Spin dynamics in general relativity

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Chapter 4 starts with summarizing the traditional method of describing spin in the curved space-time – Mathisson-Papapetrou formalism. In the spinning particle approximation, I explain an alternative complementary formalism for spin dynamics. I derive equations of motion for point-like objects in curved space-time by using the Poisson-Dirac brackets and the minimal Hamiltonian. Then our method is compared with the traditional one in a qualitative way. The conserved quantities are developed in the Schwarzschild space-time. Since the closed set of Poisson-Dirac brackets is model independent, the analysis has been extended with gravitational and electric Stern-Gerlach interactions by introducing non-minimal Hamiltonians. Also modified conservation laws emerge reflecting the spin-orbit coupling. The equations of motions are also obtained from the conservation of the energy-momentum tensor.

4

Spinning Bodies in Curved Space-time

4.1 Spinning particles

In general relativity (GR), given the metric $g_{\mu\nu}$, the motion of test (single-pole) particles is determined by the geodesic equations of motion. Thus the single-pole particle doesn't have any internal structure [2]. The dynamical equations can be obtained from the covariant conservation of the energy-momentum tensor. A spinning particle in GR is a *pole-dipole particle*. Therefore its motion is generalised on a world line rather than geodesics. The evolution equations for spinning particles were derived (similar to test particles) by applying the conservation law for the energy-momentum tensor of matter $T^{\mu\nu}$, together with the Einstein field equations; and famously known as Mathisson-Papapetrou (MP) equations [59]:

$$\frac{\mathcal{D}p^\mu}{d\tau} = -\frac{1}{2}\,R^\mu{}_{\nu\kappa\lambda}u^\nu S^{\kappa\lambda}, \tag{4.1.1}$$

$$\frac{\mathcal{D}S^{\mu\nu}}{d\tau} = p^\mu u^\nu - u^\mu p^\nu, \tag{4.1.2}$$

where p^μ is the total 4-momentum of the particle, $u^\mu = dx^\mu/d\tau$ is the time like tangent vector ($u^\mu u_\mu = -1$) to the world line along which the particle moves i.e., *centre of mass* line used to make the multipole reduction, τ is the proper time along this world line, and $R^\mu{}_{\nu\kappa\lambda}$ is the Riemann tensor.

The energy-momentum vector p^μ and the intrinsic angular-momentum tensor $S^{\mu\nu}$ can be constructed by computing integrals of components of the energy-momentum tensor and their first moments over the volume of the body, using suitable boundary conditions [60]:

$$S^{\mu\nu} = \int_{x^0=const} \left(T^{\nu 0}\delta x^\mu - T^{\mu 0}\delta x^\nu\right)\sqrt{-g}\,d^3x \tag{4.1.3}$$

$$p^\mu = m u^\mu - u_\nu \frac{\mathcal{D}S^{\mu\nu}}{d\tau} \tag{4.1.4}$$

The quantity $m \equiv -p_\sigma u^\sigma$ is the particle's mass in the rest frame and it reduces to ordinary mass when the spin vanishes. The evolution equations (4.1.1) and

(4.1.2) are not a closed set of first order differential equations i.e., the system has 10 equations, but has 13 unknown quantities: $u(3)$, $p(4)$ and $S(6)$. Therefore it needs additional spin supplementary conditions (SSC) to fix a unique world line, and such that, it makes it possible to keep track of aspects of the structure of the body. A SSC fixes a centre of reference e.g. the centre of mass and different SSC defines a different centre of mass world line. Thus the MP equations describe the evolution of p^μ and $S^{\mu\nu}$ along the centre of mass world lines u^μ.

In literature there are many supplementary conditions, but for our discussion we choose to explore with Tulczyjew-Dixon (TD) condition;

$$S^{\mu\nu}p_\nu = 0, \tag{4.1.5}$$

which is claimed to be more physical [61, 62]. For an excellent review on various supplementary conditions and their relation we refer to [63]. The MP equations along with above conditions are called as Mathisson-Papapetrou-Dixon (MPD) model [64]. Further analysis concludes that different supplementary conditions lead to the same physical motion [63].

These highly non-linear (full) equations have been studied through numerical analysis [65–67]. The analytical solutions are very difficult even in highly symmetric space-times. The physical reason is that the particle has non-zero size i.e., a small extended body, whose internal structure is described by its spin (4.1.3). But through linearising the differential equations (4.1.1) and (4.1.2), an analytical description is achieved.

The dynamical equations imply, spin-orbit coupling, i.e., spin couples to the curvature of the background space-time. Therefore the spin force pushes the particle away from the geodesic. Then the deviation from geodesic motion should be very small compared with the curvature tensor of the space-time, which enforces a *limit* on the particle's spin [64]. Under these assumptions, the back reaction of the particle and the gravitational radiation emitted by the particle in its motion are neglected. This leads to consider the linear approximation of the spin; in this limit p^μ and u^μ are parallel: $p^\mu \approx mu^\mu$. Neglecting the higher order terms, equations (4.1.1) and (4.1.2) reduce to

$$\frac{\mathcal{D}(mu^\mu)}{d\tau} = -\frac{1}{2}\, R^\mu{}_{\nu\kappa\lambda} u^\nu S^{\kappa\lambda} + \mathcal{O}(2), \tag{4.1.6}$$

$$\frac{\mathcal{D}S^{\mu\nu}}{d\tau} = \mathcal{O}(2). \tag{4.1.7}$$

Which implies the mass of the particle remains constant along the motion: $dm/d\tau = 0$ and the *spin tensor is parallel transported along the path*. Then the TD condition reduces to the so-called Pirani condition:

$$S^{\mu\nu}u_\nu = 0. \tag{4.1.8}$$

Equations (4.1.6), (4.1.7) and (4.1.8) constitutes the Mathisson-Papapetrou-Pirani (MPP) model. Note, the Pirani vector (say) $Z^\mu = S^{\mu\nu}u_\nu = 0$, throughout the evolution.

