Topics in the arithmetic of del Pezzo and K3 surfaces

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Chapter 2

Unirationality of del Pezzo surfaces of degree 2

In this section we will present some results about unirationality of del Pezzo surfaces of degree 2. In particular, we will show that all del Pezzo surfaces of degree 2 over a finite field are unirational. All the material presented in this chapter is part of joint work with Ronald van Luijk, and it can be found in [FvL15]; many of these results have already been published in [FvL16].

2.1 The main results

In Chapter 1 we have already seen that every del Pezzo surface, and so in particular every del Pezzo surface of degree 2, over an algebraically closed field is birational to the projective plane.

The same statement does not need to hold if the field is not algebraically closed, and so we look at weaker notions. Let k be any field and let X be a variety of dimension n over k. We say that X is *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$, defined over k.

Work of B. Segre, Yu. Manin, A. Knecht, J. Kollár, and M. Pieropan prove that every del Pezzo surface of degree $d \geq 3$ defined over k is unirational, provided that the set $X(k)$ of rational points is non-empty. For references, see [Seg43, Seg51] for $k = \mathbb{Q}$ and $d = 3$, see [Man86, Theorem 29.4 and 30.1] for $d \geq 3$ with the assumption that k is large enough for $d \in \{3, 4\}$. See [Kol02, Theorem 1.1] for $d = 3$ in general. See [Pie12, Proposition 5.19] and, independently, [Kne15, Theorem 2.1] for $d = 4$ in general. Since all del Pezzo surfaces over finite fields have a rational point (see [Man86, Corollary 27.1.1]), this implies that every del Pezzo surface of degree at least 3 over a finite field is unirational.

Building on work by Manin (see [Man86, Theorem 29.4]), C. Salgado, D. Testa, and A. Várilly-Alvarado prove that all del Pezzo surfaces of degree 2 over a finite field are unirational as well, except possibly for three isomorphism classes of surfaces (see [STVA14, Theorem 1]). In this chapter, we show that these remaining three cases are also unirational, thus proving our first main theorem.

Theorem 2.1.1. *Every del Pezzo surface of degree 2 over a finite field is unirational.*

More generally, we give some sufficient conditions for a del Pezzo surface of degree 2 to be unirational.

Theorem 2.1.2. *Suppose k is a field of characteristic not equal to 2, and let $\overline{k}$ be an algebraic closure of k. Let X be a del Pezzo surface of degree 2 over k. Let $B \subset \mathbb{P}^2$ be the branch locus of the anti-canonical morphism $\pi \colon X \to \mathbb{P}^2$. Let $C \subset \mathbb{P}^2$ be a projective curve that is birationally equivalent to $\mathbb{P}^1$ over k. Assume that all singular points of C that are contained in B are ordinary singular points. Then the following statements hold.*

1. *Suppose that there is a point $P \in X(k)$ such that $\pi(P) \in C - B$. Suppose that B contains no singular points of C and that all intersection points of B and C have even intersection multiplicity. Then the surface X is unirational.*

2. *Suppose that one of the following two conditions hold.*

   (a) *There is a point $Q \in C(k) \cap B(k)$ that is a double or a triple point of C. The curve B contains no other singular points of C, and all intersection points of B and C have even intersection multiplicity.*

(b) *There exist two distinct points $Q_1, Q_2 \in C(\overline{k}) \cap B(\overline{k})$ such that B and C intersect with odd multiplicity at Q_1 and Q_2 and with even intersection multiplicity at all other intersection points. Furthermore, the points Q_1 and Q_2 are smooth points or double points on the curve C, and B contains no other singular points of C.*

Then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.

Corollary 2.1.3. *Suppose k is a field of characteristic not equal to 2. Let X be a del Pezzo surface of degree 2 over k. Assume that X has a k-rational point, say P. Let $C \subset \mathbb{P}^2$ be a geometrically integral curve over k of degree $d \geq 2$ and suppose that $\pi(P)$ is a point of multiplicity $d-1$ on C. Suppose, moreover, that C intersects the branch locus B of the anti-canonical morphism $\pi\colon X \to \mathbb{P}^2$ with even multiplicity everywhere. Then the following statements hold.*

1. *If $\pi(P)$ is not contained in B, then X is unirational.*

2. *If $\pi(P)$ is contained in B, it is an ordinary singular point on C and we have $d \in \{3, 4\}$, then there exists a field extension ℓ of k of degree at most 2 for which the preimage $\pi^{-1}(C_\ell)$ is birationally equivalent with $\mathbb{P}^1_\ell$; for each such field ℓ, the surface X_ℓ is unirational.*

In the next section, we will present the three difficult surfaces and prove Theorem 2.1.1. The main tool is Lemma 2.2.2, which states that it suffices to construct a rational curve on each of the three del Pezzo surfaces.

Recall that if X is a del Pezzo surface of degree 2, then X admits 56 exceptional curves (cf. [Man86, Theorem IV.26.2]). A point on X is called a *generalised Eckardt point* if it lies on four of the 56 exceptional curves.

If a point P on a del Pezzo is not a generalised Eckardt point, and it does not lie on the ramification locus of the anti-canonical morphism, then Manin's construction, extended by C. Salgado, D. Testa, and A. Várilly-Alvarado, yields a rational curve that satisfies the assumptions of case (1) of Corollary 2.1.3 with the degree d being such that there are $4 - d$ exceptional curves through P (cf. Example 2.3.7).

The three difficult surfaces do not contain such a point. The proofs of unirationality of these three cases use a rational curve that is an example of case (2) of Corollary 2.1.3 instead (cf. Remark 2.2.4 and Example 2.3.9). Here we benefit from the fact that if k is a finite field, then any curve that becomes birationally equivalent with $\mathbb{P}^1$ over an extension of k, already is birationally equivalent with $\mathbb{P}^1$ over k itself. For two of the three cases, the rational curve we use has degree 4. For the last case, the curve we use has degree 3, but there also exist quartic curves satisfying the hypotheses of case (2) of Corollary 2.1.3. This raises the following question (cf. Question 2.4.6, Remark 2.4.8, and Example 2.4.9), which together with case (2) of Corollary 2.1.3 could help proving unirationality of del Pezzo surfaces of degree 2 over any field of characteristic not equal to 2.

