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ARAKELOV INVARIANTS OF RIEMANN SURFACES

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Abstract. We derive explicit formulas for the Arakelov-Green function and the Faltings delta-invariant of a Riemann surface. A numerical example illustrates how these formulas can be used to calculate Arakelov invariants of curves.

1. Introduction

The Arakelov-Green function and Faltings' delta-invariant are fundamental invariants attached to Riemann surfaces [3], [8]. However, they are defined in a quite implicit way. It is therefore natural to ask for explicit formulas for these invariants, and indeed in many cases such explicit formulas are known. For example, in [8] the case of elliptic curves is treated in detail, and in [4], [5] we find explicit results dealing with the case of Riemann surfaces of genus 2. In higher genera there only seem to be some scattered results; for example, [11] treats a certain family of plane quartic curves, and in [1], [17] the modular curves $X_0(N)$ are studied from the point of view of Arakelov theory. The purpose of the present note is to give general explicit formulas for the Arakelov-Green function and the delta-invariant that make it possible to calculate these invariants efficiently. We have included an explicit numerical example at the end of this note to illustrate the use of our formulas for computations.

We now describe our results. Let X be a compact and connected Riemann surface of genus $g > 0$. We recall from [3] and [8] that X carries a canonical (1,1)-form μ, giving rise to a Green-function $G : X \times X \to \mathbb{R}_{\geq 0}$ and a canonical structure of metrised line bundle on the holomorphic cotangent bundle Ω^1_X and the line bundles $O_X(D)$ associated to a divisor D on X. The line bundle $O(\Theta)$ on $\mathrm{Pic}_{g-1}(X)$ admits a metric $\| \cdot \|_\Theta$ with $\|s\|_\Theta = \|\vartheta\|$, where s is the canonical section of $O(\Theta)$ and where $\|\vartheta\|$ is the function defined on [8], p. 401.

Our first result deals with the Arakelov-Green function G. It has been observed by some authors (see the remarks on [14], p. 229) that for a generic point $P \in X$ there exists a constant $c = c(P)$ depending only on P such that for all $Q \in X$ we have $G(P,Q)^g = c(P) \cdot \|\vartheta\|(gP - Q)$. Our contribution is that we make the dependence on P of the constant $c(P)$ clear. Our formula involves the divisor $\mathcal{W}$ of Weierstrass points on X. Recall that

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this is a divisor of degree $g^3 - g$ on X, given as the divisor of a Wronskian differential formed out of a basis of the holomorphic differentials $H^0(X, \Omega_X^1)$. For each point $P \in X$, the multiplicity of P in $\mathcal{W}$ is given by a weight $w(P)$, that can also be calculated by means of the classical gap sequence at P.

Let $S(X)$ be the invariant defined by the formula

$$\log S(X) := - \int_X \log \|\vartheta\|(gP - Q) \cdot \mu(P)$$

for any $Q \in X$. We will see later that the integrand has logarithmic singularities only at the Weierstrass points of X, which are integrable. Hence the integral is well-defined. Also we will see later that the integral does not depend on the choice of Q. As an example, consider the case $g = 1$ and write $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$, with τ in the complex upper half plane. A calculation (see for example [15], p. 45) shows that in this case

$$\log S(X) = - \log((\mathrm{Im}\tau)^{1/4} |\eta(\tau)|) \,,$$

where $\eta(\tau)$ is the usual Dedekind eta-function given by $\eta(q) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$, with $q = \exp(2\pi i \tau)$.

The invariant $S(X)$ appears as a normalisation constant in the formula that we propose for the Arakelov-Green function.

Theorem 1.1. *Let $P, Q \in X$ with P not a Weierstrass point. Then the formula*

$$G(P,Q)^g = S(X)^{1/g^2} \cdot \frac{\|\vartheta\|(gP - Q)}{\prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{1/g^3}}$$

holds. Here the Weierstrass points are counted with their weights.

For P a Weierstrass point, and $Q \neq P$, both numerator and denominator in the formula of Theorem 1.1 vanish with order $w(P)$, the weight of P. The formula remains true also in this case, provided that we take the leading coefficients of the appropriate power series expansions about P in both numerator and denominator. Note that apart from the normalisation term involving $S(X)$, the Arakelov-Green function can be expressed in terms of certain values of the $\|\vartheta\|$-function. These values are very easy to calculate numerically. The (real) 2-dimensional integral involved in computing $S(X)$ is harder to carry out in general, but it is still not difficult.

Other ways of expressing the Arakelov-Green function in terms of quantities associated to X and μ have been given, for instance one might use the eigenvalues and eigenfunctions of the Laplacian (see [8], Section 3), or one might use abelian differentials of the second and third kind (see [15], Chapter II). There is also a closed formula due to Bost [4]

$$\log G(P,Q) = \frac{1}{g!} \int_{\Theta + P - Q} \log \|\vartheta\| \cdot \nu^{g-1} + A(X) \,,$$

expressing the Arakelov-Green function in terms of an integral over the translated theta divisor. Here ν is the canonical translation-invariant (1,1)-form on $\mathrm{Pic}_{g-1}(X)$, and the quantity $A(X)$ is a certain normalisation constant, perhaps comparable to our $S(X)$.

One of our motives for finding a new explicit formula was the need to have a formula that makes the efficient calculation of the Arakelov-Green function possible. The other approaches that we mentioned are perhaps less suitable for this objective. For instance, the formula given by Bost involves a (real) $2g-2$-dimensional integral over a region which seems not easy to parametrise. Also, for each new pair of points (P, Q) one has to calculate such an integral again, whereas in our approach one only has to calculate a certain integral once.

Our second result deals with Faltings' delta-invariant $\delta(X)$. It is the constant appearing in the following theorem, due to Faltings (*cf.* [8], p. 402).

