

## Abelian Higgs hair for black holes

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(Received 5 June 1995)

We find evidence for the existence of solutions of the Einstein and Abelian Higgs field equations describing a black hole pierced by a Nielsen-Olesen vortex. This situation falls outside the scope of the usual no-hair arguments due to the nontrivial topology of the vortex configuration and the special properties of its energy-momentum tensor. By a combination of numerical and perturbative techniques we conclude that the black hole horizon has no difficulty in supporting the long-range fields of the Nielsen-Olesen string. Moreover, the effect of the vortex can in principle be measured from infinity, thus justifying its characterization as black hole “hair.”

PACS number(s): 04.40.Nr, 04.70.Bw, 11.27.+d, 98.80.Cq

### I. INTRODUCTION

Some years ago a collection of results was proved which established that the only long-range information that a black hole could carry was its electromagnetic charge, mass, and angular momentum. Thus, for example, lepton or baryon numbers were not good quantum numbers for black holes, despite being defined for a neutron star. This set of results came to be known as the “no-hair” theorems and, although not justified, gave rise to the much stronger, but for a while popularly held, belief that the only nontrivial field configurations an event horizon could support were its massless spin-1 and spin-2 charges  $Q$ ,  $M$ , and  $J$ . Such a picture was not only misleading, but wrong; in spite of the structure of the original “no-hair” proofs, the no-hair theorems only claim qualified uniqueness of static or stationary black hole spacetimes (see [1] for a discussion of no-hair folklore).

The extrapolation of the no-hair folklore to include matter fields on the event horizon has been known to be false for some time, as has the extrapolation to include quantum effects. Black holes can be colored [2–5], i.e., can support long-range Yang-Mills hair. Such solutions are unstable [6,7], thereby evading the usual no-hair uniqueness theorems, but they do nonetheless exist. Black holes also carry quantum hair [8,9], which,

although not locally observable, can be inferred via an Aharonov-Bohm interference in cosmic strings scattered on either side of the hole. Of course, the no-hair theorems are classical, but one might expect some modified argument to apply to quantum fields; so how is this long-range hair mediated? More importantly, also at the semiclassical level, the hair has an effect on the thermodynamics of the black hole [10] caused by the phase shifts of virtual cosmic string loops dressing the Euclidean horizon of the black hole. The existence of these Euclidean vortices [11] seemed at first sight to be incompatible with some of the principles of the “no-hair” theorems; however, an examination of the appropriate theorem for the Abelian Higgs model [12] revealed some assumptions which were not satisfied for the Euclidean vortices [13]. Finally, it was shown that a small enough magnetically charged black hole would be unstable to the nucleation of an  $SU(2)$  ’t Hooft-Polyakov field configuration outside the horizon [14]. Technically, of course, this is not new “hair” since the magnetic charge was already measurable at infinity; however, it illustrates nicely the difference between the “hair” and the short-range massive fields which in this case can live outside and on the black hole horizon.

It seems, therefore, that one must be quite specific about what one means by *hair* on black holes, and we shall take the definition to mean a charge or property of the black hole measurable at infinity; whether or not fields can live on the horizon we shall refer to as *dressing*. The impression that a black hole horizon can support only a small number of long-range massless gauge fields is somewhat misleading; an examination of the examples cited above indicates a common theme: When nontrivial

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topology is present in the field theory, the situation is more subtle and dressing becomes a real possibility. In this paper we will examine the case of the Abelian Higgs model and show that the black hole can indeed sport long hair, namely, a U(1) vortex.

This paper was largely motivated by the work of Aryal, Ford, and Vilenkin (AFV) [15], who wrote down a “solution” for a cosmic string threading a black hole. More precisely, they wrote down an axisymmetric metric which consisted of a conical singularity centered on a Schwarzschild black hole:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2(1 - 4G\mu)^2 \sin^2 \theta d\phi^2. \quad (1.1)$$

While such a metric was extremely suggestive that a true vortex would thread a black hole, it addressed only the gravitational aspect of the problem. In view of the delicacy of dressing the event horizon, it seems necessary to analyze this part of the problem as well. The main potential obstruction to putting a vortex through a black hole is in having the vortex pierce the event horizon. Recall that the event horizon is generated by a congruence of null geodesics. If there exists a static vortex–black-hole solution, then this congruence must remain convergence and shear free throughout the core of the vortex as it touches the black hole. This in turn translates to a relation on the stress-energy tensor: namely, that  $T_0^0 - T_r^r = 0$ . This is certainly true at the center of the string, where the energy and tension balance, but it is not clear that as we move away from the center and the null vector no longer aligns itself with the string world sheet that this balance will still be maintained. If a real vortex cannot puncture the event horizon of a black hole, it raises questions as to how black holes and cosmic strings interact. Do they avoid each other entirely? Or does a black hole “swallow up” a cosmic string caught in its gravitational grasp? And if a static equilibrium solution does exist, somehow avoiding the geodesic problem, how does the vortex pierce the horizon, and how does it circumvent the original no-hair theorems for the Abelian Higgs model?

In this paper we establish that the AFV solution can indeed be viewed as a thin string limit of some physical vortex by demonstrating that an Abelian Higgs vortex can thread the black hole. The layout of the paper is as follows: We first review the self-gravitating U(1) vortex in the next section. In Sec. III we examine the question of existence of the vortex in the black hole background, in the absence of gravitational back reaction. We find an analytic approximation to the solution for strings thin compared to the black hole and present analytic and numerical results for strings of varying widths and winding numbers. We also examine global strings—a scenario for which we would not expect a gravitational solution—finding the reaction of the vortex to the event horizon to be different to that of the local string. In Sec. IV we consider the gravitational back reaction of a single thin vortex, using the analytic approximation developed in Sec. III. We derive the AFV metric, but find a subtle

difference to their work involving a renormalization of the Schwarzschild mass parameter—all physically measurable results, however, agree. We also comment on the thermodynamics of the string–black-hole system. Finally, in Sec. V we summarize and discuss our results, including an examination of more exotic systems such as a string terminating on a black hole, which has a gravitational counterpart. We also comment on the dynamical process of black hole cosmic string interaction and the second law of thermodynamics. Finally, we try to answer the question of whether the Abelian Higgs vortex is true *hair*, or whether it is just horizon *dressing* and would be more appropriately dubbed a “wig.”

## II. ABELIAN HIGGS VORTEX

We start by briefly reviewing the U(1) vortex in order to establish notation and conventions. We take the Abelian Higgs Lagrangian

$$\mathcal{L}[\Phi, A_\mu] = D_\mu \Phi^\dagger D^\mu \Phi - \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} - \frac{\lambda}{4} (\Phi^\dagger \Phi - \eta^2)^2, \quad (2.1)$$

where  $\Phi$  is a complex scalar field,  $D_\mu = \nabla_\mu + ieA_\mu$  is the usual gauge covariant derivative, and  $\tilde{F}_{\mu\nu}$  the field strength associated with  $A_\mu$ . We use units in which  $\hbar = c = 1$  and a mostly minus signature. For cosmic strings associated with galaxy formation,  $\eta \sim 10^{15}$  GeV and  $\lambda \sim 10^{-12}$ .

We shall choose to express the field content in a slightly different manner and one in which the physical degrees of freedom are made more manifest. We define the (real) fields  $X$ ,  $\chi$ , and  $P_\mu$  by

$$\Phi(x^\alpha) = \eta X(x^\alpha) e^{i\chi(x^\alpha)}, \quad (2.2a)$$

$$A_\mu(x^\alpha) = \frac{1}{e} [P_\mu(x^\alpha) - \nabla_\mu \chi(x^\alpha)]. \quad (2.2b)$$

In terms of these new variables, the Lagrangian and equations of motion become

$$\mathcal{L} = \eta^2 \nabla_\mu X \nabla^\mu X + \eta^2 X^2 P_\mu P^\mu - \frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda\eta^4}{4} (X^2 - 1)^2, \quad (2.3)$$

$$\nabla_\mu \nabla^\mu X - P_\mu P^\mu X + \frac{\lambda\eta^2}{2} X(X^2 - 1) = 0, \quad (2.4a)$$

$$\nabla_\mu F^{\mu\nu} + 2e^2 \eta^2 X^2 P^\nu = 0. \quad (2.4b)$$

Thus  $P_\mu$  is the massive vector field in the broken symmetry phase,  $F_{\mu\nu} = \nabla_\mu P_\nu - \nabla_\nu P_\mu$  its field strength, and  $X$  the residual real scalar field with which it interacts.  $\chi$  is not in itself a physical quantity; however, it can contain physical information if it is nonsingle valued, in other

words, if a vortex is present. In this case, if the line integral of  $d\chi$  around a closed loop (along which  $X = 1$ ) is nonvanishing, single valuedness of  $\Phi$  then implies

$$\oint \nabla_\mu \chi dx^\mu = [\chi] = 2\pi n \text{ for some } n \in \mathbb{Z}. \quad (2.5)$$

Continuity then demands (in the absence of nontrivial spatial topology) that  $X = 0$  at some point on any surface spanning the loop—this is the locus of the vortex. Thus the true physical content of this model is contained in the fields  $P_\mu$  and  $X$  plus boundary conditions on  $P_\mu$  and  $X$  representing vortices.

