

Reduction to H -functions in Radiative Transfer with a General Anisotropic Phase Function

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The solution of the diffusion problem in an unbounded homogeneous medium, containing the Kušcer polynomials, and the solution of the reflection function against a semi-infinite atmosphere, containing the Busbridge polynomials and the H -function, are reviewed. It is shown that the solution of the Milne problem for non-conservative scattering is expressible in the same polynomials, the characteristic root and the H -function. The derivation is simple with the help of a new integral containing a combination of these polynomials and the H -function. Some additional constants and the transition to conservative scattering are discussed. In spite of the simplification reached in this paper, the reduction to H -functions is not recommended as a practical method if the phase function contains Legendre polynomials of large order.

Key words: radiative transfer — Milne problem — diffuse reflection

1. Problem and History

In this paper we consider some well known radiative transfer problems, all dealing with homogeneous atmospheres with a given, arbitrary, anisotropic phase function and a given, arbitrary, albedo for single scattering. The three main problems, formulated for non-conservative scattering, are:

I. Diffusion: to find a solution of the form

$$I(\tau, u) = e^{-k\tau} P(u) \quad (1)$$

showing how radiation diffuses through an unbounded medium in the positive τ -direction while maintaining its angular pattern. Here I = specific intensity, τ = optical depth, u = cosine of angle with positive τ -direction, $P(u)$ = diffusion pattern, k = diffusion exponent, or root of characteristic equation, $0 < k < 1$. If, apart from the corresponding diffusion stream in opposite direction obtained by a trivial change of sign of τ and u , several such solutions exist, we consider only the one with smallest k .

II. Reflection: to find the symmetric function $R(\mu, \mu_0)$ which expresses the azimuth-independent term of the intensity reflected in direction $-\mu$ from a semi-infinite atmosphere exposed to incident radiation in direction μ_0 . In neutron scattering this is called the albedo problem for a halfspace. Here $-\mu$ and μ_0 are direction cosines with regard to the inward normal.

III. Escape: to find the function $mK(\mu)$ which expresses the intensity emerging in direction $u = -\mu$ ($0 \leq \mu \leq 1$) from a semi-infinite atmosphere in which the radiation field at large τ asymptotically has the form of an outward diffusion stream (as in problem *I* with signs reversed). Neutron physicists call this the Milne problem for a halfspace; in astrophysics the term Milne problem tends to be reserved for the corresponding problem in conservative scattering only.

The normalization conventions in all three problems adopted here will be the same as in van de Hulst (1968a). In particular it may be noted that $R(\mu, \mu_0)$ is $(4\mu\mu_0)^{-1}$ times the "scattering function" used by Chandrasekhar (1950) and Busbridge (1960) and is the same as the "brightness function" used in Sobolev's papers. All three problems can also be formulated for conservative scattering. Problem *I* then has the trivial solution $k = 0$; *II* keeps the same formulation, while *III* has to be reformulated for a linear asymptotic dependence on τ .

It is well known that, if the phase function can be expressed in a finite number of Legendre polynomials

$$\Phi(\theta) = \sum_{n=0}^N \omega_n P_n(\cos\theta) \quad (2)$$

both $R(\mu, \mu_0)$ and $K(\mu)$ can be expressed in terms

of one transcendental function $H(\mu)$. The additional factors contain polynomials of degree N , the coefficients of which have to be found separately. The purpose of this paper is to derive simple, general interrelations for these polynomials.

The history of this subject shows many successfully completed problems but also many publications in which the authors obviously got lost (or at least fed up) in a forest of algebra. Chandrasekhar, extending the H -functions introduced by Ambartsumian for isotropic scattering, solved in his book (1950) the reflection problem for $N=1$ (non-conservative) and for $N=2$ (conservative only). The Milne-problem for $N=1$ (non-conservative) was added as an afterthought in the appendix. In his work every problem was solved from scratch and the coefficients in the polynomials were derived on an ad-hoc basis. Busbridge (1960) presented more complete properties of the required polynomials in the case of general N but not a complete recipe for their solution. Later Horak and Chandrasekhar (1961) completed the solution for the reflection problem for $N=2$ (non-conservative) in a set of equations for which the letters in the alphabet did not suffice. Sobolev (1968b) indicated a simpler method for general N . Busbridge and Orchard (1969) gave a complete recipe for general N in the conservative case, together with some numerical examples for $N=3$. All this time, the complete problem had in essence been solved in a paper by Kuščer (1958), which may have escaped attention because of its concise presentation. I am grateful to Professor Sobolev for pointing this out to me.

