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Patterns in natural systems

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Long wave length vegetation patterns¹

5.1 Introduction

Human activity and climate change have stressed life on the Earth's surface. Especially in the drylands the effects are tangible, as persistent soil degradation has led to barren areas, unsuitable for agriculture. To combat this process of desertification is set as one of seventeen global goals on the United Nations' sustainable development agenda [119]. The need for a better insight in arid ecosystems is thus widely acknowledged. In the absence of grazing, vegetation growth is mainly limited by scarcity of water and nutrient. Therefore, homogeneously vegetated areas may turn into bare soil as a result of decreasing precipitation. Several intermediate stages, where the terrain is partly vegetated and partly barren, are observed, all with strikingly regular patterns [106]. In the case of a flat terrain, the transition from homogeneous vegetation to bare soil goes via hole, labyrinth, and spot patterns, [132, 138]. On a sloped terrain, however, the labyrinth patterns self-organize spatially as stripe patterns, parallel to the terrain's contours, [171]. Moreover, patterns on flat terrain have been reported to be stationary, while the stripe patterns slowly move uphill, [88]. Although the evolution from fertile to barren soil is not instantaneous, it is catastrophic in the sense that it is nearly irreversible; an increase of water availability does not automatically cultivate barren terrain to be fertile again. Therefore, vegetation patterns not only announce an early warning signal for desertification, but conversely provide a starting for the development of a homogeneously vegetated state.

The mathematical models developed to study the mechanisms responsible

¹The content of this chapter was submitted under the title *Spatially periodic multi-pulse patterns in a generalized Klausmeier-Gray-Scott model*.

for this pattern formation are mostly of reaction-diffusion type. Some models are two-component systems regarding the interaction of plant density with water density, [178, 133, 163, e.g.], while others include competition for surface water, see [75, 131], or [58, 114, e.g.]. The effect of grazing of the region in question is also expected to have a significant impact, and is taken into account in [58, 115, e.g.]. Typically, the patterns arise as a heterogeneous perturbation destabilizes the homogeneous, vegetated state; a Turing bifurcation, [156]. The catastrophe is then explained by the fact that there is a bistable parameter regime, in which both a vegetation pattern and the bare soil state are attracting [132, 138, e.g.]. As water availability decreases to some bifurcation point, the vegetation pattern destabilizes, causing it to collapse to the bare soil state; a process that is not simply reverted if water increases, because the bare soil state is also stable. The water availability should be increased as far as beyond the bistable regime to be able to configure into a vegetation pattern again. This is also referred to as hysteresis.

The focus of this article is on the striped vegetation patterns, which are observed on terrains with a gentle slope, [171]. This slope induces a downwards flow of precipitation in the form of surface water, which is then modeled by an advection term, and was introduced by C.A. Klausmeier in 1999, [88]. His model has two components, for water U , and vegetation V , and is subjected to a rescaling in [159], after which it is of the form

$$\begin{aligned} U_t &= A(1 - U) - UV^2 + CU_x, \\ V_t &= D\Delta_{x,y}V - BV + UV^2, \end{aligned} \tag{5.1}$$

where A controls water input, B is the natural death rate of vegetation, C is the rate at which water flows downhill and D is a diffusion coefficient. All coefficients are positive. The diffusion Δ models vegetation spread on a two-dimensional, infinite domain, $(x, y) \in \mathbb{R}^2$. Naturally, $t \in \mathbb{R}^+$ is the temporal coordinate.

The observation that vegetation patterns also occur in the absence of a slope, motivated an extension of model (5.1). By introducing (nonlinear) diffusion of water the spread is also modeled by a term that has no preferred direction. Extended comparisons in [159] show that nonlinear diffusion does not induce significant differences to the model. This motivates that in this article, we focus specifically on linear diffusion of water, i.e. we study

$$\begin{aligned} U_t &= \Delta_{x,y}U + A(1 - U) - UV^2 + CU_x, \\ V_t &= \delta^2\Delta_{x,y}V - BV + UV^2. \end{aligned} \tag{5.2}$$

Because it is natural to assume that water diffuses faster than vegetation, $0 < \delta \ll 1$ is a small parameter. The homogeneous background state $(U, V) = (1, 0)$ is

interpreted as a desert state; a constant water availability, yet with no vegetation. The main parameter, A , relates to precipitation, B models the vegetation's death rate and C models advection of water induced by a sloped terrain. Furthermore we assume $A, B > 0$, and $C \geq 0$. Under the assumption $A > 4B^2$, there are two more homogeneous steady states, $(U_{\pm}, V_{\pm})$ which are vegetated. We refer to (5.2) as the *generalized Klausmeier–Gray–Scott (gKGS) model*, because for $C = 0$, the model reduces to the Gray–Scott model, [62]. The model in [62] describes autocatalytic reactions, but as their setting is in a continuously stirred tank reactor, no diffusion is taken into account [62]. The Gray–Scott ODE system, extended with diffusion, is what is currently referred to as the Gray–Scott model, and was introduced in [19, 124].

It is widely known that both the Gray–Scott and the Klausmeier model exhibit a plethora of spatial patterns, [88, 123], and the gKGS-model captures patterns on both sloped (Klausmeier) and flat (Gray–Scott) terrains. To describe this phenomena mathematically, the existence of solutions with certain properties (wave number, amplitudes) is usually proved. However, stability analysis of these patterns is a necessary follow-up, as one does not expect to observe patterns which are unstable. To bridge the gap between existence of regular patterns in a mathematical setting and observable patterns, we consider the so-called *Busse balloon*, introduced in [16] and generalized to reaction-diffusion equations in [40, 117]. This balloon is a region in (wave number, parameter)-space in which patterns are stable against perturbations. In the context of vegetation patterns, the natural choice for the parameter is precipitation, A . Multistability of patterns with different wavelengths explains the hysteresis mentioned earlier. The onset of patterns, an extremum of the Busse balloon, occurs through a Turing bifurcation, while at the other end of the balloon, only patterns with very small wave numbers, are stable. The latter coincides with the homoclinic limit of periodic patterns confirmed by, for example [90, 91, 120, 157].

Research in the setting of system (5.2) has been fruitful at the interface of ecology and mathematics. Several Busse balloons for the gKGS system were derived using numerical continuation in [159]. In the same article, analytic control over the beginning of pattern formation is gained through the analysis of the complex Ginzburg–Landau equation associated with the pattern's amplitude. Although system (5.2) is written in a slightly different way in [143, 145], the instability mechanism for stripe patterns is unfolded further in these two articles. Using numerical continuation and simulations, [145] reports an extensive study of the destabilization of stripe patterns under the influence of a slowly decreasing parameter A . For flat terrains, the existence of nontrivial two-dimensional patterns of stripe and rhomb type and to some extent also their stability, is established in [143].

An analytical result is formulated in Theorem 3 of that article, which states that stripe patterns are unstable in the Gray–Scott model, i.e. on flat terrain, described by the $C = 0$ case of (5.2). For nonzero slope C , however, the stability is derived only using numerical continuation. These numerics suggest that as the slope of the terrain increases, the stability regime of stripe patterns spreads from the Turing bifurcation down. Eventually, for a slope large enough, even patterns with the smallest wave numbers are stable. Observations in nature also support this theory, because stripe patterns occur on sloped terrain [32]. Yet, analytical results on existence and stability of stripe patterns with small wave numbers have not been reported so far. To justify both numerics and to test the ability of the gKGS model to describe natural observations, this analysis is precisely the focus of the current paper. Our methods are based on those used for multiscale pulses in the Gray–Scott model, which as a long history of analytical studies that are discussed in section 5.1.1. Due to the advective term in (5.2), this truly is a nontrivial task; the symmetries that are used extensively in the proofs in for example [34, 39], break as soon as $C \neq 0$. Hence, the proof of existence of pulse patterns with small wave numbers in (5.2) – Theorem 5.3.1 of this paper – requires mathematical methods that go beyond the classical framework of geometric singular perturbation theory.

5.1.1 Existence and stability results of the Gray–Scott model

In this article, we exploit the singularly perturbed nature of (5.2), (recall that $0 < \delta \ll 1$) and use asymptotic analysis to prove existence and stability of stripe patterns. Our approach mainly follows the techniques developed for generalized Gray–Scott type systems in [34, 35, 36, 39, 117]. It involves geometric singular perturbation methods as well as an Evans function formulation, using the slow/fast structure of the problem and the evaluation of the solutions of a *nonlocal eigenvalue problem*.

In order to clearly compare the results, we summarize briefly the relevant results derived in for the Gray–Scott model, (5.2) with $C = 0$. In one spatial dimension, the existence of homoclinic N -pulses with $N \geq 1$ is largely covered in Theorem 4.1 of [39], see also Figure 5.1. These pulse solutions correspond to a single strip or N stripes of vegetation in an elsewhere bare terrain. An N -pulse is constructed as a homoclinic solution to the desert background state $(U, V) = (1, 0)$ that makes N fast excursions in a small spatial regime in which V is large. Already in the early publications on existence of pulses in the Gray–Scott model, the asymptotic scaling of the parameters and variables proves to be a crucial step in the analysis; a feature we also encounter for $C \neq 0$ throughout this article. In terms of the small parameter δ , the V -component of the pulse solutions constructed in [39] have asymptotically large amplitudes

in a small spatial interval, while being exponentially small outside the pulse region. The pulse solutions of [117] have less restrictions on relative magnitude of the U and V -component, and in [90, 91], an even broader parameter spectrum is analyzed in which pulse solutions exist. The rescaled, singularly perturbed system gives rise to a geometric singular perturbation analysis, where solutions of the limiting slow and fast system are concatenated according to classical Fenichel theory [82, 48, 49]. Periodic extensions of the N -pulses are characterized by slow components with long length scaled with a negative power of δ , in which the V -coordinate is exponentially small. That is, the slow/fast structure of these periodic solutions remains to be clearly distinguished. As stated in Theorem 4.2 of [39], there exists a 1-parameter family of stationary periodic solutions, and the proof relies highly on the reversible symmetry of (5.2) with $C = 0$. With a trivial extension in y -direction, the existence proof remains valid for two dimensions, but of course the extra spatial freedom gives way to more complex patterns. The existence of spot and multispot patterns on bounded two-dimensional domains have been proved in [168, 169]. The stability of homoclinic and periodic pulse patterns in the one-dimensional Gray–Scott is further analyzed in [34], where again scaling is essential, especially the relative magnitude of parameter B . From the stability problem, a nonlocal eigenvalue problem (NLEP) is formulated, and the eigenfunctions are constructed using matched asymptotic expansions. The NLEP is subsequently solved using hypergeometric functions, which reveals that for B small enough with respect to A , the 1-pulse is stable. There is a Hopf bifurcation through which it loses/gains stability in a specific B -regime, and the 1-pulse is unstable if B becomes larger than that. In the more general setting, the terminology of a Busse balloon for patterns in the Gray–Scott model was used as early as in [117]. Of course, as the authors are considering only one spatial dimension, the stability results do not necessarily hold for two-dimensional solutions, albeit with a trivial extension in the y -direction, as perturbations in the transverse direction are not taken into account. For spot and stripe patterns in two dimensions, the instability mechanisms of the Gray–Scott model are analyzed in [92] and in more detail in [22].

Another method to calculate the eigenvalues in a setting like this, is using the Evans function, see [1]. The embedding of this a priori formal stability analysis using the NLEP into the Evans function approach is established in [36], and studied in full detail in a more general context in [35].

As stated earlier, the one-dimensional patterns are trivially (constantly), extended in the transverse y -direction form 2D stripe patterns. To show existence of these patterns, no extra analysis is needed compared to the 1D existence proof. Stability, however, now needs to be tested against perturbations in both the x - and y -direction.

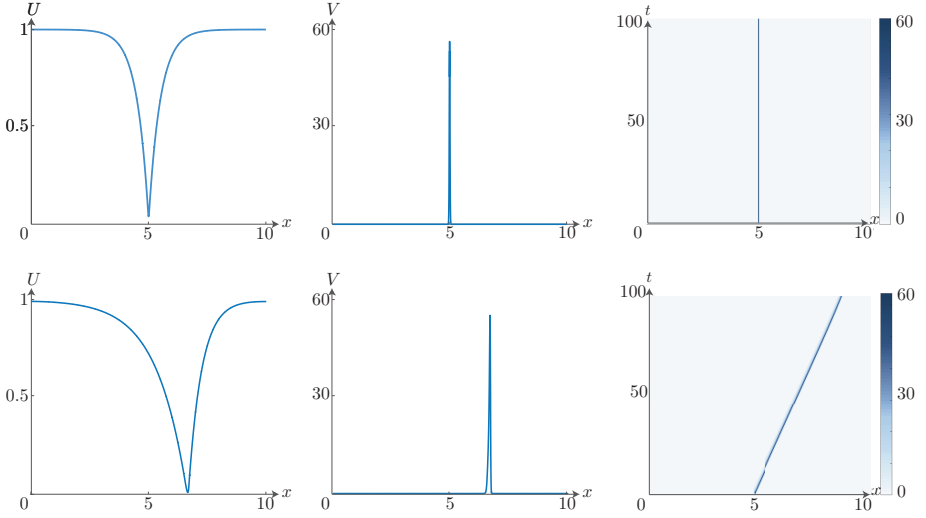


Figure 5.1: Simulations of homoclinic 1-pulses in system (5.2) in one spatial dimension x and $\delta = 0.01$ with $C = 0$ (Gray-Scott) in the top panels and $C = 1$ (gKGS) in the bottom panels. Left, a plots of U at $t = 50$ are depicted, in the middle V at $t = 50$ is depicted, and on the right we see a surface plot of V against both x and t . As parameter values we have used the sets $A = 4, B = 1.8, C = 0$ in the Gray-Scott case, and $A = 1.4, B = 0.6, C = 1$ in the gKGS case. Note that the V -pulse is stationary for $C = 0$, and travels with a constant speed to the right for $C = 1$. For the gKGS case, the values of A, B, C may, for instance, be equivalently represented by $a \approx 0.28, b = 0.25, c \approx 0.45$ with $\gamma \approx -0.174$ and $\beta \approx -0.190$ via rescaling (5.7).

This analysis is done in [143], and in the parameter regimes considered in that article, the transverse perturbations cause instability.

5.1.2 Outline of this article

In this article, we exploit the singularly perturbed nature of the system (recall δ is a small parameter), and analytically prove the existence of traveling stripe patterns on a sloped terrain, with a singular character. That is, we establish the existence of single pulses with an asymptotically large amplitude, as well as periodic extensions of these pulses. For the N -pulse, our approach goes along the lines of [39]. However, the advection term breaks the reversible symmetry so that the solutions now travel with a constant wavespeed, S , in x -direction, which we

introduce by setting $\chi = x - St$. Figure 5.1 shows a numerical simulation of the model where the symmetry breaking and constant travel speed is clearly visible. Furthermore, by focussing on stripe patterns and choosing the y -coordinate along the patterns, the spatial derivatives with respect to y vanish. Hence, we use the following system to construct stripe patterns,

$$\begin{aligned} 0 &= U_{\chi\chi} + A(1 - U) - UV^2 + (C + S)U_{\chi}, \\ 0 &= \delta^2 V_{\chi\chi} - BV + UV^2 + SV_{\chi}, \end{aligned} \tag{5.3}$$

where we make specific choices for the magnitude of the parameters and coordinates with respect to δ , in section 5.2.1. Note that the homogeneous, stationary state $(U, V) = (1, 0)$ is still a solution. Due to the loss of symmetry, the existence analysis for the spatially periodic solutions requires a novel approach. Using a contraction argument, we show that, for each given slope C , there is an interval of speeds S , for which traveling periodic pulse solutions exist.

In section 5.4, we analyze the Evans function to evaluate the eigenvalues corresponding to the stability of homoclinic and periodic pulse patterns. We require the constructed pulse patterns to have a sufficiently long wave length so that solutions to the stability problem are exponentially small in between the pulses. This implies that every family of eigenvalues one would expect in the stability analysis of spatially periodic patterns may be asymptotically approximated by a single discrete eigenvalue. See also section 5.4.1 and, for a rigorous validation, [31, 51, 52, 157]. Using the slow/fast splitting of the solutions, a leading order approximation of the eigenvalue problem is then formulated. This is equivalent with again an NLEP, that can be solved using hypergeometric functions. Similar to the results for the Gray–Scott model, the stability of patterns depends on the asymptotic magnitude of B , but the slope also affects the stability. The results are presented in Theorems 5.4.1–5.4.3. One of the findings is that perturbations with (asymptotically) large transverse wave numbers eventually destabilize any pattern. This confirms in some sense the numerical results of [143], which state that stripe patterns can only be stable on very steep slopes; a parameter regime that is not covered by the scaling restrictions of our asymptotic analysis. Moreover, one may argue that not all perturbations are representative to model ecological resilience of a pattern. Therefore, we also present Corollary 5.4.5, which analyzes the stability with respect to a more restrictive and specifically chosen function space. Since it is the transverse perturbations that form the main destabilization mechanism of stripe patterns, Theorems 5.4.2 and 5.4.3 restrict to one-dimensional patterns for which we derive saddle-node and Hopf bifurcations in several parameter regimes. Lastly, a detailed characterization of destabilization through a decrease of A or C is discussed in section 5.5. One of the conclusions that can be drawn is that indeed, a larger

value of both precipitation and slope is preferable for stripe patterns, confirming the natural observations.

5.2 Existence of homoclinic traveling multi-pulse patterns

In this section, we construct N -pulse solutions and periodic solutions of (5.3) that consist of slow and fast components. Specifically, these solutions have large χ -intervals for which the V -solution is close to zero. The building block for these solutions is the 1-pulse, i.e. a solution homoclinic to $(1, 0)$, with a localized pulse where V is $\mathcal{O}(1)$ or larger compared to δ and U small within the localized χ -interval. In this section, we will prove the following theorem.