Question 2.1.4. *Let $d \in \{3, 4\}$ be an integer. Let X be a del Pezzo surface of degree two over a field of characteristic not equal to 2, and let $P \in X(k)$ be a point on the ramification locus of the anti-canonical map $\pi\colon X \to \mathbb{P}^2$. Does there exist a geometrically integral curve of degree d in $\mathbb{P}^2$ over k that has an ordinary singular point of multiplicity $d-1$ at $\pi(P)$, and that intersects the branch locus of π with even multiplicity everywhere?*

For some d, X, and P, the answer to this question is negative (see Example 2.4.9), but in all cases we know of (all over finite fields), there do exist singular curves of degree d with a point of multiplicity at least $d-1$ at $\pi(P)$. Hence, it may be true that the answer to Question 2.1.4 is positive for X and P general enough.

In line with case (1) of Corollary 2.1.3, we can ask, in fact for any integer $d \geq 1$, an analogous question for points P that do not lie on the ramification locus, where we do not require the singular point to be ordinary. In this case, if P lies on $r \leq 3$ exceptional curves, then Manin's construction shows that the answer is positive for degree $d = 4 - r$. Therefore, this analogous question is especially interesting when P lies on four exceptional curves (cf. Remark 2.3.5 and Example 2.3.8).

In Section 2.3 we prove Theorem 2.1.2 and a generalisation, Corollary 2.1.3. In Section 2.4 we discuss how to search for curves satisfying the assumptions of Theorem 2.1.2 and in particular of Corollary 2.1.3.

2.2 Proof of the first main theorem

Set $k_1 = k_2 = \mathbb{F}_3$ and $k_3 = \mathbb{F}_9$. Let $\gamma \in k_3$ denote an element satisfying $\gamma^2 = \gamma + 1$. Note that γ is not a square in k_3. For $i \in \{1, 2, 3\}$, we define the surface X_i in $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ with coordinates x, y, z, w over k_i by

$$\begin{aligned} X_1: \ -w^2 &= (x^2+y^2)^2 + y^3z - yz^3, \\ X_2: \ -w^2 &= x^4 + y^3z - yz^3, \\ X_3: \ \gamma w^2 &= x^4 + y^4 + z^4. \end{aligned}$$

These surfaces are smooth, so they are del Pezzo surfaces of degree 2. C. Salgado, D. Testa, and A. Várilly-Alvarado proved the following result.

Theorem 2.2.1. *Let X be a del Pezzo surface of degree 2 over a finite field. If X is not isomorphic to X_1, X_2, or X_3, then X is unirational.*

Proof. See [STVA14, Theorem 1]. □

We will use the following lemma to prove the complementary statement, namely that X_1, X_2, and X_3 are unirational as well.

Lemma 2.2.2. *Let X be a del Pezzo surface of degree 2 over a field k. Suppose that $\rho\colon \mathbb{P}^1 \to X$ is a non-constant morphism; if the characteristic of k is 2 and the image of ρ is contained in the ramification divisor R_X, then assume also that the field k is perfect. Then X is unirational.*

Proof. See [STVA14, Theorem 17]. □

For $i \in \{1, 2, 3\}$, we define a morphism $\rho_i\colon \mathbb{P}^1 \to X_i$ by extending the map $\mathbb{A}^1(t) \to X_i$ given by

$$t \mapsto (x_i(t) : y_i(t) : z_i(t) : w_i(t)),$$

where

$$\begin{aligned} x_1(t) &= t^2(t^2-1), & x_2(t) &= t(t^2+1)(t^4-1), \\ y_1(t) &= t^2(t^2-1)^2, & y_2(t) &= -t^4, \\ z_1(t) &= t^8-t^2+1, & z_2(t) &= t^8+1, \\ w_1(t) &= t(t^2-1)(t^4+1)(t^8+1), & w_2(t) &= t^2(t^2+1)(t^{10}-1), \end{aligned}$$

$$\begin{aligned} x_3(t) &= (t^4+1)(t^2-\gamma^3), \\ y_3(t) &= (t^4-1)(t^2+\gamma^3), \\ z_3(t) &= (t^4+\gamma^2)(t^2-\gamma), \\ w_3(t) &= \gamma^2 t(t^8-1)(t^2+\gamma). \end{aligned}$$

It is easy to check for each i that the morphism ρ_i is well defined, that is, the polynomials x_i, y_i, z_i, and w_i satisfy the equation of X_i, and that ρ_i is non-constant. The methods used to find these curves are exposed in Section 2.4.

Theorem 2.2.3. *The del Pezzo surfaces X_1, X_2, and X_3 are unirational.*

Proof. By Lemma 2.2.2, the existence of ρ_1, ρ_2, and ρ_3 implies that X_1, X_2, and X_3 are unirational. □

Proof of Theorem 2.1.1. This follows from Theorems 2.2.1 and 2.2.3. □

Remark 2.2.4. Take any $i \in \{1,2,3\}$. Set $A_i = \rho_i(\mathbb{P}^1)$ and $C_i = \pi_i(A_i)$, where $\pi_i = \pi_{X_i} \colon X_i \to \mathbb{P}^2$ is as described in the previous section. By Remark 2 of [STVA14], the surface X_i is minimal, and the Picard group $\operatorname{Pic} X_i$ is generated by the class of the anti-canonical divisor $-K_{X_i}$. The same remark states that the linear system $|-nK_{X_i}|$ does not contain a geometrically integral curve of geometric genus zero for $n \leq 3$ if $i \in \{1,2\}$, nor for $n \leq 2$ if $i = 3$. For $i \in \{1,2\}$, the curve A_i has degree 8, so it is contained in the linear system $|-4K_{X_i}|$. The curve A_3 has degree 6, so it is contained in the linear system $|-3K_{X_i}|$. This means that the curve C_i has minimal degree among all rational curves on X_i. The restriction of π_i to A_i is a double cover $A_i \to C_i$. The curve

$C_i \subset \mathbb{P}^2$ has degree 4 for $i \in \{1,2\}$ and degree 3 for $i = 3$, and C_i is given by the vanishing of h_i, with

$$\begin{aligned} h_1 =& x^4 + xy^3 + y^4 - x^2yz - xy^2z, \\ h_2 =& x^4 - x^2y^2 - y^4 + x^2yz + yz^3, \\ h_3 =& x^2y + xy^2 + x^2z - xyz + y^2z - xz^2 - yz^2 - z^3. \end{aligned}$$

For $i \in \{1,2\}$, the curve C_i has an ordinary triple point Q_i, with $Q_1 = (0:0:1)$, $Q_2 = (0:1:1)$. The curve C_3 has an ordinary double point at $Q_3 = (1:1:1)$. For all i, the point Q_i lies on the branch locus $B_i = B_{X_i}$.