It is certain that a systematic treatment of these problems by the van Kampen-Case method based on complete eigenfunction expansions also generate the H -functions and the Busbridge polynomials. Case and Zweifel (1967) treat isotropic scattering and some simple examples of anisotropic transfer, based on the work of Mika and others. Formulations for general anisotropic scattering were presented by McCormick and Kuščer (1966) and in practical form by Shultis and Kaper (1969) and in full detail in Kaper, Shultis and Veninga (1970). Among the extensions of this theory given by Pahor (1966, 1967) is a complete recipe for determining the Busbridge polynomials. The formulae presented below could certainly be derived along that route, but those familiar with the Chandrasekhar-Busbridge approach would probably regard this a detour.

The main source of trouble in the traditional derivations is the necessity to use at the right point

non-linear relations between the moments of the H -function, which in turn may be derived from the non-linear equations satisfied by the H -functions themselves. Even for N as small as 1 or 2 this may lead to a real tangle, as an inspection of the papers cited may show.

Much work has been done on the H -functions themselves. The characteristic function $\Psi(\mu)$, in terms of which the H -function is derived, can be found from the coefficients ω_n in a manner first derived by Kuščer (1955). A similar recipe is found in Mullikin (1964a, b). See also Busbridge (1967) and Sobolev (1968b).

2. Kuščer Polynomials and Busbridge Polynomials

Two sets of polynomials, apart from the Legendre polynomials $P_n(u)$, occur in this subject. Let $h_n = 2n + 1 - \omega_n$. The polynomials $g_j(x)$ were defined by Kuščer (1955) by the recursion formula

$$g_0(x) = 1, \quad g_1(x) = h_0 x, \quad (3)$$

$$(j+1)g_{j+1}(x) + jg_{j-1}(x) = h_j x g_j(x).$$

They are called the Kuščer polynomials; it may be noted, however, that the same polynomials had already been introduced by means of a different equation in the work of Sobolev (1949). The degree of $g_j(x)$ in the non-conservative case ($h_0 \neq 0$) is j and the coefficients are rational non-linear expressions in ω_n , which are easily found by means of the recurrence relations (3). These polynomials occur in the solution of problem *I* of Section 1. The diffusion pattern defined there has the form

$$P(u) = \sum_{n=0}^{\infty} (2n+1) g_n(\gamma) P_n(u) \quad (4)$$

which may also be transformed into

$$P(u) = \frac{1}{1-ku} \sum_{n=0}^N \omega_n g_n(\gamma) P_n(u). \quad (5)$$

Here $\gamma = k^{-1}$ is the largest root of the characteristic equation $T(z) = 0$, where $T(z)$ is a function defined below in Eq. (16). If this root has been determined, problem *I* is completely solved by either Eq. (4) or (5). For small N the form (5) is usually preferred because it has only $N+1$ terms.

The polynomials $q_n(\mu)$ are defined by

$$q_n(\mu) = \frac{1}{H(\mu)} \left\{ P_n(\mu) + 2\mu (-1)^n \int_0^1 R(\mu, \nu) P_n(\nu) d\nu \right\}. \quad (6)$$

It is historically fair to call these the Busbridge polynomials because Busbridge (1960, p. 124) proved what could be conjectured from Chandrasekhar's examples, namely that the functions defined by (6) are indeed polynomials of degree $\leq N$. Please note that our definition of $q_n(\mu)$ differs from that adopted in Busbridge's book by a factor $(-1)^n$ but agrees with that adopted by Pahor (1966, 1967) and Sobolev (1968b) and with the definition in Busbridge's later papers. The coefficients of these polynomials cannot be found in a simple manner. Their determination generally requires first the determination of the moments of the *H*-function and even then the recipe is not simple.

