > S_{AA}$. Then Theorem 1.3.1(i) and Proposition 3.2.1(ii) with $0 < \gamma < 1$ yield

$$f \geq \frac{(1-\gamma)\bar{a}_{AA} S_{AA} + \gamma \bar{a}_{AB} S_{AB}}{(1-\gamma)\bar{a}_{AA} + \gamma \bar{a}_{AB}} > S_{AA} \tag{4.1.4}$$

(here it is important that $\gamma > 0$). □

We see from Proposition 4.1.1 that $\mathcal{D}$ corresponds to the situation where the polymer is fully A-delocalized ($f = S_{AA}$), while $\mathcal{L}$ corresponds to the situation where the polymer is partially AB-delocalized ($S_{AA} < f < S_{AB}$).

4.1.2 Proof of Theorem 1.4.3

Since $S_{AA}(\alpha, \beta)$ does not depend on β, and $\beta \mapsto S_{AB}(\alpha, \beta)$ is continuous and non-decreasing on $\mathbb{R}$ for every $\alpha \in \mathbb{R}$ by Lemma 2.2.5(ii), the boundary between $\mathcal{D}$ and $\mathcal{L}$ is a continuous function in CONE. We denote this function by $\alpha \mapsto \beta_c(\alpha)$.

We first show that the curve is concave. To that end, pick any $\alpha_1 < \alpha_2$, and consider the points $(\alpha_1, \beta_c(\alpha_1))$ and $(\alpha_2, \beta_c(\alpha_2))$ on the curve. Let (α_3, β_3) be the midpoint of the line connecting the two. We want to show that $\beta_3 \leq \beta_c(\alpha_3)$. By the convexity of f, stated in Theorem 1.3.1(ii), we have

$$f(\alpha_3, \beta_3) \leq \frac{1}{2}[f(\alpha_1, \beta_c(\alpha_1)) + f(\alpha_2, \beta_c(\alpha_2))]. \tag{4.1.5}$$

Since the curve itself is part of $\mathcal{D}$ (recall (1.4.4)), it follows from Propositions 2.2.1 and 4.1.1(i) that the right-hand side of (4.1.5) equals

$$\frac{1}{2}\left[\left(\frac{1}{2}\alpha_1+\kappa\right)+\left(\frac{1}{2}\alpha_2+\kappa\right)\right]=\frac{1}{2}\left(\frac{\alpha_1+\alpha_2}{2}\right)+\kappa=\frac{1}{2}\alpha_3+\kappa. \tag{4.1.6}$$

Thus, $f(\alpha_3,\beta_3)\le\frac{1}{2}\alpha_3+\kappa$. But, by Proposition 4.1.1, the reverse inequality is true always, and so equality holds. Consequently, $(\alpha_3,\beta_3)\in\mathcal{D}$, which proves the claim that $\beta_3\le\beta_c(\alpha_3)$.

The concavity in combination with the lower bound in part (i) of the following lemma show that the curve is non-decreasing.

The following lemma settles most of Theorem 1.4.3.

Lemma 4.1.2 *Fix $p\ge p_c$.*
(i) $\beta_c(\alpha)\ge\log(2-e^{-\alpha})$ for all $\alpha\ge 0$.
(ii) $\beta_c(\alpha)<8\log 3$ for all $\alpha\ge 0$.
(iii) $\beta_c(\alpha)=\alpha$ for all $0\le\alpha\le\alpha_0$, where α_0 is the number defined in the first line of (2.1.29).

Proof. (i) We have, recalling (2.2.3–2.2.4),

$$\phi^{\mathcal{I}}(\mu)=\lim_{L\to\infty}\frac{1}{L}\log Z_L^{\omega,\mathcal{I}}(\mu)\quad\omega-a.s. \tag{4.1.7}$$

with

$$\begin{aligned}Z_L^{\omega,\mathcal{I}}(\mu)&=\sum_{\pi\in\mathcal{W}_{\mu L,L}}\exp\left[-H_{\mu L}^{\omega,\mathcal{I}}(\pi)\right]\\ H_{\mu L}^{\omega,\mathcal{I}}(\pi)&=-\sum_{i=1}^{\mu L}\Big(\alpha 1\{\omega_i=A,\pi_i>0\}+\beta 1\{\omega_i=B,\pi_i\le 0\}\Big).\end{aligned} \tag{4.1.8}$$

We will derive an upper bound on $\phi^{\mathcal{I}}(\mu)$ by doing a so-called first-order annealed estimate (also referred to as a first-order Morita approximation; see Orlandini *et al* [25]). This estimate consists in writing

$$H_{\mu L}^{\omega,\mathcal{I}}(\pi)=-\sum_{i=1}^{\mu L}\alpha 1\{\omega_i=A\}-\sum_{i=1}^{\mu L}1\{\pi_i\le 0\}\Big(-\alpha 1\{\omega_i=A\}+\beta 1\{\omega_i=B\}\Big), \tag{4.1.9}$$

using that the first term is $-\mu L\frac{1}{2}\alpha[1+o(1)]$ ω-a.s. as $L\to\infty$ and is independent of π, substituting this into (4.1.8) and performing an expectation over ω. This gives

