Solving multiplication and division problems: latent variable modeling of students' solution strategies and performance

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General introduction

This dissertation concerns the mathematical strategies and performance of students and what factors affect these different aspects of problem solving. Before delving into the research on this point, I would like to invite you to take a moment to solve the multiplication and division problem presented below:

23×56	$544 \div 34$

Were you successful in obtaining the answers? For the multiplication problem, you should have found the answer 1288, and for the division problem the answer 16. And how did you go about obtaining the answers? Did you diligently take up paper and pencil and perform the algorithms you were taught in primary school, or did you perhaps take a less formal approach? Given that you are reading a dissertation, you probably enjoyed quite some years of education or even have a PhD, which means that according to Goodnow (1976), you are especially likely to solve mathematical problems using a mental approach without any external aids.

In taking such an approach, you would not be alone. The line of research that gave rise to this dissertation, comes from the observation of simultaneously declining performance in multiplication and division at the end of Dutch primary school and increasing amounts of problems that are answered without any calculations that are written down (Fagginger Auer, Hickendorff, & Van Putten, 2013; Hickendorff, Heiser, Van Putten, & Verhelst, 2009; Van Putten, 2005). In this dissertation, factors that affect students' solution strategy use and performance are therefore investigated, as well as the statistical techniques that may be used to conduct such an investigation. This introduction provides a framework for this research by discussing solution strategies from a cognitive psychology point of view, and the place of strategies in developments in mathematics education. The introduction is concluded with an outline of how the different chapters of this dissertation each contribute to the larger theme.

1.1 Solution strategies in cognitive psychology

Learning and problem solving are characterized by the use of a variety of strategies at every developmental stage (Siegler, 2007). This is already evident in children as young as infants: for example, some infants who are in their first weeks of independent walking use a stepping strategy, while others use a twisting strategy, and still others a falling strategy (Snapp-Childs & Corbetta, 2009). First graders who are asked to spell words use strategies as varied as retrieval, sounding out, drawing analogies, relying on rules, and visual checking (Rittle-Johnson & Siegler, 1999). Older children who solve transitive reasoning problems differ in their use of deductive and visual solution strategies (Sijtsma & Verweij, 1999). Solution strategies of children and adults have been a topic of continued investigation for cognitive tasks concerning diverse topics, such as mental rotation and transformation (e.g., Arendasy, Sommer, Hergovich, & Feldhammer, 2011), counting (e.g., Blöte, Van Otterloo, Stevenson, & Veenman, 2004), class inclusion (e.g., Siegler & Svetina, 2006), analogical reasoning (e.g., Tunteler, Pronk, & Resing, 2008), and digital gaming (e.g., Ott & Pozzi, 2012).

A popular topic in solution strategy research is strategy use for arithmetic problems. Many studies have been conducted on elementary addition, subtraction, multiplication and division (e.g. Barrouillet & Lépine, 2005; Barrouillet, Mignon, & Thevenot, 2008; Beishuizen, 1993; Bjorklund, Hubertz, & Reubens, 2004; Campbell & Fugelsang, 2001; Campbell & Xue, 2001; Carr & Davis, 2001; Davis & Carr, 2002; Geary, Hoard, Byrd-Craven, & DeSoto, 2004; Imbo & Vandierendonck, 2007; Laski et al., 2013; Mulligan & Mitchelmore, 1997; Van der Ven, Boom, Kroesbergen, & Leseman, 2012), which concern operations in the number domain up to 100 that are taught in the lower grades of primary school. However, while this elementary arithmetic is the subject of a rich body of literature that has identified the strategies that are used and described their characteristics, there is less research on strategy use by higher grade students on more complex arithmetic problems (though there is some; e.g., Hickendorff, 2013; Van Putten, Van den Brom-Snijders, & Beishuizen, 2005; Selter, 2001; Torbeyns, Ghesquière, & Verschaffel, 2009). This more advanced arithmetic is called multidigit arithmetic, as it involves larger numbers and decimal numbers.