Theorem 5.2.1 (Existence of 1-pulse patterns). *Let $A = \delta^{2\gamma}a$, $B = \delta^\beta b$, $C = \delta^\gamma c$, and $S = \delta^{1-\frac{3}{2}\beta+2\gamma}s$, where γ, β satisfy the following assumptions, also illustrated by Figure 5.2*

$$\mathbf{A1} \quad \beta < \gamma,$$

$$\mathbf{A2} \quad \beta > \frac{2}{3}(\gamma - 1),$$

$$\mathbf{A3} \quad \beta \geq 2(\gamma - 1),$$

and $a, b, c, s \in \mathcal{O}(1)$. Then, there exists a δ_0 , such that for all $\delta < \delta_0$ and a, b, c given, there exists a uniquely determined speed s such that the four-dimensional dynamical system (5.3) has a homoclinic solution to the critical point $(U, U_\chi, V, V_\chi) = (1, 0, 0, 0)$. This solution corresponds to a traveling wave solution $\gamma_{\text{hom}}(\chi)$ of (5.2), with speed s . Its spatial profile is in x biasymptotic to $(U, V) = (1, 0)$, and trivially extends into the y -direction. The orbit γ_{hom} consists of two slow components and a single fast excursion. The magnitude of the U and V components during the fast excursion are $\delta^{1+\frac{3}{2}\beta-\gamma}$ and $\delta^{-1-\frac{1}{2}\beta+\gamma}$, respectively. Moreover, the traveling speed of this wave is to leading order in δ given by

$$s = \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}}. \quad (5.4)$$

During the fast excursion, the U -component is constant to leading order, $U = \delta^{1+\frac{3}{2}\beta-\gamma}u_0$, with

$$u_0 = \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}. \quad (5.5)$$

In the remainder of this section, we prove this theorem using geometric singular perturbation techniques.

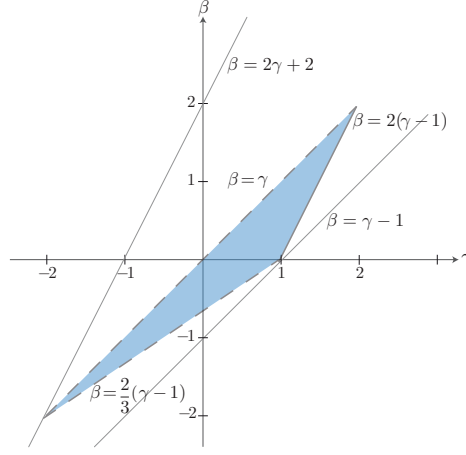


Figure 5.2: Scaling regime in the (γ, β) -plane, that satisfies assumptions **A1–A3** in Theorem 5.2.1.

5.2.1 Rescaling

The model proposed by Klausmeier, (5.1), is in dimensionalized form, meaning that it does not take into account the relative magnitudes of parameters and coordinates. However, in the case of Gray–Scott models, patterns are observed in regimes where not all quantities are $\mathcal{O}(1)$. In fact, scaling is essential for the analysis and plays an important role in the results. Hence, scaling of all parameters and coordinates is also appropriate in system (5.2), and here too, it is crucial for the analysis.

In the system (5.3), which already incorporates traveling solutions with a trivial y -extension, we introduce a new traveling coordinate,

$$\xi = \delta^{\frac{1}{2}\beta-1}\chi, \quad (5.6)$$

which we will refer to as the fast variable. The original χ will be referred to as the super slow variable. Furthermore, we introduce the following rescaled parameters and coordinates,

$$\begin{aligned} S &= \delta^{1-\frac{3}{2}\beta+2\gamma}s, & U &= \delta^{1+\frac{3}{2}\beta-\gamma}u, & V &= \delta^{-1-\frac{1}{2}\beta+\gamma}v \\ A &= \delta^{2\gamma}a, & B &= \delta^\beta b, & C &= \delta^\gamma c, \end{aligned} \quad (5.7)$$

where a, b, c, s, u, v are $\mathcal{O}(1)$. This specific scaling is motivated by arguments derived from simulations and observations, as well as earlier work on the Gray–Scott model, see [34, 39]. Most important is assumption **A1**, without which our

rescaled system would not have distinct slow and fast behavior. This assumption makes it possible to perform geometric singular perturbation analysis in the first place. The fact that A is scaled quadratically compared to C , is the most significant choice because this is the only way that neither the effects of a nor those of c (the most important parameters) are negligible. The factor $\delta^{1-\frac{3}{2}\beta+2\gamma}$ of s is chosen such that the speed has a measurable effect in the two-dimensional Hamiltonian system associated with the fast dynamics. Furthermore, we require the U -equation to exhibit slow behavior with small amplitudes in the fast interval as opposed to the fast dynamics with large amplitudes governed by the V -equation, this results in assumptions **A2** and **A3**, respectively. Lastly, we balance BV and UV^2 , so that the two-dimensional Hamiltonian system associated with the fast V -equation has a homoclinic orbit for $\delta \rightarrow 0$ and we take into account the forthcoming matching of slow and fast behavior, which leads to the specific choices of the scalings of U and V . The details of the limit behavior in the slow and fast variable, are laid out in sections 5.2.2 and 5.2.3, respectively. In the end, all quantities are scaled with only two scaling parameters, β and γ , combined with three assumptions **A1**–**A3**. Note that the calibration of the scalings is largely consistent with that of the Gray–Scott case, where only the scaling parameter β was left undetermined. Upon introducing an extra parameter, the slope C , one may expect the degree of freedom in scaling to increase by one which is indeed the case here. In Figure 5.2, the allowed regime of β, γ according to the assumptions is graphically outlined.

Rescaling system (5.3) according to (5.7) gives rise to,

$$\begin{aligned} u_{\xi\xi} &= \delta^{2(\gamma-\beta)} \left[uv^2 - \delta^{2+\beta} a(\delta^{\gamma-\frac{3}{2}\beta-1} - u) - \delta^{1-\gamma+\frac{3}{2}\beta} cu_{\xi} - \delta^2 su_{\xi} \right], \\ v_{\xi\xi} &= bv - uv^2 - \delta^{2(\gamma-\beta)} sv_{\xi}. \end{aligned} \quad (5.8)$$

Note that $\gamma > \beta$ (**A1**), so that the right hand side of u -equation is indeed asymptotically small for $0 < \delta \ll 1$. The homogeneous equilibrium state $(U, V) = (1, 0)$ is now represented by $(u, v) = (\delta^{-1-\frac{3}{2}\beta+\gamma}, 0)$. We label this desert state P . We write (5.8) as a four-dimensional system of first order differential equations, by introducing the variables p, q .

$$\begin{aligned} u_{\xi} &= \delta^{\gamma-\beta} p, \\ p_{\xi} &= \delta^{\gamma-\beta} \left[uv^2 - \delta^{2+\beta} a(\delta^{\gamma-\frac{3}{2}\beta-1} - u) - \delta^{1+\frac{1}{2}\beta} cp - \delta^{2+\gamma-\beta} sp \right], \\ v_{\xi} &= q, \\ q_{\xi} &= bv - uv^2 - \delta^{2(\gamma-\beta)} sq. \end{aligned} \quad (5.9)$$

Assumptions **A1**–**A3** guarantee that the right hand side of the equation for p_ξ is always asymptotically small. We now introduce a second slow variable, see **A1**, to write system (5.9) in the slow form,

$$\zeta = \delta^{\gamma-\beta} \xi.$$

Note that ζ is, however, faster or of the same speed as χ because of **A3**,

$$\chi = \delta^{1-\frac{1}{2}\beta} \xi \lesssim \zeta = \delta^{\gamma-\beta} \xi \ll \xi.$$

From now on, we work with ζ, ξ exclusively and refer to those as the slow and fast variable, respectively. In terms of the slow variable ζ , (5.9) becomes

$$\begin{aligned} u_\zeta &= p, \\ p_\zeta &= uv^2 - \delta^{2+\beta} a(\delta^{\gamma-\frac{3}{2}\beta-1} - u) - \delta^{1+\frac{1}{2}\beta} cp - \delta^{2+\gamma-\beta} sp, \\ \delta^{\gamma-\beta} v_\zeta &= q, \\ \delta^{\gamma-\beta} v_\zeta &= bv - uv^2 - \delta^{2(\gamma-\beta)} sq. \end{aligned} \tag{5.10}$$

Using the limiting behavior for $\delta \rightarrow 0$ of the two equivalent systems (5.9) and (5.10), we will establish the existence of solutions for $\delta \neq 0$.

5.2.2 Slow limit behavior

For $\delta \rightarrow 0$, the last two equations of system (5.10) become,

$$\begin{aligned} 0 &= q, \\ 0 &= bv - uv^2, \end{aligned} \tag{5.11}$$

which define two two-dimensional critical manifolds in $\mathbb{R}^4$, $\{v = 0, q = 0\}$ and $\{v = b/u, q = 0\}$. Since P is on the former, and as the latter is not normally hyperbolic for all v, q , we focus on the dynamics on

$$\mathcal{M} = \{u \geq 0, v = 0, q = 0\}, \tag{5.12}$$

where we restrict to $u \geq 0$ because this is the only ecologically relevant regime for u . Fenichel theory [9, 48, 49], guarantees the persistence of $\mathcal{M}$ for $0 < \delta \ll 1$ as a slow manifold. In the present case, (5.12) exactly defines the invariant slow manifold associated to the full problem. Fenichel theory and its implications are discussed in more detail in section 5.2.4.

The slow dynamics on $\mathcal{M}$ is described by

$$\begin{aligned} u_\zeta &= p, \\ p_\zeta &= -\delta^{2+\beta} a(\delta^{\gamma-\frac{3}{2}\beta-1} - u) - \delta^{1+\frac{1}{2}\beta} cp, \end{aligned} \tag{5.13}$$

from which we can derive that P is a saddle point on $\mathcal{M}$, with eigenvalues

$$\Lambda_{\pm} = \frac{1}{2} \delta^{1+\frac{1}{2}\beta} \left(-c \pm \sqrt{c^2 + 4a} \right),$$

and eigenvectors through P in the $(u, \hat{p})$ -plane,

$$\begin{aligned} \ell^u : \hat{p} &= \frac{1}{2} \left(c - \sqrt{c^2 + 4a} \right) (1 - \delta^{1+\frac{3}{2}\beta-\gamma} u), \\ \ell^s : \hat{p} &= \frac{1}{2} \left(c + \sqrt{c^2 + 4a} \right) (1 - \delta^{1+\frac{3}{2}\beta-\gamma} u). \end{aligned} \quad (5.14)$$

with a new variable $\hat{p} = \delta^{\beta-\gamma} p$. When assumption **A2** is satisfied, the eigenvectors do not depend on u in the leading order. The point P then has an asymptotically large u -value, and hence in the $(\hat{p}, u)$ -plane $\ell^{s,u}$ are vertical at leading order. This is depicted in the middle panel of Figure 5.3. However, as $\beta = \frac{2}{3}(\gamma - 1)$, the fixed point comes into frame and $U = u$. In this case the eigenvectors do depend on u , see the right panel in Figure 5.3. This case is also explained in more detail in remarks 5.1 and 5.2. Contrary to the Gray–Scott case, where $c = 0$, the behavior on $\mathcal{M}$ is in both panels of Figure 5.3 not symmetric about the u -axis.

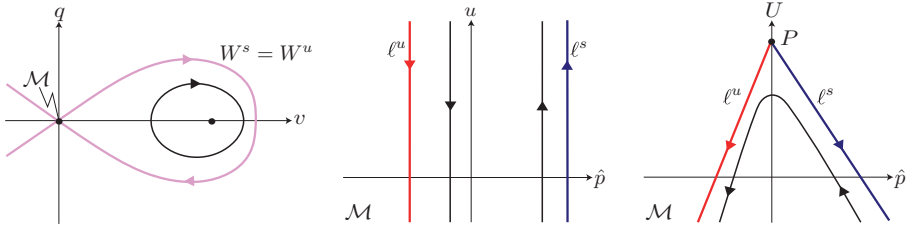


Figure 5.3: Left: The phase plane corresponding to (5.16), with in purple the homoclinic orbit to $(v, q) \in \mathcal{M}$. Center: Dynamics on $\mathcal{M}$ dictated by (5.13) in the $(\hat{p}, u)$ plane when **A2** is satisfied. The equilibrium P has an asymptotically large u -value and does not appear in this frame. To leading order, all dynamics, as well as $\ell^{s,u}$ are vertical. Right: Dynamics on $\mathcal{M}$ in the $(\hat{p}, U)$ plane, when $\beta = \frac{3}{2}(\gamma - 1)$ and $u = U$. The equilibrium P is $\mathcal{O}(1)$ and the eigenvectors vary with U .

5.2.3 Fast limit behavior

The limiting fast problem, system (5.9) with $\delta \rightarrow 0$, implies that u and p are constant, to leading order, i.e.

$$u(\xi) \equiv u_0, \quad p(\xi) \equiv p_0. \quad (5.15)$$

Hence, the fast dynamics is determined by

$$\begin{aligned} v_\xi &= q, \\ q_\xi &= bv - u_0 v^2, \end{aligned} \tag{5.16}$$

where u_0 occurs as a parameter. The fast limit problem (5.16) is Hamiltonian, with conserved quantity

$$H(v, q) = \frac{1}{2}q^2 - \frac{1}{2}bv^2 + \frac{1}{3}u_0 v^3. \tag{5.17}$$

Note that there exists a symmetry in (5.16),

$$q \rightarrow -q, \quad \xi \rightarrow -\xi. \tag{5.18}$$

The equilibrium of (5.16) of our focus, $(v, q) = (0, 0)$, has $H = 0$ and is a saddle. There is a homoclinic connection to $(0, 0)$, and we find an explicit solution by integrating $H(v, q) = 0$ with respect to ξ ,

$$\begin{aligned} v_0(\xi) &= \frac{3b}{2u_0} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{b} \xi \right), \\ q_0(\xi) &= -\frac{3b\sqrt{b}}{2u_0} \operatorname{sech}^2 \left(\frac{1}{2} \sqrt{b} \xi \right) \tanh \left(\frac{1}{2} \sqrt{b} \xi \right). \end{aligned} \tag{5.19}$$

The phase plane of (5.16) is depicted in the left panel of Figure 5.3. Since the homoclinic orbit exists for any u_0 and p_0 , the equilibrium $(v, q) = (0, 0)$ represents the entire critical manifold $\mathcal{M}$ (5.12). The critical manifold has three-dimensional stable and unstable manifolds, $W^{u,s}(\mathcal{M})$. Because the homoclinic orbit exists for all u_0, p_0 , we find $W^u(\mathcal{M}) = W^s(\mathcal{M})$ in the limit $\delta \downarrow 0$. This is represented in three dimensions in the left hand panel of Figure 5.4.

5.2.4 Persistence of solutions in the perturbed problem

With the results of sections 5.2.2 and 5.2.3, we construct a singular homoclinic orbit to P with both slow and fast parts obtained by the $\delta = 0$ limits. For $\delta \downarrow 0$, the eigenvectors $\ell^{s,u}$ tend to and eventually collide on the u -axis. In other words, the stable and unstable manifolds of P , restricted to $\mathcal{M}$ are in fact the u -axis in this limit. To construct a singular slow/fast homoclinic solution to P , we may match any point on the u -axis to the fast homoclinic excursion, because for $\xi \rightarrow \pm\infty$, the eigenvectors $\ell^{s,u}$ will connect it to P . As a result, we have a one parameter family of singular homoclinic connections to P , parametrized by the value of u at the concatenation point of the slow and fast parts, say u_0 .

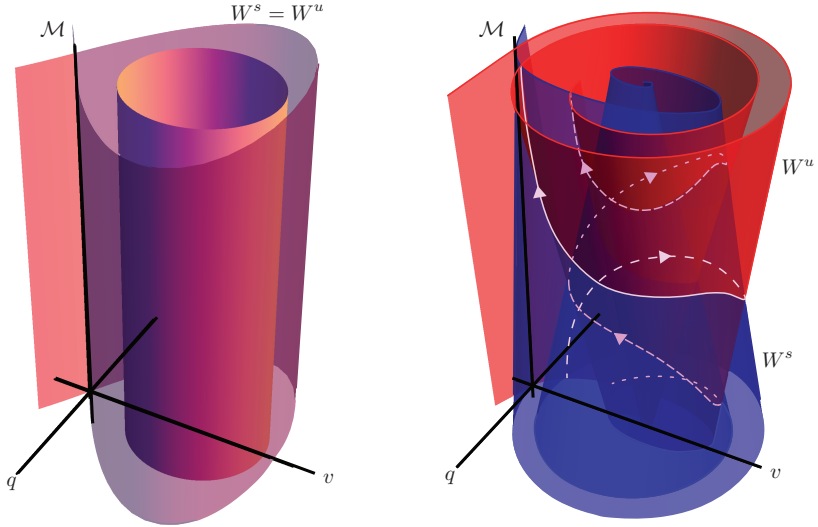


Figure 5.4: Schematic three-dimensional representation of the intersection of $W^u(\mathcal{M})$ and $W^s(\mathcal{M})$ of (5.9) for $\delta = 0$ (left) and $0 < \delta \ll 1$ in four dimensions. In reality, $\mathcal{M}$, here the one-dimensional vertical axis, is two-dimensional. Left: The stable and unstable manifolds of $\mathcal{M}$ are three-dimensional, and coincide. For every point on $\mathcal{M}$ there exists a fast homoclinic connection. Right: The stable (blue) and unstable (red) manifolds persist, but no longer coincide. Their intersections correspond with fast N -pulses, homoclinic to $\mathcal{M}$. Here, only the first two intersections are drawn: the 1-pulse, in white, and the 2-pulse in purple.

For $\delta \neq 0$, the stable and unstable manifolds of $\mathcal{M}$ persist as $W_{\delta}^{s,u}(\mathcal{M})$ according to Fenichel theory [48, 49], but they no longer coincide. Instead, the intersection of $W_{\delta}^{s,u}(\mathcal{M})$ perturbs, like depicted in Figure 5.4. A fast connection to $\mathcal{M}$ is thus no longer guaranteed for every pair $(u, \hat{p})$.

A solution homoclinic to P , with a slow/fast structure must leave P via the perturbed ℓ^u on $\mathcal{M}$, connect to a fast orbit homoclinic to $\mathcal{M}$, and returning to $\mathcal{M}$ exactly such that it enters P along the perturbed stable manifold, ℓ^s . These eigenvectors $\ell^{s,u}$ have also perturbed with δ , and no longer coincide on the u -axis. Schematically, we represent a homoclinic slow/fast solution in four dimensions in a three-dimensional illustration in the left hand panel of Figure 5.5.