Theorem 1.2. *(Faltings) There is a constant $\delta = \delta(X)$ depending only on X such that the following holds. Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega_X^1)$ provided with the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$. Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$\|\vartheta\|(P_1 + \cdots + P_g - Q) = \exp(-\delta(X)/8) \cdot \frac{\|\det \omega_k(P_l)\|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} \cdot \prod_{k=1}^{g} G(P_k, Q)$$

holds.

The significance of the delta-invariant is that it appears as an archimedean contribution in the so-called Noether formula [8], [18] for arithmetic surfaces. When viewed as a function on the moduli space $\mathcal{M}_g$ of curves of genus g, the value $\delta(X)$ can be seen as the minus logarithm of the distance of the class of X to the Deligne-Mumford boundary of $\mathcal{M}_g$. This interpretation is supported by the Noether formula.

Let $\Phi : X \times X \to \mathrm{Pic}_{g-1}(X)$ be the map sending (P, Q) to the class of $(gP - Q)$. For a fixed $Q \in X$, let $i_Q : X \to X \times X$ be the map sending P to (P, Q), and put $\phi_Q := \Phi \cdot i_Q$. Define the (fractional) line bundle L_X by

$$L_X := \left(\bigotimes_{W \in \mathcal{W}} \phi_W^* \left(O(\Theta) \right) \right)^{\otimes (g-1)/g^3} \otimes_{O_X} \left(\Phi^*(O(\Theta))|_{\Delta_X} \otimes_{O_X} \Omega_X^{\otimes g} \right)^{\otimes -(g+1)} \otimes_{O_X}$$
$$\otimes \left(\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right) \right)^{\vee} \Big)^{\otimes 2} .$$

We have then the following theorem.

Theorem 1.3. *The line bundle L_X is canonically trivial. Let $T(X)$ be the norm of the canonical trivialising section of L_X. Then the formula*

$$\exp(\delta(X)/4) = S(X)^{-(g-1)/g^2} \cdot T(X)$$

holds.

Despite appearances to the contrary, the invariant $T(X)$ admits a very concrete description. In Proposition 2.7 below we will see that the computation of $T(X)$ only involves elementary operations on special values of $\|\vartheta\|$ and of the $\|J\|$-function, a function introduced by Guàrdia in [10]. These special values are easy to calculate numerically. The significance of Theorem 1.3 is then that we have reduced the calculation of $\delta(X)$ to the calculation of two new invariants $S(X)$ and $T(X)$, the former involving a (real) 2-dimensional integral over the surface X, the latter being elementary to calculate.

It seems an important problem to relate the invariants $S(X)$ and $T(X)$ to more classical invariants. In Theorem 2.8 we state a result that does this for $T(X)$ with X a hyperelliptic Riemann surface.

The plan of this note is as follows. First in Section 2 we give the proofs of Theorems 1.1 and 1.3. The major idea will be to give Arakelov-theoretical versions of classical results on the divisor of Weierstrass points. In Section 3 we will give some applications of our results in the Arakelov intersection theory of arithmetic surfaces. We derive a lower bound for the self-intersection of the relative dualising sheaf, and we give a formula for the self-intersection of a point. In Section 4 we give a numerical example in the spirit of [5], calculating the Arakelov invariants of an arithmetic surface associated to a certain hyperelliptic curve of genus 3 and defined over $\mathbb{Q}$.

Our inspiration to study Weierstrass points in order to obtain results in Arakelov theory stems from the papers [2], [6] and [14]. Especially the latter paper has been useful. For example, our formula for the delta-invariant in Theorem 1.3 is closely related to the formula from Theorem 2.6 of that paper. Our improvement on that formula is perhaps that we give an explicit splitting of the delta-invariant in a new invariant $S(X)$ involving an integral, and a new invariant $T(X)$ which is purely 'classical'. These invariants seem to be of interest in their own right, *cf.* also our remarks at the end of Section 2.

2. PROOFS

We start by recalling the definitions of the (1,1)-form μ, the Arakelov-Green function G and the canonical metric on Ω^1_X. The (1,1)-form μ is given by $\mu = \frac{i}{2g}\sum_{k=1}^{g} \omega_k \wedge \overline{\omega}_k$, where $\{\omega_1, \ldots, \omega_g\}$ is an orthonormal basis of the holomorphic differentials $H^0(X, \Omega^1_X)$ provided with the hermitian inner product $(\omega, \eta) \mapsto \frac{i}{2}\int_X \omega \wedge \overline{\eta}$.

The Arakelov-Green function G is the unique function $X \times X \to \mathbb{R}_{\geq 0}$ such that the following three properties hold:

(i) $G(P,Q)^2$ is C^∞ on $X \times X$ and $G(P,Q)$ vanishes only at the diagonal Δ_X, with multiplicity 1;

(ii) for all $P \in X$ we have $\partial_Q \overline{\partial}_Q \log G(P,Q)^2 = 2\pi i \mu(Q)$ for $Q \neq P$;

(iii) for all $P \in X$ we have $\int_X \log G(P,Q)\mu(Q) = 0$.

These properties imply, by an application of Stokes' theorem, the symmetry $G(P,Q) = G(Q,P)$ of the function G.

The canonical metric $\| \cdot \|_{\mathrm{Ar}}$ on the cotangent bundle Ω^1_X is the unique metric that makes the canonical adjunction isomorphism $O_{X \times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega^1_X$ an isometry, the line bundle $O_{X \times X}(\Delta_X)$ being given the hermitian metric defined by $\|\mathbf{1}_{\Delta_X}\|(P,Q) := G(P,Q)$.

