Spin dynamics in general relativity
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Chapter 3 explores the motion of test particles in curved space-time in the Hamiltonian formalism. I develop the particle's dynamics with the Poisson brackets, the minimal Hamiltonian and the conserved quantities. Then applying the formalism in the Schwarzschild space-time, the circular orbits and the Innermost Stable Circular Orbit are found. By describing the effective potential, the various kinds of orbits: circular, eccentric, scattering and plunging orbits are explained. Further exploring the geodesic deviation method, the fully relativistic first order perturbation theory for eccentric orbits are obtained. From the frequency analysis and stability criterion the method of finding the Innermost Stable Circular Orbit is generalized. The chapter is concluded with the equations of motion obtained from the conservation of the energy-momentum tensor.

3

Motion in Curved Space-time

3.1 Hamiltonian Formalism

The basic machinery of GR has been described in the previous chapter. Now we want to investigate the dynamics of test particles in curved space-time with in the Hamiltonian framework. Hamiltonian formalism includes three sets of ingredients: equations of motion, phase-space and the conserved quantities.

The equations describe test particle dynamics are so-called geodesic equations. We have derived geodesic equations of motion starting from the standard variational procedure and also from the Hamiltonian dynamics. In the following sections it is further shown that, it can be obtained from the principles of energy-momentum conservation.

The phase-space formulation of motion in curved space-time is being constructed with the closed set of covariant Poisson-Dirac brackets, obeying Jacobi identities. It consists of the position co-ordinate x^μ and the covariant momentum π_μ, and therefore its anti-symmetric bracket is:

$$\{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \tag{3.1.1}$$

all other possible brackets vanish. These brackets are independent of the specific Hamiltonian. Therefore, in principle we can use varieties of covariant Hamiltonians with the brackets to obtain the equation of motion. However, here we are interested in studying the geodesic motion of the test particle in curved space-time i.e., the particle's interaction is strictly gravitational. Therefore as described in the previous chapter the appropriate Hamiltonian is

$$H = \frac{1}{2m} g^{\mu\nu}(x)\pi_\mu\pi_\nu. \tag{3.1.2}$$

Then the proper-time evolution equations for phase-space co-ordinates are obviously generated by computing the brackets. It is important to note that this Hamiltonian describes the particle's mass as a universal constant of motion for any space-time:

$$H = -\frac{m}{2} \qquad \Rightarrow \qquad g_{\mu\nu}u^\mu u^\nu = -1. \tag{3.1.3}$$

It is called the Hamiltonian constraint in the literature.

3.2 Symmetries, Killing vectors, and Constants of motion

In addition to the universal constants of motion eq. (3.1.3), there exists conserved quantities as a result of symmetries of space-time. Emmy Noether discovered that physical quantities such as energy, momentum, angular momentum, etc. which remain constant during the evolution of the system are related to symmetries of the dynamics. Thus symmetries lead to conservation laws, and knowing a conserved quantity of a dynamical system allows to reduce the dimension of the phase space in which the system is defined.

From special theory of relativity we know that suitable coordinate transformations on the Minkowski metric leaves the metric invariant, giving rise to the Poincaré group of symmetries. Similarly, the standard metrics on the two- or three-sphere have rotational symmetries because they are invariant under rotations of the sphere. We can describe this in two ways: either as an active transformation, in which we rotate the sphere and nothing changes, or as a passive transformation, in which we do not move the sphere, and we just rotate the coordinate system. These descriptions are equivalent.

In the context of geometry we define *symmetry* as an invariance of the metric under a coordinate transformation. The symmetries of a metric are called *isometries*. Quantitatively, we start with a manifold $\mathcal{M}$, with coordinates x^μ. Let the metric in these coordinates be $g_{\mu\nu}(x)$. Suppose we make an infinitesimal change of coordinates

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x) \tag{3.2.1}$$

For detecting continuous symmetries we require the invariance of the line element under infinitesimal transformations. We know that the metric tensor transforms as

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x). \tag{3.2.2}$$

Using the invariance of the metric under an isometry we can also write

$$g'_{\mu\nu}(x') = g_{\mu\nu}(x') \simeq g_{\mu\nu}(x) - \xi^\lambda \partial_\lambda g_{\mu\nu}(x). \tag{3.2.3}$$

The infinitesimal coordinate transformation also implies

$$\frac{\partial x^\alpha}{\partial x'^{\,\mu}} \simeq \delta^\alpha_\mu + \partial_\alpha \xi^\mu, \quad \frac{\partial x^\beta}{\partial x'^{\,\nu}} \simeq \delta^\beta_\nu + \partial_\alpha \xi^\nu. \tag{3.2.4}$$

Combining these results the Lie derivative of the metric w.r.t. the displacement vector ξ_μ must vanish:

$$\mathcal{L}_\xi g_{\mu\nu} \equiv \xi^\lambda \partial_\lambda g_{\mu\nu} + \partial_\mu \xi^\lambda g_{\lambda\nu} + \partial_\nu \xi^\lambda g_{\mu\lambda} = 0. \tag{3.2.5}$$

Using the metric postulate

$$\nabla_\lambda g_{\mu\nu} = 0, \tag{3.2.6}$$

this can be rewritten covariantly as

$$\mathcal{L} g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0. \tag{3.2.7}$$

Vector fields satisfying these equations are called the *Killing vectors*. Now we will establish the conserved quantities associated with these Killing vectors.

