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INTUITIONISM AND LOGICAL TOLERANCE

Tradition is classical. Surely, nothing could be more pleonastic than that? The *logical* tradition, certainly, was squarely classical from Bolzano to Carnap, with, say, Frege, Moore, Russell and the Wittgenstein of the *Tractatus* as intermediaries. Propositions are construed as being in themselves true-or-false. Indeed, in this tradition, a declarative sentence S expresses a proposition (or is a proposition, depending on what version of the theory that is adopted) by being true-or-false. So the meaningfulness of a sentence consists in its being true-or-false. But S is true-or-false, or so they say, only when S is true, or when S is false. On the classical account the presumption of bivalence is built into the very notion of meaningfulness: there is no difference between asserting that A is a proposition and asserting that A is true-or-false. The matter came to the fore in the foundations of set theory. In his first attempt at giving an application criterion for sets Cantor noted:

Eine Mannigfaltigkeit (ein Inbegriff, eine Menge) von Elementen, die irgendwelcher Begriffssphäre angehören, nenne ich wohldefiniert, wenn auf Grund ihrer Definition und infolge des logischen Prinzips vom ausgeschlossenen Dritten es als intern bestimmt angesehen werden muss, sowohl ob irgendein derselben Begriffssphäre angehöriges Objekt zu der gedachten Mannigfaltigkeit gehört oder nicht, wie auch, ob zwei zur Mannigfaltigkeit gehörige Objekte trotz formaler Unterschiede in der Art des Gegebenseins einander gleich sind oder nicht.¹

Here Cantor's reference to the Law of the Excluded Third is, in my opinion, of the above kind, where meaningfulness, rather than logicity, is at issue: when α is a set, and a an object from the right 'concept sphere', $a \in \alpha$ has to be a well-put statement, that is, it must be determinately true-or-false.²

The passage continues:

Im allgemeinen werden die betreffenden Entscheidungen nicht mit den zu Gebote stehenden Methoden oder Fähigkeiten in Wirklichkeit sicher und genau ausführbar sein; darauf kommt es aber hierdurch nicht an, sondern allein auf die *interne Determination*, welche in konkreten Fällen, wo die Zwecke fordern, durch Vervollkommnung der Hilfsmittel zu einer *aktuellen* (*externen*) *Determination* auszubilden ist.

The issue concerning the actual execution of methods of evaluation and decision was a bone of contention between Cantor and Kronecker. The Cantorian point of view was followed by Zermelo in his axiomatization, where *definite Eigenschaften* were used in the formulation of the *Axiom der Aussonderung*.³ Similarly, Frege imposed a very strict sharpness condition of complete determination on his propositional functions ("concepts"). Also he allowed for the possibility that human agents might not be able to execute the required decision.⁴

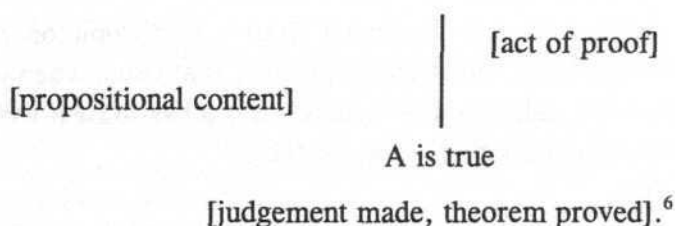
Wittgenstein stressed similar points in the *Tractatus*. A sentence must have true-or-false bipolarity in order to be able to say anything. The possibility of human

decision, at least in principle, was of central importance to him, though. For a *Satz S* it is an internal matter, that is, part of its *Satzsein*, that it fixes reality on yes-or-no (4.023). Therefore, that it does so, can be computed by syntactic calculation *am Symbol allein* (6.126): according to the *Tractatus*, the *Satz S* has a complete truth-table that can be mechanically generated and inspected in order to see whether it is a tautology or a contradiction.⁵ When *S* is neither a tautology nor a contradiction, one cannot compute from the symbol alone whether *S* is true or whether *S* is false; this can be done only for the borderline cases of tautologies and contradictions. In the other cases, that is, for real *Sätze* with bipolar true-false possibilities, we know *a priori* that *S* is true-or-false, but in order to know that *S* is true, or that *S* is false, a comparison with the world is called for (6.113).

