Vortex Duality in Higher Dimensions

Beekman, A.J.

Chapter 2

Preliminary material

Here we present some material that is not at all new, but on which later parts of this work are based. We only include discussion as far as is needed to understand the following chapters of this work.

2.1 The Ginzburg–Landau model

Here we shall very shortly recap the overly familiar Ginzburg–Landau model of superconductivity, because all of the following work will use the same order parameter language. As such it is good to set the stage such that one can always compare with well-established results, see for instance Refs. [28, 51].

2.1.1 Superfluid

In 1937 Lev Landau proposed a phenomenological field-theoretical model that was capable of capturing the essential features of continuous or second order phase transitions. It centred around the concept of an *order parameter* $\Psi(x)$, which is a function on every point in space, i.e. a field. It is capable of distinguishing between ordered and disordered phases: in the disordered phase, its average or expectation value is zero $\langle\Psi\rangle = 0$, while in the ordered phase it is non-zero $\langle\Psi\rangle = \Psi_0 \neq 0$. Landau established the simplest form that can show this behaviour,

$$E = \int \mathrm{d}^3x \ \frac{1}{2}|\nabla\Psi|^2 + \frac{1}{2}\alpha|\Psi|^2 + \frac{1}{4}\beta|\Psi|^4. \tag{2.1}$$

Here Ψ is a complex scalar field. The first term represents fluctuations in the order parameter, and can therefore be regarded as the kinetic energy. The second term is as a mass for the order parameter, and the third causes the energy to always be bounded from below. When $\alpha > 0$ the potential energy is minimized by $|\Psi| = 0$ and we are in the disordered phase. But when $\alpha < 0$, the potential energy has minima at $|\Psi| = \pm\sqrt{|\alpha|/\beta}$. This is sometimes called the 'Mexican hat potential'.

Because $\Psi = |\Psi|e^{i\varphi}$, the phase can still freely fluctuate, and in the so-called London limit where the amplitude is fixed everywhere $|\Psi|(x) = \Psi_0$, the energy reduces to,

$$E = \int d^3x \, \frac{1}{2g}(\nabla\varphi)^2, \tag{2.2}$$

modulo constant terms, and g poses as the coupling constant. This very simple model actually describes the dynamics of a superfluid, with the massless zero sound mode φ and massive density fluctuations $|\Psi|$.

The parameter α is usually taken as a function of temperature, changing sign at the critical temperature T_c. This model then also contains the scaling laws at the critical point up to the mean field level, and as such partly explains universality, the phenomenon that microscopic details are often unimportant in capturing the collective behaviour of many-body systems.

2.1.2 Superconductor

It was not until 1950 that this powerful concept was extended to charged superfluids, i.e. superconductors with the help of Vitaly Ginzburg. This was done by minimal coupling to the electromagnetic gauge potential,

$$E = \int d^3x \, \frac{\hbar^2}{2m^*}|(\nabla - i\frac{e^*}{\hbar}\mathbf{A})\Psi|^2 + \alpha|\Psi|^2 + \frac{1}{4}\beta|\Psi|^4 + \frac{1}{2\mu_0}(\nabla\times\mathbf{A})^2. \tag{2.3}$$

Here m^* and e^* are the mass and the electric charge of the charge carriers (Cooper pairs as we know now). From this energy functional, we can derive the Ginzburg–Landau equations of motion,

$$-\frac{\hbar^2}{2m^*}(\nabla - \frac{e^*}{\hbar}\mathbf{A})^2\Psi + \alpha\Psi + \beta|\Psi^2|\Psi = 0 \tag{2.4}$$

$$\frac{1}{\mu_0}\nabla\times\nabla\times\mathbf{A} - \frac{\hbar e^*}{m^*}|\Psi|^2(\nabla\varphi - \frac{e^*}{\hbar}\mathbf{A}) = 0 \tag{2.5}$$

From the first equation, when there are no external electromagnetic fields present, one can derive the *coherence length* $\xi = \frac{\hbar^2}{m^*|\alpha|}$ as the typical length scale over which the value of $|\Psi|(x)$ still fluctuates.

By action with the curl operator $\nabla\times$ on the second equation, using the definition of the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ and the Maxwell equation $\nabla \cdot \mathbf{B} = 0$, one finds,

$$\lambda^2 \nabla^2 \mathbf{B} - \mathbf{B} = -\frac{1}{2\pi} \Phi_0 (\nabla \times \nabla)\varphi. \tag{2.6}$$

Here we defined the *London penetration depth* $\lambda = \sqrt{\frac{m^*}{\mu_0 |\Psi|^2 e^{*2}}}$ and the *flux quantum* $\Phi_0 = h/e^*$. When the phase field φ is smooth the right-hand side vanishes, and this equation then tells us that the magnetic field is expelled from the superconductor, as it falls off exponentially over length scale λ. This is called the Meissner effect. We also identify,

$$\mathbf{J}_\mathrm{s} = -\frac{\delta E}{\delta \mathbf{A}} = \frac{\hbar e^*}{m^*} |\Psi|^2 (\nabla \varphi - \frac{e^*}{\hbar} \mathbf{A}), \tag{2.7}$$

as the supercurrent. Then Eq. (2.5) can also be written as,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_\mathrm{s}, \tag{2.8}$$

which is the non-dynamic part of the Ampère–Maxwell law. Furthermore, acting with the curl operator on Eq. (2.7), we find the London equation,

$$\nabla \times \mathbf{J}_\mathrm{s} = -\frac{1}{\mu_0 \lambda^2} \mathbf{B}. \tag{2.9}$$

where we again have used $(\nabla \times \nabla)\varphi = 0$ for a smooth phase field.

It was Alexei Abrikosov's great insight [52] that when $\lambda > \xi/\sqrt{2}$, it is energetically more favourable to let the magnetic field penetrate through vortex lines than to expel it altogether. Such a material is called a type-II superconductor. We will see in the next section that in the presence of vortices, φ becomes multivalued, and then we should identify $(\nabla \times \nabla)\varphi = 2\pi \delta^{(2)}(\mathbf{x}) N$, a 2-dimensional delta function in the plane orthogonal to the vortex line times the winding number N (see also §2.2.3). Eq. (2.6) then shows that the vortex line *is* magnetic field, that falls off exponentially away from the centre.

We can take a line integral of (2.5) deep within the superconductor where $\mathbf{B} = 0$ over a closed contour $\mathscr{C}$ around a vortex line. We find using Stokes' theorem,

$$\int_{\mathscr{S}} \mathrm{d}\mathbf{S} \cdot \mathbf{B} = \oint_{\mathscr{C}} \mathrm{d}\mathbf{x} \cdot \mathbf{A} = \frac{1}{2\pi} \Phi_0 \oint \mathrm{d}\mathbf{x} \cdot \nabla \varphi = \Phi_0 \int_{\mathscr{S}} \mathrm{d}\mathbf{S}\ \delta^{(2)}(\mathbf{x}) = \Phi_0 N. \tag{2.10}$$

Here $\mathscr{S}$ is the area enclosed by the contour $\mathscr{C}$. Thus we see that the magnetic flux through $\mathscr{S}$ and therefore through the vortex line is quantized in units of Φ_0.

The electrodynamics of Abrikosov vortices is derived from a relativistic field theory in chapter 4.