Next let us recall the Wronskian differential that defines the divisor of Weierstrass points on X. For proofs and more details we refer to [12], pp. 120–128. Let $\{\psi_1, \ldots, \psi_g\}$ be a basis of $H^0(X, \Omega^1_X)$. Let P be a point on X and let z be a local coordinate about P. Write $\psi_k = f_k \cdot dz$ for $k = 1, \ldots, g$. The Wronskian determinant about P is then the holomorphic function

$$W_z(\psi) := \det \left(\frac{1}{(l-1)!} \frac{d^{l-1} f_k}{dz^{l-1}} \right)_{1 \leq k,l \leq g} .$$

Let $\tilde{\psi}$ be the $g(g+1)/2$-fold holomorphic differential

$$\tilde{\psi} := W_z(\psi) \cdot (dz)^{\otimes g(g+1)/2} .$$

Then $\tilde{\psi}$ is independent of the choice of the local coordinate z, and extends to a non-zero global section of $\Omega_X^{g(g+1)/2}$. A change of basis changes the Wronskian differential by a non-zero scalar factor, so that the divisor of a Wronskian differential $\tilde{\psi}$ on X is unique: we denote this divisor by $\mathcal{W}$, the divisor of Weierstrass points.

The Wronskian differential leads to a canonical sheaf morphism

$$\left(\wedge^g H^0(X, \Omega^1_X) \otimes_{\mathbb{C}} O_X\right) \longrightarrow \Omega_X^{g(g+1)/2}$$

given by

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi} .$$

This gives a canonical section in $\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega^1_X) \otimes_{\mathbb{C}} O_X)\right)^\vee$ whose divisor is $\mathcal{W}$.

Proposition 2.1. *The canonical isomorphism*

$$\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega^1_X) \otimes_{\mathbb{C}} O_X)\right)^\vee \xrightarrow{\sim} O_X(\mathcal{W})$$

has a constant norm on X.

Proof. This follows since both sides have the same curvature form, and the divisors of the canonical sections are equal. ☐

We shall denote by $R(X)$ the norm of the isomorphism from Proposition 2.1. In more concrete terms we have $\prod_{W \in \mathcal{W}} G(P,W) = R(X) \cdot \|\tilde{\omega}\|_{\mathrm{Ar}}(P)$ for any $P \in X$, where $\{\omega_1, \ldots, \omega_g\}$ is an orthonormal basis of $H^0(X, \Omega_X^1)$, and where the norm of $\tilde{\omega}$ is taken in the line bundle $\Omega_X^{\otimes g(g+1)/2}$ with its canonical metric induced from the canonical metric on Ω_X^1. Taking logarithms and integrating against $\mu(P)$ gives, by property (iii) of the Arakelov-Green function, the formula $\log R(X) = -\int_X \log \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot \mu(P)$.

Recall from the Introduction the map $\Phi : X \times X \to \mathrm{Pic}_{g-1}(X)$ sending (P,Q) to the class of $(gP-Q)$. A classical result on the divisor of Weierstrass points is that the equality of divisors

$$\Phi^*(\Theta) = \mathcal{W} \times X + g \cdot \Delta_X$$

holds on $X \times X$, see for example [9], p. 31. Denote by $p_1 : X \times X \to X$ the projection on the first factor. Using Proposition 2.1, the above equality of divisors yields a canonical isomorphism of line bundles

$$\Phi^*(O(\Theta)) \xrightarrow{\sim} p_1^* \left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^{\vee} \right) \otimes O_{X \times X}(\Delta_X)^{\otimes g}$$

on $X \times X$. We will reprove this isomorphism in the next proposition, and show that its norm is constant on $X \times X$. After Corollary 2.4 to this proposition, the proofs of Theorems 1.1 and 1.3 are just a few lines.

Proposition 2.2. *On $X \times X$, there exists a canonical isomorphism of line bundles*

$$\Phi^*(O(\Theta)) \xrightarrow{\sim} p_1^* \left(\Omega_X^{\otimes g(g+1)/2} \otimes \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^{\vee} \right) \otimes O_{X \times X}(\Delta_X)^{\otimes g} .$$

The norm of this isomorphism is everywhere equal to $\exp(\delta(X)/8)$.

Proof. We are done if we can prove that

$$\exp(\delta(X)/8) \cdot \|\vartheta\|(gP-Q) = \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot G(P,Q)^g$$

for all $P, Q \in X$, where $\{\omega_1, \ldots, \omega_g\}$ is an orthonormal basis of $H^0(X, \Omega_X^1)$. But this follows from the formula in Theorem 1.2, by a computation which is performed in [14], p. 233. Let P be a point on X, and choose a local coordinate z about P. By definition of the canonical metric on Ω_X^1 we have then that $\lim_{Q \to P} |z(Q) - z(P)|/G(Q,P) = \|dz\|_{\mathrm{Ar}}(P)$. Letting $P_1, \ldots, P_g$ approach P in Theorem 1.2 we get

$$\begin{aligned}
\lim_{P_l \to P} \frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} G(P_k, P_l)} &= \lim_{P_l \to P} \left\{ \frac{\| \det \omega_k(P_l) \|_{\mathrm{Ar}}}{\prod_{k<l} |z(P_k) - z(P_l)|} \cdot \frac{\prod_{k<l} |z(P_k) - z(P_l)|}{\prod_{k<l} G(P_k, P_l)} \right\} \\
&= \left\{ \lim_{P_l \to P} \frac{| \det \omega_k(P_l) |}{\prod_{k<l} |z(P_k) - z(P_l)|} \right\} \cdot \|dz\|_{\mathrm{Ar}}^{g+g(g-1)/2}(P) \\
&= |W_z(\omega)(P)| \cdot \|dz\|_{\mathrm{Ar}}^{g(g+1)/2}(P) \\
&= \|\tilde{\omega}\|_{\mathrm{Ar}}(P) .
\end{aligned}$$

The required formula is therefore just a limiting case of Theorem 1.2 where all P_k approach P. $\square$

Corollary 2.3. *The formula* $S(X) = R(X) \cdot \exp(\delta(X)/8)$ *holds.*

Proof. This follows easily by taking logarithms in the formula

$$\exp(\delta(X)/8) \cdot \|\vartheta\|(gP - Q) = \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot G(P,Q)^g$$

and integrating against $\mu(P)$. Here we use again property (iii) of the Arakelov-Green function and the formula $\log R(X) = -\int_X \log \|\tilde{\omega}\|_{\mathrm{Ar}}(P) \cdot \mu(P)$, which was noted above. $\square$

Corollary 2.4. *(1) Let* $Q \in X$. *Then we have a canonical isomorphism*

$$\phi_Q^*(O(\Theta)) \xrightarrow{\sim} O_X(\mathcal{W} + g \cdot Q)$$

of constant norm $S(X)$ *on* X. *(2) We have a canonical isomorphism*

$$(\Phi^*(O(\Theta))|_{\Delta_X}) \otimes_{O_X} \Omega_X^{\otimes g} \xrightarrow{\sim} O_X(\mathcal{W})$$

of constant norm $S(X)$ *on* X.