$$\begin{aligned}&\Big\langle\log Z_L^{\omega,\mathcal{I}}(\mu)\Big\rangle\\ &\quad\le\mu L\frac{1}{2}\alpha[1+o(1)]+\log\sum_{\pi\in\mathcal{W}_{\mu L,L}}\prod_{i=1}^{\mu L}1\{\pi_i\le 0\}\left\langle e^{-\alpha 1\{\omega_i=A\}+\beta 1\{\omega_i=B\}}\right\rangle\\ &\quad\le\mu L\frac{1}{2}\alpha[1+o(1)]+\mu L[\hat\kappa(\mu)+o(1)]+\mu L\log\left(\frac{1}{2}e^{-\alpha}+\frac{1}{2}e^{\beta}\right),\end{aligned} \tag{4.1.10}$$

where $\langle\cdot\rangle$ denotes expectation over ω, we use Jensen's inequality as well as the i.i.d. property of ω in the first line, and we use Lemma 2.1.2(i) in the second line. Consequently,

$$\phi^{\mathcal{I}}(\mu)=\lim_{L\to\infty}\frac{1}{\mu L}\Big\langle\log Z_L^{\omega,\mathcal{I}}(\mu)\Big\rangle\le\frac{1}{2}\alpha+\hat\kappa(\mu)+\log\left(\frac{1}{2}e^{-\alpha}+\frac{1}{2}e^{\beta}\right). \tag{4.1.11}$$

Suppose that

$$\log\left(\frac{1}{2}e^{-\alpha}+\frac{1}{2}e^{\beta}\right)\le 0. \tag{4.1.12}$$

Then substitution of (4.1.11) into (2.3.3) gives

$$S_{AB}-S_{AA}\le \sup_{\mu\ge 1,\,\nu\ge 1}\frac{\mu[\hat\kappa(\mu)-\frac{1}{2}\log 5]-\nu[\frac{1}{2}\log 5-f(\nu)]}{\mu+\nu}. \tag{4.1.13}$$

But the right-hand side is the same as $S_{AB}-S_{AA}$ when $\alpha=\beta=0$ (as can be seen from (2.3.3) because $\phi^{\mathcal{I}}(\mu)=\hat\kappa(\mu)$ when $\alpha=\beta=0$), and therefore is equal to 0. Hence, recalling (1.3.5), we find that (4.1.12) implies that $(\alpha,\beta)\in\mathcal{D}$. Consequently,

$$\log\left(\frac{1}{2}e^{-\alpha}+\frac{1}{2}e^{\beta_c(\alpha)}\right)\ge 0 \qquad \text{for all } \alpha\ge 0, \tag{4.1.14}$$

which gives the lower bound that is claimed.

(ii) We will show that there exists a $\mu_0>1$ such that

$$\mu_0[\phi^{\mathcal{I}}(\mu_0)-S_{AA}]>\frac{1}{2}\log\frac{9}{5} \quad \text{for all } \alpha\ge 0 \text{ when } \beta\ge 8\log 3. \tag{4.1.15}$$

This will prove the claim via Theorem 1.4.2 and Proposition 2.3.1.

Consider the polymer along the single infinite interface $\mathcal{I}$ introduced in Section 2.2.2. Fix ω. In ω, look for the strings of B's that are followed by a string of at least three A's. Call these B-strings "good", and call all other B-strings "bad". Let $\pi(\omega)$ be the path that starts at $(0,0)$, steps to $(0,1)$ and proceeds as follows. Each time a good B-string comes up, the path moves down from height 1 to height 0 during the step that carries the A just preceeding the good B-string, moves at height 0 during the steps that carry the B's inside the string, moves up from height 0 to height 1 during the step that carries the first A after the string, and moves at height 1 during the step that carries the second A after the string. The third A can be used to either move from height 1 to height 0 in case the next good B-string comes up immediately, or to move at height 0 in case it is not. When a bad B-string comes up, the path stays at height 1.

Along $\pi(\omega)$, we have that all the A's lie in the upper halfplane, all the bad B-strings lie in the upper halfplane, while all the good B-strings lie in the interface. Asymptotically, the good B-strings contain $\frac{1}{4}$-th of the B's. Hence $\frac{1}{8}$-th of the steps carry a B that is in a good B-string. Moreover, the number of steps between heights 0 and 1 is $\frac{1}{8}$ times the number of steps at heights 0 and 1 (the average length of a good B-string is 2), and so $\pi(\omega)$ travels a distance L in time $\frac{9}{8}L$ for L large, which corresponds to $\mu=\mu_0=\frac{9}{8}$. Thus, the contribution of $\pi(\omega)$ to the Hamiltonian in (2.2.4) equals

$$H^{\omega,\mathcal{I}}_{\mu_0 L}(\pi(\omega))=-\mu_0 L\left(\frac{1}{2}\alpha+\frac{1}{8}\beta\right)[1+o(1)] \qquad \omega-a.s. \text{ as } L\to\infty. \tag{4.1.16}$$

Therefore, recalling (2.2.5) and (2.2.14), we have

$$\phi^{\mathcal{I}}(\mu_0)\ge\frac{1}{2}\alpha+\frac{1}{8}\beta. \tag{4.1.17}$$

Via (2.2.1) this gives

$$\mu_0[\phi^{\mathcal{I}}(\mu_0)-S_{AA}]\ge\mu_0\left(\frac{1}{8}\beta-\frac{1}{2}\log 5\right). \tag{4.1.18}$$

Consequently, the inequality in (4.1.15) holds as soon as

$$\frac{1}{8}\beta > \frac{8}{9}\log 3 + \frac{1}{18}\log 5. \tag{4.1.19}$$

Since the right-hand side is strictly smaller than $\log 3$, this proves the claim.