3.2.1 Constants of motion

In classical mechanics, the angular momentum of a particle moving in a rotationally symmetric gravitational field is conserved. In GR the concept of symmetries of a newtonian gravitational field is replaced by symmetries of the metric, and we therefore expect conserved quantities associated with the presence of Killing vectors.

Let us consider a massive particle moving along a geodesic of a spacetime which admits a Killing vector ξ_α. The geodesic equations written in terms of the particle's four-velocity $u^\alpha = dx^\alpha/d\tau$ read

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu = 0, \tag{3.2.8}$$

by contracting the above equation with ξ_α, we find

$$\xi_\alpha \left[\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu \right] \equiv \frac{d\left(\xi_\alpha u^\alpha\right)}{d\tau} - u^\alpha \frac{d\xi_\alpha}{d\tau} + \Gamma^\alpha_{\beta\nu} u^\beta u^\nu \xi_\alpha = 0 \tag{3.2.9}$$

Since

$$u^\alpha \frac{d\xi_\alpha}{d\tau} = u^\beta u^\nu \frac{\partial \xi_\beta}{\partial x^\nu} \tag{3.2.10}$$

therefore eq. (3.2.9) becomes,

$$\frac{d(\xi_\alpha u^\alpha)}{d\tau} - u^\beta u^\nu \left[\frac{\partial \xi_\beta}{\partial x^\nu} - \Gamma^\alpha_{\beta\nu} \xi_\alpha \right] \equiv \frac{d(\xi_\alpha u^\alpha)}{d\tau} - u^\beta u^\nu \xi_{\beta;\nu} = 0. \tag{3.2.11}$$

Since $\xi_{\beta;\nu}$ is antisymmetric in β and ν, while $u^\beta u^\nu$ is symmetric, the term $u^\beta u^\nu \xi_{\beta;\nu}$ vanishes, and eq. (3.2.11) finally becomes

$$\frac{d(\xi_\alpha u^\alpha)}{d\tau} = 0 \quad \Rightarrow \quad \xi_\alpha u^\alpha = g_{\alpha\mu} \xi^\mu u^\alpha = const. \tag{3.2.12}$$

Eq. (3.2.12) can re-written as $\xi^\mu \pi_\mu =$ constant $\equiv J$ (let's say), where $\pi_\mu = m g_{\mu\nu} u^\nu$. It is also straight forward to check the quantity J is a constant of the particle motion, by demanding its brackets to vanish with the Hamiltonian:

$$\{J, H\} = 0 \qquad \Rightarrow \qquad J_i = \xi^\mu_i \pi_\mu \tag{3.2.13}$$

Thus, for every Killing vector there exists an associated conserved quantity.

3.3 Spherical symmetry

The Einstein Field Equations are a complicated set of non-linear equations with 10 unknown functions of space-time. These equations are most easily solved in space-times with a maximal number of symmetries as these give rise to a maximal number of constants of motion. This accessibility makes using spherically symmetric spacetimes all the more attractive as a starting point. Birkhoff's theorem classifies all vacuum spherically symmetric spacetimes.

A spacetime is spherically symmetric if it admits an SO(3) group of isometries. In particular every point will lie on some round sphere, on which the rotation group acts transitively, which means that one can go from any point on the sphere to any other point by means of a rotation. Further a space-time is said to be stationary or static, if it exhibits the property of time-translation symmetry. Static spherically symmetric metrics admit four Killing vectors, one of which is timelike, while the remaining three are spacelike, representing the Lie algebra of the rotation group SO(3).

The most general, static and spherically symmetric metric can be expressed in spherical polar coordinates with the ansatz [16]

$$ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{3.3.1}$$

The coefficients $f(r)$ and $g(r)$ are fixed by requiring the asymptotic limit i.e., for $r \to \infty$, the metric should be Minkowskian: $ds^2 = -dt^2+dr^2+r^2\left(d\theta^2 + sin^2\theta d\varphi^2\right)$. Due to isotopy and time independence these coefficients cannot depend on (t, θ, φ) and no linear terms in $d\theta$ and $d\varphi$.
Note that this metric is diagonal. Therefore the metric and its inverse has the following components only

$$\begin{aligned} &g_{tt} = -f(r) \qquad g_{rr} = g(r) \qquad g_{\theta\theta} = r^2 \qquad g_{\varphi\varphi} = r^2 sin^2\theta \\ &g^{tt} = -\frac{1}{f(r)} \qquad g^{rr} = \frac{1}{g(r)} \qquad g^{\theta\theta} = \frac{1}{r^2} \qquad g^{\varphi\varphi} = \frac{1}{r^2 sin^2\theta}. \end{aligned} \tag{3.3.2}$$