Proof. We obtain the isomorphism in (1) by restricting the isomorphism from Proposition 2.2 to a slice $X \times \{Q\}$, and using Proposition 2.1. Its norm is then equal to $R(X) \cdot \exp(\delta(X)/8)$, which is $S(X)$ by Corollary 2.3. For the isomorphism in (2) we restrict the isomorphism from Proposition 2.2 to the diagonal, and apply the canonical adjunction isomorphism $O_{X \times X}(-\Delta_X)|_{\Delta_X} \xrightarrow{\sim} \Omega_X^1$. Again we get norm equal to $R(X) \cdot \exp(\delta(X)/8)$, since the adjunction isomorphism is an isometry. $\square$

Note that Corollary 2.4 gives an alternative interpretation to the invariant $S(X)$.

Proof of Theorem 1.1. By taking norms of canonical sections on left and right in the isomorphism from Corollary 2.4 (1) we obtain

$$G(P,Q)^g \cdot \prod_{W \in \mathcal{W}} G(P,W) = S(X) \cdot \|\vartheta\|(gP - Q)$$

for any $P, Q \in X$. Now take the (weighted) product over $Q \in \mathcal{W}$. This gives

$$\prod_{W \in \mathcal{W}} G(P,W)^{g^3} = S(X)^{g^3 - g} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)\,.$$

Plugging this in in the first formula gives

$$G(P,Q)^g \cdot S(X)^{\frac{g^3-g}{g^3}} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{1/g^3} = S(X) \cdot \|\vartheta\|(gP - Q)\,,$$

from which the theorem follows. $\square$

Proof of Theorem 1.3. From Corollary 2.4 (1) we obtain, again by taking the (weighted) product over $Q \in \mathcal{W}$, a canonical isomorphism

$$\left(\bigotimes_{W \in \mathcal{W}} \phi_W^* O(\Theta) \right) \xrightarrow{\sim} O_X(g^3 \cdot \mathcal{W})$$

of norm $S(X)^{g^3-g}$. It follows that we have a canonical isomorphism

$$\left(\bigotimes_{W \in \mathcal{W}} \phi_W^* O(\Theta) \right)^{\otimes (g-1)/g^3} \xrightarrow{\sim} O_X((g-1) \cdot \mathcal{W})$$

of norm $S(X)^{(g-1)(g^3-g)/g^3}$. From Corollary 2.4 (2) we obtain a canonical isomorphism

$$\left((\Phi^*(O(\Theta))|_{\Delta_X}) \otimes_{O_X} \Omega_X^{\otimes g} \right)^{\otimes -(g+1)} \xrightarrow{\sim} O_X(-(g+1)\mathcal{W})$$

of norm $S(X)^{-(g+1)}$. Finally from Proposition 2.1 and Corollary 2.3 we have a canonical isomorphism

$$\left(\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^{\vee} \right)^{\otimes 2} \xrightarrow{\sim} O_X(2\mathcal{W})$$

of norm $S(X)^2 \exp(-\delta(X)/4)$. It follows that indeed the line bundle L_X is canonically trivial, and that its canonical trivialising section has norm

$$S(X)^{-(g-1)(g^3-g)/g^3} \cdot S(X)^{g+1} \cdot S(X)^{-2} \cdot \exp(\delta(X)/4) = S(X)^{(g-1)/g^2} \cdot \exp(\delta(X)/4) \, .$$

By definition this is $T(X)$, so the theorem follows. □

It remains to make clear that the invariant $T(X)$ admits an elementary description in terms of classical functions.

Proposition 2.5. *Let $P \in X$ not a Weierstrass point and let z be a local coordinate about P. Define $\|F_z\|(P)$ as*

$$\|F_z\|(P) := \lim_{Q \to P} \frac{\|\vartheta\|(gP - Q)}{|z(P) - z(Q)|^g} \, .$$

Let $\{\omega_1, \ldots, \omega_g\}$ be an orthonormal basis of $H^0(X, \Omega_X^1)$. Then the formula

$$T(X) = \|F_z\|(P)^{-(g+1)} \cdot \prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)^{(g-1)/g^3} \cdot |W_z(\omega)(P)|^2$$

holds.

Proof. Let F be the canonical section of $(\Phi^*(O(\Theta))|_{\Delta_X}) \otimes \Omega_X^{\otimes g}$ given by the canonical isomorphism in Corollary 2.4 (2). For its norm we have $\|F\| = \|F_z\| \cdot \|dz\|_{\mathrm{Ar}}^g$ in the local coordinate z. The canonical section of $\bigotimes_{W \in \mathcal{W}} \phi_W^* O(\Theta)$ has norm $\prod_{W \in \mathcal{W}} \|\vartheta\|(gP - W)$ at P. Finally, the canonical section of $\Omega_X^{\otimes g(g+1)/2} \otimes_{O_X} \left(\wedge^g H^0(X, \Omega_X^1) \otimes_{\mathbb{C}} O_X \right)^{\vee}$ has norm $\|\tilde{\omega}\|_{\mathrm{Ar}} = |W_z(\omega)| \cdot \|dz\|_{\mathrm{Ar}}^{g(g+1)/2}$. The proposition follows then from the definition of $T(X)$. □