(iii) Pick $\alpha = \beta$. Then, by (2.2.17), $\phi^I(\mu) \le \alpha + \hat\kappa(\mu)$. If $\alpha \in [0, \alpha_0]$, with α_0 given by the first line of (2.1.29), then this bound in combination with Propositions 2.2.1 and 2.3.1 gives $S_{AB} = S_{AA}$. Thus, $\{(\alpha, \alpha)\colon\ \alpha \in [0, \alpha_0]\} \subset \mathcal{D}$. □

Lemma 4.1.2, together with the concavity of $\alpha \mapsto \beta_c(\alpha)$ (shown prior to Lemma 4.1.2), proves Theorem 1.4.3, except for the slope discontinuity stated in (1.4.8). But the latter follows from the fact that if the piece of β_c on $[\alpha^*, \infty)$ is analytically continued outside CONE, then it hits the vertical axis at a strictly positive value, namely, α_0 defined in the first line of (2.1.29). Indeed, for $\alpha = 0$ we have $\phi^{\mathcal{I}}(\mu) = \frac{1}{2}\beta + \hat\kappa(\mu)$, because there is zero exponential cost for the path to stay in the lower halfplane (recall (2.2.3–2.2.5) and (2.2.14)). Consequently, the criterion for delocalization in Proposition 2.3.1, $S_{AB} = S_{AA}$, reduces to

$$\sup_{\mu \ge 1} \mu \left[\hat\kappa(\mu) + \frac{1}{2}\beta - \frac{1}{2}\log 5 \right] \le \frac{1}{2}\log\frac{9}{5}. \tag{4.1.20}$$

This is true precisely when $\beta \le \alpha_0$.

4.2 Subcritical case $p < p_c$

4.2.1 Proof of Theorems 1.5.1 and 1.5.2

Proposition 4.2.1 *If $\psi_{AB} \equiv \psi_{AA}$ and $\psi_{BA} \equiv \psi_{BB}$, then $f = F(\rho^*(p))$.*

Proof. Suppose that (α, β) is such that $\psi_{AB}(a) = \psi_{AA}(a)$ and $\psi_{BA}(a) = \psi_{BB}(a)$ for all $a \ge 2$. Then the variational formula in (1.3.8) reduces to

$$f = \sup_{(a_{kl}) \in \mathcal{A}} \sup_{(\rho_{kl}) \in \mathcal{R}(p)} \frac{\rho_{AA}a_{AA}\psi_{AA}(a_{AA}) + \rho_{AB}a_{AB}\psi_{AA}(a_{AB}) + \rho_{BA}a_{BA}\psi_{BB}(a_{BA}) + \rho_{BB}a_{BB}\psi_{BB}(a_{BB})}{\rho_{AA}a_{AA} + \rho_{AB}a_{AB} + \rho_{BA}a_{BA} + \rho_{BB}a_{BB}}. \tag{4.2.1}$$

Define

$$\tilde a_{AA} = \frac{\rho_{AA}}{\rho_A}a_{AA} + \frac{\rho_{AB}}{\rho_A}a_{AB}, \qquad \tilde a_{BB} = \frac{\rho_{BA}}{\rho_B}a_{BA} + \frac{\rho_{BB}}{\rho_B}a_{BB}, \tag{4.2.2}$$

where $\rho_A = \rho_{AA} + \rho_{AB}$ and $\rho_B = \rho_{BA} + \rho_{BB}$. By Lemma 2.1.1(ii) and (2.2.2), $a \mapsto a\psi_{AA}(a)$ and $a \mapsto a\psi_{BB}(a)$ are concave on $[2, \infty)$. Hence, the numerator in (4.2.1) can be bounded above as

$$\begin{aligned} &\rho_A \left[\frac{\rho_{AA}}{\rho_A}a_{AA}\psi_{AA}(a_{AA}) + \frac{\rho_{AB}}{\rho_A}a_{AB}\psi_{AA}(a_{AB}) \right] \\ &\qquad + \rho_B \left[\frac{\rho_{BA}}{\rho_B}a_{BA}\psi_{BB}(a_{BA}) + \frac{\rho_{BB}}{\rho_B}a_{BB}\psi_{BB}(a_{BB}) \right] \\ &\quad \le \rho_A \tilde a_{AA}\psi_{AA}(\tilde a_{AA}) + \rho_B \tilde a_{BB}\psi_{BB}(\tilde a_{BB}), \end{aligned} \tag{4.2.3}$$

while the denominator in (4.2.1) equals

$$\rho_A \tilde a_{AA} + \rho_B \tilde a_{BB}. \tag{4.2.4}$$

For any choice of $(\rho_{kl}) \in \mathcal{R}(p)$, as (a_{kl}) runs through $\mathcal{A}$, $\tilde{a}_{AA}$ and $\tilde{a}_{BB}$ run through all values ≥ 2. Moreover, equality can be achieved in (4.2.3) by picking $a_{AA} = a_{AB}$ and $a_{BA} = a_{BB}$. Hence (4.2.1) reduces to

$$f = \sup_{(\rho_{kl}) \in \mathcal{R}(p)} F(\rho_A) \tag{4.2.5}$$

with $F(\rho)$ given by (1.5.3). Thus, it remains to show that the supremum is taken at $\rho^*(p) = \sup_{(\rho_{kl}) \in \mathcal{R}(p)} \rho_A$.