The next steps are standard, we first compute the non-vanishing components of affine connections $\Gamma^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\nu\lambda} = \frac{1}{2}\, g^{\mu\kappa}\left(g_{\kappa\lambda,\nu} + g_{\kappa\nu,\lambda} - g_{\lambda\nu,\kappa}\right)$:

$$\begin{aligned} &\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{f'}{f} \qquad \Gamma^r_{tt} = \frac{1}{2}\frac{f'}{g} \qquad \Gamma^r_{rr} = \frac{1}{2}\frac{g'}{g} \\ &\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \qquad \Gamma^r_{\theta\theta} = -\frac{r}{g} \qquad \Gamma^r_{\varphi\varphi} = -\frac{r}{g} sin^2\theta \\ &\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r} \qquad \Gamma^\varphi_{\theta\varphi} = \Gamma^\varphi_{\varphi\theta} = cot\,\theta \qquad \Gamma^\theta_{\varphi\varphi} = -sin\,\theta\, cos\,\theta. \end{aligned} \tag{3.3.3}$$

where $'$ stands for $\frac{\partial}{\partial r}$. Then the Riemann tensor contracted to get Ricci tensor

$$R_{\mu\nu} = \partial_\nu \Gamma^\rho_{\rho\mu} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\sigma_{\rho\mu}\Gamma^\rho_{\nu\sigma} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\rho\sigma}, \tag{3.3.4}$$

and as a result

$$\begin{aligned}
R_{tt} &= \frac{1}{2}\frac{f''}{g} + \frac{1}{4}\frac{f'^2}{fg} - \frac{1}{4}\frac{f'g'}{g^2} + \frac{1}{2r}\frac{f'}{g} \\
R_{rr} &= \frac{1}{2}\frac{f''}{f} + \frac{1}{4}\frac{f'^2}{f^2} - \frac{1}{4}\frac{f'g'}{fg} + \frac{1}{2r}\frac{g'}{g} \\
R_{\theta\theta} &= -1 + \frac{1}{g} + \frac{r}{2g}\left(\frac{f'}{f} - \frac{g'}{g}\right) \\
R_{\varphi\varphi} &= sin^2\theta \, R_{\theta\theta}
\end{aligned} \tag{3.3.5}$$

The non-diagonal components $R_{\mu\nu}$ with $\mu \neq \nu$ vanish. These geometric quantities are more general. Therefore it can be used for any static spherically symmetric space-time like Schwarzschild, Reissner-Nordstrøm etc.

3.3.1 The Schwarzschild solution

We now want to find an exact solution of Einstein's equations in vacuum $R_{\mu\nu} = 0$ (for $\mu \neq \nu$), which is spherically symmetric and static. This will be the relativistic generalization of the newtonian solution for a pointlike mass $\varphi = -M/r$ and it will describe the gravitational field in the exterior of a non-rotating body. The solution will be obviously in the form of eq. (3.3.1), where the coefficients f and g are fixed in the following way:

The linear combination of time and radial equations of Ricci tensor implies

$$\frac{R_{tt}}{f} + \frac{R_{rr}}{g} = \frac{1}{r}\frac{g'}{g^2} + \frac{1}{r}\frac{f'}{gf} = 0, \tag{3.3.6}$$

which reveals a simple relation between f and g:

$$\frac{g'}{g} = -\frac{f'}{f} \quad \Rightarrow \quad log(g) = -log(f) + constant, \quad or \quad g \propto \frac{1}{f}. \tag{3.3.7}$$

Now, we fix the proportionality constant between f and g as follows. Imagine we are extremely far away from the star (for example), then the metric should reduce to the Minkowski metric. So in the limit $r \to \infty$ we have $g = f = 1$. This fixes the proportionality constant to be 1. Therefore $g = 1/f$.

Then we only need to compute one of them from one of the differential equations (3.3.5). Let's consider $R_{\theta\theta}$ component and replace g with $1/f$. We have

$$R_{\theta\theta} = 1 - rf' - f = 0 \qquad \Rightarrow \qquad f(r) = 1 + \frac{C}{r}, \tag{3.3.8}$$

where C is some constant we want to determine. We can fix the constant by resorting to the weak-field limit which should reproduce the Newtonian gravitational potential φ. In the weak-field limit we just have

$$f(r) = 1 + 2\varphi(r), \qquad where \qquad \varphi = -\frac{M}{r}, \tag{3.3.9}$$

so the constant $C = -2M$. Then the complete line element in Droste co-ordinates; with M the mass, r the radius of the object,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + sin^2\theta d\varphi^2\right). \tag{3.3.10}$$

This is the famous *Schwarzschild metric*: a unique, static and spherically symmetric vacuum solution, according to Birkhoff's theorem; obtained by the astronomer Karl Schwarzschild [54] in 1916, the very same year that Einstein published his field equations. It was apparently discovered independently by Johannes Droste [55], a student of Lorentz at Leiden University, around the same time.