We will see later that the curve C_i intersects the branch locus B_i with even multiplicity everywhere. Of course, one could check this directly as well using the polynomial h_i. In fact, had we *defined* C_i by the vanishing of h_i, then one would easily check that C_i satisfies the conditions of part (2) of Corollary 2.1.3, which gives an alternative proof unirationality of X_i without the need of the explicit morphism ρ_i (see Example 2.3.9). Indeed, in practice we first found the curves C_1, C_2, and C_3, and then constructed the parametrisations ρ_1, ρ_2, ρ_3, which allow for the more direct proof that we gave of Theorem 2.2.3.

2.3 Proof of the second main theorem

Let k be a field of characteristic different from 2 and recall the notation introduced in Section 1.2.3. In what follows X denotes a del Pezzo surface of degree 2 over k, the map $\pi\colon X \to \mathbb{P}^2$ is its associated double covering map, with branch locus $B \subset \mathbb{P}^2$ and ramification locus $R \subseteq X$. The map $\iota\colon X \to X$ is the involution of X induced by the double covering map π. Let P be a point inside $X(k)$.

Combining Lemma 2.2.2 and Corollary 1.2.27 it is possible to relate the existence of some particular plane curves with the unirationality of a del Pezzo surface of degree 2.

Proposition 2.3.1. *Let $C \subset \mathbb{P}^2$ be a geometrically integral projective curve with $g(C) = 0$. Let $\tilde{C}$ denote its normalisation and set $n = \#b(\tilde{C}, B)$. The following statements hold.*

1. *If $n = 0$, then there exists a field extension ℓ of k of degree at most 2 such that the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent to C_ℓ. For each such ℓ for which C_ℓ is rational, the surface X_ℓ is unirational.*

2. *If $n = 0$ and C is rational and there exists a rational point $P \in X(k)$ with $\pi(P) \in C - B$, then the preimage $\pi^{-1}(C)$ consists of two rational components and X is unirational.*

3. *If $n = 2$ and the preimage $\pi^{-1}(C)$ is rational, then the surface X is unirational.*

Proof. First note that since B is a smooth quartic, it has genus 3, then by the initial hypothesis $g(C) = 0$ it follows that $C \neq B$. Let $A = \pi^*(C)$ the pull back of the curve C on the surface X. Since C is geometrically integral and $C \neq B$, the curve A is geometrically reduced. The morphism $A \to C$ induced by π has degree 2, and so $\overline{A} = A \times_k \overline{k}$ consists of at most two components. Then there is an extension ℓ of k of degree at most 2 such that the components of A_ℓ are geometrically irreducible. Let ℓ be such an extension and let D be an irreducible component of A_ℓ.

Suppose $n = 0$. Then, from Corollary 1.2.27.(1), the preimage $\pi^{-1}(C_\ell)$ consists of two irreducible components that are birationally equivalent to C_ℓ. If, moreover, C_ℓ is rational, then Lemma 2.2.2 implies the unirationality of X_ℓ, proving statement *(1)*.

Assume C is itself rational and there is a rational point $P \in X(k)$ such that $\pi(P) \in C - B$. Then we have that $P \neq \iota(P)$ and the points P and $\iota(P)$ lie in different components of $A_\ell = D_\ell \cup \iota(D_\ell)$. Since the Galois group $G = G(\ell/k)$ fixes the points P and $\iota(P)$, it follows that G also fixes D_ℓ and $\iota(D_\ell)$, so these components are defined over k. Then statement *(2)* follows from *(1)* taking $\ell = k$.

Statement *(3)* follows immediately from Corollary 1.2.27.(2) and Lemma 2.2.2. □

Remark 2.3.2. Note that statement (1) of Proposition 2.3.1 is consistent with [STVA14, Corollary 1.3], in which it is stated that if X is a del Pezzo surface of degree 2 over a finite field k, then there is a quadratic extension k'/k such that $X_{k'}$ is unirational.

Remark 2.3.3. Let D be a geometrically integral curve over a field k with $g(D) = 0$. Then there exists a field extension ℓ of k of degree at most 2 such that D_ℓ is rational. In fact, if k is a finite field, then D is rational over k. Therefore, if k is finite in Proposition 2.3.1, then C is rational; moreover, by case *(3)* we conclude that if $n = 2$, then X is unirational over k.

Remark 2.3.4. Propositions 1.2.26 and 2.3.1 imply that the geometrically integral projective curves $D \subset X$ with $g(D) = 0$ are exactly the geometrically irreducible components above geometrically integral projective curves $C \subset \mathbb{P}^2$ with $g(C) = 0$ and $\#b(\tilde{C}, B) \in \{0, 2\}$, where $\tilde{C}$ denotes the normalisation of C.

Remark 2.3.5. Suppose $P \in X(k)$ is a rational point that does not lie on the ramification curve, so $\pi(P) \notin B$. Suppose C is a geometrically integral curve of degree d that has a singular point of multiplicity $d-1$ at $\pi(P)$, and that intersects B with even multiplicity everywhere. Then Proposition 1.2.31 shows that $b(\tilde{C}, B)$ is empty, so, by Corollary 1.2.27, the pull back $\pi^*(C)$ splits into two components.

If X is general enough, then the Picard group $\operatorname{Pic} X$ of X is generated by the canonical divisor K_X, and the automorphism group of X acts trivially on $\operatorname{Pic} X$, so these two components would be linearly equivalent to the same multiple of K_X; as their union is linearly equivalent to $-dK_X$, we find that d is even. Hence, for odd d, the answer to the analogous question mentioned below Question 2.1.4 is negative for X general enough.

It is possible, however, that, even for odd d, a variation of this analogous question still has a positive answer. If we forget the del Pezzo surface, and only consider the quartic curve $B \subset \mathbb{P}^2$ with a point $Q \in \mathbb{P}^2$ that does not lie on B, we could ask for the existence of a curve of degree d that intersects B with even multiplicity everywhere, and on which Q is a point of multiplicity $d - 1$. The argument above merely shows that if such a curve exists for odd d and Q lifts to a rational point on the del Pezzo surface, then the surface does not have Picard number one.

Proof of Theorem 2.1.2. Assume that the assumptions of statement (1) hold. This implies that $C^s \cap B = \emptyset$ and $b(C, B) = \emptyset$. Therefore, by

Proposition 1.2.31, we have $\#b(\tilde{C}, B) = 0$. Statement *(1)* follows from applying part (2) of Proposition 2.3.1.