For $\rho \in (0,1)$, let $\bar{x} = \bar{x}(\rho)$ and $\bar{y} = \bar{y}(\rho)$ denote the unique maximers of (1.5.3). Then, for any $\rho_1, \rho_2 \in (0,1)$,

$$F(\rho_1) - F(\rho_2) = R(\rho_1, \bar{x}(\rho_1), \bar{y}(\rho_1)) - R(\rho_2, \bar{x}(\rho_2), \bar{y}(\rho_2)) \tag{4.2.6}$$

with

$$R(\rho, x, y) = \frac{\rho x u(x) - (1-\rho) y u(y)}{\rho x + (1-\rho) y}. \tag{4.2.7}$$

Since $R(\rho_1, \bar{x}(\rho_1), \bar{y}(\rho_1)) \geq R(\rho_1, \bar{x}(\rho_2), \bar{y}(\rho_2))$, we have

$$\frac{\partial}{\partial \rho} F(\rho) \geq \left(\frac{\partial}{\partial \rho} R \right) (\rho, \bar{x}(\rho), \bar{y}(\rho)). \tag{4.2.8}$$

Now compute

$$\left(\frac{\partial}{\partial \rho} R \right) (\rho, x, y) = \frac{xy[u(x) - v(y)]}{[\rho x + (1-\rho) y]^2} \tag{4.2.9}$$

and use Proposition 2.5.1(ii), to conclude from (4.2.8) that $\frac{\partial}{\partial \rho} F(\rho) > 0$. Hence (4.2.5) reduces to $F(\rho^*(p))$. □

The following proposition is the analogue of Proposition 4.1.1.

Proposition 4.2.2 *Fix $p < p_c$.*
(i) If $\psi_{AB}(\bar{x}) = \psi_{AA}(\bar{x})$ and $\psi_{BA}(\bar{y}) = \psi_{BB}(\bar{y})$, then $f = F(\rho^*(p))$.
(ii) If $\psi_{AB}(\bar{x}) > \psi_{AA}(\bar{x})$ or $\psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})$, then $f > F(\rho^*(p))$.

Proof. It follows from (1.3.8) and (2.4.1) that f is bounded below by the right-hand side of (4.2.1). The latter equals $F(\rho^*(p))$, as shown in the proof of Proposition 4.2.1, and so

$$f \geq F(\rho^*(p)). \tag{4.2.10}$$

(i) Abbreviate $\theta_{kl}(a) = a\psi_{kl}(a)$. We know that $\theta_{AB} \geq \theta_{AA}$ and $\theta_{BA} \geq \theta_{BB}$ (by (2.4.1)), and that all four functions are concave (by Lemma 2.2.5(i)). Since θ_{AA} and θ_{BB} are both differentiable (by (2.1.3) and (2.2.2)), the assumption of equality at $\bar{x}$, respectively, $\bar{y}$ implies that θ_{AB} and θ_{BA} are differentiable at $\bar{x}$, respectively, $\bar{y}$ and that the equality carries over to the derivatives. Thus, we have

$$\theta'_{AB}(\bar{x}) = \theta'_{AA}(\bar{x}) \quad \text{and} \quad \theta'_{BA}(\bar{y}) = \theta'_{BB}(\bar{y}). \tag{4.2.11}$$

Fix $(\rho_{kl}) \in \mathcal{A}$. Abbreviate

$$\bar{N} = \rho_A \theta_{AA}(\bar{x}) + \rho_B \theta_{BB}(\bar{y}) \quad \text{and} \quad \bar{D} = \rho_A \bar{x} + \rho_B \bar{y}. \tag{4.2.12}$$

The fact that $\bar{x}$ and $\bar{y}$ are the maximisers of

$$\sup_{a_{AA},a_{BB}\geq 2} \frac{\rho_A\theta_{AA}(a_{AA})+\rho_B\theta_{BB}(a_{BB})}{\rho_A a_{AA}+\rho_B a_{BB}} \tag{4.2.13}$$

implies that

$$\theta'_{AA}(\bar{x})=\theta'_{BB}(\bar{y})=\frac{\bar{N}}{\bar{D}}. \tag{4.2.14}$$

Hence, all four derivatives in (4.2.11) are equal to $\bar{N}/\bar{D}$. Next, abbreviate

$$N=\sum_{kl}\rho_{kl}\theta_{kl}(a_{kl}) \quad \text{and} \quad D=\sum_{kl}\rho_{kl}a_{kl}. \tag{4.2.15}$$

By the concavity of $a\mapsto\theta_{kl}(a)$ (recall Lemma 2.2.5(i)), we have that for all (a_{kl}),

$$\begin{aligned} N &\leq \bar{N}+\rho_{AA}(a_{AA}-\bar{x})\theta'_{AA}(\bar{x})+\rho_{AB}(a_{AB}-\bar{x})\theta'_{AB}(\bar{x}) \\ &\quad +\rho_{BA}(a_{BA}-\bar{y})\theta'_{BA}(\bar{y})+\rho_{BB}(a_{BB}-\bar{y})\theta'_{BB}(\bar{y}) \\ &= \bar{N}+V\,\frac{\bar{N}}{\bar{D}}, \end{aligned} \tag{4.2.16}$$

where

$$V=\rho_{AA}(a_{AA}-\bar{x})+\rho_{AB}(a_{AB}-\bar{x})+\rho_{BA}(a_{BA}-\bar{y})+\rho_{BB}(a_{BB}-\bar{y}). \tag{4.2.17}$$