The polynomials thus introduced may be combined into two polynomial expressions of two variables, which play a somewhat parallel role in the following derivation. These are

$$G(\nu, \mu) = \sum_{n=0}^N \omega_n g_n(\nu) P_n(\mu) \tag{7}$$

and

$$F(\nu, \mu) = \sum_{n=0}^N \omega_n (-1)^n q_n(\nu) q_n(\mu) \tag{8}$$

We have already seen that $G(\nu, \mu)$ occurs in the solution of the diffusion problem, Eq. (5). It also is a key function in Mullikin's and Sobolev's papers and in the Case method. The function $F(\nu, \mu)$ occurs in the reflection function

$$R(\mu, \nu) = \frac{H(\mu) H(\nu) F(\mu, \nu)}{4(\mu + \nu)}, \tag{9}$$

which is again a well known equation (Ambartsumyan, 1943; Busbridge, 1960, Eq. (48.51)).

Both functions defined by (7) and (8) lead to the characteristic function $\Psi(\mu)$, as follows:

$$\frac{1}{2} G(\mu, \mu) = \frac{1}{2} F(-\mu, \mu) = \Psi(\mu), \tag{10}$$

again a combination of known results. The characteristic function is even; $P_n(\mu)$ and $g_n(\mu)$ are even or odd, according as n is even or odd.

3. A New Integral Relation

The two sets of polynomials obey the following integral equations

$$q_n(\mu) = \frac{P_n(\mu)}{H(\mu)} + \frac{1}{2} \mu (-1)^n \int_0^1 \frac{F(\mu, \nu) H(\nu) P_n(\nu) d\nu}{\mu + \nu}, \tag{11}$$

$$g_n(z) = \frac{q_n(z)}{H(-z)} - \frac{1}{2} z \int_0^1 \frac{G(z, \nu) H(\nu) q_n(\nu) d\nu}{\nu - z}, \tag{12}$$

which have a remarkable similarity. Equation (11) emerges upon inserting the expression (9) for the reflection function into the defining Eq. (6); it is a transcription of Equation (48.4) of Busbridge (1960). Equation (12) is less obvious. Mullikin (1964b) derives a linear integral equation for the function

$$\psi_n(z) = H(z) q_n(z), \tag{13}$$

which in our notation, and reduced to the case of a semi-infinite atmosphere and the azimuth-independent term ($m = 0$), reads

$$T(z) \psi_n(z) = g_n(z) + \frac{1}{2} z \int_0^1 \frac{G(z, \nu)}{\nu - z} \psi_n(\nu) d\nu. \tag{14}$$

The same equation is found in Sobolev (1968b, Eq. (27)).

Here

$$T(z) = 1 - 2z^2 \int_0^1 \frac{\Psi(\mu) d\mu}{z^2 - \mu^2}. \tag{15}$$

It is known from earlier work (Chandrasekhar, 1950; Sec. 39.1) that

$$T(z) = \frac{1}{H(z) H(-z)} \tag{16}$$

so that Eq. (12) simply follows.

In the derivations we invariably meet integrals of the form

$$I(\mu, z) = \int_0^1 \frac{H(x) F(\mu, x) G(-z, x) x dx}{(\mu + x)(z + x)}. \tag{17}$$

In making sample calculations for simple cases it is usually with this type of integral that we are led into the algebraic labyrinth. This integral can, for $z \neq \mu$, surprisingly be reduced to the simple form

$$I(\mu, z) = \frac{2}{z - \mu} \left\{ \frac{G(z, \mu)}{H(\mu)} - \frac{F(\mu, -z)}{H(z)} \right\}. \tag{18}$$

The proof starts with the separation

$$\frac{x}{(\mu + x)(z + x)} = \frac{1}{z - \mu} \left(\frac{z}{z + x} - \frac{\mu}{\mu + x} \right).$$

In the term with denominator $z + x$ we keep $G(-z, x)$ intact and write $F(\mu, x)$ out as (8). In the term with denominator $\mu + x$ we keep $F(\mu, x)$ intact and write $G(-z, x)$ out as

$$G(-z, x) = \sum_{n=0}^N \omega_n (-1)^n g_n(-z) P_n(-x),$$

which is a consequence of (7). A straight application of relations (11) and (12) then leads to the desired result (18).