Assume statement *(2a)* holds. This means that $C^s \cap B = \{Q\}$ and $b(C, B) = \emptyset$. Since Q is a double or triple point of C, Proposition 1.2.31 implies that $\#b(\tilde{C}, B) = 2$. The conclusion of statement *(2)* follows from applying part (3) of Proposition 2.3.1 and Remark 2.3.3.

Assume statement *(2b)* holds. It means that $b(C, B) = \{Q_1, Q_2\}$ and $C^s \cap B \subseteq \{Q_1, Q_2\}$. Since the points Q_1 and Q_2 are distinct, Proposition 1.2.31 implies that $\#b(\tilde{C}, B) = 2$. As before, the conclusion of statement *(2)* follows from part (3) of Proposition 2.3.1 and Remark 2.3.3. This concludes the proof of the theorem. □

Proof of Corollary 2.1.3. Set $Q = \pi(P)$. Let $\mathfrak{L}_Q$ denote the line in the dual of $\mathbb{P}^2$ consisting of all lines $L \subset \mathbb{P}^2$ going through Q, and note that $\mathfrak{L}_Q$ is isomorphic to $\mathbb{P}^1$. Since C has degree d and $\pi(P)$ is a point of multiplicity $d-1$, each line in $\mathfrak{L}_Q$ intersects C in a unique d-th point, counting with multiplicity. It follows that C is smooth at all points $T \neq Q$. It also follows that the rational map $C \dashrightarrow \mathfrak{L}_Q$ that sends a point $T \in C$ to the line through T and Q is birational, so C is birationally equivalent with $\mathbb{P}^1$. By hypothesis, all intersection points of B and C have even intersection multiplicity.

Assume that Q is not contained in B. Since C is smooth away from Q, the curve B contains no singular points of C. Then X is unirational by part (1) of Theorem 2.1.2. This proves part (1).

Assume that Q is contained in B, that Q is an ordinary singularity of C, and $d \in \{3, 4\}$. Then Q is a double or a triple point of C. Since Q is the only singularity of C, the curve B contains no other singular points of C. Then X is unirational by part (2) of Theorem 2.1.2. This proves part (2). □

We now give some examples of curves that satisfy the conditions of Theorem 2.1.2 or Corollary 2.1.3.

Example 2.3.6. If C is a bitangent to the branch curve B that is defined over k, and $C(k)$ contains a point $Q \notin B$ that lifts to a k-rational point on X, then Theorem 2.1.2 implies that X is unirational. We can also prove this directly. Indeed, in this case the pull back $\pi^{-1}(C)$ consists of two exceptional curves that are defined over k, so X is not minimal.

Blowing down one of these exceptional curves yields a del Pezzo surface Y of degree 3 with a rational point. This implies that Y, and therefore also X, is unirational.

Example 2.3.7. Suppose the point $P \in X(k)$ is not a generalised Eckardt point and P is not on the ramification curve. Set $Q = \pi(P)$, let $\rho\colon \mathfrak{L}_Q \to X$ be as in [FvL15, Section 4, p.6], and set $C = \pi(\rho(\mathfrak{L}_Q))$. Then by [FvL15, Proposition 4.14], the map $\rho(\mathfrak{L}_Q) \to C$ has degree 1, so by Propositions 1.2.26 and 1.2.31, the intersection multiplicity of C and the branch curve B is even at all intersection points. Also by [FvL15, Proposition 4.14], the curve C has a point Q off the branch curve B of multiplicity $\deg C - 1$, so the curves of Manin's construction are examples of the curves described in Corollary 2.1.3. For further discussion on this see [FvL15, Remark 4.15].

Example 2.3.8. Consider the surface $X \subset \mathbb{P}(1,1,1,2)$ over $\mathbb{F}_3$, defined by the equation

$$w^2 = x^4 + y^4 + z^4.$$

The surface X is a del Pezzo surface of degree 2. All its rational points either are on the ramification curve, or they are generalised Eckardt points. In fact, the surface X has 154 rational points over $\mathbb{F}_9$, with 28 of those lying on the ramification locus. The remaining 126 are generalised Eckardt points, which is also the maximum number of generalised Eckardt points a del Pezzo surface of degree two can have (see [STVA14, before Example 7]). It follows that Manin's method does not apply to this surface. Let P be the point $(0:0:1:1)$ on X. Then P is a generalised Eckardt point and its image $Q = \pi(P) = (0:0:1) \in \mathbb{P}^2$ does not lie on the branch locus B, which is given by $x^4 + y^4 + z^4 = 0$. Consider the curve $C \subset \mathbb{P}^2$ given by $x^3y + xy^3 = z(x+y)^2(y-x)$. The curve C is a geometrically integral quartic plane curve that has a triple point at Q and that intersects B with even multiplicity everywhere. Therefore, by case (1) of Corollary 2.1.3, the surface X is unirational.

Of course, unirationality of X was already known: it follows for instance from Lemma 20 in [STVA14] (cf. Example 2.3.10 below). It is nice to see, though, that, even though Manin's construction and the generalisation in [STVA14] do not produce a curve in $\mathbb{P}^2$ of some degree d with a point of multiplicity $d-1$ at Q, and even intersection multiplicity with B everywhere, such curves do still exist, and then case (1) of

Corollary 2.1.3 implies unirationality of X. This gives a positive answer to the question below Question 2.1.4 for $d = 4$ and this particular surface X and this generalised Eckardt point P.

One might ask whether there are curves of lower degree satisfying the hypotheses of case (1) of Corollary 2.1.3. Indeed, there are conics that do, for example the one given by $y^2 = xz$. An exhaustive computer search, based on Proposition 2.3.1.(2), and Corollary 2.4.2, shows that there are no cubic curves with a double point at Q satisfying the hypotheses of Corollary 2.1.3 and its case (1).

Example 2.3.9. Let X_1, X_2, X_3 be the three del Pezzo surfaces defined as in Section 2.2 and let B_i be their branch locus, for $i = 1, 2, 3$. For $i = 1, 2, 3$, all rational points of the surface X_i lie on the ramification locus. Consider the rational points $P_1 = (0 : 0 : 1 : 0) \in X_1$, $P_2 = (0 : 1 : 1 : 0) \in X_2$, and $P_3 = (1 : 1 : 1 : 0) \in X_3$, and set $Q_i = \pi(P_i)$. Clearly, we have $Q_i \in B_i$. Set $d_1 = d_2 = 4$ and $d_3 = 3$. Let $C_i \subset \mathbb{P}^2$ be the projective plane curve of degree d_i given by the polynomial h_i defined as in Remark 2.2.4. The curve C_i is geometrically irreducible and it has an ordinary singular point at Q_i of multiplicity $d_i - 1$. Given that the curve C_i pulls back to the geometrically irreducible rational curve A_i of Remark 2.2.4, we find from Corollary 1.2.27 and Proposition 1.2.31 that C_i intersects B_i with even multiplicity everywhere.