In [10], Guàrdia introduced a function $\|J\|$ on $\mathrm{Sym}^g X$ which involves the first order partial derivatives of the theta function. We claim that it can be used to give a formula for $T(X)$ which is especially well-suited for concrete calculations. Let $\tau \in \mathcal{H}_g$, the Siegel upper half space of complex symmetric $g \times g$-matrices with positive definite imaginary part, be a period matrix associated to X. Consider then the analytic jacobian $\mathrm{Jac}(X) := \mathbb{C}^g/\mathbb{Z}^g + \tau\mathbb{Z}^g$. Then for $w_1, \ldots, w_g \in \mathbb{C}^g$ we put

$$\begin{aligned} J(w_1, \ldots, w_g) &:= \det\left(\frac{\partial\vartheta}{\partial z_k}(w_l)\right), \\ \|J\|(w_1, \ldots, w_g) &:= (\det \mathrm{Im}\tau)^{\frac{g+2}{4}} \cdot \exp(-\pi \textstyle\sum_{k=1}^g {}^t y_k (\mathrm{Im}\tau)^{-1} y_k) \cdot |J(w_1, \ldots, w_g)| . \end{aligned}$$

Here $y_k = \mathrm{Im} w_k$ for $k = 1, \ldots, g$. The latter definition depends only on the classes in $\mathrm{Jac}(X)$ of the vectors w_k. For a set of g points $P_1, \ldots, P_g$ on X we let, under the usual correspondence $\mathrm{Pic}_{g-1}(X) \leftrightarrow \mathrm{Jac}(X)$, the divisor $\sum_{\substack{l=1 \\ l \neq k}}^g P_l$ correspond to the class $[w_k] \in \mathrm{Jac}(X)$ of a vector $w_k \in \mathbb{C}^g$. We then define $\|J\|(P_1, \ldots, P_g) := \|J\|(w_1, \ldots, w_g)$; one may check that this does not depend on the choice of the period matrix τ at the beginning. The following theorem is Corollary 2.6 in [10].

Theorem 2.6. *Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$\|\vartheta\|(P_1 + \cdots + P_g - Q)^{g-1} = \exp(\delta(X)/8) \cdot \|J\|(P_1, \ldots, P_g) \cdot \frac{\prod_{k=1}^g G(P_k, Q)^{g-1}}{\prod_{k<l} G(P_k, P_l)}$$

holds.

Proposition 2.7. *Let $P_1, \ldots, P_g, Q$ be generic points on X. Then the formula*

$$\begin{aligned} T(X) = & \left(\frac{\|\vartheta\|(P_1 + \cdots + P_g - Q)}{\prod_{k=1}^g \|\vartheta\|(gP_k - Q)^{1/g}} \right)^{2g-2} \cdot \\ & \cdot \left(\frac{\prod_{k \neq l} \|\vartheta\|(gP_k - P_l)^{1/g}}{\|J\|(P_1, \ldots, P_g)^2} \right) \cdot \prod_{W \in \mathcal{W}} \prod_{k=1}^g \|\vartheta\|(gP_k - W)^{(g-1)/g^4} \end{aligned}$$

holds.

Proof. The formula follows from Theorem 2.6, using Theorem 1.1 to eliminate the occurring values of the Arakelov-Green function G, and using Theorem 1.3 to eliminate the factor $\exp(\delta(X)/8)$. The factors involving $S(X)$ that are introduced in this way cancel out. □

For example, if $g = 1$ and X is given as $X = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ with τ in the complex upper half plane, we obtain

$$T(X) = (\mathrm{Im}\tau)^{-3/2} \exp(\pi \mathrm{Im}\tau/2) \cdot |\frac{\partial\vartheta}{\partial z}(\frac{1+\tau}{2}; \tau)|^{-2} .$$

By Jacobi's derivative formula we have then

$$T(X) = (2\pi)^{-2} \cdot ((\mathrm{Im}\tau)^6 |\Delta(\tau)|)^{-1/4}$$

where $\Delta(\tau)$ is the discriminant modular form $\Delta(q) := \eta(q)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24}$. It follows that Faltings' delta-invariant is given by

$$\delta(X) = -\log((\mathrm{Im}\tau)^6|\Delta(\tau)|) - 8\log(2\pi)$$

which is well-known, see [8], p. 417.

The formula for $T(X)$ for an elliptic curve X can be generalised to hyperelliptic Riemann surfaces of arbitrary genus. In [13] the following result is proven. For any integer $g \geq 2$, let φ_g be the discriminant modular form on $\mathcal{H}_g$ as defined in [16], Section 3. This is a modular form on $\Gamma_g(2) := \{\gamma \in \mathrm{Sp}(2g,\mathbb{Z}) : \gamma \equiv I_{2g} \mod 2\}$ of weight $4r$, where $r := \binom{2g+1}{g+1}$.

Theorem 2.8. *Let X be a hyperelliptic Riemann surface of genus $g \geq 2$. Choose an ordering of the Weierstrass points on X and a canonical symplectic basis of the homology of X given by this ordering (cf.* [19]*, Chapter IIIa, §5). Let $\tau \in \mathcal{H}_g$ be the period matrix of X associated to this canonical basis and put $\Delta_g(\tau) := 2^{-(4g+4)n} \cdot \varphi_g(\tau)$ where $n := \binom{2g}{g+1}$. Then the formula*

$$T(X) = (2\pi)^{-2g} \cdot ((\mathrm{Im}\tau)^{2r}|\Delta_g(\tau)|)^{-\frac{3g-1}{8ng}}$$

holds.

The proof of Theorem 2.8 is quite complicated, and unfortunately we do not know how to generalise the proof to arbitrary Riemann surfaces of genus g. We leave it as an open question whether in general the invariant $T(X)$ can be naturally expressed in terms of Siegel modular forms on $\mathcal{H}_g$.