Although the fast connection to $\mathcal{M}$ generally does not persist, a perturbed

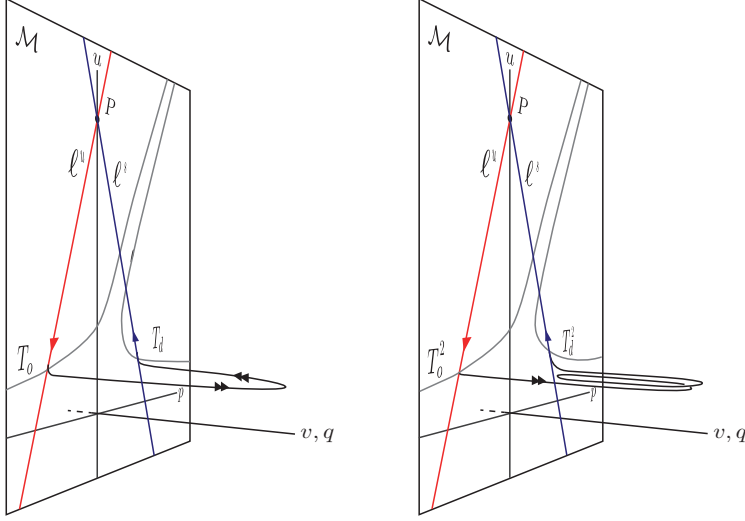


Figure 5.5: Schematic representation of a 1-pulse (left) and a 2-pulse (right); a slow/fast homoclinic connection to P , with a fast component that has one (left) or two (right) maxima. On $\mathcal{M}$, the solution follows ℓ^u before taking off into the fast field, where v and q become large. After that, the solution lands back on $\mathcal{M}$ at ℓ^s , which limits to P .

homoclinic orbit is still approximated by the $\delta = 0$ solution in the fast field, (5.19). To prove the statement of Theorem 5.2.1, we show that there still exists a homoclinic solution to P for $0 < \delta \ll 1$. We label this solution $\gamma_{\text{hom}}(\xi) = (u(\xi), p(\xi), v(\xi), q(\xi))$. Without a loss of generality, we choose $\gamma_{\text{hom}}(0) = (u_0, p_0, v_{\text{max}}, 0)$. That is, we select the $\xi = 0$ point exactly there where $q = 0$; it is at the maximum value of the V -pulse.

We first show that there exists a one-dimensional manifold for which $W_\delta^s(\mathcal{M})$ and $W_\delta^u(\mathcal{M})$ intersect transversally in the hyperplane $\{q = 0\}$. Then, following the approach developed in [35, 39], we construct take-off and touch-down curves that dictate how to connect the intersection $W_\delta^s(\mathcal{M}) \cap W_\delta^u(\mathcal{M}) \cap \{q = 0\}$ to the slow stable and unstable manifolds of P restricted to $\mathcal{M}$, which are the eigenvectors $\ell^{s,u}$.

Lemma 5.2.2. *Assume the conditions on a, b, c from Theorem 5.2.1 are satisfied. Then there exists a $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ the stable and unstable manifold of $\mathcal{M}$ in system (5.9) intersect transversally in $\{q = 0\}$; the (u_0, p_0)*

coordinates of this one-dimensional intersection are at leading order given by

$$p_0 = \delta^{\gamma-\beta} s u_0. \quad (5.20)$$

In other words, the homoclinic connection of (5.16) to $\mathcal{M}$ persists for $\delta > 0$ in (5.9) if (5.20) is satisfied.

For $0 < \delta \ll 1$, the critical manifold $\mathcal{M}$, as well as the stable and unstable manifolds $W^{s,u}(\mathcal{M})$ persist as a slow manifold $\mathcal{M}_\delta$ and $W_\delta^{s,u}(\mathcal{M})$ in an $\mathcal{O}(\delta^{\gamma-\beta})$ -neighborhood. This follows directly from Fenichel's first and second theorem, see [48, 82]. In fact, we may choose $\mathcal{M}_\delta = \mathcal{M}$, because $v = q = 0$ is still invariant under the flow for $0 < \delta \ll 1$. This is not the case for $W_\delta^{s,u}(\mathcal{M})$; the stable and unstable manifolds that used to coincide (see the left hand side of Figure 5.4), no longer do so. In general, the two three-dimensional manifolds intersect in two-dimensional surfaces, and in these intersections lie the only trajectories that are biasymptotic (homoclinic) to $\mathcal{M}$, see an illustration of such a perturbation and a persisting connection in the right-hand panel of Figure 5.4.

To detect these intersections, and hence proving Lemma 5.2.2, we apply a Melnikov method, much like [35, 39]. We will work with the fast variable ξ and define the fast interval I_f , as

$$I_f = \left[-\frac{1}{\delta^\mu}, \frac{1}{\delta^\mu} \right], \quad \text{with} \quad 0 < \mu < \gamma - \beta. \quad (5.21)$$

If $\gamma(\xi)$ is a solution of (5.9), then, for $\xi \in I_f$, u and p are constant and equal to u_0, p_0 at leading order. We measure the distance between $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ in the hyperplane $\{q = 0\}$, using the Hamiltonian (5.17) and the fact that on $\mathcal{M}$, $H \equiv 0$ even though $\delta \neq 0$, see (5.17). We now determine the first order corrections of u and p in the fast variable ξ . That is, we write

$$\begin{aligned} u(\xi) &= u_0 + \delta^{\gamma-\beta} u_1(\xi) + \text{h.o.t.}, \\ p(\xi) &= p_0 + \delta^{\gamma-\beta} p_1(\xi) + \text{h.o.t.}, \end{aligned} \quad (5.22)$$

and determine $u_1(\xi), p_1(\xi)$. We assume $u(0) = u_0$, and set $u_j(0) = 0$ for all $j \geq 1$. Using a standard asymptotic analysis and the boundary conditions that U and V must remain bounded on the real line, we find that $u_1(\xi) \equiv 0$. From the leading order analysis of section 5.2.2, we derive that $p = \mathcal{O}(\delta^{\gamma-\beta})$, so $p_0 = 0$. As a matter of fact, $p_1 = \hat{p}$, the variable introduced in section 5.2.2.

We compute $\hat{p}$ from the first order terms of system (5.9),

$$\begin{aligned}\hat{p} = \int_0^\xi \hat{p}_\xi d\xi + \hat{p}(0) &= \int_0^\xi u_0 v_0^2(\xi) d\xi + \hat{p}(0), \\ &= \frac{3b\sqrt{b}}{2u_0} \tanh\left(\frac{1}{2}\sqrt{b}\xi\right) \left(2 + \operatorname{sech}^2\left(\frac{1}{2}\sqrt{b}\xi\right)\right) + \hat{p}(0),\end{aligned}\tag{5.23}$$

to leading order in δ . Now we can measure the change in the Hamiltonian H in I_f over the fast homoclinic orbit. For $\delta \neq 0$, it is given by

$$H_\xi = \delta^{2(\gamma-\beta)} \left(\frac{1}{3} v^3 \hat{p} - s q^2 \right), \tag{5.24}$$

which implies

$$\begin{aligned}\Delta H &= \delta^{2(\gamma-\beta)} \int_{I_f} \left(\frac{1}{3} v^3 \hat{p} - s q^2 \right) d\xi, \\ &= \delta^{2(\gamma-\beta)} \frac{6b^2\sqrt{b}}{5u_0^2} \left(\frac{2\hat{p}(0)}{u_0} - s \right),\end{aligned}\tag{5.25}$$

where ΔH means the change in H over I_f . Hence, for given $p_0 = \delta^{\gamma-\beta} \hat{p}(0)$, ΔH changes sign once and in a transversal way at

$$\hat{p}(0) = \frac{1}{2} s u_0, \tag{5.26}$$

which implies that $W_\delta^s(\mathcal{M})$ and $W_\delta^u(\mathcal{M})$ indeed intersect transversally. This proves Lemma 5.2.2.

Note that equation (5.25) is actually the first moment we explicitly focus on the construction of a 1-pulse, by measuring the change in H over exactly one fast excursion. In Corollary 5.2.4, we explain how this step changes to establish the existence of N -pulses with $N > 1$.

Lemma 5.2.3 (Take-off and touch-down curves). *Let $\Gamma(\xi; u_0)$ be the one-parameter family of solutions of (5.9) with $\Gamma(0; u_0)$ in $W_\delta^s(\mathcal{M}) \cap W_\delta^u(\mathcal{M}) \cap \{q = 0\}$. In other words, let $\Gamma(0; u_0) = (u_0, \frac{1}{2}\delta^{\gamma-\beta} s u_0, v(0), 0)$ at leading order, see (5.26). Moreover, let $\Gamma^{o,d}(\xi; u^{o,d}, p^{o,d})$ be trajectories strictly on $\mathcal{M}$ with initial condition at $(u^{o,d}, p^{o,d})$. Then, there exist two curves $T_o(u)$ and $T_d(u)$ on $\mathcal{M}$, such that there are $\Gamma^{o,d}$ satisfying*

$$\begin{aligned}|\Gamma(\xi; u_0) - \Gamma^o(\xi; u_0, T_o(u_0))| &< k_1 e^{k_3 \delta^{\beta-\gamma}}, \quad \text{for } -\xi \geq \mathcal{O}(\delta^{\beta-\gamma}), \\ |\Gamma(\xi; u_0) - \Gamma^d(\xi; u_0, T_d(u_0))| &< k_2 e^{k_3 \delta^{\beta-\gamma}}, \quad \text{for } \xi \geq \mathcal{O}(\delta^{\beta-\gamma}),\end{aligned}\tag{5.27}$$

for some $\mathcal{O}(1)$ constants $k_{1,2,3} > 0$. The curves $T_o(u)$ and $T_d(u)$ are called *take-off* and *touch-down curve*, respectively, and are given by

$$\begin{aligned} T_o(u) &= \frac{1}{2} \delta^{\gamma-\beta} \left(su - \frac{6b\sqrt{b}}{u} \right), \\ T_d(u) &= \frac{1}{2} \delta^{\gamma-\beta} \left(su + \frac{6b\sqrt{b}}{u} \right), \end{aligned} \tag{5.28}$$

to leading order.

Thus, this lemma explicitly describes trajectories on $\mathcal{M}$, for which the distance to a solution homoclinic to $\mathcal{M}$ gets exponentially small as ξ gets large in either the negative or the positive half-line.

Proof. The existence of the take-off and touch-down curves is established by Fenichel theory, where the points in T_o and T_d are referred to as base points of the Fenichel fibers, see [48, 49]. The geometry of $W^s(\mathcal{M}) \cap W^u(\mathcal{M})$ implies that T_o and T_d can indeed be characterized as graphs over u . Their quantification is obtained by explicitly determining the change of p from the $\xi = 0$ point in the intersection $W_\delta^s(\mathcal{M}) \cap W_\delta^u(\mathcal{M}) \cap \{q = 0\}$, over half a fast pulse, to an exponentially small neighborhood of $\mathcal{M}$ for both negative and positive ξ ,

$$\int_{-\frac{1}{\delta^\mu}}^0 p_\xi d\xi \quad \int_0^{\frac{1}{\delta^\mu}} p_\xi d\xi,$$

for $\xi < 0$ and $\xi > 0$, respectively. The boundaries of the integrals are the boundaries of I_f , see (5.21). The fast behavior is symmetric to leading order, see (5.18), so both integrals are half of the integral over the full fast interval, and equal to

$$\begin{aligned} \frac{1}{2} \int_{I_f} p_\xi d\xi &= \frac{1}{2} \delta^{\gamma-\beta} \int_{I_f} u_0 v_0(\xi)^2 d\xi + \text{h.o.t.}, \\ &= \frac{1}{2} \delta^{\gamma-\beta} \frac{6b\sqrt{b}}{u_0} + \text{h.o.t.} \end{aligned} \tag{5.29}$$

So, combining this rate of change of p with $p(0)$ from (5.23), we arrive at (5.28). The estimates of (5.27) are derived from Fenichel theory, and the constant k_3 is related to the largest eigenvalue of the fast field. $\square$

Using Lemmas 5.2.2 and 5.2.3, we now prove Theorem 5.2.1.

Proof. (of Theorem 5.2.1). From all the solutions $\gamma(\xi)$ that are biasymptotic to $\mathcal{M}$, the homoclinic 1-pulse γ_{hom} is by construction exactly that one that connects to ℓ^u for $\xi < 0$ and to ℓ^s for $\xi > 0$. Thus, we need

$$T_o(u_0) = \ell^u(u_0) \quad \text{and} \quad T_d(u_0) = \ell^s(u_0).$$

This happens when

$$u_0 = \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}, \quad \text{and} \quad s = \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}}, \quad (5.30)$$

to leading order in δ , see the left panel of Figure 5.5. $\square$

Corollary 5.2.4 (Existence of N -pulse patterns). *Let A, B, C satisfy the assumptions A1–A3.*

Then there exists a δ_0 , such that for all $\delta < \delta_0$, and for each N of $\mathcal{O}(1)$, there exists a solution $\gamma_{N,\text{hom}}$ of (5.3) homoclinic to $(U, U_\chi, V, V_\chi) = (1, 0, 0, 0)$ for a uniquely defined speed s_N . The orbit $\gamma_{N,\text{hom}}$ corresponds to a traveling wave solution of (5.2). Its spatial profile that is in x biasymptotic to $(U, V) = (1, 0)$, and trivially extends into the y -direction. It consists of two slow components and N fast excursions. The traveling speed of this wave is given by

$$s_N = \frac{c\sqrt{c^2 + 4a}}{6Nb\sqrt{b}}. \quad (5.31)$$

During the fast excursions, the U -component is constant to leading order, $U = \delta^{1+\frac{3}{2}\beta-\gamma}u_0$, with

$$u_{0,N} = \frac{6Nb\sqrt{b}}{\sqrt{c^2 + 4a}}. \quad (5.32)$$

For $N = 2$ the homoclinic orbits to P are schematically drawn in the right hand panels of Figures 5.4 and 5.5. The key principle to realize is that manifolds $W^s(\mathcal{M})$ and $W^u(\mathcal{M})$ may intersect more than once in $\{q = 0\}$. In [35], it is shown, using the symmetry of the fast system, that indeed, there may be many intersections $W^s(\mathcal{M}) \cap W^u(\mathcal{M}) \cap \{q = 0\}$, all of dimension 1. Following ξ in forward and backward time along W^s and W^u , starting from $\mathcal{M}$, the first intersection corresponds to the 1-pulse of Theorem 5.2.1 and has $v = \mathcal{O}(1)$. The second intersection happens for $0 < v \ll 1$, and corresponds to the 2-pulse, the third intersection has again $v = \mathcal{O}(1)$ and corresponds to the 3-pulse, et cetera.

Proof. The condition (5.23) must still hold, but since the trajectory makes N circuits in the fast regime, we must measure the change of p a total of N times.

Symmetry (5.18) makes that we can construct take-off and touch-down curves for every N , as follows

$$T_o^N(u) = \frac{1}{2}\delta^{\gamma-\beta} \left(su - \frac{6Nb\sqrt{b}}{u} \right), \quad T_d^N(u) = \frac{1}{2}\delta^{\gamma-\beta} \left(su + \frac{6Nb\sqrt{b}}{u} \right).$$

Again, an intersection of T_o^N, T_d^N with $\ell^{s,u}$ constructs the homoclinic N -pulse, and we arrive at the conditions for $u_{0,N}$ and s_N as stated in the theorem. $\square$

As opposed to the homoclinic 1-pulse, we will not dive into the details of the existence of periodic extensions of these N -pulses nor their stability. The main reason for this is the fact they can be argued to be unstable by a simple argument, explained in section 5.4.1.

Remark 5.1. Even in the case that $\beta = \frac{2}{3}(\gamma - 1)$, when **A2** is violated, homoclinic pulse solutions continue to exist. In that case, the expressions for $\ell^{s,u}$ from (5.14) depend on u in the leading order and the behavior on $\mathcal{M}$ is as described by the right panel of Figure 5.3. The fast limit results remain unchanged but the fine tuning for the values (5.30) is slightly different. In both cases, the touch-down curve intersects ℓ^s twice, while the take-off curve intersects ℓ^u just once. By tuning s and u_0 , there are two solutions

$$\begin{aligned} u_0^- &= \frac{1 - \sqrt{1 - 24\delta^{\gamma-\frac{2}{3}\beta-1}b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2\delta^{\gamma-\frac{2}{3}\beta-1}}, & s^+ &= c\delta^{\gamma-\frac{2}{3}\beta-1} \left(\frac{1}{u_0^-} - 1 \right), \\ u_0^+ &= \frac{1 + \sqrt{1 - 24\delta^{\gamma-\frac{2}{3}\beta-1}b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2\delta^{\gamma-\frac{2}{3}\beta-1}}, & s^- &= c\delta^{\gamma-\frac{2}{3}\beta-1} \left(\frac{1}{u_0^+} - 1 \right). \end{aligned} \quad (5.33)$$

When **A2** is satisfied, only the pair (u_0^-, s^-) is an eligible solution because u_0^+ is not $\mathcal{O}(1)$. To leading order in δ , this results exactly in (5.30).

However, as $\beta = \frac{2}{3}(\gamma - 1)$, both intersections $u_0^\pm$ are $\mathcal{O}(1)$, so there are two solutions under the condition that

$$\frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} \leq 1. \quad (5.34)$$

$$\begin{aligned} u_0^- &= \frac{1 - \sqrt{1 - 24b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2}, & s^+ &= c \left(\frac{1}{u_0^-} - 1 \right), \\ u_0^+ &= \frac{1 + \sqrt{1 - 24b\sqrt{b}(c^2 + 4a)^{-1/2}}}{2}, & s^- &= c \left(\frac{1}{u_0^+} - 1 \right). \end{aligned} \quad (5.35)$$

Both solutions correspond to a homoclinic 1-pulse solution of (5.3), and (5.34) defines the parameter combination at which the solutions merge in a saddle-node bifurcation, see also [39].

5.3 Existence of traveling multiscale periodic solutions

Given that the origin of system (5.3) is the description of vegetation patterns, the existence of periodic patterns is perhaps more relevant than the homoclinic N -pulse patterns. After all, homoclinic patterns represent N relatively nearby stripes that are isolated in an elsewhere completely bare desert. The stripe patterns, as they are observed in the field, are naturally much better represented by periodic pulse patterns, [143]. The periodic patterns still have a slow/fast structure, and we will construct periodic patterns as depicted in Figure 5.6. That is, the slow parts of the period patterns lie close to $\mathcal{M}$ and connect to a fast excursion. Furthermore, we use that the fast parts of the constructed solutions are nearly homoclinic to the slow manifold $\mathcal{M}$; solutions stay in a neighborhood of $\mathcal{M}$ for an $\mathcal{O}(\delta^{-1+\frac{1}{2}\beta-\gamma})$ amount of time in ξ . By definition, a periodic orbit cannot be homoclinic to $\mathcal{M}$, but we use in our construction that it is exponentially close to $W^u(\mathcal{M}) \cap W^s(\mathcal{M})$, see also [38]. We focus on the periodic solution that consists of one fast excursion from the slow manifold $\mathcal{M}$, and one long, slow segment near $\mathcal{M}$, per period. This periodic orbit may be distinguished from an N -pulse by the distance between the fast pulses. After their fast excursions, periodic solutions constructed in this section remain exponentially close to $\mathcal{M}$ for a long time in ξ before making another excursion that actually marks a new period of the pattern. Therefore, the wavelength of the constructed patterns is algebraically large in δ^{-1} . The N -pulses, however, do not return to an exponentially small neighborhood of $\mathcal{M}$ after a fast excursion, since N -pulses only approach $\mathcal{M}$ to an algebraically small distance in δ between consecutive fast excursions. The distance between pulses of an N -pulse is thus only logarithmically large in δ . It is only after the N -th excursion that the N -pulse returns to $\mathcal{M}$, indefinitely.