Of course, one could also check directly that C_i intersects B_i with even multiplicity everywhere. Then Corollary 2.1.3 and Remark 2.3.3 give an alternative proof that the surface X_i is unirational (cf. Remark 2.2.4). There is a quartic alternative for C_3 as well. The curve $C_3' \subset \mathbb{P}^2$ given by the vanishing of

$$\begin{aligned} h_3' = &\gamma^2 x^4 + x^3 y + \gamma x^2 y^2 + \gamma^3 x y^3 - y^4 + x^3 z + \gamma x^2 y z + x y^2 z \\ &- \gamma y^3 z + \gamma x^2 z^2 + x y z^2 + \gamma^3 y^2 z^2 + \gamma^3 x z^3 - \gamma y z^3 - z^4 \end{aligned}$$

is geometrically integral, has an ordinary triple point at $(-1 : 1 : 1)$, and intersects B with even multiplicity everywhere.

Example 2.3.10. Let k be a field with characteristic different from 2. Let $a_1, \ldots, a_6 \in k$ be such that the variety X in the weighted projective space $\mathbb{P} = \mathbb{P}(1, 1, 1, 2)$ defined by

$$w^2 = a_1^2 x^4 + a_2^2 y^4 + a_3^2 z^4 + a_4 x^2 y^2 + a_5 x^2 z^2 + a_6 y^2 z^2$$

is a del Pezzo surface of degree 2. This is the surface of Lemma 20 in [STVA14], where it is noted that the surface in $\mathbb{P}$ given by the equation $w = a_1x^2 + a_2y^2 + a_3z^2$ intersects the surface X in a curve D, which the anti-canonical map $\pi\colon X \to \mathbb{P}^2$ sends isomorphically to the plane quartic curve $C \subset \mathbb{P}^2$ given by

$$(a_4 - 2a_1a_2)x^2y^2 + (a_5 - 2a_1a_3)x^2z^2 + (a_6 - 2a_2a_3)y^2z^2 = 0.$$

They also note that this curve C is birationally equivalent to a conic under the standard Cremona transformation, so C and D are rational over an extension of k of degree at most 2. If they are rational over k, then X is unirational.

Indeed, one checks that the curve C satisfies the conditions of part (1) of Proposition 2.3.1, and if C is rational over k, then it also satisfies the conditions of part (1) of Theorem 2.1.2, where one can take P to be any of the points on X above any of the singular points $(0 : 0 : 1)$, $(0 : 1 : 0)$, and $(1 : 0 : 0)$ of C.

2.4 Finding appropriate curves

In this section, we assume that the characteristic of k is not 2, and we give sufficient easily-verifiable conditions for a curve C to satisfy the hypotheses of Corollary 2.1.3. This is also how we found the three curves, C_1, C_2, and C_3 of Remark 2.2.4, whose existence implies unirationality of the three difficult surfaces X_1, X_2, X_3 (see Example 2.3.9 and Remark 2.4.7).

Let $X \subset \mathbb{P}(1, 1, 1, 2)$ be a del Pezzo surface of degree 2, given by $w^2 = g$ with $g \in k[x, y, z]$ homogeneous of degree 4. Let $B \subset \mathbb{P}^2(x, y, z)$ be the branch curve of the projection $\pi\colon X \to \mathbb{P}^2$. Then B is given by $g = 0$. Let $P \in X(k)$ be a rational point and set $Q = \pi(P)$. Without loss of generality, we assume $Q = (0 : 0 : 1)$. Let $C \subset \mathbb{P}^2$ be a geometrically irreducible curve of degree $d \geq 2$ on which Q is a point of multiplicity $d - 1$.

There are coprime homogeneous polynomials $f_{d-1}, f_d \in k[x, y]$ of degree $d - 1$ and d, respectively, such that C is given by $zf_{d-1} = f_d$. The projection away from Q induces a birational map from C to the family $\mathfrak{L}_Q$ of lines in $\mathbb{P}^2$ through Q. Its inverse is a morphism ϑ that sends a line $L \in \mathfrak{L}_Q$ to the d-th intersection point of L with C. If we identify $\mathfrak{L}_Q$ with $\mathbb{P}^1$, where $(s:t) \in \mathbb{P}^1$ corresponds to the line given by $sy = tx$, then $\vartheta\colon \mathbb{P}^1 \to C$ sends $(s:t)$ to

$$(sf_{d-1}(s,t) : tf_{d-1}(s,t) : f_d(s,t)).$$

The curve C has no singularities outside Q, and we may identify the morphism $\vartheta\colon \mathbb{P}^1 \to C$ with the normalisation of C. The points on $\mathbb{P}^1$ above the point Q are exactly the points where $f_{d-1}(s,t)$ vanishes. The curve C has an ordinary singularity at Q if and only if $d > 2$ and $f_{d-1}(s,t)$ vanishes at $d-1$ distinct $\overline{k}$-points of $\mathbb{P}^1(s,t)$.

The pull back $\pi^*(C)$ is birationally equivalent with the curve given by $w^2 = G$ in the weighted projective space $\mathbb{P}(1,1,2d)$ with coordinates s, t, w, and with

$$G = g\big(sf_{d-1}(s,t), tf_{d-1}(s,t), f_d(s,t)\big) \in k[s,t].$$

Proposition 2.4.1. *For any point $T \in \mathbb{P}^1(\overline{k})$, the intersection multiplicity $\mu_T(\mathbb{P}^1, B)$ equals the order of vanishing of G at T.*