The simplest vortex solution is the Nielsen-Olesen (NO) vortex [16], an infinite, straight static solution with cylindrical symmetry. In this case, we can choose a gauge in which

$$\Phi = \eta X_0(R) e^{i\phi}, \quad P_\mu = P_0(R) \nabla_\mu \phi, \quad (2.6)$$

where  $R = \sqrt{\lambda} \eta r_c$  in cylindrical polar coordinates  $(r_c, \phi)$ . The equations for  $X$  and  $P_\mu$  greatly simplify to

$$-X_0'' - \frac{X_0'}{R} + \frac{P_0'^2 X_0}{R^2} + \frac{1}{2} X_0 (X_0^2 - 1) = 0, \quad (2.7a)$$

$$-P_0'' + \frac{P_0'}{R} + \beta^{-1} X_0^2 P_0 = 0, \quad (2.7b)$$

where  $\beta = \lambda/2e^2 = m_{\text{scalar}}^2/m_{\text{vector}}^2$  is the Bogomolnyi parameter [17] ( $\beta = 1$  corresponds to the vortex being supersymmetrizable). Note that in these rescaled coordinates the string has width of order unity. This string has winding number 1; for winding number  $N$ , we replace  $\chi$  by  $N\chi$  and hence  $P$  by  $NP$ .

This solution can be readily extended to include self-gravity by using Thorne's cylindrically symmetric coordinate system [18]

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - dr_c^2) - e^{2\psi} dz^2 - \tilde{\alpha}^2 e^{-2\psi} d\phi^2 \quad (2.8)$$

(where  $\gamma, \psi, \tilde{\alpha}$  are independent of  $z, \phi$ ), with the string energy-momentum tensor as the source:

$$T_{\mu\nu} = 2\eta^2 \nabla_\mu X \nabla_\nu X + 2\eta^2 X^2 P_\mu P_\nu - \frac{2\beta}{\lambda} F_{\mu\sigma} F_\nu^\sigma - \mathcal{L} g_{\mu\nu} \quad (2.9)$$

in unrescaled coordinates. To rescale the coordinates, we set  $R = \sqrt{\lambda} \eta r_c$ ,  $\alpha = \sqrt{\lambda} \eta \tilde{\alpha}$ , and for future comparison, we write the rescaled version of the energy and stresses  $\hat{T}_{ab} = T_{ab}/(\lambda\eta^4)$ :

$$\hat{T}_0^0 = \mathcal{E} = e^{-2(\gamma-\psi)} X'^2 + \frac{e^{2\psi} X^2 P^2}{\beta^{-1} \alpha^2} + e^{2(\gamma-2\psi)} P'^2 / \alpha^2 + (X^2 - 1)^2 / 4, \quad (2.10a)$$

$$\hat{T}_R^R = -\mathcal{P}_R = -e^{-2(\gamma-\psi)} X'^2 + \frac{e^{2\psi} X^2 P^2}{\beta^{-1} \alpha^2} - e^{2(\gamma-2\psi)} P'^2 / \alpha^2 + (X^2 - 1)^2 / 4, \quad (2.10b)$$

$$\hat{T}_\phi^\phi = -\mathcal{P}_\phi = e^{-2(\gamma-\psi)} X'^2 - \frac{e^{2\psi} X^2 P^2}{\beta^{-1} \alpha^2} - e^{2(\gamma-2\psi)} P'^2 / \alpha^2 + (X^2 - 1)^2 / 4, \quad (2.10c)$$

$$\hat{T}_z^z = -\mathcal{P}_z = \hat{T}_0^0. \quad (2.10d)$$

Also, for future reference, the Bianchi identity gives

$$\mathcal{P}'_R + (\mathcal{P}_R - \mathcal{P}_\phi) \left( \frac{\alpha'}{\alpha} - \psi' \right) + \gamma' \mathcal{P}_R + \gamma' \mathcal{E} = 0. \quad (2.11)$$

To zeroth order (flat space),

$$\alpha = R, \quad \psi = \gamma = 0, \quad X = X_0, \quad P = P_0, \quad (2.12)$$

and (2.11) gives

$$(R\mathcal{P}_{0R})' = \mathcal{P}_{0\phi}. \quad (2.13)$$

To first order in  $\epsilon = 8\pi G\eta^2$ , the string metric is given by [19]

$$\alpha = \left[ 1 - \epsilon \int_0^R R(\mathcal{E}_0 - \mathcal{P}_{0R}) dR \right] R + \epsilon \int_0^R R^2(\mathcal{E}_0 - \mathcal{P}_{0R}) dR, \quad (2.14a)$$

$$\gamma = 2\psi = \epsilon \int_0^R R\mathcal{P}_{0R} dR, \quad (2.14b)$$

where the subscript zero indicates evaluation in the flat space limit. Note that when the radial stresses do not vanish, there is a scaling between the time,  $z$ , and radial coordinates for an observer at infinity and those for an observer sitting at the core of the string. The only case in which these stresses do vanish is when  $\beta = 1$ . In this case, the field equations reduce to

$$X' = XP/\alpha,$$

$$P' = \frac{1}{2}\alpha(X^2 - 1),$$

$$\alpha' = 1 - \epsilon[(X^2 - 1)P + 1],$$

$$\gamma = \psi = 0,$$

a first order set of coupled differential equations as one might expect from the fact that the solution is supersymmetrizable.

We conclude this section by demonstrating the asymptotically conical nature of the corrected metric [19,20] (see also [21–24] for discussions involving model cores rather than vortices). Note that since the string functions  $X$  and  $P$  rapidly fall off to their vacuum values outside the core, the integrals in (2.14) rapidly converge

to their asymptotic, constant, values. Let

$$\begin{aligned} \epsilon \int_0^R R(\mathcal{E}_0 - \mathcal{P}_{0R})dR &= A, \\ \epsilon \int_0^R R^2(\mathcal{E}_0 - \mathcal{P}_{0R})dR &= B, \\ \epsilon \int_0^R R\mathcal{P}_{0R} &= C; \end{aligned} \quad (2.16)$$

then, the asymptotic form of the metric is

$$\begin{aligned} ds^2 &= e^C [dt^2 - dr_c^2 - dz^2] - r_c^2(1 - A + B/r_c)^2 e^{-C} d\phi^2 \\ &= \hat{t}^2 - \hat{r}_c^2 - \hat{z}^2 - \hat{r}_c^2(1 - A)^2 e^{-2C} d\phi^2, \end{aligned} \quad (2.17)$$

where  $\hat{t} = e^{C/2}t$ ,  $\hat{z} = e^{C/2}z$ , and  $\hat{r}_c = e^{C/2}[r_c + B/(1 - A)]$ . This is seen to be conical with a deficit angle

$$\begin{aligned} \Delta &= 2\pi(A + C) = 2\pi\epsilon \int R\mathcal{E}_0 dR = 16\pi^2 G \int r_c T_0^0 dr_c \\ &= 8\pi G\mu, \end{aligned} \quad (2.18)$$

where  $\mu$  is the energy per unit length of the string. Notice that the deficit angle is independent of the radial stresses, but that there is a redshift or blueshift of time between infinity and the core of the string if they do not vanish.

### III. STRING IN BACKGROUND SCHWARZSCHILD METRIC

In solving a Schwarzschild background, there are two coordinate systems we could consider.

(i) Spherical (Schwarzschild) coordinates

$$\begin{aligned} ds^2 &= \left(1 - \frac{2GM}{r_s}\right) dt^2 - \left(1 - \frac{2GM}{r_s}\right)^{-1} dr_s^2 \\ &\quad - r_s^2(d\theta^2 + \sin^2\theta d\phi^2). \end{aligned} \quad (3.1)$$

This is the usual Schwarzschild metric. This coordinate system is good for analyzing the existence of a background solution, but, not being tailored to the symmetries of the full problem, it does not deal with gravitational back reaction well.

(ii) Axisymmetric (Weyl) coordinates [25]. Here

$$\begin{aligned} ds^2 &= \frac{R_1 + R_2 - 2GM}{R_1 + R_2 + 2GM} - \frac{(R_1 + R_2 + 2GM)^2}{4R_1 R_2} (dz^2 + dr_c^2) \\ &\quad - r_c^2 \frac{(R_1 + R_2 + 2GM)}{(R_1 + R_2 - 2GM)} d\phi^2, \end{aligned} \quad (3.2)$$

where

$$R_1^2 = (z - GM)^2 + r_c^2, \quad R_2^2 = (z + GM)^2 + r_c^2. \quad (3.3)$$

The transformation between the two systems is given by

$$z = (r_s - GM) \cos\theta, \quad r_c^2 = r_s(r_s - 2GM) \sin^2\theta. \quad (3.4)$$

Weyl coordinates are appropriate for analyzing the grav-

itational back reaction, but are rather cumbersome for the background solution problem.