3. Applications to intersection theory

In this section we use Proposition 2.1 to give a formula for the relative dualising sheaf on a semi-stable arithmetic surface (Proposition 3.2). As consequences we derive a lower bound for the self-intersection of the relative dualising sheaf (Proposition 3.3) and a formula for the self-intersection of a point (Proposition 3.6).

Let $p : \mathcal{X} \to B$ be a semi-stable arithmetic surface over the spectrum B of the ring of integers in a number field K. We assume that the generic fiber $\mathcal{X}_K$ is a geometrically connected, smooth proper curve of genus $g > 0$. Denote by $\mathcal{W}$ the Zariski closure in $\mathcal{X}$ of the divisor of Weierstrass points on $\mathcal{X}_K$, and denote by $\omega_{\mathcal{X}/B}$ the relative dualising sheaf of p.

The next lemma is an analogue of Lemma 3.3 in [2].

Lemma 3.1. *There exists an effective vertical divisor V on $\mathcal{X}$ such that we have a canonical isomorphism*

$$\omega_{\mathcal{X}/B}^{\otimes g(g+1)/2} \otimes_{O_{\mathcal{X}}} \left(p^*(\det p_*\omega_{\mathcal{X}/B})\right)^{\vee} \xrightarrow{\sim} O_{\mathcal{X}}(V+\mathcal{W})$$

of line bundles on $\mathcal{X}$.

Proof. We have on $\mathcal{X}$ a canonical sheaf morphism $p^*(\det p_*\omega_{\mathcal{X}/B}) \longrightarrow \omega_{\mathcal{X}/B}^{\otimes g(g+1)/2}$ given locally by

$$\xi_1 \wedge \ldots \wedge \xi_g \mapsto \frac{\xi_1 \wedge \ldots \wedge \xi_g}{\psi_1 \wedge \ldots \wedge \psi_g} \cdot \tilde{\psi}$$

for a K-basis $\{\psi_1, \ldots, \psi_g\}$ of the differentials on the generic fiber of $\mathcal{X}$. Multiplying by $(p^*(\det p_*\omega_{\mathcal{X}/B}))^\vee$ we obtain a morphism

$$O_{\mathcal{X}} \longrightarrow \omega_{\mathcal{X}/B}^{\otimes g(g+1)/2} \otimes_{O_{\mathcal{X}}} \left(p^*(\det p_*\omega_{\mathcal{X}/B})\right)^\vee .$$

The image of 1 is a section whose divisor is an effective divisor $V + \mathcal{W}$ where V is vertical. This gives the required isomorphism. □

We will now turn to the Arakelov intersection theory on $\mathcal{X}$. Our references are, once more, [3] and [8]. For a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$ we denote by F_σ the "fiber at infinity" associated to σ. The corresponding Riemann surface of genus g is denoted by X_σ.

Proposition 3.2. *Let V be the effective vertical divisor from Lemma 3.1. Then we have*

$$\frac{1}{2}g(g+1)\omega_{\mathcal{X}/B} = V + \mathcal{W} + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) \cdot F_\sigma + p^*(\det p_*\omega_{\mathcal{X}/B})$$

as Arakelov divisors on $\mathcal{X}$. Here the sum runs over the embeddings of K in $\mathbb{C}$.

Proof. Consider the canonical isomorphism from Lemma 3.1. The restriction of this isomorphism to X_σ is the isomorphism of Proposition 2.1. In particular it has norm $R(X_\sigma)$. The proposition follows. □

We shall deduce two consequences from this proposition. We assume for the moment that $g \geq 2$. We define R_b for a closed point $b \in B$ by the equation $(2g-2) \cdot \log R_b = (V_b, \omega_{\mathcal{X}/B})$, where the intersection is taken in the sense of Arakelov. The assumption that $p : \mathcal{X} \to B$ is semi-stable implies that the quantity $\log R_b$ is always non-negative.

Proposition 3.3. *Assume that $g \geq 2$. Then the lower bound*

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) \geq \frac{8(g-1)}{(2g-1)(g+1)} \cdot \left(\sum_b \log R_b + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_*\omega_{\mathcal{X}/B} \right)$$

holds. Here the first sum runs over the closed points $b \in B$, and the second sum runs over the embeddings of K in $\mathbb{C}$.

Proof. Intersecting the equality from Proposition 3.2 with $\omega_{\mathcal{X}/B}$ we obtain

$$\frac{1}{2}g(g+1)(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B}) =$$

$$= (\mathcal{W},\omega_{\mathcal{X}/B}) + (2g-2)\cdot\left(\sum_b \log R_b + \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log R(X_\sigma) + \widehat{\deg}\det p_*\omega_{\mathcal{X}/B}\right).$$

Now since the generic degree of $\mathcal{W}$ is g^3-g we obtain by Theorem 5 of [8] the lower bound

$$(\mathcal{W},\omega_{\mathcal{X}/B}) \geq \frac{g^3-g}{2g(2g-2)}(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B}).$$

Using this in the first equality gives the result. □

One should compare the above lower bound for $(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})$ with the lower bounds for $(\omega_{\mathcal{X}/B},\omega_{\mathcal{X}/B})$ given in [6], Section 3.3. The contributions at infinity $\log R(X_\sigma)$ have properties similar to the terms $A_{k,\sigma}$ occurring in [6]. In particular, the right-hand side of the inequality in Proposition 3.3 may be negative.

We refer to the author's thesis for a proof of the following result.

Proposition 3.4. *Let X_t be a holomorphic family of compact and connected Riemann surfaces of genus $g \geq 2$ over the punctured disk, degenerating to the union of two Riemann surfaces of positive genera g_1, g_2 with two points identified. Suppose that neither of these two points was a Weierstrass point on each of the two separate Riemann surfaces. Then the formula*

$$\log R(X_t) = -\frac{g_1g_2}{2g}\log|t| + O(1) \quad \text{as} \quad t \to 0$$

holds.