Proof. Since C either has degree 2 or it is singular, it is not equal to B. As C is irreducible, it has no irreducible components in common with B. By symmetry between s and t, we may assume $T = (\alpha : 1)$ for some $\alpha \in \overline{k}$. Then the local ring $\mathcal{O}_{T,\mathbb{P}^1}$ is isomorphic to the localisation of $\overline{k}[s]$ at the maximal ideal $(s-\alpha)$. Let $\ell \in k[x,y,z]$ be a linear form that does not vanish at $\vartheta(T)$. Then locally around $\vartheta(T) \in \mathbb{P}^2$, the curve B is given by the vanishing of the element g/ℓ^4, whose image in $\mathcal{O}_{T,\mathbb{P}^1}$ is $G(s,1)/L(s,1)^4$ with $L(s,t) = \ell\big(sf_{d-1}(s,t), tf_{d-1}(s,t), f_d(s,t)\big)$. Since $L(s,1)$ does not vanish at α, we find that $\mu_T(\mathbb{P}^1, B)$ equals the order of vanishing of $G(s,1)$ at α, which equals the order of vanishing of G at T. □

Corollary 2.4.2. *We have $b(\mathbb{P}^1, B) = \emptyset$ if and only if G is a square in $\overline{k}[s,t]$.*

Proof. By Proposition 2.4.1, we have $b(\mathbb{P}^1, B) = \emptyset$ if and only if the order of vanishing of G is even at every point $T \in \mathbb{P}^1(\overline{k})$. This is equivalent with G being a square in $\overline{k}[s,t]$. □

If B does not contain the unique singular point Q of C, then ϑ induces a bijection $b(\mathbb{P}^1, B) \to b(C, B)$, so in this case we also have $b(C, B) = \emptyset$ if and only if G is a square in $\overline{k}[s, t]$. The following proposition gives an analogue of this statement when Q is contained in B.

Proposition 2.4.3. *Suppose that Q is contained in B. Then the rational polynomial $H = G/f_{d-1}(s, t)$ is in fact contained in $k[s, t]$. Suppose, furthermore, that the tangent line to B at Q is given by $h = 0$ with $h \in k[x, y]$, and that Q is an ordinary singular point on the curve C. Then the following statements hold.*

1. *Suppose d is odd. Then the set $b(C, B)$ is empty if and only if H is a square in $\overline{k}[s, t]$.*
2. *Suppose d is even. If h divides f_{d-1}, then $H/h(s, t)$ is contained in $k[s, t]$. The set $b(C, B)$ is empty if and only if h divides f_{d-1} and $H/h(s, t)$ is a square in $\overline{k}[s, t]$.*

Proof. Write $g = \sum_{i=0}^{4} g_i z^{4-i}$, where $g_i \in k[x, y]$ is homogeneous of degree i for all $0 \leq i \leq 4$. If $g(Q) = g_0$ vanishes, then each monomial of g is divisible by x or y, which implies that G is divisible by f_{d-1}, which in turn shows $H \in k[s, t]$. Suppose that all hypotheses hold. By $g(Q) = 0$ we find $g_0 = 0$. The tangent line to B at Q is given by $g_1 = 0$, so h is a scalar multiple of g_1. Note that all statements are invariant under the action of $\mathrm{GL}_2(k)$ on $\mathbb{P}^1$ and $\mathbb{P}^2$ given on their respective homogeneous coordinate rings $k[s, t]$ and $k[x, y, z]$ by $\gamma(s) = as + bt, \gamma(t) = cs + dt$ and $\gamma(x) = ax + by, \gamma(y) = cx + dy, \gamma(z) = z$ for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

After applying an appropriate element $\gamma \in \mathrm{GL}_2(k)$ and rescaling h, we assume, without loss of generality, that $h = g_1 = y$.

If y divides f_{d-1}, then t divides $f_{d-1}(s, t)$; since all monomials in g besides y are divisible by x^2, xy, or y^2, it follows that in this case G is divisible by $tf_{d-1}(s, t)$, so H/t is contained in $k[s, t]$. This does not depend on d being even.

In the open neighbourhood of Q given by $z \neq 0$, the curve B is given by the vanishing of $g/z^4 = g(x/z, y/z, 1) = \sum_i g_i(x/z, y/z)$. The maximal ideal $\mathfrak{m}$ of the local ring $\mathcal{O}_{Q,C}$ is generated by x/z and y/z, so the image of g/z^4 in $\mathcal{O}_{Q,C}/\mathfrak{m}^2$ is $g_1(x/z, y/z) = y/z$. Let $T \in \mathbb{P}^1(\overline{k})$ be a point with $\vartheta(T) = Q$, and let $\mathfrak{n}$ be the maximal ideal of the local ring $\mathcal{O}_{T,\mathbb{P}^1}$. Then the image of g in $\mathcal{O}_{T,\mathbb{P}^1}/\mathfrak{n}^2$ equals the image of y/z, which is $tf_{d-1}(s,t)/f_d(s,t)$. The point T corresponds to a linear factor of $f_{d-1}(s,t)$. Since $f_d(s,t)$ does not vanish at T, we find that the valuation $v_T(g)$ of g in $\mathcal{O}_{T,\mathbb{P}^1}$ is at least 2 if t vanishes at T, that is, $\mu_T(\mathbb{P}^1, B) \geq 2$ if $T = (1:0)$. We have $\mu_T(\mathbb{P}^1, B) = v(g) = 1$ if $T \neq (1:0)$. From Lemma 1.2.24 we conclude

$$\mu_Q(C,B) = \begin{cases} d - 2 + \mu_{(1:0)}(\mathbb{P}^1, B) & \text{if } y \text{ divides } f_{d-1}, \\ d-1 & \text{otherwise.} \end{cases} \tag{2.1}$$

We now consider the two cases.

1. Suppose d is odd. From (2.1) it follows that $\mu_Q(C,B)$ is even if and only if either y divides f_{d-1} and $\mu_{(1:0)}(\mathbb{P}^1, B)$ is odd, or y does not divide f_{d-1}. This happens if and only if $\mu_T(\mathbb{P}^1, B)$ is odd for all $T \in \mathbb{P}^1$ at which $f_{d-1}(s,t)$ vanishes. For all other points $R \in C$ with $R \neq Q$, the multiplicity $\mu_R(C,B)$ is even if and only if $\mu_{\vartheta^{-1}(R)}(\mathbb{P}^1, B)$ is even. From Proposition 2.4.1, we conclude that $b(C,B)$ is empty if and only if the order of vanishing of G is odd at all points $T \in \mathbb{P}^1$ at which $f_{d-1}(s,t)$ vanishes and even at all other points. This is equivalent with H being a square in $\overline{k}[s,t]$.