In this section, we prove the following theorem.

Theorem 5.3.1. *Let $a, b, c, s \in \mathcal{O}(1)$ be given and let **A1–A3** and*

$$s < \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}}, \quad (5.36)$$

be satisfied. Then, there exists a δ_0 , such that for all $\delta < \delta_0$, there exists a unique periodic solution $\gamma_{p,s}(\xi)$ of (5.8) with a slow/fast structure. That is, for any s satisfying (5.36), there exists a traveling wave solution of (5.2), with a spatial profile which is periodic.

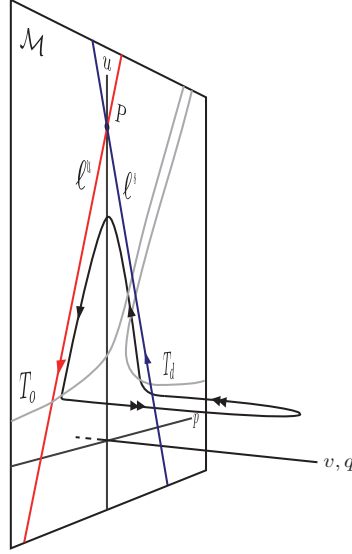


Figure 5.6: Schematic representation of a periodic slow/fast solution of which the slow part is close to the manifold $\mathcal{M}$. During the fast excursion the coordinate u is constant to leading order.

Figure 5.7 shows a simulation of (5.2) representing a solution like $\gamma_{p,s}$.

Because v, q are exponentially small during the largest part of the period of the constructed solution, the leading order behavior of the slow parts is still governed by (5.13), and the derivation of the take-off and touch-down curves remains valid. We use the explicit formulas we have for T_o and T_d to construct periodic solutions and prove Theorem 5.3.1. As an illustrative guide to the proof, we use Figure 5.8.

First, note that the slow segments of the periodic solution must lie in the area enclosed by ℓ^s and ℓ^u , because they are separatrices in the slow manifold. Within this area, the take-off curve is monotonous as a function of p , while the touch-down curve is not. This fact is key in the existence of periodic solutions.

Second, note that the geometry of $T_{o,d}$ and $\ell^{s,u}$ determines the eligible u -values for which the periodic pattern will make the fast excursion. For a given s , the maximum of the interval of eligible u -values is always

$$\tilde{u} = \sqrt{\frac{6b\sqrt{b}}{s}} \quad (5.37)$$

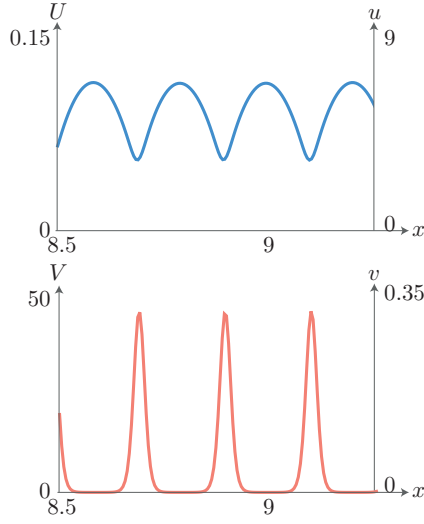


Figure 5.7: Simulation of (5.2) with $A = 15$, $B = 1.8$, $C = 1$ and $\delta = 0.01$, or, $a = 3$, $b = 0.75$, $c \approx 0.447$ with $\gamma \approx -0.174$, $\beta \approx -0.190$. The simulation was done on a spatial domain of length 10, and the patterns travel to the right over time. The left y -axes denote the unscaled U_0 (blue) and V - (red) values while the right y -axes denote the values of the scaled u and v for our choices of β, γ .

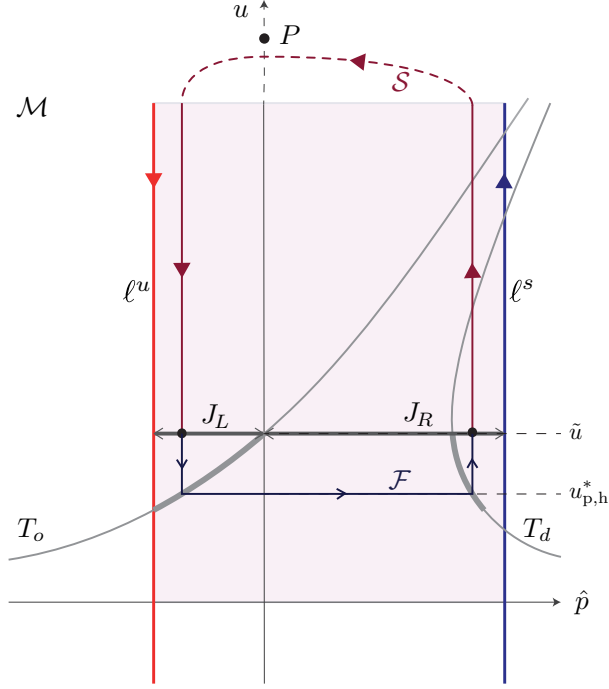


Figure 5.8: Schematic illustration of the construction of a periodic solution of (5.8). Depicted is the slow manifold $\mathcal{M}$ with the stable and unstable eigenvectors $\ell^{s,u}$ and take-off and touch-down curves $T_{o,d}$. In dark red, a schematic representation of the slow function $\mathcal{S}$ is sketched, and the fast function $\mathcal{F}$ is dark blue. The point P is at an $\mathcal{O}(\delta^{-1-\frac{3}{2}\beta+\gamma})$ -distance, hence the dashed lines.

, i.e. the intersection of T_o with the $\{p = 0\}$ -axis. Condition (5.36) implies that the minimum is always the u -value corresponding to $\ell^u \cap T_o$, see Figure 5.8.

Due to the lack of reversible symmetry, the methods developed in [38] do not provide sufficient control over the slow system to prove Theorem 5.3.1. We therefore employ a different approach making use of a contraction argument.

We define a horizontal line segment in the $(\hat{p}, u)$ -plane, i.e. on $\mathcal{M}$ enclosed by ℓ^u and ℓ^s , and at the $u = \tilde{u}$ level.

$$J = \left\{ \hat{p} \in \left(c - \sqrt{c^2 + 4a}, c + \sqrt{c^2 + 4a} \right), \quad u = \tilde{u} \right\}. \quad (5.38)$$

and J_R and J_L as the parts of J , where $\hat{p}$ is positive, resp. negative, see Figure 5.8. On these intervals, we define two functions. Firstly, a map $\mathcal{S} : J_R \rightarrow J_L$.

Let J_R represent a line of initial conditions of the slow flow, see (5.13). Note that $u_\xi = 0$ for $p = 0$. As the slow flow is linear and P is a saddle point on $\mathcal{M}$, we know that all orbits with an initial point in J_R have an intersection with J_L as well. In fact, this is a one-to-one correspondence. The map $\mathcal{S}$ assigns to every point $(\tilde{u}, p) \in J_R$, the corresponding intersection in J_L , and this is a bijection.

Secondly, in the parameter regime where (5.36) holds, define the map

$$\mathcal{F} : J_L \rightarrow J_R, \quad \mathcal{F}(\tilde{u}, p) = \left(\tilde{u}, \quad p + \frac{24sb\sqrt{b}}{p + \sqrt{p^2 + 24sb\sqrt{b}}} \right). \quad (5.39)$$

This map represents the fast flow indicated by the take-off and touch-down curves. For initial values in J_L , the map is based on a concatenation of three steps. To leading order, p does not change by the evolution of the slow flow. Furthermore, T_o is monotonous, so by following the slow flow the initial points on J_L correspond bijectively to points T_o . This is the first step of $\mathcal{F}$. As the second step, dictated by the fast flow, the point on T_o that results from the first step is sent to a point on T_d , which is also well-defined since the u -value remains, at leading order, constant during the fast excursion. Lastly, the slow flow takes all these touch-down points to J_R . Condition (5.36) implies that the range of $\mathcal{F}$ is guaranteed to be within J_R . Hence, the map $\mathcal{F}$ keeps solutions in the bounded area of $\mathcal{M}$, shaded in Figure 5.8. By inverting the expressions for T_o and T_d , it can be verified that $\mathcal{F}$ indeed acts as given in (5.39).

Lemma 5.3.2. *Both maps $\mathcal{S} : J_R \rightarrow J_L$ and $\mathcal{F} : J_L \rightarrow J_R$ are contractions.*

Proof. Naturally, we use the Euclidian metric on J_L and J_R . As explained, the map $\mathcal{S}$ is a bijection. However, the size of the domain, J_R is $c + \sqrt{c^2 + 4a}$, which is smaller than the size of codomain J_L , which is $c - \sqrt{c^2 + 4a}$. Since P is a saddle, the flow on $\mathcal{M}$ is linear and orbits cannot intersect, the initial points in J_R remain ordered after they intersect the nullcline $p = 0$. By explicitly evaluating $\mathcal{S}$, we indeed find that for every pair $X, Y \in J_R$,

$$|\mathcal{S}(X) - \mathcal{S}(Y)| < |X - Y|,$$

and $\mathcal{S}$ is a contraction.

For the map $\mathcal{F}$ it suffices to check that

$$\frac{\partial}{\partial p} \left(p + \frac{24sb\sqrt{b}}{p + \sqrt{p^2 + 24sb\sqrt{b}}} \right) = \frac{p}{\sqrt{p^2 + 24sb\sqrt{b}}} < 1.$$

A derivative smaller than 1 implies contraction, so the function $\mathcal{F}$ is a contraction, too. $\square$

Lemma 5.3.2 allows us to establish the first ingredient of the proof of Theorem 5.3.1 about the existence of periodic solutions. The composition $\mathcal{F} \circ \mathcal{S} : J_R \rightarrow J_R$ is a contraction as well. By Banach's fixed point theorem, this contraction has a unique fixed point in J_R , say $\hat{p} = \hat{p}_{p,h}^* \in J_R$. This implies that $(\tilde{u}, \hat{p}_{p,h}^*)$ is an initial point for a slow segment in $\mathcal{M}$ which intersects the take-off and touch-down curves in the same u -value. We label this u -value $u_{p,h}^*$ so that

$$(u_{p,h}^*, \hat{p}_{p,h}^*) = \left(u_{p,h}^*, \frac{1}{2} \left(s u_{p,h}^* + \frac{6b\sqrt{b}}{u_{p,h}^*} \right) \right). \quad (5.40)$$

Note that $u_{p,h}^*$ is always larger than the u_0 of (5.30), because u_0 marks the left boundary of J_L , in which T_o is monotonically increasing. In sections 5.4 and 5.5, we show that this has implications for the stability of the solutions and links directly to Ni's conjecture [120].

A concatenation of the slow and fast parts of the constructed orbit associated with $(u_{p,h}^*, \hat{p}_{p,h}^*)$ does not immediately result in a periodic orbit, because the orbit is derived using the intersection of $W^s(\mathcal{M}) \cap W^u(\mathcal{M})$ for $T_{o,d}$. The constructed orbit, we label it $\gamma_{p,h}(\xi)$, is depicted in Figure 5.9, and is rather than being periodic, homoclinic to the manifold $\mathcal{M}$. That means that it makes only one excursion in the fast field, after it has returned to a neighborhood of $\mathcal{M}$ it only gets closer and closer. Note that actually, $\gamma_{p,h}(\xi)$ depends on the choice of s . We label the speed corresponding to $\gamma_{p,h}$ as $s_{p,h}$.

We reset the initial point of $\gamma_{p,h}$ so that it is a the point where v is maximal, i.e. $q = 0$. That is, we write

$$\gamma_{p,h}(0) = (u_{p,h}(0), p_{p,h}(0), v_{p,h}(0), 0).$$

We will show that a true periodic orbit exists with initial point exponentially close to $\gamma_{p,h}(0)$. This orbit is $\gamma_{p,s}(\xi)$ that is defined in the statement of theorem 5.3.1.

The proof of Theorem 5.3.1 is inspired by that of Theorem 3.1 in [38], but cannot be carried over immediately. The advection term in (5.2) breaks the reversible symmetry, which is crucial to the proofs in [38]. In the present, not symmetric case, more delicate arguments are developed to resolve that issue.

Proof. (of Theorem 5.3.1)

We prove the existence of periodic orbits by using arguments of intersecting manifolds, similar to the proof of Theorem 5.2.1. The aim is to show (in four dimensions) that there is a one-dimensional periodic orbit that lies close to $\gamma_{p,h}(\xi)$.

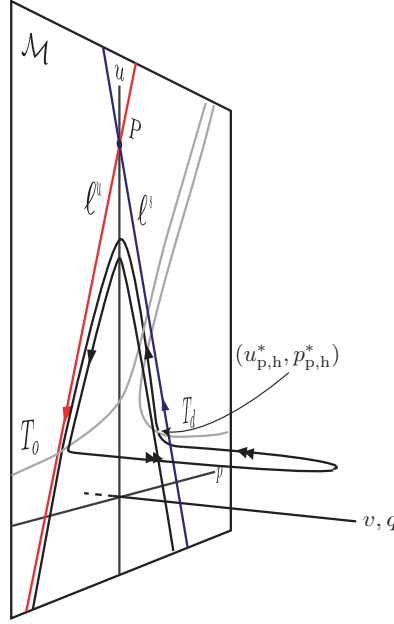


Figure 5.9: Schematic representation of the four-dimensional orbit $\gamma_{p,h}$ in three dimensions. The orbit makes one fast excursion and is homoclinic to the manifold $\mathcal{M}$.

Define the exponentially small, two-dimensional rectangle j as follows,

$$\begin{aligned}
 j = \{u = u_{p,h}(0)\} \times & \left(p_{p,h}(0) - k_1 e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma}, p_{p,h}(0) + k_2 e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma} \right) \\
 & \times \left(v_{p,h}(0) - k_3 e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma}, v_{p,h}(0) - k_4 e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma} \right) \\
 & \times \{q = 0\},
 \end{aligned} \tag{5.41}$$

with $k, k_i > 0$ and $k_3 > k_4$. The value k is associated with the eigenvalues of the fast limit system. Figure 5.10 shows the rectangle j in the rescaled $e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma}(v, p)$ -plane. Note that $-1 + \frac{1}{2}\beta - \gamma < 0$ in the regime defined by **A1–A3** (Figure 5.2), so $-k\delta^{-1} + \frac{1}{2}\beta - \gamma$ is very large and negative, and j is indeed exponentially small.

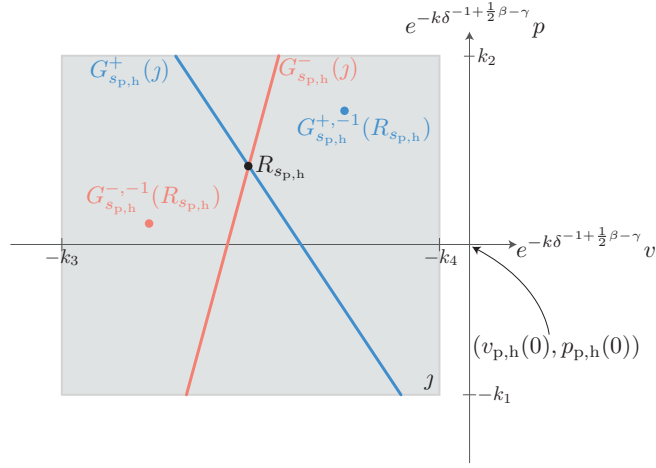


Figure 5.10: Schematic representation of the rectangle j (gray) with $G_{s_{p,h}}^+(j)$ and $G_{s_{p,h}}^-(j)$ as one-dimensional curves in j in red and blue, respectively. The point $G_{s_{p,h}}^{+,-1}(R_{s_{p,h}})$ is the original in j that was sent to $R_{s_{p,h}}$ following the forward flow and is therefore red, while $G_{s_{p,h}}^{-,-1}(R_{s_{p,h}})$ is the original under G^- and is therefore associated with the blue curve and colored accordingly.

We use j as a set of initial conditions that lie exponentially close to the intersection $W^u(\mathcal{M}) \cap W^s(\mathcal{M}) \cap \{q = 0\}$. All orbits formed by flowing initial condition in j forward, remain exponentially close to $W^s(\mathcal{M})$ as long as ξ does not become larger than $\mathcal{O}(\delta^{1-\frac{1}{2}\beta+\gamma})$. Because of the exact closeness estimate of $\mathcal{O}(e^{-k\delta^{-1} + \frac{1}{2}\beta - \gamma})$, it remains close to $W^s(\mathcal{M})$ exactly long enough to make an $\mathcal{O}(\delta^{\gamma-\beta})$ change in p . This is the same order of magnitude that p changes during the slow part of $\gamma_{p,h}$. The three-dimensional manifold is obtained from flowing j forward is labeled J^+ . The flow depends directly on the speed of $\gamma_{p,h}$, i.e. $s_{p,h}$, so J^+ does so as well.

By choosing k_i appropriately, J^+ will return to an exponentially small neighborhood of j , since it is exponentially close to $W^u(\mathcal{M})$. Thus, J^+ intersects j in a one-dimensional manifold $j \cap J^+$; a curve in j . We define the map $G_{s_{p,h}}^+ : j \rightarrow j$ that maps points in j to their next intersection with j when the forward flow induced by (5.8) is followed. Hence we find

$$G_{s_{p,h}}^+(j) = J^+ \cap j,$$

see Figure 5.10.

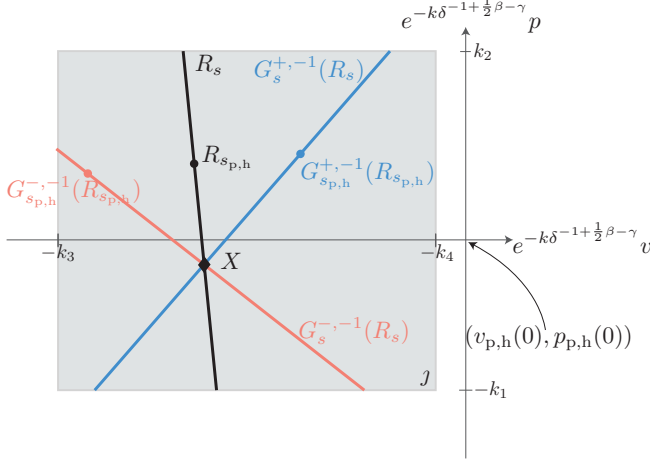


Figure 5.11: Representation of the rectangle j (gray) with $G_s^{-, -1}(R_s)$ and $G_s^{+, -1}(R_s)$ as one-dimensional curves in j in red and blue, respectively. $R_{s_{p,h}}$ is the original intersection $G_s^+(j) \cap G_s^-(j)$ in j associated with the speed $s_{p,h}$ of orbit $\gamma_{p,h}(\xi)$. The points $G_s^{\pm, -1}(R_{s_{p,h}})$ are the pre-images of $R_{s_{p,h}}$ and the curves are parametrized by s . The intersection of the curves is the point X , which is associated to an initial condition for the periodic orbit $\gamma_p(\xi)$, and is indicated with a black diamond.