Finally we conclude this section with some additional references for generalisation of this method. A similar analysis has been extended for charged spinning particle (pole-dipole approx.) in a given gravitational as well as electromagnetic field, by Dixon and Souriau [68–71]. The dynamical equations of motion for an extended body in a given gravitational field were deduced by Dixon in *multipole approximation* to any order [61, 62, 72]. The MP model has been extended for *massless particles* i.e., null multipole reduction world line by Mashhoon [73].

4.2 Spinning-particle approximation

In addition to the above well-known method, there is an other complementary approach to the subject [74] . It constructs effective equations of motion for point-like objects, which is an idealization of a compact body, at the price of neglecting details of the internal structure by assigning the point-like object an overall position, momentum and spin. This is also known as the spinning-particle approximation, and is used for the semi-classical description of elementary particles as well. A large variety of models for spinning particles is found in the literature [75–85].

We take the second point of view for the description of spinning test masses in curved space-time, using an *effective hamiltonian formalism* similar to the one introduced in ref. [86]. One of the advantages of this description is that it can be applied to compact bodies with different types of spin dynamics, such as different gravimagnetic ratios. In this way specific aspects of the structure can still be accounted for.

4.3 Covariant Hamiltonian Formalism

Hamiltonian dynamical systems are specified by three sets of ingredients: *the phase space*, identifying the dynamical degrees of freedom, the *Poisson-Dirac brackets* defining a symplectic structure, and the *hamiltonian* generating the evolution of the system with given initial conditions by specifying a curve in the phase space passing through the initial point. The parametrization of phase-space is not unique, as is familiar from the Hamilton-Jacobi theory of dynamical systems. Changes in the parametrization can be compensated by redefining the brackets and the hamiltonian. A convenient starting point for models with gauge-field interactions is the use of covariant, i.e. kinetic, momenta rather than canonical momenta; see [87] and references cited there for a general discussion, and [86] for the application to spinning particles.

The spin degrees of freedom are described by an antisymmetric tensor $\Sigma^{\mu\nu}$, which can be decomposed into two space-like four-vectors by introducing a time-like unit vector u: $u_\mu u^\mu = -1$, and defining

$$S^\mu = \frac{1}{2\sqrt{-g}}\,\varepsilon^{\mu\nu\kappa\lambda}\,u_\nu \Sigma_{\kappa\lambda}, \qquad Z^\mu = \Sigma^{\mu\nu} u_\nu. \tag{4.3.1}$$

By construction both four-vectors S and Z are space-like:

$$S^\mu u_\mu = 0, \qquad Z^\mu u_\mu = 0. \tag{4.3.2}$$

In the following we take u to be the proper four-velocity of the particle. Then S is the Pauli-Lubanski pseudo-vector, from which a magnetic dipole moment can be constructed, whilst the components of Z, which will be referred to as the Pirani vector, can be used to define an electric dipole moment [88, 89]. Observe that we can invert the relations (4.3.1) to write

$$\Sigma^{\mu\nu} = -\frac{1}{\sqrt{-g}}\,\varepsilon^{\mu\nu\kappa\lambda}\,u_\kappa S_\lambda + u^\mu Z^\nu - u^\nu Z^\mu. \tag{4.3.3}$$

Therefore, if the Pirani vector vanishes: $Z = 0$ [90], the full spin tensor can be reconstructed from S. However, in general this is not the case in our formalism. It is also interesting to note that in addition one can define a third space-like vector

$$W^\mu = -\frac{1}{\sqrt{-g}}\,\varepsilon^{\mu\nu\kappa\lambda} u_\nu S_\kappa Z_\lambda = \left(\Sigma^{\mu\nu} - u^\mu Z^\nu\right) Z_\nu, \tag{4.3.4}$$

orthogonal to the other ones:

$$W \cdot u = W \cdot S = W \cdot Z = 0. \tag{4.3.5}$$

Together (u, S, Z, W) form a set of independent vectors, one time-like and three space-like, which can be used to define a frame of basis vectors carried along the particle world-line.

4.3.1 Covariant phase-space structure

The full set of phase-space co-ordinates of a spinning particle thus consists of the position co-ordinate x^μ, the covariant momentum π_μ and the spin tensor $\Sigma^{\mu\nu}$, with anti-symmetric Dirac-Poisson brackets

$$\begin{aligned}
&\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \qquad \{\pi_\mu, \pi_\nu\} = \frac{1}{2}\,\Sigma^{\kappa\lambda} R_{\kappa\lambda\mu\nu},\\
&\{\Sigma^{\mu\nu}, \pi_\lambda\} = \Gamma_{\lambda\kappa}^{\;\;\;\mu}\,\Sigma^{\nu\kappa} - \Gamma_{\lambda\kappa}^{\;\;\;\nu}\,\Sigma^{\mu\kappa},\\
&\left\{\Sigma^{\mu\nu}, \Sigma^{\kappa\lambda}\right\} = g^{\mu\kappa}\Sigma^{\nu\lambda} - g^{\mu\lambda}\Sigma^{\nu\kappa} - g^{\nu\kappa}\Sigma^{\mu\lambda} + g^{\nu\lambda}\Sigma^{\mu\kappa}.
\end{aligned} \tag{4.3.6}$$

The brackets imply that π represents the generator of covariant translations, whilst the spin degrees of freedom Σ generate internal rotations and Lorentz transformations. It is straightforward to check that these brackets are closed in the sense that they satisfy the Jacobi identities for triple bracket expressions. Thus they define a consistent symplectic structure on the phase space.