The rest of this section is devoted to arguing the existence of a vortex solution in a black hole background. Formally, this means taking the somewhat artificial limit  $G\eta^2 \rightarrow 0$  keeping  $GM$  fixed. It is straightforward to show that it is consistent to take

$$P_\mu = P\nabla_\mu\phi \quad (3.5)$$

and letting

$$r = \sqrt{\lambda}\eta r_s, \quad (3.6)$$

$$E = \sqrt{\lambda}\eta GM,$$

in the Schwarzschild metric, the equations of motion for  $X$  and  $P$  are

$$\begin{aligned} -\frac{1}{r^2}\partial_r[r(r - 2E)]\partial_r X - \frac{1}{r^2\sin\theta}\partial_\theta[\sin\theta\partial_\theta X] \\ + \frac{1}{2}X(X^2 - 1) + \frac{XP^2}{r^2\sin^2\theta} = 0, \end{aligned} \quad (3.7a)$$

$$\begin{aligned} \partial_r[(1 - 2E/r)\partial_r P] + \frac{\sin\theta}{r^2}\partial_\theta[\csc\theta\partial_\theta P] - \beta^{-1}X^2 P = 0. \end{aligned} \quad (3.7b)$$

We proceed at first by taking a ‘‘thin string limit’’; in other words, we assume  $2E \gg 1$ . This is equivalent to requiring  $M \gg 1000$  kg for the parameters of the grand unified theory (GUT) string. We will then consider thicker and higher winding number strings.

First of all, let us try

$$X = X(r \sin\theta), \quad P = P(r \sin\theta); \quad (3.8)$$

substituting these forms into (3.7) and writing (suggestively)  $r \sin\theta = R$ , we get

$$\begin{aligned} -X'' - X'/R + \frac{1}{2}X(X^2 - 1) + XP^2/R^2 \\ + \frac{2E \sin^2\theta}{r}[X'' + X'/R] = 0, \end{aligned} \quad (3.9a)$$

$$P'' - P'/R - \beta^{-1}X^2 P + \frac{2E \sin^2\theta}{r}[-P'' + P'/R] = 0, \quad (3.9b)$$

where a prime denotes a derivative with respect to  $R$ . If we were to input  $X, P = X_0, P_0$ , then these equations would be satisfied up to errors of the form

$$\frac{2E \sin^2\theta}{r} \times [\text{other terms in equation}]. \quad (3.10)$$

However, since  $r \sin\theta = R \lesssim 1$  in the core of the string,  $\sin\theta = O(1/r)$ , and the errors are  $O(E/r^3) < O(1/E^2) \ll 1$ . Thus the Nielsen-Olesen solution is in fact a good solution throughout and beyond the core of the string,

whether or not it is near the event horizon. By the time the premultiplying term in the errors is significant, we are well into the exponential falloff of the vortex and essentially in vacuum. However, since we are interested in showing that the event horizon can support the vortex, for completeness we include the solution to order  $1/E$  on the horizon:

$$1 - X \sim e^{-2E\theta} + e^{-2E(\pi-\theta)}, \quad (3.11)$$

$$P \sim e^{-2E\sqrt{\beta^{-1}\theta}} + e^{-2E\sqrt{\beta^{-1}(\pi-\theta)}}.$$

Of course, these are only approximate forms and do not prove the existence of a solution to (3.7). However, they will provide a good approximation to the true solution, if such can be shown to exist. We will provide such evidence in the form of numerical solutions later in this section. For the moment, we conclude this description of the thin string limit by examining the equations of motion in the extended Schwarzschild spacetime in terms of Kruskal coordinates.

One problem with using Schwarzschild coordinates for our analysis is that they are singular on the event horizon. This is, of course, purely a coordinate singularity, but since demonstrating a dressing of the event horizon is central to this paper, we will examine the thin vortex solution in coordinates that are not singular at the event horizon, namely, Kruskal coordinates, to convince the reader that the analytic approximation really does hold true at the event horizon. Kruskal coordinates are based on the incoming and outgoing radial null congruences of the Schwarzschild spacetime, but we shall instead use a Kruskal “time” and “space” coordinate defined as

$$T = (r_s - 2GM)^{1/2} e^{r_s/4GM} \sinh\left(\frac{t}{4GM}\right), \quad (3.12)$$

$$Y = (r_s - 2GM)^{1/2} e^{r_s/4GM} \cosh\left(\frac{t}{4GM}\right),$$

in terms of which the metric is

$$ds^2 = \frac{16G^2 M^2}{r_s} e^{-r_s/2GM} (dT^2 - dY^2) - r_s^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (3.13)$$

This metric is clearly regular away from  $r_s = 0$ , and the future and past event horizons are presented by  $T = Y > 0$  and  $-T = Y > 0$ , respectively. What we would like to show is that  $X = X_0(r \sin\theta)$  and  $P = P_0(r \sin\theta)$  are good solutions (expressed in Kruskal coordinates) in the vicinity of the horizon,  $|T| = |Y|$ .

First note that near the horizon

$$r = \sqrt{\lambda\eta} r_s = 2E - e^{-1} \sqrt{\lambda\eta} (T^2 - Y^2) + O((T^2 - Y^2)^2), \quad (3.14)$$

where here  $e = 2.718\dots$  is the natural number. This implies that

$$\partial_T X = \frac{-2T\sqrt{\lambda\eta}}{e} \sin\theta X'(R), \quad (3.15)$$

$$\partial_Y X = \frac{2Y\sqrt{\lambda\eta}}{e} \sin\theta X'(R)$$

and hence

$$\left[ \frac{1}{\sqrt{g}} \partial_T g^{TT} \sqrt{g} \partial_T + \frac{1}{\sqrt{g}} \partial_R g^{RR} \sqrt{g} \partial_R \right] X$$

$$= \lambda\eta^2 \left[ \frac{2E - r}{2E} \sin^2\theta X''(R) + \frac{E - r}{2E^2} X'(R) \sin\theta \right],$$

$$\left[ \frac{1}{\sqrt{g}} \partial_T \frac{g^{TT} \sqrt{g}}{r_s^2 \sin^2\theta} \partial_T + \frac{1}{\sqrt{g}} \partial_R \frac{g^{RR} \sqrt{g}}{r_s^2 \sin^2\theta} \partial_R \right] P$$

$$= \lambda\eta^2 \left[ \frac{2E - r}{2E^3} P'' - \frac{P'}{E^2 R} \right]. \quad (3.16)$$

In other words, as in Schwarzschild coordinates, the Nielsen-Olesen solution solves the equations of motion to  $O(E^{-2})$  near, on, and even beyond the event horizon. Indeed, replacing

$$r = \sqrt{\lambda\eta} r^{*-1} \{2GM \ln(Y^2 - T^2)\}, \quad (3.17)$$

where  $r^*(r_s)$  is the tortoise coordinate, indicates that the approximation holds true well within the event horizon for larger black holes.

Having established that it is possible for the horizon to support  $N = 1$  vortices, we now turn to the large- $N$  case, where analytic approximations are available [26]. First of all, note that a string with  $N \gg 1$  has a core radius of order  $\sqrt{N}$ . To see this, consider the rather special case of  $\beta = 1$  (for general  $\beta$ , see [26]). Setting  $X = \xi^N$ , we have from (2.15),

$$NP' = \frac{1}{2} R (\xi^{2N} - 1), \quad \frac{\xi'}{\xi} = \frac{P}{R}, \quad (3.18)$$

in the absence of gravity. These are in fact the same as the large- $N$  equations for general  $\beta$ ; it is in the subleading terms that the two cases differ. These are solved in the core by

$$P = 1 - \frac{R^2}{4N}, \quad \xi \propto R e^{-R^2/2N}, \quad (3.19)$$

using  $0 < \xi < 1$ . The transition to vacuum, and hence a different approximate solution to the above, can be seen to occur quite abruptly at  $R = O(\sqrt{N})$ . Thus the condition that the string be thin compared with the black hole is now  $2E \gg \sqrt{N}$ , and in that case the previous arguments still apply, since

$$\frac{2E \sin^2\theta}{r} = \frac{2ER^2}{r^3} < \frac{2EN}{r^3} \leq \frac{N}{(2E)^2} \ll 1;$$

i.e., the Nielsen Olesen solution for a large- $N$  string is good also on the event horizon.

Now consider the opposite limit, where the string is much bigger than the black hole,  $\sqrt{N} \gg 2E$ . The black hole sits well inside the core of the string, in a region where  $X \approx 0$ ,  $P \approx 1 - pR^2$ . In the absence of a black hole, the  $P$  equation simply states that the magnetic field is constant throughout the core of the string. Notice that since we are ignoring  $X^2$  in the large- $N$  expansion, (3.9b) implies that the presence of the black hole does not affect the large- $N$   $P$  equation, and so we still find  $P(R) \sim 1 - pR^2$ . The magnetic field will still be constant (and equal to  $-2p$ ) in the string core. However, its value may change due to the black hole. Notice that we may expect a slight “squeezing” of the string core due to the black hole. To see this, consider rewriting the  $P$  equation (3.9b) in the form

$$P'' - (P'/R) - \beta^{-1}(r, \theta)PX^2 = 0. \quad (3.20)$$

On the equatorial plane of the black hole,  $\sin \theta = 1$  and  $\beta^{-1}(R) = \beta^{-1}/(1 - 2E/R)$ . For  $R \gg 2E$  we have the vacuum solution  $X = 1$ ,  $P = 0$ . As we come in toward the horizon,  $P$  has to leave its vacuum value—however, the effective value of  $\beta^{-1}$  (which measures an “effective mass” for  $P$ ) is increasing. Compared with the situation where there is no black hole,  $P$  should be more reluctant to leave its vacuum value. The magnetic field will remain zero for as long as possible, and as a result, the string core is somewhat smaller around the black hole. Note that this argument does not apply for global strings, where numerical simulations indeed show that it is a much smaller effect.

