Vortex Duality in Higher Dimensions

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Chapter 3

Vortex duality in 3+1 dimensions

The vortex–boson or Abelian-Higgs duality as pertaining to many-body physics in 2+1 dimensions is by now well established and has been researched for over three decades [27–33, 35, 36]. One can wonder why this has almost exclusively been restricted to planar physics, while many systems of interest are in fact three-dimensional. The reason is quite simple: vortices in two dimensions are pointlike and trace out world lines, whereas in three dimensions they are linelike and trace out world sheets in spacetime. As such the dual objects are more complicated as they have more internal degrees of freedom. Although a single vortex world sheet is still quite tractable, for a rigorous description of a condensate of such extended objects, a "string foam", one needs string field theory [38, 39], which is as of yet still in early stages of development.

Surely, several authors have made progress on the condensation of vortex world sheets, in the context of string theory [47, 48] and condensed matter theory [46]. However in this chapter we shall discover that the proposed methods do not apply for the case at hand, the quantum phase transitions in 3+1 dimensional condensed matter. The reason is that they do not yield the proper mode content for the disordered phase (the Bose-Mott insulator) as they ascribe too many degrees of freedom to the vortex condensate as a compressible liquid. In part this can be explained by the fact that the vortices in condensed matter are so-called Nielsen–Olesen strings [60] which have a finite core size and core energy and no internal conformal symmetry. This is different from fundamental or 'critical' strings[1]. Nevertheless one

[1] I thank Dr Soo-Jong Rey for pointing this out.

encounters the difficulty that second quantization cannot be formulated for stringy matter. Accordingly, different from matter formed out of particles, an algorithm is lacking to compute the properties of such string condensates directly. The only example of a precise duality involving stringy topological excitations is the transversal field global Ising model in 2+1d [3]. The strong coupling phase can be viewed as Bose condensate of Ising domain walls in space time [61]; remarkably, the Wegner duality [4] demonstrates that this string condensate is actually the ordered (deconfining) phase of Ising gauge theory, while the ordered Ising phase corresponds with the confining phase of the gauged theory.

In this chapter we develop the effective theory governing the condensation of vortex world sheets in superfluids. In the ordered phase the vortices interact by exchanging 2-form gauge fields instead of 1-form or vector fields. We will show that these 2-form gauge fields undergo a Higgs mechanism in the disordered phase much like regular vector fields do. Guided by the knowledge that the disordered superfluid must correspond to the Bose-Mott insulator and its two gapped doublon and holon excitations, we argue that the string foam should add only a single degree of freedom, contrary to earlier claims. As a result, not the gauge fields but rather the physical supercurrents are to be regarded the fundamental quantities, and the phase transition is in this context at that point where supercurrents are no longer conserved. The results are generalizable to any dimension higher than two. Systems more complicated than the superfluid should undergo a similar mechanism, for instance the superconductor that will be investigated in chapter 5.

We include a discussion about vortices in the disordered phase, and two appendices on the counting of degrees of freedom and the application of this current formalism to Maxwell electromagnetism.

3.1 Dualization of the phase mode

Let us start right away by repeating as much as possible the exercise of dualizing the superfluid phase mode. The starting point is again Eq. (2.40).

$$Z = \int \mathscr{D}\varphi \; \mathrm{e}^{-\int \mathscr{L}} = \int \mathscr{D}\varphi \; \mathrm{e}^{-\int -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}}\varphi)^2}. \tag{3.1}$$

Introduce auxiliary variables, the canonical momentum or the supercurrent,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial^{\rm ph}_\mu \varphi)} = \frac{1}{g}\partial^{\rm ph}_\mu \varphi. \tag{3.2}$$

The partition sum after a Hubbard–Stratonovich transformation is now,

$$Z_{\rm dual} = \int \mathscr{D}\varphi \mathscr{D} w_\mu \ {\rm e}^{-\int \frac{1}{2} g w_\mu w_\mu - w_\mu \partial^{\rm ph}_\mu \varphi}. \tag{3.3}$$

We split the phase field into smooth (phase mode) and multivalued (vortices) parts, $\varphi = \varphi_{\rm smooth} + \varphi_{\rm MV}$. In 3+1 dimensions, the contour integral around the multivalued part will still yield the winding number N times 2π, but the vortices are now linelike, because otherwise we could close the contour by pulling it 'over' the point, see §2.2. The smooth part can be integrated out as a Lagrange multiplier for the constraint $\partial^{\rm ph}_\mu w_\mu = 0$, the conservation of supercurrent.

3.1.1 2-form gauge fields

Now comes the first deviation from the treatment in 2+1 dimensions. The constraint can be explicitly enforced by expressing the supercurrent as the curl of a gauge field, but since in four dimensions the Levi-Civita symbol has four indices, the gauge field is an antisymmetric 2-form field,

$$w_\mu = \epsilon_{\mu\nu\kappa\lambda} \partial^{\rm ph}_\nu b_{\kappa\lambda}. \tag{3.4}$$

There are six independent components in $b_{\kappa\lambda}$. This expression is invariant under the addition of the gradient of any smooth vector field $\varepsilon_\lambda(x)$,

$$b_{\kappa\lambda}(x) \to b_{\kappa\lambda}(x) + \partial_\kappa \varepsilon_\lambda(x) - \partial_\lambda \varepsilon_\kappa(x). \tag{3.5}$$

The addition of the gradient of any smooth scalar field $\varepsilon_\lambda(x) \to \varepsilon_\lambda(x) + \partial_\lambda \eta(x)$ will lead to the exact same gauge transformation for $b_{\kappa\lambda}$, so there is a redundancy in the gauge redundancy itself. This is sometimes referred to as "gauge in the gauge", and is of importance in the counting of degrees of freedom as described in the appendix 3.A. The result is that a free massless 2-form field has one propagating degree of freedom, which we already know since we derived it from the superfluid phase mode. Substituting the gauge field in the generating functional we find,

$$Z_{\text{dual}} = \int \mathscr{D}\varphi_{\text{MV}} \mathscr{D} b_{\kappa\lambda} \mathscr{F}(b_{\kappa\lambda})\, \mathrm{e}^{-\int \mathscr{L}_{\text{dual}}} \tag{3.6}$$

$$\begin{aligned} \mathscr{L}_{\text{dual}} &= \frac{1}{2} g (\epsilon_{\mu\nu\kappa\lambda} \partial_\nu^{\text{ph}} b_{\kappa\lambda})^2 - \epsilon_{\mu\nu\kappa\lambda} \partial_\nu^{\text{ph}} b_{\kappa\lambda} \partial_\mu^{\text{ph}} \varphi_{\text{MV}} \\ &= \frac{1}{2} g (\epsilon_{\mu\nu\kappa\lambda} \partial_\nu^{\text{ph}} b_{\kappa\lambda})^2 - b_{\kappa\lambda} J^{\text{V}}_{\kappa\lambda}. \end{aligned} \tag{3.7}$$

