Spin dynamics in general relativity

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Chapter 2 presents the Equivalence Principle and builds up the necessary differential geometry to describe General Relativity. The geodesic equations of motion are developed starting from the action principle and also from the Hamiltonian dynamics. I conclude by describing the field equation and its significance.

2

Gravity and General Relativity

2.1 The Equivalence Principle

The Equivalence Principle is a corner stone of Einstein's theory of gravity, General Relativity (GR) [50].

In its original form it refers to the equivalence between gravitational and inertial mass, as demonstrated experimentally by various scientists from the late 16th century onward. This was the starting point for Newton's theory of gravity in which the gravitational pull of the earth gives the same acceleration to an apple and to the moon. In GR it holds as the acceleration of objects results from the geometry of space-time, independent of mass or composition. In this context it is usually referred to as the *Weak Equivalence Principle.*

The Equivalence Principle formulated by Einstein, the *Einstein Equivalence Principle* is slightly stronger. It says that a reference system in free fall is a local Lorentz frame, in which the laws of special relativity hold. In such a system non-interacting objects fall at the same rate with no relative acceleration. It is a *local* Lorentz frame as these statements only hold in the limit that distances are small compared to the local scale of space-time curvature, otherwise there would be tidal accelerations.

There also exists a third version of the Equivalence Principle, which states that the equivalence of gravitational and inertial mass includes all possible contributions to the mass including gravitational binding energy (self-energy). This is called the *Strong Equivalence Principle* and is most difficult to test, as usually gravitational binding energy is extremely weak. The only objects in which gravitational contribution to mass is significant are compact bodies like neutron stars and black holes.

2.2 Coordinates, metric and motion

In the absence of gravity, in a local Lorentz frame, free particles move with constant velocities on straight trajectories. When different sections of these trajectories in overlapping Lorentz frames are glued together, one gets trajectories which are still in a generalized sense the "shortest path" between two space-time points: *geodesics* [51].

An invariant measure of distance along a particle's space-time trajectory, or *world line*, is the proper time τ. It is the time measured during any short sections of the worldline by a clock at rest w.r.t. the particle. Let (x) be a coordinates for the patch of space-time where the trajectory is located, and consider two points on the trajectory with coordinates (x^μ) and $(x^\mu + dx^\mu)$. Then the proper time interval $d\tau$ is determined from a quadratic expression in the coordinate intervals dx^μ:

$$-\,d\tau^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{2.2.1}$$

The coefficients $g_{\mu\nu}(x)$ define the metric for the coordinate system x^μ at the given point. As it is an invariant, the same quantity measured in terms of a different coordinate system (x') is

$$-\,d\tau^2 = g'_{\mu\nu}(x') dx'^{\,\mu} dx'^{\,\nu}. \tag{2.2.2}$$

Now the *diffeomorphism* $x^\mu \to x'^{\,\mu}(x)$ if it is smooth allows us to write the last expression also as

$$g'_{\mu\nu}(x') \frac{\partial x'^{\,\mu}}{\partial x^\kappa} \frac{\partial x'^{\,\nu}}{\partial x^\lambda} dx^\kappa dx^\lambda. \tag{2.2.3}$$

Comparing with expression (2.2.1) gives

$$g_{\kappa\lambda}(x) = g'_{\mu\nu}(x') \frac{\partial x'^{\,\mu}}{\partial x^\kappa} \frac{\partial x'^{\,\nu}}{\partial x^\lambda}. \tag{2.2.4}$$

This shows how the metric coefficients change between different coordinate systems. The inverse metric is written $g^{\mu\nu}(x)$, such that at the same point in the same coordinate system

$$g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu.$$

It changes between coordinate systems by the inverse transformation

$$g^{\mu\nu}(x) = g'^{\,\kappa\lambda}(x') \frac{\partial x^\mu}{\partial x'^{\,\kappa}} \frac{\partial x^\nu}{\partial x'^{\,\lambda}}.$$

Now the total proper time along a curve $x^\mu(\tau)$ between two space-time points (a, b) with time-like separation is

$$\int_a^b d\tau. \tag{2.2.5}$$

Notice that we can introduce an arbitrary parameter λ labeling the points on the curve, as long as it is monotonic between (a, b). Therefore, the time interval

$$d\tau \, = \, \left(-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{1/2} d\lambda, \tag{2.2.6}$$

where $d\lambda$ is the displacement on space-time. We are varying the paths

$$x^{\mu}(x) \rightarrow x^{\mu}(x) + \delta x^{\mu}(x) \tag{2.2.7}$$

keeping the end-points fixed, and will denote the τ-derivatives by $\dot{x}(\tau)$ and $\partial_{\lambda} \equiv \frac{\partial}{\partial x^{\lambda}} =,_{\lambda}$. By the standard variational procedure one then finds