The modern classical tradition in logic was inaugurated by Bolzano, who sought for a firm basis on which to found the distinction between the subjective and the objective. He availed himself of such a norm in terms of *Sätze an sich* (propositions) and their being true or false, also *an sich*. This is brought out clearly with respect to the crucial epistemological notions: proposition(al content), judgement (made), and act of judgement (proof). To each corresponds a correctness notion, as must be the case if objectivity is to be upheld:

- (1) a proposition is true (*wahr*);
- (2) a judgement made is correct (*richtig*);
- (3) an act of proof/judgement is right [or valid] (*gültig*).

The notions themselves are related as in the following diagram:



The classical logical theory opts for (1) as the crucial, objectivity-conferring notion of correctness: a proposition is, determinately, true or false. The rightness of the act (3) is reduced to the correctness of the judgement made (2):

the act of proving is right, only when the theorem proved is correct.

The correctness of the judgement made (2) is further reduced to (1):

the judgement *A is true* is correct, only when the proposition *A* really is true.⁷

The central role of (1) may well have been motivated ontologically: propositions are true or false according to how things are in (mathematical or logical) reality. Cantor and Wittgenstein, for example, adhere to such an ontological reduction.⁸ The position within the foundations of mathematics that is concomitant upon the classical logical paradigm operates according to the equation:

(*) Mathematics = Mathematical axioms + logical inference,

where logical inference proceeds according to the rules of *classical* logic. The realist, or Platonist, ontology treats of mathematics as fixed, clear-cut, determined and ready. The mathematical axioms hold because they are basic, primitive truths about how matters stand within the mathematical ontology. The use of classical logic, furthermore, is permitted within mathematics just because of the sharpness of the mathematical ontology. The picture has been immortalised in Hardy's [1929].

However, neither the classical view of the dependency of mathematics upon logic, nor the classical order of logical priorities, is necessary. Brouwer reversed the order of dependence, as well as the logical order of priority [1907], and he criticised the unlimited use of classical logic [1908].

The three stages of constructivist criticism can be seen to follow a common pattern. According to Kronecker, definitions must be decidable:

Die ... aufgestellte Definition der Irreduktibilität entbehrt so lange einer sicheren Grundlage, als nicht eine Methode angegeben ist, mittels deren bei einer bestimmten vorgelegten Funktion entscheiden werden kann, ob dieselbe der aufgestellten Definition gemäss irreduktibel ist oder nicht.⁹

In place of this complex Kronecker case of defining a property (propositional function) of algebraic functions, I consider the simpler case of functions on the natural numbers. Clearly, we can come up with examples that do not allow for the execution of the resulting number terms. Consider

$$(I) \quad f(k) =_{\text{def}} \begin{cases} 1 & \text{if the Riemann Hypothesis is true} \\ 0 & \text{if the Riemann Hypothesis is false.} \end{cases}$$

At present, though, no *Auswertung* of f can be effected, irrespective of whatever argument is supplied, since any completed evaluation would enable us to decide the Riemann Hypothesis. Accordingly we here have an example of alleged number terms, such as $f(14)$, that cannot be evaluated to the canonical form of an Arabic numeral. This is too high a price to pay for the use of definitions by means of undecided cases: in a stipulative definition it must be possible to eliminate the definiendum in favour of the definiens. Strictly speaking, under present circumstances one is not even allowed to assert that $f(14) \in \mathbb{N}$. One can attempt to improve upon this uncomfortable situation by considering the relation

$$R(x, y) \in \text{Prop, for } x \in \mathbb{N} \text{ and } y \in \mathbb{N},$$

where

$$R(x, y) =_{\text{def}} (RH \ \& \ y =_N 1) \vee (\neg RH \ \& \ y =_N 0).^{10}$$