To get a well-defined dynamical system we need to complete the phase-space structure with a hamiltonian generating the proper-time evolution of the system. In principle a large variety of covariant expressions can be constructed; however if we impose the additional condition that the *particle interacts only gravitationally* and that in the limit of vanishing spin the motion reduces to geodesic motion, the variety is reduced to hamiltonians

$$H = H_0 + H_\Sigma, \qquad H_0 = \frac{1}{2m}\, g^{\mu\nu}\pi_\mu\pi_\nu, \tag{4.3.7}$$

where $H_\Sigma = 0$ whenever $\Sigma^{\mu\nu} = 0$. In the following sections we focus first on the dynamics generated by the minimal hamiltonian H_0. However, we also consider extensions with gravitational and electric Stern-Gerlach forces [80]. Thus the choice of hamiltonians can be enlarged further by including spin-spin interaction via space-time curvature and charges coupling the particle to vector fields like the electromagnetic field [86, 88].

Eqs. (4.3.6) and (4.3.7) specify a complete and consistent dynamical scheme for spinning particles. Note that the choice of hamiltonian is fixed by further physical requirements, and can differ for different compact objects. In that sense the hamiltonian is an *effective* hamiltonian, suitable to describe the motion of various types of objects in so far as the role of other internal degrees of freedom can be restricted to their effects on overall position, linear momentum and spin.

4.3.2 Minimal equations of motion

The simplest model for a massive free spinning particle in the absence of Stern-Gerlach forces and external fields is obtained by restricting the hamiltonian to the minimal geodesic term H_0. By itself this hamiltonian generates the following set of proper-time evolution equations:

$$\dot{x}^\mu = \{x^\mu, H_0\} \quad \Rightarrow \quad \pi_\mu = m g_{\mu\nu}\dot{x}^\nu, \tag{4.3.8}$$

stating that the covariant momentum π is a tangent vector to the world line, proportional to the proper four-velocity $u = \dot{x}$. Next

$$\dot{\pi}_\mu = \{\pi_\mu, H_0\} \quad \Rightarrow \quad D_\tau\pi_\mu \equiv \dot{\pi}_\mu - \dot{x}^\lambda\Gamma_{\lambda\mu}^{\;\;\;\nu}\pi_\nu = \frac{1}{2m}\,\Sigma^{\kappa\lambda}R_{\kappa\lambda\mu}^{\quad\;\;\nu}\,\pi_\nu, \tag{4.3.9}$$

which specifies how the world line curves in terms of the evolution of its tangent vector. Finally the rate of change of the spin tensor is

$$\dot{\Sigma}^{\mu\nu} = \{\Sigma^{\mu\nu}, H_0\} \quad \Rightarrow \quad D_\tau\Sigma^{\mu\nu} \equiv \dot{\Sigma}^{\mu\nu} + \dot{x}^\lambda\Gamma_{\lambda\kappa}^{\;\;\;\mu}\Sigma^{\kappa\nu} + \dot{x}^\lambda\Gamma_{\lambda\kappa}^{\;\;\;\nu}\Sigma^{\mu\kappa} = 0. \tag{4.3.10}$$

In these equations the overdot denotes an ordinary derivative w.r.t. proper time τ, whereas D_τ denotes the pull-back of the covariant derivative along the world line $x^\mu(\tau)$. By substitution of eq. (4.3.8) into eq. (4.3.9) one finds that

$$D_\tau^2 x^\mu = \ddot{x}^\mu + \Gamma_{\lambda\nu}{}^{\mu} \dot{x}^\lambda \dot{x}^\nu = \frac{1}{2m}\, \Sigma^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu}{}_{\nu}\, \dot{x}^\nu, \tag{4.3.11}$$

which reduces to the geodesic equation in the limit $\Sigma = 0$. The world line is the solution of the combined equations (4.3.11) and (4.3.10) satisfying some initial conditions. This *world line is a curve in space-time along which the spin tensor is covariantly constant* (Fig. 4.1).

It has been remarked by many authors [86, 91–93], that the spin-dependent force (4.3.9) exerted by the space-time curvature on the particle is similar to the Lorentz force with spin replacing the electric charge and curvature replacing the electromagnetic field strength. In this analogy the covariant conservation of spin along the world line is the natural equivalent of the conservation of charge.