$$\begin{aligned} \delta S \, &= \, \frac{1}{2} \int d\lambda \left(-g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right)^{-1/2} \left[-\delta g_{\mu\nu} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} - 2 g_{\mu\nu} \frac{d\delta x^{\mu}}{d\lambda} \frac{dx^{\nu}}{d\lambda} \right] \\ &= \, \frac{1}{2} \int d\tau \left[-g_{\mu\nu},_{\lambda} \, \dot{x}^{\mu} \dot{x}^{\nu} \delta x^{\lambda} + 2 g_{\mu\nu} \ddot{x}^{\nu} \delta x^{\mu} + 2 g_{\mu\nu},_{\lambda} \, \dot{x}^{\lambda} \dot{x}^{\nu} \delta x^{\mu} \right] \\ &= \, \int d\tau \left[g_{\mu\nu} \ddot{x}^{\nu} + \frac{1}{2} \left(g_{\mu\nu},_{\lambda} + g_{\mu\lambda},_{\nu} - g_{\nu\lambda},_{\mu} \right) \dot{x}^{\nu} \dot{x}^{\lambda} \right] \delta x^{\mu} \end{aligned} \tag{2.2.8}$$

Here the factor of 2 in the first equality is a consequence of the symmetry of the metric, the second equality follows from an integration by parts, the third from relabelling the indices in one term and using the symmetry in the indices of $\dot{x}^{\lambda} \dot{x}^{\nu}$ in the other.

We set the variation of action to zero, $\delta S = 0$. Further re-naming $\mu \rightarrow \kappa$ and multiplying by $g^{\mu\kappa}$, we obtain the equations for a timelike geodesic in an arbitrary gravitational field:

$$\frac{d^2 x^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\tau} \frac{dx^{\lambda}}{d\tau} = 0, \tag{2.2.9}$$

where $\Gamma^{\mu}_{\nu\lambda}$ is the Christoffel connection or Levi-Civita symbol, which is symmetric in the second and third indices:

$$\Gamma^{\mu}_{\lambda\nu} = \Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\kappa} \left(g_{\kappa\lambda,\nu} + g_{\kappa\nu,\lambda} - g_{\lambda\nu,\kappa} \right). \tag{2.2.10}$$

2.3 Hamiltonian dynamics

The equation (2.2.9) for geodesic motion was derived from the geometric principle of extremizing the amount of proper time along the curve. However the same equation of motion can also be obtained in a canonical phase-space approach with an appropriate hamiltonian. This approach introduces next to the particle's co-ordinates $x^{\mu}(\tau)$ also the canonical momenta $\pi_{\mu}(\tau)$. The appropriate hamiltonian is

$$H = \frac{1}{2m} g^{\mu\nu}(x) \pi_{\mu} \pi_{\nu}. \tag{2.3.1}$$

Hamilton's equations then imply the following equations of motion:

$$\begin{aligned}\dot{x}^{\mu} &= \frac{\partial H}{\partial \pi_{\mu}} = \frac{1}{m}\, g^{\mu\nu}\pi_{\nu},\\ \dot{\pi}_{\mu} &= -\frac{\partial H}{\partial x^{\mu}} = \frac{1}{m} g_{\kappa\lambda,\mu} g^{\kappa\rho} g^{\lambda\sigma}\pi_{\rho}\pi_{\sigma} = m g_{\kappa\lambda,\mu}\dot{x}^{\kappa}\dot{x}^{\lambda}.\end{aligned} \tag{2.3.2}$$

These equations can be rewritten in the form

$$\pi_{\mu} = m g_{\mu\nu}\dot{x}^{\nu}, \quad \ddot{x}^{\mu} + \Gamma_{\lambda\nu}^{\mu}\dot{x}^{\lambda}\dot{x}^{\nu} = 0. \tag{2.3.3}$$

Thus equation (2.2.9) is reobtained. Although less geometric, this method is entirely equivalent and is useful if more interactions than just with the curved background geometry are to be included, like electric charge or spin. This will become clear in the chapters to follow.