Now consider the  $X$  equation, to leading order in  $N$  ( $r > 2E$ ):

$$\begin{aligned} \left(1 - \frac{2E \sin^2 \theta}{r}\right) \left(\frac{\xi'}{\xi}\right)^2 &= \left(1 - \frac{2ER^2}{(R^2 + z^2)^{3/2}}\right) \left(\frac{\xi'}{\xi}\right)^2 \\ &= \frac{P^2}{R^2} + O\left(\frac{1}{N^2}\right). \end{aligned} \quad (3.21)$$

We want to show that the solutions to this equation are regular at the horizon. The equation becomes singular at  $\sin \theta = 1$ ,  $r = 2E$  (or  $z = 0$ ,  $R = 2E$ ), the equatorial plane of the horizon, and so let us integrate the equations to leading order in  $1/N$  and  $z/R$ :

$$\frac{\xi'}{\xi} \sqrt{(1 - 2E/R)} = \frac{P}{R}. \quad (3.22)$$

We find

$$\begin{aligned} \xi &= K[R - E + \sqrt{R(R - 2E)}]^{(1-3pE^2/2)} \\ &\quad \times \exp\left[-p\left(\frac{3E + R}{2}\right)\sqrt{R(R - 2E)}\right] \end{aligned} \quad (3.23)$$

which is finite at  $R = 2E$ . The constant  $K$  can be fixed by the requirement that  $\xi \sim 1$  when  $P \sim 0$ , i.e., at  $R = 1/\sqrt{p}$ .

Finally, note that the horizon seems to be capable of supporting global strings as well, in spite of the fact that

we do not expect to be able to account for gravitational back reaction consistently, since the energy per unit length of a global string diverges and the self-gravitating global string spacetime is singular [27,28]. To find the global string solution in the black hole background, we simply set  $P = 1$  everywhere, to find

$$\begin{aligned} -X'' - X'/R + \frac{1}{2}X(X^2 - 1) + NX/R^2 \\ + \frac{2E \sin^2 \theta}{r}[X'' + X'/R] = 0 \end{aligned} \quad (3.24)$$

The case where the global string is thin compared to the black hole works as before—the vortex is essentially undisturbed. In the case where the string is bigger than the black hole, we can take  $p \rightarrow 0$  in (3.23), and so the solution is again regular at the horizon. We conclude that the presence of the black hole is, if anything, less noticeable than in the local string case, as can be seen in the numerical simulations described next.

## A. Numerical results

We will now provide confirmation of the previous analytic arguments by means of a numerical solution of the equations of motion outside the event horizon. To this end, we note that the equations are elliptic outside the event horizon, parabolic on it, and hyperbolic inside it. Some care is therefore required with specification of the boundary conditions.

At large radii we want to recover the NO solutions, while the symmetry axes outside the horizon must form the core of the string:

$$(X, P) = \begin{cases} (1, 0), & r \rightarrow \infty \\ (0, 1), & r \geq 2E, \theta = 0, \pi. \end{cases} \quad (3.25)$$

On the horizon the equation turns parabolic, taking the form

$$\begin{aligned} \frac{1}{2E} \partial_r X \Big|_{r=2E} &= -\frac{1}{4E^2 \sin^2 \theta} \partial_\theta [\sin \theta \partial_\theta X] + \frac{1}{2}X(X^2 - 1) \\ &\quad + \frac{N^2 X P^2}{4E^2 \sin^2 \theta}, \end{aligned} \quad (3.26a)$$

$$-\frac{1}{2E} \partial_r P \Big|_{r=2E} = \frac{\sin \theta}{4E^2} \partial_\theta [\csc \theta \partial_\theta P] - \beta^{-1} X^2 P. \quad (3.26b)$$

A given choice of  $X(2E, \theta)$  and  $P(2E, \theta)$  on the horizon therefore directly implies  $\partial_r X(2E, \theta)$  and  $\partial_r P(2E, \theta)$ . Generally, boundary values and normal derivatives cannot be specified simultaneously when solving an elliptic equation: Either suffices to determine the solution completely. A particular choice of  $X$  and  $P$  on the horizon and the corresponding normal derivatives will not normally generate the same solution outside the horizon. The boundary conditions on the event horizon are thus determined by the requirement that both solutions coincide.

A practical algorithm for solving the equations of motion in a Schwarzschild background numerically then is as follows. We employ a uniformly spaced polar grid  $\{(r_i, \theta_j)\}$ , with boundaries at  $r = 2E$ , a large radius  $r_L \gg 2E$ , and  $\theta$  ranging from 0 to  $\pi$ . Then we approxi-

mate the derivatives with finite-difference expressions on the grid. Writing  $F_{00}$  for the value of the field  $F$  at the grid point  $(r_i, \theta_j)$  and similarly  $F_{\pm 0}$  for  $F(r_{i\pm 1}, \theta_j)$  and  $F_{0\pm}$  for  $F(r_i, \theta_{j\pm 1})$ , we obtain the finite difference equations

$$X_{00} = \frac{\frac{2}{r} \left(1 - \frac{E}{r}\right) \frac{X_{+0} - X_{-0}}{2\Delta r} + \frac{\cot \theta}{r^2} \frac{X_{0+} - X_{0-}}{2\Delta \theta} + \left(1 - \frac{2E}{r}\right) \frac{X_{+0} + X_{-0}}{\Delta r^2} + \frac{X_{0+} + X_{0-}}{r^2 \Delta \theta^2}}{\left(1 - \frac{2E}{r}\right) \frac{2}{\Delta r^2} + \frac{2}{r^2 \Delta \theta^2} + \frac{1}{2}(X_{00}^2 - 1) + \left(\frac{NF_{00}}{r \sin \theta}\right)^2}, \quad (3.27a)$$

$$P_{00} = \frac{\left(\frac{2E}{r^2}\right) \frac{P_{+0} - P_{-0}}{2\Delta r} - \cot \theta \frac{P_{0+} - P_{0-}}{2\Delta \theta} + \left(1 - \frac{2E}{r}\right) \frac{P_{+0} + P_{-0}}{\Delta r^2} + \frac{P_{0+} + P_{0-}}{r^2 \Delta \theta^2}}{\left(1 - \frac{2E}{r}\right) \frac{2}{\Delta r^2} + \frac{2}{r^2 \Delta \theta^2} + \frac{2}{r^2 \Delta \theta^2} + \beta^{-1} X_{00}^2} \quad (3.27b)$$

inside the grid and

$$X_{00} = \frac{\frac{EX_{+0}}{\Delta r} + \frac{X_{0+} + X_{0-}}{2\Delta \theta^2} + \cot \theta \frac{X_{0+} - X_{0-}}{4\Delta \theta}}{\frac{E}{\Delta r} + \frac{1}{\Delta \theta^2} + E^2(X_{00}^2 - 1) + \frac{1}{2} \left(\frac{NF_{00}}{\sin \theta}\right)^2}, \quad (3.27c)$$

$$P_{00} = \frac{\frac{EP_{+0}}{\Delta r} + \frac{P_{0+} + P_{0-}}{2\Delta \theta^2} - \cot \theta \frac{P_{0+} - P_{0-}}{4\Delta \theta}}{\frac{E}{\Delta r} + \frac{1}{\Delta \theta^2} + 2E^2\beta^{-1}X_{00}^2} \quad (3.27d)$$

on the horizon.

Initial values for  $X$  and  $P$  are assigned on the boundaries according to (3.25); on the horizon, we initially set  $X = 0$ ,  $P = 1$ .  $X$  and  $P$  are then iteratively adjusted on the interior grid points according to (3.27a) and (3.27b), analogous to the Gauss-Seidel scheme for linear elliptic equations [29]. After each pass through the interior grid points, the  $r$  gradients of  $X$  and  $P$  just outside the horizon are calculated and Eqs. (3.27c) and (3.27d) iter-

ated to derive new values for  $X$  and  $P$  on the horizon (for given  $r$  gradients, the equations on the horizon are one-dimensional elliptic equations). The whole process is then iterated to convergence. In order to speed up convergence, the grid is overrelaxed: Instead of replacing  $X$  and  $P$  by the right-hand sides (RHS's) of Eqs. (3.27),  $wX_{\text{new}} + (1-w)X$ , with  $1 < w < 2$ , is used. The optimal value for the over relaxation parameter  $w$  is found by trial and error, and depends on the number of grid points and on the differential equation.

Sample results are presented in Figs. 1–7 and confirm the analytic arguments above. Figures 1–4 show a sequence of solutions with increasing winding number (and therefore string thickness) threading an  $E = 10$  black hole. Qualitatively, the string simply continues regardless of the black hole, though some mild pinching of the magnetic flux does take place. Figures 5 and 6 compare a local and global string with the same winding number and “width”; the global string is apparently fatter due to the power law, as opposed to exponential falloff in the fields. Figure 7 shows a comparison between the numerically obtained solutions and the Nielsen-Olesen analytic approximation that will be used in the next section. As

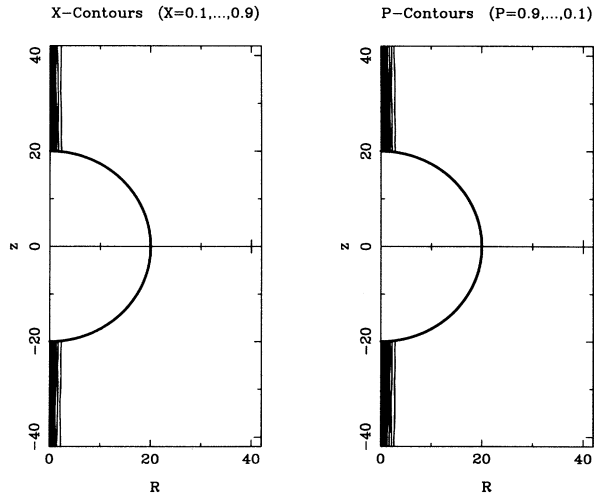


FIG. 1. Numerical solution of the Nielsen-Olesen equations with  $N = 1$ ,  $\beta = \frac{1}{2}$  in a Schwarzschild metric ( $E = 10$ ) background. The event horizon is indicated by a semicircle. Evidently, the presence of the black hole horizon hardly affects the string structure at all. This solution was calculated with 100 radial and 100 azimuthal grid points, out to radius  $r_L = 60$ .