Here $\mathscr{F}(b_{\kappa\lambda})$ is a suitable gauge-fixing factor, and in the last step we defined the vortex current,

$$J^{\text{V}}_{\kappa\lambda}(x) = \epsilon_{\kappa\lambda\mu\nu} \partial_\mu^{\text{ph}} \partial_\nu^{\text{ph}} \varphi_{\text{MV}}(x). \tag{3.8}$$

The interpretation of Eq. (3.7) is the following: in the superfluid there are vortex lines which trace out world sheets, built up out of surface elements $J^{\text{V}}_{\kappa\lambda}$, spanned by two non-parallel directions κ and λ. These vortices are sources in the sense of Schwinger [62], and therefore interact by exchanging two-form gauge fields $b_{\kappa\lambda}$. This gauge field corresponds to the zero sound or Goldstone boson of the superfluid. The first term is the kinetic energy or dynamics of the gauge field. Just as before, because of the long-range interactions, we call this the Coulomb phase for the vortices.

3.1.2 Mode content of the Coulomb phase

To examine the mode content explicitly, it is useful to go to the (τ, L, θ, ϕ) coordinate system, in which L is the spatial-longitudinal direction, and θ, ϕ are two arbitrarily chosen orthogonal transversal directions (see Fig. 1.3). We can use the gauge freedom Eq. (3.5) to impose the generalized Coulomb gauge $\partial_k b_{k\lambda} = q b_{L\lambda} = 0$, which removes all longitudinal components. The Lagrangian can now be expanded in the remaining components to find,

$$\mathscr{L} = \frac{1}{2} g q^2 b_{\tau\theta}^2 + \frac{1}{2} g q^2 b_{\tau\phi}^2 + \frac{1}{2} g (\omega^2 + q^2) b_{\theta\phi}^2. \tag{3.9}$$

Here we clearly identify the purely transversal component $b_{\theta\phi}$ as the single propagating mode. This makes sense as in 2+1 dimensions it was the transversal polarization of the dual gauge field, b_T, that represented the Goldstone mode. Furthermore there are now *two* temporal components $b_{\tau\theta}$ and $b_{\tau\phi}$ that communicate static Coulomb interactions between two vortex lines. The number of Coulomb forces increases because of the higher dimensionality of space: the relative orientation of vortex line sources allows for more diverse interactions. Except for this little surprise, we observe that the Coulomb phase of this stringy 2-form gauge theory is coding precisely for the physics of the 3+1d superfluid with its single propagating mode.

3.2 Vortex proliferation

Now it is time to try and increase the coupling constant g, let the vortex world sheets grow to the system size and let the vortices proliferate to effect the phase transition. We anticipate a kind of 'string foam' as the analogue of the 'tangle of vortex world lines'. As mentioned, there is presently no 'second quantized' way to do this, and all we can hope to achieve is an effective theory that captures the collective behaviour of the vortex liquid. The problem is to find a (dis)order parameter to which the dual 2-form gauge fields couple minimally. This was attempted in earlier works [46–48], and we now shall review their approach (a different path with some ideas similar to ours was taken in Refs. [49, 63]).

3.2.1 Naive generalization of the vortex proliferation

The defect world sheet is parametrized by $\sigma = (\sigma_1, \sigma_2)$ and $X(\sigma)$ is the map from the world sheet to real space. Hence each point on the world sheet σ is mapped to a specific point in real space $X(\sigma)$. A surface element of the world sheet is given by,

$$\Sigma_{\kappa\lambda}[X(\sigma)] = \frac{\partial X_\kappa}{\partial \sigma_1}\frac{\partial X_\lambda}{\partial \sigma_2} - \frac{\partial X_\lambda}{\partial \sigma_1}\frac{\partial X_\kappa}{\partial \sigma_2}. \tag{3.10}$$

The dynamics of the world sheet is given by the Nambu–Goto action,

$$S_{\text{worldsheet}} = \int \mathrm{d}^2\sigma \; T\sqrt{\Sigma_{\mu\nu}\Sigma_{\mu\nu}}, \tag{3.11}$$

where the integral is over the entire world sheet and T is the string tension, comparable to our $1/g$.

The source term $J_{\kappa\lambda} = \epsilon_{\kappa\lambda\mu\nu}\partial_\mu\partial_\nu\varphi_{\text{MV}}$ is related to the world sheet by,

$$J_{\kappa\lambda}(x) \sim \int \mathrm{d}^2\sigma \; \Sigma_{\kappa\lambda}[X(\sigma)]\delta(X(\sigma) - x). \tag{3.12}$$

According to figure 2.2(b), the gauge field $b_{\kappa\lambda}(x)$ couples to the world sheet surface element $\Sigma_{\kappa\lambda}[X(\sigma)]$. Suppose that a condensate of these vortex strings has formed, giving rise to a collective variable $\Phi[X(\sigma)]$ which is now a functional of the coordinate function $X(\sigma)$. The fluctuations of the condensate are given by the functional derivative,

$$\partial_\mu \Phi \to \frac{\delta}{\delta \Sigma_{\kappa\lambda}[X(\sigma)]} \Phi[X(\sigma)]. \tag{3.13}$$

When a condensate has formed, the amplitude $|\Phi|$ acquires a vacuum expectation value. The amplitude fluctuations freeze out as in the particle condensate and only the phase of the string condensate field is left as a dynamical variable. The phase fluctuations enumerate the collective motions of the string condensate but in the absence of an automatic formalism it is guess work to find out what these are. Marshall & Ramond, Rey and Franz [46–48] find inspiration in the analogy with the particle condensate. The phase degrees of freedom have to be matched through the covariant derivative with the 2-form gauge fields and they conjecture the seemingly obvious generalization,

$$\Phi[X(\sigma)] = |\Phi| e^{i \int dX_\mu(\sigma) C_\mu[X(\sigma)]}, \tag{3.14}$$

which implies that the collective motions of the string condensate are parametrized in a vector valued phase. The functional derivative (3.13) yields,

$$\frac{\delta}{\delta \Sigma_{\kappa\lambda}} \Phi[X(\sigma)] = |\Phi| (\partial_\kappa C_\lambda - \partial_\lambda C_\kappa), \tag{3.15}$$

reducing in turn to a natural minimal coupling form,

$$\left|\frac{\delta}{\delta \Sigma_{\kappa\lambda}} \Phi\right| \to \left|\left(\frac{\delta}{\delta \Sigma_{\kappa\lambda}} - i b_{\kappa\lambda}\right)\Phi\right| = |\Phi| (\partial_\kappa C_\lambda - \partial_\lambda C_\kappa - b_{\kappa\lambda}), \tag{3.16}$$