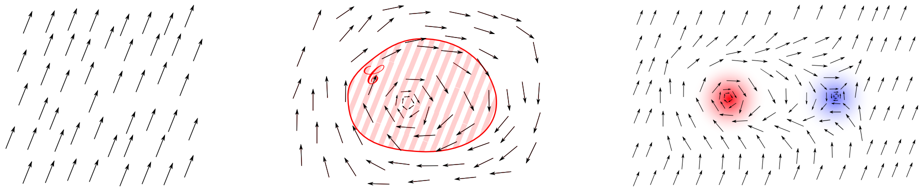
2.2 Topological defects

Once one finds oneself in an ordered state, a natural question is how it can be made disordered. Disorder is caused by defects, a simple example of which would be an interstitial atom or ion in an otherwise perfectly regular crystal lattice. Such defects cost energy to make, but usually only a fixed amount independent of the system size. As such their disordering capabilities are also not that great. It turns out that most forms of disorder are due to *topological* defects, the energy of which increases with the system size. They are thus energetically very expensive, and will in strongly ordered systems only appear in confined combinations, often pairs, which are said to be *topologically neutral*. Increasing disorder amounts to deconfining such pairs (see §1.1.4).

To understand what topological defects are and how they are classified for a specific ordered medium, one needs the mathematical machinery of homotopy theory. It explores the concept of continuity, which turns out to be *the* property of relevance in describing ordered states and the topological defects they can support. We shall not delve deeply into these matters; a good introduction is found in the review by David Mermin [53]. Here we will quote some of the results as needed for the Abelian $U(1)$-symmetry we are exclusively interested in.

2.2.1 Order parameter space

As explained above in §2.1.1, an order parameter is a continuous function on every point in space. If there are long-range correlations between the values of this function, the state is said to be ordered. The domain of the function is called "order parameter space" $\mathscr{M}$, and it can be a number, vector or any continuous manifold. We are interested in superfluids and superconductors, with order parameter a complex scalar field $\Psi = |\Psi| \mathrm{e}^{\mathrm{i}\varphi}$. In the completely ordered state the amplitude obtains a so-called vacuum expectation value



(a) Phase ordered (b) Single vortex (c) Vortex pair

Figure 2.1: Configurations of the phase field in the plane. (a) A trivial state with the phase perfectly ordered. (b) Configuration with a $N = 1$ vortex present. Taking the line integral around the contour $\mathscr{C}$ will give 2π. The contour and therefore the hatched area are arbitrary along as they comprise the vortex core. (c) A vortex–anti-vortex pair. Far away from these vortices the phase is ordered, and therefore this configuration is topological neutral.

(VEV) that is non-zero and constant throughout the medium. The phase of Ψ has long-range correlations. Small fluctuations around this VEV cost some energy but vanishingly little as the fluctuations die out. These fluctuations are actually the Goldstone modes, and it is easy to see that they can only arise for continuous order parameters, as there is no such thing as a small fluctuation in a discrete set. The Goldstone modes communicate the rigidity of the order parameter.

Let us first take the example of the $U(1)$ order parameter to illustrate the principles. When the amplitude $|\Psi|$ has obtained an expectation value, then only the phase φ is left, which can be pictured as an arrow on every point in space. If the system is spatially 2-dimensional, the order parameter space can be conveniently drawn just on real space. Consider the configurations in figure 2.1. Without a defect present, the phase is perfectly ordered, barring small fluctuations. When however the phase around a closed contour makes a 2π rotation, there must be a singular point where the phase is not well defined. This point is the topological defect, called a vortex for a $U(1)$-field. Wherever we draw this contour, the phase change is always 2π, which is the reason for the denomination 'topological'. We also see that a configuration of a vortex and an anti-vortex together is topologically neutral.

2.2.2 Homotopy groups

In the general case, due to thermal or quantum fluctuations, the system is free to explore part of configuration space by small perturbations around the present, ordered, state. As such we can define configurations to be equivalent if they differ by *continuous deformations* only. All of configuration space is then divided up in equivalence classes, and one class cannot be transformed into another continuously. There is one trivial class, and all the others are said to contain topological defects. It turns out that the equivalence classes are classified by the *homotopy groups* of the order parameter space. Mathematically, the nth homotopy group $\pi_n(\mathscr{M})$ has as elements all the different ways in which an n-sphere S_n can be mapped onto the space $\mathscr{M}$. For instance the first homotopy group (or fundamental group) $\pi_1(\mathscr{M})$ classifies how 'lassos' can or cannot be contracted into a point.

From the drawings in figure 2.1, we see that such lassos characterize *point* defects in a 2-dimensional plane. But in 3 dimensions, we would be able to pull the lasso 'over' the singular point. If we had a singular line, the lasso cannot be contracted. For this reason, the n-th homotopy group classifies $D-n-1$-dimensional defects in D-dimensional space. Thus π_1 classifies point defects in 2D and line defects in 3D; and π_2 classifies point defects in 3D. Now it is a result of homotopy theory that $\pi_n\big(U(1)\big)$ is isomorphic to the trivial group except for $n=1$, where it is the set of integers representing the winding numbers. Therefore the only topological defects possible are point defects in 2D and line defects in 3D, both characterized by the winding number N.

2.2.3 Multivalued fields

Almost all of the properties of vortices (or topological defects in general) can be ascribed to the singular point or line in the vortex core. The singularity is by definition not well-behaved analytically. Yet it turns out to be very fruitful to try and apply field-theoretical techniques as much as we can. In fact this is the central topic of Kleinert's textbooks [28, 41, 42]. For us it suffices to establish the following identity. The phase winds in units of 2π around the vortex core, by traversing contour $\mathscr{C}$. Thus the change of of the phase adds up to $2\pi N$,

$$\oint_{\mathscr{C}} \mathrm{d}\varphi = \oint_{\mathscr{C}} \mathrm{d}x^m\ \partial_m \varphi = 2\pi N. \tag{2.11}$$

Let $\mathscr{S}$ be the area enclosed by $\mathscr{C}$. In 3D it has a normal k that is parallel to the vortex line. Then we formally apply Stokes' theorem,

$$2\pi N = \oint_{\mathscr{C}} \mathrm{d}x^m \; (\partial_m \varphi) = \int_{\mathscr{S}} \mathrm{d}S^k \; \epsilon_{knm} \partial_n (\partial_m \varphi). \tag{2.12}$$

Thus, if there is a vortex present $N \neq 0$ the left-hand side is not zero, and we must conclude that the derivatives of the singular field φ do not commute. Therefore we are led to identify,

$$\epsilon_{knm} \partial_n \partial_m \varphi(x) = 2\pi N \delta_k^{(2)}(x). \tag{2.13}$$

Here $\delta_k^{(2)}(x)$ is a 2-dimensional delta function in the plane orthogonal to k centred around the vortex core. Since away from the core the phase field is smooth, the non-vanishing contribution is indeed purely attributable to the singular point itself. In the sequel, we shall often split the phase field in a smooth and a multivalued part,

$$\varphi = \varphi_{\mathrm{smooth}} + \varphi_{\mathrm{MV}}, \tag{2.14}$$

where $\epsilon_{knm} \partial_n \partial_m \varphi_{\mathrm{smooth}}(x) = 0 \;\forall x$, whereas the multivalued part satisfies the relation above. Even though the derivatives of a multivalued field do not commute, it does satisfy the integrability condition, [28, 42],

$$\partial_k (\epsilon_{knm} \partial_n \partial_m \varphi) = 0. \tag{2.15}$$

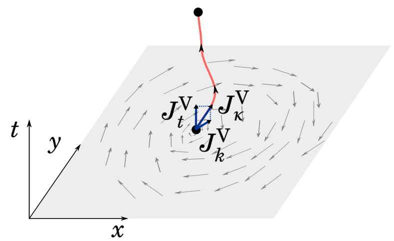
Regarded as a physical field, we define,

$$J_k^{\mathrm{V}} = \epsilon_{knm} \partial_n \partial_m \varphi = 2\pi N \delta_k^{(2)}(x), \tag{2.16}$$