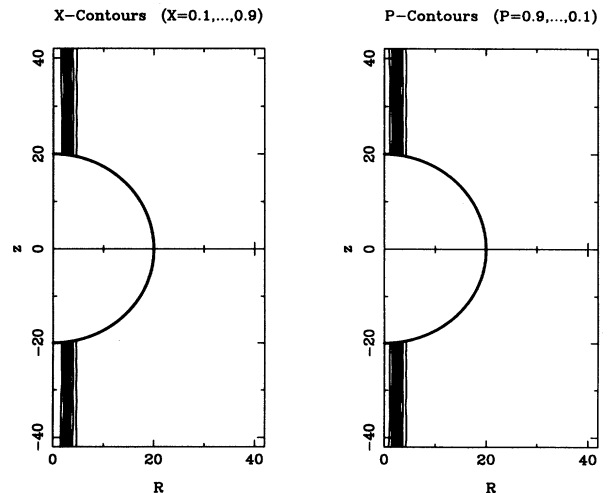


FIG. 2. As in Fig. 1, but for winding number 5.

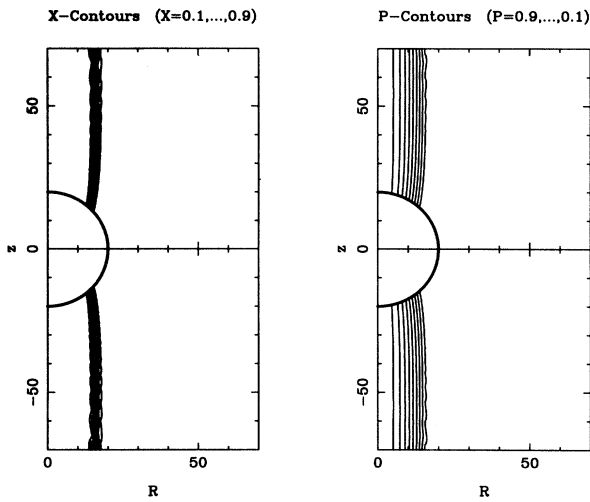


FIG. 3. As in Fig. 1, but for winding number 100. The string is noticeably pinched ( $r_L = 100$  for this calculation). The undulations in the outer contours occur on the scale of one grid cell and are an artifact of the contouring package's conversion from polar to rectangular coordinates.

can be seen, even for a string of non-negligible thickness, this is still an excellent approximation.

IV. GRAVITATING STRINGS

In order to get the gravitational effect of the string superimposed on the black hole, we need to consider a general static axially symmetric metric

$$ds^2 = e^{2\psi} dt^2 - e^{2(\gamma-\psi)}(dz^2 + dr_c^2) - \tilde{\alpha}^2 e^{-2\psi} d\phi^2, \quad (4.1)$$

where  $\psi, \gamma, \tilde{\alpha}$  are independent of  $t, \phi$ . Notice that this is related to (2.8) through  $z \rightarrow it, t \rightarrow iz$ . We then apply an iterative procedure to solving equations, starting with the background solutions (3.2) and the Nielsen-

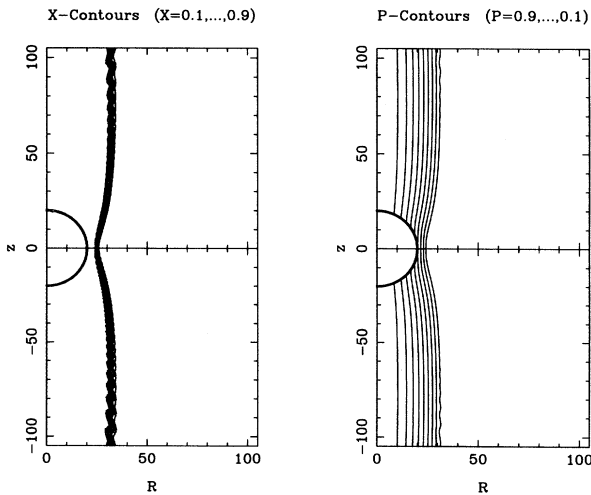


FIG. 4. As in Fig. 1, but for winding number 400. The event horizon is now entirely inside the core of the string, which is slightly pinched. ( $r_L = 150$ ).

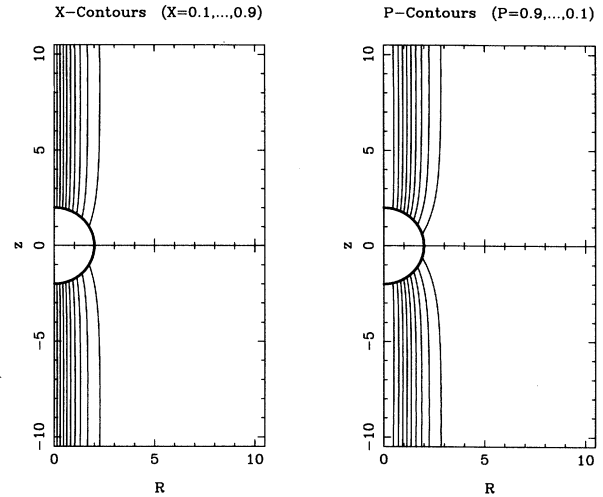


FIG. 5. As in Fig. 1, but with  $E = 1, N = 15$ . The string and black hole have comparable radii, but distortion of the string by the background is still rather mild.

Olesen forms of  $X$  and  $P$ , and expanding the equations of motion in terms of  $\epsilon = 8\pi G\eta^2$ , which is assumed small. ( $\epsilon \leq 10^{-6}$  for GUT strings.) We first rescale coordinates to bring them into line with the rescaled Schwarzschild coordinates used in the previous section:

$$\begin{aligned} \rho &= \sqrt{\lambda\eta} r_c, \\ \zeta &= \sqrt{\lambda\eta} z, \\ \alpha &= \sqrt{\lambda\eta} \tilde{\alpha}, \end{aligned} \quad (4.2)$$

and rewrite

$$\begin{aligned} \hat{R}_1^2 &= (\zeta - E)^2 + \rho^2, \\ \hat{R}_2^2 &= (\zeta + E)^2 + \rho^2. \end{aligned} \quad (4.3)$$

In terms of the rescaled coordinates and energy-momentum tensor, the Einstein equations become

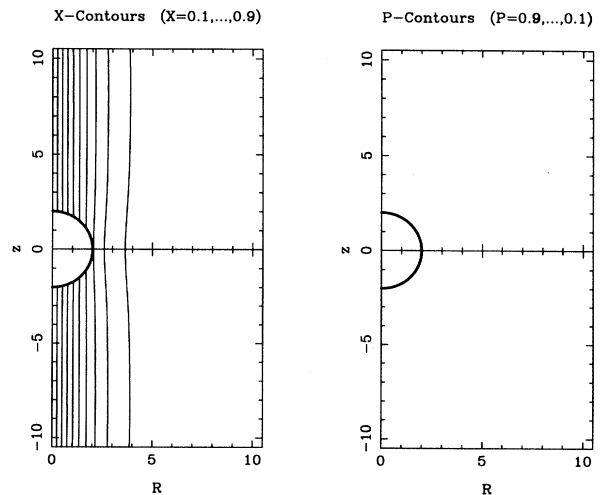


FIG. 6. As in Fig. 5, but for a global string. In agreement with our analytical arguments, the effect of the black hole on a global string is weaker than on the corresponding local string.