Similarly, we can define J^- by flowing j backwards in ξ . Again, appropriate k_i will make sure that J^- intersects j and we label this analogous map $G_{s_{p,h}}^- : j \rightarrow j$, so that we have

$$G_{s_{p,h}}^-(j) = J^- \cap j.$$

See again Figure 5.10 for an illustration. The curves $G_{s_{p,h}}^{\pm}(j)$ are one-dimensional and generically intersect in a point in j , say $R_{s_{p,h}}$. The intersection point $R_{s_{p,h}}$ is the point in j that in both in forward and backward time returns to j . Since $R_{s_{p,h}}$ is on $G_{s_{p,h}}^+(j)$, there is a point in j that is flown forward to $R_{s_{p,h}}$, say $G_{s_{p,h}}^{+, -1}(R_{s_{p,h}})$, as it is a the pre-image of $R_{s_{p,h}}$. On the other hand, since $R_{s_{p,h}}$ is on $G_{s_{p,h}}^-(j)$ as well, there is a point in j that is flown backward to $R_{s_{p,h}}$, which we label $G_{s_{p,h}}^{-, -1}(R_{s_{p,h}})$. However, $G_{s_{p,h}}^{-, -1}(R_{s_{p,h}})$ and $G_{s_{p,h}}^{+, -1}(R_{s_{p,h}})$ are both zero-dimensional in j and hence a priori do not intersect. The point $R_{s_{p,h}}$ generally is thus not an initial value for a periodic orbit.

We do know, however, that if we consider exponentially small deviations in s from $s_{p,h}$, the points $G_{s_{p,h}}^{-, -1}(R_{s_{p,h}})$ and $G_{s_{p,h}}^{+, -1}(R_{s_{p,h}})$ move in j , along curves

parametrized by s , see Figure 5.11. These curves, $G_s^{\pm,-1}(R_{s_{p,h}})$, generically intersect in J for a specific s_p , say in the point X . Exactly when this happens, both the forward flow map G^+ and backward flow map G^- have X as a fixed point. Of course, that also implies that the intersection $G_{s_p}^+(J) \cap G_{s_p}^-(J)$ occurs exactly in X so $R_{s_p} = X$.

This point $X = (v^X, p^X)$ is associated to a periodic orbit with an initial value $(u_{p,h}(0), p^X, v^X, 0)$. This periodic orbit is $\gamma_p(\xi)$ and it has a speed that is exponentially close to $s_{p,h}$. Note that, by our choice of rectangle J , we do fix the initial condition of the periodic solution at the maximum value of v (since $q = 0$) and that the u -value at that point is equal to the u -value of $\gamma_{p,h}$ where $v_{p,h}$ is maximal.

Using the direct method as developed in [37], or the exchange lemma approach with exponential errors of [83], it is possible to show that indeed this X exists for a certain s_p . Moreover, this s_p is unique, but we refrain from going into the details any further. $\square$

Remark 5.2. As for the existence of homoclinic singular pulse solution, see 5.2.1, we have assumed through **A2** that $\beta > \frac{2}{3}(\gamma - 1)$. When $\beta = \frac{2}{3}(\gamma - 1)$ the construction of a homoclinic singular pulse solution is still possible, see Remark 5.1. The same holds for periodic patterns. As the assumption **A2** attains equality, the construction of periodic solutions remains valid as the rectangle J remains exponentially small. The only difference to take into account is that condition (5.36) changes into

$$u_{\min} < \frac{2c}{2c + s},$$

where $u_{\min}$ is the u -value corresponding to the intersection of T_o with ℓ^u in $\mathcal{M}$.

5.4 Stability of singular multiscale patterns

Although the existence of many types of stripe patterns in system (5.3) is guaranteed by the analysis of sections 5.2 and 5.3, not all will be relevant in light of vegetation patterns. Of course, an unstable solution of a simplified model like (5.3) can never be observed in a natural system. In this section, we test the linear stability of the constructed patterns from the previous sections against two-dimensional perturbations. We formulate the Evans function corresponding to the linear stability problem, and make extensive use of the slow/fast structure of our solutions to evaluate it. This method is developed and described in full detail for reversible systems in [34, 35, 36]. However, like in the existence problem, the advection term breaks the symmetry, so the theory cannot be carried over completely. Nevertheless, we show that this approach can be extended and we decompose the Evans function in a fast and a slow component. As in the

literature, the slow component of the Evans function is determined explicitly in terms of hypergeometric functions.

Let (U_0, V_0) be a traveling solution of (5.2) corresponding to a single stripe $(U_{\text{hom}}, V_{\text{hom}})$ of vegetation or a periodic pattern (U_p, V_p) of vegetation stripes in $\mathbb{R}$, trivially extended in a second spatial dimension. Since N -pulse solutions with $N > 1$ are unstable (this is motivated in section 5.4.1), we do not consider the stability of these patterns in the current section. The solutions γ_{hom} and $\gamma_{p,s}$ that have been established in Theorems 5.2.1 and 5.3.1 may be rescaled back into U, V coordinates according to (5.7) to obtain $(U_{\text{hom}}, V_{\text{hom}})$ and (U_p, V_p) , respectively; both solutions of (5.2) could be substituted for (U_0, V_0) . Due to the trivial structure of (U_0, V_0) in y -direction we can use a Fourier ansatz and perturb the solutions as,

$$(U(\chi, y, \tau), V(\chi, y, \tau)) = \left(U_0 + e^{\hat{\lambda}\tau + i\hat{\ell}y} \hat{u}(\chi), V_0 + e^{\hat{\lambda}\tau + i\hat{\ell}y} \hat{v}(\chi) \right), \quad (5.42)$$

where χ is the super slow traveling coordinate. Here we must make the following clarification. For periodic patterns, any perturbation in the direction transverse to the pattern (that is the y -direction), may be represented by $\hat{\ell} \in \mathbb{R}$ and a $\hat{\gamma} \in S^1$, taking into account the so-called ‘ $\hat{\gamma}$ -eigenvalues’ [31]. We may then speak of eigenvalues $\hat{\lambda}(\hat{\gamma}, \hat{\ell})$ for each $\hat{\gamma}, \hat{\ell} \in (\mathbb{R}, S^1)$. However, since the fundamental interval of our periodic solutions is asymptotically large and the exponential decay of U and V is fast enough so that they are both exponentially small outside I_f , the entire family $\hat{\lambda}(\hat{\gamma}, \hat{\ell})$ is exponentially close to one specific value $\hat{\lambda}(\hat{\ell})$ for each $\hat{\ell}$. We quantify this in section 5.4.1. Then, for every fixed $\hat{\ell} \in \mathbb{R}$ there exists an eigenvalue $\hat{\lambda}(\hat{\ell})$ that determines the stability of the solution (U_0, V_0) , also in the case that (U_0, V_0) is a spatially periodic pattern. For more details on this approach, see [31, 51, 52, 157].

If $\hat{\lambda}$ has negative real part for every $\hat{\ell} \in \mathbb{R}$ – apart from the trivial translation eigenvalue $\hat{\lambda}(0) = 0$ – the pattern is (spectrally) 2D-stable. Conversely, if there is one $\hat{\ell}$ for which an eigenvalue $\hat{\lambda}(\hat{\ell})$ has positive real part, the pattern is 2D-unstable. Note that the reduction $\hat{\ell} = 0$ in (5.42) implies only perturbations in the χ -direction. The results for $\hat{\ell} = 0$ therefore correspond to 1D-stability. In what follows, we derive the stability problem of (U_0, V_0) and subsequently formulate three theorems that summarize the results. In section 5.4.1, the Evans function framework is established and analyzed for 1-pulse patterns. These results are expanded to also be valid for periodic patterns, so that the proofs of the theorems can be presented for all stripe patterns constructed in this article in section 5.4.2.

Substituting (5.42) into (5.3) and linearizing, yields the linear stability prob-

lem for $(\hat{u}, \hat{v})$,

$$\begin{aligned} 0 &= \hat{u}_{\chi\chi} + (C + S)\hat{u}_\chi - (A + \hat{\ell}^2 + \hat{\lambda} + V_0^2)\hat{u} - 2U_0V_0\hat{v}, \\ 0 &= \delta^2\hat{v}_{\chi\chi} + S\hat{v}_\chi + (2U_0V_0 - \delta^2\hat{\ell}^2 - \hat{\lambda} - B)\hat{v} + V_0^2\hat{u}. \end{aligned} \quad (5.43)$$

Rescaling according to (5.7), and introducing

$$\begin{aligned} U_0 &= \delta^{1-\gamma+\frac{3}{2}\beta}u_0, & \hat{u} &= \delta^{1-\gamma+\frac{3}{2}\beta}u, & \hat{\lambda} &= \delta^\beta b\lambda, \\ V_0 &= \delta^{-1+\gamma-\frac{1}{2}\beta}v_0, & \hat{v} &= \delta^{-1+\gamma-\frac{1}{2}\beta}v, & \hat{\ell} &= \delta^\theta \sqrt{b}\ell, \end{aligned} \quad (5.44)$$

we derive a linear, non-autonomous, four-dimensional system of first order differential equations in the fast variable ξ , much like system (5.9),

$$\begin{aligned} u_\xi &= \delta^{\gamma-\beta}p, \\ p_\xi &= \delta^{\gamma-\beta} \left[2u_0v_0v + v_0^2u + \delta^{2+\beta}au + (\delta^{2+2\theta+\beta-2\gamma}\ell^2 + \delta^{2+2\beta-2\gamma}\lambda)bu \right. \\ &\quad \left. - \delta^{1+\frac{1}{2}\beta}cp - \delta^{2+\gamma-\beta}sp \right] \\ v_\xi &= q, \\ q_\xi &= [b(\lambda + 1) - 2u_0v_0]v + v_0^2u - \delta^{2+2\theta-\beta}b\ell^2v - \delta^{2(\gamma-\beta)}sq. \end{aligned} \quad (5.45)$$

Here, the new scaling parameter θ is a dummy parameter to make the magnitude of $\hat{\ell}$ explicit. In principle, 2D-stability is only guaranteed if we can make a stability statement for all $\theta \in \mathbb{R}$. In order to perform our slow/fast analysis, however, we restrict to a subclass of transverse perturbations by assuming,

$$\theta > \gamma - \frac{1}{2}\beta - 1. \quad (5.46)$$

That is, we test stability against perturbations with a transverse wave number that may be arbitrarily small, but is not larger than $\mathcal{O}(\delta^{\gamma-\frac{1}{2}\beta-1})$. Since we have assumed **(A3)** that $\gamma - \frac{1}{2}\beta - 1 \geq 0$, this is an asymptotically large bound. Restricting to this subclass in general weakens the stability statements. However, we find that also for θ within this bound, there always exist unstable perturbations. That implies that the restriction to this subclass does not change anything for the 2D-stability of (U_0, V_0) .

Concerning the stability of solutions $(U_0(\xi), V_0(\xi))$ of (5.2), we formulate the following theorems.

Theorem 5.4.1. *Let the assumptions **A1**–**A3** be satisfied and let δ be small enough. Let (U_0, V_0) be a slow/fast solution of (5.2), either of a 1-pulse type, or a spatially periodic pattern with asymptotically large wave length of $\mathcal{O}(\delta^{\frac{1}{2}\beta-1-\gamma})$ as established in Theorems 5.2.1 and 5.3.1. Then there is an $\hat{\ell}$ such that there is a bounded solution to (5.42) for a $\hat{\lambda}(\hat{\ell})$ with positive real part. That is, the 1-pulse and periodic stripe patterns constructed in this article, are 2D-unstable.*

For $\hat{\ell} = 0$, the stability of a solution (U_0, V_0) is tested for perturbations without a component in the y -component. Since we have assumed the constructed solutions in this article have a trivial extension in the y -direction, results for $\hat{\ell} = 0$ correspond to the stability of one-dimensional patterns. For this 1D-stability of a spatially one-dimensional pattern, which we will, slightly abusive, also label (U_0, V_0) , we formulate Theorem 5.4.2.

For a clear presentation of the results, we introduce,

$$C_1 = \frac{\bar{u}^2}{6b}, \quad C_2 = \frac{c^2 + 4a}{b}. \quad (5.47)$$

where $\bar{u}$ is the value of $\delta^{\gamma-\frac{3}{2}\beta-1}U_0$ in the fast regime I_f , where U_0 is to leading order constant, see 5.9. In the case that $(U_0, V_0) = (U_{\text{hom}}, V_{\text{hom}})$, the value of $\bar{u}$ is in fact reported in (5.30).

Theorem 5.4.2. *Let the assumptions **A1** and **A3** be satisfied and let $\beta \geq \frac{2}{3}(\gamma - 1)$. Let (U_0, V_0) be a slow/fast 1-pulse solution of (5.2) in one spatial dimension. That is, let (U_0, V_0) be $(U_{\text{hom}}, V_{\text{hom}})$ established in Theorem 5.2.1 without the trivial extension in the y -direction. The 1D-stability of this pattern can be summarized as follows.*

- (i) *If $\beta < \gamma - \frac{1}{2}$, there is a bounded solution to (5.42) with $\Re(\hat{\lambda}) > 0$, so (U_0, V_0) is 1D-unstable.*
- (ii) *If $\beta > \gamma - \frac{1}{2}$ and **A2** is satisfied, all nontrivial eigenvalues $\hat{\lambda}$ corresponding to pattern (U_0, V_0) have negative real part, so the pattern is 1D-stable.*
- (iii) *If $\beta = \gamma - \frac{1}{2}$ and **A2** is satisfied, then a pair of eigenvalues $\hat{\lambda}_{1,2}$ passes through the imaginary axis (i.e. a Hopf bifurcation occurs) if $C_1 = 2H^*$. Here, H^* is given as the explicit solution of an expression in terms of hypergeometric functions, and $H^* \approx 0.661$. Using (5.30), we equivalently formulate that for $\sqrt{\frac{b}{c^2+4a}} < 2H^*$, the pattern (U_0, V_0) is 1D-stable.*
- (iv) *If $\beta > \gamma - \frac{1}{2}$ and $\beta = \frac{2}{3}(\gamma - 1)$, then an eigenvalue $\hat{\lambda}$ passes through zero if*

$$C_1 \sqrt{C_2} = 1. \quad (5.48)$$

The pattern (U_0, V_0) is 1D-stable if $C_1\sqrt{C_2} < 1$. This implies, using (5.35) that the pattern with $\bar{u} = u_0^+$ is 1D-unstable, and the pattern with $\bar{u} = u_0^-$ is 1D-stable. The bifurcation occurs exactly at the saddle-node bifurcation, at which $u_0^\pm$ collide.

(v) If $\beta = \gamma - \frac{1}{2}$, $\beta = \frac{2}{3}(\gamma - 1)$ and $C_2 > C_2^*$, which is a explicit solution of an equation in terms of hypergeometric functions and $C_2^* \approx 1.333$, then an eigenvalue $\hat{\lambda}$ passes through zero if (5.48) is satisfied. This implies, using (5.35) that the pattern with $\bar{u} = u_0^+$ is 1D-unstable, and the pattern with $\bar{u} = u_0^-$ is 1D-stable. The bifurcation occurs exactly when (5.34) is an equality; where $u_0^\pm$ collide.

(vi) If $\beta = \gamma - \frac{1}{2}$, $\beta = \frac{2}{3}(\gamma - 1)$ and $C_2 \leq C_2^*$ as defined in case (v), then a pair of eigenvalues $\hat{\lambda}_{1,2}$ passes through the imaginary axis (i.e. a Hopf bifurcation occurs) if

$$C_1 = Z(C_2) := \frac{1}{\sqrt{4i\bar{\kappa}(C_2) + C_2}} \left(\frac{9}{K(2\sqrt{i\bar{\kappa}(C_2) + 1})} - 1 \right). \quad (5.49)$$

Here, the function $\bar{\kappa}(C_2)$ is the function that assigns to every $C_2 \in (0, C_2^*)$ the imaginary part of the eigenvalues at the Hopf bifurcation, and $K(P)$ is defined in (5.77). The bifurcation never occurs for patterns with u_0^+ of (5.35), and the pattern with $\bar{u} = u_0^+$ is 1D-unstable. The pattern with $\bar{u} = u_0^-$ may undergo the Hopf bifurcation and is 1D-stable if $C_1 < Z(C_2)$.

For a schematic representation of the stability results in different scaling regimes from Theorem 5.4.2, see Figure 5.12.

Theorem 5.4.3. *Let the assumptions **A1** and **A3** be satisfied and let $\beta \geq \frac{2}{3}(\gamma - 1)$. Let (U_0, V_0) be a slow/fast periodic solution of (5.2) in one spatial dimension with wave length of $\mathcal{O}(\frac{1}{\delta^{1+\gamma-\frac{1}{2}\beta}})$. That is, let (U_0, V_0) be $(U_{p,s}, V_{p,s})$ as established in Theorem 5.3.1 without the trivial extension in the y -direction. The 1D-stability of this pattern can be summarized as follows.*

- (i) If $\beta < \gamma - \frac{1}{2}$, there is a solution to (5.42) with $\Re(\hat{\lambda}) > 0$, so (U_0, V_0) is 1D-unstable.
- (ii) If $\beta > \gamma - \frac{1}{2}$ and $\beta < 2\gamma$, all eigenvalues $\hat{\lambda}$ corresponding to pattern (U_0, V_0) have negative real part, so the pattern is 1D-stable.
- (iii) If $\beta = \gamma - \frac{1}{2}$ and $\beta < 2\gamma$, then a pair of eigenvalues $\hat{\lambda}_{1,2}$ passes through the imaginary axis (i.e. a Hopf bifurcation occurs) if $C_1 = 2H^*$. Here, H^* is

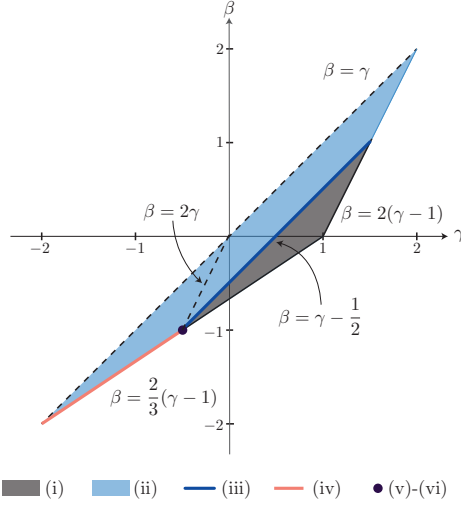


Figure 5.12: Schematic representation of 1D-stability regimes as described in Theorems 5.4.2 and 5.4.3. The shaded regimes are the regimes in the (γ, β) -plane where stripe patterns exist, defined by assumptions **A1**, **A3** and $\beta \geq \frac{2}{3}(\gamma - 1)$. Stripe patterns with scaling parameters in the gray shaded areas are 1D-unstable. Stripe patterns with scaling parameters in the blue shaded area are 1D-stable. Stripe patterns with scaling parameters on the borders may undergo a bifurcation when parameters change as described in Theorem 5.4.2. The regimes (i)–(vi) correspond to the subresults with the same labels in Theorem 5.4.2, and the regimes (i)–(iii) with those of Theorem 5.4.3.

given as the explicit solution of an expression in terms of hypergeometric functions, and $H^* \approx 0.661$. For $C_1 < 2H^*$, the pattern (U_0, V_0) is 1D-stable.