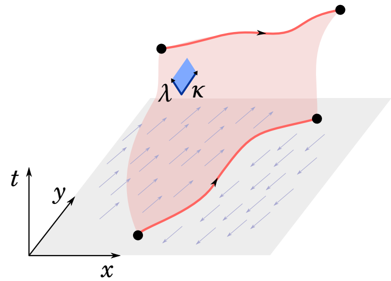
as the *vortex current*. It is conserved $\partial_k J_k^{\mathrm{V}} = 0$, because of the integrability condition above. These vortex currents are the central topic of this thesis.

2.2.4 Vortex world lines and world sheets

We have seen that for $U(1)$-fields there are pointlike vortices in 2-dimensional and linelike vortices in 3-dimensional space. Now we regard these objects as physical entities as moving in spacetime. The 2D vortex point (vortex pancake in superconductivity parlance) then traces out a world line in spacetime, just as any particle would. But the 3D vortex line traces out a world sheet. This is pictured in figure 2.2. In 2+1d the direction orthogonal to



(a) Vortex world line



(b) Vortex world sheet

Figure 2.2: Vortices in 2+1d and 3+1d (a) A point vortex will trace out a world line. The line element J^V_κ can be decomposed in a temporal density part J^V_t and a spatial current part J^V_k. (b) In 3D we have a vortex line, here in the xy-plane, since the third spatial dimension cannot be drawn. The world sheet is built up out of surface elements $J^V_{\kappa\lambda}$. The temporal components J^V_{tl} represents the density of vorticity of the line along l, and the spatial components J^V_{kl} are the flow in direction k of the line along l.

the plane is always the time direction, but in a relativistic treatment we consider the vortex current $J^V_\kappa = \epsilon_{\kappa\nu\mu}\partial_\nu\partial_\mu\varphi_{\mathrm{MV}}$ where the indices take values in (t,x,y). $J^V_\kappa(x)$ is just the line element of the vortex world line at x. The temporal component J^V_t is the density of vorticity defined in Eq. (2.16). The spatial components are the 'current' related to this density, such that the conservation law $\partial_\kappa J^V_\kappa = 0$ is in fact the continuity equation $\partial_t J^V_t + \partial_k J^V_k = 0$.

It is now obvious how to generalize to 3+1 dimensions. The singular field φ_{MV} has the same properties as before, and since in four dimensions the Levi-Civita symbol has four indices, our vortex current becomes an antisymmetric 2-form field,

$$J^V_{\kappa\lambda} = \epsilon_{\kappa\lambda\nu\mu}\partial_\nu\partial_\mu\varphi_{\mathrm{MV}}. \tag{2.17}$$

The field $J^V_{\kappa\lambda}(x)$ locally represents a surface element of the vortex world sheet, defined by two non-parallel directions κ and λ. Similar as before, the temporal components J^V_{tl} are the density of vorticity of the vortex line along l. A spatial line integral around this component will result in $2\pi N$; the normal of the area enclosed by this contour is set by the two directions t and l. The purely spatial components J^V_{kl} represent the flow in the direction k of the vortex line along l. There are three independent continuity equations $\partial_\kappa J^V_{\kappa\lambda} = 0$.

This interpretation of the vortex current as field-theoretical objects will turn out to be especially useful for vortices in superconductors (chapter 4) and Mott insulators (chapter 5).

2.3 The Bose–Hubbard model

The study of quantum phase transitions concerns the collective behaviour of quantum matter at zero temperature. In many respects they resemble thermal phase transitions where one just has to replace thermal fluctuations by quantum zero-point fluctuations. Yet time plays a special role, and it is useful to consider extremely simple models that do feature the basic properties of quantum phase transitions. The simplest one would be the quantum Ising model where the dynamical variable can take only two values. One step further is to take a continuous variable and this is called the XY-model or the quantum rotor model. These systems are studied in depth in Sachdev's textbook [17]. It turns out that the latter model in the ordered state is just the quantum field theory of a free scalar field, and as such describes Goldstone modes such as the phase mode in a superfluid. The quantum phase transition arises when this phase, ordered in the superfluid, fluctuates so wildly that the long-range correlations disappear. We will see in the next section that this is equivalent to the formation of a condensate of vortices.

Here we will show how another simple model, called the Bose–Hubbard model [54], reduces to the quantum XY-model. The reason for this is twofold. Firstly, this model describes bosons hopping on a lattice but repelling each other locally. This is a realistic approximation of some real-world systems, and is in fact almost perfectly realized in cold atom experiments in optical lattices [50]. Furthermore the phase dynamics is also seen in arrays of Josephson junctions [55, 56]. The second reason is that it shows explicitly that the state across the phase transition is a Bose-Mott insulator. Therefore the disordered state after unbinding of the vortices must be equivalent to this insulating state. We will use this argument in chapter 3 to lead us to the understanding of the vortex unbinding transition in higher dimensions.

2.3.1 Bose–Hubbard Hamiltonian

We will start out from a simple Hamiltonian model for lattice bosons, and map it onto the Euclidean action of a continuum field theory, which is the most useful form for the quantum phase transition. The Hamiltonian of the Bose-Hubbard model on a D-dimensional hypercubic lattice is,

$$H_{\mathrm{BH}} = -\frac{t}{2}\sum_{\langle ij\rangle}(b_i^\dagger b_j + b_j^\dagger b_i) - \mu\sum_i n_i + \frac{U}{2}\sum_i (n_i - 1)n_i. \tag{2.18}$$

Here $b_i^\dagger$ and b_i are boson creation and annihilation operators on lattice site i, that satisfy the commutation relation $[b_i, b_j^\dagger] = \delta_{ij}$. The sum over $\langle ij\rangle$ is over nearest-neighbour sites. The number operator is $n_i = b_i^\dagger b_i$. Furthermore, the energy scales are the boson hopping t, the on-site repulsion U and the chemical potential μ. We shall assume that the chemical potential is tuned so that there is a large integer number n_0 of bosons per site. We call this "zero chemical potential". The commutation relation for n and b is,

$$[n_i, b_j] = [b_i^\dagger b_i, b_j] = 0 + [b_i^\dagger, b_j]b_i = -\delta_{ij} b_i. \tag{2.19}$$

Similarly $[n_i, b_j^\dagger] = \delta_{ij} b_i^\dagger$. To recognize quantum phase dynamics consider the substitution,

$$b_i^\dagger = \sqrt{n_i}\,\mathrm{e}^{\mathrm{i}\varphi_i}, \qquad\qquad b_i = \mathrm{e}^{-\mathrm{i}\varphi_i}\sqrt{n_i}. \tag{2.20}$$