being gauge invariant under the combined transformations,

$$b_{\kappa\lambda} \to b_{\kappa\lambda} + \partial_\kappa \varepsilon_\lambda - \partial_\lambda \varepsilon_\kappa, \tag{3.17}$$

$$C_\kappa \to C_\kappa + \varepsilon_\kappa. \tag{3.18}$$

While this conjecture seems elegant and natural it is actually wrong, at least for the string field theory as of relevance to the 3+1d vortex string condensate. The flaw is in the overcounting of the degrees of freedom of the Mott-insulator/dual superconductor: the vector phase field ascribes too many collective degrees of freedom to the string condensate. Relying on the gauge invariance in the previous paragraph, we choose the unitary gauge $C_\kappa \equiv 0$ (cf. (2.58)). The action then reduces to that of a massive 2-form, which is known to have three propagating degrees of freedom. These can be identified by noting that we have 'spent' all gauge freedom in this gauge fix, such that all components of $b_{\kappa\lambda}$ become physical degrees of freedom. The three components $b_{\tau\lambda}$ are Coulomb forces, the other three are propagating. But we know that we should end up with two propagating degrees of freedom from the correspondence to the Bose-Mott insulator of section 2.3. Another view on this is that without interactions, this vortex condensate carries the two propagating degrees of freedom of a free massless vector field C_κ in four dimensions (just like a photon). In the unitary gauge these two get transferred to the gauge field $b_{\parallel\kappa}$, just as the ϕ-degree of freedom was transferred to $b_\parallel$ in (2.58). So if the vortex condensate were described by (3.14), it would carry two degrees of freedom, instead of only a single pressure mode.

The fallacy of this guess becomes even more obvious extending matters to higher dimensions. Generalizing this minimal coupling guess to d spacetime dimensions,

$$|\partial_\mu \phi - b_\mu| \to |\partial_{[\mu}\phi_{\nu_1\cdots\nu_{d-3}]} - b_{\mu\nu_1\cdots\nu_{d-3}}|, \tag{3.19}$$

One easy way is to count the number of propagating degrees of freedom of the phase field $\phi_{\nu_1\cdots\nu_{d-3}}$ if it were not coupled to the gauge field $b_{\mu\nu_1\cdots\nu_{d-3}}$. All of these modes transfer to the gauge field via the Higgs mechanism, adding their degrees of freedom to the single spin-wave mode. The number of propagating modes for an antisymmetric form field is given by all possible spatial-transversal polarizations [cf. (3.9)]. In d spacetime dimensions there are $d-2$ transversal directions, which must be accommodated in the $d-3$ indices of the phase field ϕ. Therefore, the number of degrees of freedom is

$$\begin{pmatrix} d-2 \\ d-3 \end{pmatrix} = \frac{(d-2)!}{(1)!(d-3)!} = d-2, \qquad d \geq 3. \tag{3.20}$$

This must be added to the single spin-wave mode, so in d spacetime dimensions, the naive prescription (3.19) would yield $d-1$ massive degrees of freedom, overcounting the modes of the Mott insulator by $d-3$. In this regard, d=2+1 is quite special indeed!

The fact that the usual minimal coupling procedure for the Higgs phenomenon is failing so badly in the higher dimensional cases indicates that it is subtly flawed in a way that does not become obvious in the 2+1d duality case, or even the 3+1d electromagnetic Higgs condensate. What is then the correct description of the string condensate? It surely has to correspond to the Bose-Mott insulator, which implies that the string condensate can only add one additional mode. One way to establish its nature is by invoking a general physics principle: the neutral string condensate would surely represent some form of compressible quantum liquid—which is not necessarily the case for fundamental strings—and such an entity has to carry pressure and thereby a zero sound mode. There is just no room for anything else given the mode counting that we know from the Bose-Mott insulator and we can already conclude that a Nielsen–Olesen string superfluid is at macroscopic distances indistinguishable from a particle superfluid!

3.2.2 Fate of the supercurrent

We need a different approach to guide us through the phase transition. Remember that in the duality transformation, we started out with regarding the supercurrent as the central object instead of the phase mode. The supercurrent is conserved in the superfluid $\partial^{\mathrm{ph}}_\mu w_\mu = 0$, which was the reason we could express it in terms of a dual gauge field $w_\mu = \epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_\nu b_{\kappa\lambda}$. There is a one-to-one correspondence between the components of the supercurrent and of the gauge field when expressed in the $(\|,\perp,\theta,\phi)$ coordinate system,

$$w_\perp \leftrightarrow b_{\theta\phi} \qquad w_\theta \leftrightarrow b_{\perp\phi} \qquad w_\phi \leftrightarrow b_{\perp\theta}. \tag{3.21}$$

In the superfluid the conservation of supercurrent eliminates $w_\|$ as a degree of freedom, and for the gauge fields we can remove $b_{\|\lambda}\ \forall\lambda$ by a suitable gauge transformation $\partial^{\mathrm{ph}}_\kappa b_{\kappa\lambda} = 0$. This choice, called the (generalized) Lorenz gauge, is very natural as these components are not sourced by the vortex current, as it is also conserved $\partial^{\mathrm{ph}}_\kappa J^{\mathrm{V}}_{\kappa\lambda} = 0$.

But in the dual superconductor we have seen that there is an additional degree of freedom due to the vortex condensate. How is this reflected by the supercurrent?

The Helmholtz theorem, familiar from vector analysis in electrodynamics, states that a sufficiently smooth vector field can always be separated into a irrotational (curl-free) and a solenoidal (divergence-free) part. This theorem can be generalized to dimensions other than three [64]. Thus we can split any vector field, in particular the supercurrent, into,

$$w_\mu = \partial^{\mathrm{ph}}_\mu \chi + \epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_\nu b_{\kappa\lambda}. \tag{3.22}$$

It it easy to see that the curl of the first term and the divergence of the second term both vanish. Now in the superfluid the current is conserved, $\partial_\mu^{\mathrm{ph}} w_\mu = 0$, which imposes a constraint on the irrotational part, namely $(\partial^{\mathrm{ph}})^2 \chi = 0$. Clearly this irrotational part, corresponding to $w_\parallel$, is removed as a dynamic degree of freedom in the superfluid. But what is the situation for the vortex condensate?

Recall that the formation of a vortex line induces supercurrent to flow around it. In other words, a vortex is a source of supercurrent. In the vortex condensate vortices and anti-vortices can form and disappear freely, and as they are sources and sinks of supercurrent, the latter is no longer conserved anywhere. This is equivalent to the statement that there are now only short-range correlations of the supercurrent due to the Higgs mechanism, and the local conservation law no longer holds. The constraint $\partial_\mu^{\mathrm{ph}} w_\mu = 0$ is removed, and in view of the above this also implies the release of the irrotational, longitudinal component as an additional degree of freedom. The compressional mode of the vortex condensate is reflected by the longitudinal component of the superfluid.