Note that the cases (i)–(iii) of Theorem 5.4.3 do not cover the entire triangular scaling regime defined by assumptions **A1**–**A3** and Figure 5.2, due to the extra assumptions $\beta < 2\gamma$ in cases (ii) and (iii). This is associated with the validity regime of the slow/fast approximation of the Evans function, and is explained in more detail in section 5.4.1.

Remark 5.3. In the case of the homoclinic pulses of Theorem 5.4.2, the fact that the operator is sectorial immediately establishes nonlinear stability if the pattern is spectrally stable, [36, 74]. We refrain from going into details about the nonlinear stability of the periodic patterns and two-dimensional patterns: stability statements in Theorem 5.4.3 and upcoming Corollary 5.4.5 concern only

spectral stability with respect to $\mathcal{O}(1)$ eigenvalues. This follows the approach of [34], and more details are reported in [31].

5.4.1 The Evans function and associated nonlinear eigenvalue problem

For the proof of Theorems 5.4.1–5.4.3, we establish an Evans function framework for the 1-pulse patterns, $(U_{\text{hom}}, V_{\text{hom}})$ constructed in section 5.2. In section 5.4.1 we will show how this framework can be in essence carried over to periodic patterns with periods of sufficient asymptotic length, after which we may proceed to section 5.4.2 for the proofs of the Theorems 5.4.1, 5.4.2. To make use of the slow/fast structure of the system in an efficient way, we will use the four-dimensional formulation of the stability problem, (5.45).

Homoclinic 1-pulse solutions

As an equivalent representation of (5.45), we write

$$\frac{\partial}{\partial \xi} \phi(\xi; \lambda, \ell) = \mathcal{A}(\xi; \lambda, \ell) \phi(\xi; \lambda, \ell), \quad (5.50)$$

where $\phi(\xi; \lambda, \ell) = (u(\xi), p(\xi), v(\xi), q(\xi))$, and $\mathcal{A}$ is a 4×4 matrix. We will lay out the stability analysis in terms of the fast variable ξ . For ξ outside I_f , the solution v_0 is exponentially close to zero. Moreover, every term of (5.45) that involves u_0 , which varies outside the fast regime, is multiplied with v_0 . Hence, the matrix $\mathcal{A}(\xi; \lambda, \ell)$ approaches a constant matrix outside that fast regime;

$$\lim_{\xi \rightarrow \pm\infty} \mathcal{A}(\xi; \lambda, \ell) = \mathcal{A}_\infty(\lambda, \ell),$$

with

$$\mathcal{A}_\infty(\lambda, \ell) = \begin{pmatrix} 0 & \delta^{\gamma-\beta} & 0 & 0 \\ \delta^{2-\gamma+\beta} [\delta^{2\gamma-\beta} a + b(\delta^{2\theta-\beta} \ell^2 + \lambda)] & -\delta^{1+\gamma-\beta} (\delta^{\frac{1}{2}\beta} c + \delta^{1+\gamma-\beta} s) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & b(\lambda + 1 + \delta^{2+2\theta-\beta} \ell^2) & -\delta^{2(\gamma-\beta)} s & \end{pmatrix} \quad (5.51)$$

The matrix $\mathcal{A}_\infty$ has four eigenvalues $\Lambda_{1,2,3,4}$. Two of those eigenvalues are asymptotically small and two are $\mathcal{O}(1)$. Their leading order approximations are,

$$\begin{aligned}\Lambda_1 &= \sqrt{b(\lambda + 1 + \delta^{2+2\theta-\beta}\ell^2)} + \text{h.o.t.} \\ \Lambda_2 &= \frac{1}{2}\delta^{1+\gamma-\frac{1}{2}\beta} \left[-c + \sqrt{c^2 + 4a + 4\delta^{\beta-2\gamma}(\delta^{2\theta-\beta}b\ell^2 + b\lambda)} \right] + \text{h.o.t.} \\ \Lambda_3 &= \frac{1}{2}\delta^{1+\gamma-\frac{1}{2}\beta} \left[-c - \sqrt{c^2 + 4a + 4\delta^{\beta-2\gamma}(\delta^{2\theta-\beta}b\ell^2 + b\lambda)} \right] + \text{h.o.t.} \\ \Lambda_4 &= -\sqrt{b(\lambda + 1 + \delta^{2+2\theta-\beta}\ell^2)} + \text{h.o.t.}\end{aligned}\tag{5.52}$$

Assumptions **A1**–**A3** and (5.46), imply that indeed $\Lambda_{2,3} \ll 1$, while $\Lambda_{1,4} = \mathcal{O}(1)$. Furthermore, the eigenvalues $\Lambda_{1,2}$ have by definition positive real part and are unstable, while $\Lambda_{3,4}$ have negative real part and are stable. The corresponding eigenvectors are

$$\begin{aligned}E_{1,4} &= (0, 0, 1, \Lambda_{1,4})^T, \\ E_{2,3} &= (1, \delta^{\beta-\gamma}\Lambda_{2,3}, 0, 0)^T.\end{aligned}\tag{5.53}$$

The essential spectrum associated with (5.50) coincides with all $\lambda(\ell)$ for which $\mathcal{A}_\infty$ has an eigenvalue $\Lambda_i \in i\mathbb{R}$, that is,

$$\sigma_{\text{ess}} = \bigcup_{k, \ell \in \mathbb{R}} \left\{ -1 - \delta^{2+2\theta-\beta}\ell^2 - k^2, \frac{1}{4\delta^{\beta-2\gamma}b}(-4a - 4\delta^{2\theta-2\gamma}b\ell^2 + 2cik - k^2) \right\}\tag{5.54}$$

Depending on whether $\beta \gtrless 2\gamma$, the maximal real part of the essential spectrum is -1 or $-\delta^{2\gamma-\beta}\frac{a}{b}$, see a schematic representation of both cases in $\mathbb{C}$ in 5.13. The full linear stability problem (5.50), has four independent (vector-)solutions $\phi_j(\xi; \lambda, \ell)$. The theory in [35, 36] explains that we may introduce eigenvectors ϕ_j such that,

$$\begin{aligned}\lim_{\xi \rightarrow -\infty} \phi_j(\xi, \lambda, \ell)e^{-\Lambda_j\xi} &= E_j \quad \text{for } j = 1, 2, \\ \lim_{\xi \rightarrow \infty} \phi_j(\xi, \lambda, \ell)e^{-\Lambda_j\xi} &= E_j \quad \text{for } j = 3, 4.\end{aligned}\tag{5.55}$$

In particular, this implies that $\phi_{1,2} \rightarrow (0, 0, 0, 0)$ as $\xi \rightarrow -\infty$ and $\phi_{3,4} \rightarrow (0, 0, 0, 0)$ as $\xi \rightarrow \infty$. Because Λ_1 is the largest, positive eigenvalue, a general solutions ϕ

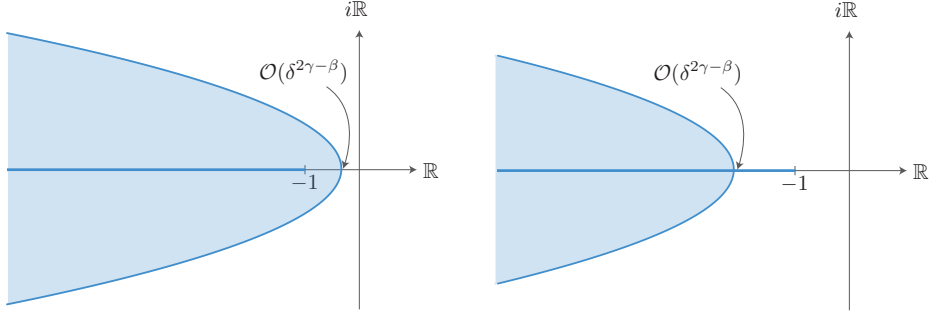


Figure 5.13: Schematic representation of the essential spectrum, (5.54) in $\mathbb{C}$. On the left, the case that $2\gamma > \beta$, so the maximal real part of the essential spectrum is $-\delta^{2\gamma-\beta} \frac{a}{b}$. On the right the case $2\gamma < \beta$, so the maximal real part of the essential spectrum is -1 .

will grow as $e^{\Lambda_1 \xi}$ as $\xi \rightarrow \infty$. We define the *fast transmission function* $t_f(\lambda, \ell)$, which is an analytic function over λ as,

$$\lim_{\xi \rightarrow \infty} \phi_1(\xi; \lambda, \ell) e^{-\Lambda_1 \xi} = t_f(\lambda, \ell) E_1(\lambda, \ell). \quad (5.56)$$

Not all ϕ_i necessarily grow with the largest rate, though. In the case that $t_f(\lambda, \ell) \neq 0$, we can define ϕ_2 uniquely by assuming it does not grow like $e^{\Lambda_1 \xi}$, i.e.

$$\lim_{\xi \rightarrow \infty} \phi_2 e^{-\Lambda_1 \xi} = (0, 0, 0, 0)^T. \quad (5.57)$$

In other words, the only vector solution that grows with the fast rate as $\xi \rightarrow \infty$ is $\phi_1(\xi)$, because (5.55) has already put a boundary condition on ϕ_2 and ϕ_3 . A detailed justification of this procedure follows especially from Lemma 3.7 in [35]. Generically, the behavior of the second solution ϕ_2 is dominated by the slow growth rate Λ_2 for $\xi > 0$. Hence, we define a *slow transmission function*, $t_s(\lambda, \ell)$.

$$\lim_{\xi \rightarrow \infty} \phi_2(\xi; \lambda, \ell) e^{-\Lambda_2 \xi} = t_s(\lambda, \ell) E_2(\lambda, \ell). \quad (5.58)$$

The Evans function is the determinant of the four independent solutions of $\mathcal{A}_\infty$,

$$\mathcal{D}(\lambda, \ell) = \det [\phi_1, \phi_2, \phi_3, \phi_4] e^{-\int_0^\xi \text{Tr}(\mathcal{A}(\eta; \lambda, \ell)) d\eta}, \quad (5.59)$$

see [35]. The Evans function is only defined for λ outside of the essential spectrum, (5.54). Since

$$\text{Tr}(\mathcal{A}(\eta; \lambda, \ell)) = \sum_j \Lambda_j = -\delta^{2(\gamma-\beta)} s,$$

and because the Evans function is independent of ξ , see [1], (5.59) is equivalent to,

$$\begin{aligned} \mathcal{D}(\lambda, \ell) &= \lim_{\xi \rightarrow \infty} \det [\phi_1 e^{-\Lambda_1 \xi}, \phi_2 e^{-\Lambda_2 \xi}, \phi_3 e^{-\Lambda_3 \xi}, \phi_4 e^{-\Lambda_4 \xi}] \\ &= \det [t_f(\lambda, \ell) E_1(\lambda, \ell), t_s(\lambda, \ell) E_2(\lambda, \ell), E_3(\lambda, \ell), E_4(\lambda, \ell)] \quad (5.60) \\ &= \delta^{\beta-\gamma} t_s(\lambda, \ell) t_f(\lambda, \ell) ((\Lambda_4 - \Lambda_1)(\Lambda_3 - \Lambda_2)), \end{aligned}$$

to leading order. Zeros of the Evans function coincide with eigenvalues $\lambda(\ell)$. Since outside of the essential spectrum σ_{ess} , see (5.54), the Λ_i never coincide, we conclude that the Evans function is zero only if the product of the transmission functions, $t_s t_f$ is zero. For the fast transmission function, the zeros can simply be found in literature. The associated fast reduced stability problem, that can be obtained from (5.45) by taking the limit $\delta \rightarrow 0$ and setting $u \equiv 0$, is

$$v_{\xi\xi} + 2\bar{u}v_0v - b(\lambda + 1)v = 0. \quad (5.61)$$

Which equivalent to those reported in [35, 36, 143]. The fast isolated eigenvalues are

$$\lambda_f^0 = \frac{5}{4}, \quad \lambda_f^1 = 0, \quad \lambda_f^2 = -\frac{3}{4}. \quad (5.62)$$

Paradoxically, the positive eigenvalue λ_f^0 does not immediately imply instability of (u_0, v_0) , because the slow transmission is not analytic but merely meromorphic and has a pole at λ_f^0 , as we will show below. This zero-pole cancellation is explained in full detail in [35, 36]. In [35], it is also shown that N -pulses pick up the same fast eigenvalues but with multiplicity N , which are not canceled by the order 1 poles of the slow transmission function. Hence, N -pulses are unstable. For more details on this mechanism, see [38].

Zeros of the slow transmission function

We determine the zeros of the slow transmission function by matching the values of u and p in- and outside the fast interval I_f , see (5.21). The change in u and p is measured by the uniquely determined fundamental solution ϕ_2 . We know that it does not grow with the fast rate, see (5.57), so outside I_f and with $\xi > 0$,

$$\phi_2(\xi) = t_s e^{\Lambda_2 \xi} E_2 + \tilde{t}_s e^{\Lambda_3 \xi} E_3 + \tilde{t}_f e^{\Lambda_4 \xi} E_4,$$

where $\tilde{t}_{s,f}$ are also transmission functions of λ and ℓ . The Λ_4 -term does not contribute to the leading order behavior, however, because Λ_4 is large and negative. In fact, outside I_f , $e^{\Lambda_4 \xi}$ is already exponentially small. On the other side outside of I_f , where $\xi < 0$, we know by (5.55)

$$\phi_2(\xi) = e^{\Lambda_2 \xi} E_2,$$

up to exponentially small corrections. We measure the change of ϕ_2 over I_f from outside of I_f (in the slow regime) first. We define the slow difference function Δ_s as follows,

$$\Delta_s \begin{pmatrix} u \\ p \\ v \\ q \end{pmatrix} = \lim_{\xi \downarrow \delta^{-\mu}} (t_s e^{\Lambda_2 \xi} E_2 + \tilde{t}_s e^{\Lambda_3 \xi} E_3 + \tilde{t}_f e^{\Lambda_4 \xi} E_4) - \lim_{\xi \uparrow -\delta^{-\mu}} e^{\Lambda_2 \xi} E_2. \quad (5.63)$$

Upon assuming **A1–A3** and (5.46), we can always choose a $\mu > 0$ (recall (5.21)) such that

$$\begin{aligned} \Delta_s u &= t_s e^{\Lambda_2 \delta^{-\mu}} E_2 + \tilde{t}_s e^{\Lambda_3 \delta^{-\mu}} E_3 + \tilde{t}_f e^{\Lambda_4 \delta^{-\mu}} E_4 - e^{\Lambda_2 \delta^{-\mu}} E_2, \\ &= t_s + \tilde{t}_s - 1, \end{aligned} \quad (5.64)$$

to leading order. We know that during the fast transition, the u -component does not change (recall (5.45)), therefore

$$\tilde{t}_s = 1 - t_s.$$

Although u is constant even up to the second order corrections during the fast transition, p is not constant in the first correction, see (5.9). We also measure the change in p with the slow difference function,

$$\begin{aligned} \Delta_s p &= t_s e^{\Lambda_2 \delta^{-\mu}} \delta^{\beta-\gamma} \Lambda_2 + \tilde{t}_s e^{\Lambda_3 \delta^{-\mu}} \delta^{\beta-\gamma} \Lambda_3 + \tilde{t}_f e^{\Lambda_4 \delta^{-\mu}} \cdot 0 - e^{\Lambda_2 \delta^{-\mu}} \delta^{\beta-\gamma} \Lambda_2, \\ &= \delta^{\beta-\gamma} (t_s - 1) (\Lambda_2 - \Lambda_3), \end{aligned} \quad (5.65)$$

to leading order. On the other hand, we can measure the change in p over the fast regime, much like section 5.2, equation (5.29). We define the fast difference function Δ_f as,

$$\begin{aligned}
 \Delta_f p &= \int_{I_f} p_\xi(\xi) d\xi, \\
 &= \delta^{\gamma-\beta} \int_{I_f} \left(v_0^2 u + 2\bar{u} v_0 v + \delta^{2+\beta} a u + \delta^{2+2\theta-2\gamma+\beta} b \ell^2 u + \delta^{2-2\gamma+2\beta} b \lambda u \right. \\
 &\quad \left. - (\delta^{1+\frac{1}{2}\beta} c + \delta^{2+\gamma-\beta} s) u_\xi \right) d\xi,
 \end{aligned} \tag{5.66}$$

We use the fact that, at the boundary of I_f , the two difference functions must be equal; $\Delta_f p = \Delta_s p$. Moreover, as we are interested in zeros (and poles) of the slow transmission function, we set $t_s = 0$, so that

$$\delta^{\beta-\gamma} (\Lambda_3 - \Lambda_2) = \Delta_f p. \tag{5.67}$$

In the evaluation of $\Delta_f p$, we may choose $u = 1$, because it must be a constant and we use the one-parameter freedom of choice in determining an eigenfunction. If **A1**–**A3** and (5.46) are satisfied, we can always choose a μ as defined in (5.21) such that $\Delta_f p$ is, at leading order, purely associated with the fast variables, that is,

$$\Delta_f p = \delta^{\gamma-\beta} \int_{I_f} 2\bar{u} v_0(\xi) v(\xi) + v_0^2(\xi) d\xi, \tag{5.68}$$

In particular, this implies we must choose

$$0 < \mu < \min\{2 + 2\theta - 2\gamma + \beta, 2 - 2\gamma + 2\beta, \gamma - \beta, 1 + \gamma - \frac{1}{2}\beta\}. \tag{5.69}$$

See Figure 5.2 and take into account assumption (5.46) to verify that in our restricted regime for γ and β , we can always choose μ within this range.