Here φ_i is a real scalar variable. The commutation relation for n and φ follows,

$$\begin{aligned} [n_i, b_j] = \delta_{ij} b_i \qquad &\Rightarrow \qquad [n_i, \mathrm{e}^{-\mathrm{i}\varphi_j}\sqrt{n_j}] = -\delta_{ij}\mathrm{e}^{-\mathrm{i}\varphi_j}\sqrt{n_j} \\ &\Rightarrow \qquad [n_i, \mathrm{e}^{-\mathrm{i}\varphi_j}] = -\delta_{ij}\mathrm{e}^{-\mathrm{i}\phi_j}. \end{aligned} \tag{2.21}$$

This commutation relation corresponds to $[n_i, \varphi_j] = -\mathrm{i}\delta_{ij}$, which one can check via the Taylor expansion of the exponential. In this way we have switched from a description in terms of the conjugate variables b and $b^\dagger$ into the conjugate variables n and φ. Substituting this definition in Eq. (2.18) leads to,

$$H = -J\sum_{\langle ij\rangle}(1 - \cos(\varphi_i - \varphi_j)) + \frac{U}{2}\sum_i (n_i - 1)n_i. \tag{2.22}$$

Here we have defined $J = tn_0$ and added a constant term for later convenience. The physics of the weak- and strong-coupling limits is immediately clear: for large t/U, we have a superfluid where spatial fluctuations in the phase φ are very costly; for small t/U the on-site repulsion dominates, the bosons spread out evenly to minimize $U\sum_i n_i^2$ and are thereafter confined to their lattice sites: the Mott insulator.

2.3.2 Legendre transformation and continuum limit

Since we are pursuing a relativistic quantum calculation, we shall move from a Hamiltonian to a Lagrangian formalism. The commutation relation $[\varphi_i, n_j] = \mathrm{i}\delta_{ij}$ is to be compared to the canonical commutation relation $[\varphi_i, \pi_j] = \mathrm{i}\hbar\delta_{ij}$. We can therefore regard as the canonical momentum $\pi_j = \hbar n_j$. The velocity is defined by,

$$\partial_t \varphi_j = \frac{\partial H}{\partial \pi_j} = \frac{U}{\hbar^2}\pi_j. \tag{2.23}$$

From this we find the Lagrangian by Legendre transformation,

$$L = \sum_i \pi_i \partial_t \varphi_i - H = \frac{\hbar^2}{2U}\sum_i (\partial_t \varphi_i)^2 - J\sum_{\langle i,j\rangle}\big(1 - \cos(\varphi_i - \varphi_j)\big), \tag{2.24}$$

which also has units of energy. Now we can take the continuum limit in D space dimensions,

$$a^D \sum_i \mapsto \int \mathrm{d}^D x, \qquad \varphi_i - \varphi_j \to a\nabla\varphi(x), \tag{2.25}$$

where a is the lattice constant. After this and expanding the cosine to leading order we find,

$$L = \frac{1}{a^D}\frac{\hbar^2}{2U}\int \mathrm{d}^D x\, (\partial_t\varphi)^2 - \frac{J}{2}\frac{1}{a^D}\int \mathrm{d}^D x\, a^2(\nabla\varphi)^2. \tag{2.26}$$

The partition function is $Z = \int \mathscr{D}\varphi\ \mathrm{e}^{\mathrm{i}/\hbar S}$, with S the action,

$$S = \int \mathrm{d}t\, L = \frac{1}{a^D}\int \mathrm{d}t\mathrm{d}^D x\, \Big[\frac{\hbar^2}{2U}(\partial_t\varphi)^2 - \frac{J}{2}a^2(\nabla\varphi)^2\Big]. \tag{2.27}$$

Thus, the Bose-Hubbard model at zero chemical potential is equal to the XY-model. We proceed to imaginary time $t = \mathrm{i}\tau$ to give the partition function with the Euclidean action $Z = \int \mathscr{D}\varphi\ \mathrm{e}^{-\frac{1}{\hbar}S_\mathrm{E}}$ where,

$$\begin{aligned} S_\mathrm{E} &= \frac{1}{a^D}\int \mathrm{d}\tau\mathrm{d}^D x\, \Big[-\frac{\hbar^2}{2U}(\partial_\tau\varphi)^2 - \frac{J}{2}a^2(\nabla\varphi)^2\Big] \\ &\equiv \int \mathrm{d}\tau\mathrm{d}^D x\, \frac{1}{2}Ja^{2-D}\, \Big[-\frac{1}{c_\mathrm{ph}^2}(\partial_\tau\varphi)^2 - (\nabla\varphi)^2\Big]. \end{aligned} \tag{2.28}$$

2.3.3 Equivalence to superfluid/Mott insulator transition

This is to be compared with the quantum action for a superfluid (cf. Eq. (3.13) in Ref. [54]),

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \left[-\frac{1}{2}\hbar^2 \kappa (\partial_\tau \varphi)^2 - \frac{1}{2}\hbar^2 \frac{\rho_{\mathrm{s}}}{m^*} (\nabla \varphi)^2 \right]. \tag{2.29}$$

Hence we identify the compressibility $\kappa = \frac{1}{Ua^D}$, the superfluid density divided by the boson mass $\frac{\rho_{\mathrm{s}}}{m^*} = \frac{Ja^{2-D}}{\hbar^2}$ and the superfluid velocity $c_{\mathrm{ph}} = \frac{a}{\hbar}\sqrt{UJ}$. Defining the relativistic derivative $\partial_\mu^{\mathrm{ph}} = (\frac{1}{c_{\mathrm{ph}}}\partial_\tau, \nabla)$, we find a convenient form of the action,

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2} J a^{2-D} \; (\partial_\mu^{\mathrm{ph}} \varphi)^2. \tag{2.30}$$

One can worry what happened to the on-site repulsion term $\sim U$? In fact, in the relativistic picture everything is contained in the fluctuations of the phase variable φ. In the superfluid the fluctuations are suppressed. But for small values of $J/U \sim J^2/c_{\mathrm{ph}}^2$, the temporal correlations $\partial_\tau \varphi$ fluctuate heavily, signalling the arbitrary creation and annihilation of vortex excitations. Thus, the destroying the superfluid takes us across the phase transition, and the disordered superfluid is equivalent to the Bose-Mott insulating state.

Indeed, this model has two stable fixed points, separated by a continuous phase transition governed by XY-universality in $D+1$ dimensions [17, 28, 32, 33, 54]. The scaling limit physics of the two stable states can be discerned by inspecting the $g \sim \sqrt{U/J} \to 0$ (weak coupling) and $g \sim \sqrt{U/J} \to \infty$ limits. In the weak coupling limit the $U(1)$ field breaks symmetry spontaneously and the theory describes the superfluid state. The small fluctuations in the phase field φ correspond either with a single Goldstone boson corresponding with the zero sound mode of the superfluid, or with the spin-wave of the quantum XY model. The strong coupling limit has an integer number of bosons n^0 per site as imposed by the choice of chemical potential. The effect of the hopping will be to create a 'doublon' n^0+1 and 'holon' n^0-1 pair on two different sites i and j: $n_i^0 n_j^0 \to (n^0-1)_i (n^0+1)_j$. This will cost an energy U: the system turns into a Bose-Mott insulator.