3.2.3 Supercurrent Higgs action

For another viewpoint, let us step back to the 2+1 dimensional case. Using the definition $w_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda$ and by integrating out the phase field ϕ, Eq. (2.58) can be formally rewritten as,

$$\mathscr{L} = \frac{1}{2} g w_\mu w_\mu + \frac{1}{2}\Phi_\infty^2 w_\mu \frac{1}{-\partial^2} w_\mu. \tag{3.23}$$

Here the first term is just the kinetic energy of the supercurrent as in Eq. (2.41), and the second is the Meissner term indicative of the now short-range interactions, and it is sometimes referred to as the "gauge-invariant Higgs term". But since this is the Higgs phase, there must also be the additional degree of freedom coming from the vortex condensate compressibility. We now know that this role is taken up by the longitudinal component of the supercurrent.

This expression is true for any dimensionality! And we have already provided the interpretation, the components of the supercurrent are classified as follows: the component $w_\perp$ corresponds to the purely transversal component of the gauge field and represent the superfluid zero sound or Goldstone mode; the transversal components w_{T_i} correspond to temporal components of the gauge fields, and therefore represent the static Coulomb forces; and the longitudinal component $w_\parallel$ couples to the vortex condensate, and is a dynamical degree of freedom only in the Higgs phase.

In light of these considerations, it is almost always best to choose the Lorenz gauge fix. Then using Eq. (3.22), the Higgs Lagrangian in 3+1 dimensions Eq. (3.23) reads,

$$\mathscr{L} = \frac{1}{2}(gp^2 + \Phi_\infty^2)(\chi^2 + b_{\perp\theta}^2 + b_{\perp\phi}^2 + b_{\theta\phi}^2). \tag{3.24}$$

Here the first two terms are the degenerate doublet of propagating modes, whereas the last two are the static Coulomb forces—their static nature with propagator $\sim q^2$ is seen only explicitly in the Coulomb gauge. All terms acquire a Higgs mass and therefore represent short-range interactions.

3.2.4 Summary of the results

The take-home message of this section is as follows. The conventional way of deriving the duality has a 'materialistic' attitude, invoking the vortices as a form of matter while the gauge fields enter much in the way as fundamental gauge fields code for the way that matter interacts. As we discussed, it is however also possible to reformulate the duality in terms of the physical currents, focussing on the way their continuity is lost—in phase representation this turns into the emergent gauge invariance of the Mott insulator. In the next section we will show that the ingredients of the vortex duality in the gauge language are strongly dependent on the dimensionality of spacetime, actually posing some problem of principle associated with the nature of string field theory.

However, when formulated in terms of the gauge invariant currents the dependence on dimensionality disappears, just as in the canonical Bose-Hubbard language of section 2.3. It leads to the correct mode counting as detailed in table 3.1. The 'current language' is still closely tied to the vortex language and this gives us the hold to control the duality in higher dimensions. The explicit statement is:

The neutral superfluid–charged superconductor duality of the 2+1d global $U(1)$ theory is equally valid in $D+1$ dimensional systems with $D > 2$, where the dual superconductor describes a $D-1$ form gauge theory Higgsed by a $p = D-2$ Nielsen–Olesen brane condensate that supports one massive compressional mode.

	Coulomb phase		Higgs phase	
	Coul. forces	propagating	Coul. forces	propagating
2+1d	1 long-range	1 massless	1 short-range	2 massive
3+1d	2 long-range	1 massless	2 short-range	2 massive

Table 3.1: Mode counting in the XY-model. Vortex proliferation in terms of the demise of supercurrents leads to the correct mode counting. Furthermore it contains as well the static Coulomb forces, which increase with the dimensionality of the system.

The derivation goes as follows. For each broken symmetry generator, there is a Goldstone mode that communicates the rigidity of that order parameter. The set of Goldstone modes $\{\varphi^a\}$ is labelled by an index a. Because these modes are massless and non-interacting, the canonical momenta $w^a_\mu = \frac{\partial \mathscr{L}}{\partial(\partial_\mu \varphi^a)}$ are conserved $\partial_\mu w^a_\mu = 0$. They are in fact the Noether currents under the global symmetry transformations $\varphi^a(x) \to \varphi^a(x) + \alpha^a$. As current carries energy, the action is of the form $S \sim \int w^a_\mu w^a_\mu$. Topological defects are regions where the Goldstone variable is not well-defined; consequently, the current is no longer conserved in that region. Each flavour a of current w^a_μ can be generated by the appropriate topological defect. A condensate of such defects Φ^a will have two effects:

i) they generate current everywhere, so that the current is conserved nowhere $\partial_\mu w^a_\mu \neq 0$ which introduces a new degree of freedom;

ii) the current–current correlations are destroyed by the defects, causing them to be exponentially decay with scale set by the Higgs mass Φ^a_∞.

The action in the Higgs phase is of the form,

$$S \sim \int w^a_\mu \big(1 + \frac{(\Phi^a_\infty)^2}{-\partial^2}\big) w^a_\mu. \tag{3.25}$$

3.3 Minimal coupling to 2-form gauge fields

The Lagrangian Eq. (3.23) contains all the dynamical information, and is valid for any dimension. Still, since the gauge fields are interpreted as the force carriers of the interaction between vortices, it would be nice if there were a description in terms of the gauge fields as well. In other words, we want a minimal coupling description that supersedes Eq. (3.16), and that incorporates the 2-form gauge fields while still leading to the correct mode content. The central problem is how to match the 1-form gradient of the phase field $\partial_\mu\phi$ to the 2-form gauge field $b_{\kappa\lambda}$.

We shall here present two proposals that accomplish this task. The first is valid in any dimension, but in fact leads to a slightly different definition of the gauge field, which in turn has an effect on the vortices of the disordered phase. The second avoids this last complication, but is as of yet only valid in 3+1 dimensions, and has no obvious way in which the "duality squared"-procedure of §2.4.6 follows. Let us first describe the two proposals, and address these issues when they present themselves.