Equation (5.68) does not stand on itself but is directly linked to the stability problem (5.50). Our slow eigenvalue problem thus becomes

$$\begin{cases} \delta^{2(\gamma-\beta)} \int_{I_f} 2\bar{u} v_0(\xi) v_{\text{in}}(\xi) + v_0^2(\xi) d\xi = \Lambda_3 - \Lambda_2, \\ v_{\text{in},\xi\xi} + (2\bar{u} v_0 - b(\lambda + 1)) v_{\text{in}} = -v_0^2 \end{cases} \tag{5.70}$$

We can rewrite (5.70) into one equation: the so-called nonlinear eigenvalue problem (NLEP) as is used in [34],

$$v_{\xi\xi} + (2\bar{u} v_0 - b(\lambda + 1)) v = \frac{2\delta^{2(\gamma-\beta)} \bar{u} v_0^2}{\delta^{2(\gamma-\beta)} \int_{I_f} v_0^2(\xi) d\xi - \Lambda_3 + \Lambda_2} \int_{I_f} v_0(\xi) v(\xi) d\xi. \tag{5.71}$$

Using (5.19), we derive

$$\int_{I_f} v_0^2(\xi) d\xi = \frac{6b\sqrt{b}}{\bar{u}^2}, \quad (5.72)$$

which simplifies equation (5.71),

$$\begin{aligned} v_{\text{in},\xi\xi} + \left(\frac{3b}{\cosh^2\left(\frac{\sqrt{b}\xi}{2}\right)} - b(\lambda + 1) \right) v_{\text{in}} \\ = \frac{\frac{27}{4}\delta^{2(\gamma-\beta)}b^3\operatorname{sech}^2\left(\frac{\sqrt{b}}{2}\xi\right)}{\delta^{2(\gamma-\beta)}b\sqrt{b} - (\Lambda_3 - \Lambda_2)\bar{u}^2} \int_{I_f} \frac{v_{\text{in}}}{\cosh^2\left(\frac{\sqrt{b}}{2}\xi\right)} d\xi. \end{aligned} \quad (5.73)$$

Following [34], we transform equation (5.71) using the following substitutions,

$$\xi = \frac{2}{\sqrt{b}}t, \quad v_{\text{in}}(\xi) = y(t), \quad \lambda = \frac{1}{4}P^2 - 1. \quad (5.74)$$

We only consider $\operatorname{Re}(P) > 0$, to stay away from the essential spectrum, see (5.54) and Figure 5.13. To convert P into λ , we use the principal square root. Substituting (5.74) into (5.73) yields

$$y_{tt} + \left(\frac{12}{\cosh^2(t)} - P^2 \right) y = \frac{K}{\cosh^4(t)} \int_{I_f} \frac{y(t)}{\cosh^2(t)} dt, \quad (5.75)$$

with

$$K = \frac{9\delta^{2(\gamma-\beta)}}{\delta^{2(\gamma-\beta)} + \frac{\bar{u}^2(\Lambda_2 - \Lambda_3)}{6b\sqrt{b}}}. \quad (5.76)$$

Following [34], the differential equation for y can be modified to a hypergeometric differential equation for $G(x)$ with a second substitution,

$$y(t) = G(t)/(\cosh(t))^P, \quad t = \tanh^{-1}(2x - 1).$$

The exact derivation of the solutions of that hypergeometric differential equation is detailed in [34], and an alternative expression for K is derived,

$$K(P) = \frac{P(P-1)(P-2)(P-3)}{16\mathcal{R}(P)}, \quad (5.77)$$

where

$$\mathcal{R}(P) = \frac{-1}{(P+3)(P+2)(P+1)} \int_0^1 \int_0^\xi \xi(1-\xi) \left(\frac{x(1-\xi)}{\xi(1-x)} \right)^{P/2} k(P, 1-x) k(P, \xi) dx d\xi, \quad (5.78)$$

where $k(P, \xi)$ is defined as,

$$k(P, \xi) = (P-3)(P-2)(P-1) + 12(P-3)(P-2)\xi + 60(P-3)\xi^2 + 120\xi^3. \quad (5.79)$$

For consistency in K , we must equate (5.76) and (5.77), and we arrive at the final formulation of the eigenvalue problem.

$$\delta^{-2(\gamma-\beta)} (\Lambda_2 - \Lambda_3) \frac{\bar{u}^2}{6b\sqrt{b}} = \frac{9}{K(P)} - 1, \quad (5.80)$$

with $K(P)$ as in the formulation (5.77). The right hand side of (5.80) has poles of $\mathcal{O}(1)$ for $P = 1$ and $P = 3$, see Figure 5.14 and [34, 35]. They link directly to two of the fast eigenvalues (5.62), $\lambda_f^0 = \frac{5}{4}$ and λ_f^2 and hence the positive eigenvalue λ_f^0 does not necessarily destabilize the pattern. This phenomenon is often referred to as the ‘NLEP-paradox’ see [35].

Periodic solutions

In the previous section, we have derived equation (5.80) which determines the stability properties of a 1-pulse $(U_{\text{hom}}, V_{\text{hom}})$. In this subsection, we briefly comment on how the same equation determines stability of periodic solutions $(U_{\text{p},s}, V_{\text{p},s})$. For homoclinic solutions, which are localized, the eigenfunctions are localized as well. Hence, the full spectrum consists of discrete eigenvalues $\hat{\lambda}$ for given $\hat{\ell}$, united with the essential spectrum corresponding to the background state. Perturbations of periodic solutions are in general represented by both a wave number $\hat{\ell}$ as well as a $\hat{\gamma} \in S^1$, because the eigenfunctions are not localized. This implies that the eigenvalues $\hat{\lambda}$ outside σ_{ess} , see (5.54), are not discrete for a given $\hat{\ell}$. Instead, there exist curves of essential spectrum parametrized by $\hat{\gamma}$, [51].

For periodic solutions with a sufficient length, these $\hat{\gamma}$ -parametrized curves of spectrum may be exponentially approximated by the discrete eigenvalue corresponding to a localized pulse, because the periodic solution is ‘nearly’ localized. The period of the solutions constructed in section 5.3, has a length of $\mathcal{O}(\frac{1}{\delta^{\gamma-\frac{1}{2}\beta+1}})$ in ξ . If we assume that $q(0) = 0$ we can construct a fundamental interval for the periodic solution as

$$I_{\text{per}} = \left(-\frac{l}{\delta^{\gamma-\frac{1}{2}\beta+1}}, \frac{l}{\delta^{\gamma-\frac{1}{2}\beta+1}} \right),$$

with $l > 0$. The eigenfunctions (U, V) of (5.42) corresponding to a periodic solution $(U_{\text{per},s}, V_{\text{per},s})$ may then be called nearly localized if both U and V are exponentially small on the boundaries of I_{per} . For the V -pulse, this is clear, because V decays exponentially fast to zero already within the fast interval I_f , which is smaller than I_{per} . What is left is to verify that the decay rate of U is also fast enough. About this decay rate and the validity of the approximation, we formulate the following lemma.

Lemma 5.4.4. *For periodic solutions of (5.2) as constructed in section 5.3, with wave length of $\mathcal{O}\left(\frac{1}{\delta^{1+\gamma-\frac{1}{2}\beta}}\right)$, we may approximate the spectrum outside σ_{ess} by the discrete values that are solutions to (5.80) in any of the following regimes.*

- (i) $\beta < 2\gamma$;
- (ii) $\beta \geq 2\gamma$ and $\gamma - \frac{1}{2}\beta - 1 < \theta < \gamma$.

Proof. To show this, we will work with the rescaled coordinates and system (5.45). Outside the fast interval I_f , the equation for u decouples because v is exponentially small. We obtain,

$$u_{\xi\xi} = \delta^{2\gamma-2\beta} \left[\delta^{2+\beta} a u + (\delta^{2+2\theta+\beta-2\gamma} \ell^2 + \delta^{2+2\beta-2\gamma} \lambda) b u - \left(\delta^{1+\frac{3}{2}\beta-\gamma} c + \delta^2 s \right) u_{\xi} \right], \quad (5.81)$$

which is a second order ODE for u with constant coefficients. The solution for u is a linear combination of exponentials in variable $\delta^r \xi$, where

$$r = \min\left\{1, 1 + \gamma - \frac{1}{2}\beta, 1 + \theta - \frac{1}{2}\beta\right\}, \quad (5.82)$$

is the decay rate of u . If we require u to be exponentially small on the boundaries of I_{per} , which is $\mathcal{O}\left(\frac{1}{\delta^{1+\gamma-\frac{1}{2}\beta}}\right)$, we must satisfy

$$1 + \gamma - \frac{1}{2}\beta > r = \min\left\{1, 1 + \gamma - \frac{1}{2}\beta, 1 + \theta - \frac{1}{2}\beta\right\} \quad (5.83)$$

In other words, the length of the fundamental interval I_{per} must be asymptotically strictly larger than the decay of u . If we want to exponentially approximate the spectrum corresponding to periodic solutions by discrete eigenvalues, condition (5.83) must be satisfied. Note that the strict inequality (5.83) can never be satisfied if $r = 1 + \gamma - \frac{1}{2}\beta$, the same parameter combination is stated on the left side of the inequality. This implies that if $\beta \geq 2\gamma$, a restriction must be put on θ so that $r = 1 + \theta - \frac{1}{2}\beta$, in order to satisfy (5.83). Combining this with condition

(5.46), this yields the result stated in (ii). If, on the other hand, $\beta < 2\gamma$, then $r \neq 1 + \gamma - \frac{1}{2}\beta$ and this immediately implies that (5.83) is satisfied, so we arrive at the statement of (i). $\square$

Since θ is only a dummy parameter restrictions on θ change the function space of perturbations against which we will test the (in)stability of periodic patterns. By condition (5.46), we put an upper bound on the transverse wave number of the perturbation. A consequence of Lemma 5.4.4 is that if $\beta \geq 2\gamma$, the stability statements derived from equation (5.80) for periodic solutions are only valid for a smaller subclass of perturbations. Namely, those perturbations that have a transverse wave number that is not only bounded below by (5.46), but also bounded above by requiring $\theta < \gamma$.

5.4.2 Proof of stability theorems

In this section, we analyze (5.80), to prove Theorems 5.4.1 and 5.4.2. The right-hand side of (5.80) is plotted as a graph over $P \in \mathbb{R}$ in Figure 5.14. Note

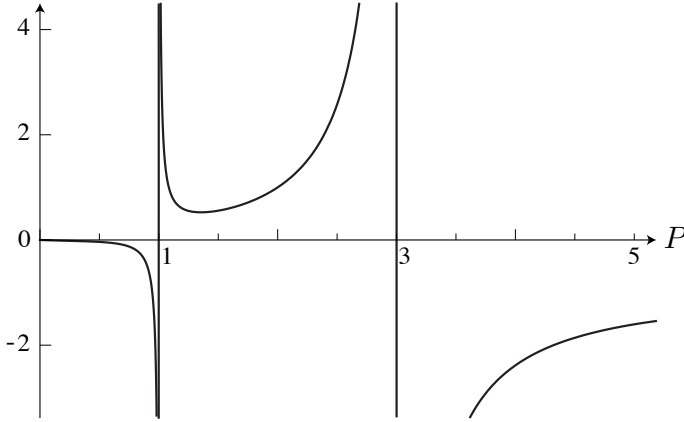


Figure 5.14: Black: Graph of $\frac{9}{K(P)} - 1$ with $P > 0$ and real-valued. Note that there are vertical asymptotes at $P = 1$ and $P = 3$. These are the poles of the slow transmission function, and they cancel two of the zeros of the fast transmission function, λ_f^0 and λ_f^2 , see (5.62).

that, as $\lambda \in \mathbb{C}$, so is P . The representation of Figure 5.14 is therefore not exhaustive. Note that, indeed, the graph is singular at $P = 1$ and $P = 3$, where the function has poles of order 1. In the cases that λ is real, however, the solutions of (5.80) may be visualized as the intersections of the graph in Figure 5.14 and a real-valued curve of the left hand side of (5.80) as a function over P . We clarify

this in the proofs of the Theorems 5.4.1–5.4.3. As was already obvious from their formulations, the magnitude of $\Lambda_{2,3}$ makes a crucial difference. From (5.52) which we derive

$$\Lambda_2 - \Lambda_3 = \delta^{1+\gamma-\frac{1}{2}\beta} \sqrt{c^2 + 4a + 4b(\delta^{2\theta-2\gamma}\ell^2 + \delta^{\beta-2\gamma}\lambda)}. \quad (5.84)$$

Note that

$$\mathcal{O}(\Lambda_2 - \Lambda_3) = \mathcal{O}(\delta^r), \quad (5.85)$$

where r was defined in (5.82).

Proof. (of Theorem 5.4.1).

If the left hand side of equation (5.80) is much larger than $\mathcal{O}(1)$, there are solutions for $P \gtrsim 1$ and $P \lesssim 3$, because the right hand side has poles for $P = 1$ and $P = 3$. By choosing θ appropriately, namely

$$\theta < 2\gamma - \frac{3}{2}\beta - 1, \quad (5.86)$$

we obtain $r < 2(\gamma - \beta)$, so that the left hand side of (5.80) is asymptotically large. That implies that there is always a $\hat{\ell}$ such that there exists an solution $P \lesssim 3$, which corresponds to $\lambda = \frac{5}{4}$. Note that condition (5.46) is automatically satisfied in this case. Hence, perturbations with a transverse wave number that is small enough grow exponentially, so the patterns are 2D-unstable. This result holds for both the 1-pulse and periodic solutions, because (5.86) implies by assumption **A2** that $\theta < \gamma$, see Lemma 5.4.4. $\square$

Proof. (of Theorem 5.4.2)

This theorem concerns only patterns in one spatial dimension. Hence, we do not consider transverse perturbations so we may set $\ell = 0$ in equation (5.80) or, more specifically, (5.84). This simplifies equation (5.80) to

$$\delta^{1-\gamma+\frac{3}{2}\beta} \sqrt{c^2 + 4a + 4\delta^{\beta-2\gamma}b\lambda} \frac{\bar{u}^2}{6b\sqrt{b}} = \frac{9}{K(P)} - 1. \quad (5.87)$$

- (i) If $\beta < \gamma - \frac{1}{2}$ and $\beta \geq \frac{2}{3}(\gamma - 1)$, then the left hand side of (5.80) is much larger than $\mathcal{O}(1)$, so the same argument as in the proof of Theorem 5.4.1 applies. There is an eigenvalue $\lambda \lesssim \frac{5}{4}$, so the solution is 1D-unstable.
- (ii) If $\beta > \gamma - \frac{1}{2}$ and $\beta > \frac{2}{3}(\gamma - 1)$, the left hand side of (5.80) is asymptotically small. Thus, we solve

$$0 = \frac{9}{K(P)} - 1.$$

By a numerical evaluation of the analytic expressions (5.76)–(5.79), we find that this occurs for $P \approx 0.56 \pm 0.52i$ or $\lambda \approx -0.99 \pm 0.14i$. Hence, the pattern is 1D-stable.

- (iii) If $\beta = \gamma - \frac{1}{2}$ and $\beta > \frac{2}{3}(\gamma - 1)$, recall the definition of C_1 in (5.47), so that the leading order of equation (5.87) becomes

$$\frac{1}{2}C_1 = \frac{1}{\sqrt{P^2 - 4}} \left(\frac{9}{K(P)} - 1 \right), \quad (5.88)$$

which is equivalent to equation (5.15) of [34]. Thus, there is a Hopf bifurcation when this equation is solved for a purely imaginary set of eigenvalues $\lambda_{\pm}$. Using numerical evaluation we can derive immediately that this occurs when

$$\frac{1}{2}C_1 = H^* \approx 0.661,$$

at which $\lambda \approx \pm 0.535i$. This implies that the pattern is 1D-stable for $C_1 < 2H^*$ and 1D-unstable for $C_1 > 2H^*$. Since we know that in this case $\bar{u} = u_0 = \frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}$ (see 5.30), we can simplify this to

$$\sqrt{\frac{b}{c^2 + 4a}} < 2H^*.$$

This proves this part of the theorem.

- (iv) If $\beta > \gamma - \frac{1}{2}$ and $\beta = \frac{2}{3}(\gamma - 1)$ equation (5.80) reduces to leading order to,

$$C_1\sqrt{C_2} = \frac{9}{K(P)} - 1. \quad (5.89)$$

The left hand side of this equation does not depend on λ and is always real-valued. The stability of a pattern changes if the real part of an eigenvalue λ passes zero. The special case $\lambda = 0$ corresponds to $P = 2$, and a simple check with [34] yields

$$\lim_{P \rightarrow 2} \frac{9}{K(P)} - 1 = 1. \quad (5.90)$$

Which implies that (5.89) is solved for a zero eigenvalue if

$$C_1\sqrt{C_2} = 1.$$

Hence, this is where a bifurcation occurs. A straightforward verification yields that the pattern is 1D-stable if $C_1\sqrt{C_2} < 1$, and 1D-unstable if

$C_1\sqrt{C_2} > 1$. Since this is the case that $\beta = \frac{2}{3}(\gamma - 1)$, we use solutions $u_0^\pm$ of (5.35) for $\bar{u}$. That implies that

$$C_1^\pm = \frac{\left(1 \pm \sqrt{1 - \frac{24b\sqrt{b}}{\sqrt{c^2+4a}}}\right)^2}{24b},$$

so the pattern is stable if,

$$\frac{\left(1 \pm \sqrt{1 - \frac{24b\sqrt{b}}{\sqrt{c^2+4a}}}\right)^2}{24b\sqrt{b}} \sqrt{c^2 + 4a} < 1,$$

which is always true for u_0^- , and never true for u_0^+ . Hence, the pattern with $\bar{u} = u_0^-$ is in this case always 1D-stable, and the pattern with $\bar{u} = u_0^+$ is always 1D-unstable. This concludes the proof of part (iv) of the theorem.

For cases (v) and (vi) we have $\beta = -1$ and $\gamma = -\frac{1}{2}$, we rewrite the leading order terms of equation (5.87) to

$$C_1 = \frac{1}{\sqrt{P^2 - 4 + C_2}} \left(\frac{9}{K(P)} - 1 \right). \quad (5.91)$$

Recall also the values (5.35) for $\bar{u}$ and the existence requirement (5.34). While C_1 is real-valued, $P \in \mathbb{C}$. We use MATHEMATICA to solve for which values of P the right hand side of (5.91) is real-valued, and plot its values against the real part of these values of P . For several values of C_2 , we have illustrated this in Figure 5.15. The P -values of intersections of the left and right side of (5.91), correspond to eigenvalues λ via (5.74). Regardless of the value of C_2 , there is an asymptote at $P = 3$ ($\lambda = \frac{5}{4}$). Hence, a stability criterion is always an upper limit for C_1 . On the other hand, as C_1 approaches 0, equation (5.91) approaches case (ii) of this theorem and the pattern is stable for every C_2 of $\mathcal{O}(1)$. Hence, there is always a bifurcation, although the nature of it may vary. A transition between a Hopf bifurcation or a zero eigenvalue occurs when the minimum of the right hand side of (5.91) for real-valued P occurs exactly at $P = 2$. That happens for

$$C_2 = C_2^* = -\frac{9}{2K'(2)} \approx 1.333,$$

which separates the cases (v) and (vi).