One additional result may be mentioned, because we shall presently need it. The form $I(-\gamma, -\gamma)$ cannot be reduced to the form (18) because of the equal arguments. But it may be written in the form

$$I(-\gamma, -\gamma) = 4 \Psi(\gamma) H(\gamma) T'(\gamma), \quad (19)$$

where Ψ is the characteristic function, Eq. (10), and T' is the derivative of the function introduced by Eq. (16).

The derivation of Eq. (19), which need not be given in detail, may be performed by reducing $I(-\gamma, -\gamma + \varepsilon)$ by means of (18) and then making the transition $\varepsilon \rightarrow 0$, using Eq. (10) as well as the fact that

$$1/H(-\gamma) = 0 \quad (20)$$

which means that $H(z)$ has a pole at $z = -\gamma$.

The equation equivalent to Eq. (19) for isotropic scattering is traditionally derived from a contour integral, in which the residue of this pole gives rise to the factor $T'(\gamma)$. The more direct way chosen here makes it clear why this derivative enters and why we do not have to worry about the existence of other poles in deriving this particular quantity.

4. The Solution of the Milne Problem

Van de Hulst (1968a) derived in symbolic form the simple relations

$$0 = Q - RP,$$

$$mK = P - RQ$$

which in full notation read:

$$0 = P(-\mu) - \int_0^1 R(\mu, \nu) P(\nu) 2\nu d\nu, \quad (21)$$

$$mK(\mu) = P(\mu) - \int_0^1 R(\mu, \nu) P(-\nu) 2\nu d\nu. \quad (22)$$

These equations describe simple fictitious experiments (van de Hulst and Terhoeve, 1964) and special forms of these equations are found in many places in the literature (e.g. Busbridge, 1960, Eq. (49.7)).

It is obvious that, with $P(\mu)$ known from (5) and $R(\mu, \nu)$ from (9), Eq. (21) can be used for a check and Eq. (22) for a general determination of $mK(\mu)$. Performing the integrations we do indeed meet integrals of type (17). Replacing these by the solution (18), the check of (21) follows at once and

Eq. (22) yields

$$mK(\mu) = \frac{\gamma H(\mu) F(\mu, -\gamma)}{H(\gamma)(\gamma - \mu)} = \frac{H(\mu) F(\mu, -\gamma)}{H(\gamma)(1 - k\mu)}, \quad (23)$$

The factor $H(\mu)/(1 - k\mu)$ in this result is well known (Ambartsumyan, 1944). The added factor is a polynomial. Its form, $F(\mu, -\gamma)/H(\gamma)$, which by (8) is a linear combination of the Busbridge polynomials, was believed to be a new result until, upon completion of this paper, it turned out that the proportionality to $F(\mu, -\gamma)$ had already been derived by Kuščer (1958).

Although the main problem posed is thus solved, it is useful to derive two further constants. The constant m is defined by (van de Hulst, 1968a)

$$m = \int_{-1}^1 \{P(u)\}^2 2u du \quad (24)$$

I find that it can be most conveniently reduced by applying the identity which in symbolic and complete form reads

$$m = (mK) P = \int_0^1 mK(\mu) P(\mu) 2\mu d\mu.$$

Substitution of (5) and (23) then leads again to an integral of form (17), which combined with (18) and (19) gives the result

$$m = 2\gamma^2 I(-\gamma, -\gamma)/H(\gamma) = 8\gamma^2 \Psi(\gamma) T'(\gamma). \quad (25)$$

The constant l , which is related to the extrapolation length q_∞ by

$$l = \exp(-2kq_\infty), \quad (26)$$

and is defined by

$$l = \int_0^1 K(\mu) P(-\mu) 2\mu d\mu, \quad (27)$$

reduces, similarly, to

$$l = \frac{2\gamma F(-\gamma, -\gamma)}{m \{H(\gamma)\}^2}. \quad (28)$$

Finally, just like Eq. (6) suffices to express the various moments of the reflection function, the moments of the escape function, for instance, the density and flux at the surface, may most easily be read from

$$\int_0^1 mK(\mu) P_n(\mu) d\mu = 2(-1)^n q_n(-\gamma)/H(\gamma) \quad (29)$$

which follows from (23), (11) and (20).