The Schwarzschild metric (3.3.10) looks divergent at $r = 2M$, the *Schwarzschild radius*. As can be seen by switching to other co-ordinates this is actually a co-ordinate singularity, not a physical singularity of space-time. But the Schwarzschild radius defines a characteristic gravitational scale for any celestial object, related to the formation of a horizon. For the earth or even for the sun the radius is actually very smaller than the radius of the object itself. To compute the radius we need to insert G and c back to the expression and find

$$R_s = \frac{2GM}{c^2} \tag{3.3.11}$$

which is about $3\,km$ for the sun. So, for most astronomical objects this number is so small that we don't need to consider it. However, objects smaller than their Schwarzschild radius disappear behind the horizon and become black holes.

3.3.2 Geodesic equations of motion and effective potential

We now want to consider the motion of a freely falling particle in the Schwarzschild space-time. The analysis can be simplified by using the constants of motion as implied by the Noether's theorem; because of the spherical symmetry of the Schwarzschild metric, there exists four constants associated with the Killing vectors (3.2.13): $J_i = \xi_i^\mu \pi_\mu$. Then

$$E = \xi^0 \pi_0, \qquad J_j = \xi_j^\alpha \pi_\alpha \tag{3.3.12}$$

where E is the particle's energy; $J_j = (J_1, J_2, J_3)$ is the total angular momentum of the system. Without loss of generality we can choose the coordinate system such that $\theta = \pi/2 \Rightarrow u^\theta = 0$, this way the trajectory lies on the plane perpendicular to the orbital angular momentum. Here the total angular momentum is strictly orbital, and the direction chosen to be z-axis. Then we write the equations of motion (3.2.8) in the component form [56]:

$$\frac{du^t}{d\tau} = -\frac{2M}{r\,(r-2M)}\,u^r u^t, \tag{3.3.13}$$

$$\frac{du^r}{d\tau} = -\frac{M\,(r-2M)}{r^3}\,u^{t\,2} + \frac{M}{r(r-2M)}\,u^{r\,2} + (r-2M)\,u^{\varphi\,2}, \tag{3.3.14}$$

$$\frac{du^\varphi}{d\tau} = -\frac{2}{r}\,u^r u^\varphi. \tag{3.3.15}$$

With z-axis being the choice of the angular momentum, the constants J_1 and J_2 turns out to be zero i.e., $J_1 = J_2 = 0$. Then we are left with the remaining two constants from (3.3.12); $\varepsilon = E/m$ energy per unit mass and $\ell = J_3/m$ angular momentum per unit mass,

$$\varepsilon = \left(1-\frac{2M}{r}\right)u^t, \qquad\qquad \ell = r^2 sin^2\theta\, u^\varphi = r^2 u^\varphi. \tag{3.3.16}$$

To establish the particle's orbits, we investigate the equations (3.3.13), (3.3.14) and (3.3.15). Eq. (3.3.13) can be re-written as

$$\frac{d}{d\tau}\left[ln(u^t) + ln\left(1-\frac{2M}{r}\right)\right] = 0, \tag{3.3.17}$$

which can be integrated as $ln\left[u^t\left(1-\frac{2M}{r}\right)\right] =$ constant or

$$u^t\left(1-\frac{2M}{r}\right) = constant \tag{3.3.18}$$

Similarly eq. (3.3.15) is re-written as

$$\frac{1}{r^2}\frac{d}{d\tau}\left(r^2 u^\varphi\right) = 0, \qquad \Rightarrow \qquad r^2 u^\varphi = constant. \tag{3.3.19}$$

From the Killing constants (3.3.16), we interpret (3.3.18) and (3.3.19) as ε and ℓ. This implies geodesic equations (3.3.13) and (3.3.15) doesn't give any new result. Thus we are left with the radial geodesic equation (3.3.14) only. Upon using the Killing constants (3.3.16) for u^t and u^φ, it turns out be

$$\frac{du^r}{d\tau} = -\frac{M\varepsilon^2}{r\,(r-2M)} + \frac{M}{r(r-2M)}\,u^{r\,2} + \frac{\ell^2}{r^4}\,(r-2M). \tag{3.3.20}$$

a relation for (ε, ℓ). A second relation between these quantities are given by the Hamiltonian constraint: $g_{\mu\nu}u^\mu u^\nu = -1$, similarly by using the Killing constants (3.3.16) we express

$$\left(1-\frac{2M}{r}\right)u^{t\,2} - \frac{u^{r\,2}}{1-\frac{2M}{r}} - r^2\,u^{\varphi\,2} = 1 \quad\Rightarrow\quad u^{r\,2} + \left(1-\frac{2M}{r}\right)\left(\Delta + \frac{l^2}{r^2}\right) = \varepsilon^2. \tag{3.3.21}$$

or

$$E = \frac{1}{2}u^{r\,2} + \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(\Delta + \frac{l^2}{r^2}\right) - 1\right] \tag{3.3.22}$$

where, $E = (\varepsilon^2 - 1)/2$ is the total energy. Note, Δ is 1 for massive particles and 0 for massless particles. If the particle is massless, the geodesic equation cannot be parametrized with the proper time. In this case the particle worldline has to be parametrized using an affine parameter λ such that the geodesic equation takes the form (3.2.8), and the particle tangent vector is $u^\alpha = dx^\alpha/d\lambda$. The derivation of the constants of motion associated to a spacetime symmetry, i.e. to a Killing vector, is similar as for massive particles, recalling that by a suitable choice of the parameter along the geodesic $J = \{E, J_i\}$. Then since for massless particles $m^2 = 0$, the Killing constants ε and ℓ are identified as energy and angular momentum.