3.3.1 Orthogonal projection

Since we know that the Lagrangian in gauge field components Eq. (3.24) is correct, we would be satisfied with any minimal coupling form that results in this expression. Now this Lagrangian is explicitly gauge fixed by $\partial^{\mathrm{ph}}_\kappa b_{\kappa\lambda}$ to project out the longitudinal components. We can also collect these three components in vector form by contracting with the Levi-Civita symbol where one of the indices is fixed to be this longitudinal direction. Consequently, we propose the minimal coupling to be,

$$\partial^{\mathrm{ph}}_\mu \phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda}. \tag{3.26}$$

The second term is non-zero only when μ,κ and λ take values in $(\perp,\theta,\phi)$ exclusively. Now since the derivative operator has only a longitudinal component, any crossterms automatically vanish, and indeed we find,

$$|\Phi|^2(\partial^{\mathrm{ph}}_\mu \phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda})^2 = |\Phi|^2\big((\partial_\mu\phi)^2 + b^2_{\theta\phi} + b^2_{\perp\theta} + b^2_{\perp\phi}\big). \tag{3.27}$$

Several remarks are in order. Firstly, this minimal coupling does not seem to be explicitly gauge fixed, as the gauge-variant components are projected out. However after taking the square as above, one cannot help to think that the Lorenz gauge fix is still in place. This should not concern us too much: we can contend ourselves with this gauge-fixed form, knowing that the ultimate truth is represent by the "gauge-invariant Higgs action" Eq. (3.23) anyway.

Secondly and more importantly, the gauge fields $b_{\kappa\lambda}$ in this expression are not precisely the same as those we used before in e.g. Eq. (3.22). This becomes clear when we step back to 2+1 dimensions. The analogue of Eq. (3.26) is,

$$\partial_\mu \phi - \epsilon_{\mu \| \lambda} b_\lambda, \tag{3.28}$$

which is clearly different from the standard minimal coupling Eq. (2.57). In fact, the directions in the transversal directions have been shuffled by the Levi-Civita symbol. This is the reason why I refer to this minimal coupling as "orthogonal projection". In 2+1 dimensions the relationship between the two forms for the gauge fields is clear, but in higher dimensions there is no immediate way of doing this. This does not seem to matter much now as the gauge fields are secondary variables anyway, but it has in fact bearing on the definition of the dual vortices as we will see in the next section.

Finally, this prescription can be generalized to any dimension $d \geq 2+1$,

$$\partial_\mu \phi - \epsilon_{\mu \| \lambda_1 \cdots \lambda_{d-2}} b_{\lambda_1 \cdots \lambda_{d-2}}. \tag{3.29}$$

The only surviving components of the gauge field are the single superfluid phase mode with only spatial-transversal components, and the Coulomb forces which have one index with temporal direction $\perp$.

3.3.2 Sum over vortex world sheet components

There is another form of the minimal coupling that results in Eq. (3.24), namely,

$$\frac{1}{2} \sum_\alpha \delta_{\kappa\alpha} \partial^{\mathrm{ph}}_\lambda \phi - b_{\kappa\lambda}. \tag{3.30}$$

Indeed,

$$\begin{aligned} |(\frac{1}{2} \sum_\alpha \delta_{\kappa\alpha} \partial^{\mathrm{ph}}_\lambda - b_{\kappa\lambda})\Phi|^2 &= |\Phi|^2 \big(\frac{1}{4} (\sum_\alpha \delta_{\kappa\alpha} \sum_\beta \delta_{\kappa\beta})(\partial^{\mathrm{ph}}_\lambda \phi)^2 - \sum_\alpha \delta_{\kappa\alpha} (\partial_\lambda \phi) b_{\kappa\lambda} + {b_{\kappa\lambda}}^2 \big) \\ &= |\Phi|^2 \big((\partial^{\mathrm{ph}}_\lambda \phi)^2 + {b_{\kappa\lambda}}^2 \big). \end{aligned} \tag{3.31}$$

In the last line we have imposed the Lorenz gauge so that the crossterms vanish. The expression Eq. (3.30) looks rather awkward. Nevertheless there is a concrete physical example where the minimal coupling has to be of this form, namely the vortices in a disordered superconductor. There the summation causes all κ-components of the dual vortex current $\mathscr{J}^{\mathrm{V}}_{\kappa\mu}$ to contribute to the current w_μ. This will be argued extensively in chapter 5.

Again, one could be satisfied by the correct outcome for the Lagrangian in gauge field components, always able to fall back on Eq. (3.23) when doubt arises. The specialization back to 2+1 dimensions is straightforward, by just leaving out the κ-components, avoiding the summation altogether. However it is not clear how to generalize to dimensions higher than four, but that is of no practical concern. Finally, these gauge fields $b_{\kappa\lambda}$ here are the same as used throughout this chapter, contrary to the previous construction Eq. (3.26).

3.3.3 Discussion

Exactly because the demise of the supercurrent is the defining feature of the dual Higgs condensate, there is no automatic way to derive the expression in terms of the dual gauge field. What is clear is that all of the gauge-invariant components (namely $b_{\theta\phi}$, $b_{\perp\theta}$ and $b_{\perp\phi}$) should be included and gain a Higgs mass. We are free to rotate between these components, or redefine them as we see fit. Therefore, even though the expressions Eqs. (3.26) and (3.30) look very different, we know they contain the same physics as far as the Lagrangian is concerned.

It may even be possible to define an explicit mapping between the two formulations, which would clear up the confusion that is presented here. As of yet I have not been able to find such a mapping. In the next section we will see that naively proceeding from these formulation leads to two very different interpretations of the dual vortex currents. Perhaps it is wisest to accept both forms just as different models, to be called upon in the suitable physical situation.

3.4 Vortices in the disordered phase

One of the appealing features of the vortex duality is that we have complete control over the disordered side. Indeed, in dual language it is just a Ginzburg–Landau theory of its own, with disorder parameter Φ, condensate phase fluctuations ϕ and coupling to a gauge field $b_{\kappa\lambda}$. The disordered phase is just a superconductor, albeit in 3+1 dimensions one with 2-form gauge fields.

This raises the immediate question of whether there are also dual topological defects (dual Abrikosov vortices) in the disordered phase. Since we have at hand just the theory of a (dual) superconductor, the answer is: of course there are. We already alluded to this in §2.4.6. But remembering that the disordered state is in fact the Bose-Mott insulator the appearance of such vortices is actually quite surprising. The Bose-Mott insulator is generally regarded as an exceedingly boring state of matter, where all particles are localized, everything is gapped, and there are only the two propagating doublon and holon modes. Even the dynamic spin system active in the fermionic Mott insulator is absent here.

Apparently, the state is richer and does allow for vortex excitations. For clarity I shall refer to these as Mott vortices for now on. The reason that they have not been suggested before is that usually one considers the so-called atomic or strong-coupling limit $U/t \gg 1$. But just as for superconductors, things become more interesting when the condensate is not so strong. Recall that Abrikosov vortices can appear when the penetration depth λ exceeds the coherence length, and the penetration depth is inversely proportional to the superfluid density $\lambda^2 \sim 1/|\Psi|^2$, see §2.1.2. Similarly, we expect vortices to arise in the Mott insulator when the (dis)order parameter $|\Phi|$ is not very big, so that the dual penetration depth $\tilde{\lambda}$ is large. The order parameter shrinks when one approaches the phase transition, and that would be the first place to look for them. We will have much more to say about these matters in chapter 5. Here we just show how the vortices arise in the calculation.