One then readily verifies – using constructive means only – that

$$x \in N, y \in N, R(x, y) \text{ true}, R(x, y') \text{ true} \Rightarrow y =_N y' \text{ true}.^{11}$$

This consequence is one way of to view the *internal determination* demanded by Cantor: a non-constructive proof of

$$(\forall x \in N)(\exists y \in N)R(x, y) \text{ true},$$

is readily obtained from the definition of $R(x, y)$ and an application of the law of excluded middle to RH . In order to proceed to the corresponding external determination, however, one would need a constructive proof, which cannot be given without deciding the Riemann Hypothesis.

In his [1906] criticism of impredicative definitions Poincaré noted that “les définitions non prédictives ne peuvent pas être substituées au terme défini.”¹²

A meaning-explanation for the second-order quantifier begins by stipulating that $(\forall X \in \text{Prop})A$ has to be proposition under the assumption that A is a propositional function from Prop to Prop , that is, that $A \in \text{Prop}$, provided that $X \in \text{Prop}$. One then has to explain, still under the same assumption, which proposition it is:

$$(\forall X \in \text{Prop})A \text{ is true if and only if } A[P/X] \text{ is true, for each proposition } P.$$

In the special case of $(\forall X \in \text{Prop})X$ one obtains

$$(**) (\forall X \in \text{Prop})X \text{ is true} =_{\text{def}} P \text{ is true, for each proposition } P.^{13}$$

But $(\forall X \in \text{Prop})X$ is (meant to be) a proposition, so it has to be considered on the right-hand side. Accordingly $(**)$ cannot serve as a definition of what it is for $(\forall X \in \text{Prop})X$ to be true: it does not allow for the elimination, whether effective or not, of

... is true

when applied to the alleged proposition $(\forall X \in \text{Prop})X$. Note that this rejection of $(**)$ does not presuppose that the truth in question is classical. The argument applies with equal force to second-order *intuitionistic* quantification.¹⁴

Of course Poincaré's point about the non-eliminability remains valid in a classical framework, say, that of Frege [1893]. There the explanation of the second-order quantifier can be given a definition-by-cases form analogous to (I):

$$(II) \quad (\forall X \in \text{Prop}) A =_{\text{def}} \begin{cases} \text{The True, if } A[P/X] = \text{The True,} \\ \text{provided } P \in \text{Prop.} \\ \\ \text{The False, otherwise.} \end{cases}$$

The specialisation to $(\forall X \in \text{Prop}) X$ will yet again be underdetermined along the lines indicated above. Accordingly, the right-hand side of (II) does not determine a meaning for $(\forall X \in \text{Prop}) A$. The definiendum cannot be eliminated – whether effectively or not – in favour of its alleged definiens. Furthermore, in addition to this impredicative underdetermination, the definition-by-cases has a non-decidable condition. Thus, even if (II) had served to determine a fixed meaning, it still would yield non-eliminable defined expressions that resist effective evaluation to primitive form, just as in the simpler Kroneckerian case of (I), or the Brouwerian (III) below.

Also the criticism of classical first-order quantification in Brouwer [1908] follows this pattern. When propositions are understood classically, that is, as ways of presenting elements of the set of the truth-values {The True, The False}, it is not clear that Frege's [1893] explanation of the universal quantifier does yield a proposition. $(\forall x \in D) A$ has to be a proposition when D is a set and A is a propositional function over D , that is, $A \in \text{Prop}$, provided that $x \in D$. Under the same assumptions, Frege then (essentially) defines

$$(III) \quad (\forall x \in D) A =_{\text{def}} \begin{cases} \text{The True, if } A[a/x] = \text{The True, provided } a \in D. \\ \\ \text{The False, otherwise.}^{15} \end{cases}$$