Even though the spin tensor is covariantly constant, this does not hold for the Pauli-Lubanski and Pirani vectors S and Z individually. Indeed, due to the gravitational Lorentz force

$$\begin{aligned} D_\tau S^\mu &= \frac{1}{4m\sqrt{-g}}\, \varepsilon^{\mu\nu\kappa\lambda} \Sigma_{\kappa\lambda} \Sigma^{\alpha\beta} R_{\alpha\beta\nu\rho} u^\rho, \\ D_\tau Z^\mu &= \frac{1}{2m}\, \Sigma^{\mu\nu} \Sigma^{\alpha\beta} R_{\alpha\beta\nu\rho} u^\rho, \end{aligned} \tag{4.3.12}$$

where $\Sigma^{\mu\nu}$ is the linear expression in terms of S^μ and Z^μ given in eq. (4.3.3). We observe that the rate of change of both spin vectors is of order $\mathcal{O}[\Sigma^2]$. In particular, as Z is not conserved in non-flat space-times the condition $Z = 0$ cannot be imposed during the complete motion in general. Indeed, the evolution of the system is completely determined by eqs. (4.3.8, 4.3.9, 4.3.10), and leaves *no room for additional constraints.*

We close this section by remarking that the gravitational Lorentz force for unit mass $1/2\, \Sigma^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu}{}_{\nu} u^\nu$ can be interpreted geometrically as the change in the unit vector u^μ generated by transporting it around a closed loop with area projection in the x^κ-x^λ-plane equal to $\Sigma^{\kappa\lambda}$.

4.4 *Effective* Hamiltonian and MPD formalism: a comparison

The dynamical equations in the MP formalism are not a closed set of first order differential equations. The system has 10 equations, but has 13 unknown quantities: $u(3)$, $p(4)$ and $S(6)$. Thus one needs spin supplementary conditions to solve them. SSC define world lines traced by differently defined centres of mass. The most commonly used SSC is TD condition (4.1.5). The system is commonly known as MPD formalism. The full MPD equations are very difficult to solve even in the highly symmetric space-times. Therefore one linearizes MPD formalism which leads to MPP model. Then the coupled equations (4.1.6), (4.1.7) and (4.1.8) constitutes the closed system.

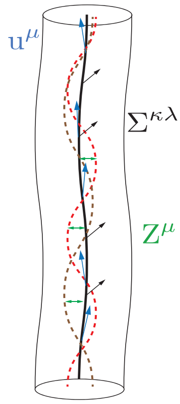

Figure 4.1. Compares the world lines [94] traced by *effective* hamiltonian formalism and MPD model. The thick black line is the world line along which the spin tensor ($\Sigma^{\kappa\lambda}$) is covariantly constant, and u^μ is the tangent vector to this world line. Dotted lines are the world lines followed by some preferred centre of mass in the MPD model. Green double arrows represents the dipole vector Z^μ which quantifies the difference between centre of mass world lines and the world line of spin tensor (thick, black line).

Where as in the hamiltonian dynamics, the system is described by a set of 2N phase-space variables satisfying first-order differential equations in the evolution parameter (proper time). Then by fixing initial conditions the evolution of the system is completely and uniquely determined. For spinning particles the phase-space is 14-dimensional: $x(4)$, $\pi(4)$, $\Sigma(6)$. These are subject to 14 first-order differential equations (4.3.8), (4.3.9) and (4.3.10). Thus the evolution of the system is completely determined by the initial conditions. Therefore we don't need any SSC in our formalism.

The linearized form of MPD formalism precisely coincides with the original equations of motion (4.3.10) and (4.3.11) in our formalism, whose solution is the spin tensor parallelly transported along the world line. Then if we consider the linearized form of our equations of motion, neglecting quadratic or higher order terms in the spin-tensor $\Sigma^{\mu\nu}$, the right-hand side of equations (4.3.12) vanishes, and it is possible to require the Pirani condition $Z^\mu = 0$ at all times. Thus the usual MPP dynamics can be recovered in linearized form from our equations.

In the MPD formalism the equations of motion are constructed in a such a way that, if one fixes an initial condition a constraint on the dipole, such as the Pirani constraint $Z^\mu = 0$; it holds true throughout the evolution. In other words, the equation of motion for Z^μ has been replaced by an algebraic constraint $Z^\mu = 0$. This is because of the fact that the canonical momentum p^μ in the MPD model is different from the kinetic momentum π^μ in our formalism. Thus the analysis of the evolution equations are complicated in MPD model.

In contrast, in our formulation the momentum is always strictly kinetic: $\pi^\mu = mu^\mu$, but the dipole Z^μ is dynamical and non-vanishing in general. Therefore the mass-dipole constraint has been replaced by a proper equation of motion (4.3.12), which determines how Z^μ evolves even if it vanishes initially.

The two formulations are not necessarily contradicting each other. In the MP case the solution of the dynamical equations is the world line on which the SSC is always true and so it traces the centre of mass (dotted lines in Fig. 4.1). Since it accounts for the internal structure of the particle, the spin dynamics is complicated. In our case the spin-dynamics is simple and straightforward, but the center of mass is not necessarily located on the world-line, as signalled by the non-zero mass dipole. Therefore the solution of our equations of motion is the world line in which the spin tensor is covariantly constant (thick, black line). Thus its a matter of choice.

One of the major advantage in our formalism is that the back reaction of the particle motion on the space-time geometry can be calculated unambiguously. This is accounted by the energy-momentum tensor which exhibits the effect of the mass dipole.

4.5 Conservation laws

The Hamiltonian formalism we have developed is also convenient for deriving constants of motion. There are two classes of constants in the theory. The universal constants which exist for any space-time geometry and the constants of motion emerging as a result of symmetries of the space-time. These constants commute with the hamiltonian in the sense of the brackets.