When solving mathematical problems, especially more complex multidigit problems, there is an array of possible solution strategies. Lemaire and Siegler (1995) proposed a general framework for charting the strategy use for a given domain, consisting of four aspects of strategic competence. The first aspect of the framework is the strategy repertoire, or in other words, which strategies are used. The second aspect concerns the frequency with which each of the strategies in that repertoire is chosen for use. The third aspect is strategy efficiency, which describes the performance of each strategy. The fourth aspect is the adaptivity of the choices that are made between strategies, which can be judged based on task, subject and context variables. Combining these different factors, Verschaffel, Luwel, Torbeyns, and Van Dooren (2009) defined the choice for a strategy as adaptive when the chosen strategy is most appropriate for a particular problem for a particular individual, in a particular sociocultural context.

An important aspect of adaptivity is the degree to which choices between strategies are adapted to the relative performance of those strategies. This performance entails both accuracy and speed, which can be considered simultaneously by defining the best performing strategy as the one that results in the correct solution the fastest (Luwel, Onghena, Torbeyns, Schillemans, & Verschaffel, 2009; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009; Kerkman & Siegler, 1997). Performance depends on both the person using the strategy and on the problem the strategy is applied to. In the Adaptive Strategy Choice Model (ASCM; Siegler & Shipley, 1995), a strategy is selected for a problem using individual strategy accuracy and speed information for both problems in general and problems with the specific features of the problem at hand. Another important aspect of adaptivity is the degree to which strategy choices are adapted to the context in which they are made (Verschaffel et al., 2009). Both the direct task context (e.g., demands on working memory, time restrictions, or the characteristics of preceding items) and the sociocultural context can be considered. Examples of influential aspects of the sociocultural context are whether mental strategies are valued over using external aids, whether speed or accuracy is more important, whether using conventional procedures or original approaches is preferred, and whether asking for help in problem solving is desirable (Ellis, 1997).

1.2 Solution strategies in mathematics education

An essential element of the context for mathematical solution strategies is of course the educational system. The educational systems for mathematics underwent quite some changes in the second half of the twentieth century in many Western countries, among which the Netherlands, where the research for this dissertation took place (see descriptions by Klein, 2003, and the Royal Netherlands Academy of Arts and Sciences, 2009). Already prior to this period, there was discontent with mathematics education and its outcomes, but no real changes occurred until the U.S.S.R. launched the first space satellite Sputnik in 1957. This caused a shock in the Western world and an international conference was held in Royaumont in 1959, with the aim of reforming education to advance economical and technological development. Here, a radically different approach to mathematics education was envisioned with the name of 'New Math', which de-emphasized algorithms in light of the uprise of computers and calculators, and focused on set theory and logic instead.

New Math was adopted in various European countries and in the U.S., and mathematics education followed its own course of development after that in each country. For example, in the U.S. (Klein, 2003), New Math's scant attention for basic skills and applications and its sometimes overly formal and abstract nature led to criticisms, and by the early 1970s, New Math programs were discontinued there. During the 1980s, progressivist changes to the curriculum were proposed in the U.S., that revolved around student-centered, discovery-based learning through 'real world' problem solving. Increased attention was prescribed for topics such as cooperative work, mental computation and use of calculators, whereas direct teacher instruction, algorithms (long division in particular) and paper-and-pencil computations were to receive decreased attention (National Council of Teachers of Mathematics, 1989). In the 1990s, these changes were implemented throughout the country, but they also met with resistance from parents and mathematicians, resulting in so-called 'math wars'.

In the Netherlands (Royal Netherlands Academy of Arts and Sciences, 2009), a committee was set to work in 1961 to translate the ideas of New Math into changes of the curriculum, which resulted in publications on a new curriculum in the late 1970s. Though New Math was the starting point for the committee, the end result was something quite different: basic skills remained important (though algorithms to a lesser extent), and clever strategies, estimation, measurement, and geometry were added to the curriculum (Freudenthal, 1973). This new curriculum was labeled 'realistic mathematics', because contexts familiar to students were used that should make mathematics meaningful. Five core principles were established for realistic mathematics (Treffers, 1987b): students construct their own knowledge, making students' own strategies the starting point; models are used to advance from informal to more formal approaches; students reflect on their own approaches; students learn from their own and others' approaches through interaction; and students are stimulated find connections between what they have learned. By 2002, there were only realistic mathematics textbooks on the market for primary schools. Following a talk that heavily criticized realistic mathematics at a mathematics education conference in 2007 (Van de Craats, 2008), a national debate started.