2.4 Differential geometry

The quantities that appeared in the previous sections can be introduced in a more general way not only on curves (geodesics) but as fields of geometric objects on the space-time manifold at large [52]. Vector fields $A_{\mu}(x)$ are sets of functions transforming under a change of coordinates as

$$A_{\mu}(x) = A'_{\nu}(x')\frac{\partial x'^{\,\nu}}{\partial x^{\mu}},$$

and similarly for higher-rank tensors $A_{\mu\nu\ldots}$. Taking the derivative of a vector or tensor is somewhat delicate, as in general it produces an new object which is not a tensor itself. However, one can define a *covariant* derivative using the Christoffel connection introduced before. Indeed one can construct a proper rank-2 tensor from a vector by taking

$$\mathcal{D}_{\lambda}A_{\mu} = \partial_{\lambda}A_{\mu} - \Gamma_{\lambda\mu}^{\nu}A_{\nu}. \tag{2.4.1}$$

Similarly a rank-2 tensor is lifted to a rank-3 tensor by taking

$$\mathcal{D}_{\lambda}A_{\mu\nu} = \partial_{\lambda}A_{\mu\nu} - \Gamma_{\lambda\mu}^{\kappa}A_{\kappa\nu} - \Gamma_{\lambda\nu}^{\kappa}A_{\mu\kappa},$$

etc. To prove the statement one has to check the transformation properties of the connection coefficients:

$$\Gamma_{\lambda\nu}^{\mu}(x) = \Gamma'^{\,\kappa}_{\rho\sigma}(x')\frac{\partial x'^{\,\rho}}{\partial x^{\lambda}}\frac{\partial x'^{\,\sigma}}{\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x'^{\kappa}} - \frac{\partial^2 x'^{\,\kappa}}{\partial x^{\lambda}\partial x^{\nu}}\frac{\partial x^{\mu}}{\partial x'^{\,\kappa}}.$$

For proofs we refer to the literature [3, 50].

Clearly in contrast to ordinary partial derivatives, covariant derivatives do not commute. Indeed

$$\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu] V_\lambda &= \mathcal{D}_\mu \left(\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda} V_\rho\right) - (\mu \leftrightarrow \nu) \\
&= \partial_\mu \left(\partial_\nu V_\lambda - \Gamma^\rho_{\nu\lambda} V_\rho\right) - \Gamma^\sigma_{\mu\nu} \left(\partial_\sigma V_\lambda - \Gamma^\rho_{\sigma\lambda} V_\rho\right) - \Gamma^\sigma_{\mu\lambda} \left(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma} V_\rho\right) \\
&\quad - (\mu \leftrightarrow \nu) \\
&= -\partial_\mu \left(\Gamma^\rho_{\nu\lambda} V_\rho\right) - \Gamma^\sigma_{\mu\lambda} \left(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma} V_\rho\right) - (\mu \leftrightarrow \nu) \\
&= -\partial_\mu \Gamma^\rho_{\nu\lambda} V_\rho + \Gamma^\sigma_{\mu\lambda} \Gamma^\rho_{\nu\sigma} V_\rho - (\mu \leftrightarrow \nu) \\
&= R_{\mu\nu\lambda}{}^{\rho} V_\rho
\end{aligned} \tag{2.4.2}$$

where

$$R_{\mu\nu\lambda}{}^{\rho} = -\partial_\mu \Gamma^\rho_{\nu\lambda} + \partial_\nu \Gamma^\rho_{\mu\lambda} - \Gamma^\sigma_{\nu\lambda} \Gamma^\rho_{\mu\sigma} + \Gamma^\sigma_{\mu\lambda} \Gamma^\rho_{\nu\sigma}. \tag{2.4.3}$$

Here although each single term in $R_{\mu\nu\lambda}{}^{\rho}$ is not a tensor, under a diffeomorphsim, we can prove the following transformation properties [3] for the resulting combination

$$R'_{\sigma\alpha\xi}{}^{\beta} = \frac{\partial x^\mu}{\partial x'^\sigma} \frac{\partial x^\nu}{\partial x'^\alpha} \frac{\partial x^\lambda}{\partial x'^\xi} \frac{\partial x'^\beta}{\partial x^\rho} R_{\mu\nu\lambda}{}^{\rho}, \tag{2.4.4}$$

and therefore, it is a (1, 3)-tensor; called as the *Riemann tensor*. It includes second-order derivatives of the metric: it does not vanish therefore in a locally inertial frame. It vanishes if and only if a manifold is flat. It is therefore the curvature tensor. In particular, if the Riemann tensor vanishes, we can always construct a coordinate system in which the metric components are constant.

The Riemann tensor (2.4.3) satisfies a number of symmetry properties. It is anti-symmetric in the first two or last two indices and symmetric in the first and last pairs of indices:

$$R^\mu{}_{\nu\rho\sigma} = -R^\mu{}_{\nu\sigma\rho}, \quad R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}, \tag{2.4.5}$$

and sum of cyclic permutation are zero:

$$R^\mu{}_{\nu\rho\sigma} + R^\mu{}_{\rho\sigma\nu} + R^\mu{}_{\sigma\nu\rho} = 0, \tag{2.4.6}$$

where the first index has been lowered using the metric: $R_{\mu\nu\rho\sigma} = g_{\mu\eta} R^\eta{}_{\nu\rho\sigma}$. It can be shown that these constraints reduce the number of independent components of the Riemann tensor in n dimensions from n^4 to $n^2\left(n^2-1\right)/12$, i.e. 20 in 4 dimensions, and only 1 in two dimensions.