5. The Conservative Case

The conservative case, in which $\omega_0 = 1$, $g_1(x) = 0$, $k = 0$, $\gamma = \infty$, $m = 0$, $l = 1$, has been excluded from the previous discussion. It can be solved from scratch or by transition to the limit $k \rightarrow 0$. The only important change is that in the conservative case Eq. (22) is replaced by

$$K(\mu) = \frac{3}{4}\mu + \frac{3}{2} \int_0^1 R(\mu, \nu) \nu^2 d\nu, \quad (30)$$

which upon introduction of (9) and the separation

$$\nu^2/(\mu + \nu) = \nu - \mu + \mu^2/(\mu + \nu)$$

leads to the simple result

$$K(\mu) = \{q_0(\mu) + 2q_2(\mu)\} H(\mu)/4\mu. \quad (31)$$

This shows that the solution of the Milne problem in the conservative case contains only the zero and second-order Busbridge polynomials. This result, although not explicitly pointed out by Busbridge and Orchard (1969), can be checked at once from their formulae. It is also found, with a printing error in the sign, in Sobolev (1968a, Eq. (60)).

The extrapolation length, defined by (26) and (27), degenerates upon the transition $k \rightarrow 0$ into

$$q_\infty = (1 - 1/3 \omega_1)^{-1} \int_0^1 K(\mu) 2\mu^2 d\mu \quad (32)$$

(van de Hulst, 1968a, Eq. (39)). Combining this with Eq. (31) and with the fact that the Busbridge polynomials in the conservative case have a degree $\leq N - 1$ (provided $N > 0$), we find that q_∞ is a linear combination of the moments α_n of $H(\mu)$ of degree $1 \leq n \leq N$, the coefficients of which follow from the coefficients in $q_0(\mu)$ and $q_2(\mu)$.

6. Conclusions

The preceding sections describe the complete reduction of the problems mentioned in Sec. 1 to a H -function and two sets of polynomials. Most of the results rest on known equations, but Eq. (23) fills a gap in the existing astrophysical literature.

The assessment of these equations on their merit for fast and convenient computation is an entirely different problem. The following remarks, though somewhat sketchy, may serve to indicate when these equations can be most useful.

1. Generally, a numerical computation along the lines of this paper would involve the following key steps. Determination of $g_n(x)$ by the recurrence

relation (3), of the characteristic function $\Psi(\mu)$ by (10), and solution of the root γ from $T(\gamma) = 0$. We show elsewhere (van de Hulst, 1970b) that a very convenient alternative is to base the computation of γ on the requirement

$$\lim_{n \rightarrow \infty} g_n(\gamma) = 0. \quad (33)$$

The diffusion pattern follows from (5) and the constant m from (25). Again, alternative equations based on (3), (24) and (33) may be suggested, namely

$$m = 8 \sum_{n=0}^{\infty} (n+1) g_n(\gamma) g_{n+1}(\gamma) = 4\gamma \sum_{n=0}^{\infty} h_n \{g_n(\gamma)\}^2. \quad (34)$$

The fact that both (33) and (34) suppose $g_n(\gamma)$ to be computed for all n , and not only for $n \leq N$, seems to make these equations unattractive for practical use. However, the convergence is so fast that a computational method based on these equations may be preferable over one dealing with finite series, especially if N is not very small.

The methods for finding the H -function have been sufficiently discussed in the literature. A complete method for finding the polynomials $q_n(\mu)$ in the conservative case has been spelled out by Busbridge and Orchard (1969). A method for finding these polynomials in the non-conservative case was partially explained in Busbridge's book (1960, Sec. 48). It is based on the non-linear integral equations for $q_n(\mu)$, i.e. our Eq. (11). Here a really simpler method is to obtain the solution of Eq. (12), which is linear in $q_n(z)$ and can be solved separately for each n . The same method has been recommended by Pahor (1966, 1967) and by Sobolev (1968b). A direct recipe for finding the polynomial occurring in the solution of the Milne problem, Eq. (23), was given by Russman (1965).