1.2.1 Strategy use and performance

As can be seen from this short history description, solution strategies were an important aspect of the reforms of mathematics education. Algorithms were de-emphasized in the light of technological advances, while attention for students' problem solving strategies increased. In realistic mathematics, the informal strategies that students invent themselves are used as the building blocks for formalization. Problems do not have a single standardized approach; instead, the multitude of possible strategies is emphasized through interaction, and students have to make choices between strategies when they solve a problem. This makes the adaptivity of strategy choices highly important: selecting the best performing strategy is vital to performance.

That students do not always choose the optimal strategy from the array at their disposal is illustrated by Dutch students' strategy choices for multidigit multiplication and division problems. These are problems with larger or decimal numbers (e.g., 23×56 or $31.2 \div 1.2$), that were typically solved with algorithms in traditional mathematics education. Given the challenging nature of the numbers in these problems, often a variety of informal strategies can be applied (e.g., Fagginger Auer & Scheltens, 2012), and realistic mathematics also introduced new standardized approaches (Treffers, 1987a). Whereas in the traditional algorithms numbers are broken up into digits that can be handled without an appreciation of their magnitude in the whole number, in these new approaches numbers are considered as a whole. The different approaches have therefore been labeled digit-based and whole-number-based respectively (Van den Heuvel-Panhuizen, Robitzsch, Treffers, & Köller, 2009; see Table 1.1 for examples). For multiplication, the digit-based algorithm is usually learned after the whole-number-based approach, but for quite some time this was not the case for division (Buijs, 2008; J. Janssen, Van der Schoot,

Table 1.1: Examples of written work for different multiplication and division strategies for the problems 23×56 and $544 \div 34$.

	digit-based algorithm	whole-number-based algorithm	non-algorithmic written	no written work
23×56	56 23 × 168 1120 + 1288	56 23 × 18 150 120 1000 + 1288	1120 + 3 × 56 1120 + 168 1288	1288
$544 \div 34$	34/544\16 34 204 204 0	544 : 34 = 340 - 10× 204 102 - 3× 102 102 - 3×+ 0 16×	10 × 34 = 340 15 × 34 = 510 16 × 34 = 544	16

& Hemker, 2005). The newest editions of some textbooks do include digit-based division.

The development of students' strategy use in a context of changing mathematics education can be followed through national large-scale assessments, of which five have taken place in the Netherlands since the late 1980s (Wijnstra, 1988; Bokhove, Van der Schoot, & Eggen, 1996; J. Janssen, Van der Schoot, Hemker, & Verhelst, 1999; J. Janssen et al., 2005; Scheltens, Hemker, & Vermeulen, 2013). Students write down their calculations in the assessment booklets, and from this written work strategy use can be inferred (Fagginger Auer, Hickendorff, & Van Putten, 2015). Analyses of strategy use (Fagginger Auer et al., 2013; Hickendorff et al., 2009) showed that from 1997 to 2004, the use of digit-based algorithms for multidigit multiplication and division decreased considerably, as might be expected given the changes in the curriculum (see Figure 1.1 for strategy use in the assessments of 1997, 2004 and 2011; Table 1.1 provides an example of each of the strategies). However, use of the whole-number-based algorithms and more informal written approaches did not increase accordingly; instead, there was a large increase in the number of problems that were solved without any calculations that were written down. From 2004 to 2011 strategy use remained largely stable, with high levels of answering without written work. Follow-up research indicated that this answering without any written work should be interpreted as mental calculation (Hickendorff, Van Putten, Verhelst, & Heiser, 2010).