(v) This case occurs if $C_2 > C_2^*$ and (5.34) is satisfied. That implies

$$\frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} < \min \left\{ \frac{24b}{\sqrt{C_2^*}}, 1 \right\}. \quad (5.92)$$

If $C_2 > C_2^*$, the right hand side of (5.91) is real-valued in a neighborhood of $P = 2$. Hence, there is a bifurcation with a zero eigenvalue. We use (5.90) to derive that the pattern is 1D-stable if

$$C_1 < \frac{1}{\sqrt{C_2}},$$

and 1D-unstable if $C_1 > \frac{1}{\sqrt{C_2}}$. A straightforward computation yields that the pattern with $\bar{u} = u_0^+$ never satisfies this case but undergoes a saddle-node bifurcation when it collides with u_0^- , i.e. when (5.34) is equality. The pattern with $\bar{u} = u_0^-$, is always stable, apart from that same bifurcation point.

- (vi) Using both conditions $C_2 \leq C_2^*$ and (5.34), we derive that case (vi) occurs for

$$\frac{24b}{\sqrt{C_2^*}} \leq \frac{24b\sqrt{b}}{\sqrt{c^2 + 4a}} \leq 1,$$

which is only a nonempty range if $b \leq \frac{\sqrt{C_2^*}}{24} \approx 0.048$. In this case, destabilization occurs through a Hopf bifurcation. For $\text{Re}(P) > 0$, P has nonzero imaginary part if and only if λ has nonzero imaginary part. If $\lambda = i\kappa$ with $\kappa \in \mathbb{R}$, i.e. at the bifurcation point, $P = 2\sqrt{i\kappa + 1}$, see (5.74). We substitute $P = 2\sqrt{i\kappa + 1}$ in (5.91) and for every C_2 , we solve for which κ the right hand side of (5.91) has zero imaginary part,

$$\text{Im} \left(\frac{1}{\sqrt{4i\kappa(C_2) + C_2}} \left(\frac{9}{K(2\sqrt{i\kappa(C_2) + 1})} - 1 \right) \right) = 0. \quad (5.93)$$

This yields a curve that assigns to every $C_2 \in (0, C_2^*)$ the value of the right hand side of (5.91) at the Hopf bifurcation, see the right panel of Figure 5.14, which is exactly the function $Z(C_2)$ defined in (5.49). A 1D-stability requirement is then,

$$C_1 < Z(C_2),$$

where equality coincides with the Hopf bifurcation. The value of the eigenvalues as function of C_2 is depicted in Figure 5.16, where we see that $C_2 \downarrow 0$ returns the same result as described in case (iii) of this theorem.

□

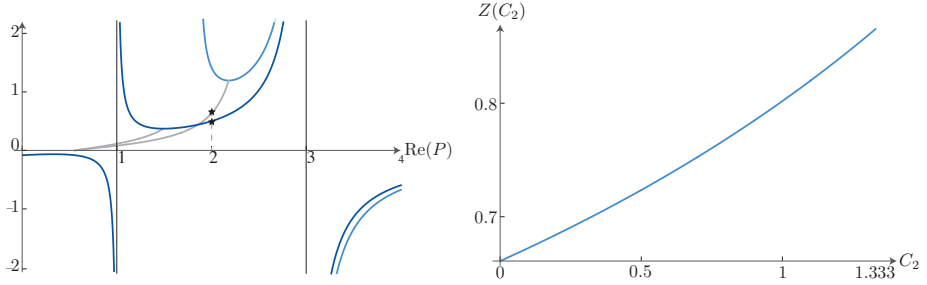


Figure 5.15: Left: Two times the graph of the right hand side of (5.91) for two values of C_2 , for values of $P \in \mathbb{C}$ for which the expression equals a value in $\mathbb{R}$, plotted against the real part of the corresponding P . When P has nonzero imaginary part, the graph is gray. Dark blue corresponds to $C_2 = 4$, while lighter blue corresponds to $C_2 = 0.5$. The stars indicate where the intersections with $P = 2$ are, i.e. at the bifurcation point. Right: The function $Z(C_2)$ (see (5.49)) for those values of C_2 for which $Z(C_2)$ is real-valued, and with $\bar{\kappa}$ for $C_2 \in (0, C_2^*)$.

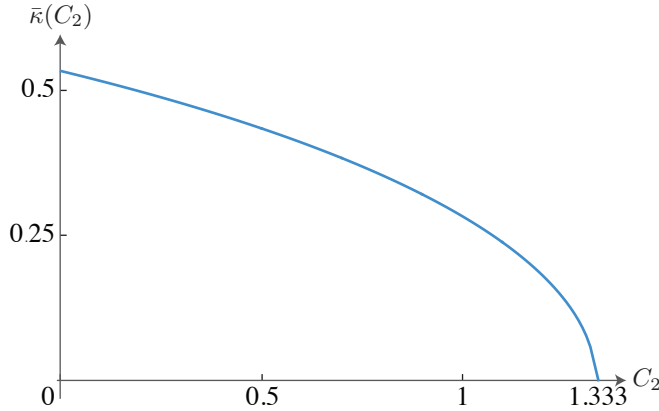


Figure 5.16: The function $\bar{\kappa}$ for $C_2 \in (0, C_2^*)$, which assigns to every C_2 the imaginary part of the eigenvalues at the Hopf bifurcation.

Proof. (of Theorem 5.4.3). In Theorem 5.4.3, the only change compared to Theorem 5.4.2 is that this concerns periodic solutions with long wave lengths. Following Lemma 5.4.4, the results for the 1-pulse may be carried over to this situation exactly if $\beta < 2\gamma$. The case (ii) of that lemma does not apply, because for one spatial dimension there is no parameter θ . Selecting exactly those parameter

regimes that satisfy $\beta < 2\gamma$, yields the results stated in the theorem. $\square$

The results of the Theorems 5.4.1 is derived from testing the stability of solutions (U_0, V_0) of (5.2) against perturbations in two dimensions. Of these perturbations, the wave number of the transverse perturbation factor may be arbitrarily small, but has an upper bound defined by (5.46). The Theorems 5.4.2 and 5.4.3, however, consider only patterns with one spatial dimension and hence perturbations in only one space variable. If, however, we consider a more restrictive function space for perturbations in two spatial dimensions, we may carry over the results of Theorems 5.4.2 and 5.4.3 to two-dimensional stability.

Corollary 5.4.5. *Let the assumptions **A1** and **A3** be satisfied and let $\beta \geq \frac{2}{3}(\gamma - 1)$. Let (U_0, V_0) be a slow/fast 1-pulse solution of (5.2), trivially extended in the y direction. Let X be the function space spanned by perturbations*

$$e^{i\ell}(\hat{u}(\chi), \hat{v}(\chi)),$$

with $\hat{\ell} \ll \delta^{2\gamma - \frac{3}{2}\beta - 1}$. Then, the statements of Theorem 5.4.2 summarize the two-dimensional stability of (U_0, V_0) against functions space X .

Let (U_0, V_0) be a slow/fast periodic solution of (5.2), trivially extended in the y direction. Then, the statements of Theorem 5.4.3 summarize the two-dimensional stability of (U_0, V_0) against functions space X .

Proof. The proof follows immediately from the observation that if $\hat{\ell} \ll \delta^{2\gamma - \frac{3}{2}\beta - 1}$, $\theta \geq 2\gamma - \frac{3}{2}\beta - 1$, which is more restrictive than (5.46). The proof of Theorem 5.4.1 can no longer be applied. In fact, if the wave number is not too large, the leading order equation that determines the stability, (5.80), is (5.87). Hence, the results from Theorems 5.4.2 and 5.4.3 follow. $\square$

The Theorems 5.4.1–5.4.3 particularize the 2D-instability and 1D-stability results for a range of parameter and scaling regimes. Since the results are concluded from the Evans function that was derived in section 5.4.1 for 1-pulses and where it was explicitly used that the function has a slow/fast splitting, our results are not exhaustive. If wave numbers become too large (i.e. when (5.46) is violated), equation (5.80) does not describe the leading order approximation of the eigenvalues. In fact, it can be shown that the unstable eigenvalue $\lambda(\ell)$ of (the proof of) Theorem 5.4.1 become negative as ℓ increases further, see also [42]. Similarly, if $\beta > 2\gamma$, the constructed periodic solutions cannot be considered ‘nearly’ localized, and therefore the associated spectrum cannot be approximated by discrete values united with σ_{ess} . In that case, the stability problem is similar to that of the periodic solutions of the Gierer-Meinhardt equations, see [157]. A full analysis of this case is, however, not part of the present article.

5.5 Conclusions and ecological implications

The need for a mathematical framework to study vegetation patterns has been acknowledged at least since [88, 133, 58]. So far, in various mathematical models, both analytic and numerical results have been derived, [58, 88, 133, 142, e.g]. However, the trade-off between manageable analytics and a realistic model is ever apparent. In the original Klausmeier, (5.1), or the Gray-Scott model, the existence of patterns has been thoroughly studied and reported in [88] and [34, 39, e.g]. With the introduction of the generalized Klausmeier-Gray-Scott model in [159], the combination of water diffusion (Gray-Scott) with the advective term induced by a gentle slope (Klausmeier) came forth to be quite effective. In more recent times, both analytic and numerical results about solutions and pattern formation in the gKGS system have been published, [159, 145, 143]. Apart from the intrinsic value of homoclinic or periodic pulse solutions of the gKGS system, they also form the foundation for complex pattern dynamics as, for example, pulse interactions. Still, rigorous results on the existence and stability of traveling slow/fast solutions of neither homoclinic nor periodic type have been reported in the literature.

In this article, we have explored the existence and stability of traveling stripe patterns of the generalized Klausmeier-Gray-Scott(5.2). They arise as multiscale pulse patterns in one spatial dimension, trivially extended in the other direction. The scaling of parameters and coordinates in this derivation is nontrivial and crucial to our analysis, and we present it in its most general form.

The existence of traveling single pulse solutions is established using geometric singular perturbation theory. Such a traveling homoclinic solution corresponds to a single strip of vegetation in an elsewhere endless desert: a traveling oasis. Perhaps more realistic are the periodic patterns that were constructed in section 5.3. The stripe patterns that correspond to these solutions are widely observed and in the field of ecology also known as tiger bushes [171]. Both types of solutions have a positive speed, indicating that the patterns travel uphill. This phenomenon is confirmed by observations, and can be explained by a surplus of water on the upper side of a vegetation strip. For homoclinic traveling waves, the values (5.30) imply that as the slope c decreases, the speed of the homoclinic traveling wave also decreases. This is in agreement with the Gray-Scott results, see section 5.1.1, where stationary pulses exist on flat terrain ($c = 0$). Note that as the rainfall, parametrized by a , decreases, the speed decreases. Furthermore, the water density value in the vegetation strip is smaller in the case of a traveling oasis (u_0) compared to traveling periodic patterns for the same set of parameters.

There are several destabilization mechanisms described by the Theorems 5.4.1–5.4.3. Theorem 5.4.1 describes that in the validity regime of our analysis,

defined by assumptions **A1–A3**, there is always a range transverse wave number of perturbations that destabilize the pattern. In other words, stripe patterns will cease to exist, as soon as the perturbations along the vegetation strip have a transverse wave number that is within this regime. Since the scaling regime bounds the slope of the terrain, c , this is in agreement with [143], where two-dimensional stripe patterns are numerically found to be stable only above a certain threshold for the slope. Since we only consider patterns that are trivially extended, it is not yet clear if the patterns will in fact collapse to a desert state. Considering the observations of rhombic or spot patterns, one may suggest that a large wave number perturbation could generate nontrivial two-dimensional patterns, as is also confirmed by the numerical simulations in [143]. If we consider a more ‘mild’ definition of stability, with perturbations in a well-chosen function space, we can make more detailed conclusions about the resilience of stripe patterns, formulated in Corollary 5.4.5.

The Theorems 5.4.2 and 5.4.3 describe stability of homoclinic and periodic patterns, with respect to one-dimensional (non-transversal) perturbations, respectively. In absence of transverse perturbations, the patterns may indeed be stable and several destabilization mechanisms take place as parameters vary. In the cases (iii)–(vi) in Theorem 5.4.2 and case (iii) in 5.4.3, destabilization occurs as C_1 (defined in (5.47)) grows. Reversely, the patterns are stable as long as C_1 is small enough. Since C_1 is minimal for the homoclinic 1-pulse, because u_0 is smaller than $u_{p,h}^*$, the homoclinic 1-pulse is the last pattern to destabilize, i.e. ‘the most stable pattern’. This is completely in agreement with Ni’s conjecture, which was formulated for the Gierer-Meinhardt system in [120], see also [42, 157, 40]. This implies that, as either precipitation (a) or slope (c) decreases, periodic solutions with the larger water density in the vegetation strip destabilize first and the traveling oasis is the last observable pattern. Although this also confirms the numerical observations of [143, 145, 159], it should be remarked that the present analysis only holds for the validity regime of our method, which is characterized by the splitting of fast and slow behavior in the system and defined by **A1–A3**. Furthermore, our analysis is only valid for patterns that are ‘nearly localized’, i.e. that the pattern is a homoclinic pattern or that Lemma 5.3.2 holds. Outside this regime, extra measures are necessary to draw conclusions about the stability (and the existence) of stripe patterns.

The value of $\bar{u}$ – the water density within a vegetated area – for periodic patterns is not explicitly determined in section 5.3, which makes the interpretation of the stability results of section 5.4 less straightforward. However, we do have a lower and upper bound for $\bar{u}$, namely,

$$\bar{u} \in (u_0, \tilde{u}) = \left(\frac{6b\sqrt{b}}{\sqrt{c^2 + 4a}}, \sqrt{\frac{6b\sqrt{b}}{s}} \right), \quad (5.94)$$

see (5.30) and (5.37). The upper bound can be carried over to C_1 to obtain, by Theorem 5.4.2(iii), a sufficient but not necessary condition for the stability of periodic patterns, namely

$$\frac{\sqrt{b}}{s} < 1.322, \quad (5.95)$$

see (5.88). In other words, it implies a lower bound for the migration speed of the periodic patterns. Together with the upper bound for s defined in (5.36), we have a range of s -values for which one-dimensional periodic patterns exist and are stable,

$$s \in \left(0.756\sqrt{b}, \frac{c\sqrt{c^2 + 4a}}{6b\sqrt{b}} \right). \quad (5.96)$$

This range is nonempty if $4.539b^2 \leq c\sqrt{c^2 + 4a}$. We conclude that even if a or c decreases, this does not have to destabilize the pattern, as long as it adapts to a suitable speed s , to satisfy the stability condition.

For all patterns constructed in this manuscript, we conclude that if a or c decreases the parameter regime in which patterns are stable shrinks, making them less resilient. A general conclusion we may draw is that decreasing rainfall or slope may have a serious negative impact on the ecological resilience of both homoclinic and periodic patterns. This is also confirmed by [32], where field observations show that stripe patterns disappear as the slope decreases.

In mathematical terms, we have considered the existence of homoclinic and spatially periodic slow/fast patterns in a reaction-advection-diffusion model. The advection term breaks the reversibility symmetry of the original reaction-diffusion system. Since this symmetry is a key feature of the establishment of especially the existence of these types of patterns in reaction-diffusion equations – Gray-Scott, Gierer-Meinhardt – several novel ideas have been introduced to incorporate the skew effects of the advection term in the geometric singular perturbation approach. We found that the symmetry breaking effect could be incorporated in the Evans function framework for spectral stability of the patterns in a natural way. Our stability results on one-dimensional patterns provide direct generalizations of previous results in the literature on reaction-diffusion models, see [34, 80, 157, 164]. Moreover, our result on the instability of stripes to the

transverse perturbations, Theorem 5.4.1, is similar to the findings in [42, 118, 143], to a class of reaction-advection-diffusion systems.

5.6 Discussion

In any article where mathematical models are used to describe a natural phenomenon, many simplifications need to be done to keep the work manageable. This, in turn, implies the need for a brief discussion of the conclusions drawn according to the simplified model. As mentioned before, our analysis does not extend beyond stripe patterns, and hence does not describe the widely observed gap or labyrinth patterns described in, for example, [61, 60]. Many of the assumptions made in this article cannot be made in the case where the pattern is not trivially extended in one spatial dimension. Perhaps another type of symmetry could resolve this. Furthermore, we did not incorporate nonlinear diffusion, nor did we examine a spatially dependent slope $C(x, y)$ or a nonconstant speed $S(t)$.

A more general, but prudent remark that should be made, is that the comparison of mathematical results with observation from the field remains to be a delicate task. We did not attempt to estimate the magnitude of parameters, let alone the values of the parameters themselves, from field data. The difficulties related to this data validation are also discussed in [144]. Of course, this makes it difficult to distinguish which of our conclusions drawn are relevant to the natural system it applies to, and which are interesting for the mathematical audience. Especially in the case of stability, it is not easily identified whether stability should indeed be tested against perturbations with all possible transverse wave numbers. Moreover, case (vi) of Theorem 5.4.3 is only valid for $b \lesssim 0.048$. It is not clear if this can be regarded as a valid result if b is assumed to be $\mathcal{O}(1)$ with respect to δ . A necessary continuation of this research should therefore analyze data, to confirm or reject the mathematical theory.

Finally, it should be explicitly noted that our analysis relies heavily on the scaling regimes. The clear advantages of this asymptotic approach is that we can explicitly determine the existence and stability regimes. The major drawback is, however, that we were not able to capture all types of stripe patterns. This could also be an explanation for the fact that we do not recover the stability results of [143], in which 2D-stable stripe patterns were reported for (a slightly rescaled version of) system (5.2) with large slopes. These large slopes may be described by parameters outside the triangular regime defined by assumptions **A1–A3**, and therefore cannot be fully understood by our analysis. However, since this [143] is to our awareness the first mathematical result in the literature that states that singular, far-from-equilibrium stripes may be stable, this challenging problem is intriguing, [42, 118]. Solving it requires a new mathematical approach that goes beyond the currently developed method for reaction-advection-diffusion systems.

This is the topic of ongoing research.

5.7 Acknowledgments

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