3.4.1 Dual vortex current

Vortices arise when there is a non-trivial winding of the dual phase field,

$$\oint \mathrm{d}\phi = \oint \mathrm{d}x^\mu \partial_\mu \phi = 2\pi N. \tag{3.32}$$

As before, we split the phase field in a smooth and a multivalued part, $\phi = \phi_{\mathrm{smooth}} + \phi_{\mathrm{MV}}$. Then we define the dual vortex current as (cf. Eq. (2.17)),

$$\mathscr{J}^{\mathrm{V}}_{\kappa\lambda} = \epsilon_{\kappa\lambda\mu\nu} \partial^{\mathrm{ph}}_\mu \partial^{\mathrm{ph}}_\nu \phi_{\mathrm{MV}}. \tag{3.33}$$

These vortices communicate via the dual currents, the fluctuations in the Mott order parameter (just as the original superfluid vortices interact via the zero sound mode). What is the nature of these vortices? The well-understood central physical quantity in all of our treatment here is the supercurrent w_μ. If we can see how the dual vortex current couples to the supercurrent, we have a clear interpretation of what the dual vortices really are.

It is possible to derive this relationship at the level of the Lagrangian, by introducing new variables that couple to the multivalued phase in the disordered phase. Then we define yet another gauge field that couples to the Mott vortices, and integrating out that gauge field will show the coupling between the Mott vortices and the original supercurrent. But we shall not take this route because i) the calculation is rather involved and yields no further insight, and ii) the current will seem to couple non-locally to the Mott vortices, while it is in fact a local coupling. It is more fruitful to simply inspect the equations of motion, and identify the physical properties from there.

3.4.2 Equation of motion: orthogonal projection

When taking the minimal coupling prescription of Eq. (3.26), the action reads,

$$\mathscr{L} = \frac{1}{2}g(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu}b_{\kappa\lambda})^2 + \frac{1}{2}|(\partial_\mu - \mathrm{i}\epsilon_{\mu\|\kappa\lambda}b_{\kappa\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4. \tag{3.34}$$

Varying with respect to $b_{\kappa\lambda}$ leads to the equation of motion,

$$-g\epsilon_{\kappa\lambda\nu\mu}\partial^{\mathrm{ph}}_{\nu}w_\mu + \Phi^2_\infty\epsilon_{\mu\|\kappa\lambda}(\partial^{\mathrm{ph}}_{\mu}\phi - \epsilon_{\mu\|\rho\sigma}b_{\rho\sigma}) = 0. \tag{3.35}$$

Acting on this expression with the operator $\epsilon_{\alpha\beta\kappa\lambda}\partial^{\mathrm{ph}}_{\beta}$, contracting repeated indices and substituting (3.33) leads to,

$$g\partial^2 w_\mu - \Phi^2_\infty w_\mu = -\Phi^2_\infty \epsilon_{\mu\|\kappa\lambda}\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}. \tag{3.36}$$

This is to be compared to the Ginzburg–Landau expression Eq. (2.6) for the magnetic field sourced by an Abrikosov vortex. Without any vortices the right-hand side is zero, and the left-hand side indicates that the supercurrent decays exponentially over characteristic length scale $\sqrt{g/\Phi^2_\infty}$, which is the expected behaviour for a (Mott) insulating state. Conversely, a Mott vortex current $\mathscr{J}^{\mathrm{V}}_{\kappa\lambda}$ is here a source of supercurrent locally. If we neglect the first term, this expression says that there is current wherever there is a Mott vortex.

Perhaps puzzling at first sight, this makes perfect sense: recall that a superconductor expels magnetic field, but an Abrikosov vortex consists of magnetic field permeating the superconductor through tubes, or rather vortex lines. Here the "type-II Mott insulator" expels current, but the current can penetrate locally through a vortex line.

This equation also illustrates our earlier objections to the minimal coupling prescription Eq. (3.26). On would expect that the current flows parallel to the vortex line, just as the magnetic field does in a type-II superconductor. In chapter 5 we see that this is indeed the case. However, Eq. (3.36) would set the current orthogonal to the vortex world sheet. One could argue that the vortex world sheet components are just wrongly defined, and need an additional rotation. However, one then loses the intuitive identification of the relation to the multivalued phase in real space as in Eq. (3.33). Furthermore, there is no natural way to perform this additional rotation. This form does however generalize to any higher dimension.

3.4.3 Equation of motion: sum over vortex components

When taking the minimal coupling prescription of Eq. (3.30), the action reads,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\kappa\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\kappa\lambda})^2 + \frac{1}{2}|(\frac{1}{2}\sum_{\alpha}\delta_{\alpha\kappa}\partial_{\lambda} - \mathrm{i}b_{\kappa\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4. \tag{3.37}$$

Varying with respect to $b_{\kappa\lambda}$ leads to the equation of motion,

$$-g\epsilon_{\kappa\lambda\nu\mu}\partial^{\mathrm{ph}}_{\nu} w_{\mu} + \Phi^2_{\infty}\big(\frac{1}{2}\sum_{\alpha}(\delta_{\alpha\kappa}\partial_{\lambda}\phi - \delta_{\alpha\lambda}\partial_{\kappa}\phi) - b_{\kappa\lambda}\big) = 0. \tag{3.38}$$

Acting on this expression with the operator $\epsilon_{\alpha\beta\kappa\lambda}\partial^{\mathrm{ph}}_{\beta}$, contracting repeated indices and substituting (3.33) leads to,

$$g\partial^2 w_{\mu} - \Phi^2_{\infty} w_{\mu} = -\Phi^2_{\infty}\sum_{\kappa}\mathscr{J}^{\mathrm{V}}_{\kappa\mu}. \tag{3.39}$$

The left-hand side is the same as Eq. (3.36), but the right-hand side is rather different. The interpretation is as follows: a vortex line $\mathscr{J}^{\mathrm{V}}_{\kappa\mu}$ sources (super)current in the direction μ. All of the components κ contribute to this current. This may seem awkward now, but has a very natural interpretation when it represents a moving line of electric current. We will elaborate on this extensively in §5.2.