In particular, the value $\log R(X_t)$ goes to plus infinity under the conditions described in the theorem. It would be interesting to have a more precise, quantitative version of Proposition 3.4.

Our second result is a formula for the self-intersection of a point. In the proof of the next lemma, we make use of the Deligne bracket (see [7]). This is a rule that assigns to a pair L, M of line bundles on $\mathcal{X}$ a line bundle $\langle L, M\rangle$ on B such that the following properties hold: (i) we have canonical isomorphisms $\langle L_1\otimes L_2, M\rangle \xrightarrow{\sim} \langle L_1, M\rangle\otimes\langle L_2, M\rangle$, $\langle L, M_1\otimes M_2\rangle \xrightarrow{\sim} \langle L, M_1\rangle\otimes\langle L, M_2\rangle$ and $\langle L, M\rangle \xrightarrow{\sim} \langle M, L\rangle$; (ii) for a section $P: B\to\mathcal{X}$ we have a canonical isomorphism $\langle O_{\mathcal{X}}(P), M\rangle \xrightarrow{\sim} P^*M$; (iii) (adjunction formula) for a section $P: B\to\mathcal{X}$ we have a canonical isomorphism $\langle P, \omega_{\mathcal{X}/B}\rangle \xrightarrow{\sim} \langle P, P\rangle^{\otimes -1}$; (iv) (Riemann-Roch) for a line bundle L on $\mathcal{X}$ we have a canonical isomorphism $(\det Rp_*L)^{\otimes 2} \xrightarrow{\sim} \langle L, L\otimes\omega_{\mathcal{X}/B}^{-1}\rangle\otimes(\det p_*\omega)^{\otimes 2}$ relating the Deligne bracket to the determinant of cohomology.

Assume that $g \geq 1$ again.

Lemma 3.5. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then we have a canonical isomorphism*

$$P^*(O_{\mathcal{X}}(V+\mathcal{W}))^{\otimes 2} \xrightarrow{\sim} (\det Rp_* O_{\mathcal{X}}(gP))^{\otimes 2}$$

of line bundles on B.

Proof. Applying Riemann-Roch to the line bundle $O_{\mathcal{X}}(gP)$ we obtain a canonical isomorphism

$$(\det Rp_* O_{\mathcal{X}}(gP))^{\otimes 2} \xrightarrow{\sim} \langle O_{\mathcal{X}}(gP), O_{\mathcal{X}}(gP) \otimes \omega_{\mathcal{X}/B}^{-1} \rangle \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes 2}$$

of line bundles on B. The line bundle at the right hand side is, by the adjunction formula, canonically isomorphic to the line bundle $\langle P, P \rangle^{\otimes g(g+1)} \otimes (\det p_* \omega_{\mathcal{X}/B})^{\otimes 2}$. On the other hand, pulling back the isomorphism from Lemma 3.1 along P and using once more the adjunction formula gives a canonical isomorphism

$$\langle P, P \rangle^{\otimes -g(g+1)/2} \xrightarrow{\sim} \langle V+\mathcal{W}, P \rangle \otimes \det p_* \omega_{\mathcal{X}/B} \, .$$

The lemma follows by a combination of these observations. □

Proposition 3.6. *Let P be a section of p, not a Weierstrass point on the generic fiber. Then $-\frac{1}{2}g(g+1)(P,P)$ is given by the expression*

$$-\sum_{\sigma : K \hookrightarrow \mathbb{C}} \log G(P_\sigma, \mathcal{W}_\sigma) + \log \# R^1 p_* O_{\mathcal{X}}(g \cdot P) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \, ,$$

where σ runs through the complex embeddings of K.

Proof. Intersecting the equality from Proposition 3.2 with P, and using the adjunction formula $(\omega, P) = -(P,P)$, we obtain the equality

$$-\frac{1}{2}g(g+1)(P,P) = (V+\mathcal{W}, P) + \sum_{\sigma : K \hookrightarrow \mathbb{C}} \log R(X_\sigma) + \widehat{\deg} \det p_* \omega_{\mathcal{X}/B} \, .$$

It remains therefore to see that $(V+\mathcal{W}, P)_{\mathrm{fin}} = \log \# R^1 p_* O_{\mathcal{X}}(g \cdot P)$. For this we consider the isomorphism in Lemma 3.5. Note that $p_* O_{\mathcal{X}}(g \cdot P)$ is canonically trivialised by the function 1. This gives a canonical section at the right hand side with norm the square of $\# R^1 p_* O_{\mathcal{X}}(g \cdot P)$. Under the isomorphism, it is identified with the canonical section on the left-hand side, which has norm the square of $\exp((V+\mathcal{W}, P)_{\mathrm{fin}})$. The required equality follows. □

We see that minus the self-intersection of a point P is large if P is close to a Weierstrass point, either in the p-adic or in the complex topology.

4. A NUMERICAL EXAMPLE

In this final section we wish to illustrate the practical significance of our Theorems 1.1 and 1.3 by exhibiting a concrete example dealing with a hyperelliptic curve of genus 3. The propositions below can be proved by methods similar to those in [5], Section 3.

Let K be a number field, and A its ring of integers. Let $F \in A[x]$ be monic of degree 5 with $F(0)$ and $F(1)$ a unit in A. Put $R(x) := x(x-1) + 4F(x)$. Suppose that the following holds for R: the discriminant Δ of R is non-zero; for every prime $\wp$ of residue characteristic $\mathrm{char}(\wp) \neq 2$ of A we have $v_\wp(\Delta) = 0$ or 1; if $\mathrm{char}(\wp) \neq 2$ and $v_\wp(\Delta) = 1$, then $R(\mathrm{mod}\,\wp)$ has a unique multiple root, and its multiplicity is 2.