Eq. (3.3.22) has the form of an energy equation with a "kinetic energy" term, $\dot{r}^2$ plus a function of r, "potential energy" equalling a constant. Thus the motion in the radial coordinate is exactly equivalent to a particle moving in an effective potential $V_{eff}(r)$ where

$$V_{eff}(r) = \frac{1}{2}\left[\left(1-\frac{2M}{r}\right)\left(\Delta + \frac{l^2}{r^2}\right) - 1\right]. \tag{3.3.23}$$

Then the simplest orbits one can start with are circular orbits i.e., $r = R$, for which we can differentiate the potential and set it to zero: $\partial_r V_{eff}(r) = 0$, which results:

$$\ell^2(R - 3M) = \Delta\, MR^2 \tag{3.3.24}$$

Thus we conclude, the circular geodesics exists only for $R > 3M$, for massive particles ($\Delta = 1$) and $R = 3M$ implies null geodesic which is interpreted as *light ring* for massless particles ($\Delta = 0$). Further evaluating the second derivative of the potential yields

$$\frac{\partial^2 V_{eff}(r)}{\partial r^2} = 2\Delta\frac{M}{R^3}\frac{(R-6M)}{(R-3M)} \tag{3.3.25}$$

we observe the circular orbits for $R \geq 6M$ are stable and positive; $R = 6M$ implies the flex point. Then the circular orbits between the radius $3M \leq R < 6M$ are necessarily unstable.

We show the above results qualitatively in Fig. 3.1 [50]. Massive test particles obey four kinds of orbits in Schwarzschild space-time. The Schwarzschild potential has one maximum and one minimum if $\ell/M > 12$. The following 4 points describes the Fig. 3.1 from the top:

(i). The circular orbits exists at the radii, when the potential has minimum or maximum. The orbit at maximum will be in unstable equilibrium, because a small perturbation will through the particle to infinity or the particle will reach the singularity at $r = 0$.

(ii). For $E < 0$, the particle bounds between two turning points. The cross symbols are the turning points: the closest approach to the centre is the perihelion and the farthest approach is the aphelion.

(iii). When E is positive and less than the maximum of the effective potential, then the orbit is scattering. That is the particle comes from infinity and orbits the centre and then moves out to infinity.

(iv). If E is greater than the maximum, then the particle comes from infinity and plunges into the centre.

Now re-writing eq. (3.3.24) for R results,

$$R = \frac{\ell^2}{2M}\left(1 + \sqrt{1 - \frac{12M^2}{\ell^2}}\right) \tag{3.3.26}$$

which relates the radius of the orbits to angular momentum per unit mass ℓ. Thus the minima of the potential lies at a special value of $\ell = 2\sqrt{3}M$, called as the *Innermost Stable Circular Orbit (ISCO)*. *ISCO* can also be predicted through a more general method: *stability criterion*, obtained by evaluating the geodesic deviations between the neighbouring geodesics. This is presented in the next section.
Then returning to the radial geodesic equation (3.3.20); for circular orbits, it yields

$$\varepsilon^2 = \frac{\ell^2}{MR}\left(1 - \frac{2M}{R}\right)^2 \qquad \Leftrightarrow \qquad \frac{u^\varphi}{u^t} = \sqrt{\frac{M}{R^3}} \tag{3.3.27}$$

which is a well known Kepler's result. Thus the geodesic equation (3.2.8) can be viewed as a generalisation of the Kepler's law. Finally, using the above result (3.3.27) along with the normalisation condition (3.3.21), we uniquely express, for circular orbits the Killing constants (ε, ℓ) in terms of the mass M and the radius R