2. For small values of N we can retrace the solutions given in the papers cited in the introduction. Employing the notation of Horak and Chandrasekhar (1961) for general non-conservative scattering with $N = 2$ (see Sobolev, 1968b, for somewhat simpler expressions) we have

$$\left. \begin{aligned} q_0(\mu) &= 1 + x\mu + z\mu^2, \quad x = \frac{r+t}{\xi}, \quad z = \frac{v+s}{\xi} \\ q_1(\mu) &= p\mu + q\mu^2 \\ q_2(\mu) &= -\frac{1}{2}q_0(\mu) + 3/2(r\mu + s\mu^2) \end{aligned} \right\} \quad (34)$$

These expressions still hold if we put $\omega_2 = 0$, i.e., go to $N = 1$. The value of $q_2(\mu)$ then becomes

irrelevant and $z = q = 0$. Writing

$$a = \frac{1}{2} \omega_0, \quad b = \frac{1}{2} (1 - \omega_0) \omega_1, \quad (35)$$

and α_n for the moments of the H -functions as usual, we have in this case

$$\left. \begin{aligned} x &= -\frac{b\alpha_1}{1-a\alpha_0} \text{ (called } -c \text{ in Chandrasekhar's} \\ &\quad \text{book Secs. 46 and 96)} \\ p &= \frac{1-2a}{1-a\alpha_0} \text{ (there called } q) \\ \Psi(\mu) &= a + b\mu^2 \\ F(\mu, \nu) &= \omega_0 + x\omega_0(\mu + \nu) - (1 - \omega_0) \omega_1 \mu \nu \\ F(-\gamma, \mu) &= 2 \{ (a - ax\gamma) + (ax + b\gamma) \mu \}. \end{aligned} \right\} (36)$$

In combination with Eq. (9) this reproduces the result found in Chandrasekhar's book Sec. 46, Eq. (49) [which should be multiplied by $\omega_0/4\mu\mu_0$ in order to agree with our $R(\mu, \mu_0)$]. Likewise, with Eq. (23) we reproduce the result in Sec. 96, Eq. (16).

In non-conservative isotropic scattering ($N = 0$) the further reduction $\omega_1 = b = x = 0$ occurs and only one constant term

$$F(\mu, \nu) = 2\Psi(\mu) = \omega_0$$

remains.

3. For larger values of N the reduction to a H -function as a method of computation is not very practical. The theoretical elegance of having to determine only one transcendental function remains. But the job of determining the Busbridge polynomials starts to overshadow that of finding $H(\mu)$. In the limit $N \rightarrow \infty$ even the one theoretical advantage is lost, because also the function $F(\mu, \nu)$, defined by (8), then becomes transcendental.

We recommend that in those cases it is practical to forget the H -function altogether and, instead, to find the solution of all problems mentioned in the introduction by the process of doubling and asymptotic fitting (van de Hulst and Grossman, 1968; van de Hulst, 1968b), whereby accuracies of the order of 10^{-5} can equally well be reached.

4. There must be a close similarity between the actual computation based on the Case-method and that based on the Chandrasekhar-Busbridge-Mullikin-Sobolev equations presented here, although the derivation of these equations is rather different. It seems likely, therefore, that the remarks made above on the computation for large N hold equally well

for the Case-method. This is only a conjecture until detailed comparisons have been made, e.g., with Kaper *et al.* (1970). It should further be kept in mind that the Case-method at once gives the field everywhere, whereas the equations presented in this paper give the emerging intensity and require certain extensions before the radiation field at arbitrary optical depth can be computed.

5. The availability of numerical results based on two fundamentally different methods (via the H -function and by asymptotic fitting) yields an excellent opportunity to check the numerical accuracy by which either method has been executed. The results of this comparison are extremely gratifying. Isotropic scattering with $\omega_0 \geq 0.8$ gave agreement within 10^{-5} (van de Hulst, 1968b). The extrapolation length for anisotropic conservative scattering depends on ω_2 and ω_3 in a manner again reproducible to an accuracy of 10^{-4} (van de Hulst, 1970a). More examples will be cited in a forthcoming book.

In a subject like this, where the same results are often reached along 2, 3 or 4 largely independent lines, by authors from different countries, with a view to different field of application, and in different notations, it is almost impossible to be fair in selecting the references. I wish to acknowledge in particular the stimulating discussions with Dr. H. G. Kaper and Professor Sobolev and the correspondence with Miss Busbridge in the early stage of this study.

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