2. Suppose d is even. From (2.1) it follows that $\mu_Q(C,B)$ is even if and only if y divides f_{d-1} and $\mu_{(1:0)}(\mathbb{P}^1, B)$ is even. As in the case for odd d, this implies that $b(C,B)$ is empty if and only if the order of vanishing of G is odd at all points $T \neq (1:0)$ at which $f_{d-1}(s,t)$ vanishes, and even at all other points, including $(1:0)$. Since the order of vanishing of $tf_{d-1}(s,t)$ at $(1:0)$ is 2, this is equivalent to $G/(tf_{d-1}(s,t)) = H/t$ being a square in $\overline{k}[s,t]$.

This finishes the proof. □

We have already seen that the pull back $\pi^*(C)$ is birationally equivalent with the curve given by $w^2 = G$ in $\mathbb{P}(1,1,2d)$. This curve splits into two k-rational components if and only if G is a square in $k[s,t]$. If Q is an ordinary singular point of C that lies on B, then this never happens. However, the curve $\pi^*(C)$ may itself be k-rational, in which case G factors as a square times a quadric.

We will now focus on the case $d = 4$, so Q is a triple point. The following corollary says that if Q is an ordinary triple point, then we do not need to factorise G, as we know exactly which part should be the square, and which the quadric.

Corollary 2.4.4. *Suppose that Q is an ordinary singular point of C that lies on B. If the pull back $\pi^*(C) \subset X$ is k-rational, then we have $d \leq 4$.*

Moreover, suppose $d = 4$, and let the tangent line to B at Q be given by $h = 0$ with $h \in k[x, y]$. Then the pull back $\pi^*(C) \subset X$ is k-rational if and only if there is a constant $c \in k^*$ such that the following statements hold:

1. *the polynomial h divides f_3;*
2. *the polynomial $cH(s,t)/h(s,t)$ is a square in $k[s,t]$;*
3. *the conic given by $cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$ is k-rational.*

Proof. Suppose $\pi^*(C)$ is k-rational. Then $\pi^*(C)$ is geometrically integral and has genus $g(\pi^*(C)) = 0$. From Proposition 1.2.26 we obtain $b(\mathbb{P}^1, B) = 2$. From Proposition 1.2.29 we conclude that the contribution $c_Q(C, B)$ is at most 2. Moreover, this proposition also gives $c_Q(C, B) \geq d - 2$ with equality if and only if $\mu_Q(C, B)$ is even. We conclude $d \leq 4$.

Suppose $d = 4$. Then we have equality $c_Q(C, B) = 2 = \#b(\mathbb{P}^1, B)$, so $\mu_Q(C, B)$ is even, and we find that $b(C, B)$ is empty. From Proposition 2.4.3 we find that h divides f_3, and $m = H(s,t)/h(s,t)$ is a square in $\overline{k}[s,t]$. Let c be the main coefficient of $m(s, 1)$. Then cm is a square in $k[s,t]$. Therefore, the k-rational curve given by $w^2 = G$ with

$$G = cm \cdot h^2(s,t) \cdot c^{-1} f_3(s,t)/h(s,t) \tag{2.2}$$

in $\mathbb{P}(1, 1, 2d)$ is birationally equivalent with the conic given by the equation $cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$, which is therefore also k-rational.

Conversely, if there is a constant c such that $cH(s,t)/h(s,t)$ is a square in $k[s,t]$, then it follows from (2.2) that the conic given by

$cw^2 = f_3(s,t)/h(s,t)$ in $\mathbb{P}^2(s,t,w)$ is birationally equivalent with the curve in $\mathbb{P}(1,1,2d)$ given by $w^2 = G$, which is birationally equivalent with $\pi^*(C)$. Hence, if this conic is k-rational, then so is $\pi^*(C)$. □

Remark 2.4.5. Corollary 2.4.4 helps us in finding all curves C of degree $d = 4$ that satisfy the conditions of case (2) of Corollary 2.1.3 with $\ell = k$. More explicitly, after a linear transformation of $\mathbb{P}^2$, we may assume that $Q = (0 : 0 : 1)$, and the tangent line to B at Q is given by $y = 0$. Then we claim that every curve C of degree $d = 4$ that satisfies the conditions of case (2) of Corollary 2.1.3 with $\ell = k$ is given by

$$yz\phi_2 = x^4 + y\phi_3$$

for some homogeneous $\phi_2, \phi_3 \in k[x,y]$ of degree 2 and 3, respectively, with ϕ_2 squarefree and not divisible by y. Indeed, we find that f_3 is divisible by y, so there is a $\phi_2 \in k[x,y]$ such that $f_3 = y\phi_2$; since C is irreducible, the polynomial f_4 is not divisible by y, so the coefficient of x^4 in f_4 is nonzero, and after scaling ϕ_2, f_3, and f_4, we may assume that there exists a $\phi_3 \in k[x,y]$ such that $f_4 = x^4 + y\phi_3$. Moreover, Q is an ordinary singularity if and only if ϕ_2 is squarefree and not divisible by y.

Hence, to find all such curves C, we are looking for all pairs (ϕ_2, ϕ_3) with $\phi_i \in k[x,y]$ homogeneous of degree i, such that

1. the polynomial ϕ_2 is squarefree and y does not divide ϕ_2,

2. the curve given by $yz\phi_2 = x^4 + y\phi_3$ is geometrically integral,

3. there is a constant $c \in k^*$ such that polynomial $c{\cdot}G(s,t)/(t^2\phi_2(s,t))$ with
$$G = g\big(st\phi_2(s,t),\ t^2\phi_2(s,t),\ s^4 + t\phi_3(s,t)\big)$$
is a square,

4. the conic given by $cw^2 = \phi_2(s,t)$ in $\mathbb{P}^2(s,t,w)$, with c as in (3), is k-rational.

Note for (3) that, because the characteristic is not 2, a homogeneous polynomial $H \in k[s,t]$ of even degree is a square in $\overline{k}[s,t]$ if and only if there is a constant $c \in k^*$ such that cH is a square in $k[s,t]$, which happens if and only if $\gamma^{-1}H(s,1)$ is a square in $k[s]$, where γ is the main coefficient of $H(s,1)$. This follows from the fact that a monic polynomial in $k[s]$ is a square in $k[s]$ if and only if it is a square in $\overline{k}[s]$. Moreover, the $c \in k^*$ for which cH is a square, form a coset in k^*/k^{*2}, so whether or not (4) holds does not depend on the choice of c.

Question 2.1.4 for $d = 4$ can be rephrased using Remark 2.4.5. It is equivalent to the following question.