4.5.1 Universal conserved quantities

For the spinning body in curved space-time, there exist universal constants of motion, irrespective of the specific geometry of the space-time manifold. By construction the time-independent hamiltonian represented by (4.3.7) is a constant of motion. In particular for the minimal geodesic hamiltonian H_0 we have

$$H_0 = -\frac{m}{2}, \tag{4.5.1}$$

defining the particles mass. The above equation is equivalent to normalizing proper time such that $u_\mu u^\mu = -1$. In addition there are two constants of motion for the spin: the total spin I as a result of local Lorentz invariance

$$I = \frac{1}{2}\, g_{\kappa\mu} g_{\lambda\nu} \Sigma^{\kappa\lambda} \Sigma^{\mu\nu} = S_\mu S^\mu + Z_\mu Z^\mu, \tag{4.5.2}$$

and the pseudo-scalar spin-dipole product:

$$D = \frac{1}{8}\,\sqrt{-g}\,\varepsilon_{\mu\nu\kappa\lambda}\Sigma^{\mu\nu}\Sigma^{\kappa\lambda} = S\cdot Z. \tag{4.5.3}$$

Note, I and D are quadratic expressions in spin. In the Hamiltonian formalism all these three constants obey obviously

$$\{H_0, H_0\} = 0, \qquad \{I, H_0\} = 0, \qquad \{D, H_0\} = 0. \tag{4.5.4}$$

4.5.2 Geometrical conserved quantities

Furthermore, there may exist conserved quantities $J(x, \pi, \Sigma)$ resulting from symmetries of the background geometry, as implied by Noether's theorem [61, 95, 96]. They are solutions of the generic equation

$$\{J, H_0\} = \frac{1}{m}\,g^{\mu\nu}\pi_\nu\left[\frac{\partial J}{\partial x^\mu} + \Gamma_{\mu\lambda}^{\;\;\;\kappa}\,\pi_\kappa\frac{\partial J}{\partial \pi_\lambda} + \frac{1}{2}\,\Sigma^{\alpha\beta}R_{\alpha\beta\lambda\mu}\frac{\partial J}{\partial \pi_\lambda} + \Gamma_{\mu\alpha}^{\;\;\;\kappa}\,\Sigma^{\lambda\alpha}\frac{\partial J}{\partial \Sigma^{\kappa\lambda}}\right] = 0. \tag{4.5.5}$$

The symmetries of the space-time manifest themselves as Killing vectors. Here due to spin-orbit coupling, the conserved quantities implied by Noether's theorem are linear combinations of momentum [96] and spin components:

$$J = \alpha^\mu\pi_\mu + \frac{1}{2}\,\beta_{\mu\nu}\,\Sigma^{\mu\nu}, \tag{4.5.6}$$

with

$$\nabla_\mu\alpha_\nu + \nabla_\nu\alpha_\mu = 0, \qquad \nabla_\lambda\beta_{\mu\nu} = R_{\mu\nu\lambda}^{\;\;\;\;\;\;\kappa}\alpha_\kappa. \tag{4.5.7}$$

These equations imply that α is a Killing vector on the space-time, and β is its anti-symmetrized gradient:

$$\beta_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu\alpha_\nu - \nabla_\nu\alpha_\mu\right). \tag{4.5.8}$$

Similarly constants of motion quadratic in momentum [97] are of the form:

$$J = \frac{1}{2}\,\alpha^{\mu\nu}\pi_\mu\pi_\nu + \frac{1}{2}\beta_{\mu\nu}^{\;\;\;\;\lambda}\,\Sigma^{\mu\nu}\pi_\lambda + \frac{1}{8}\,\gamma_{\mu\nu\kappa\lambda}\Sigma^{\mu\nu}\Sigma^{\kappa\lambda}, \tag{4.5.9}$$

where the coefficients have to satisfy the ordinary partial differential equations

$$\begin{aligned}
&\nabla_\lambda\alpha_{\mu\nu} + \nabla_\mu\alpha_{\nu\lambda} + \nabla_\nu\alpha_{\lambda\mu} = 0, \\
&\nabla_\mu\beta_{\kappa\lambda\nu} + \nabla_\nu\beta_{\kappa\lambda\mu} = R_{\kappa\lambda\mu}^{\;\;\;\;\;\;\rho}\alpha_{\nu\rho} + R_{\kappa\lambda\nu}^{\;\;\;\;\;\;\rho}\alpha_{\mu\rho}, \\
&\nabla_\rho\gamma_{\mu\nu\kappa\lambda} = R_{\mu\nu\rho}^{\;\;\;\;\;\;\sigma}\beta_{\kappa\lambda\sigma} + R_{\kappa\lambda\rho}^{\;\;\;\;\;\;\sigma}\beta_{\mu\nu\sigma}.
\end{aligned} \tag{4.5.10}$$

Thus α is a symmetric rank-two Killing tensor, and the coefficients (β, γ) satisfy a hierarchy of inhomogeneous Killing-like equations determined by the $\alpha_{\mu\nu}$. In the case of Grassmann-valued spin tensors $\Sigma^{\mu\nu} = i\psi^\mu \psi^\nu$ the coefficient γ is completely anti-symmetric and the equations are known to have a solution in terms of Killing-Yano tensors [98].