The accuracy of mental strategies was found to be much lower than that of written strategies (see percentage correct rates in Figure 1.1). The increasing choices for an inaccurate strategy rather than for the much more accurate alternatives suggest that the important educational goal of adaptivity is not attained for a substantial part of the students. Especially lower ability students and boys appear at risk in this respect (Hickendorff et al., 2009). The changing strategy choices also appear to have had considerable consequences for performance: the overall performance level for the domain of multidigit multiplication and division decreased sharply from 1997 to 2004 (J. Janssen et al., 2005), and remained at that lower level in 2011 (Scheltens et al., 2013).

This also raises the question of how instruction affects students' performance. As illustrated by the endings of the paragraphs on the history of mathematics reforms in the U.S. and the Netherlands, this is a topic that inspires (sometimes heated) debate. An important contribution to the discussion can be made by empirical investigations that evaluate the actual effects that the prescribed curriculum and different instructional practices have on performance. The existing research on the effects of the curriculum (usually operationalized as the mathematics textbook that is used) finds those effects to be very limited, though studies often lack proper experimental design (Royal Netherlands Academy of Arts and Sciences, 2009; Slavin & Lake, 2008). However, there are considerable effects of teachers' actual instructional behaviors (e.g., positive effects of cooperative learning methods and programs targeting teachers' skills in classroom management, motivation, and effective time use; Slavin & Lake, 2008).

1.3 Contents of this dissertation

This dissertation is an investigation of factors that affect students' mathematical strategy use and performance. Both instruction (in daily practice and special interventions) and students' and teachers' characteristics are considered. This investigation is carried out in the context of multidigit multiplication and division at the end of Dutch primary school. This context has special theoretical and practical relevance: theoretical because it is an interesting case of developments in strategy use in reform mathematics; and practical because it constitutes a direct problem in