2.3.4 Emergent gauge invariance

The localization of the bosons implies a phenomenon that is well-known in condensed matter physics [57, 58]. This simple Mott localization has in fact

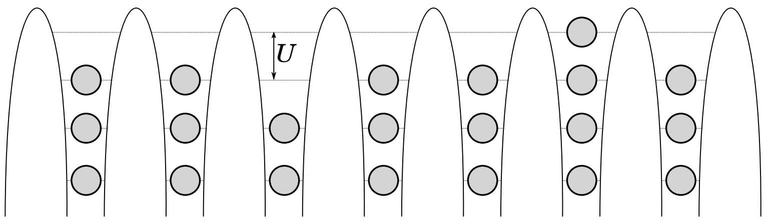


Figure 2.3: Cartoon picture of a Mott insulator. Because of the integer number of particles per site, and their strong mutual repulsion, the ground state has the same number of particles on each site. The Mott gap energy must be paid both for adding and for removing a particle. An elementary excitation is creating a doublon–holon pair; both the doublon and the holon can then propagate throughout the system without further energy penalty. This is the doublet of gapped modes.

a profound consequence: it causes a 'dynamical' emergence of a gauge symmetry. The global $U(1)$ symmetry controlling the weak coupling limit gets 'spontaneously' gauged into a compact $U(1)$ local symmetry. In the superfluid $b_i^\dagger \to \sqrt{n^0} e^{i\varphi_i}$ and the phase φ_i can undergo the global $U(1)$ symmetry transformation of the superfluid. However, in the strongly-coupled Mott insulator the number operator of the bosons is sharply quantized on every site,

$$\hat{n}_i |\Psi(\text{Mott})\rangle = n_0 |\Psi(\text{Mott})\rangle \tag{2.31}$$

and this in turn implies a gauge invariance under the multiplication by an arbitrary phase α_i,

$$\begin{aligned} b_i^\dagger &\to e^{i\alpha_i} b_i^\dagger \\ b_i &\to e^{-i\alpha_i} b_i \\ \hat{n}_i = b_i^\dagger b_i &\to \hat{n}_i. \end{aligned} \tag{2.32}$$

This is the celebrated 'stay-at-home' $U(1)$ gauge invariance that has played a prominent role in the various gauge theories for high-T_c superconductivity developed for the fermionic incarnation of the Hubbard model [58]. We will return to this interesting feature in chapter 6.

2.3.5 Mode content of the Bose-Mott insulator

One can also immediately read off the nature of the collective modes of the Bose-Mott insulator from the strong coupling limit. One can either remove or add a boson and the holon and doublon that are created can just freely delocalize on the lattice giving rise to massive excitations with a mass $\simeq U/2$ given that the chemical potential is in the middle of the Mott gap (see Fig 2.3). The continuum theory we are dealing with requires that the length scales are large compared to the lattice constant, a regime that is quite different from the lattice cut-off regime exposed here. The continuum description becomes literal close to the quantum phase transition but given adiabatic continuity we know that the strong coupling limits are still representative for the mode counting and so forth. Starting close to the critical coupling on the Mott side, the Mott physics takes over from the critical regime at the correlation length (or time). At larger scales the stay-at-home gauge invariance takes over, although it now involves a volume with a dimension set by the correlation length. Accordingly, one will find the pair of degenerate propagating holon/doublon modes which appear as bound states that are pulled out of the critical continuum [31]. Similarly one finds on the superfluid side of the quantum critical point the single zero sound Goldstone boson at energies less than the scale set by the renormalized superfluid stiffness that disappears at the quantum critical point.

The simple features we have discussed in this section are generic and completely independent of the dimensionality of spacetime. Although perhaps unfamiliar, they are easily identified in the context of the standard Abelian-Higgs duality in 2+1d as discussed in the next section. In turn, they will be quite helpful in giving a firm hold in our construction of the duality in higher dimensions.

2.3.6 Charged superfluid

If we are interested in charged superfluids, i.e. superconductors, we must minimally couple to the electromagnetic potential, or photon field. Now we must recall that the gauge-covariant derivative acts on the superfluid order parameter, which is a complex scalar field $\Psi = \sqrt{\rho_\mathrm{s}}\mathrm{e}^{\mathrm{i}\varphi}$. Hence, the minimal coupling prescription in the London limit (ρ_s constant), is,

$$|\partial_\mu^{\mathrm{ph}}\Psi|^2 \to |(\partial_\mu^{\mathrm{ph}} - \mathrm{i}\frac{e^*}{\hbar}A_\mu^{\mathrm{ph}})\Psi|^2 = \rho_\mathrm{s}(\partial_\mu^{\mathrm{ph}}\varphi - \frac{e^*}{\hbar}A_\mu^{\mathrm{ph}})^2. \tag{2.33}$$

Here e^* is the electric charge of one boson, so of one Cooper pair. To preserve gauge invariance, the temporal component of the gauge potential should have the same velocity factor as the covariant derivative, and therefore we define $A_\mu^{\rm ph} = (-{\rm i}\frac{1}{c_{\rm ph}}V, \mathbf{A})$. Then we include the Maxwell action for the dynamics of the electromagnetic field, which is governed of course by the speed of light c. Defining the electromagnetic field tensor $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ where $\partial_\mu = (\frac{1}{c}\partial_\tau, \nabla)$ and $A_\mu = (-{\rm i}\frac{1}{c}V, \mathbf{A})$, the total action is,

$$S_{\rm E} = \int {\rm d}\tau {\rm d}^D x \; [-\frac{1}{2} J a^{2-D} \; (\partial_\mu^{\rm ph}\varphi - \frac{e^*}{\hbar}A_\mu^{\rm ph})^2 - \frac{1}{4\mu_0}F_{\mu\nu}^2]. \tag{2.34}$$

The identification of the dimensionful constant μ_0 as the permeability of the vacuum in units of N/A^2 is accurate only in 3+1 dimensions, but that is the case we will be mostly interested in anyway. We have established the Euclidean action of the superconductor. The equations of motion obtained by variation with respect to A_n are for instance (in real time, and substituting $Ja^{2-D} = \hbar^2\rho/m^*$),

$$\frac{1}{c^2}\partial_t(-\partial_t A_n - \partial_n V) - \frac{1}{\mu_0}\partial_m(\partial_m A_n - \partial_n A_m) - \frac{e^*\hbar\rho}{m^*}(\partial_m\varphi - \frac{e^*}{\hbar}A_m) = 0, \tag{2.35}$$

which is one of the Ginzburg–Landau equations.

2.3.7 Dimensionless variables

It is sometimes useful to rescale all variables to be dimensionless. For our purposes this pertains especially to the charge of the dual gauge field (see next section) which has to be 1 in these dimensionless units. Starting from Eq. (2.34), we define the dimensionless variables denoted by a prime,

$$S_{\rm E} = \hbar S'_{\rm E}, \quad x = ax', \quad \tau = \frac{a}{c_{\rm ph}}\tau', \quad A_m = \frac{\hbar}{e^* a}A'_m. \tag{2.36}$$

We shall suppress the primes from now on. The dimensionless version of the action Eq. (2.34) reads,

$$S_{\rm E} = \int {\rm d}\tau {\rm d}^D x \; -\frac{1}{2g}(\partial_\mu^{\rm ph}\varphi - A_\mu)^2 - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.37}$$

Here the coupling constants are,

$$\frac{1}{g} = \frac{Ja}{\hbar c_{\rm ph}}, \qquad \frac{1}{\mu} = \frac{\hbar a^{D-3}}{\mu_0 c_{\rm ph} e^{*2}}. \tag{2.38}$$

The first is always dimensionless, the last is dimensionless if $D = 3$, in other dimensions one has to come up with a suitable replacement for the magnetic constant μ_0. For the chargeless superfluids one lets $e^* \to 0$, which will leave only,

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \ -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}} \varphi)^2. \tag{2.39}$$

2.4 Vortex duality in 2+1 dimensions

We will now perform the duality transformation of the superfluid action in 2+1 dimensions, and show how the phase transition is described as the proliferation of vortices. In 2+1 dimensions the vortices are pointlike, and trace out world lines in spacetime. Therefore their collective behaviour is captured by just a quantum field theory as for ordinary point particles. For simplicity we will proceed for the uncharged superfluid; the extension to a superconductor is straightforward by having the photon field tag along the duality transformation, the results of which are briefly mentioned at the end of this section. Here we show that vortices in a superfluid are just like charged particles with Coulomb interactions mediated by a dual gauge field. The phase transition is the proliferation of the vortices, causing the interactions to become short-ranged due to the Anderson–Higgs mechanism, which is exactly like a superconductor in this analogy.