Either form of the dual vortex current, Eqs. (3.36) and (3.39), clearly couples to supercurrent. In this regard the dual vortices exactly mirror the behaviour of Abrikosov vortices in type-II superconductors: just as superconductors expel magnetic field, the Bose-Mott insulator expels supercurrent. And just as Abrikosov vortices let magnetic field permeate the superconductor in local flux lines, the dual vortices are lines of supercurrent that

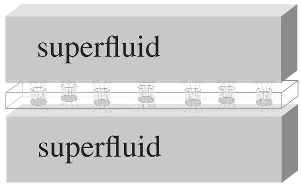

Figure 3.1: Proposed setup to show vortex lines in the Bose-Mott insulator. The Mott insulator (white) should be sandwiched between two regions with superfluid order (grey). The order parameter extends outside of the superfluid itself to pierce through the Mott insulator, in the form of vortex lines.

penetrate the insulator. Therefore we name such systems "type-II Mott insulators". The correspondence is even more striking when it is a Mott insulator made out of Cooper pairs, and that is the subject of chapter 5.

3.4.4 Tunnelling experiment

Because the superfluid is charge-neutral, the range of experimental tools that can probe these materials is limited. On the other hand, cold atoms on an optical lattice can be tuned at will to the superfluid to Mott-insulating state [50]. Furthermore, Josephson tunnelling between two superfluids has also been observed [65, 66]. Let us therefore sketch the outlines of an experiment that would create vortices in a Bose-Mott insulator.

A Josephson junction is a weak link, that can be an insulating barrier, a strip of vacuum, or just a constriction between to 'reservoirs' of superconducting order. As mentioned above, the same phenomenon has been observed in superfluids with different chemical potential. We now propose to make the barrier out of a Bose-Mott insulator near the quantum phase transition, see figure 3.1. In the regular Josephson effect, the supercurrent would flow homogeneously through the barrier, the energy cost of which grows with the volume of the barrier. But in type-II Bose-Mott insulator, the system can let the supercurrent flow through vortex lines, the energy cost of which grows with barrier width only. It is exactly like preferring the Abrikosov lattice above the fully magnetized Meissner state in type-II superconductors.

In the charged Mott insulator there is a plethora of possibilities to prove the existence of the Mott vortices, see §5.6.

3.4.5 Duality squared

For completeness, let us show that the "duality squared" procedure of §2.4.6 can also be repeated in 3+1 dimensions. As for now, I only know how to do this for the "orthogonal projection" minimal coupling prescription Eq. (3.26). But we argued that this must capture the essential physics, so we shall proceed accordingly.

We will write down only the most important steps. The minimal coupling term is linearized,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda\kappa}\partial_\nu^{\mathrm{ph}} b_{\kappa\lambda})^2 - \frac{1}{2}\frac{1}{\Phi_\infty^2} v_\mu^2 - v_\mu(\partial_\mu \phi - \epsilon_{\mu\|\kappa\lambda} b_{\kappa\lambda}). \tag{3.40}$$

The condensate phase ϕ is split into a smooth and a multivalued part. The smooth part is integrated out to give the constraint $\partial_\mu^{\mathrm{ph}} v_\mu = 0$, which is enforced by expressing $v_\mu = \epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\mathrm{ph}} z_{\kappa\lambda}$. After several partial integrations and rescaling $b_{\kappa\lambda} \to \frac{1}{\sqrt{g}} b_{\kappa\lambda}$, this leads to,

$$\mathscr{L} = \frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\mathrm{ph}} b_{\kappa\lambda})^2 - \frac{1}{2}\frac{1}{\Phi_\infty^2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\mathrm{ph}} z_{\kappa\lambda})^2 + z_{\kappa\lambda}\mathscr{J}_{\kappa\lambda}^{\mathrm{V}} - \frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\mu\nu}\partial_\nu^{\mathrm{ph}}\epsilon_{\mu\|\rho\sigma} b_{\rho\sigma}, \tag{3.41}$$

where $\mathscr{J}_{\kappa\lambda}^{\mathrm{V}} = \epsilon_{\kappa\lambda\mu\nu}\partial_\mu^{\mathrm{ph}}\partial_\nu^{\mathrm{ph}}\phi_{\mathrm{MV}}$ is the Mott vortex current. For contractions in the last term we use the identity

$$\epsilon_{\kappa\lambda\mu\|}\epsilon_{\mu\|\rho\sigma} = \delta_{\kappa\rho}\delta_{\lambda\sigma} - \delta_{\kappa\sigma}\delta_{\lambda\rho}, \tag{3.42}$$

where the indices on the right-hand side take values orthogonal to $\|$ only. The coupling of the z-gauge field to the b-gauge field then looks like,

$$\frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\|\mu}(\epsilon_{\mu\nu\rho\sigma}\partial_\nu^{\mathrm{ph}} b_{\rho\sigma}) = \frac{1}{\sqrt{g}} z_{\kappa\lambda}\epsilon_{\kappa\lambda\|\mu} w_\mu. \tag{3.43}$$

The gauge field $b_{\rho\sigma}$ only shows up in the combination $w_\mu = \epsilon_{\mu\nu\rho\sigma}\partial_\nu^{\mathrm{ph}} b_{\rho\sigma}$, which can be integrated out to yield a Meissner term for $z_{\kappa\lambda}$,

$$\mathscr{L} = -\frac{1}{2}\frac{1}{\Phi_\infty^2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\mathrm{ph}} z_{\kappa\lambda})^2 - \frac{1}{2g} z_{\kappa\lambda}^2 + z_{\kappa\lambda}\mathscr{J}_{\kappa\lambda}^{\mathrm{V}}, \tag{3.44}$$

which is valid in the Lorenz gauge $\partial_\kappa^{\mathrm{ph}} z_{\kappa\lambda} = 0$. Here we have a theory of Abrikosov vortex strings $\mathscr{J}_{\kappa\lambda}^{\mathrm{V}}$ that have short-range interactions with each other through the exchange of massive two-form fields $z_{\kappa\lambda}$. When vortices proliferate, they are described by a collective field Ψ, minimally coupled to the gauge field that we have rescaled $z_{\kappa\lambda} \to \Phi_\infty z_{\kappa\lambda}$,

$$\mathscr{L} = -\frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu^{\mathrm{ph}} z_{\kappa\lambda})^2 - \frac{\Phi_\infty^2}{2g} z_{\kappa\lambda}^2 \\ - \frac{1}{2}|(\partial_\mu^{\mathrm{ph}} - \mathrm{i}\Phi_\infty \epsilon_{\mu\|\kappa\lambda} z_{\kappa\lambda})\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{3.45}$$

Through the phase transition, the Mott vortices destroy the dual superconducting order so that Φ_∞ vanishes. Then the gauge field $z_{\kappa\lambda}$ decouples and we are left with the action of a neutral superfluid Eq. (2.1), exactly our starting point. In this way duality$^2 = 1$ also holds in 3+1 dimensions.