The constants of motion (4.5.6) linear in momentum are special in that they define a Lie algebra: if J and J' are two such constants of motion, then their bracket is a constant of motion of the same type. This follows from the Jacobi identity

$$\left\{ \left\{ J, J' \right\}, H_0 \right\} = \left\{ \left\{ J, H_0 \right\}, J' \right\} - \left\{ \left\{ J', H_0 \right\}, J \right\} = 0. \tag{4.5.11}$$

Thus, if $\{e_i\}_{i=1}^r$ is a complete basis for Killing vectors:

$$\alpha^\mu = \alpha^i e_i^\mu, \qquad e_j^\nu \nabla_\nu e_i^\mu - e_i^\nu \nabla_\nu e_j^\mu = f_{ij}{}^k e_k^\mu,$$

the constants of motion define a representation of the same algebra:

$$J_i = e_i^\mu \pi_\mu + \frac{1}{2}\, \nabla_\mu e_{i\nu}\, \Sigma^{\mu\nu} \quad \Rightarrow \quad \left\{ J_i, J_j \right\} = f_{ij}{}^k J_k. \tag{4.5.12}$$

Evidently such constants of motion are helpful in the analysis of spinning particle dynamics [51,95,99].

4.6 Non-minimal hamiltonian: gravitational Stern-Gerlach force

So far we have studied the dynamics of compact spinning objects generated by the minimal geodesic hamiltonian H_0. In this section we consider the non-minimal extension including the *spin-spin interaction* via space-time curvature:

$$H = H_0 + H_\Sigma, \qquad H_\Sigma = \frac{\kappa}{4}\, R_{\mu\nu\kappa\lambda} \Sigma^{\mu\nu} \Sigma^{\kappa\lambda}. \tag{4.6.1}$$

The Dirac-Poisson brackets (4.3.6) remain the same (obviously). It is straightforward to derive the equations of motion:

$$\begin{aligned}
\dot{x}^\mu &= \left\{ x^\mu, H \right\} \quad \Rightarrow \quad \pi_\mu = m g_{\mu\nu} \dot{x}^\nu, \\
\dot{\pi}_\mu &= \left\{ \pi_\mu, H \right\} \quad \Rightarrow \quad D_\tau \pi_\mu = \frac{1}{2m}\, \Sigma^{\kappa\lambda} R_{\kappa\lambda\mu}{}^\nu \pi_\nu - \frac{\kappa}{4}\, \Sigma^{\kappa\lambda} \Sigma^{\rho\sigma} \nabla_\mu R_{\kappa\lambda\rho\sigma}, \\
\dot{\Sigma}^{\mu\nu} &= \left\{ \Sigma^{\mu\nu}, H \right\} \Rightarrow \quad D_\tau \Sigma^{\mu\nu} = \kappa \Sigma^{\kappa\lambda} \left(R_{\kappa\lambda}{}^\mu{}_\sigma \Sigma^{\nu\sigma} - R_{\kappa\lambda}{}^\nu{}_\sigma \Sigma^{\mu\sigma} \right).
\end{aligned} \tag{4.6.2}$$

Comparing again with the electro-magnetic force, the middle equation implies that in addition to the gravitational Lorentz force there is a *gravitational Stern-Gerlach force*, coupling spin to the gradient of the curvature. Therefore the coupling parameter κ has been termed the gravimagnetic ratio [81,100]. Like in the electromagnetic case [101] the Pauli-Lubanski and Pirani-vectors are affected by this Stern-Gerlach force:

$$D_\tau S^\mu = \frac{1}{4m\sqrt{-g}}\,\varepsilon^{\mu\nu\kappa\lambda}\Sigma_{\kappa\lambda}\Sigma^{\alpha\beta}\left(R_{\alpha\beta\nu\sigma}u^\sigma - \frac{\kappa}{2}\,\Sigma^{\rho\sigma}\nabla_\nu R_{\rho\sigma\alpha\beta}\right),$$

$$D_\tau Z^\mu = -\kappa\Sigma^{\kappa\lambda}R_{\kappa\lambda\ \ \nu}^{\ \ \ \mu}Z^\nu + \left(\kappa + \frac{1}{2m}\right)\Sigma^{\mu\nu}\Sigma^{\kappa\lambda}R_{\kappa\lambda\nu\sigma}u^\sigma - \frac{\kappa}{4m}\,\Sigma^{\mu\nu}\Sigma^{\kappa\lambda}\Sigma^{\rho\sigma}\nabla_\nu R_{\kappa\lambda\rho\sigma}. \tag{4.6.3}$$

The second equation simplifies strongly for the special value

$$\kappa = -\frac{1}{2m}. \tag{4.6.4}$$

In that case an initial condition $Z^\mu = 0$ is conserved up to terms of cubic order in spin.

4.6.1 Extension of conservation laws to non-minimal dynamics

For the extended hamiltonian the conditions for the existence of constants of motion are modified. The total spin I defined in (4.5.2) is still conserved, but the conserved hamiltonian now is of course $H = H_0 + H_\Sigma$. Finally we prove that the constants of motion J of the form (4.5.6) are preserved under this modification of the hamiltonian. To see this, observe that

$$\{J, H_\Sigma\} = -\kappa\Sigma^{\mu\nu}\Sigma^{\rho\sigma}\left(\frac{1}{4}\,\alpha^\lambda\nabla_\lambda R_{\mu\nu\rho\sigma} + \beta_{\mu\lambda}R^\lambda_{\ \ \nu\rho\sigma}\right). \tag{4.6.5}$$