2.4.1 Dual variables

The quantum partition sum associated with the Euclidean action Eq. (2.39) is the path integral,

$$Z = \int \mathscr{D}\varphi \ \mathrm{e}^{-\int \mathscr{L}} = \int \mathscr{D}\varphi \ \mathrm{e}^{-\int -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}} \varphi)^2}. \tag{2.40}$$

For small g fluctuations of the phase φ are costly and will be much suppressed. This is the superfluid, and ϕ is the zero sound or phase mode. Even though this is already a free theory, we can still linearize for the variable φ by the introduction of an auxiliary variable w_μ through a Hubbard–Stratonovich transformation,

$$Z_{\mathrm{dual}} = \int \mathscr{D}\varphi \mathscr{D} w_\mu \ \mathrm{e}^{-\int \frac{1}{2} g w_\mu w_\mu - w_\mu \partial_\mu^{\mathrm{ph}} \varphi}, \tag{2.41}$$

The auxiliary field w_μ is a dual variable, in the sense that for this field the coupling constant is g instead of $1/g$. In canonical language going from φ

to w_μ amounts to a Legendre transform; the dual variables are in fact the canonical momenta,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial_\mu^{\mathrm{ph}}\varphi)} = \frac{1}{g}\partial_\mu^{\mathrm{ph}}\varphi. \tag{2.42}$$

The w_μ is also the Noether current related to the tranformation $\varphi(x) \to \varphi(x)+\alpha$ under which (2.40) is invariant. In the superfluid we identify this as the supercurrent. Integrating out the auxiliary field w_μ from Eq. (2.41) will give back the original partition sum Eq. (2.40).

2.4.2 Dual gauge field

When vortices are present in the superfluid, the otherwise smooth phase variable φ is singular inside the core region (see Fig. 2.1(b)). We therefore split it into a smooth and a multivalued part: $\varphi = \varphi_{\mathrm{smooth}} + \varphi_{\mathrm{MV}}$. The multivalued part denotes vortices of winding number N via,

$$\oint \mathrm{d}\varphi_{\mathrm{MV}} = 2\pi N. \tag{2.43}$$

We have,

$$Z_{\mathrm{dual}} = \int \mathscr{D}\varphi_{\mathrm{MV}}\mathscr{D}\varphi_{\mathrm{smooth}}\mathscr{D}w_\mu \; \mathrm{e}^{-\int \mathscr{L}_{\mathrm{dual}}}, \tag{2.44}$$

$$\mathscr{L}_{\mathrm{dual}} = \frac{1}{2}g w_\mu w_\mu - w_\mu \partial_\mu^{\mathrm{ph}}\varphi_{\mathrm{MV}} - w_\mu \partial_\mu^{\mathrm{ph}}\varphi_{\mathrm{smooth}}. \tag{2.45}$$

We can perform partial integration on the term with the smooth part of the phase field to find,

$$\mathscr{L}_{\mathrm{dual}} = \frac{1}{2}g w_\mu w_\mu - w_\mu \partial_\mu^{\mathrm{ph}}\varphi_{\mathrm{MV}} - (\partial_\mu^{\mathrm{ph}} w_\mu)\varphi_{\mathrm{smooth}}. \tag{2.46}$$

Now we can integrate out $\varphi_{\mathrm{smooth}}$ as a Lagrange multiplier for the constraint $\partial_\mu^{\mathrm{ph}} w_\mu = 0$. This constraint expresses the conservation of supercurrent and is in fact the continuity equation for the supercurrent $\partial_t w_t + \nabla \cdot \mathbf{w} = 0$. Thus we see that the conservation of supercurrent is due to the smoothness of the phase field. We can explicitly enforce this constraint by expressing it as the curl of a non-compact $U(1)$ gauge field,

$$w_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda, \tag{2.47}$$

which is invariant under the addition of the gradient of any smooth scalar field $\varepsilon(x)$,

$$b_\lambda(x) \to b_\lambda(x) + \varepsilon(x). \tag{2.48}$$

If we substitute this into Eq. (2.46) we find,

$$Z_{\text{dual}} = \int \mathscr{D}\varphi_{\text{MV}} \mathscr{D} b_\lambda \mathscr{F}(b_\lambda)\, \mathrm{e}^{-\int \mathscr{L}_{\text{dual}}}, \tag{2.49}$$

$$\mathscr{L}_{\text{dual}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda)^2 - \epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda \partial_\mu^{\text{ph}} \varphi_{\text{MV}}. \tag{2.50}$$

Here $\mathscr{F}(b_\lambda)$ is a gauge-fixing factor which we leave implicit from now on. Because the gauge field is smooth everywhere, we can perform integration by parts to leave,

$$\mathscr{L}_{\text{dual}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda)^2 + b_\lambda \epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} \partial_\mu^{\text{ph}} \varphi_{\text{MV}} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda)^2 - b_\lambda J_\lambda^{\text{V}}. \tag{2.51}$$

Here we have defined the vortex current $J_\lambda^{\text{V}} = \epsilon_{\lambda\nu\mu}\partial_\nu^{\text{ph}}\partial_\mu^{\text{ph}}\varphi_{\text{MV}}$ as in Eq. (2.16). If we use the identity $(\epsilon_{\mu\nu\lambda}\partial_\nu^{\text{ph}} b_\lambda)^2 = \frac{1}{2}(\partial_\mu b_\nu - \partial_\nu b_\mu)^2 \equiv \frac{1}{2} f_{\mu\nu}^2$, this becomes,

$$\mathscr{L}_{\text{dual}} = \frac{1}{4} g f_{\nu\lambda}^2 - b_\lambda J_\lambda^{\text{V}}. \tag{2.52}$$

This looks exactly like Maxwell electromagnetism in 2+1 dimensions, with the fluctuating dual gauge fields b_λ playing the role of the photon fields, and the vortex currents J_λ^{V} are like electrically charged monopole sources. Note that in these dimensionless units the charge of the coupling is 1. Because of this correspondence we call the superfluid in this context the Coulomb phase of the dual gauge fields. This equivalence is accidental in 2+1 dimensions, as we shall discover in the next chapter.