Moreover,

$$D=\bar{D}+V. \tag{4.2.18}$$

Combining (4.2.16) and (4.2.18), we obtain

$$\frac{N}{D}\leq\frac{\bar{N}+V\frac{\bar{N}}{\bar{D}}}{\bar{D}+V}=\frac{\bar{N}}{\bar{D}}. \tag{4.2.19}$$

Thus we have proved that

$$\sup_{(a_{kl})\in\mathcal{A}}\frac{N}{D}\leq\frac{\bar{N}}{\bar{D}}=F(\rho_A). \tag{4.2.20}$$

Optimising over $(\rho_{kl})\in\mathcal{R}$, we arrive at $f\leq F(\rho^*(p))$. Together with (4.2.10) this gives the claim.

(ii) Suppose that $\psi_{AB}(\bar{x})>\psi_{AA}(\bar{x})$ or $\psi_{BA}(\bar{y})>\psi_{BB}(\bar{y})$. Then, by picking $a_{AA}=a_{AB}=\bar{x}$, $a_{BA}=a_{BB}=\bar{y}$ and $\rho_A=\rho^*(p)$ in (1.3.8), we get

$$f>\frac{\rho^*(p)\bar{x}\psi_{AA}(\bar{x})+(1-\rho^*(p))\bar{y}\psi_{BB}(\bar{y})}{\rho^*(p)\bar{x}+(1-\rho^*(p))\bar{y}}=F(\rho^*(p)), \tag{4.2.21}$$

where we use that $0<\rho^*(p)<1$ for all $0<p<p_c$ (recall Fig. 6). □

4.2.2 Proof of Theorem 1.5.3

The following proposition is the analogue of Proposition 4.1.2.

Proposition 4.2.3 *Fix $p < p_c$.*
(i) $\partial\mathcal{D}$ lies on or below the supercritical curve $\alpha \mapsto \beta_c(\alpha)$.
(ii) $\{(\alpha,\alpha)\colon\ \alpha \in [0,\alpha^*]\} \subset \partial\mathcal{D}$, where α^* is the same constant as for the supercritical curve, but lies below the diagonal elsewhere.
(iii) For every $p < p_c$ there exists an $\alpha^*(p) \in (0,\infty)$ such that the intersection of $\mathcal{D}$ with the lower half of CONE *is the line segment $\{(\beta+\alpha^*(p),\beta)\colon\ \beta \in [-\frac{1}{2}\alpha^*(p),0]\}$.*
(iv) $\mathcal{D}$ is continuous and each line $\{(\beta+C,\beta)\colon\ \beta \geq 0\}$, $C \in [-1,1)$, intersects $\partial\mathcal{D}$ in at most one point.
(v) As $p \downarrow 0$, the part of $\mathcal{D}$ not containing the diagonal segment of (ii) converges to the mirror image of the analytic continuation of the supercritical curve outside CONE. *In particular, $\lim_{p\downarrow 0}\alpha^*(p) = \alpha_0$, with α_0 the constant defined in the first line of* (2.1.29).
(vi) $\lim_{p\uparrow p_c}\alpha^*(p) = \alpha_1$, with α_1 the constant defined in the second line of (2.1.29).

Proof. (i) Abbreviate

$$G(\mu,a) = \frac{1}{2}\left(\frac{\mu-1}{\mu}\right)\log\left(\frac{a}{a-2}\right) + \frac{1}{\mu}\log[2(a-1)]. \tag{4.2.22}$$

Then Proposition 2.4.1 can be rewritten as the statement that $\psi_{AB}(a) > \psi_{AA}(a)$ if and only if

$$\phi^{\mathcal{I}}(\mu) > \frac{1}{2}\alpha + G(\mu,a) \quad \text{ for some } \mu \geq 1, \tag{4.2.23}$$

and similarly for $\psi_{BA}(a) > \psi_{BB}(a)$ with $\frac{1}{2}\beta$ instead of $\frac{1}{2}\alpha$. To prove the claim, we must show that, for all $(\alpha,\beta) \in$ CONE and $p < p_c$, the following is true: For all $\mu \geq 1$,

$$\phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\alpha + G(\mu,\bar{x}) \quad \text{ and } \quad \phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\beta + G(\mu,\bar{y}) \tag{4.2.24}$$

imply

$$\phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\alpha + G(\mu,a^*). \tag{4.2.25}$$

Indeed, by Theorem 1.5.2, Proposition 2.2.1 and Proposition 2.3.1, this yields that $(\alpha,\beta) \in \mathcal{D}$ for $p < p_c$ implies $(\alpha,\beta) \in \mathcal{D}$ for $p \geq p_c$.