$$\alpha_{,\zeta\zeta} + \alpha_{,\rho\rho} = -\epsilon\sqrt{-g}(\hat{T}_\zeta^\zeta + \hat{T}_\rho^\rho), \quad (4.4a)$$

$$(\alpha\psi_{,\zeta})_{,\zeta} + (\alpha\psi_{,\rho})_{,\rho} = \frac{1}{2}\epsilon\sqrt{-g}(\hat{T}_0^0 - \hat{T}_\zeta^\zeta - \hat{T}_\rho^\rho - \hat{T}_\phi^\phi), \quad (4.4b)$$

$$(\alpha_{,\zeta}^2 + \alpha_{,\rho}^2)\gamma_{,\rho} = \epsilon\sqrt{-g}(\alpha_{,\rho}\hat{T}_\zeta^\zeta - \alpha_{,\zeta}\hat{T}_\rho^\rho) + \alpha\alpha_{,\rho}(\psi_{,\rho}^2 - \psi_{,\zeta}^2) + 2\alpha\alpha_{,\zeta}\psi_{,\zeta}\psi_{,\rho} + \alpha_{,\rho}\alpha_{,\rho\rho} + \alpha_{,\zeta}\alpha_{,\zeta\rho}, \quad (4.4c)$$

$$(\alpha_{,\zeta}^2 + \alpha_{,\rho}^2)\gamma_{,\zeta} = -\epsilon\sqrt{-g}(\alpha_{,\zeta}\hat{T}_\zeta^\zeta - \alpha_{,\rho}\hat{T}_\rho^\rho) - \alpha\alpha_{,\zeta}(\psi_{,\rho}^2 - \psi_{,\zeta}^2) + 2\alpha\alpha_{,\rho}\psi_{,\zeta}\psi_{,\rho} - \alpha_{,\zeta}\alpha_{,\rho\rho} + \alpha_{,\rho}\alpha_{,\zeta\rho}, \quad (4.4d)$$

$$\gamma_{,\rho\rho} + \gamma_{,\zeta\zeta} = -\psi_{,\rho}^2 - \psi_{,\zeta}^2 - \epsilon e^{2(\gamma-\psi)}\hat{T}_\phi^\phi, \quad (4.4e)$$

where the energy-momentum tensor is given by

$$\begin{aligned} \hat{T}_0^0 &= V(X) + \frac{X^2 P^2}{\alpha^2 e^{-2\psi}} + \left[ \frac{P_{,\rho} + P_{,\zeta}^2}{\beta^{-1}\alpha^2 e^{-2\psi}} + (X_{,\rho}^2 + X_{,\zeta}^2) \right] e^{-2(\gamma-\psi)}, \\ \hat{T}_\phi^\phi &= V(X) - \frac{X^2 P^2}{\alpha^2 e^{-2\psi}} + \left[ -\frac{P_{,\rho}^2 + P_{,\zeta}^2}{\beta^{-1}\alpha^2 e^{-2\psi}} + (X_{,\rho}^2 + X_{,\zeta}^2) \right] e^{-2(\gamma-\psi)}, \\ \hat{T}_\rho^\rho &= V(X) + \frac{X^2 P^2}{\alpha^2 e^{-2\psi}} + \left[ \frac{P_{,\zeta}^2 - P_{,\rho}^2}{\beta^{-1}\alpha^2 e^{-2\psi}} + (X_{,\zeta}^2 - X_{,\rho}^2) \right] e^{-2(\gamma-\psi)}, \\ \hat{T}_\zeta^\zeta &= V(X) + \frac{X^2 P^2}{\alpha^2 e^{-2\psi}} - \left[ \frac{P_{,\zeta}^2 - P_{,\rho}^2}{\beta^{-1}\alpha^2 e^{-2\psi}} + (X_{,\zeta}^2 - X_{,\rho}^2) \right] e^{-2(\gamma-\psi)}, \\ \hat{T}_\zeta^\rho &= -2e^{-2(\gamma-\psi)} \left[ X_{,\zeta} X_{,\rho} + \frac{P_{,\zeta} P_{,\rho}}{\beta^{-1}\alpha^2 e^{-2\psi}} \right]. \end{aligned} \quad (4.5)$$

We now write  $\alpha = \alpha_0 + \epsilon\alpha_1$ , etc., and solve the Einstein equations (4.4) and the string equations (2.4) iteratively. To zeroth order, we have the background solutions

$$\alpha_0 = \rho, \quad \psi_0 = \frac{1}{2} \ln \frac{R_1 + R_2 - 2GM}{R_1 + R_2 + 2GM}, \quad \gamma_0 = \frac{1}{2} \ln \frac{(R_1 + R_2 - 2GM)(R_1 + R_2 + 2GM)}{4R_1 R_2}, \quad (4.6a)$$

$$X = X_0(R), \quad P = P_0(R), \quad (4.6b)$$

where  $R_1$  and  $R_2$  were defined in (3.3). In these coordinates,  $R = r \sin \theta = \rho e^{-\psi_0}$ , and so (4.6b) indicates that many of the terms in  $\hat{T}_b^a$  are simply functions of  $R$ .

Before proceeding to calculate the back reaction, however, it is prudent to check that the energy-momentum tensor (4.5) will admit a geodesic shear-free event horizon. Recall that we require  $T_0^0 - T_r^r = 0$  on the horizon in Schwarzschild coordinates. This is clearly satisfied at  $\theta = 0$ , where the energy and tension balance, but what about  $\theta \neq 0$ ? In Weyl coordinates, this corresponds to  $\hat{T}_0^0 - \hat{T}_\rho^\rho = 0$  for  $\rho \rightarrow 0$ ,  $\zeta \neq \pm E$ . From (4.5) we see that this is given by

$$\begin{aligned} \hat{T}_0^0 - \hat{T}_\rho^\rho &= 2e^{2(\gamma-\psi)} \left[ \frac{\beta P_{,\rho}^2}{\alpha e^{-2\psi}} + X_{,\rho}^2 \right] \\ &= \frac{8\hat{R}_1 \hat{R}_2}{(\hat{R}_1 + \hat{R}_2 + 2E)^2} \left( \frac{dR}{d\rho} \right)^2 \\ &\quad \times \left[ \frac{\beta P'(R)^2}{R^2} + X'(R)^2 \right]. \end{aligned} \quad (4.7)$$

All terms in this expression remain finite and nonzero as  $\rho \rightarrow 0$  ( $R \rightarrow 0$ ) except for  $dR/d\rho$ . Using the transformation (3.4) between the Schwarzschild and Weyl coordinates, we have

dinates, we have

$$\frac{\partial R}{\partial \rho} = \frac{\rho(r - E \sin^2 \theta)}{\hat{R}_1 \hat{R}_2 \sin \theta}, \quad (4.8)$$

$$\frac{\partial R}{\partial \zeta} = -\frac{Er \sin \theta \cos \theta}{\hat{R}_1 \hat{R}_2};$$

hence,  $dR/d\rho \rightarrow 0$  as  $\rho \rightarrow 0$  and  $\hat{T}_0^0 - \hat{T}_\rho^\rho$  is indeed zero on the horizon. Thus there is no gravitational obstruction, at least in this linearized method, of painting the vortex onto the horizon.

Proceeding with the analysis of the energy-momentum tensor, we use  $\hat{R}_2^2 - \hat{R}_1^2 = 4E\zeta$ , which gives  $\hat{R}_1 \hat{R}_2 = E^2 \sin^2 \theta + r(r - 2E)$ ; thus, near the core of the string, where  $\sin \theta = O(E^{-1})$ ,

$$\begin{aligned} R_{,\zeta}^2 + R_{,\rho}^2 &= \frac{r^2}{\hat{R}_1 \hat{R}_2} \left( 1 - \frac{2E}{r} \sin^2 \theta \right) \\ &\sim \frac{r^2}{\hat{R}_1 \hat{R}_2} = e^{2(\gamma_0 - \psi_0)}. \end{aligned} \quad (4.9)$$

Therefore, in and near the core of the string, the zeroth order rescaled energy-momentum tensor now reads

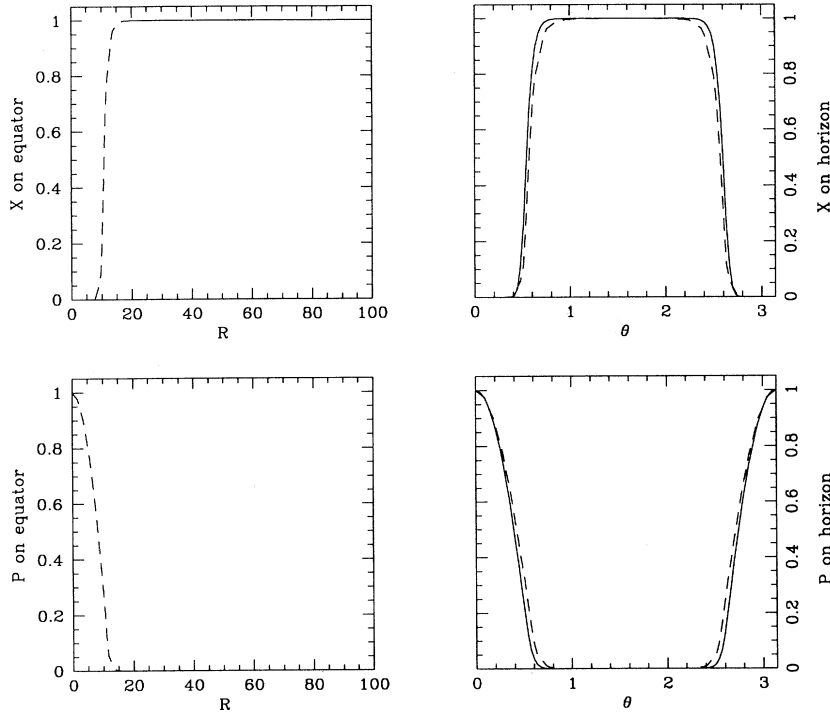


FIG. 7. Illustration of the relatively small effect of a black hole horizon ( $E = 10$ ) on a local ( $\beta = 0.5$ ,  $N = 50$ ) string. In these panels, solid lines show values of the field in the black hole background and dashed lines the values at corresponding positions in a flat metric. Upper and lower panels  $X$  and  $P$ , respectively, while left and right panels show cuts along the equator ( $\theta = \pi/2$ ) and around the horizon ( $r = 2E$ ).