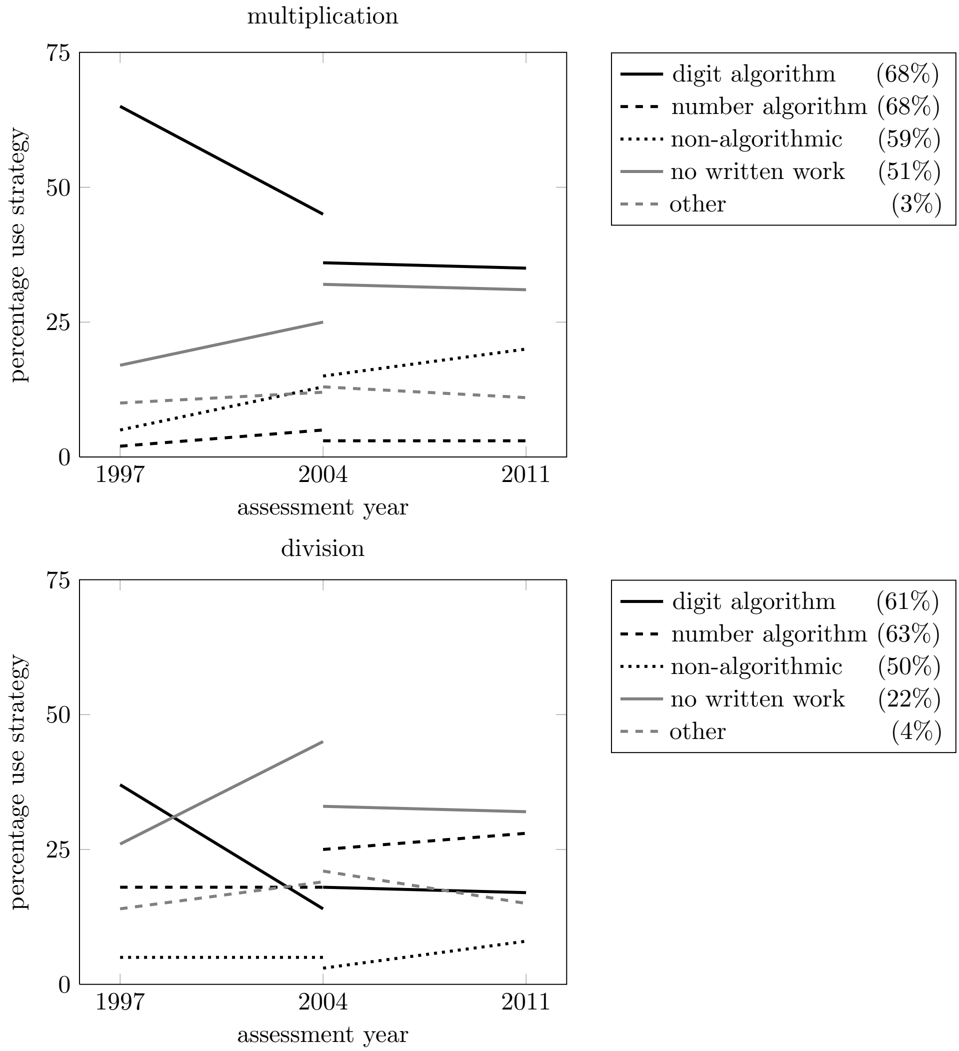

Figure 1.1: Use of the different multiplication and division strategies on the assessments in 1997, 2004 and 2011 (percentage correct per strategy in 2011 is given between brackets). The lines are broken because the items that are compared for 1997 and 2004 are different from those compared for 2004 and 2011.

students' mathematical performance that needs to be addressed. Two approaches to investigating relations with strategy use and performance are taken: secondary analyses of large-scale assessment data and experiments in primary schools.

The first approach is taken in Chapter 2 and Chapter 3, which contain secondary analyses of data from the most recent Dutch large-scale assessment of mathematical ability at the end of primary school. Many of the students participating in this assessment solved several multidigit multiplication and division problems, and the accuracy and strategy use for each of these solutions was coded based on students' written work. The students' teachers filled out a questionnaire on their mathematics instruction: both on general aspects of this instruction and on multiplication and division instruction more specifically. These teacher reports, and student characteristics, were related to students' strategy use (Chapter 2) and to their performance (Chapter 3).

Investigating these relations posed several statistical challenges: how to deal with the large number of items from the teacher questionnaire; the multilevel structure of the data (item responses within students, who are within classes); the nominal measurement level of the strategies; and the incomplete assessment design, in which students do not complete all items but only systematically varying subsets of items. These issues are addressed with latent variable models. In Chapter 2, a first application of multilevel latent class analysis (MLCA) to large-scale assessment data is demonstrated, and several issues in applying this technique are discussed. In Chapter 3, a new combination of LASSO penalization and explanatory item response theory (IRT) is introduced to deal with the large number of teacher variables.

The second approach to investigating the relation between instruction and strategy use and performance is taken in Chapter 4 and Chapter 5, which describe experiments in primary schools. Whereas analyses of large-scale assessments only allow for the investigation of correlational relations, experiments enable causal inference. The experiments in both chapters focus on mental versus written strategy use, given the large performance difference between the two, and consider the effects of student characteristics.

In Chapter 4, it is investigated whether instructing students to write down their calculations actually improves their performance. In a choice/no-choice experiment (Siegler & Lemaire, 1997), students first solved a set of division problems with free strategy choice as usual, but this choice phase of the experiment was followed by a no-choice phase, in which students were required to write down calculations for another version of the set of division problems, and to not do so for a third version. This experimental set-up allowed for an unbiased assessment of the differences in accuracy and speed between mental and written strategies, and for an investigation of the adaptivity of students' strategy choices. In Chapter 5, it is evaluated what the effects on spontaneous strategy choices and performance are of a training program that features instruction in writing down calculations, using a pretest-posttest design with a control training condition and a no training condition.

Finally, in Chapter 6, a particular aspect of the comparability of results from the first approach in Chapters 2 and 3 and the second approach in Chapters 4 and 5 is considered. It is investigated to which extent strategy and performance results can be generalized from tasks that only concern one mathematical operation (typical in experiments) to tasks in which multiple operations are mixed together (typical in assessments and educational practice). This generalization could be hindered by task switching costs and strategy perseveration, and the occurrence of these phenomena is investigated with an experimental comparison of a single-task and a mixed-task condition.