Then the invariant parts of the Riemann tensor are defined as the Ricci tensor (a symmetric tensor) $R_{\mu\nu}$ and Ricci or curvature scalar R:

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu}, \qquad R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{2.4.7}$$

In addition to these algebraic identities, the Riemann tensor obeys a differential identity:

$$\nabla_{\gamma} R^{\mu}{}_{\nu\rho\sigma} + \nabla_{\sigma} R^{\mu}{}_{\nu\gamma\rho} + \nabla_{\rho} R^{\mu}{}_{\nu\sigma\gamma} = 0, \tag{2.4.8}$$

also called as Bianchi identity. Further contracting the Bianchi identity gives

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R. \tag{2.4.9}$$

This allows to define a "conserved" tensor, the *Einstein tensor*:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{2.4.10}$$

i.e., the Bianchi identity implies that the divergence of this tensor vanishes identically,

$$\nabla^{\mu} G_{\mu\nu} = 0. \tag{2.4.11}$$

This is sometimes called the contracted Bianchi identity.

2.5 Einstein's Field Equation

Einstein field equations [3, 53] describe the physical universe as a 4-dimensional Lorentzian manifold. It is the relation between curvature and energy-momentum content in the universe. This allows us to view the curvature tensor as a physical property of the universe, as a function of mass, momentum and energy.

The curvature of the Lorentzian manifold of space-time is caused by energy-momentum. Since geodesics on this manifold are motions of particles in free fall; that is, only affected by the force of gravity, curvature and gravitation are linked. The source of gravity is energy-momentum, and the source of curvature in this manifold is gravity. The precise equation for this relation is formulated as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G T_{\mu\nu}}{c^4}, \tag{2.5.1}$$

where the left hand side is the Einstein tensor $G_{\mu\nu}$ as we defined in (2.4.10), $G = 6.674 \times 10^{-11} Nm^2/kg^2$ is the Newton's constant, $c = 3 \times 10^8$ is the speed of light and $T_{\mu\nu}$ is the *Energy-momentum tensor* of all gravitating matter.

Now, number of observations can be made: As a consequence of Bianchi identity, the Einstein's tensor is covariantly conserved as shown in (2.4.11). Then the consistency of the Einstein's field equation (2.5.1) implies that the Energy momentum tensor $T_{\mu\nu}$ must also be covariantly conserved,

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{2.5.2}$$

Einstein's field equations constitutes a set of non-linear coupled partial differential equations whose general solution is not known. Usually one makes some assumptions, for instance spherical symmetry. Because the Ricci tensor is symmetric, the Einstein's field equations constitute a set of 10 algebraically independent second order differential equations for $g_{\mu\nu}$. Then the general covariant nature of Einstein equations makes us to expect only 6 independent equations for the metric.

Observe that the Riemann curvature tensor (2.4.3) contains terms, which are of the form a single derivative acting on the Christoffel connection, and terms which are quadratic forms in the connection. The Christoffel connection (2.2.10) is in turn expressed in terms of single derivatives acting on the metric tensor. This then implies that the Einstein's field equation (2.5.1) contains derivatives of the metric tensor up to second order in space-time, and in that sense it resembles the Maxwell equations.

The principal difference between the electrodynamics and the dynamics of gravitational field in GR are the nonlinear terms, contained in the quadratic forms in the Christoffel connection, which makes the theory more complicated. These terms are dynamically very relevant in strong gravitational fields. A second difference is that, in GR the dynamical field is the metric tensor, which is a rank two symmetric tensor field, while in the electrodynamics there are vector fields.

Finally, the coupling constant, $8\pi G/c^4 \sim 2 \times 10^{-43} s^2\, kg^{-1}\, m^{-1}$ is dimensionfull, but extremely small on any other physical scale, such that only in the presence of matter under extreme conditions (large energy densities), the matter effects on space-time can be strong. Such extreme conditions are found in compact objects like black holes and neutron stars.

Thus, GR models the effects of gravity as the curvature of Lorentzian manifold. It of course also generalizes the special relativity by using an in general non-flat metric tensor, and in fact is required to approximate to special relativity locally. Special relativity is a special case of GR, where there is no gravitational force acting on the particle. Further, when the motion is non-relativistic and in the weak gravitational field, we can recover Newton's theory of gravity: $\nabla^2 \phi = 4\pi G \rho$ (ϕ is the gravitational potential and ρ is the matter density).