We first show that the first half of (4.2.24) is redundant. Indeed,

$$\begin{aligned}
&\left[\frac{1}{2}\alpha + G(\mu,\bar{x})\right] - \left[\frac{1}{2}\beta + G(\mu,\bar{y})\right] \\
&\quad = \frac{\alpha-\beta}{2} + [G(\mu,\bar{x}) - G(\mu,\bar{y})] \\
&\quad = -\frac{1}{2}\log\left(\frac{\bar{x}(\bar{y}-2)}{\bar{y}(\bar{x}-2)}\right) + \frac{1}{2}\left(\frac{\mu-1}{\mu}\right)\log\left(\frac{\bar{x}(\bar{y}-2)}{\bar{y}(\bar{x}-2)}\right) + \frac{1}{\mu}\log\left(\frac{\bar{x}-1}{\bar{y}-1}\right) \\
&\quad = \frac{1}{2\mu}\left[\log\left(\frac{(\bar{x}-2)(\bar{x}-1)^2}{\bar{x}}\right) - \log\left(\frac{(\bar{y}-2)(\bar{y}-1)^2}{\bar{y}}\right)\right] \\
&\quad \geq 0,
\end{aligned} \tag{4.2.26}$$

where in the third line we use the second line of (2.5.4), and in the fifth line we use that $\bar{x} \geq \bar{y}$ (recall Proposition 2.5.1(i)).

Thus, it remains to show that, for all $\mu \geq 1$,

$$\phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\beta + G(\mu, \bar{y}) \quad \text{implies} \quad \phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\alpha + G(\mu, a^*). \tag{4.2.27}$$

Indeed,

$$\begin{aligned}
&\left[\frac{1}{2}\alpha + G(\mu, a^*)\right] - \left[\frac{1}{2}\beta + G(\mu, \bar{y})\right] \\
&\quad = \frac{\alpha - \beta}{2} + [G(\mu, a^*) - G(\mu, \bar{y})] \\
&\quad = -\frac{1}{2}\log\left(\frac{\bar{x}(\bar{y}-2)}{\bar{y}(\bar{x}-2)}\right) + \frac{1}{2}\left(\frac{\mu-1}{\mu}\right)\log\left(\frac{a^*(\bar{y}-2)}{\bar{y}(a^*-2)}\right) + \frac{1}{\mu}\log\left(\frac{a^*-1}{\bar{y}-1}\right) \\
&\quad = \frac{1}{2}\left[\log\left(\frac{a^*}{a^*-2}\right) - \log\left(\frac{\bar{x}}{\bar{x}-2}\right)\right] \\
&\qquad + \frac{1}{2\mu}\left[\log\left(\frac{(a^*-2)(a^*-1)^2}{a^*}\right) - \log\left(\frac{(\bar{y}-2)(\bar{y}-1)^2}{\bar{y}}\right)\right] \\
&\quad \geq 0,
\end{aligned} \tag{4.2.28}$$

where in the third line we use the second line of (2.5.4), and in the fifth line we use that $\bar{x} \geq a^* \geq \bar{y}$ (recall Proposition 2.5.1(i)) to get that both terms between square brackets are ≥ 0.

Remark: The redundancy of the first half of (4.2.24) shows that, as the critical curve is crossed from $\mathcal{D}$ to $\mathcal{L}$, localization occurs in the BA-blocks rather than in the AB-blocks. This is why the first criterion in Proposition 4.2.2(i) is redundant (as was claimed in (1.5.7)).

(ii) If $\alpha = \beta$, then $\bar{x} = \bar{y} = a^*$ by Proposition 2.5.1(ii). Therefore the second criterion in Proposition 4.2.2(i) reduces to $\psi_{BA}(a^*) = \psi_{BB}(a^*)$ (while the first criterion in Proposition 4.2.2(i) is redundant). But if $\alpha = \beta$, then $\psi_{BA} = \psi_{AB}$ and $\psi_{BB} = \psi_{AA}$. Hence, the criterion for delocalization on the diagonal reads $\psi_{AB}(a^*) = \psi_{AA}(a^*)$, which is the same as the criterion for delocalization in the supercritical case (recall Proposition 2.2.1 and 4.1.1).

This also shows that $\partial\mathcal{D}$ must leave the diagonal at the same point as the supercritical curve, i.e., at (α^*, α^*). (Incidentally, note that $\alpha = \beta$ does not imply $\psi_{BA} = \psi_{BB}$ or $\psi_{AB} = \psi_{AA}$, because only matches of the polymer and the emulsion receive an energy.)

(iii) Let $\alpha \geq 0$ and $\beta \leq 0$. By (2.2.17), $\phi^{\mathcal{I}}(\mu; \alpha, \beta) = \frac{1}{2}\alpha + \hat{\kappa}(\mu)$. It therefore follows from (1.5.7) and the line below (4.2.23) that $(\alpha, \beta) \notin \mathcal{D}$ if and only if

$$\exists\, \mu \geq 1\colon \quad \hat{\kappa}(\mu) > G(\mu, \bar{y}) - \frac{1}{2}C, \tag{4.2.29}$$

where we recall from Section 2.5 that $\bar{y}$ is a function of $C = \alpha - \beta$ and $\rho^*(p)$ only. Combining the second line of (2.5.4) with (4.2.22), we have

$$\begin{aligned}
G(\mu, \bar{y}) - \frac{1}{2}C &= \frac{1}{2}\left(\frac{\mu-1}{\mu}\right)\log\left(\frac{\bar{y}}{\bar{y}-2}\right) + \frac{1}{\mu}\log[2(\bar{y}-1)] - \frac{1}{2}C \\
&= -\frac{1}{2\mu}\log\left(\frac{\bar{y}}{\bar{y}-2}\right) + \frac{1}{\mu}\log[2(\bar{y}-1)] + \frac{1}{2}\log\left(\frac{\bar{x}}{\bar{x}-2}\right).
\end{aligned} \tag{4.2.30}$$

By Proposition 2.5.1(iv), the right-hand side of (4.2.30) is strictly decreasing in C for fixed μ and $\rho^*(p)$. Hence there is a unique critical value C^*, which we call $\alpha^*(p)$, above which (4.2.29) holds. Since $\bar{x} \uparrow \infty$ and $\bar{y} \downarrow 2$ as $C \to \infty$ by Proposition 2.5.1(vii), the right-hand side tends to $-\infty$ as $-(1/2\mu)[\log(1/2(\bar{y}-2)) + o(1)] + o(1)$. Since $\hat{\kappa}(\mu) \geq 0$ for all $\mu \geq 1$, it follows that C^* is finite.