Proposition 4.1. *The equation*

$$C_F \,:\, y^2 = x(x-1)R(x)$$

defines a hyperelliptic curve of genus 3 over K. It extends to a semi-stable arithmetic surface $p : \mathcal{X} \to B = \mathrm{Spec}(A)$. We have that $\mathcal{X}$ has bad reduction at $\wp$ if and only if $\mathrm{char}(\wp) \neq 2$ and $v_\wp(\Delta) = 1$. In this case, the bad fiber is an irreducible curve with a single double point. The differentials $dx/y, xdx/y, x^2dx/y$ form a basis of the O_B-module $p_*\omega_{\mathcal{X}/B}$. The points W_0, W_1 on C_F given by $x = 0$ and $x = 1$ extend to disjoint σ-invariant sections of p.

As for the Arakelov invariants of C_F, we have the following result.

Proposition 4.2. *At a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$, let $\Omega_\sigma = (\Omega_{1\sigma}|\Omega_{2\sigma})$ be a period matrix for the Riemann surface corresponding to $C_F \otimes_{\sigma,K} \mathbb{C}$, formed on the basis $dx/y, xdx/y, x^2dx/y$. Further, let $\tau_\sigma = \Omega_{1\sigma}^{-1}\Omega_{2\sigma}$. Then*

$$\widehat{\deg} \det p_*\omega_{\mathcal{X}/B} = -\frac{1}{2}\sum_\sigma \log\left(|\det\Omega_{1\sigma}|^2(\det \mathrm{Im}\tau_\sigma)\right) \,,$$

where the sum runs over the complex embeddings of K. Further, the formula

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = 24\sum_\sigma \log G_\sigma(W_0, W_1)$$

holds.

For our example, we choose the polynomial $F(x) = x^5 + 6x^4 + 4x^3 - 6x^2 - 5x - 1$ defined over $\mathbb{Q}$. Then the corresponding $R(x) = x(x-1) + 4F(x)$ satisfies the conditions described above. The corresponding hyperelliptic curve (which we will call X from now on) of genus 3 has bad reduction at the primes $p = 37, p = 701$ and $p = 14717$. An equation is given by

$$X \,:\, y^2 = x(x-1)(4x^5 + 24x^4 + 16x^3 - 23x^2 - 21x - 4)\,.$$

We choose an ordering of the Weierstrass points of X. As in [19], Chapter III, §5 we construct from this a canonical symplectic basis of the homology of (the Riemann surface corresponding to) X. Using Mathematica, we compute the periods of the differentials $dx/y, xdx/y, x^2dx/y$. This leads to an explicit value of $\Omega = (\Omega_1|\Omega_2)$ and the numerical approximation

$$\widehat{\deg} \det p_* \omega_{\mathcal{X}/B} = -1.280295247656532068...$$

Using the Riemann vector given by [19], p. 3.82 we can make the identification $\mathrm{Pic}_2(X) \leftrightarrow \mathbb{C}^3/\mathbb{Z}^3 + \tau\mathbb{Z}^3$ explicit. With Theorem 2.8 we find then the following numerical approximation to $T(X)$:

$$\log T(X) = -4.44361200473681284...$$

The values of the theta function that are needed for this computation are approximated by the defining summation formula, which consists of rapidly decreasing exponential terms. An elementary *a priori* calculation shows how much terms we need to compute in order to approximate a value of the theta function with a prescribed accuracy.

It remains then to calculate the invariant $\log S(X)$. Recall the definition

$$\log S(X) := -\int_X \log \|\vartheta\|(3P - Q) \cdot \mu(P) \, .$$

Note that the integrand diverges at infinity, so we would rather want to make use of the formula

$$\log S(X) = -9 \int_X \log \|\vartheta\|(3P - Q) \cdot \mu(Q) + \frac{1}{3} \cdot \sum_{W \in \mathcal{W}} \log \|\vartheta\|(3P - W) \, ,$$

valid for any $P \in X$ which is not a Weierstrass point. This formula can be easily derived from Theorem 1.1 by taking logarithms and integrating against $\mu(Q)$. The integrand has now only a (logarithmic) singularity at $Q = P$. Write $x = u + iv$ with u, v real. We want to express $\mu(Q)$ in terms of the coordinates u, v. This is done by the following lemma.

Lemma 4.3. *Let h be the 3×3-matrix given by*

$$h = \left(\overline{\Omega}_1 (\mathrm{Im} \tau)^t \Omega_1 \right)^{-1} \, .$$

Then we can write

$$\mu = \left(h_{11} + 2h_{12}u + 2h_{13}(u^2 - v^2) + h_{22}(u^2 + v^2) + 2h_{23}u(u^2 + v^2) + h_{33}(u^2 + v^2)^2 \right) \cdot \frac{dudv}{3|f|}$$

in the coordinates u, v.

Proof. Let $\omega_k = x^{k-1} dx/y$ for $k = 1, 2, 3$. By Riemann's bilinear relations, the fundamental (1,1)-form μ is given by $\mu = \frac{i}{6} \sum_{k,l=1}^{3} h_{kl} \cdot \omega_k \wedge \overline{\omega_l}$. Expanding this expression gives the result, where we note that the matrix h is real symmetric, since our defining equation for X is defined over the reals. □

We can now effectuate the required integral, choosing an arbitrary point P and taking care of the singularity of the integrand at this point P. We find the approximation

$$\log S(X) = 17.57...$$

In order to check this result, we have taken several choices for P. By Theorem 1.3 we have

$$\delta(X) = -33.40...$$

and using Theorem 1.1 we can approximate, by taking $Q = W_1$ and letting P approach W_0,

$$G(W_0, W_1) = 2.33...$$

By Proposition 4.2 we finally find

$$(\omega_{\mathcal{X}/B}, \omega_{\mathcal{X}/B}) = 20.32...$$

The running times of the computations were negligible, except for the computation of the integral involved in $\log S(X)$, which took about 7 hours on the author's laptop.

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