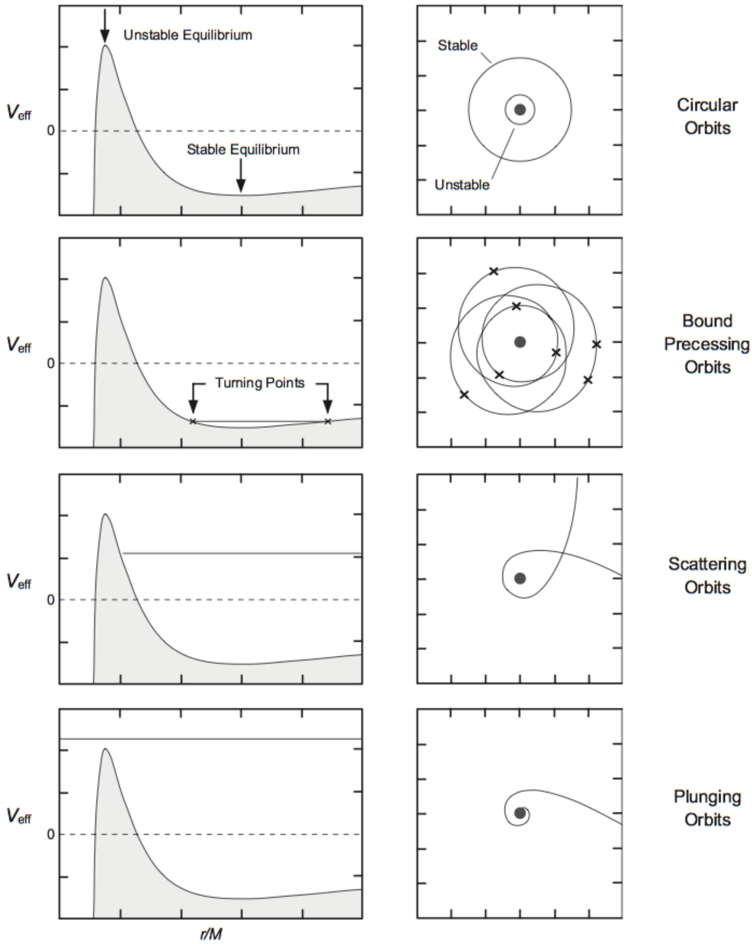

Figure 3.1. The effective potential $V_{eff}(r)$ and its relation to the total energy E is shown in the left, where the vertical axis is $V_{eff}(r)$ and the horizontal axis is r/M. Horizontal lines indicate the vale of E. The shapes of the corresponding orbits are plotted in polar coordinates r and φ, in the plane. The dark region (dot) in each plot is $r < 2M$.

of the black hole

$$\varepsilon_{circ} = \frac{(1 - \frac{2M}{R})}{\sqrt{(1 - \frac{3M}{R})}}, \qquad \ell_{circ} = \sqrt{\frac{MR}{(1 - \frac{3M}{R}.)}}. \tag{3.3.28}$$

We conclude this section with re-writing (3.3.16); the angular frequency of the circular orbits in terms of (M, R) by using (3.3.28)

$$u^{\varphi} \equiv \omega_{circ} = \frac{1}{R^2}\sqrt{\frac{MR}{(1-\frac{3M}{R})}}. \tag{3.3.29}$$

3.3.3 Geodesic Deviation: Tidal forces

The equivalance principle is only valid locally, at each point. Two neighbouring mass points which are each in free fall will fall differently. Hence if two such points are physically connected, they will feel a force coming from difference in the way that they free fall. These forces are known as tidal forces.

Thus we are interested in the rate of change of the displacement between the two curves along the geodesic, i.e. the acceleration of the separation [57,58]. Therefore, we consider two geodesic paths traced by the near by test particles, with coordinate vectors, $x^{\lambda}(\tau)$ and $x'^{\lambda}(\tau)$. Then $\delta x^{\lambda}(\tau) = x'^{\lambda}(\tau) - x^{\lambda}(\tau)$ is the difference of two nearby geodesics. If $v^{\nu} = dx^{\nu}/d\tau$ is the tangent vector to a curve $x^{\nu}(\tau)$, then $u^{\lambda} = v^{\nu}\mathcal{D}_{\nu}\delta x^{\lambda}$ is the velocity of the displacement. Thus from the geodesics analysis we have that

$$u^{\lambda} = v^{\nu}\mathcal{D}_{\nu}\delta x^{\lambda} = \frac{d\delta x^{\lambda}}{d\tau} + \Gamma^{\lambda}_{\mu\nu}v^{\nu}\delta x^{\mu}. \tag{3.3.30}$$

This leads to the acceleration,

$$\begin{aligned} a^{\lambda} = v^{\nu}\mathcal{D}_{\nu}(u^{\lambda}) &= \frac{d}{d\tau}\left(\frac{d\delta x^{\lambda}}{d\tau} + \Gamma^{\lambda}_{\mu\nu}v^{\nu}\delta x^{\mu}\right) + \Gamma^{\lambda}_{\mu\nu}u^{\nu}v^{\mu} \\ &= \frac{d^2\delta x^{\lambda}}{d\tau^2} + \partial_{\rho}\Gamma^{\lambda}_{\mu\nu}v^{\nu}v^{\rho}\delta x^{\mu} - \Gamma^{\mu}_{\rho\sigma}\Gamma^{\lambda}_{\mu\nu}\delta x^{\mu}v^{\rho}v^{\sigma} \\ &\quad + \Gamma^{\lambda}_{\mu\nu}v^{\nu}\frac{d\delta x^{\mu}}{d\tau} + \Gamma^{\lambda}_{\mu\nu}\left(\frac{d\delta x^{\nu}}{d\tau} + \Gamma^{\nu}_{\rho\sigma}v^{\rho}\delta x^{\sigma}\right)v^{\mu} \end{aligned} \tag{3.3.31}$$