(iv) The continuity of $\mathcal{D}$ is immediate from Lemma 2.2.5, Proposition 2.5.1(iv), Proposition 2.5.2 and Proposition 4.2.2.

To prove the remainder of the claim, we need the following.

Lemma 4.2.4 *$\beta \mapsto \phi^{\mathcal{I}}(\mu; \beta + C, \beta) - \frac{1}{2}\beta$ is non-decreasing on $[0, \infty)$ for all $\mu \geq 1$ and $C \geq 0$.*

Proof. The function $(\alpha, \beta) \mapsto \phi^{\mathcal{I}}(\alpha, \beta; \mu)$ is convex on $\mathbb{R}^2$ for all $\mu \geq 1$, by an argument similar to that given in the proof of Theorem 1.3.1(ii) in Section 3.1 (recall (2.2.3–2.2.5) and (2.2.14)). Fix $\mu \geq 1$ and $C = \alpha - \beta \geq 0$. Abbreviate $\Delta(\beta) = \phi^{\mathcal{I}}(\beta + C, \beta; \mu) - \frac{1}{2}\beta$. Then $\beta \mapsto \Delta(\beta)$ is convex. Moreover, by (2.2.17), $\Delta(0) = \phi^{\mathcal{I}}(C, 0; \mu) = \frac{1}{2}C + \hat{\kappa}(\mu)$ and $\Delta(\beta) \geq \frac{1}{2}(\beta + C) + \hat{\kappa}(\mu) - \frac{1}{2}\beta = \Delta(0)$ when $\beta \geq 0$. Hence $\beta \mapsto \Delta(\beta)$ is non-decreasing on $[0, \infty)$. □

To prove the claim, use that $\bar{x}$ and $\bar{y}$ are functions of $C = \alpha - \beta$ only, repeat the same argument as in the proof of (iii), and use Lemma 4.2.4.

(v) The limit $p \downarrow 0$ corrsponds to $\rho \downarrow 0$ (recall Fig. 6). By Proposition 2.5.1(vi), $\bar{y} \uparrow a^*$ when $\alpha < \beta + \log 5$ and $\bar{y} \uparrow 2/(1 - e^{-(\alpha-\beta)})$ when $\alpha \geq \beta + \log 5$. In the first case, the criterion for delocalization reduces to

$$\sup_{\mu \geq 1} \mu \left[\phi^{\mathcal{I}}(\mu) - \frac{1}{2}\beta - \frac{1}{2}\log 5 \right] \leq \frac{1}{2}\log \frac{9}{5}. \tag{4.2.31}$$

Recalling Propositions 2.2.1 and 2.3.1, we see that this is precisely the criterion for delocalization in the superciritcal case but with α and β interchanged. The second case is ruled out by the observation that $\alpha_0 \leq \log 5$, as is immediate from the first line of (2.1.29).

(vi) The limit $p \uparrow p_c$ corresponds to $\rho \uparrow 1$. Let $\beta = 0$. Then, by Proposition 2.5.1(v), $\bar{y} \downarrow 10/(5 - e^{-\alpha})$ as $\rho \uparrow 1$. Therefore, with the help of Proposition 2.4.1 and (2.2.17), the criterion for delocalization when $\alpha \geq 0$ reduces to

$$\sup_{\mu \geq 1} \mu \left[\hat{\kappa}(\mu) - \frac{1}{2}\log 5 \right] \leq \frac{1}{2}\log \left[\frac{4e^{-\alpha}(5 + e^{-\alpha})^2}{5(5 - e^{-\alpha})^2} \right]. \tag{4.2.32}$$

This inequality holds if and only if $\alpha \leq \alpha_1$. □

The redundancy of the first half of (4.2.24), which is strict when $\alpha > \beta$, shows that, as the critical curve is crossed from $\mathcal{D}$ to $\mathcal{L}$ off the diagonal, localization occurs in the BA-blocks rather than in the AB-blocks.

4.2.3 Critical lines on the diagonal

In this section we explain why in Figs. 9 and 10 the diagonal segment $\{(\alpha, \alpha) \colon \alpha \in [-\alpha^*, \alpha^*]\}$ is a critical line. This is not obvious from our earlier considerations, because the segment lies on the boundary of CONE.

Take Fig. 9 ($p \geq p_c$). We have $f = \psi_{AA}(a^*)$ in the phase $\mathcal{D}_A$ and $f = \psi_{AA}(\bar{y})$ in the phase $\mathcal{D}_{A+B}$, where $\bar{y} = \bar{y}(\beta, \alpha; \rho^*(1-p))$ is the y-maximiser of (1.5.3) with $\alpha \leftrightarrow \beta$ and $p \leftrightarrow 1-p$. Since a^* is the unique maximiser of ψ_{AA} (by Lemma 2.1.1(iv) and (2.2.2)) and $\bar{y} \neq a^*$ when $\beta > \alpha$ (by Proposition 2.5.1(i)), this shows that the free energy is non-analytic along the separation line between $\mathcal{D}_A$ and $\mathcal{D}_{A+B}$.