This definitional equality, however, is not *auswertbar* to primitive form. To my mind, this way of framing the intuitionistic undecidability criticism of classical logic is the most satisfactory: what is wrong with the classical position is that it allows for non-eliminable defined notions. In other words, it is not clear that the formation-rule for the quantifier preserves meaningfulness.¹⁶

Intuitionism is a *foundationalism*.¹⁷ Mathematical language has content.¹⁸ This holds true also for the *practice* of Brouwer: mathematical language reports the carrying out of constructions within "mathematical intuition".¹⁹ Thus, language is held accountable to the mathematics it describes. In particular, logic is not prior to the acts of construction in which theorems are proved. It is not so that a set of fixed and ready principles are all-applicable, irrespective of subject-matter, or chosen domain of quantification. The laws of logic are not applied in mathematics: on the contrary, the principles of logic are read off from the practice of constructive mathematics. This view, together with the fact that intuitionistic mathematical practice eschews the use of non-constructive methods, constitutes a negative criticism of the traditional *classical* mathematical foundationalism as set out in the equation (*) above. In the early thirties Heyting and Kolmogorov provided (what

turned out to be) equivalent elucidations of the notion of a proposition within constructive mathematics: a proposition is a set of proof-objects and a proof of the proposition is an element of the set of proof-objects.²⁰ The truth of the proposition is equated with the existence of a proof, that is

proposition A is true = there exists a proof of A.

One has the right to assert the theorem

A is true

only when one has found a proof-object

$c \in A$.

What is here at issue is a *proof of a proposition*. From a philosophical point of view this is one of the foremost innovations that we owe to Brouwer's intuitionism (and the explicit semantical formulations of Heyting). Previously, also in the non-constructive tradition, a proof was always a proof of a *theorem*, that is, an *assertion* that a certain proposition is true. Accordingly, in view of this novel conception of proof, we might do well to observe also a terminological distinction and use 'proof(-object)' for proofs of propositions and 'demonstration' for proofs of theorems.

Both the classical position and its intuitionistic rival are "foundationalisms": mathematical theorems have content. Indeed, as was already noted, the classical tradition operated according to the equation (*) above. However, also the intuitionist can accept (*), but with a changed order of priorities. For him, mathematics cannot be held accountable to a fixed and prior (classical) logic. On the contrary, the validity of logical principles depend on the possibilities of constructing mathematical objects. In foundations of mathematics, when formal languages were used, from Frege onwards, until Heyting, they were invariably provided with careful meaning-explanations, or at least such explanations were attempted. Frege [1893, §§ 29-31] wanted to show that every expression that can be formed within his "concept script" has meaning and that the demonstrated theorems are true. Similarly, one fruitful way to view Wittgenstein's *Tractatus* is as an attempt to provide the missing semantical framework for *Principia Mathematica*. These attempts at a semantical foundation for mathematical language were clearly extra-mathematical; owing to the universality of the mathematical languages for which semantical treatments had to be given, there was no place inside mathematics where this could be done. Indeed, Frege is aware of a 'peculiar obstacle' at this point which forces him to ask his reader to be satisfied with pinches of salt (*Körnchen Salz*) and hints (*Winke*), whereas Wittgenstein superimposes an elaborate framework of internal relations in order to deal with the ineffability of semantics. Only with the advent of the model-theoretical perspective during the 1920's does this change and the study of formalisms without a content becomes the order of the day. Jean van Heijenoort [1967] first drew attention to the distinction between these two conceptions of logic. Jaakko Hintikka has redeployed van Heijenoort's distinction

as one applying primarily to language. His version of the distinction holds between Language as the Universal Medium and Language as Calculus and he has been much concerned to propagate its importance.²¹

<i>Logic as Language</i>	<i>Logic as Calculus</i>
(1) Foundational aims	Metamathematical aims
(2) Interpreted object-language for proving theorems <i>in</i>	Metamathematical object-language for proving theorems <i>about</i>
(3) \vdash is an "assertion-sign".	\vdash is a theorem-predicate. ²²