For the Killing-vector solutions (4.5.7) the right-hand side takes the form

$$\begin{aligned}\Sigma^{\mu\nu}\Sigma^{\rho\sigma}\left(\frac{1}{4}\,\alpha^\lambda\nabla_\lambda R_{\mu\nu\rho\sigma} + \beta_{\mu\lambda}R^\lambda_{\ \ \nu\rho\sigma}\right) &= \frac{1}{2}\,\Sigma^{\mu\nu}\Sigma^{\rho\sigma}\left(\nabla_\mu\nabla_\rho\nabla_\sigma + \nabla_\rho\nabla_\mu\nabla_\sigma\right)\alpha_\nu \\ &= \frac{1}{2}\,\Sigma^{\mu\nu}\Sigma^{\rho\sigma}\left(\nabla_\mu\nabla_\rho + \nabla_\rho\nabla_\mu\right)\beta_{\sigma\nu} = 0,\end{aligned} \tag{4.6.6}$$

due to the anti-symmetry of the tensor $\beta_{\sigma\nu}$.

4.7 Non-minimal hamiltonian: electric Stern-Gerlach force

In this section we further extend our formalism with the non-minimal hamiltonian generating electric Stern-Gerlach forces. The spinning particle with charge q, in the presence of external fields subject to spin-dependent forces coupling to gradients in the fields like the well-known Stern-Gerlach force [82,86,88,102] in electrodynamics. Such forces can be modeled in our approach by additional spin-dependent terms in the hamiltonian:

$$H = H_0 + H_{SG}, \qquad H_{SG} = \frac{\kappa}{4}\,R_{\mu\nu\kappa\lambda}\Sigma^{\mu\nu}\Sigma^{\kappa\lambda} + \frac{\lambda}{2}\,F_{\mu\nu}\Sigma^{\mu\nu}. \tag{4.7.1}$$

Here the electromagnetic coupling term $\frac{\lambda}{2} F_{\mu\nu}\Sigma^{\mu\nu}$ requires modification of the Poisson-Dirac brackets. Therefore

$$\{\pi_\mu, \pi_\nu\} = \frac{1}{2}\,\Sigma^{\kappa\lambda} R_{\kappa\lambda\mu\nu} + qF_{\mu\nu}. \tag{4.7.2}$$

The remaining brackets are same as in (4.3.6). Then using this non-minimal hamiltonian in the brackets to construct equations of motion we get

$$\pi_\mu = mg_{\mu\nu}u^\nu,$$

$$mg_{\mu\nu}D_\tau u^\nu = \frac{1}{2}\,\Sigma^{\kappa\lambda} R_{\kappa\lambda\mu\nu}u^\nu + qF_{\mu\nu}u^\nu - \frac{\kappa}{4}\,\Sigma^{\rho\sigma}\Sigma^{\kappa\lambda}\,\nabla_\mu R_{\rho\sigma\kappa\lambda} - \frac{\lambda}{2}\,\Sigma^{\kappa\lambda}\,\nabla_\mu F_{\kappa\lambda}, \tag{4.7.3}$$

and

$$D_\tau \Sigma^{\mu\nu} = \left(\kappa\Sigma^{\rho\sigma} R_{\rho\sigma\ \ \lambda}^{\ \ \ \mu} + \lambda F^{\mu}_{\ \ \lambda}\right)\Sigma^{\nu\lambda} - \left(\kappa\Sigma^{\rho\sigma} R_{\rho\sigma\ \ \lambda}^{\ \ \ \nu} + \lambda F^{\nu}_{\ \ \lambda}\right)\Sigma^{\mu\lambda}. \tag{4.7.4}$$

4.7.1 Extension of conservation laws to non-minimal dynamics

The universal constants of motion (4.5.1), (4.5.2) and (4.5.3), hold true for charged spinning particles as well. But the constants of motion depending on the symmetries of the geometry are altered because of the presence of charge. They are constructed in terms of Killing vectors and tensors. In particular constants of motion J of the form

$$J = \gamma + \alpha^\mu \pi_\mu + \frac{1}{2}\,\beta_{\mu\nu}\Sigma^{\mu\nu}, \tag{4.7.5}$$

exist if

$$\nabla_\mu \alpha_\nu + \nabla_\nu \alpha_\mu = 0, \quad \nabla_\lambda \beta_{\mu\nu} = R_{\mu\nu\lambda\kappa}\alpha^\kappa, \quad \partial_\mu \gamma = qF_{\mu\nu}\alpha^\nu. \tag{4.7.6}$$

Thus α^μ is a Killing vector and $\beta_{\mu\nu}$ its curl:

$$\beta_{\mu\nu} = \frac{1}{2}\left(\nabla_\mu \alpha_\nu - \nabla_\nu \alpha_\mu\right), \tag{4.7.7}$$

whilst a solution for γ can be found if the Lie-derivative of the vector potential with respect to α vanishes:

$$\alpha^\nu \partial_\nu A_\mu + \partial_\mu \alpha^\nu A_\nu = 0 \quad \Rightarrow \quad \gamma = qA_\mu \alpha^\mu. \tag{4.7.8}$$

This requirement in fact states that the electromagnetic and gravitational fields must both exhibit the same symmetries for an associated constant of motion to exist.