$$\begin{aligned}\hat{T}_{00}^0 &= V(X_0) + \frac{X_0^2 P_0^2}{R^2} + \frac{P_0'^2}{\beta^{-1} R^2} + X_0'^2 + O(E^{-2}), \\ \hat{T}_{0\phi}^\phi &= V(X_0) - \frac{X_0^2 P_0^2}{R^2} - \frac{P_0'^2}{\beta^{-1} R^2} + X_0'^2 + O(E^{-2}), \\ \hat{T}_{0\rho}^\rho &= V(X_0) + \frac{X_0^2 P_0^2}{R^2} + \left[ \frac{P_0'^2}{\beta^{-1} R^2} + X_0'^2 \right] (R_{,\zeta}^2 - R_{,\rho}^2) e^{-2(\gamma_0 - \psi_0)}, \\ \hat{T}_{0\zeta}^\zeta &= V(X_0) + \frac{X_0^2 P_0^2}{R^2} - \left[ \frac{P_0'^2}{\beta^{-1} R^2} + X_0'^2 \right] (R_{,\zeta}^2 - R_{,\rho}^2) e^{-2(\gamma_0 - \psi_0)}, \\ \hat{T}_{0\zeta}^\rho &= -2e^{-2(\gamma_0 - \psi_0)} R_{,\zeta} R_{,\rho} \left[ \frac{P_0'^2}{\beta^{-1} R^2} + X_0'^2 \right] + O(E^{-2}),\end{aligned}\tag{4.10}$$

and the combinations used in Eqs. (4.4a), (4.4b), and (4.4c) are all purely functions of  $R$ . This strongly suggests looking for metric perturbations as functions of  $R$ . However, we must check that the left-hand sides of these equations can be written as appropriate functions of  $R$ .

Consider  $R = \rho e^{-\psi_0}$ ; then,

$$R_{,\zeta} = -R\psi_{0,\zeta} \Rightarrow R_{,\zeta\zeta} = -R\psi_{0,\zeta\zeta} + \frac{R_{,\zeta}^2}{R}\tag{4.11}$$

and

$$R_{,\rho} = \left( \frac{1}{\rho} - \psi_{0,\rho} \right) R \Rightarrow R_{,\rho\rho} = -(\rho\psi_{0,\rho})_{,\rho} \frac{R}{\rho} - R_{,\rho} \left( \frac{1}{\rho} - \frac{R_{,\rho}}{R} \right)\tag{4.12}$$

imply

$$R_{,\zeta\zeta} + R_{,\rho\rho} = \frac{R_{,\zeta}^2 + R_{,\rho}^2}{R} - \frac{R_{,\rho}}{\rho} - [(\rho\psi_{0,\rho})_{,\rho} + \rho\psi_{0,\zeta\zeta}] \frac{R}{\rho} = -\frac{E \sin \theta}{r \hat{R}_1 \hat{R}_2} = O(E^{-3})e^{2(\gamma_0 - \psi_0)}, \quad (4.13)$$

where we have used the zeroth order equation of motion for  $\psi_0$ . Therefore

$$\partial_\zeta^2 + \partial_\rho^2 = (R_{,\zeta}^2 + R_{,\rho}^2) \frac{d^2}{dR^2} + (R_{,\zeta\zeta} + R_{,\rho\rho}) \frac{d}{dR} = e^{2(\gamma_0 - \psi_0)} \left[ \frac{d^2}{dR^2} + O(E^{-3}) \right] \quad (4.14)$$

in the core of the string. Exterior to the core, the vacuum equations will apply. We now solve (4.4a), (4.4b), and (4.4c) to first order in  $\epsilon$ : namely,

$$\alpha_{1,\zeta\zeta} + \alpha_{1,\rho\rho} = -2\rho e^{2(\gamma_0 - \psi_0)} \left( V(X_0) + \frac{X_0^2 P_0^2}{R^2} \right) = -\rho e^{2(\gamma_0 - \psi_0)} (\mathcal{E}_0 - \mathcal{P}_{0R}) \quad (4.15a)$$

$$\alpha_{1,\zeta}\psi_{0,\zeta} + \alpha_{1,\rho}\psi_{0,\rho} + \rho\psi_{1,\zeta\zeta} + (\rho\psi_{1,\rho})_{,\rho} = \rho e^{2(\gamma_0 - \psi_0)} \left[ \frac{P_0^2}{R^2} - V(X_0) \right] = \frac{1}{2}\rho e^{2(\gamma_0 - \psi_0)} (\mathcal{P}_{0R} + \mathcal{P}_{0\phi}) \quad (4.15b)$$

$$\gamma_{1,\zeta\zeta} + \gamma_{1,\rho\rho} + 2\psi_{0,\rho}\psi_{1,\rho} + 2\psi_{0,\zeta}\psi_{1,\zeta} = -e^{2(\gamma_0 - \psi_0)} \hat{T}_\phi^\phi = e^{2(\gamma_0 - \psi_0)} \mathcal{P}_{0\phi} \quad (4.15c)$$

where  $\mathcal{E}$  and the  $\mathcal{P}$ 's are given by (2.10).

We first solve for  $\alpha_1$ . Note that there is an  $\alpha_0 = \rho$  in the  $\sqrt{-g}$  on the RHS of (4.15a). This suggests that we write

$$\alpha_1 = \rho a(R). \quad (4.16)$$

$a(R)$  then satisfies

$$a''(R) + \frac{2}{R}a'(R) = -[\mathcal{E}_0 - \mathcal{P}_{0R}]. \quad (4.17)$$

Thus

$$\begin{aligned} a(R) &= -\int \frac{1}{R^2} \int R^2 [\mathcal{E}_0 - \mathcal{P}_{0R}] dR \\ &= -\int R [\mathcal{E}_0 - \mathcal{P}_{0R}] dR + \frac{1}{R} \int R^2 [\mathcal{E}_0 - \mathcal{P}_{0R}] dR; \end{aligned} \quad (4.18)$$

this is readily seen to have the asymptotic form

$$a(R) \sim -\frac{A}{\epsilon} + \frac{B}{\epsilon R} \quad (4.19)$$

[where  $A, B$  are given by (2.16)] and solves the vacuum equations.

Setting  $\psi_1 = \psi_1(R)$  and using the form of  $\alpha_1$  given by (4.16) and (4.18), we see that (4.15b) becomes

$$\psi_1''(R) + \frac{1}{R}\psi_1'(R) = \frac{1}{2}(\mathcal{P}_{0R} + \mathcal{P}_{0\phi}), \quad (4.20)$$

which is solved by

$$\psi_1 = \frac{1}{2} \int \frac{1}{R} \int R (\mathcal{P}_{0R} + \mathcal{P}_{0\phi}) = \frac{1}{2} \int R \mathcal{P}_{0R} \quad (4.21)$$

using the zeroth order equations of motion (2.12). Thus  $\psi_1$  tends to a constant ( $C/2\epsilon$ ) at infinity, which is also a vacuum solution.

Finally, setting  $\gamma_1 = \gamma_1(R)$  and using the form of  $\psi_1$  given above, (4.15c) reduces to

$$\gamma_1''(R) = \mathcal{P}_{0\phi} \quad (4.22)$$

and

$$\gamma_1 = \int \int \mathcal{P}_{0\phi} dR = \int R \mathcal{P}_{0R} dR = 2\psi_1. \quad (4.23)$$

Thus the corrections to the metric written in this form are almost identical to the self-gravitating vortex solution. In fact, using these corrections, we see that the asymptotic form of the metric given by

$$\begin{aligned} ds^2 &\rightarrow e^C [e^{2(\psi_0)} dt^2 - e^{2(\gamma_0 - \psi_0)} (dr_c^2 + dz^2)] \\ &\quad - r_c^2 \left( 1 - A + \frac{B}{\sqrt{\lambda\eta r_c} e^{-\psi_0}} \right)^2 e^{-C} e^{-2\psi_0} d\phi^2 \end{aligned} \quad (4.24)$$

in the Weyl metric or

$$\begin{aligned} e^C \left[ \left( 1 - \frac{2GM}{r_s} \right) dt^2 - \left( 1 - \frac{2GM}{r_s} \right)^{-1} dr_s^2 - r_s^2 d\theta^2 \right] \\ - r_s^2 \left( 1 - A + \frac{B}{\sqrt{\lambda\eta r_s} \sin \theta} \right)^2 e^{-C} \sin^2 \theta d\phi^2 \end{aligned} \quad (4.25)$$

in the Schwarzschild metric. Note that although the  $B$  term appears to distort the event horizon,  $B/\sqrt{\lambda\eta r_s} = O(G_\mu) \times O(E^{-1})$ , and hence represents an effect outside the regime of applicability of our approximation. We therefore drop this term, rescale the metric so that time asymptotically approaches proper time at infinity,  $\hat{t} = e^{C/2} t$ , etc., and setting  $\hat{M} = e^{C/2} M$ , we have

$$ds^2 = \left(1 - \frac{2G\hat{M}}{\hat{r}_s}\right) dt^2 - \left(1 - \frac{2G\hat{M}}{\hat{r}_s}\right)^{-1} d\hat{r}_s^2 - \hat{r}_s^2 d\theta^2 - \hat{r}_s^2 (1-A)^2 e^{-2C} \sin^2 \theta d\phi^2. \quad (4.26)$$

We thus see that our spacetime is asymptotically locally flat with deficit angle  $2\pi(A+C) = 8\pi G\mu$ . Thus, by using a physical vortex model, we have confirmed the results of AFV. However, note that the presence of the radial pressure term  $e^{-C}$  has modified the Schwarzschild mass parameter at infinity to  $\hat{M} = e^{C/2}M$ . The gravitational mass of the black hole has therefore shifted to

$$M_g = \hat{M} = e^{C/2}M. \quad (4.27)$$

The inertial mass of the black hole, or its internal energy, can be found by considering the black hole as being formed by a spherical shell of matter infalling from infinity. Because of deficit angle, this has mass

$$M_I = \hat{M}(1-A)e^{-C} = M_g(1-4G\mu); \quad (4.28)$$

thus, the inertial mass of the black hole is actually less than its gravitational mass. However, since we cannot accelerate the black hole without accelerating the string, it is perhaps more correct to refer to this as the internal energy of the black hole. We conclude this section on the gravitating string-black hole system by remarking on the thermodynamics of the system.