In order to get a picture of how the distinction works out it is convenient to locate a number of well-known logicians and foundational researchers within the two paradigms.²³

<i>Logic as Language</i>	<i>Logic as Calculus</i>
Frege*	Hilbert
Russell*	
Wittgenstein*	
Carnap* (<i>Abriss</i>)	Carnap (<i>Syntax</i>)
Heyting*	Kreisel, Goodman
Church [1932]	
Leśniewski	Łukasiewicz
Tarski <i>Wahrheitsbegriff</i> (1933)	Tarski <i>Nachwort</i> [1935]
Scholz, Friedrich Bachmann	
Quine [1934]	

The traditional view of the mathematical *Grundlagenstreit* is that it was a three-cornered fight between three independent positions – logicism, intuitionism, and formalism – without much common ground. Consideration of the two themes that have been dealt with above, namely, the controversy concerning the unlimited use of classical logic, and the division form/content with respect to mathematical language, suggest another, perhaps more realistic picture. It is obtained by crossing the two dichotomies.

Three of the four options are readily filled; the fourth, which would study contentless languages while rejecting classical logic, appears lacking in motivation. After all, the rejection of classical reasoning is based, not on knee-jerk revulsion, but on the reasoned insight that the intuitionistic meaning-explanations do not validate the classical laws, together with the fact that at present *no* meaning explanations are known that serve to make evident the laws of classical mathematics. But if one abandons content, why make life more difficult than it is? If meaning no longer imposes inconvenient constraints, why force oneself into an intuitionistic strait-jacket?

The Grundlagenstreit

View of Language	Classical Logic	
	/ Acceptance	\ Rejection
Universal Medium	<i>Logicism</i>	<i>Intuitionism</i>
Calculus	<i>Formalism</i>	

The abandonment of logicism as a viable position within the foundations of mathematics, after the last-ditch stand by Ramsey [1926] and Carnap [1931], comes as no surprise: their valiant efforts, as well as that of the *Tractatus*, simply fail to make evident the rules and axioms of *Principia Mathematica*. The axioms of infinity, choice and reducibility remain problematic. Formalism is able to take over logicism's role as torch-bearer for classical logic only by means of a strategic retreat into the contentless pure forms of the calculus-conception of language.

Karl Menger spent some years at Amsterdam as Brouwer's assistant and he wrote a revealing memoir [1979] concerning his experiences there. In thought-provoking articles [1928, 1930], written after his return to Vienna as Associate Professor of Geometry, Menger noted that constructivity comes in degrees – a point subsequently elaborated by Bernays [1935] – and objected strongly to founding mathematics on “intuition”, in place of which he preferred “implicationism”:

All that matters is into which statements certain others can be transformed by the use of given transformation rules. Attempts to found the acceptance or rejection of propositions ... on intuition are ultimately empty words.²⁴

Menger's hostility is surely caused in part by the ambiguity of the terms *Intuition*, and *intuitive*. Three relevant readings, at least, can be discerned:

- (i) Vague, non-scientific, founded upon feeling rather than reason;
- (ii) An English rendition of Kantian *Anschauung* ;

- (iii) Non-discursive, immediate rather than mediate, not grounded in reasoning.²⁵

Menger thinks in terms of meaning (i), whereas Brouwer's meaning is a combination of (ii) and (iii): mathematical theorems must be proved directly in the Kantian intuition of time by means of evident construction-steps, and not via a detour through logic.