2.4.3 Mode content of the Coulomb phase

To see that we indeed retrieve electromagnetism for the dual fields, let us examine the two-point functions for the dual gauge field. In this context it is most convenient to go to a coordinate system in which the spatial directions are rotated to a longitudinal and a transversal component, see Fig. 1.3 on page 14. In this (τ, L, T) coordinate system, the momentum vector reads $p_\mu = (\frac{1}{c_{\text{ph}}}\omega, q, 0)$. We are free to choose the Coulomb gauge $\partial_l b_l = q b_L = 0$, such that the longitudinal component is removed. The Lagrangian for the remaining components is,

$$\mathscr{L}_{\text{dual}} = \frac{g}{2} q^2 b_\tau b_\tau + \frac{g}{2}\left(\frac{1}{c_{\text{ph}}^2}\omega^2 + q^2\right) b_T b_T - b_\tau J_\tau^{\text{V}} - b_T J_T^{\text{V}}. \tag{2.53}$$

We see that the vortex sources emit gauge fields with propagators,

$$\langle\langle b_\tau(p) b_\tau(0)\rangle\rangle = \frac{1}{gq^2}, \tag{2.54}$$

$$\langle\langle b_T(p) b_T(0)\rangle\rangle = \frac{1}{g(\frac{1}{c_{\mathrm{ph}}^2}\omega^2 + q^2)} = \frac{1}{gp^2}. \tag{2.55}$$

We recover the static long-range Coulomb force with a $\frac{1}{|\mathbf{r}|}$-potential, and the single, transversely polarized massless propagating photon of 2+1d EM, respectively. The static 'photon' reflects the well known fact that static vortices in 2D interact via a Coulomb potential, and the transversal photon is just zero sound while in the dual 'force' language it becomes explicit that this Goldstone boson can propagate forces between sources and sinks of supercurrent. We stress again that this correspondence between the 'XY universe' and 2+1d EM with scalar matter is quite accidental for the 2+1d case.

2.4.4 Vortex proliferation

The description above is suitable for one or several remote vortices in the superfluid that have long-range interactions. Upon increasing the coupling constant g, the phase fluctuations in Eq. (2.40) increase, which implies also that the spontaneous creation and annihilation of vortex–anti-vortex pairs becomes more frequent. These pairs are also longer-lived. The best description is in terms of spacetime loops of the world lines of vortex–anti-vortex pairs. The coupling constant is then as the inverse line tension, and an increasing coupling constant allows the loops to become larger and larger. At the critical point g_c the loops will have grown of the system size, and vortex lines permeating the system can freely form and disappear. This is characteristic for a condensate of particles, just as Cooper pairs can be freely extracted from the superconducting vacuum. Thus, such a "tangle of vortex world lines" is indeed equivalent to a "condensate of vortices".

This statement can be made very precise, and is in fact the central topic of Kleinert's textbooks [28, 42]. It is easiest to go to the lattice, and calculate the energy cost of meandering vortex world lines as chains of lattice links. We will not repeat this treatment here, but only cite the result, which is also established in [31, 40]. From the dual perspective it is immediately clear what will happen: the vortex condensate forms a medium (liquid) to which the dual gauge fields are minimally coupled. This just follows Ginzburg–

Landau theory of §2.1. This collective vortex condensate field is represented by a complex (dis)order parameter $\Phi = |\Phi|e^{i\phi}$, the amplitude of which corresponds to the density of the vortex fluid. The disorder parameter is related to the vortex current as,

$$J^{\mathrm{V}}_{\lambda} = \mathrm{i}\bar{\Phi}\partial_{\lambda}\Phi - \mathrm{i}(\partial_{\lambda}\bar{\Phi})\Phi. \tag{2.56}$$

The minimal coupling to the dual gauge field $\sim b_{\lambda} J^{\mathrm{V}}_{\lambda}$ is now reflected by the new Lagrangian,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\lambda})^2 + \frac{1}{2}|(\partial^{\mathrm{ph}}_{\lambda} - \mathrm{i}b_{\lambda})\Phi|^2 + \frac{\tilde{\alpha}}{2}|\Phi|^2 + \frac{\tilde{\beta}}{4}|\Phi|^4. \tag{2.57}$$

Here we have added Ginzburg–Landau potential terms. The dual gauge field b_{κ} clearly acts just as the electromagnetic field would in a superconductor. Thus, when $\tilde{\alpha} < 0$, the disorder parameter obtains a vacuum expectation value $|\Phi| = \sqrt{\frac{|\tilde{\alpha}|}{\tilde{\beta}}} \equiv \Phi_{\infty}$. Only the phase ϕ remains as a degree of freedom, it represents the density fluctuations of the vortex condensate, i.e. the compression mode of the vortex liquid.

2.4.5 Mode content of the vortex condensate

How to count the modes of the dual superconductor? It is just the usual business for the Anderson–Higgs mechanism. Choose coordinates $(\|, \perp, T)$ with $\|$ parallel to the spacetime momentum p_{μ}, and $\perp$ perpendicular to both $\|$ and T (Fig. 1.3). In this system the momentum becomes $p_{\mu} = (p, 0, 0)$. We see that the condensate phase ϕ couples only to the parallel direction,

$$\begin{aligned} \mathscr{L} &= \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_{\nu} b_{\lambda})^2 + \frac{1}{2}|(\partial^{\mathrm{ph}}_{\lambda} - \mathrm{i}b_{\lambda})\Phi|^2 \\ &\to \frac{1}{2}(p^2 + \Phi^2_{\infty})(b^2_{\perp} + b^2_{T}) + \frac{1}{2}\Phi^2_{\infty}(p\phi - b_{\|})^2. \end{aligned} \tag{2.58}$$

This action is invariant under the combined gauge transformations $b_{\|} \to b_{\|} + p\varepsilon$ and $\phi \to \phi + \varepsilon$. One possible gauge fix is the unitary gauge $\phi \equiv 0$ and in this way one shuffles the condensate mode into the "longitudinal photon" $b_{\|}$, which then becomes a true physical degree of freedom. This is sometimes referred to as the gauge field "eating the Goldstone boson". Alternatively, we can choose the Lorenz gauge $pb_{\|} \equiv 0$, in which this degree of freedom is indeed seen to originate in the condensate field ϕ. The field $b_{\perp}$ corresponds to the now short-ranged Coulomb force, and A_T and $A_{\|}$ form a degenerate

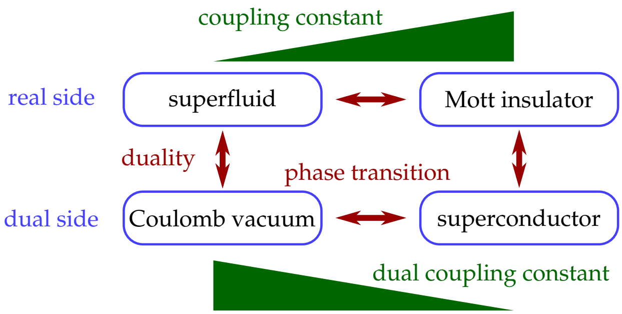


Figure 2.4: Overview of duality relations. The vertical correspondence is the duality; the horizontal is the phase transition. The dual side is in terms of the interactions between vortices: individual sources interacting via the Coulomb law; or as a superconducting condensate that effects a Higgs mechanism for the dual gauge fields. When the real coupling constant is small (the superfluid), the dual coupling constant, which is the string tension of the vortex world lines, is large and vice versa.

pair of massive propagating modes. This matches precisely the expectations that follow from the Bose-Hubbard model; in the superfluid/Coulomb phase a single massless propagating mode is present corresponding with the phase mode/photon. In the dual superconductor one finds a pair of massive propagating modes corresponding with the Higgsed transversal and longitudinal photons: these correspond with the holon and doublon excitations of the Bose-Mott insulator while the Higgs mass of the dual superconductor just codes for the Mott gap. The fate of the second mode when going to the superfluid phase was discussed in Ref. [59].