3.5 Discussion

This chapter comprises the main result of this thesis: the vortex-boson duality that is so well known in condensed matter physics holds in (at least) all dimensions larger than two. The reason is that the fundamental physical quantities are the Noether currents in the ordered phase, and their conservation law imposes exactly one constraint. In the disordered phase the vortex condensate enters as a featureless fluid, whose compression mode is the additional single degree of freedom, simultaneously responsible for the demise of the currents, releasing the constraint. Related to this, all correlation functions become short-ranged due to the disorder induced by the vortices. This last statement has a very nice interpretation in terms of emergent gauge symmetry, which is the topic of chapter 6.

Even if the currents are the principal objects, the gauge fields that can be defined because of the conservation law have a natural interpretation as the force carriers of the interaction between vortices. They are the dual of the Goldstone modes. Precisely because the gauge fields couple to the vortices, they also couple minimally to the vortex condensate disorder parameter field, and are therefore instrumental in the (mathematical) construction of the dual phase transition. We have noticed that there are at least two ways to define a suitable minimal coupling, which seem equivalent at the level of the Lagrangian. But we will see in §5.2 that the precise form *is* of importance. There is room for improvement here, also considering that our proposals Eqs. (3.26) and (3.30) are not strictly gauge invariant.

These details left aside, the formalism developed here is general and should be applicable in more complex situations than the simple $U(1)$-symmetry here. The next two chapters are about charged superfluids, superconductors, in which this global symmetry is coupled to a vector gauge field, the photon. Chapter 6 alludes to its relevance for quantum liquid crystals. At the end of the thesis, chapter 7, I will contemplate some further susceptible cases.

3.A Degrees of freedom counting

We have determined the degrees of freedom by explicit examination of the action and propagators. There is a more general and formal way of deriving the *propagating* degrees of freedom given an action (Coulomb forces do not fall into this general scheme). It precisely determines the gauge degrees of freedom and the influence of constraints. This is exhaustively explained in Ref. [37]. We will very briefly discuss this procedure for free Abelian 1- and 2-forms (*op. cit.* ch.19).

The Maxwell Lagrangian in d spacetime dimensions is,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2. \tag{3.46}$$

The vector field A_μ has d components, so we start out with d degrees of freedom. The action is invariant under gauge transformation $A_\mu \to A_\mu + \partial_\mu \varepsilon$; furthermore this gauge transformation corresponds to a so-called *first-class constraint*, which means it removes two degrees of freedom in total. The reason for this is that we fix the vector field not only in space at one moment in time (a time slice), but also its evolution using $\partial_t \varepsilon$. Another point of view is that the temporal component A_t is set by the scalar electrostatic potential, which is zero everywhere for a free field; the temporal component is completely fixed by the equation of motion $\nabla^2 A_\tau = 0$.

Therefore a free vector field in d dimensions has $d-2$ propagating degrees of freedom, exactly the transversal polarizations of the photon.

The generalization of (3.46) for an anti-symmetric 2-form field $b_{\mu\nu}$ in 4 dimensions is,

$$\mathscr{L} = -\frac{1}{2}(\epsilon_{\mu\nu\kappa\lambda}\partial_\nu b_{\kappa\lambda})^2. \tag{3.47}$$

The field has six independent components. The action is invariant under gauge transformations,

$$b_{\kappa\lambda}(x) \to b_{\kappa\lambda}(x) + \partial_\kappa \varepsilon_\lambda(x) - \partial_\lambda \varepsilon_\kappa(x). \tag{3.48}$$

Here $\varepsilon_\lambda(x)$ is any smooth real vector field with 4 components; but there are only three independent gauge transformations since $\delta_{\lambda\kappa}(\partial_\kappa\varepsilon_\lambda - \partial_\lambda\varepsilon_\kappa) = 0$ always. As explained above each gauge transformation removes two degrees of freedom. The transformations are however redundant, since another vector field,

$$\varepsilon'_\lambda(x) = \varepsilon_\lambda(x) + \partial_\lambda\eta(x), \tag{3.49}$$

where η is any smooth scalar field gives exactly the same transformation in (3.48). A free 2-form field in 4 dimensions therefore has $6-(6-1)=1$ propagating degree of freedom.

3.B Current conservation in electromagnetism

We apply the conservation-of-current considerations to the most famous example of the Higgs mechanism: the photon field in 3+1 dimensions coupled to a complex scalar condensate field. This is variously known as the Abelian–Higgs model, Ginzburg–Landau theory or scalar QED. It describes the basic physics of the electromagnetic field in the vacuum and in a superconductor.

The electromagnetic field is a vector field $A_\mu(x)$. Its dynamics is governed by the field strength $F_{\mu\nu} = \partial_\mu A_\mu - \partial_\nu A_\mu$ and the Maxwell action,

$$S = \int -\frac{1}{4}F_{\mu\nu}^2. \tag{3.50}$$

The field strength is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu\varepsilon$. The vector field with gauge fix $\partial_\mu A_\mu = 0$ has three degrees of freedom: the two transversal photon polarizations A_θ and A_ϕ, and the part mediating static Coulomb interactions $A_\perp$.

The field strength $F_{\mu\nu}$ has six independent components and is therefore overcounting the degrees of freedom. This can be cured by imposing the homogeneous Maxwell equations or Bianchi identities,

$$\mathsf{d}\mathsf{F} = \epsilon_{\mu\nu\kappa\lambda}\partial_\nu F_{\kappa\lambda} = 0. \tag{3.51}$$

In $(\parallel,\perp,\theta,\phi)$-coordinates (see figure 1.3) this implies that the only non-zero components of the field strength are $F_{\parallel\nu}$, which we collect in a vector field $f_\nu \equiv F_{\parallel\nu}$ (the 'current'). From this point we act as if the field strength $F_{\parallel\nu}$ were not necessarily anti-symmetric; still the longitudinal component is set to zero as long as there are no external sources: $\partial_\nu f_\nu = \partial_\nu F_{\parallel\nu} = J^{\mathrm{ext}}_\parallel \to 0$ (inhomogeneous

Maxwell equations). The other three components of f_ν correspond to the three physical degrees of freedom identified above via,

$$f_\nu = pA_\nu. \tag{3.52}$$

Now we couple the photon field to a complex scalar Higgs field via $|\partial_\mu \Psi| \to |(\partial_\mu - \mathrm{i}A_\mu)\Psi|$ as in (2.33). The Higgs field describes a condensate destroying the current conservation, so that the longitudinal component $f_\parallel$ is released. Indeed, from (3.52) this corresponds to the longitudinal polarization of the photon: $f_\parallel = pA_\parallel$. In terms of the field strength, it is seen to correspond to the symmetric component $F_{\parallel,\parallel}$, which is normally not taken into consideration.