For the Vienna Circle the dichotomies analytic/synthetic and *a priori/a posteriori* had to coincide. In particular, according to them, there was no synthetic *a priori* in mathematics, whether founded in intuition or not. Brouwer's criticism of classical reasoning, and his providing a (more or less) viable alternative, with its different conception of logic, rendered an uncritical foundationalism with respect to classical mathematics untenable. Similarly, to found analyticity on what is true in (virtue of) classical logic is no longer very attractive, when the basis on which the whole foundational edifice would presumably rest is no longer undisputed. Carnap's first way out of his dilemma [1931] was to meet the challenge by biting the Brouwerian bullet, but his attempt at providing an adequate foundation for classical logic remained unconvincing, though. Shortly afterwards [1934, p. xv], therefore, he abandons foundationalism for mathematical language – be it classical or not – and follows the path taken by Menger. The conception of language as a universal medium is firmly rejected and the calculus perspective on logic and language is adopted:

Up to now, in constructing a language the procedure *has usually been*, first assign a meaning to the fundamental mathematico-logical symbols, and then to consider what sentences and inferences are seen to be logically correct in accordance with this meaning. ...

The connection will only become clear when approached from the opposite direction: let any postulates and rules be chosen arbitrarily; then this choice, whatever it may be, will determine what meaning is to be assigned to the fundamental logical symbols.

How, we may ask, does Carnap know that every choice, 'whatever it may be', does indeed determine meaning? Why cannot simply nonsense result from arbitrary postulation? Carnap expressed his admiration for the early, but old-fashioned, foundationalist attempts at laying down meanings and asserting the mathematical axioms and logical principles of inference that are analytically self-evident according to the meanings thus explained. These attempts

were certainly bold ones, But they were hampered by the striving after 'correctness'. Now, however, that impediment has been overcome, and before us lies the boundless ocean of unlimited possibilities.

The result is the "Language as Calculus" conception with a vengeance: "By a language we mean here in general any sort of calculus, ..." ²⁶

Meaning is no longer of primary importance for mathematical language; what matters are the rules of transformation only, and, if meanings there be, they have to be read off from, or are superimposed by, those very rules of transformation. The *Grundlagenstreit* is resolved by telling the protagonists to go away:

Principle of Tolerance. It is not our business to set up prohibitions, but to arrive at conventions. ...

In logic, there are no morals. Everyone is at liberty to build up his own logic ... as he wishes.²⁷

It is interesting to note that Carnap makes heavy play with the notion of analyticity. In the traditional sense axioms have to be analytic in the sense of self-evident truths. The evidence of a traditional axiom is immediate, rather than mediate. The knowledge expressed rests upon nothing else: it must be grounded solely in the concepts that occur in the axiom in question. Kant's analytic judgements of the form $[S \text{ is } P]$, where the predicate is contained in the subject, are prime examples here. Such analyticity is unthinkable without content. According to the scholastic tags axioms have to be *propositio per se nota* and their evidence is *evidentia ex vi terminorum*. A close analogue in the *Tractatus* is that of elucidatory ascriptions of internal properties and relations: when P is an internal property of a , a grasp of a and P alone suffices for getting to the insight that a has the property P .²⁸ Carnap is concerned to show that the laws of logic are analytic in a certain formalistic sense involving suitable transformation rules.²⁹ This notion is not the formalistic counterpart of the above traditional notion of analyticity. It is patterned on another notion which goes back to Bolzano and was used by Wittgenstein in the *Tractatus*, namely that of a proposition which is logically true, come what may, independently of what is the case.

Carnap's Principle of Tolerance, in my opinion, simply does not even begin to engage the issues; it merely avoids them. Such formalistic tolerance leaves everything the way it is. A genuinely interesting case of *contentual* tolerance is provided by Gödel, who could move back and forth between intuitionism and platonist set theory, apparently without effort.³⁰ The list of his contributions to intuitionism is long and impressive: the double-negation interpretation, the modal translation, no finitely many-valued truth-tables, the Zilsel-lecture, and the Dialectica-interpretation.