Take Fig. 10 ($1 - p_c < p < p_c$). We have $f = F(\alpha, \beta; \rho^*(p))$ in the lower half of the phase $\mathcal{D}_{A+B}$ and $f = F(\beta, \alpha; \rho^*(1-p))$ in the upper half of the phase $\mathcal{D}_{A+B}$. Similarly as in Theorem 1.3.1(iv), there is the symmetry

$$F(\beta, \alpha; \rho^*(1-p)) = F(\alpha, \beta; 1 - \rho^*(1-p)) \tag{4.2.33}$$

(as is also evident from (1.5.3) and (2.5.4)). However,

$$\rho^*(p) > 1 - \rho^*(1-p) \qquad \forall\, p \in (0,1) \tag{4.2.34}$$

because the curve in Fig. 6 lies strictly above the diagonal. Since $\rho \mapsto F(\alpha, \beta; \rho)$ is strictly increasing (as we saw below (4.2.9)), this shows that free energy is non-analytic along the separation line between the two halves of $\mathcal{D}_{A+B}$.

It might be that in Fig. 10 also inside the two phases $\mathcal{L}_{AB+BA}$ there is a critical line on the diagonal. We do not expect this, but we lack the tools to decide.

4.3 Further observations about $\mathcal{L}$ for $p < p_c$

We close by making some observations about the second subcritical curve, lying inside $\mathcal{L}$. By Proposition 4.2.2(ii), the criterion for $\mathcal{L}$ is $\psi_{BA}(\bar{y}) > \psi_{BB}(\bar{y})$, corresponding to BA-localization. The phase $\mathcal{L}$ splits further into two parts, one where AB-localization does not occur and one where it does. The criterion for AB-localization reads

$$\psi_{AB}(\tilde{x}) > \psi_{AA}(\tilde{x}) \tag{4.3.1}$$

where we denote by $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ the values of the minimisers $a_{AB} (= a_{AA})$, a_{BA}, a_{BB}, repectively, when in the variational expression for the free energy (1.3.8) we replace ψ_{AB} by ψ_{AA}. Indeed, this is in complete analogy with the argument in Section 4.2.1 identifying the first subcritical curve as the one arising when in the variational expression for the free energy (1.3.8) we replace ψ_{AB} by ψ_{AA} *and* ψ_{BA} by ψ_{BB}. Unfortunately, whereas the latter reduction leads to a computable supremum (as shown in Section 4.2.1), the former reduction does not (because we have no closed form expression for ψ_{BA}). Consequently, we have little information on $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ (unlike for $\bar{x}$, $\bar{y}$), which is why (4.3.1) is hard to exploit. However, we can use (4.3.1) to obtain a lower bound on the second subcritical curve.

Lemma 4.3.1 *If $\alpha \geq 0$ and $\beta \leq \log(2 - e^{-\alpha})$, then $\psi_{AB} \equiv \psi_{AA}$ and hence* (4.3.1) *fails.*

Proof. From (4.1.11) we know that if $\beta \leq \log(2 - e^{-\alpha})$, then

$$\phi^{\mathcal{I}}(\mu) \leq \frac{1}{2}\alpha + \hat{\kappa}(\mu) \tag{4.3.2}$$

and hence

$$\sup_{\mu \geq 1} \mu \left[\phi^{\mathcal{I}}(\mu) - \frac{1}{2}\alpha - \frac{1}{2}\log\left(\frac{a}{a-2}\right) \right] \leq \sup_{\mu \geq 1} \mu \left[\hat{\kappa}(\mu) - \frac{1}{2}\log\left(\frac{a}{a-2}\right) \right]. \tag{4.3.3}$$

But

$$\sup_{\mu\geq 1}\mu\left[\hat{\kappa}(\mu)-\frac{1}{2}\log\left(\frac{a}{a-2}\right)\right]\leq\frac{1}{2}\log\left[\frac{4(a-2)(a-1)^2}{a}\right], \tag{4.3.4}$$

as can be seen from Proposition 2.4.1, because $\phi^{\mathcal{I}}\equiv\hat{\kappa}$ and $\psi_{AB}\equiv\psi_{AA}$ when $\alpha=\beta=0$. Combining (4.3.3) and (4.3.4) with Proposition 2.4.1, we find that if $\beta\leq\log(2-e^{-\alpha})$, then $\psi_{AB}\equiv\psi_{AA}$. □

Lemma 4.3.1 shows that the second critical curve is bounded below by the curve $\alpha\mapsto\log(2-e^{-\alpha})$.

The second subcritical curve splits off the diagonal at the same point (α^*,α^*) as the first subcritical curve. Indeed, if $\alpha=\beta$, then $\psi_{AB}\equiv\psi_{BA}$, $\psi_{AA}\equiv\psi_{BB}$, and $\bar{x}=\bar{y}$. Therefore, on the diagonal the criteria for AB-localization ($\psi_{AB}(a^*)=\psi_{AA}(a^*)$) and BA-localization ($\psi_{BA}(a^*)=\psi_{BB}(a^*)$) coincide.

We believe that the second critical curve has a finite horizontal asymptote, as argued on physical grounds in Section 1.6, and that it lies above the supercritical curve. We are unable to prove this.

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