This is a good point to reflect on the correspondences in the vortex duality, see figure 2.4. The superfluid is dual to the Coulomb vacuum where the vortices take the role of the monopole charges, and the dual gauge fields are like photons. The phase transition is from the superfluid to the Bose-Mott insulator which has two gapped modes. On the dual side this is the superconductor with two massive dual photons. In duality parlance, it is sometimes said that the superfluid is dual to a superconductor; strictly speaking this is incorrect, but the since the strength of the dualities is in phase transitions, one often compares the weak-coupling phases of the dual sides.

In the next chapter we shall explore how this generalizes to higher dimensions. It turns out that not the dual gauge field but rather the supercurrent itself is the quantity containing the important information.

2.4.6 Duality squared equals unity

Just to complete the duality exercise, we can ask the question whether there can also be the analogues of Abrikosov vortices in the dual superconductor? This is indeed the case, and it goes in exactly the same way as above. First, introduce an auxiliary field v_μ, such that,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda)^2 + \frac{1}{2}\Phi^2_\infty(\partial^{\mathrm{ph}}_\lambda \phi - b_\lambda)^2, \tag{2.59}$$

turns into,

$$\mathscr{L} = \frac{1}{2} g(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda)^2 - \frac{1}{2\Phi^2_\infty} v^2_\mu - v_\mu(\partial^{\mathrm{ph}}_\mu \phi - b_\mu). \tag{2.60}$$

If there are dual vortices, we should split the dual phase field into a smooth and a multivalued part, $\phi = \phi_{\mathrm{smooth}} + \phi_{\mathrm{MV}}$. The smooth part can be integrated out as a Lagrange multiplier for the constraint $\partial^{\mathrm{ph}}_\mu v_\mu = 0$. This constraint can be enforced explicitly by introducing yet another gauge field $v_\mu = \epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda$. The Lagrangian now reads,

$$\mathscr{L} = \frac{1}{2} g w^2_\mu - \frac{1}{2\Phi^2_\infty}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 + z_\lambda \mathscr{J}^{\mathrm{V}}_\lambda + z_\lambda w_\lambda. \tag{2.61}$$

Here $\mathscr{J}^{\mathrm{V}}_\lambda = \epsilon_{\lambda\nu\mu}\partial^{\mathrm{ph}}_\nu \partial^{\mathrm{ph}}_\mu \phi_{\mathrm{MV}}$ is the vortex current, and we have resubstituted $w_\mu = \epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu b_\lambda$; the last term indicates how the original supercurrent couples to the z-degrees of freedom. It is at this point possible to integrate out the supercurrents w_μ, to leave a Meissner/Higgs term for the gauge fields $\frac{1}{2g} z^2_\lambda$. This indicates that the interactions between vortices $\mathscr{J}^{\mathrm{V}}_\lambda$ are Meissner screened, as it should be in a (dual) superconductor.

Instead, suppose that the vortices proliferate, then they form a condensate with order parameter Ψ, with its own Ginzburg–Landau potential,

$$\mathscr{L} = -\frac{1}{2g} z^2_\lambda - \frac{1}{2\Phi^2_\infty}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 - \frac{1}{2}|(\partial_\lambda - \mathrm{i} z_\lambda)\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.62}$$

We can now rescale the gauge field $z_\lambda \to \Phi_\infty z_\lambda$, to leave,

$$\mathscr{L} = -\frac{\Phi^2_\infty}{2g} z^2_\lambda - \frac{1}{2}(\epsilon_{\mu\nu\lambda}\partial^{\mathrm{ph}}_\nu z_\lambda)^2 - \frac{1}{2}|(\partial_\lambda - \mathrm{i}\Phi_\infty z_\lambda)\Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.63}$$

The vortex condensate will destroy the dual order, with the effect that the dual superfluid density $\Phi_\infty \to 0$. In the above Lagrangian the order parameter Ψ then decouples from the dual gauge field, and we end up with just the

Landau action for a superfluid,

$$\mathscr{L} = -\frac{1}{2}|\partial_\lambda \Psi|^2 - \frac{1}{2}\alpha|\Psi|^2 - \frac{1}{4}\beta|\Psi|^4. \tag{2.64}$$

Concluding, the phase transition from the dual superconductor to *its* disordered phase is again the superfluid with which we started out. Thus indeed "duality$^2 = 1$".

Note that we have seen above that vortices can form in the dual superconductor, so there are vortices in the Bose-Mott insulator. This is a bit surprising result, that has been overlooked for quite a while. It will be a topic of interest in chapter 3 and moreover 5.

2.4.7 Charged vortex duality

For charged superfluids, i.e. superconductors, one can do the same calculation, without many changes. The starting point is the Ginzburg–Landau action Eq. (2.34), which in dimensionless units reads,

$$S_{\mathrm{E}} = \int \mathrm{d}\tau \mathrm{d}^D x \; -\frac{1}{2g}(\partial_\mu^{\mathrm{ph}}\varphi - A_\mu^{\mathrm{ph}})^2 - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.65}$$

Here $1/\mu = \frac{\hbar a^{D-3}}{\mu_0 c_{\mathrm{ph}} e^{*2}}$. The chargeless supercurrent is defined as the canonical momentum,

$$w_\mu = -\frac{\partial \mathscr{L}}{\partial(\partial_\mu^{\mathrm{ph}}\varphi)} = \frac{1}{g}(\partial_\mu^{\mathrm{ph}}\varphi - A_\mu), \tag{2.66}$$

and is related in dimensionful units to the familiar charged supercurrent as $J_\mu^{\mathrm{s}} = \frac{e^*}{\hbar} w_\mu$. Separating the multivalued part of the phase field, integrating out the smooth part, and enforcing the conservation of supercurrent by introducing the dual gauge fields leads to the equivalent of Eq. (2.51),

$$\mathscr{L}_{\mathrm{dual}} = \frac{1}{2}g(\epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda)^2 - b_\lambda J_\lambda^{\mathrm{V}} + A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.67}$$

Here we see that the photon field simply couples to the supercurrent $w_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda$ as it should. One could integrate out the dual gauge field to find an interaction between the vortex currents J_λ^{V} that is Meissner screened due to the electromagnetic field. But instead we proceed with the duality, where basically we just keep around the last two terms in the above expression. Thus, after proliferation of the vortices we have,

$$\mathscr{L} = \frac{1}{2}g(\epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda)^2 + \frac{1}{2}\Phi_\infty^2(\partial_\lambda^{\mathrm{ph}}\phi - b_\lambda)^2 + A_\mu \epsilon_{\mu\nu\lambda}\partial_\nu^{\mathrm{ph}} b_\lambda - \frac{1}{4\mu}F_{\mu\nu}^2. \tag{2.68}$$

Here we have assumed the dual London limit $|\Phi| = \Phi_\infty$ everywhere. One could again integrate out the dual gauge field to find the electromagnetic response for the Mott insulator. We will see in §5.A.4 that we indeed find gapped poles for the conductivity instead of the delta-function response of the superconductor.