Carnap's struggle with classical content was witnessed at close distance by the young Gödel, and according to Menger, the latter was sympathetic to the earlier tolerance-related ideas of Menger.³¹ Content, however, seems to have been central to Gödel's philosophical position as evidenced by his (late) affinity to Husserl's phenomenology. Be that as it may: the mature logician wanted no truck with the tolerance principle. Menger's implicationism is refuted in the Gibbs lecture [1951, p. 310, fn. 16], whereas the aborted Schilpp-paper on Carnap can be seen as one long attempt to bury the calculus-conception of language decisively.³²

Gödel's early reactions were not less impressive. I know of no better way to conclude this inventory of the intuitionistic provocation, which made Carnap seek refuge in anodyne tolerance, than to contrast Carnap's own reaction with Gödel's response to Brouwer's Vienna lecture [1929]. Carnap's report on Gödel's views 'about the inexhaustibility of mathematics' takes the form of a diary-entry. His extraordinary passage epitomises most 20th century lines of development within

logic and the foundations of mathematics with marvellous clarity and precision, and I leave it to speak for itself without further comment:

We admit as legitimate mathematics certain reflections on the grammar of a language that contains the empirical. If one seeks to formalize such a mathematics, then with each formalization there are problems, which one can understand and express in ordinary language, but cannot express in the given formalized language. It follows (Brouwer) that mathematics is inexhaustible: one must always again draw afresh from the 'fountain of intuition'. There is therefore no *characteristica universalis* for the *whole* of mathematics, and no decision procedure for the whole of mathematics. In each and every *closed language* there are only countably many expressions. The *continuum* appears only in 'the whole of mathematics.'... If we have *only one language*, and can only make 'elucidations' about it, then these elucidations are inexhaustible, they always require some new intuition again.³³

NOTES

1. Cantor [1882, p. 114] (my italics). I am indebted to Per Martin-Löf for drawing my attention to this passage. The link – *Art des Gegebenseins* – from Cantor to Frege should not be overlooked.
2. The term *Begriffssphäre* occurs in Kant's (Jäsche-) *Logik*, §8: the "sphere" of a concept is explained as its extension. Accordingly it seems that what Cantor is here explaining is what it is for a set of natural numbers, or of reals, or of complex numbers, or whatever the case may be, according to the choice of the relevant concept-sphere. Alternatively, the concept-sphere might be seen as the set-theoretic universe, which, as Cantor reported in some famous letters to Dedekind [1899], is not a set, but *eine inkonsistente Vielheit*.
3. [1908, § 1.3, § 1.6, Axiom III, pp. 263-264].
4. [1903, § 56, p. 69].
5. That one runs into the Church-Turing undecidability theorem at this point was unknown when the *Tractatus* was written.
6. This diagram is an elaboration of one found in Martin-Löf [1987].
7. Bolzano [1837, § 34] is quite explicit on this point.
8. For instance, in the *Tractatus*, Wittgenstein reduced propositional truth yet one step further, to the obtaining of the ontological state-of-affairs (*Sachverhalt*) that is presented by the (elementary) proposition in question, and that, in turn, to "reality" (*Wirklichkeit*) (2.06).
9. Kronecker [1882].
10. RH is Kreisel's [1958] arithmetical Π_1 formulation of the Riemann Hypothesis.
11. The verification consists of repeated applications of \vee -elimination: one major with respect to $R(x, y)$, with subsidiary applications with respect to $R(x, y')$, using (the constructively valid) *modus tollens* in order to reject the unwanted alternative in each case.
12. [1906, p. 316].
13. Here $A := X$.
14. The classical notion of a proposition (Fregean *Gedanke* [= way of presenting a truth-value]) is unclear. In particular, the relevant notion of identity between propositions is problematic (cf. my "Proof-theoretical semantics and Fregean identity criteria for propositions", *The Monist* 77 (1994), pp. 294-314.). Per Martin-Löf pointed out to me that on the classical Ramsey semantics, which puts

$$\text{prop} = \text{Boole} = \{\text{true}, \text{false}\} \in \text{Set},$$
 there is, constructively speaking, nothing amiss with quantification over all propositions, since propositional quantification comes down to quantification over a finite, in fact two-element, set:

$$(\forall X \in \text{prop}) A = A[\text{true}/X] \ \& \ A[\text{false}/X].$$
 Thus the problem is to provide a joint validation for quantification over propositions as well as quantification with respect to an infinite domain. Constructively, quantification with respect to infinite domains is made good sense of, but quantification over propositions has to be rejected