Either by Euclideanization or by considering the wave function of a quantum field propagating on the black hole background, one can see that the temperature of the black hole is  $T = \tilde{\beta}^{-1} = 1/8\pi GM_g$ . We denote the thermodynamic quantity  $T^{-1}$  as  $\tilde{\beta}$  to distinguish it from the Bogomolnyi parameter; additionally, we have set the Boltzmann constant  $k$  to unity. Such the spacetime is no longer asymptotically flat, Euclidean arguments must be interpreted with care; nonetheless, by a somewhat non-rigorous partition function calculation, we confirm the AFV result that the entropy of the string black hole system is

$$S = \frac{\tilde{\beta}^2}{16\pi G} (1-A)e^{-C} = \frac{1}{4G}A. \quad (4.29)$$

Thus, although the temperature of the black holes is unchanged in terms of the gravitational mass measured at infinity and although the area-entropy relationship is unchanged, since the internal and gravitational masses are no longer equal, the entropy of the black hole with the string is less than that of a black hole of the same temperature (i.e., gravitational mass) without the string.

It is interesting to use these thermodynamical results to examine the dynamical situation of a cosmic-string-black-hole merger. If one demands that the gravitational mass of the black hole is fixed, then the temperature of the black hole remains unchanged, but its entropy decreases. If one demands conservation of internal energy, then the temperature decreases and the entropy increases. Clearly, thermodynamics indicates that conservation of internal energy is the correct condition to use. It is interesting to note that in this case the change

in gravitational mass is

$$\delta M_g = M_g - M_I = 2 \times 2GM_g \times \mu; \quad (4.30)$$

i.e., the change in gravitational mass is equivalent to the length of string swallowed up by the black hole as seen from infinity times its energy per unit length. In this sense, the vortex at infinity is direct hair, conveying exact information as to the last  $4GM\mu$  units of matter that the black hole swallowed. We will take up this theme further in the next section.

## V. SUMMARY AND DISCUSSION

In this paper we have provided evidence, both analytical and numerical, that  $U(1)$  Abelian Higgs vortices can pierce a black hole horizon. We have shown that there is no gravitational obstruction to this solution, and at this point it is perhaps worthwhile detailing how our solution avoids the revamped Abelian Higgs no-hair theorem [30]. A simple answer would be that the string system is not spherically symmetric; however, many of the steps in [30] can be generalized to include more generic situations. Indeed, recent interesting results of Ridgway and Weinberg [31], who show nonspherically symmetric dressing (although not hair) of black hole event horizons, indicate that spherical symmetry should not be a prerequisite of no-hair theorems. The main reason our solution evades such a no-hair ‘‘proof’’ is that a vortex mandates a nonzero spatial gauge field, which then destroys the inequalities on which the no-hair results are based. In particular, the fact that the field  $P_\mu$  has lines of singularity (corresponding to the vortex cores) explicitly breaks the argument given in [30] as to the vanishing of  $P_i$ .

But is the vortex dressing on hair? The thermodynamical argument seems to indicate that it is hair, telling us about the last  $4GM\mu$  units of mass the black hole swallowed. Can we make this argument stronger? Suppose instead of considering a vortex threading a black hole we consider a single vortex terminating on a black hole. This has a gravitational counterpart in the guise of a uniformly accelerating black hole connected to infinity by a conical singularity [32]; therefore, we can ask whether there exists a particle physics vortex counterpart to this setup. One immediate difference with the previous situation is that the metric here is nonstatic; however, that can be remedied by introducing a second black hole attached to infinity by a second string placed so that its gravitational attraction neutralizes the uniform acceleration [33]. This leaves us with the topological question of how to paint a single semi-infinite vortex onto a black hole event horizon. Recall from Sec. II, when the transformation to the real variables  $X$  and  $P_\mu$  was performed, that the phase of the Higgs field,  $\chi$ , was purely gauge and only acquired physical significance via boundary conditions on  $P_\mu$ . It was remarked that if spatial topology was trivial, any surface spanning a loop with a winding of  $\chi$  would have a vortex on it. If the space is nontrivial, then the question of whether any surface spanning such a loop must have a vortex reduces to a question of topology: If the

first Chern class of the  $U(1)$  bundle is trivial, then any spanning surface must have a vortex.

Recall that in Kruskal coordinates the extended Schwarzschild spacetime contains a wormhole: the  $t = \text{const}$  surface. This has topology  $S^2 \times \mathbb{R}$  with two asymptotically flat regions. Thus the spatial topology of the Schwarzschild black hole is nontrivial and is homotopically equivalent to a sphere. The issue of whether a vortex can terminate on a black hole therefore reduces to that of placing vortices on two-spheres, a well-studied problem (see [34] and references therein). In our case, the answer is to take two gauge patches, e.g.,

$$I_1 = \{\theta, \phi | \theta < \pi/2 + \delta\},$$

$$I_2 = \{\theta, \phi | \theta > \pi/2 - \delta\},$$

for any  $\delta < \pi/2$ . Then define the (gauge) transition function

$$g_{12} = e^{i\phi},$$

such that

$$\Phi_1 = g_{12}\Phi_2, \quad A_{1\mu} = A_{2\mu} - \frac{g_{12}^{-1}}{e}\partial_\mu g_{12},$$

on the overlap, and we take  $\Phi_2 = A_2 = 0$ . Thus we have a vacuum on the southern hemisphere and a vortex on the northern hemisphere connected via a nonsingular gauge transformation on the overlap. Of course, if this two-sphere could be shrunk to zero radius, then this would not be an allowed gauge transformation, but since the two-spheres in the Schwarzschild spacetime have a minimum radius  $2GM$ , there is no topological obstruction to this definition, and we can therefore have just a single vortex connected to the black hole. In terms of the extended Schwarzschild spacetime, this vortex enters the black hole via the North Pole, goes down the wormhole, and emerges from the North Pole of the black hole in the other asymptotic regime. The string world sheet itself looks like a two-dimensional black hole, but occupies only the  $\theta = 0$  portion of the full four-dimensional Penrose diagram. We have not verified that the Nielsen-Olesen solution can be painted on to the nonstatic accelerating black hole spacetime; however, based on the static evidence and the lack of a topological obstruction, it would be very surprising if it could not be.

This now leaves us with the question of how a black hole might have got just a single semi-infinite vortex in the first place. Certainly, it cannot happen as the result of interaction between an infinite vortex and a black hole, and so let us consider what the presence of the

vortex actually means. When a single vortex is present on the two-sphere, more than one gauge patch is necessary for a nonsingular description of the physics. This is analogous to the Wu-Yang [35] description of a Dirac monopole. Indeed, given that in each case we are dealing with the same mathematical object [a  $U(1)$  bundle over the sphere], the only real difference between the two cases is the spontaneously broken symmetry. Thus the interpretation of the vortex is that it is localized “magnetic” flux emanating from the black hole. In terms of the dynamics of phase transitions in the early universe, one is led to a picture of a magnetically Reissner-Nordström black hole prior to the phase transition having its flux localized in the vortex after the phase transition. Thus the information (namely, “magnetic” charge), which one would not normally expect to be able to measure corresponding as it does to a massive field, is indeed preserved for external observers to see in the form of the long vortex hair stretching to infinity. We can correspondingly imagine a charge-2 Reissner-Nordström hole becoming a Schwarzschild hole with two vortices extending to infinity from its opposite poles, which would then be of the AFV form described in Sec. IV, where it was the energy momentum rather than the orientation of the vortices that was relevant.

In the light of this evidence, we claim that the Abelian Higgs vortex is not simply dressing of the black hole, as the  $SU(2)$  monopole is, but is true hair, carrying information from the black hole to infinity.

*Note added in proof.* After this work was completed, we were informed that Eardley *et al.* [36] had also developed the gauge patch description (Sec. V) in order to argue the instability of NO vortices to black hole nucleation. Additionally, the conjecture in Sec. V that the NO vortex could be painted on to the nonstatic  $C$  metric has since been verified in [37].

## ACKNOWLEDGMENTS

We have benefited from discussions with many colleagues, in particular Fay Dowker, Iñigo Egusquiza, Gary Horowitz, Patricio Letelier, Nick Manton, and Bernd Schroers. We wish to thank the Isaac Newton Institute and the University of Utrecht for their hospitality. This work was partially supported by the Isaac Newton Institute, by NSF Grant No. PHY-9309364, CICYT Grant No. AEN-93-1435, and University of the Basque Country Grant No. UPV-EHU 063.310-EB119/92.

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