Question 2.4.6. *Let k be a field of characteristic not equal to 2, and $g \in k[x,y,z]$ a homogeneous polynomial of degree 4 such that the curve $B \subset \mathbb{P}^2(x,y,z)$ given by the equation $g = 0$ is smooth, it contains the point $Q = (0:0:1)$, and the tangent line to B at Q is given by $y = 0$. Do there exist homogeneous polynomials $\phi_2, \phi_3 \in k[x,y]$ of degree 2 and 3, such that conditions (1)–(3) of Remark 2.4.5 are satisfied?*

Remark 2.4.7. If k is a ("small") finite field, then we can list all pairs (ϕ_2, ϕ_3) with $\phi_i \in k[x,y]$ homogeneous of degree i, and check for each whether the conditions (1)–(4) of Remark 2.4.5 are satisfied. In fact, condition (4) is automatically satisfied over finite fields. Indeed, this is how we found the curves C_1, C_2 given in Remark 2.2.4, whose existence implies unirationality of the three difficult surfaces X_1, X_2 (see Example 2.3.9). Finding the rational cubic curve C_3 on X_3, as given in Remark 2.2.4, was easier, based on part (1) of Proposition 2.4.3.

Remark 2.4.8. For any integer i, let $k[x,y]_i$ denote the $(i+1)$-dimensional space of homogeneous polynomials of degree i. In general, over any field, we can describe the set of pairs $(\phi_2, \phi_3) \in k[x,y]_2 \times k[x,y]_3$ satisfying condition (3) of Remark 2.4.5 as follows.

Identify $k[x,y]_2 \times k[x,y]_3$ with the affine space $\mathbb{A}^7$ and let R denote the coordinate ring of $\mathbb{A}^7$, that is, R is the polynomial ring in the $3 + 4 = 7$ coefficients of ϕ_2 and ϕ_3. Let $Z \subset \mathbb{A}^7$ be the locus of all (ϕ_2, ϕ_3) that satisfy condition (3).

For *generic* ϕ_2, ϕ_3, that is, with the variables of R as coefficients, the coefficients of the polynomial

$$G' = G(s,t)/(t^2\phi_2(s,t))$$

of condition (3) of Remark 2.4.5 lie in R. For general enough g, the coefficient $c \in R$ of s^{12} in the polynomial $G' \in R[s,t]$ is nonzero. On the open set U of $\mathbb{A}^7$ given by $c \neq 0$, we may complete $G'(s, 1)$ to a square in the sense that there are polynomials $G_1, G_2 \in R[c^{-1}][s]$ with G_1 monic of degree 6 in s and G_2 of degree at most 5 in s such that $G'(s, 1) = cG_1^2 - G_2$. The vanishing of the six coefficients in R of G_2 determines the locus $Z \cap U$ inside U of all pairs (ϕ_2, ϕ_3) at which cG' is a square. Note that we have $c = G'(1, 0)$. For each point $(s_0 : t_0) \in \mathbb{P}^1$, we can use an automorphism of $\mathbb{P}^1$ that sends $(s_0 : t_0)$ to $(1 : 0)$, to similarly describe the intersection of Z with the open subset of $\mathbb{A}^7$ where $G'(s_0, t_0)$ is nonzero; it is also given by the vanishing of six polynomials in R. We can cover $\mathbb{A}^7$ with open subsets of this form, thus describing Z completely.

A naive dimension count suggests that the locus Z has dimension $7 - 6 = 1$. This is consistent with the following, similarly naive, dimension count. The family of quartic curves in $\mathbb{P}^2$ is 14-dimensional, as it is the projective space $\mathbb{P}(k[x, y, z]_4)$, where $k[x, y, z]_4$ is the 15-dimensional vector space of polynomials of degree 4. The codimension of the subset of those curves having a triple point at Q is 6, and demanding that the intersection multiplicity $\mu_Q(C, B)$ is at least 4 cuts down another dimension. Since B is also a quartic curve, by Bezout's theorem it follows that B and C have 16 intersection points, counted with multiplicity. Hence, generically, the curves in the remaining 7-dimensional family intersect B, besides in Q, in $16 - 4 = 12$ more points. One might expect the subfamily of those curves where this degenerates to six points with multiplicity 2 to have codimension 6, in which case this would leave a 1-dimensional family of quartic curves with a triple point at $Q \in B$ and intersecting B with even multiplicity everywhere.

However, the locus Z also contains some degenerate components that we are not interested in. For example, the locus of all $(0, \phi_3)$ for which $f_4 = x^4 + y\phi_3$ is a square is contained in Z and has dimension 2. Also, for any smooth conic Γ that contains Q, that has its tangent line at Q given by $y = 0$, and that has even intersection multiplicity with B everywhere, we get a 1-dimensional subset of Z consisting of pairs (ϕ_2, ϕ_3) that correspond with the union of Γ with any double line through Q (these lines are parametrised by $\mathbb{P}^1$). Note that in all these degenerate cases the curve C is reducible. Another degenerate case is the limit of Manin's construction. By [FvL15, Remark 4.11], this limit curve is the non-reduced curve $\pi_*\pi^*(2L) = 4L$, where L is the tangent line to B at Q, given by $y = 0$. Hence, this quartic curve is given by $y^4 = 0$, which does not correspond to a point on the affine set Z, as the coefficient of x^4 is zero.

Let Z_0 denote the affine subset of Z corresponding to curves C that are geometrically integral and on which Q is an ordinary triple point. Then Questions 2.1.4 (for $d = 4$) and 2.4.6 can be rephrased by asking whether the subset Z_0 contains a k-rational point.

Example 2.4.9. Let $B \subset \mathbb{P}^2$ be the smooth curve given by

$$y^4 - x^4 - x^3y - xy^3 + y^3z + yz^3 = 0$$

over $k = \mathbb{F}_3$, and let Q be the point $(0 : 0 : 1) \in B(k)$. The tangent line to B at Q is given by $y = 0$. Running through all the homogeneous polynomials in x, y of degree 2 and 3 over k one can find that there is no pair (ϕ_2, ϕ_3) of polynomials satisfying conditions (1)–(3) of Remark 2.4.5; there do exist pairs satisfying only conditions (2) and (3). This means that Questions 2.1.4 (for $d = 4$) and 2.4.6 have negative answer in this specific case. It could, however, still be true that the answer is positive for X and P general enough.

Notice that the curve B is isomorphic to the Fermat curve of degree four, that is, the curve given by $x^4 + y^4 + z^4 = 0$, via the following linear change of variables:

$$(x : y : z) \mapsto (x - z : x + y + z : z).$$