- owing to impredicativity. For classical semantics, on the other hand, owing to the very narrow concept of a proposition as a truth-value, quantification over all propositions is also constructively acceptable, but the validation of, say, universal quantifier-formation (with respect to an infinite set) has not been given.
15. Frege's does not include the set D as domain of quantification since he quantifies over all individual objects.
 16. This way of understanding Brouwer's criticism was hinted at by Martin-Löf [1985, p. 33]. I have come to appreciate it through the lucid advocacy of Aarne Ranta [1994, p. 38]. See also my [1994a, p. 153].
 17. The term 'foundationalism' has been used by Shaprio [1991], but then in a slightly pejorative sense, which I cannot share. To my mind, 'foundations *without* foundationalism' are no foundations.
 18. Van Heijenoort [1976, p. 46] notes that 'Brouwer was certainly concerned with meaning' (in which opinion I concur, even though I would prefer *content* to 'meaning'), and continues: 'In fact, a great part of his polemics against Hilbert turned around the notion of *Inhalt*. But neither he nor Heyting thought of systematising the notion.' The first sentence of this quote is certainly correct. Van Heijenoort does not give references, but Brouwer [1927, §1, p. 375] is as good a place as any. I want to take exception to the second sentence, though. Heyting, did not merely think of systematising *content* from an intuitionistic point of view, but, as witnessed by his semantical works, actually carried out such a systematisation.
 19. Brouwer's *theory* of language, as set out, e.g., in [1929] was less charitable. There mathematical activity is deemed to be 'languageless'. So often, deeds speak more than words, though. In *practice*, also that of Brouwer, language is indispensable. For instance, how to deal with, say, the theory of countable large constructive ordinals in a constructive, but *languageless*, fashion, without recourse to elaborate linguistic notation-systems, is completely unclear.
 20. I have dealt with the formulations of Heyting and Kolmogorov in a number of places, for instance, [1983, 1993, 1994].
 21. See Hintikka [1988], and also the recent [1996], which collects his deployments of the distinction. I share Hintikka's view concerning its importance and applaud his efforts to make it known and appreciated; only, my sympathies lie with Language as a Universal Medium rather than as a Calculus.
 22. Kusch [1988] gives a very clear presentation of the dichotomy between the two conceptions of language in terms of eight characteristic theses *Universal Medium* 1 – *UM* 8, that are opposed to a matching Calculus-series of theses.
 23. An asterisk here indicates that the logician in question used the turnstile essentially as a Fregean assertion sign.
 24. Menger [1928, p.57]
 25. A fourth meaning, that might be relevant to current discussions of mathematical intuition by Charles Parsons and others, is that of "non-reflective skill, or insight, obtained through thorough practice within an area."
 Cooking at the highest level is almost purely intuitive,
 would be an example of such a use.
 26. Carnap [1934, p. 167].
 27. Carnap [1934, p. 51-52].
 28. See my [1990] for an elaboration of this point.
 29. Carnap's [1934] treatment shows considerable constructivist influence. See, for example §§ 15, 16, 43, 44.
 30. Gödel is one of the few logicians that uphold the traditional notions of analyticity [1944, p.151] and evidence [1972, p. 275, note h, 1].
 31. Gödel 'supported me with a nod of assent', Karl Menger [1979, p.13] reports.
 32. [1953/9-III] and [1953/9-V].
 33. Carnap, *Diary*, December 23, 1929. Quoted from Wang [1987, pp. 50, 84].

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