where we have used the geodesic equation $\frac{dv^{\mu}}{d\tau} = -\Gamma^{\mu}_{\rho\sigma}v^{\rho}v^{\sigma}$ in the third term on the second line. Then expanding the geodesic equations

$$\begin{aligned} \frac{d^2x^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}(x)\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} &= 0, \\ \frac{d^2x'^{\lambda}}{d\tau^2} + \Gamma^{\lambda}_{\mu\nu}(x')\frac{dx'^{\mu}}{d\tau}\frac{dx'^{\nu}}{d\tau} &= 0, \end{aligned} \tag{3.3.32}$$

to lowest order in $x'^{\nu}(\tau) - x^{\nu}(\tau) = \delta x^{\nu}(\tau)$ to find an equation for $\delta x^{\nu}(\tau)$;

$$\frac{d^2\delta x^{\lambda}}{d\tau^2} + 2\Gamma^{\lambda}_{\nu\rho}v^{\nu}\frac{d\delta x^{\rho}}{d\tau} + \partial_{\rho}\Gamma^{\lambda}_{\nu\sigma}\delta x^{\rho}v^{\nu}v^{\sigma} = 0, \tag{3.3.33}$$

inserting this equation into (3.3.31), we obtain

$$
\begin{aligned}
a^\lambda &= -\partial_\rho \Gamma^\lambda_{\nu\sigma} \delta x^\rho v^\nu v^\sigma - \Gamma^\mu_{\rho\sigma}\Gamma^\lambda_{\mu\nu}\delta x^\mu v^\rho v^\sigma + \partial_\rho \Gamma^\lambda_{\mu\nu} v^\nu v^\rho \delta x^\mu + \Gamma^\lambda_{\mu\nu}\Gamma^\nu_{\rho\sigma} v^\rho v^\mu \delta x^\sigma \\
&= -R_{\rho\mu\nu}{}^\lambda v^\mu v^\nu \delta x^\rho .
\end{aligned}
\tag{3.3.34}
$$

The *Riemann curvature tensor* $R_{\rho\mu\nu}{}^\lambda$ in the above equation implies that we can measure the curvature by examining the proper acceleration of the separation of two nearby geodesics. With this definition we can re-write the equation of *geodesic deviation* eq. (3.3.33) in the simple form,

$$
\mathcal{D}^2_\tau \delta x^\mu - R_{\lambda\kappa}{}^\mu{}_\nu v^\kappa v^\nu \delta x^\lambda = 0. \tag{3.3.35}
$$

The equation of geodesic deviation controls the congruence of nearby geodesics. In a flat space-time, the curvature tensor vanishes, and hence $\mathcal{D}^2_\tau \delta x^\mu = d^2_\tau \delta x^\mu = 0$. This means that two initially parallel geodesics remain parallel at all times. In curved space-times however, the Riemann tensor is non-vanishing, and as a consequence a freely moving observer sees a relative acceleration of nearby freely moving test particles, which manifests as *tidal effect*. This method can also be used to obtain the eccentric bound orbits of a test mass in the Schwarzschild space-time.

3.3.4 Stability of bound orbits and ISCO

The horizon of a Schwarzschild black hole is located at R = 2M, and the *ISCO* is found at a larger value of R. By analysing the effective potential we concluded it is at $R = 6M$. Here we start from the above described geodesic deviation method and analyse the *ISCO* in the Schwarzschild space-time, in a more generalised way.

The circular orbits found in the previous sections can be used as a special reference orbits to solve the geodesic deviation equations (3.3.33) to obtain the stability criterion for bound orbits. Note, it is easy to work with non-covariant variations (3.3.33) rather than the covariant ones (3.3.35). The conservation of angular momentum implies the motion of the particles in the equatorial plane: $\theta = \pi/2$ i.e., $\delta\theta = 0$. Thus the allowed deviations from the circular orbits are parametrized by $\delta x^\mu = (\delta t, \delta r, \delta\varphi)$ only. Then the deviation equations are written in the compact form

$$
\begin{pmatrix}
\frac{d^2}{d\tau^2} & \alpha \frac{d}{d\tau} & 0 \\
\beta \frac{d}{d\tau} & \frac{d^2}{d\tau^2} - \kappa & -\gamma \frac{d}{d\tau} \\
0 & \eta \frac{d}{d\tau} & \frac{d^2}{d\tau^2}
\end{pmatrix}
\begin{pmatrix}
\delta t \\ \delta r \\ \delta\varphi
\end{pmatrix} = 0 \tag{3.3.36}
$$

where the coefficients are evaluated on the circular reference orbit and are given by

$$\begin{aligned}
\alpha &= \frac{2M}{R(R-2M)}\sqrt{\frac{R}{R-3M}} \qquad \beta = \frac{2M(R-2M)}{R^3}\sqrt{\frac{R}{R-3M}},\\
\gamma &= \frac{2(R-2M)}{R}\sqrt{\frac{M}{R-3M}}, \qquad \eta = \frac{2}{R^2}\sqrt{\frac{M}{R-3M}},\\
\kappa &= \frac{3M}{R^3}\frac{R-2M}{R-3M}.
\end{aligned} \tag{3.3.37}$$