Remarkably, using eqs. (4.7.7, 4.7.8) and the Bianchi identities for $F_{\mu\nu}$ and $R_{\mu\nu\kappa\lambda}$ it is straightforward to generalize the theorem of ref. [47], that any constant of motion (4.7.5) remains a constant of motion in the presence of Stern-Gerlach forces:

$$\begin{aligned}\{J, H_{SG}\} &= \kappa\, \Sigma^{\mu\nu}\Sigma^{\rho\sigma}\left(-\frac{1}{4}\,\alpha^{\lambda}\nabla_{\lambda}R_{\rho\sigma\mu\nu} + R_{\rho\sigma\mu}{}^{\lambda}\beta_{\lambda\nu}\right)\\ &\quad +\lambda\,\Sigma^{\mu\nu}\left(-\frac{1}{2}\,\alpha^{\lambda}\nabla_{\lambda}F_{\mu\nu} + F_{\mu}{}^{\lambda}\beta_{\lambda\nu}\right) = 0.\end{aligned} \tag{4.7.9}$$

4.8 Equations of motion from energy-momentum conservation

In the previous sections the equations of motion for a relativistic spinning particle were obtained starting from a closed set of brackets (4.3.6) and the choice of a hamiltonian. The same equations can be derived by energy-momentum conservation using an appropriate energy-momentum tensor [103, 104]. This tensor then also defines the source term in the Einstein equations to compute the back reaction of the particle on the space-time geometry; indeed, the Einstein equations require the energy momentum to be divergence-free

$$G_{\mu\mu} = R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R = -8\pi G T_{\mu\nu} \quad\Rightarrow\quad \nabla^{\mu}G_{\mu\nu} = -8\pi\nabla^{\mu}T_{\mu\nu} = 0. \tag{4.8.1}$$

This identity is to be guaranteed by the equations of motion. For a neutral particle described by the minimal hamiltonian this follows by taking (3.4.3)

$$T_0^{\mu\nu} = m\int d\tau\, u^{\mu}u^{\nu}\,\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right)+\frac{1}{2}\,\nabla_{\lambda}\int d\tau\,\left(u^{\mu}\Sigma^{\nu\lambda}+u^{\nu}\Sigma^{\mu\lambda}\right)\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right). \tag{4.8.2}$$

The covariant divergence of $T_0^{\mu\nu}$ is

$$\begin{aligned}\nabla_{\mu}T_0^{\mu\nu} &= \int d\tau\left(m\frac{Du^{\nu}}{D\tau} - \frac{1}{2}\,\Sigma^{\kappa\lambda}R_{\kappa\lambda}{}^{\nu}{}_{\mu}u^{\mu}\right)\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right)\\ &\quad +\frac{1}{2}\,\nabla_{\lambda}\int d\tau\frac{D\Sigma^{\nu\lambda}}{D\tau}\,\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right) = 0.\end{aligned} \tag{4.8.3}$$

and vanishes upon applying the equations of motion (4.3.10, 4.3.11) with $q = 0$. Similarly, for a particle subject to the gravitational Stern-Gerlach force with the hamiltonian $H_0 + H_{SG}$ the correct expressions is

$$T^{\mu\nu} = T_0^{\mu\nu} + \kappa\, T_1^{\mu\nu}, \tag{4.8.4}$$

where

$$\begin{aligned}T_1^{\mu\nu} &= \frac{1}{2}\,\nabla_{\kappa}\nabla_{\lambda}\int d\tau\,\left(\Sigma^{\mu\lambda}\Sigma^{\kappa\nu}+\Sigma^{\nu\lambda}\Sigma^{\kappa\mu}\right)\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right)\\ &\quad +\frac{1}{4}\int d\tau\,\Sigma^{\rho\sigma}\left(R_{\rho\sigma\lambda}{}^{\nu}\Sigma^{\lambda\mu}+R_{\rho\sigma\lambda}{}^{\mu}\Sigma^{\lambda\nu}\right)\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right).\end{aligned} \tag{4.8.5}$$

Again performing standard operations from tensor calculus including Ricci- and Bianchi-identities leads to the result

$$\begin{aligned}\nabla_\mu T_1^{\mu\nu} &= \frac{1}{4}\int d\tau\, \nabla^\nu R_{\rho\sigma\kappa\lambda}\, \Sigma^{\rho\sigma}\Sigma^{\kappa\lambda}\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right)\\ &\quad+\frac{1}{2}\,\nabla_\lambda\int d\tau\,\Sigma^{\rho\sigma}\left(R_{\rho\sigma\kappa}{}^{\lambda}\Sigma^{\kappa\nu}-R_{\rho\sigma\kappa}{}^{\nu}\Sigma^{\kappa\lambda}\right)\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right).\end{aligned} \tag{4.8.6}$$

Combining this with the expression (4.8.3) for $\nabla_\mu T_0^{\mu\nu}$ it follows that the divergence of the full energy-momentum tensor vanishes

$$\nabla_\mu\left(T_0^{\mu\nu}+\kappa T_1^{\mu\nu}\right)=0, \tag{4.8.7}$$

provided the non-minimal equations of motion (4.6.2) hold. Finally, one can also take into account the electro-magnetic Lorentz- and Stern-Gerlach forces by additional contributions

$$T_{\mu\nu}^{em}=F_\mu{}^\lambda F_{\nu\lambda}-\frac{1}{4}\,g_{\mu\nu}F_{\kappa\lambda}F^{\kappa\lambda}-\frac{\lambda}{2}\,g_{\mu\nu}\int d\tau F_{\kappa\lambda}\,\Sigma^{\kappa\lambda}\frac{1}{\sqrt{-g}}\,\delta^4\left(x-X\right). \tag{4.8.8}$$