Then solving the operator (3.3.36) for its eigen frequency ω_d of the oscillations; λ being its eigen values and are related through, $\lambda_{\pm} = \pm i\omega$,

$$\omega_d = \sqrt{\eta\gamma - \alpha\beta - \kappa} = \sqrt{\frac{M}{R^3}\frac{R-6M}{R-3M}}. \tag{3.3.38}$$

The real eigenvalues corresponds to stable circular orbits and the imaginary ones leads to unstable orbits [58]. Then it is straight forward to conclude from eq. (3.3.38), the eigenvalues are real only for $R \geq 6M$. Therefore, we predict $R = 6M$ is the $ISCO$. Thus the value of $ISCO$ obtained from the stability criterion and from minimising the effective potential are the same. Though the machinery to arrive at $ISCO$ through stability criterion is apparently tedious, this method has advantage when we include additional degrees of freedom; spin and/or charge to the test particles [48].

The generic solution for geodesic deviated bound orbits are periodic and the detailed analysis are given in the references [57]. The frequency ω_d of those orbits can be interpreted as the relativistic generalization of an epicycle and it differs from that of circular orbits frequency (3.3.29) ω_{circ}

$$\omega_d \approx \omega_{circ}\left(1 - \frac{3M}{R}\right) \tag{3.3.39}$$

Therefore the point of closest approach – the periastron, shifts during each orbit by a fixed amount $\delta\varphi$,

$$\delta\varphi = 2\pi\left(\frac{\omega_{circ}}{\omega_d} - 1\right) \approx 2\pi\left(\frac{3M}{R}\right) \tag{3.3.40}$$

In the geodesic deviation method, we develop an approximate analytical solution to the equations of motion and study the generic bound orbits close to circular orbits. Then by analysing the frequency of such orbits, we predict the $ISCO$. The difference in the frequency of bound orbits to that of circular orbits results in periastron shift. The periastron shift calculated through this approximation [57] and the one which is obtained directly by integrating the conservation of the absolute 4-velocity (3.3.21) (a standard exercise [50] for the well-known precession of the periastron in general relativity) are exactly the same.

3.4 Energy-momentum conservation: equations of motion

The equations of motion has been derived from the standard variational procedure and from the Hamiltonian formalism. Here we present an independent proof of the equations of motion from an appropriate energy-momentum tensor.

The Einstein's tensor is covariantly conserved as a result of contraction of the Bianchi identity,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}, \qquad \nabla^{\mu}G_{\mu\nu} = \nabla^{\mu}R_{\mu\nu} - \frac{1}{2}\nabla_{\nu}R = 0, \tag{3.4.1}$$

This implies, Einstein field equation is consistent only if the energy-mometum tensor $T_{\mu\nu}$ is also covariantly conserved. Thus,

$$\nabla^{\mu}T_{\mu\nu} = 0. \tag{3.4.2}$$

the source term $T_{\mu\nu}$ has the same property as $G_{\mu\nu}$. Then the energy-mometum tensor $T_{\mu\nu}$ for a test particle moving on a world-line $X^{\mu}(\tau)$ is defined by the proper-time integral[1]

$$T_0^{\mu\nu} = \frac{m}{\sqrt{-g}} \int d\tau \, u^{\mu} u^{\nu} \delta^4 \left(x - \xi(\tau)\right) \tag{3.4.4}$$

where $\xi(\tau)$ is the position coordinate of the particle in the phase-space. Then the covariant divergence of $T_0^{\mu\nu}$ vanishes for the particle moving on geodesics:

$$\nabla_{\mu}T_0^{\mu\nu} = \frac{m}{\sqrt{-g}} \int d\tau \, \frac{Du^{\nu}}{D\tau} \delta^4 \left(x - \xi(\tau)\right) = 0. \tag{3.4.5}$$

Therefore,

$$\frac{Du^{\nu}}{D\tau} = 0, \qquad \Rightarrow \qquad \ddot{x}^{\mu} + \Gamma^{\mu}_{\lambda\nu}\dot{x}^{\lambda}\dot{x}^{\nu} = 0. \tag{3.4.6}$$

Thus we have obtained the geodesic equations of motion in another alternate way. This is an obvious result in GR. In the following chapters we prove the similar computation is also possible for non-trivial cases, like including the spin-dependent forces.

[1] The square root is included because we define the delta-function as a scalar density of weight 1/2, such that for scalar functions $f(x)$

$$\int d^4y \, \delta^4(x - y) f(y) = f(x). \tag{3.4.3}$$